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Interplays between variations of arbitrarily partitionable graphs under minimality constraints

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Abstract

An arbitrarily partitionable (AP) graph is a graph that can be partitioned into arbitrarily many connected graphs with arbitrary orders. Since independent seminal works by Barth, Baudon, and Puech, and Horňák and Woźniak, AP graphs have been receiving increasing attention in the literature, dedicated to understanding several of their aspects, including structural aspects, algorithmic aspects, and their connections with Hamiltonian graphs. Other aspects of interest cover variants of AP graphs, such as AP graphs that can be partitioned in an online way (OLAP graphs), AP graphs that can be partitioned in a recursive way (RAP graphs), and AP graphs that are edge-minimal (minAP graphs).

In the current work, we initiate the study of the latter notion of minimality for OLAP and RAP graphs. That is, we wonder about OLAP and RAP graphs that are not spanned by any strictly smaller OLAP or RAP graph, respectively, leading to the notions of minOLAP and minRAP graphs. We prove that such non-trivial graphs exist, and explore connections between minAPness, minOLAPness, and minRAPness. In particular, we prove that some fundamental connections between APness, OLAPness, and RAPness do not generalise to their minimal counterparts. We also investigate small minOLAP and minRAP graphs, as well as sufficient conditions guaranteeing an OLAP or RAP graph is not minimal, thereby generalising known results on AP graphs. This work also includes many open questions and problems for further work on the topic, that are disseminated throughout.

Keywords: arbitrarily partitionable graph; partition into connected subgraphs; online arbitrarily partitionable graph; recursively arbitrarily partitionable graph; minimality.

1. Introduction

In this work, we investigate connections and properties of several **variants** of so-called **arbitrarily partitionable graphs**, in particular of those being **minimal** w.r.t. some metric. Before going into the details, we start by recalling the main notions involved.

Let G be an n -graph, *i.e.*, a graph of order n . A subset $S \subseteq V(G)$ of vertices of G is said *connected* if $G[S]$ is indeed a connected graph. Now, an n -partition $\pi = (\lambda_1, \dots, \lambda_k)$ is a partition of n ; that is, $n = \lambda_1 + \dots + \lambda_k$. A *realisation* of π in G is a partition (V_1, \dots, V_k) of $V(G)$ such that $G[V_i]$ is a connected graph of order λ_i for every $i \in \{1, \dots, k\}$. That is, every V_i is a connected part, and the cardinalities of V_1, \dots, V_k meet the values in π . Last, we say G is *arbitrarily partitionable* (AP) if every n -partition is realisable in G .

AP graphs were introduced independently by Barth, Baudon, and Puech in [1] and Horňák and Woźniak in [12] to model a practical network sharing problem. By definition, AP graphs are part of the more general field of connected partitions of graphs, and, as such, have received some attention in the literature. In particular, structural and algorithmic

aspects of AP graphs have been studied, as well as their relationship with Hamiltonian graphs (see below). Since surveying the whole field would be rather lengthy, we instead refer the reader to the references in [7], for the issues not given in what follows.

Among other aspects of interest, a few variants of AP graphs have also been considered in the literature. To introduce the first of the two variants to be considered in this work, we need to define what a picking process is. Let G be an n -graph. In a *picking process* (in G), several rounds take place, during each of which a connected subgraph with given order of G must be picked, so that, eventually, once the process ends (assuming it does end properly), a realisation of some n -partition is obtained. Formally, each round, denoting by R the subgraph of G induced by the vertices that have not been picked earlier ($R = G$ in the first round), an integer $\lambda \in \{1, \dots, |V(R)|\}$ is given, and one must pick a connected subgraph of R with order λ . Depending on the λ 's provided each round, and on the connected parts picked, a picking process might end up properly or not (*i.e.*, at some point, it might become impossible to pick the requested connected part, for instance if R is not connected and $\lambda = |V(R)|$). In case any picking process, regardless of the successive λ 's, can be led properly in G , we say G is *online arbitrarily partitionable* (OLAP).

The second variant of AP graphs to be considered in this work, are the recursive ones. For an n -graph G and an n -partition $\pi = (\lambda_1, \dots, \lambda_k)$, a *RAP-realisation* of π in G is a realisation (V_1, \dots, V_k) such that each $G[V_i]$ is not connected only, but also can be recursively partitioned at will (see below for a thorough definition). We say G is *recursively arbitrarily partitionable* (RAP) if every n -partition admits a RAP-realisation in G .

To be more formal, OLAP and RAP graphs can be defined recursively. Namely, as a base case, the 1-vertex graph is both OLAP and RAP. Now, regarding any n -graph G :

- G is OLAP if and only if, for every $\lambda \in \{1, \dots, n\}$, there is a connected part $S \subseteq V(G)$ of cardinality λ such that $G - S$ is OLAP;
- G is RAP if and only if, for every n -partition $\pi = (\lambda_1, \dots, \lambda_k)$, there is a RAP-realisation (V_1, \dots, V_k) of π in G ; that is, every V_i is a *RAP part* of size λ_i .

Due to the definitions involved, it is not too hard to see that AP graphs, OLAP graphs, and RAP graphs are quite related, as every RAP graph is OLAP, and every OLAP graph is AP. It is worth mentioning, however, that these three variants of AP graphs are rather different, as there exist OLAP graphs that are not RAP, and similarly there exist AP graphs that are not OLAP (examples will be provided in Section 2). It is worth mentioning also that AP graphs, OLAP graphs, and RAP graphs are comprised in between two important classes of graphs, namely graphs admitting (quasi-) perfect matchings and traceable/Hamiltonian graphs. Indeed, it can be checked from the definitions that any AP n -graph must admit a (quasi-) perfect matching (*i.e.*, a set of $\lfloor \frac{n}{2} \rfloor$ independent edges), while any traceable graph (*i.e.*, having a Hamiltonian path, traversing all vertices exactly once) is AP. And, again, it is not too complicated to come up with examples of AP graphs that are not traceable, and of graphs admitting (quasi-) perfect matchings that are not AP. Thus, if, for some $n \geq 1$, we denote by $\text{PM}(n)$, $\text{AP}(n)$, $\text{OLAP}(n)$, $\text{RAP}(n)$, and $\text{TRACE}(n)$ the sets of all n -graphs having a (quasi-) perfect matching, being AP, being OLAP, being RAP, and being traceable, respectively, then the following holds:

Theorem 1.1 (Baudon, Gilbert, Woźniak [4]). *For general values of n ,*

$$\text{TRACE}(n) \not\subseteq \text{RAP}(n) \not\subseteq \text{OLAP}(n) \not\subseteq \text{AP}(n) \not\subseteq \text{PM}(n).$$

Among other properties of interest, it is also worth mentioning that all RAP graphs, OLAP graphs, and AP graphs are obviously connected. Also, adding edges to a RAP

graph, OLAP graph, or AP graph results clearly in a RAP graph, OLAP graph, or AP graph, respectively. In other words, RAPness, OLAPness, and APness are closed under adding edges. Towards understanding APness better, quite a lot of the initial efforts were thus focused on understanding AP trees, as one could legitimately assume that, perhaps, every AP graph is spanned by an AP tree. This presumption was proved wrong, however, as Ravaux came up in [20] with an AP 20-graph that is not a tree, and that is not spanned by any AP graph with fewer edges. This led to the definition of AP graphs that are minimal, which we call *minAP* throughout, in the sense that they are not spanned by any non-trivial AP graph (*i.e.*, by an AP graph with strictly less edges). Later on, Baudon, Przybyło, and Woźniak, in [5], came up with a construction showing that a minAP n -graph can have its number of edges being arbitrarily larger than $n - 1$, the size of a tree on n vertices. Still, it is believed that minAP n -graphs should be somewhat sparse, meeting a conjecture of Ravaux from [20] stating they should have linear size (of order $\mathcal{O}(n)$). A few more properties of minAP graphs have later been investigated by Bensmail in [7], including their maximum degree and their clique number.

In this work, we initiate the study of the latter notion of minimality for OLAP and RAP graphs. Generalising minAPness to OLAPness and RAPness in the obvious way, we say a graph G is *minOLAP* if G is OLAP and $G - e$ is not OLAP for every edge e of G , while G is said *minRAP* if G is RAP and $G - e$ is not RAP for every edge e of G . Obviously, minOLAP and minRAP graphs exist, as every OLAP tree is clearly minOLAP while every RAP tree is also minRAP. However, similarly as for minAPness, OLAP and RAP trees are not that interesting to consider in this context, as their OLAP and RAP minimality follow trivially from their low connectivity, and not quite from their partition properties. Mainly for this reason, similarly as for minAPness, our concerns regarding minOLAPness and minRAPness will mostly focus on graphs that are not trees.

This work is organised as follows. In Section 2, we start by recalling a few facts from the literature on AP, OLAP, and RAP graphs, and raise a few early observations and remarks on minOLAP and minRAP ones. In Section 3, we then build upon results of Bensmail on small minAP graphs from [7], exhibiting minOLAP and minRAP graphs on at most 10 vertices, and establishing some of their properties. Then, in Section 4, we investigate connections between minAPness, minOLAPness, and minRAPness. In particular, we establish that there are infinitely many non-tree minOLAP graphs that are not minAP, and that there are non-tree minRAP graphs that are not minOLAP, thereby showing that the hierarchy in Theorem 1.1 does not generalise in the obvious way in the context of minAP, minOLAP, and minRAP graphs. Next, in Section 5, we adapt a proof of a result from the literature on OLAP graphs to provide a sufficient condition for a graph to be RAP, and actually RAP but not minRAP. We finish off in Section 6 with open questions and problems for further work on this topic.

2. Previous results and early observations

We start by recalling some existing results on AP, OLAP, and RAP graphs, which mainly deal with important classes of graphs in the field.

We consider trees first. It turns out that AP trees are rather restricted, see [2]. In particular, all AP trees have maximum degree at most 4, can have arbitrarily many degree-3 vertices and degree-4 vertices, and satisfy peculiar structural properties (for instance, every degree-4 vertex must be adjacent to a leaf). In contrast, the structure of OLAP and RAP trees is even more restricted. In particular, these trees have maximum degree at most 3 and can have at most one degree-3 vertex, as they are mostly particular caterpillars and

a	b
2, 4	$\equiv 1 \pmod 2$
3	$\equiv 1, 2 \pmod 3$
5	$\in \{6, 7, 9, 11, 14, 19\}$
6	$\equiv 1, 5 \pmod 6$
7	$\in \{8, 9, 11, 13, 15\}$
8	$\in \{11, 19\}$
9, 10	$= 11$
11	$= 12$

(a) OLAP caterpillars

a	b
2, 4	$\equiv 1 \pmod 2$
3	$\equiv 1, 2 \pmod 3$
5	$\in \{6, 7, 9, 11, 14, 19\}$
6	$= 7$
7	$\in \{8, 9, 11, 13, 15\}$

(b) RAP caterpillars

Table 1: Exhaustive list of all OLAP (a) and RAP (b) caterpillars $\text{Cat}(a, b)$, where $a \leq b$.

so-called tripodes. Namely, throughout this work, for any two $a, b \geq 2$, we denote by $\text{Cat}(a, b)$ the *caterpillar* on $a + b$ vertices consisting of a main path $v_1 \dots v_{a+b-1}$ and of a unique pendant vertex outside this path, adjacent to v_a ; and, for any $k \geq 3$ and integers $a_1, \dots, a_k \geq 1$, we denote by $\text{Pode}(a_1, \dots, a_k)$ the *k-pode* (or more generally *multipode*) obtained from a star with k edges by considering each $i \in \{1, \dots, k\}$ and subdividing one of the original edges $a_i - 1$ times. In other words, $\text{Pode}(a_1, \dots, a_k)$ has one (degree- k) center vertex to which are attached k pendant paths of length a_1, \dots, a_k , respectively. Any caterpillar $\text{Cat}(a, b)$ is actually the 3-pode (or *tripode*) $\text{Pode}(1, a - 1, b - 1)$.

