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# VARIOUS QUESTIONS AROUND FINITELY POSITIVELY EXPANSIVE DYNAMICAL SYSTEMS

SILVÈRE GANGLOFF, PIERRE GUILLON, AND PIOTR OPROCHA

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ABSTRACT. It is well-known that when a positively expansive dynamical system is invertible then its underlying space is finite. C.Morales has introduced a decade ago a natural way to generalize positive expansiveness, by introducing other properties that he called positive  $n$ -expansiveness, for all  $n \geq 1$ , positive 1-expansiveness being identical to positive expansiveness. Contrary to positive expansiveness, positive  $n$ -expansiveness for  $n > 1$  does not enforce that the space is finite when the system is invertible. In the present paper we call finitely positively expansive dynamical systems as the ones which are positively  $n$ -expansive for some integer  $n$ , and prove several results on this class of systems. In particular, the well-known result quoted above is true if we add the constraint of shadowing property, while it is not if this property is replaced with minimality. Furthermore, finitely positively expansive systems cannot occur on certain topological spaces such as the interval, when the system is assumed to be invertible finite positive expansiveness implies zero topological entropy. Overall we show that the class of finitely positively expansive dynamical systems is quite rich and leave several questions open for further research.

## 1. INTRODUCTION

In this text, we study a class of properties of dynamical systems, introduced in 2012 by C.Morales [Mor12], which generalize positive expansiveness, called positive  $n$ -expansiveness, for  $n \geq 1$ , and is defined as follows. For every  $n \geq 1$  and a metric space  $(X, d)$ , a dynamical system  $(X, f)$  is called **positively  $n$ -expansive** with constant  $\varepsilon > 0$  when there exist some  $\varepsilon > 0$  such that for every  $x \in X$ ,  $|W_\varepsilon(x)| \leq n$ , where:

$$W_\varepsilon(x) := \{y \in X \mid \forall t \in \mathbb{N}, d(f^t(x), f^t(y)) < \varepsilon\}.$$

Let us notice that in particular the classical notion of positive expansiveness is identical to positive 1-expansiveness. A dynamical system is called **finitely positively expansive** when there exists some  $n \geq 1$  and  $\varepsilon > 0$  such that it is positively  $n$ -expansive with constant  $\varepsilon$ . In his article, C.Morales has proved that certain statements which are true for positively expansive dynamical systems are still true on finitely positively expansive systems while others are not anymore. It is the case of the following one:

**Theorem 1.1.** *Any invertible positively expansive dynamical system is finite.*

C.Morales proved that it is not true for finitely positively expansive systems by constructing explicit examples:

**Theorem 1.2** (Morales). *For each  $k \geq 1$ , there exists an infinite metric space  $(X_k, d_k)$  on which there is a positively  $2^k$ -expansive dynamical system  $(X_k, f_k)$  such that  $f$  is a homeomorphism and which is not  $(2^k - 1)$ -expansive.*

During the last decade, the properties defined by C.Morales have received a lot of attention and several formal statements have been proved. Most of them fall into two categories: **1.** Ones which recover Theorem 1.1 for systems under additional constraints. The proof relies on the fact that all the properties are equivalent under these constraints. For instance, C.Good, S.Macías, J. Meddaugh, J. Mitchell and J. Thomas [GMM<sup>+</sup>21] proved that topological mixing, shadowing property and  $n$ -expansiveness imply together that the dynamical system is finite when assumed to be a homeomorphism. B. Carvalho and W. Cordeiro [CC19] proved a similar statement, where mixing is replaced by transitivity, generalizing the result of A.Artigue, M.J.Pacifico, J.Vieitez [APV17] that if a homeomorphism defined on a compact surface is transitive and positively 2-expansive, then it is positively expansive. **2.** Ones which provide examples of dynamical systems which are homeomorphisms and positively  $n$ -expansive, for each  $n \geq 1$ , under various sets of constraints. For instance one can find in [CC16] an example of homeomorphism with the shadowing property which is  $n$ -expansive (not positively) and not  $(n - 1)$ -expansive for each  $n > 1$ . This can be found as well in [GMM<sup>+</sup>21], without the homeomorphism condition.