As suggested earlier, already in caterpillars (as just defined above), and more generally in trees, there is a neat difference between APness, OLAPness, and RAPness. In particular:

Theorem 2.1 (Barth, Baudon, Fournier [1], and Horňák, Woźniak [12]). *A caterpillar $\text{Cat}(a, b)$ is AP if and only if a and b are coprime.*

Theorem 2.2 (Horňák, Tuza, Woźniak [11]). *A tree is OLAP if and only if it is a path, a caterpillar $\text{Cat}(a, b)$ with a and b given in Table 1 (a), or the tripodode $\text{Pode}(2, 4, 6)$.*

Theorem 2.3 (Baudon, Gilbert, Woźniak [4]). *A tree is RAP if and only if it is a path, a caterpillar $\text{Cat}(a, b)$ with a and b given in Table 1 (b), or the tripodode $\text{Pode}(2, 4, 6)$.*

We now turn to *cyclic graphs*, *i.e.*, graphs having cycles, and more particularly to *unicyclic graphs*, *i.e.*, graphs having only one cycle. Following the previous studies on AP, OLAP, and RAP trees, a few works have then been dedicated to so-called suns, being unicyclic graphs having a unique cycle dominating degree-1 vertices. More formally, for any $k \geq 1$ and k integers $a_1, \dots, a_k \geq 0$, the *k-sun* $\text{Sun}(a_1, \dots, a_k)$ is obtained from k independent edges $v_0 v'_0, \dots, v_{k-1} v'_{k-1}$ by considering every $i \in \{0, \dots, k - 1\}$ and joining v_i and $v_{(i+1) \bmod k}$ by a new path with a_i inner vertices (which path thus has length $a_i + 1$).

Some results on AP suns can be found in [15]. In the current work, we are more particularly interested in OLAP and RAP suns, which have been characterised (Theorems 2.4 and 2.5 below). Note that 1-suns are traceable, and thus AP, OLAP, and RAP.

Theorem 2.4 (Kalinowski, Piłśniak, Woźniak, Ziolo [16]).

- A 2-sun $\text{Sun}(a, b)$ is OLAP if and only if a and b are as given in Table 2 (a).
- A 3-sun $\text{Sun}(a, b, c)$ is OLAP if and only if a, b , and c are as given in Table 2 (b).

a	b
0	anything
1, 3	$\equiv 0 \pmod 2$
2	$\not\equiv 3 \pmod 6$, or $\in \{3, 9, 21\}$
4	$\equiv 2, 4 \pmod 6$, or $\in \{4, \dots, 19\} \setminus \{15\}$
5	$\equiv 2, 4 \pmod 6$, or $\in \{6, 18\}$
6	$\in \{6, 7, 8, 10, 11, 12, 14, 16\}$
7	$\in \{8, 10, 12, 14, 16\}$
8	$\in \{8, 9, 10, 11, 12\}$
9	$\in \{10, 12\}$

(a) 2-suns

a	b	c
0	0	$\equiv 1, 2 \pmod 3$
0	1	$\equiv 0 \pmod 2$
0	2	$\equiv 2, 4 \pmod 6$, or $\in \{3, 6, 7, 11, 18, 19\}$
0	3	$\equiv 2, 4 \pmod 6$
0	4	$\in \{4, 5, 6, 8, 10, 11, 12, 14, 16\}$
0	5	$\in \{6, 8, 16\}$
0	6, 7	$\in \{8, 10\}$
0	8	$\in \{8, 9\}$
1	2	$\equiv 2, 4 \pmod 6$, or $\in \{6, 18\}$
2	3	$\in \{4, 8, 16\}$

(b) 3-suns

Table 2: Exhaustive list of all OLAP 2-suns $\text{Sun}(a, b)$ (a) and 3-suns $\text{Sun}(a, b, c)$ (b), where $a \leq b \leq c$.

a	b
0	anything
1	$\equiv 0 \pmod 2$
2	$\not\equiv 0 \pmod 3$, or $\in \{3, 6, 9, 12, 18, 21, 24, 36\}$
3	$\equiv 0 \pmod 2$
4	$\equiv 2, 4 \pmod 6$ and $\in \{20, \dots, 46\}$, or $\in \{4, \dots, 19\} \setminus \{15\}$
5	$\equiv 2, 4 \pmod 6$ and $\in \{8, \dots, 32\}$, or $\in \{6, 18\}$
6	$\in \{6, 7, 8, 10, 11, 12, 14, 16\}$

(a) 2-suns

a	b	c
0	0	$\equiv 1, 2 \pmod 3$
0	1	$\equiv 0 \pmod 2$
0	2	$\in \{2, 3, 4, 6, 7, 8, 10, 11, 14, 16, 18, 19\}$
0	3	$\in \{4, 8, 10\}$
0	4	$\in \{4, 5, 6, 8, 10, 11, 12, 14, 16\}$
0	5	$= 6$
1	2	$\in \{2, 4, 6, 8, 10, 14, 16, 18\}$
2	3	$= 4$

(b) 3-suns

Table 3: Exhaustive list of all RAP 2-suns $\text{Sun}(a, b)$ (a) and 3-suns $\text{Sun}(a, b, c)$ (b), where $a \leq b \leq c$.

- A 4-sun $\text{Sun}(a, b, c, d)$ is OLAP if and only if $a = b = 0$, $c = 1$, and $d \equiv 2, 4 \pmod 6$.
- A k -sun with $k \geq 5$ is never OLAP.

Theorem 2.5 (Baudon, Gilbert, Woźniak [3]).

- A 2-sun $\text{Sun}(a, b)$ is RAP if and only if a and b are as given in Table 3 (a).
- A 3-sun $\text{Sun}(a, b, c)$ is RAP if and only if a , b , and c are as given in Table 3 (b).
- A 4-sun $\text{Sun}(a, b, c, d)$ is RAP if and only if $(a, b, c, d) \in \{(0, 0, 1, 2), (0, 0, 1, 4)\}$.
- A k -sun with $k \geq 5$ is never RAP.

We finish off with a few remarks and results on AP, OLAP, and RAP graphs. First, when it comes to determining whether a given graph is RAP, the next result is handy.

Theorem 2.6 (Baudon, Gilbert, Woźniak [3]). *An n -graph G is RAP if and only if, for every $\lambda \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$, there is a subset $S \subset V(G)$ of vertices of G such that:*

- $|S| = \lambda$,
- $G[S]$ is RAP, and

- $G - S$ is RAP.

By all definitions involved, any RAP graph is also OLAP, and any OLAP graph is also AP (recall Theorem 1.1). Also, adding edges to a RAP, OLAP, or AP graph results in a RAP, OLAP, or AP graph, respectively. The other way round, we also have:

Observation 2.7. *Every OLAP minAP graph is also minOLAP. Likewise, every RAP minOLAP graph is also minRAP. Conversely, if a RAP graph is not minRAP, then it is not minOLAP, and if an OLAP graph is not minOLAP, then it is not minAP.*

Proof. This is because OLAPness implies APness and RAPness implies OLAPness, by Theorem 1.1. Thus, if an OLAP minAP graph G is not minOLAP, then G is spanned by a strictly smaller AP graph, a contradiction. Likewise, if a RAP minOLAP graph G is not minRAP, then G is spanned by a strictly smaller OLAP graph, another contradiction. The last part of the statement is the contraposition of the first part. \square

However, since RAPness is stronger than OLAPness, and OLAPness is stronger than APness, intuitively it is not clear whether the converse of Observation 2.7 also holds. Actually, as will be proved later on in Section 4, this is not the case. That is, there exist minRAP graphs that, although necessarily OLAP, are not minOLAP, and similarly there exist minOLAP graphs that, although necessarily AP, are not minAP.

3. Small minOLAP and minRAP graphs

In this section, we aim at exhibiting small minOLAP and minRAP graphs, that is, with at most 10 vertices, so that we can point out some of their properties. For that, we build upon the work of Bensmail [7], in which was exhibited the exhaustive list of all minAP graphs with order at most 10. In particular, cyclic minAP graphs have order at least 9. The only two cyclic minAP 9-graphs are depicted in Figure 2, while the only fourteen cyclic minAP 10-graphs are given in Figure 1¹. Caterpillars and paths apart, let us mention that there also exist other AP 7-trees and AP 9-trees, depicted in Figure 3. However, since such trees are not OLAP by Theorem 2.2, we do not deal with them in what follows.

We will not go through all graphs in Figures 1 and 2 in what follows as it would be a bit tedious to the reader (however, for completeness, formal proofs can be found in an appendix section). Instead, in what follows we illustrate how we can check whether a graph is OLAP/RAP or not, so that the reader gets an idea of the possible arguments, before listing the status of all graphs in Figure 1 w.r.t. OLAPness and RAPness.

Observation 3.1. *The graph in Figure 2 (a) is not OLAP, and thus not RAP.*

Proof. Let G denote the graph in Figure 2 (a). We deal with the vertices of G using the terminology from the figure. Recall that $|V(G)| = 9$. Consider a picking process where, on the first round, we are asked to pick a connected part S of size 1. The only options, leaving a connected graph, are to consider the parts $\{a\}$, $\{i\}$, $\{h\}$, and $\{f\}$. For S being either $\{a\}$ or $\{i\}$, it can be noted that $G - S$ admits no perfect matching, and thus, from the second round, the picking process cannot be led properly if we are repeatedly been asked to pick connected parts of size 2. Likewise, if, on the first round, we pick any of the

¹Be aware that the graph (m) exhibited in [7] is actually not AP, since it admits no perfect matching. This is due to a drawing flaw, which we have corrected in Figure 1 (m). It can be checked that this corrected graph is indeed minAP, as desired.