In the present text, we study finitely positively expansive dynamical systems and their properties. In Section 2, we mainly prove two theorems about dynamical systems which are homeomorphisms with the finite positive expansiveness in a general setting: they have zero entropy; if such a system has the shadowing property, it is finite. In Section 3 we consider dynamical systems on topological graphs. On the circle, there are known examples of positively expansive systems, which are the  $z^n$  maps for  $n \notin \{-1, 0, 1\}$ . We prove that on the other hand, other topological graphs, the interval and the Hawaiian earring, do not have finitely positively expansive systems, for different reasons. In Section 4 we consider two-sided one-dimensional shifts. In this context, we have found that S-adic shifts provide several examples of finitely positively expansive dynamical systems. In particular, Sturmian and Toeplitz shifts are positively 2-expansive. We provide and prove some sufficient conditions for a sequence of substitutions to generate a finitely positively expansive shift. Using these results, we are able to strengthen Morales' result by proving that for each  $n \geq 2$  there exists a **minimal** dynamical system which is positively  $n$ -expansive and not positively  $(n - 1)$ -expansive.

We believe that the main outcomes of this study are natural connections between the property of finite positive expansiveness and low complexity symbolic dynamics and dynamics on topological graphs, as witnessed by non trivial results formulated in Section 3 and Section 4.

In the whole text, we denote by  $\mathbb{N}$  the set of non-negative integers and by  $\mathbb{Z}_+$  the set of positive integers, and  $\mathbb{Z}_-$  the set of negative integers.

## 2. GENERAL PROPERTIES OF FINITELY POSITIVELY EXPANSIVE SYSTEMS

In this section, we study finitely positively expansive systems in a general setting. In particular we prove that some properties of dynamical systems are 'orthogonal' to finite positive expansiveness when  $f$  is a homeomorphism: the shadowing property [Section 2.5], and positive entropy [Section 2.4].

In this section,  $(X, d)$  is a compact metric space. We denote  $\mathcal{B}_\varepsilon(x)$  the ball of radius  $\varepsilon$  around  $x \in X$ . A **dynamical system** is a pair  $(X, f)$ , where  $f$  is a continuous self-map of  $X$ . We will denote by  $\mathcal{O}_f(x)$  the set  $\{f^k(x) \mid k \in \mathbb{N}\}$  and call it the *orbit* of  $x$  under the action of  $f$ . When  $f$  is a homeomorphism, we will also denote  $\mathcal{O}_f^{\mathbb{Z}}(x) := \{f^t(x) \mid t \in \mathbb{Z}\}$ . When the context is unambiguous, we will simplify these notation into respectively  $\mathcal{O}(x)$  and  $\mathcal{O}^{\mathbb{Z}}(x)$ . An  **$\omega$ -limit set** of  $(X, f)$  is a set of the form

$$\omega_f(x) := \bigcap_{n \geq 1} \overline{\bigcup_{k \geq n} \{f^k(x)\}},$$

where  $x \in X$ . We say that the dynamical system  $(X, f)$  is **transitive** when for any open sets  $u, v$  of  $X$ , there exists an integer  $n > 0$  such that  $f^n(u) \cap v \neq \emptyset$ .

**2.1. Elementary properties.** Let us notice first that the class of finitely positively expansive dynamical systems is stable by product, union, and subsystem:

**Remark 2.1.** For two dynamical systems which are, respectively, positively  $n$ -expansive and positively  $m$ -expansive, we have the following:

- (1) The disjoint union of these dynamical systems is positively  $\max(m, n)$ -expansive;
- (2) Their product is positively  $mn$ -expansive.
- (3) Any iteration of the first one is positively  $n$ -expansive.
- (4) Any of its subsystems is positively  $n$ -expansive.

**Lemma 2.2.** A system  $(X, f)$  which is positively  $n$ -expansive is locally at-most- $n$ -to-one (i.e. there exists  $\varepsilon > 0$  such that for every  $x \in X$ , and all  $z \in X$ ,  $|f^{-1}(z) \cap \mathcal{B}_\varepsilon(x)| \leq n$ ).