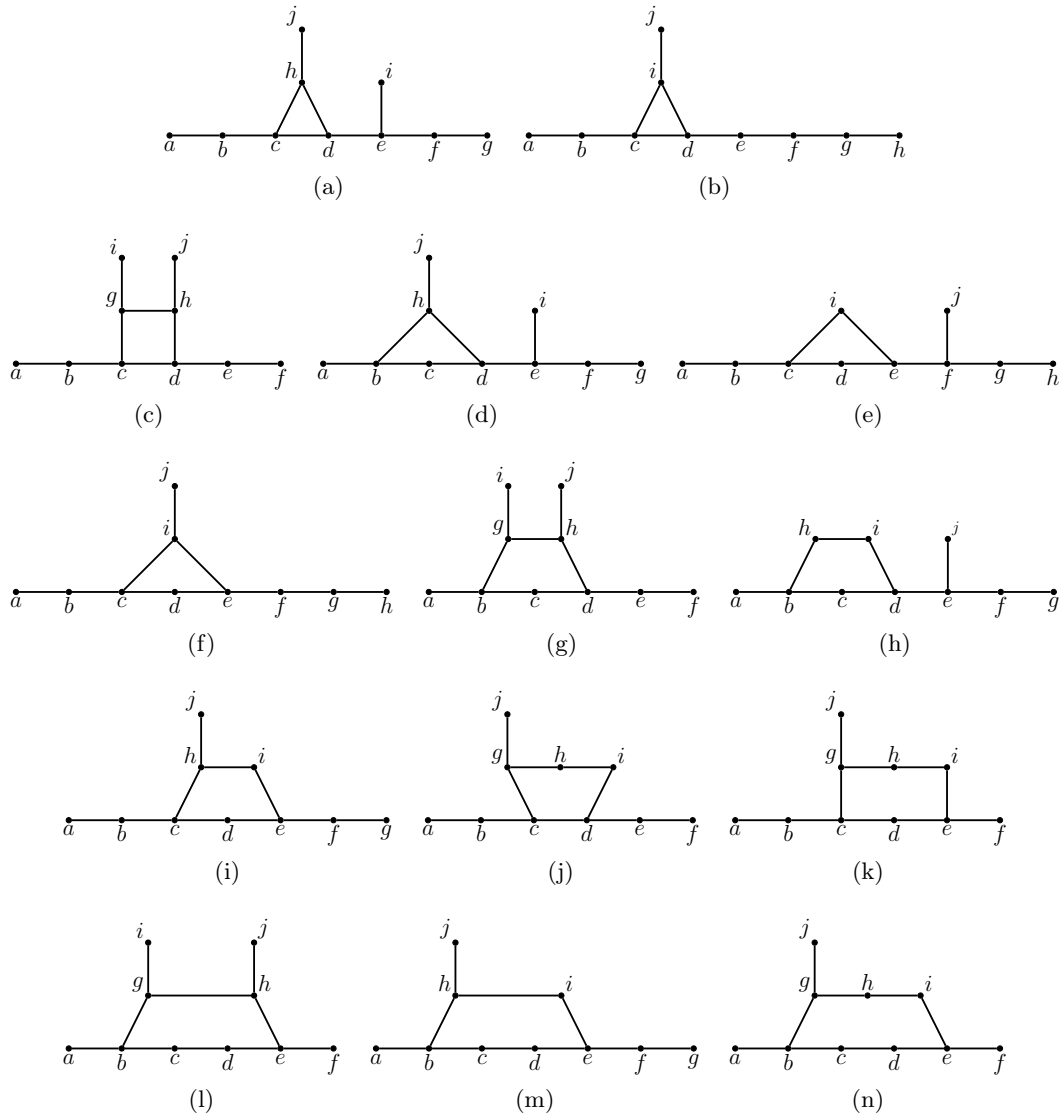


Figure 1: The fourteen cyclic minAP 10-graphs.

sets $\{h\}$ and $\{f\}$ as S , then we note that, from the second round on, $G - S$ is a graph that admits no realisation of $(4, 4)$, and thus the picking process cannot be led properly if we are repeatedly asked to pick connected parts of size 4. Thus, G is not OLAP (and hence not RAP by Theorem 1.1). \square

Observation 3.2. *The graph in Figure 2 (b) is RAP, and thus OLAP.*

Proof. Let G be the graph depicted in Figure 2 (b). We deal with the elements of G through the terminology given in the figure. Again, remember that $|V(G)| = 9$. Thus, according to Theorem 2.6, in order to prove that G is RAP it suffices to show that, for every $\lambda \in \{1, \dots, 4\}$, there is a RAP part S of size λ such that $G - S$ is RAP.

- For $\lambda = 1$, we consider $S = \{h\}$. Then $G[S]$ is the 1-graph, which is RAP, while $G - S$ is spanned by $\text{Cat}(3, 5)$, which is RAP by Theorem 2.3.
- For $\lambda = 2$, we consider $S = \{g, i\}$. Then $G[S]$ is a path, which is RAP, while $G - S$ is spanned by $\text{Cat}(2, 5)$, which is RAP by Theorem 2.3.

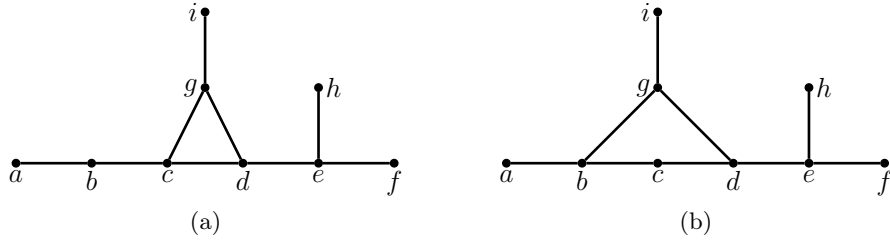


Figure 2: The only two cyclic minAP 9-graphs.

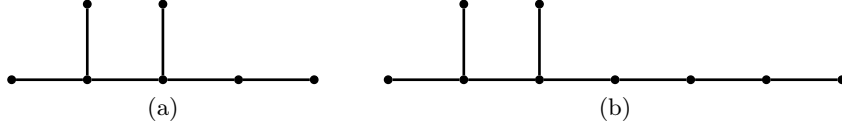


Figure 3: Two AP trees.

- For $\lambda = 3$, we consider $S = \{h, e, f\}$. Then $G[S]$ is a path, which is RAP, while $G - S$ is traceable, and thus RAP by Theorem 1.1.
- For $\lambda = 4$, we consider $S = \{a, b, g, i\}$. Then $G[S]$ is a path, which is RAP, while $G - S$ is spanned by $\text{Cat}(2, 3)$, which is RAP by Theorem 2.3.

Thus, G is RAP (and thus OLAP by Theorem 1.1), as claimed. \square

Repeating such arguments with all graphs in Figure 1, and taking a look at Theorems 2.2 and 2.3, we can refine the exhaustive list from [7] of all minAP graphs on at most 10 vertices to determine their minOLAPness and minRAPness (the interested reader is referred to the appendix section for thorough details). In particular, recall that a minAP graph that is OLAP is always minOLAP, and that a minAP graph that is RAP is always minRAP, by Observation 2.7. Also, keep in mind that every AP tree is minAP, and be aware that one needs to invoke Observation 3.2 in order to show that the graph in Figure 1 (d) is RAP. Below, for any $n \geq 1$, we denote by P_n the path of order n .

Theorem 3.3.

- For every $n \in \{3, 4\}$, the only minAP n -graph is the n -path P_n , which is both minOLAP and minRAP.
- For $n = 5$, the only two minAP n -graphs are the caterpillar $\text{Cat}(2, 3)$ and the 5-path P_5 , which are both minOLAP and minRAP.
- For $n = 6$, the only minAP n -graph is the 6-path P_6 , which is both minOLAP and minRAP.
- For $n = 7$, there are four minAP n -graphs, being the caterpillar $\text{Cat}(2, 5)$, the caterpillar $\text{Cat}(3, 4)$, the 7-path P_7 , and the tree in Figure 3 (a). The former three graphs are both minOLAP and minRAP, while the latter one is neither OLAP nor RAP.
- For $n = 8$, the only two minAP n -graphs are the caterpillar $\text{Cat}(3, 5)$ and the 8-path P_8 , which are both minOLAP and minRAP.
- For $n = 9$, there are six minAP n -graphs, being the caterpillar $\text{Cat}(2, 7)$, the caterpillar $\text{Cat}(4, 5)$, the 9-path P_9 , the tree depicted in Figure 3 (b), and the two 9-graphs depicted in Figure 2. The former three graphs and the graph in Figure 2 (b) are both minOLAP and minRAP, while the other two graphs are neither OLAP nor RAP.

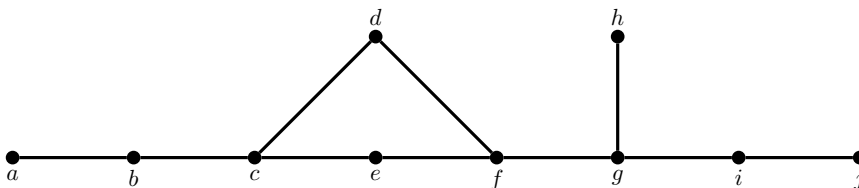


Figure 4: A minAP 10-graph.

- For $n = 10$, there are sixteen minAP n -graphs, being the caterpillar $\text{Cat}(3, 7)$, the 10-path P_{10} , and the fourteen 10-graphs depicted in Figure 1. The former two graphs are both minOLAP and minRAP, and similarly for all graphs in Figures 1 (a), (b), (c), (d), (f), (g), (h), (i), (j), (k), (m), and (n). The other two graphs, in Figure 1 (e) and (l), are neither OLAP nor RAP.

Regarding the list in Theorem 3.3, a few things are worth observing. First, the smallest graph that is AP only (*i.e.*, that is not also OLAP) is the 7-tree depicted in Figure 3 (a). Second, observe that the smallest cyclic minOLAP and minRAP graphs of the list have order 9, which meets the minimum order of a cyclic minAP graph. Third, in this sample of small graphs, every OLAP graph is also RAP, and there is actually an equivalence between minOLAPness and minRAPness. There also exist minAP graphs that are not OLAP. As will be established in the upcoming section, this phenomenon is not pathological.

Let us conclude by recalling that, though it perhaps does, we have not established that the list in Theorem 3.3 contains all minOLAP and minRAP graphs on at most 10 vertices. Indeed, recall that this list was established from the full list of all minAP graphs on at most 10 vertices, but, as will be seen later on, it is not true that all minOLAP and minRAP graphs are necessarily minAP. Thus, it would be daring to consider that list exhaustive.

4. Cyclic minAP, minOLAP, and minRAP graphs and their connections

As mentioned earlier, the notions of minAP, minOLAP, and minRAP graphs make more sense when considered in cyclic graphs. Through Theorem 3.3, we noticed that there exist small cyclic minAP graphs that are not OLAP, and that, in the restricted universe of cyclic graphs on at most 10 vertices, minOLAPness and minRAPness might be equivalent notions. Our goal in this section is to understand these relationships better. A first result we establish, is that the former phenomenon generalises to arbitrarily large graphs. As a second result, we then prove that the latter phenomenon is not true in general, since there exist arbitrarily large cyclic minOLAP graphs that are not RAP.