*Proof.* Since  $(X, f)$  is positively  $n$ -expansive, there exists  $\varepsilon > 0$  such that for every  $x \in X$ ,  $|W_{2\varepsilon}(x)| \leq n$ . We claim that for every  $x \in X$ ,  $z \in X$  and  $z' \in f^{-1}(z) \cap \mathcal{B}_\varepsilon(x)$ ,

$$f^{-1}(z) \cap \mathcal{B}_\varepsilon(x) \subset W_{2\varepsilon}(z').$$

Indeed, consider  $z'' \in f^{-1}(z) \cap \mathcal{B}_\varepsilon(x)$ . By the triangular inequality,

$$d(z'', z') \leq d(z'', x) + d(x, z') \leq 2\varepsilon.$$

Furthermore, since  $f(z'') = f(z') = z$ , for every  $n \geq 1$ , we have  $d(f^n(z''), f^n(z')) = 0 \leq 2\varepsilon$ . Therefore  $z'' \in W_{2\varepsilon}(z')$ . Since this is true for every  $z'' \in f^{-1}(z) \cap \mathcal{B}_\varepsilon(x)$ , we have  $f^{-1}(z) \cap \mathcal{B}_\varepsilon(x) \subset W_{2\varepsilon}(z')$ . This shows that the claim holds and therefore we have  $|f^{-1}(z) \cap \mathcal{B}_\varepsilon(x)| \leq n$ . This ends the proof.  $\square$

**Corollary 2.3.** A system  $(X, f)$  which is finitely positively expansive is bounded-to-one (i.e. the function  $x \mapsto |f^{-1}(x)|$  is bounded).

*Proof.* Considering some  $\varepsilon > 0$  provided by Lemma 2.2, and a finite cover  $\mathcal{U}$  by balls of the form  $\mathcal{B}_\varepsilon(x)$ , Lemma 2.2 gives that for every  $z \in X$ ,  $|f^{-1}(z)| \leq n|\mathcal{U}|$ .  $\square$

**Proposition 2.4.** For every integer  $n \geq 1$ , any dynamical system conjugate to a positively  $n$ -expansive dynamical system is also positively  $n$ -expansive.

*Proof.* Let us consider two dynamical systems  $(X, f)$  and  $(Y, g)$  respectively on metric spaces  $(X, d)$  and  $(Y, d')$ . Assume that  $(X, f)$  is positively  $n$ -expansive and consider  $\varepsilon > 0$  such that for every  $x \in X$ ,  $|W_\varepsilon(x)| \leq n$ . Let us also consider  $\sigma : X \rightarrow Y$  a homeomorphism such that  $\sigma \circ f \circ \sigma^{-1} = g$ . Since  $\sigma^{-1}$  is continuous, there exists some  $\epsilon > 0$  such that if  $d(x, y) < \epsilon$  then  $d'(\sigma^{-1}(x), \sigma^{-1}(y)) < \varepsilon$ . Then for every  $y \in Y$ ,

$$W_\epsilon(y) = \{z \in Y \mid \forall t \in \mathbb{N}, d(\sigma \circ f^t(\sigma^{-1}(z)), \sigma \circ f^t(\sigma^{-1}(y))) < \epsilon\} \subset \sigma(W_\varepsilon(\sigma^{-1}(y))).$$

As a consequence  $|W_\epsilon(y)| \leq n$ . Since this is verified for every  $y$ , this means that  $(Y, g)$  is positively  $n$ -expansive.  $\square$

Two elements  $x, y$  of  $X$  are **asymptotic** when  $d(f^n(x), f^n(y)) \rightarrow 0$  when  $n \rightarrow +\infty$ . This defines an equivalence relation, and we denote by  $\text{Asym}(x)$  the equivalence class of  $x$  for this relation. Clearly, for every  $x$  and  $\varepsilon > 0$ , and any integer  $n$  such that  $n \leq |\text{Asym}(x)|$ , there exists some  $t \in \mathbb{N}$  such that  $|W_\varepsilon(f^t(x))| \geq n$ . In other words, we have the following.

**Remark 2.5.** If a system  $(X, f)$  which is a homeomorphism is positively  $n$ -expansive, then for every  $x \in X$ , we have  $|\text{Asym}(x)| \leq n$ .