We then observe two phenomena we could not observe through restricted Theorem 3.3, focusing on the notions of minimality, that sort of concern the other directions. Namely, we prove that there exist arbitrarily large cyclic minOLAP graphs that, although necessarily AP, are not minAP, and that there exist cyclic minRAP graphs that, although necessarily OLAP, are not minOLAP.

4.1. Non-OLAP minAP graphs, and non-RAP minOLAP graphs

Our first result here is thus:

Theorem 4.1. *There exist arbitrarily large cyclic minAP graphs that are not OLAP.*

Proof. This follows from a generalisation of the graph H depicted in Figure 4, which was proved to be minAP in [7] (this graph is actually the one depicted in Figure 1 (e)). In

what follows, we deal with the vertices of H , and of part of every graph constructed from H , using the terminology displayed in the figure.

The generalisation of H is the following. For any $k \geq 0$, we denote by G_k the graph obtained from H by adding a disjoint path $v_1 \dots v_{15k}$ of order $15k$, and adding the edge $v_{15k}a$. In other words, we just prolong the pendant path attached at c so that it contains $15k$ more vertices. Note that $G_0 = H$, and that every G_i is cyclic.

We claim that any G_i with $i \geq 0$ is minAP and not OLAP. To establish this formally, by all definitions involved we need to prove three things:

- We first prove that any G_i , for some $i \geq 0$, is AP. Assume $i \geq 0$ is fixed, and set $G = G_i$ and $n = |V(G)| = 10 + 15i$. Note that $n \equiv 0 \pmod{5}$ and $n \equiv 1 \pmod{3}$. Consider now any n -partition π . We need to prove that π is realisable in G , which we do by considering the possible values λ that π can contain.
 - If π contains some value $\lambda \geq 6$, then, as a connected part of size λ , one can consider the set S containing $\{j, i, g, h, f, d\}$ and the first $\lambda - 6$ vertices of $(e, c, b, a, v_{15k}, \dots, v_1)$. Note that $G - S$ is a path, in which the remaining connected parts of π can be picked to form, with S , a realisation of π in G .
 - If $4 \in \pi$, then, as a connected part of size 4, one can consider $S = \{j, i, g, h\}$. Note then that $G - S$ is traceable. Thus, we can pick the remaining connected parts of π in $G - S$, so that, together with S , we get a realisation of π in G .
 - By the same arguments, if π contains value 2 twice, then one can consider $\{j, i\}$ and $\{g, h\}$ as two connected parts of size 2, and then deduce the rest of a realisation of π in the rest of the graph. Likewise, if π contains value 1 twice, then one can consider the two connected parts $\{d\}$ and $\{h\}$ of size 1, while, if π contains value 5 twice, then one can consider the two connected parts $\{j, i, g, h, f\}$ and $\{d, e, c, b, a\}$ of size 5.
 - Similarly, if $1 \in \pi$ and $2 \in \pi$, then one can consider the two connected parts $\{d\}$ and $\{i, j\}$, which, when removed from G , yield a traceable graph. Likewise, if $1 \in \pi$ and $5 \in \pi$, then one can consider the connected parts $\{d\}$ and $\{j, i, g, h, f\}$. Last, if $2 \in \pi$ and $5 \in \pi$, then one can consider $\{j, i\}$ and $\{g, h, f, e, d\}$.
 - It remains to consider cases where π does not contain 4 or any value at least 6, and at most one value of π lies in $\{1, 2, 5\}$. Thus, apart from at most one value in $\{1, 2, 5\}$, we get to the point where π can be assumed to contain 3's only (at least one since $n \geq 10$). Actually, since $n \equiv 1 \pmod{3}$, note that π must be of the form $(1, 3, \dots, 3)$. Then, when removing the connected parts $\{h\}$ and $\{j, i, g\}$ from G , the rest of the graph is traceable and thus we can deduce the other connected parts of size 3 to get a realisation of π in G .

In all cases, we thus deduce a realisation of π in G , and G is AP.

- We now prove that any G_i with $i \geq 0$ is actually minAP. Assume $i \geq 0$ is fixed, and set $G = G_i$ and $n = |V(G)| = 10 + 15i$. Recall that $n \equiv 0 \pmod{5}$ and $n \equiv 1 \pmod{3}$. Note that the only four edges which, when removed from G , leave a connected graph, are cd , ce , df , and ef . Actually, note that $G - cd$ and $G - ce$ are isomorphic, and similarly for $G - df$ and $G - ef$. Thus, to prove that G is minimal for APness, it suffices to show that $G - cd$ and $G - df$ are not AP.
 - Regarding $G - cd$, we claim it admits no realisation of $(5, \dots, 5)$. Indeed, such a realisation would have to contain the connected part $\{j, i, g, h, f\}$, which, when

- removed from $G - cd$, contains two connected components, one of which contains d only. From this, we get that $G - cd$ cannot be partitioned following $(5, \dots, 5)$.
- Regarding $G - df$, we claim it cannot be partitioned following $(1, 3, \dots, 3)$. Indeed, in a realisation of $(1, 3, \dots, 3)$ in $G - df$, either $\{j, i, g\}$ and $\{h\}$ would have to be connected parts, or $\{j\}$ and $\{i, g, h\}$ would have to be connected parts. Both ways, the rest of the realisation would stand as a realisation of $(3, \dots, 3)$ in $G - df - \{j, i, g, h\}$, which is clearly impossible since, in this graph, removing the connected part $\{f, e, c\}$ yields a non-connected graph in which d is isolated. Thus, $G - df$ admits no realisation of $(1, 3, \dots, 3)$.

Thus, removing any edge from G makes it lose its APness, and G is minAP.

- It now remains to prove that any G_i with $i \geq 0$ is not OLAP. We prove this by induction on i . Regarding G_0 , this follows from the fact that if, during the first round of a picking process, we are asked to pick a connected part of size 3, then we must consider connected parts leaving the graph connected (as, otherwise, during the second round it would be impossible to pick a connected part of size $|V(G_0)| - 3 = 7$), the only option being to pick $\{a, b, c\}$. Note that this option yields a tree with two degree-3 vertices, which is thus not OLAP (by Theorem 2.2). Thus, G_0 is not OLAP. Now assume we have proved G_x is not OLAP for all $x \geq 0$ up to some value $i - 1$, and now consider G_i . By similar arguments as in the base case, note that, if, during the first five rounds of a picking process, we are asked to pick connected parts of size 3 in G_i , then one must consider $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6\}$, $\{v_7, v_8, v_9\}$, $\{v_{10}, v_{11}, v_{12}\}$, and $\{v_{13}, v_{14}, v_{15}\}$. Then, at the beginning of the sixth round, what remains of G_i is necessarily G_{i-1} , which is not OLAP by the induction hypothesis. Thus, G_i itself is not OLAP, as claimed.

Thus, any G_i is cyclic, minAP, and not OLAP, as claimed. \square

Our second result reads as follows:

Theorem 4.2. *There exist arbitrarily large cyclic minOLAP graphs that are not RAP.*

Proof. Consider the following general construction. Start from H , any (cyclic or not) OLAP graph that is not RAP (assuming, for now, that such a graph exists). Set $n = |V(H)|$. Now add to the graph a new path $v_1 \dots v_n$ of order n , and join v_1 and every vertex of H . We denote by G the resulting graph of order $2n$.

First, we claim that G is not RAP. This follows from the fact that, because v_1 is a cut-vertex whose removal yields a connected component of order $n - 1$, the only partition of G into two connected parts of size n is that having parts $V(H)$ and $\{v_1, \dots, v_n\}$. However, recall that $G[V(H)] = H$ is not RAP, by the properties of H . Thus, by Theorem 2.6, we get that G admits no RAP-realisation of (n, n) , and hence G is not RAP.

On the other hand, we claim that G is OLAP. Indeed, we claim we can lead a picking process properly from start to finish following the following strategy. Essentially, we pick connected parts in $G[V(H)]$ the same way we would pick them in H , until we reach the point where the connected part to be picked exceeds $G[V(H)]$, at which moment we then repeatedly pick consecutive vertices of (v_1, \dots, v_n) . Formally, at the beginning of a new round, let us denote by R the subgraph of $G[V(H)]$ induced by the vertices that have not been picked in connected parts during the previous rounds (initially, $R = G[V(H)]$). By the way we lead the picking procedure, we assume R is OLAP. Clearly, this is true at the beginning of the first round, since, then, R is isomorphic to H , which is OLAP. Assume now that, in a new round, we are asked to pick a connected part of size λ .

- If $|V(R)| > 0$ and $\lambda \geq |V(R)|$, then we pick all vertices of R together with the first $\lambda - |V(R)|$ vertices of (v_1, \dots, v_n) , which induce a connected graph (since v_1 is joined to all vertices of R), and leaves a path for the next rounds.
- If $|V(R)| > 0$ and $\lambda < |V(R)|$, then we pick a connected part of size λ the same way we would pick it in R , that is, so that what remains of R is OLAP (and thus connected).
- If $|V(R)| = 0$, then we pick the first λ remaining vertices in (v_1, \dots, v_n) .

Clearly, these steps allow to lead the process from start to end, regardless of the successive λ 's, and we deduce that G is indeed OLAP, as claimed.

We now prove that, while G might not be minOLAP itself, it must contain a cyclic minOLAP subgraph, provided $2n$ fulfils particular properties. Note that since G is not RAP, none of its spanning subgraphs can be RAP. If G is minOLAP, then we have our conclusion, since H is connected and contains at least two vertices (as otherwise H would be RAP), implying that v_1 is contained in at least one cycle. Thus, we can assume now G is not minOLAP. If G is spanned by a cyclic minOLAP graph, then we have our conclusion as well. Otherwise, all OLAP spanning subgraphs of G are trees.

Let us analyse any OLAP spanning tree G' of G . According to Theorem 2.2, G' is either a path (meaning G is traceable) or a tripod (caterpillar or not).

- Regarding the former case, if G' is a path, then G contains a Hamiltonian path P . Due to the structure of G , in particular the fact that v_1 is a cut-vertex, P must start with the sequence (v_n, \dots, v_1) , which extends to a Hamiltonian path of (a subgraph of) $G[V(H)]$, implying H is traceable. This is a contradiction, however, to the fact that H is not RAP. Thus, G' cannot be a path.
- Let us now focus on the latter case. Paying attention to the list of OLAP tripodes in Theorem 2.2, it can be observed that if $2n$ is large enough, then the only options for G' are to be a caterpillar of the form $\text{Cat}(2, a)$, $\text{Cat}(4, a)$, $\text{Cat}(3, b)$, or $\text{Cat}(6, c)$, where $a \equiv 1 \pmod{2}$, $b \equiv 1, 2 \pmod{3}$, and $c \equiv 1, 5 \pmod{6}$. That is, it must be that $2n \equiv 1 \pmod{2}$, $2n \equiv 1, 2 \pmod{3}$, or $2n \equiv 1, 5 \pmod{6}$. The other way round, this means that if $2n$ is large enough, and $2n$ does not fulfil any of these properties, then G' cannot be an OLAP tripod.

A consequence of these arguments is that if $2n$ is large enough, and satisfies some rest properties modulo 2, 3, and 6, then G cannot be spanned by an OLAP tree, implying that, if G is not minOLAP, then it is spanned by a cyclic minOLAP graph.

Now, apply the construction above for $H = \text{Cat}(8, 19)$. By Theorem 2.2, we have that H is an OLAP tree on 27 vertices, that is not RAP by Theorem 2.3. From this H , the resulting G thus has order 54, where $54 \equiv 0 \pmod{2}$, $54 \equiv 0 \pmod{3}$, and $54 \equiv 0 \pmod{6}$. Looking at Theorem 2.2, we thus get that $2n$ is large enough so that no OLAP tree spans G . By arguments above, we thus have that G is spanned by a cyclic OLAP graph. By now iterating the construction from G , we get a series of graphs that are OLAP, not RAP, and, since their orders are multiples of 2, 3, and 6 (since the construction produces graphs with order doubled each step), spanned by cyclic minOLAP graphs. Hence, the result holds. \square

Note that, in the construction provided in the proof of Theorem 4.2, in each step we can, instead of an n -path (v_1, \dots, v_n) , add a whole clique of order n (what matters is that the structure we add is traceable, and that the vertex dominating the rest of the graph is the first vertex of a Hamiltonian path). With this modification, at each step of the

a	b	c
0	2	$\equiv 2, 4 \pmod 6$ at least 20
0	3	$\equiv 2, 4 \pmod 6$ at least 14
0	5	$\in \{8, 16\}$
0	6, 7	$\in \{8, 10\}$
0	8	$\in \{8, 9\}$
1	2	$\equiv 2, 4 \pmod 6$ at least 20
2	3	$\in \{8, 16\}$

Table 4: Exhaustive list of all OLAP 3-suns $\text{Sun}(a, b, c)$ that are not RAP, where $a \leq b \leq c$.

construction we then get a minOLAP graph G_i that is not RAP, which, compared to the graph G_{i-1} from the previous step, satisfies $|V(G_i)| = 2|V(G_{i-1})|$ and $|E(G_i)| = |E(G_{i-1})| + \frac{|V(G_{i-1})|(|V(G_{i-1})|-1)}{2} + |V(G_{i-1})|$. Since the initial graph G_0 we start the construction with is $\text{Cat}(8, 19)$, which has 27 vertices and 26 edges, by iterating the construction eventually we get graphs with density tending to $\frac{1}{3}$, where, recall, the density of a graph H is defined as $\frac{|E(H)|}{|E(K_{|V(H)|})|}$ (where, for any $n \geq 1$, we denote by K_n the complete graph of order n). Thus, applying the construction with denser graphs as G_0 (such exist, see below) would provide even denser minOLAP graphs that are not RAP.

Theorem 4.2 can also be established by considering suns that are OLAP, not RAP, and which cannot be spanned by an OLAP tree. Recall indeed that an OLAP unicyclic graph, if not minOLAP, must be spanned by an OLAP tree, while OLAP trees, in general, are rather rare (recall Theorem 2.2). To be more precise, we provide in Table 4 the list of all OLAP 3-suns that are not RAP (which can essentially be obtained by comparing Theorems 2.4 and 2.5). For an OLAP 3-sun $G = \text{Sun}(a, b, c)$ that is not RAP, by remarks above, the only way for G to be not minOLAP is to be spanned by an OLAP tripod G' . More precisely, in order to get G' from G by removing an edge uv , we must decrease the number of degree-3 vertices from three to one, which is possible only if G has adjacent degree-3 vertices u and v , which requires that $0 \in \{a, b, c\}$. In particular, any 3-sun $\text{Sun}(a, b, c)$ with $a = 1$, $b = 2$, and $c \equiv 2, 4 \pmod 6$ and $c \geq 20$ is thus minOLAP and not RAP.

Note that we would get a similar conclusion from 4-suns that are OLAP and not RAP. Indeed, looking at Theorems 2.4 and 2.5, we get that every 4-sun $\text{Sun}(a, b, c, d)$ with $a = b = 0$, $c = 1$, and $d \equiv 2, 4 \pmod 6$ with $d \geq 8$, is OLAP but not RAP, but, in such a 4-sun, it is not possible to remove an edge to get a connected graph with at most one degree-3 vertex. Thus, such a graph is actually minOLAP and not RAP.

4.2. Non-minAP minOLAP graphs, and non-minOLAP minRAP graphs

We now prove that minimality properties do not have to imply weaker ones. We first prove that minOLAPness, though it implies APness, does not necessarily imply minAPness.

Theorem 4.3. *There exist arbitrarily large cyclic minOLAP graphs that are not minAP.*

Proof. This can be seen through any sun $\text{Sun}(2, a)$ where $a = 6k$ for some $k \geq 5$ with $k \not\equiv 4 \pmod 5$. Such values of a include, for instance, 30, 36, 42, 48, 60, \dots . Indeed, for such a value of a , let $G = \text{Sun}(2, a)$. We denote by x and y the two degree-3 vertices of G , by x' and y' their unique degree-1 neighbours, respectively, and by v_1v_2 and $w_1 \dots w_a$ the two paths of $G - \{x, x', y, y'\}$, where v_1x , w_1x , v_2y , and $w_a y$ are edges of G .

According to Theorem 2.4, for the value of a we chose, G is indeed OLAP. Actually, G is even minOLAP. Indeed, note that removing the edge incident to x' , or the edge

incident to y' , disconnects G , and, thus, does not result in an OLAP graph. If we remove from G any other edge not in $\{xv_1, xw_1, yv_2, yw_a\}$, then note that we get a tree with two degree-3 vertices, which cannot be OLAP by Theorem 2.2. Now, regarding the edges in $\{xv_1, xw_1, yv_2, yw_a\}$, note that removing xv_1 and removing yv_2 from G result in isomorphic graphs, and, similarly, removing xw_1 and removing yw_a result in isomorphic graphs. Thus there are only two remaining cases to consider. On the one hand, note that $G - xv_1$ is $\text{Cat}(3, a + 3)$, which is not OLAP by Theorem 2.2 since $a + 3 = 6k + 3 \equiv 0 \pmod{3}$. On the other hand, note also that $G - xw_1$ is $\text{Cat}(5, 6k + 1)$, which is not OLAP by Theorem 2.2 since $a + 1 = 6k + 1 \geq 31$. Thus, no edge can be removed from G without losing the OLAP property, and G is minOLAP.

To be done with the proof, it remains to show that G is not minAP. We claim that $G - xw_1$ is AP. Indeed, note that $G - xw_1$ is $\text{Cat}(5, a + 1)$, with $a + 1 = 6k + 1$. By our choice of k , it follows that 5 and $6k + 1$ are coprime, and by Theorem 2.1 we get our conclusion. \square

Ideally, we would like to prove a result similar to Theorem 4.3 for minRAPness and minOLAPness. However, due to how close OLAP and RAP caterpillars and suns are, we are only able to observe that non-minOLAP cyclic minRAP graphs do exist.

Theorem 4.4. *There exist cyclic minRAP graphs that are not minOLAP.*

Proof. An example is $G = \text{Sun}(5, 8)$. According to Theorem 2.5, we have that G is RAP. Actually, G is even minRAP. Indeed, removing from G any edge of the unique cycle incident to a degree-3 vertex results either in $\text{Cat}(6, 11)$ or in $\text{Cat}(8, 9)$, which are not RAP by Theorem 2.3; while removing any other edge of G results either in a non-connected graph or in a tree with two degree-3 vertices (which is not RAP, again by Theorem 2.3). On the other hand, by Theorem 2.2, $\text{Cat}(6, 11)$ is OLAP. Thus, G is spanned by an OLAP tree, and hence G is not minOLAP. \square

5. A sufficient condition for (non-minimal) RAPness

As mentioned earlier in this work (recall Theorem 1.1), every traceable graph is AP, OLAP, and RAP, and thus APness, OLAPness, and RAPness can be perceived as a weaker form of traceability. As such, an interesting thread of research in the literature is weakening to APness sufficient conditions for traceability. As an example, let us mention one of the first works on this topic, by Marczyk [17], who provided an Ore-type condition for graphs to be AP. Namely, Ore proved in [19] that any connected n -graph in which the degree sum of every two non-adjacent vertices is at least $n - 1$ is traceable, and Marczyk proved that, roughly speaking, this metric being only $n - 3$ is sufficient to guarantee APness. Since the seminal work of Marczyk, several other works dedicated to this topic appeared in the literature, see *e.g.* [8, 9, 10, 14, 18], and established other interesting results in this very line.

In this section, we build upon a proof scheme of Kalinowski in [13] to derive an Ore-type sufficient condition for (non-minimal) RAPness from one for OLAPness, which itself was derived from one for APness due to Marczyk in [18]. So that this sufficient condition can be expressed properly, we need to recall a few definitions.

For an n -graph G , we define $\sigma_2(G)$ as the minimum degree sum of any two non-adjacent vertices of G ; that is,

$$\sigma_2(G) = \min \{d(u) + d(v) : uv \notin E(G)\}.$$

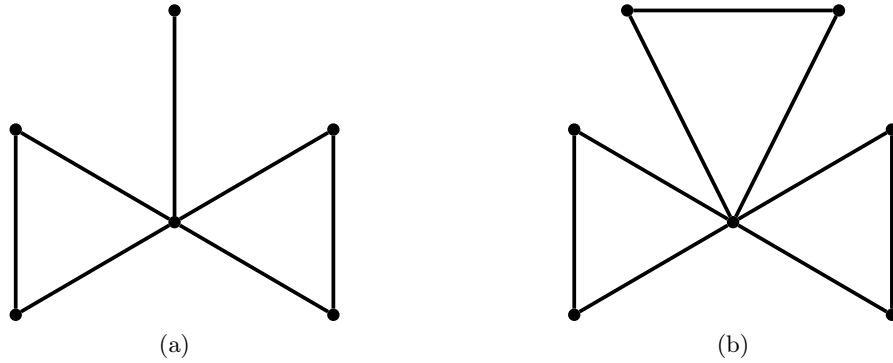


Figure 5: The two exceptional graphs in Marczyk's Theorem 5.1.

We also denote by $\alpha(G)$ the size of a maximum independent set of G . Note that a necessary condition for G to have a (quasi-) perfect matching (and thus to be AP, OLAP, or RAP) is $\alpha(G) \leq \lceil \frac{n}{2} \rceil$.