**Example 2.6.** It is straightforward to see that even if a factor of a system is positively finitely expansive, the original system is not necessarily positively finitely expansive: one can consider for instance that the shift  $\{\infty 0^\infty\}$  is a factor of the full shift  $\{0, 1\}^{\mathbb{Z}}$ .

The above example leads by Remark 2.5 to the following simple observation.

**Corollary 2.7.** Suppose that an invertible dynamical system  $(X, f)$  is finitely positively expansive, and  $x \in X$  is not periodic. Then  $\text{Asym}(x)$  does not contain any periodic orbit.

## 2.2. Equicontinuity.

**Definition 2.8.** Let  $(X, f)$  be a dynamical system. If  $\varepsilon > 0$ , a point  $x$  is  $\varepsilon$ -**stable** if there exists  $\delta_x > 0$  such that for every  $y \in X$  such that  $d(x, y) < \delta_x$  we have  $d(f^t(x), f^t(y)) < \varepsilon$  for every  $t \in \mathbb{N}$ . A point  $x$  is **equicontinuous** if it is  $\varepsilon$ -stable for every  $\varepsilon > 0$ . The system  $(X, f)$  is said to be **equicontinuous** when all points are equicontinuous (in which case compactness allows to chose  $\delta_x$  uniformly). If  $\varepsilon > 0$ ,  $(X, f)$  is said to be  $\varepsilon$ -**sensitive** when no point is  $\varepsilon$ -stable. It is **sensitive** if it is  $\varepsilon$ -sensitive for some  $\varepsilon > 0$ .

In the following, we will say that a dynamical system  $(X, f)$  is **finite** (resp. **countable**, **uncountable**) when  $X$  is so.

**Proposition 2.9.** Every finitely positively expansive system with constant  $\varepsilon > 0$  is the disjoint union of a discrete set of equicontinuous points and a (perfect)  $\varepsilon$ -sensitive subsystem.

*Proof.* Let  $x \in X$  be  $\varepsilon$ -stable: the ball of radius  $\delta_x$  around  $x$  is included in the finite set  $W_\varepsilon(x)$ . This implies that  $x$  is isolated, hence equicontinuous. By continuity of  $f$ , the set of  $\varepsilon$ -unstable points is invariant by  $f$ . Moreover, it is closed because its complement is a union of isolated points, hence it yields a  $\varepsilon$ -sensitive subsystem.  $\square$

### Corollary 2.10.

- (1) In any finitely positively expansive system, every closed set of equicontinuous points is finite.
- (2) Any finitely positively expansive equicontinuous system is finite.
- (3) Any finitely positively expansive system  $(X, f)$  over a perfect space  $X$  is sensitive.
- (4) Any finitely positively expansive system admits only finitely many periodic points of each period  $p \in \mathbb{Z}_+$ .
- (5) Any finitely positively expansive system is either finite or uncountable.

*Proof.*

- (1) Every closed discrete set is finite.
- (2) This is clear from (1).
- (3) This is clear.
- (4) The restriction of  $f$  to the set of periodic points of period  $n$  is an equicontinuous subsystem.
- (5) It is folklore that every non-empty sensitive dynamical system is uncountable (as is every non-empty perfect Hausdorff space).  $\square$

**2.3. Transitivity and periodicity.** A dynamical system  $(X, f)$  is said to be **aperiodic** when it has no periodic point. We also say that a point  $x$  is **preperiodic** if its orbit  $\mathcal{O}_f(x)$  is finite, and **asymptotically periodic** if there exist an integer  $p \geq 1$  and  $y \in X$  such that  $\lim_{t \rightarrow \infty} f^{pt}(x) = y$ .

**Proposition 2.11.** Assume that  $X$  is such that all open balls of radius  $\varepsilon$  are closed, and that  $(X, f)$  is an invertible finitely positively expansive system with constant  $\varepsilon > 0$ . If  $x \in X$  is not periodic, then  $\overline{\mathcal{O}_f(x)}$  is aperiodic. In particular, every asymptotically periodic point is periodic.

The assumptions of the proposition above are satisfied when  $X$  is totally disconnected, or more generally when  $\varepsilon$  is larger than the diameter of every connected component.

*Proof.* Assume that  $\overline{\mathcal{O}_f(x)}$  contains a point  $y$  with some period  $p \in \mathbb{Z}_+$ . First, suppose that there exists  $q \in \llbracket 0, p \rrbracket$  such that for every  $t \in \mathbb{N}$ ,  $d(f^t(x), f^{q+tp}(y)) < \varepsilon$ . In particular, for every  $k \in \mathbb{N}$ ,  $d(f^{kp+q}(x), f^{q+tp}(y)) < \varepsilon$ . This means that  $\mathcal{O}_{f^p}(x) \subset W_\varepsilon(f^q(y))$ . Since this set is finite,  $x$  is periodic, which is a contradiction. Now, suppose that for every  $q \in \llbracket 0, p \rrbracket$ , there exists  $t \in \mathbb{N}$  such that  $d(f^t(x), f^{q+tp}(y)) \geq \varepsilon$ . Since  $y \in \overline{\mathcal{O}_f(x)}$ , for every  $\ell > t$ , there exists  $t_\ell$  such that  $f^{t_\ell}(x)$  is in the open neighborhood  $\bigcap_{i=0}^{\ell-1} f^{-i}(\mathcal{B}_\varepsilon(f^{q+i}(y)))$  of  $f^q(y)$ , for some  $q \in \llbracket 0, p \rrbracket$ . Thanks to our assumption, one can assume  $t_\ell > t$  to be minimal, so that  $d(f^{t_\ell-1}(x), f^{q-1 \bmod p}(y)) \geq \varepsilon$ . By compactness of  $X$  (and finiteness of  $\llbracket 0, p \rrbracket$ ), there exists  $q \in \llbracket 0, p \rrbracket$  such that  $f^{t_\ell-1}(x)$  admits a limit point  $z$  such that  $d(z, f^q(y)) \geq \varepsilon$  and for every  $i \in \mathbb{N}$ ,  $d(f^i(z), f^{q+i}(y)) \leq \varepsilon$ . But by assumption,  $\mathcal{B}_\varepsilon(f^q(y))$  is a closed set, hence the last inequality becomes  $d(f^i(z), f^{q+i}(y)) < \varepsilon$ . Again, this means that  $\mathcal{O}_{f^p}(z) \subset W_\varepsilon(f^q(y))$ . Since this set is finite,  $z$  is periodic, which contradicts that  $d(f^{-1}(z), f^{q-1}(y)) \geq \varepsilon$ .  $\square$

The conclusion of Proposition 2.11 does not imply that, in the case when  $f$  is invertible, the closure of the twosided orbit  $\overline{\mathcal{O}_f^{\mathbb{Z}}(x)}$  is aperiodic, though. For instance, consider the configuration  $x$  such that

$x_{\mathbb{Z}_-} = {}^\infty 0$  and  $x_{\mathbb{N}}$  is the concatenation of all finite iterates of the Thue-Morse substitution from symbol 0.

**Remark 2.12.** *Proposition 2.11 implies that transitive finitely positively expansive systems over a totally disconnected space are aperiodic or finite. Indeed, every transitive dynamical system admits a point  $x \in X$  such that  $\mathcal{O}_f(x)$  is dense. This is actually a characterization of transitivity, in the case when  $f$  is surjective or when  $X$  is perfect (see for instance in [KS97]).*

**2.4. Entropy.** While finitely positively expansive invertible dynamical systems can be act on infinite spaces, they must have low complexity. Strictly speaking, after presenting some definitions and known facts in Section 2.4.1, we prove in Section 2.4.2 that every such system has zero topological entropy.

**2.4.1. Definitions.** In this section we recall a few facts about topological entropy which will be used later. We direct the reader who is not familiar with this notion to any standard textbook on ergodic theory, e.g. [Wal82].