As mentioned earlier, a well-known result of Ore states that every connected n -graph G with $\sigma_2(G) \geq n-1$ is traceable [19]. This condition was weakened to APness by Marczyk to the following:

Theorem 5.1 (Marczyk [18]). *If G is a connected n -graph with $\sigma_2(G) \geq n-3$ and $\alpha(G) \leq \lceil \frac{n}{2} \rceil$, then G is AP, unless G is one of the two graphs depicted in Figure 5.*

In [13] and [6], Kalinowski and Bednarz *et al.* adapted the proof of Marczyk's Theorem 5.1 from AP graphs to OLAP graphs (where the former author considered values of n with $n \leq 7$ or $n \geq 15$, while the latter authors considered the remaining values of n). Namely, they proved the following result and its immediate consequence.

Theorem 5.2 (Kalinowski [13], Bednarz *et al.* [6]). *If G is a connected n -graph with $\sigma_2(G) \geq n-3$ and $\alpha(G) \leq \lceil \frac{n}{2} \rceil$, then G is OLAP, unless G is one of the two graphs depicted in Figure 5.*

Corollary 5.3. *If G is an n -graph with $\sigma_2(G) \geq n-3$, then G is AP if and only if G is OLAP.*

Another way to perceive Corollary 5.3 is that, in the class of n -graphs G with $\sigma_2(G) \geq n-3$, APness and OLAPness are equivalent notions. Our main goal in our upcoming result, is to prove that RAPness is equivalent to APness and OLAPness in the same class of graphs. For transparency, let us mention that our proof mainly follows the lines of the proof of Theorem 5.2 in [13] (which we refine by a bit), and, thus, to some extent, of that of Theorem 5.1 in [18]. In our proof, we also consider minRAPness; in particular, we observe that the class of graphs in question is so dense that, although RAP, they are never minRAP. It is worth recalling, as seen through Observation 2.7, that this implies these graphs are neither minAP nor minOLAP, although they are both AP and OLAP. Let us finally mention that such concerns for minimality have already been considered by Bensmail in [7]; in particular, he proved that an AP n -graph G with $\sigma_2(G) \geq n-2$ cannot be minAP. So, our next result (for which the condition on n in the statement will be discussed later) actually goes beyond that one.

Theorem 5.4. *If G is a connected n -graph with $\sigma_2(G) \geq n-3$, $\alpha(G) \leq \lceil \frac{n}{2} \rceil$, and $n \geq 10$, then G is RAP and not minRAP.*

Proof. Assume G is connected, $\sigma_2(G) \geq n - 3$, $\alpha(G) \leq \lfloor \frac{n}{2} \rfloor$, and $n \geq 10$ hold. We consider two main cases, depending on whether G is 2-connected or not.

- G is 2-connected.

Under this hypothesis, by a well-known result of Bermond and Linial (see [13]), either G is Hamiltonian, or G contains a cycle of length at least $n - 3$. If G contains a Hamiltonian cycle, then we are done, as, then, G has a Hamiltonian path, showing that G is RAP by Theorem 1.1 and that G is not minRAP. So, from now on, we can suppose the longest cycle C of G has length in $\{n - 3, n - 2, n - 1\}$. Actually, if C has length $n - 1$, then, because G is connected, there is actually a Hamiltonian path in G , implying G is RAP, and that C has an edge which, when removed from G , does not break RAPness, implying G is not minRAP. So, two cases remain:

- C has length $n - 2$.

Let u and v denote the two vertices of G not in C . Since G is connected, if $uv \in E(G)$, then G is actually traceable, and it contains a Hamiltonian path omitting an edge of C ; thus, again, G is RAP and not minRAP. So, we can now assume $uv \notin E(G)$, and, thus, $d(u) + d(v) \geq \sigma_2(G) \geq n - 3$. Now, since C has maximum length, note that none of u and v can be adjacent to two consecutive vertices of C . This implies $d(u), d(v) \leq \frac{n-2}{2}$. Now, since $d(u) + d(v) \geq n - 3$ and $d(u), d(v) \leq \frac{n-2}{2}$, altogether we deduce that $d(u), d(v) \geq \frac{n-4}{2}$.

So, to summarise, $\frac{n-4}{2} \leq d(u), d(v) \leq \frac{n-2}{2}$. Now:

- * If n is odd, then, by the inequalities above, we actually have $d(u) = d(v) = \frac{n-3}{2}$, where, recall, $n \geq 10$. Since G is 2-connected, we have $\delta(G) \geq 2$.

If there is no vertex of C not in $N(u) \cup N(v)$, then, denoting x any neighbour of, say, u on C , we have that u cannot be adjacent to both x^- and x^+ (the vertices preceding and succeeding x on C), meaning that v is adjacent to both x^- and x^+ , and we deduce that $uxx^+x^{++} \dots x^-v$ is a Hamiltonian path of G , thus not traversing x^+v , and hence G is RAP but not minRAP.

Now, assume there is a vertex w of C not in $N(u) \cup N(v)$. Denoting $C = wx_1 \dots x_{n-3}w$, since $d(u) = d(v) = \frac{n-3}{2}$, each of $N(u)$ and $N(v)$ contains either all x_i 's with even index or all x_i 's with odd index. Recall both u and v have at least two neighbours on C each (since $\delta(G) \geq 2$). If $N(u) \neq N(v)$, then, again, there must be a vertex x of C such that, say, $x \in N(u)$ and $x^-, x^+ \in N(v)$, and we can deduce a Hamiltonian path of G (and our conclusions) as earlier. Now, if $N(u) = N(v)$, then, assuming w.l.o.g. that $N(u)$ and $N(v)$ contain all x_i 's with odd index, we deduce that G is spanned by the 2-sun $\text{Sun}(1, n - 5)$, which is RAP by Theorem 2.5. More precisely, there is such a 2-sun spanning G in which u has degree 1 with unique neighbour x_1 , v has degree 1 with unique neighbour x_3 , and the unique cycle is C . Thus this 2-sun does not contain ux_3 , and G is not minRAP.

- * If n is even, then we have $d(u), d(v) \in \{\frac{n-4}{2}, \frac{n-2}{2}\}$, a set of two values, and, because $\sigma_2(G) \geq n - 3$, it must be that one of u and v has degree $\frac{n-2}{2}$ (as otherwise we would have $d(u) + d(v) = n - 4 < \sigma_2(G)$). Assume w.l.o.g. that $d(u) = \frac{n-2}{2}$. Since u cannot be adjacent to two consecutive vertices of C , setting $C = x_1 \dots x_{n-2}x_1$, we have, say, that $N(u)$ contains all x_i 's with odd index. For similar reasons as earlier, we would get our conclusions from a Hamiltonian path of G if v is adjacent to an x_i with even i ; thus, we can assume $N(v) \subseteq N(u)$. Note now that if there are two distinct even values

i, j such that $x_i x_j$ is an edge, then, again, a Hamiltonian path of G can be deduced to get the desired conclusions (for instance, assuming $i < j$, one can consider the path $u x_i^- x_i^- \dots x_j^+ x_j x_i x_i^+ \dots x_j^- v$, that omits $x_i^- x_i$). Otherwise, the x_i 's with even index are independent, and, with u and v , we have a set of $\frac{n-2}{2} + 2 = \frac{n}{2} + 1$ independent vertices, which contradicts that $\alpha(G) \leq \lfloor \frac{n}{2} \rfloor$.

– C has length $n - 3$.

Let u, v , and w be the three vertices of G not in C . If $G[\{u, v, w\}]$ is connected, then, say, uv and vw are edges, and, because G is 2-connected and thus $\delta(G) \geq 2$, either u or w has an edge to C , from which we get our conclusions from a Hamiltonian path of G , or uw is an edge and v has a neighbour in C , from which we deduce another Hamiltonian path yielding our conclusion.

We can now assume $G[\{u, v, w\}]$ is not connected; there are two cases:

- * Assume first that the only edge of $G[\{u, v, w\}]$ is, say, uv . Again, since C is a longest cycle of G , it cannot be that w is adjacent to two consecutive vertices of C , and thus $d(w) \leq \frac{n-3}{2}$. Since u and v are not adjacent to w , we have $d(u) + d(w) \geq n - 3$ and $d(v) + d(w) \geq n - 3$ (since $\sigma_2(G) \geq n - 3$), from which we deduce $d(u), d(v) \geq \frac{n-3}{2} = \frac{n-5}{2} + 1$. Thus, omitting the edge uv , each of u and v has $\frac{n-5}{2}$ neighbours on C . Recall also that each of u and v cannot be adjacent to two consecutive vertices of C , by the maximality of C . Similarly, if there is a vertex x of C such that $x^- \in N(u)$ and $x^+ \in N(v)$ then we would deduce that $u x^- x^- \dots x^{++} x^+ v u$ is a cycle of G longer than C , a contradiction. Of course, we would reach a similar contradiction if $x^- \in N(v)$, and also if u is adjacent to some x on C and either x^- or x^+ belongs to $N(v)$. From this, we deduce that, for each x of the $\frac{n-5}{2}$ neighbours of u on C , vertices x^+ and x^{++} cannot belong to $N(v)$. So, u has $\frac{n-5}{2}$ neighbours on C , which yields that $n-5$ vertices of C cannot belong to $N(v)$. Thus, v has at most $n-3 - (n-5) = 2$ neighbours on C , and since we observed earlier that v has $\frac{n-5}{2}$ neighbours on C , we must have $n = 9$, which contradicts our initial hypothesis on n .
- * Assume then that $\{u, v, w\}$ is an independent set. Because $\sigma_2(G) \geq n - 3$, the two vertices with largest degree in $\{u, v, w\}$ must have degree at least $\frac{n-3}{2}$. Assume $d(u), d(v) \geq \frac{n-3}{2}$ w.l.o.g. Again, since C is a longest cycle of G , none of u, v , and w can be adjacent to two consecutive vertices of C , and so $d(u), d(v), d(w) \leq \frac{n-3}{2}$. Thus, we must have $d(u) = d(v) = \frac{n-3}{2}$, and, now that these degrees are revealed, since $\sigma_2(G) \geq n - 3$, we also have $d(w) = \frac{n-3}{2}$. In particular, n is odd. Also, since G is 2-connected, we have $d(u), d(v), d(w) \geq 2$. So, setting $C = x_1 \dots x_{n-3} x_1$, we have, say, that each of $N(u), N(v)$, and $N(w)$ contains either all x_i 's with odd index, or all x_i 's with even index. Similarly as in a previous case, we get our conclusions if any two of $N(u), N(v)$, and $N(w)$ are different. Thus, we can now assume $N(u) = N(v) = N(w)$, say the three sets contain the x_i 's with odd index. Now, if there are two distinct even values i, j such that $x_i x_j$ is an edge with $i < j$, then we can deduce that G is spanned by some caterpillar $\text{Cat}(2, n-2)$, which is RAP by Theorem 2.3 since n is odd and shows G is not minRAP (for instance, consider the spanning caterpillar where x_i^- has degree 3, is adjacent to both u and v which are of degree 1, and the remaining path attached to x_i^- is $x_i^- x_i x_j x_j^- \dots x_i^+ w x_j^+ x_j^{++} \dots x_i^-$; in particular, $x_i^- x_i^-$ is not part of the caterpillar). Otherwise, if all x_i 's with even index are pairwise

independent, then, together with u , v , and w , we deduce an independent set of G of size $\frac{n-3}{2} + 3 = \frac{n+3}{2}$, which contradicts that $\alpha(G) \leq \lfloor \frac{n}{2} \rfloor$.

- G has a cut-vertex z .

In this case, we use a theorem of Pósa (see [13]), which, since $\sigma_2(G) \geq n-3$, says any longest path P of G has order at least $n-2$, *i.e.*, there are at most two vertices not covered by P . If P is a Hamiltonian path of G with ends u and v , then G is RAP, and G is not minRAP. Indeed, either $uv \in E(G)$ but $uv \notin E(P)$, or $uv \notin E(G)$ and $d(u) \geq 2$ or $d(v) \geq 2$, due to the fact that $\sigma_2(G) \geq n-3$ and $n \geq 10$, implying an edge incident to u or v does not belong to P . So, we can now assume P covers $n-2$ or $n-1$ vertices of G . In what follows, we consider the possible values of q , the number of connected components of $G-z$. Recall that $q \geq 2$ since z is a cut-vertex of G .

- Assume first that $q \geq 3$. In that case, note that P must go through z , as otherwise there would be at least three vertices not in P , while P contains all vertices of G but at most two of them. Actually, we must have $q = 3$. Indeed, since z is a cut-vertex of G , if $q \geq 5$ then clearly there are at least three vertices not in P , while, if $q = 4$, then we would deduce that two connected components of $G-z$ not traversed by P contain exactly one vertex each, each adjacent to z , and that the degree sum of these two independent vertices would thus be 2, which since $\sigma_2(G) \geq n-3$ would yield $n \leq 5$, a contradiction. Thus, $q = 3$; let us denote by C_1, C_2, C_3 the three connected components of $G-z$.

- * If P has order $n-1$, then P goes through all vertices of G but some v . Assume v belongs to C_3 w.l.o.g. Then v is the sole neighbour of C_3 . Now, for every vertex $u \in V(C_1) \cup V(C_2)$, note that $uv \notin E(G)$. Since $\sigma_2(G) \geq n-3$ and $d(v) = 1$, we deduce $d(u) \geq n-4$. Since there is no edge from $V(C_1)$ to $V(C_2)$, we deduce, assuming $u \in V(C_2)$, that $|V(C_1)| \leq 2$, and similarly, assuming $u \in V(C_1)$, that $|V(C_2)| \leq 2$. Altogether, we thus deduce that $n \leq 6$, which contradicts our initial assumption on n .

- * If P has order $n-2$, then let u and v denote the two vertices not in P . Assuming P goes through C_1 and C_2 , we can assume both u and v are part of C_3 . Indeed, if, say, u belongs to C_2 while v belongs to C_3 , then $d(v) = 1$ again, and we must have $d(u) \geq n-4$; but then u is not adjacent to the end of P in C_2 , to v , and to any vertex in C_1 , which would imply that $|V(C_1)| = 1$, and by considering the unique vertex of C_1 and v we would again deduce that $n \leq 5$. So we can assume $V(C_3) = \{u, v\}$ and $uv \in E(G)$ (since $q = 3$). Now, every vertex w in $V(C_1) \cup V(C_2)$ is not adjacent to any of u and v , and since $d(u), d(v) \leq 2$, we have $d(w) \geq n-5$ since $\sigma_2(G) \geq n-2$. However, assuming $w \in V(C_i)$ for any $i \in \{1, 2\}$, all neighbours of w lie in $V(C_i) \cup \{z\}$, and thus $d(w) \leq |V(C_i)|$. The other way round, for $j \in \{1, 2\} \setminus \{i\}$, we must thus have $|V(C_j)| \leq 2$. Thus we deduce $|V(C_1)| = |V(C_2)| = |V(C_3)| = 2$, and thus $n \leq 7$, a contradiction.

- Assume last $q = 2$. We denote by C_1 and C_2 the two connected components of $G-z$, where we set $n_1 = |V(C_1)|$ and $n_2 = |V(C_2)|$, and we assume $n_1 \leq n_2$. Since any two independent vertices u and v of C_1 have, in G , their neighbours in $V(C_1) \cup \{z\}$, then, since $\sigma_2(G) \leq n-3$, we have $d_{C_1}(u) + d_{C_2}(v) \geq n-5$. Now, since $n = n_1 + n_2 + 1$ and $n_1 \leq n_2$, we have $\sigma_2(C_1) \geq 2n_1 - 4$, where $2n_1 - 4 \geq n_1$ when $n_1 \geq 4$. So, by Ore's Theorem [19], we deduce that C_1 is Hamiltonian when $n_1 \geq 4$. Meanwhile, if $n_1 \leq 3$, then either $n_1 \leq 2$ and, since G is connected,

there is a Hamiltonian path of $C_1 + z$ with end-vertex z , or $n_1 = 3$ and we get the same conclusion as otherwise C_1 would contain two vertices of degree 1, and we could repeat previous arguments with another cut-vertex of G whose removal results in three connected components. To sum up, in any case, $C_1 + z$ contains a Hamiltonian path starting from z .

If $n_1 = n_2$, then we get the same conclusion regarding $C_2 + z$, and G is traceable and thus RAP (and, because $n \geq 10$, one of C_1 and C_2 admits a Hamiltonian cycle, whose one edge does not have to be traversed by some Hamiltonian path of G , implying G is not minRAP). So, it remains to consider when $n_1 < n_2$. In this case, for any two vertices $u \in V(C_1)$ and $v \in V(C_2)$, we have $uv \notin E(G)$, and thus

$$d_{C_2}(v) \geq \sigma_2(G) - d_G(u) - 1 \geq n - 3 - n_1 - 1 = n_2 - 3.$$

In other words, $\delta(C_2) \geq n_2 - 3$. If $n_2 \geq 6$, then C_2 is Hamiltonian by Ore's Theorem [19], and we get the same conclusions as earlier. Now, if $n_2 \leq 5$, then, since $n_1 < n_2$, we have $n_1 \leq 4$ and $n \leq 10$. Since $n \geq 10$, we must have $n_2 = 5$ and $n_1 = 4$. In particular, by Ore's Theorem we have that C_2 is traceable. If C_2 has a Hamiltonian path Q starting or ending with a vertex adjacent to z , then we are done (in particular, a Hamiltonian path of G not traversing an edge of C_1 can be deduced, since, because $n_1 \geq 4$, there is a Hamiltonian cycle in C_1 ; thus G is not minRAP in this case). Thus, we can assume the two ends of Q are not adjacent to z , and similarly we can assume these two ends are not adjacent. Thus, these two ends, in G , have degree at most 3, and their degree sum is at most $6 < 7 = n - 3$. So we get a contradiction to the fact that $\sigma_2(G) \geq n - 3$.

We thus get the desired conclusion in all cases. □

Corollary 5.5. *In the class of n -graphs G with $\sigma_2(G) \geq n - 3$ and $n \geq 10$, APness, OLAPness, and RAPness are equivalent notions. Also, no AP, OLAP, or RAP graph in that class is minAP, minOLAP, or minRAP, respectively; so, minAPness, minOLAPness, and minRAPness are also equivalent notions in that class.*

Note that the condition on n in the statement of Theorem 5.4 is to make the proof less technical, through avoiding small pathological cases, such as the graphs in Figure 5 which are necessary for the more general statement to hold (recall Theorems 5.1 and 5.2). These conditions on n also permit to have the part of the statement on non-minRAPness. For instance, note indeed that the path P_5 on 5 vertices is clearly minRAP, while $\sigma_2(P_5) = 2 = |V(P_5)| - 3$ and $\alpha(P_5) = 3 = \lceil |V(P_5)|/2 \rceil$. Thus, the statement of Theorem 5.4, in any case, requires at least $n \geq 6$.

6. Conclusion

In this work, we have introduced the notions of minOLAPness and minRAPness, generalising to OLAP and RAP graphs the notion of minimality, minAPness, that has already been studied for AP graphs. In Section 3, we mainly exhibited small minOLAP and minRAP graphs, already observing discrepancies with minAP graphs. In Section 4, we got to observe other discrepancies through several graph constructions. In particular, a cyclic graph that is minimal w.r.t. some partition property neither has to possess the stronger partition property, nor has to be minimal for the weaker partition property. In Section 5, we then adapted to RAPness existing sufficient conditions for APness and OLAPness in

graphs with large degree sums of independent vertices, and derived a side non-minRAPness property for the class of graphs considered.

Following our results, there are several aspects which we think could be worth investigating further. For instance:

- We were not able to come up with a general construction so that Theorem 4.4 holds for arbitrarily large graphs (just as in Theorem 4.3). As explained earlier, this is mainly because the discrepancies between general trees, caterpillars, and suns under the OLAP and RAP properties are more shallow than when considering the AP and OLAP properties. So we would be interested in knowing whether Theorem 4.4 generalises to arbitrarily large graphs.
- As seen through Theorem 3.3, for graphs on up to at most 10 vertices, minOLAPness and minRAPness seem to coincide. As far as we can tell, the smallest known (minimal) OLAP graph that is not (minimal) RAP is $\text{Cat}(6, 11)$, which has order 17. For cyclic graphs, from Table 4, we know there is such a cyclic graph, $\text{Sun}(0, 5, 8)$, with order 19. An interesting question would thus concern the minimum order of a (minimal) OLAP graph that is not (minimal) RAP. Recall that, by Theorem 3.3, the smallest cyclic minOLAP and minRAP graphs seem to have order 9.
- In [5], Baudon, Przybyło, and Woźniak exhibited a construction for a minAP graph having arbitrarily many cycles, thus being arbitrarily far from trees. We were not able to exhibit similar constructions for minOLAP and minRAP graphs, and we believe determining whether such graphs exist would be crucial for this field. Similarly, one can wonder about other properties of minOLAP and minRAP graphs, inspired by important ones for minAP graphs. In particular, following [7], it would be interesting to wonder about their connectivity, their minimum degree, their density, or their clique number. Especially, it would be interesting to establish significant differences between minAP graphs, and minOLAP and minRAP graphs.
- Finally, we think it would be interesting to establish more results of the sort of Theorem 2.6 for minAPness, minOLAPness, and minRAPness, or for their absence. That is, sufficient conditions inspired by ones for Hamiltonicity/traceability, weakened so that they fit (or not) with these properties. In particular, in light of Corollary 5.5, we wonder whether there is some metric for which the properties differ by a lot.

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Appendix: Proof of Theorem 3.3

In this section, we consider all graphs in Figure 1, and prove formally whether they are RAP (and thus also OLAP) or not OLAP (and thus also not RAP). To prove that a graph is RAP, we mainly make use of Theorem 2.6, while, to prove that a graph is not OLAP, we exhibit a picking process that cannot be led properly from start to finish. Recall also that an AP, OLAP, or RAP graph must be connected, and that traceable graphs are RAP. Recall also that Theorems 2.2 and 2.3 provide a full list of all OLAP and RAP trees.

In each of the upcoming proofs, we deal with the vertices of a given graph using the terminology we have given in Figure 1.

Observation 6.1. *The graph in Figure 1 (a) is RAP, and thus OLAP.*

Proof. Let G be the graph depicted in Figure 1 (a). Since $|V(G)| = 10$, by Theorem 2.6 it suffices (here and similarly in the next proofs) to show that, for every $\lambda \in \{1, \dots, 5\}$, there is a RAP part S of size λ such that $G - S$ is RAP.

- $\lambda = 1$. Consider $S = \{i\}$; then $G[S]$ is the 1-graph, while $G - S$ is spanned by $\text{Cat}(4, 5)$.
- $\lambda = 2$. Consider $S = \{h, j\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(3, 5)$.
- $\lambda = 3$. Consider $S = \{a, b, c\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(3, 4)$.
- $\lambda = 4$. Consider $S = \{e, f, g, i\}$; then $G[S]$ is a path, while $G - S$ is traceable.
- $\lambda = 5$. Consider $S = \{a, b, c, h, j\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(2, 3)$. \square

Observation 6.2. *The graph in Figure 1 (b) is RAP, and thus OLAP.*

Proof. Let G be the graph depicted in Figure 1 (b).

- $\lambda = 1$. Consider $S = \{j\}$; then $G[S]$ is the 1-graph, while $G - S$ is traceable.
- $\lambda = 2$. Consider $S = \{i, j\}$; then $G[S]$ and $G - S$ are paths.
- $\lambda = 3$. Consider $S = \{a, b, c\}$; then $G[S]$ and $G - S$ are paths.
- $\lambda = 4$. Consider $S = \{e, f, g, h\}$; then $G[S]$ is a path, while $G - S$ is traceable.
- $\lambda = 5$. Consider $S = \{a, b, c, i, j\}$; then $G[S]$ and $G - S$ are paths. \square

Observation 6.3. *The graph in Figure 1 (c) is RAP, and thus OLAP.*

Proof. Let G be the graph depicted in Figure 1 (c).

- $\lambda = 1$. Consider $S = \{j\}$; then $G[S]$ is the 1-graph, while $G - S$ is spanned by $\text{Cat}(4, 5)$.
- $\lambda = 2$. Consider $S = \{a, b\}$; then $G[S]$ is a path, while $G - S$ is spanned by $\text{Cat}(3, 5)$.
- $\lambda = 3$. Consider $S = \{a, b, c\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(3, 4)$.
- $\lambda = 4$. Consider $S = \{g, h, i, j\}$; then $G[S]$ and $G - S$ are paths.
- $\lambda = 5$. Consider $S = \{a, b, c, g, i\}$; then $G[S]$ and $G - S$ are paths. \square

Observation 6.4. *The graph in Figure 1 (d) is RAP, and thus OLAP.*

Proof. Let G be the graph depicted in Figure 1 (d).

- $\lambda = 1$. Consider $S = \{g\}$; then $G[S]$ is the 1-graph, while $G - S$ is the graph depicted in Figure 2 (b) (which is RAP by Observation 3.2).
- $\lambda = 2$. Consider $S = \{h, j\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(3, 5)$.
- $\lambda = 3$. Consider $S = \{a, b, c\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(3, 4)$.
- $\lambda = 4$. Consider $S = \{e, f, g, i\}$; then $G[S]$ is a path, while $G - S$ is traceable.
- $\lambda = 5$. Consider $S = \{d, e, f, g, i\}$; then $G[S]$ and $G - S$ are $\text{Cat}(2, 3)$. \square

Observation 6.5. *The graph in Figure 1 (e) is not OLAP, and thus not RAP.*

Proof. Let G be the graph depicted in Figure 1 (e). If, during the first round of a picking process, we are asked to pick a connected part S of size 3, then the only option, leaving $G - S$ connected, is to consider $S = \{a, b, c\}$. But then $G - S$ is a tree with two degree-3 vertices, which is not OLAP by Theorem 2.2. Thus, there are no proper options, implying G is not OLAP at all. \square

Observation 6.6. *The graph in Figure 1 (f) is RAP, and thus OLAP.*

Proof. Let G be the graph depicted in Figure 1 (f).

- $\lambda = 1$. Consider $S = \{j\}$; then $G[S]$ is the 1-graph, while $G - S$ is spanned by $\text{Cat}(4, 5)$.
- $\lambda = 2$. Consider $S = \{a, b\}$; then $G[S]$ is a path, while $G - S$ is traceable.
- $\lambda = 3$. Consider $S = \{a, b, c\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(3, 4)$.
- $\lambda = 4$. Consider $S = \{a, b, c, d\}$; then $G[S]$ and $G - S$ are paths.
- $\lambda = 5$. Consider $S = \{a, b, c, i, j\}$; then $G[S]$ and $G - S$ are paths. \square

Observation 6.7. *The graph in Figure 1 (g) is RAP, and thus OLAP.*

Proof. Let G be the graph depicted in Figure 1 (g).

- $\lambda = 1$. Consider $S = \{i\}$; then $G[S]$ is the 1-graph, while $G - S$ is spanned by $\text{Cat}(4, 5)$.
- $\lambda = 2$. Consider $S = \{h, j\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(3, 5)$.
- $\lambda = 3$. Consider $S = \{a, b, c\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(3, 4)$.
- $\lambda = 4$. Consider $S = \{g, h, i, j\}$; then $G[S]$ and $G - S$ are paths.
- $\lambda = 5$. Consider $S = \{d, e, f, h, j\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(2, 3)$. \square

Observation 6.8. *The graph in Figure 1 (h) is RAP, and thus OLAP.*

Proof. Let G be the graph depicted in Figure 1 (h).

- $\lambda = 1$. Consider $S = \{j\}$; then $G[S]$ is the 1-graph, while $G - S$ is spanned by $\text{Cat}(2, 7)$.
- $\lambda = 2$. Consider $S = \{h, i\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(3, 5)$.
- $\lambda = 3$. Consider $S = \{a, b, c\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(3, 4)$.
- $\lambda = 4$. Consider $S = \{e, f, g, j\}$; then $G[S]$ is a path, while $G - S$ is traceable.
- $\lambda = 5$. Consider $S = \{d, e, f, g, j\}$; then $G[S]$ and $G - S$ are $\text{Cat}(2, 3)$. \square

Observation 6.9. *The graph in Figure 1 (i) is RAP, and thus OLAP.*

Proof. Let G be the graph depicted in Figure 1 (i).

- $\lambda = 1$. Consider $S = \{d\}$; then $G[S]$ is the 1-graph, while $G - S$ is $\text{Cat}(4, 5)$.
- $\lambda = 2$. Consider $S = \{f, g\}$; then $G[S]$ is a path, while $G - S$ is traceable.

- $\lambda = 3$. Consider $S = \{h, i, j\}$; then $G[S]$ and $G - S$ are paths.
- $\lambda = 4$. Consider $S = \{a, b, c, d\}$; then $G[S]$ and $G - S$ are paths.
- $\lambda = 5$. Consider $S = \{a, b, c, h, j\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(2, 3)$. \square

Observation 6.10. *The graph in Figure 1 (j) is RAP, and thus OLAP.*

Proof. Let G be the graph depicted in Figure 1 (j).

- $\lambda = 1$. Consider $S = \{j\}$; then $G[S]$ is the 1-graph, while $G - S$ is traceable.
- $\lambda = 2$. Consider $S = \{e, f\}$; then $G[S]$ is a path, while $G - S$ is traceable.
- $\lambda = 3$. Consider $S = \{a, b, c\}$; then $G[S]$ and $G - S$ are paths.
- $\lambda = 4$. Consider $S = \{g, h, i, j\}$; then $G[S]$ and $G - S$ are paths.
- $\lambda = 5$. Consider $S = \{a, b, c, g, j\}$; then $G[S]$ and $G - S$ are paths. \square

Observation 6.11. *The graph in Figure 1 (k) is RAP, and thus OLAP.*

Proof. Let G be the graph depicted in Figure 1 (k).

- $\lambda = 1$. Consider $S = \{f\}$; then $G[S]$ is the 1-graph, while $G - S$ is traceable.
- $\lambda = 2$. Consider $S = \{a, b\}$; then $G[S]$ is a path, while $G - S$ is spanned by $\text{Cat}(3, 5)$.
- $\lambda = 3$. Consider $S = \{a, b, c\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(2, 5)$.
- $\lambda = 4$. Consider $S = \{g, h, i, j\}$; then $G[S]$ and $G - S$ are paths.
- $\lambda = 5$. Consider $S = \{a, b, c, g, j\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(2, 3)$. \square

Observation 6.12. *The graph in Figure 1 (l) is not OLAP, and thus not RAP.*

Proof. Let G be the graph depicted in Figure 1 (l). Suppose that, during the first round of a picking process, we are asked to pick a connected part S of size 3. Then the only proper options are $S = \{a, b, c\}$ and $S = \{d, e, f\}$, since $G - S$ is not connected for any other connected part of size 3 as S . However, for both options, $G - S$ is a tree with two degree-3 vertices, which cannot be OLAP by Theorem 2.2. Thus, we cannot pick S as desired, and G is not OLAP. \square

Observation 6.13. *The graph in Figure 1 (m) is RAP, and thus OLAP.*

Proof. Let G be the graph depicted in Figure 1 (m).

- $\lambda = 1$. Consider $S = \{a\}$; then $G[S]$ is the 1-graph, while $G - S$ is spanned by $\text{Cat}(2, 7)$.
- $\lambda = 2$. Consider $S = \{c, d\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(3, 5)$.
- $\lambda = 3$. Consider $S = \{h, i, j\}$; then $G[S]$ and $G - S$ are paths.
- $\lambda = 4$. Consider $S = \{a, b, c, d\}$; then $G[S]$ and $G - S$ are paths.
- $\lambda = 5$. Consider $S = \{c, d, e, f, g\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(2, 3)$. \square

Observation 6.14. *The graph in Figure 1 (n) is RAP, and thus OLAP.*

Proof. Let G be the graph depicted in Figure 1 (n).

- $\lambda = 1$. Consider $S = \{j\}$; then $G[S]$ is the 1-graph, while $G - S$ is spanned by $\text{Cat}(4, 5)$.
- $\lambda = 2$. Consider $S = \{g, j\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(3, 5)$.
- $\lambda = 3$. Consider $S = \{a, b, c\}$; then $G[S]$ is a path, while $G - S$ is $\text{Cat}(2, 5)$.
- $\lambda = 4$. Consider $S = \{g, h, i, j\}$; then $G[S]$ and $G - S$ are paths.
- $\lambda = 5$. Consider $S = \{c, d, e, f, i\}$; then $G[S]$ and $G - S$ are $\text{Cat}(2, 3)$. □