Any set of subsets of  $X$ , denoted by  $\mathcal{U}$ , is a **cover** of  $X$  if:

$$\bigcup_{u \in \mathcal{U}} u = X.$$

A cover is said to be **open** when all its elements are open sets. Provided a cover  $\mathcal{U}$ , we denote, for every  $n \in \mathbb{N}$ , by  $\bigvee_{i=0}^{n-1} f^{-i}\mathcal{U}$ , the set of all non-empty subsets of  $X$  of the form:

$$u_0 \cap f^{-1}(u_1) \cap \dots \cap f^{-n-1}(u_{n-1}),$$

where for every  $i \in \llbracket 0, n \llbracket$ ,  $u_i \in \mathcal{U}$ . Clearly, for every  $n \in \mathbb{N}$ ,  $\bigvee_{i=0}^{n-1} f^{-i}\mathcal{U}$  is also a cover of  $X$  and it is open when  $\mathcal{U}$  was open.

**Notation 2.13.** *For every  $K \subset X$  compact and  $\mathcal{U}$  open cover of  $K$ , we denote by  $\mathcal{N}(K, \mathcal{U})$  the minimal number of elements of  $\mathcal{U}$  which form a cover of  $K$  (because  $K$  is compact, this number is well defined). When  $K = X$ , we simplify the notation  $\mathcal{N}(X, \mathcal{U})$  and simply write  $\mathcal{N}(\mathcal{U})$ .*

The sequence  $n \mapsto \mathcal{N}\left(\bigvee_{i=0}^{n-1} f^{-i}\mathcal{U}\right)$  is sub-multiplicative. As a consequence, the sequence  $n \mapsto \log_2\left(\mathcal{N}\left(\bigvee_{i=0}^{n-1} f^{-i}\mathcal{U}\right)\right)/n$  converges. We denote this limit by  $h(\mathcal{U}, X, f)$  and call it **entropy of the cover  $\mathcal{U}$** . The **topological entropy**  $h(X, f)$  of the system  $(X, f)$  is defined as the supremum, over finite open covers  $\mathcal{U}$  of  $X$ , of  $h(\mathcal{U}, X, f)$ .

For any open cover  $\mathcal{U}$ , we call **closure** of  $\mathcal{U}$  and denote it by  $\overline{\mathcal{U}}$ , the set of  $\overline{u}$  for  $u \in \mathcal{U}$ . The following is straightforward:

**Lemma 2.14.** *The topological entropy of  $(X, f)$  is also equal to the supremum of  $h(\overline{\mathcal{V}}, X, f)$ , where  $\mathcal{V}$  ranges over the open covers.*

A set  $E \subset X$  is  **$(n, \epsilon)$ -separated**, if for any distinct  $x, y \in E$  there is some  $0 \leq i < n$  such that  $d(f^i(x), f^i(y)) > \epsilon$ . Let us denote  $s(n, \epsilon)$  the maximal cardinality of an  $(n, \epsilon)$ -separated set. We will also need the following well known fact, proven first by Bowen:

**Theorem 2.15.** *The entropy of the system  $(X, f)$  is given by the following formula:*

$$h(X, f) = \lim_{\epsilon \rightarrow 0} \limsup_n \frac{1}{n} \log(s(n, \epsilon))$$

**2.4.2. Finitely positively expansive systems have zero entropy.**

**Lemma 2.16.** *Let us consider any finite open cover  $\mathcal{U}$  of  $X$ , and  $K_n$  a decreasing sequence of compact subsets of  $X$ . Then  $\mathcal{N}(K_n, \mathcal{U})$  converges towards  $\mathcal{N}(K, \mathcal{U})$ , where*

$$K = \bigcap_n K_n.$$

*Proof.* Clearly for every  $n$ ,  $\mathcal{N}(K_n, \mathcal{U}) \geq \mathcal{N}(K, \mathcal{U})$  and the sequence  $n \mapsto \mathcal{N}(K_n, \mathcal{U})$  is non-increasing. It is thus sufficient to prove that for some  $n$ ,  $\mathcal{N}(K_n, \mathcal{U}) = \mathcal{N}(K, \mathcal{U})$ . Let us assume that for every subset  $\mathcal{V}$  of  $\mathcal{U}$  of cardinality  $k := \mathcal{N}(K, \mathcal{U})$  and for every  $n$ , there exists some point  $x_n$  of  $K_n$  outside of the set

$$\mathcal{E}(\mathcal{V}) := \bigcup_{u \in \mathcal{V}} u.$$

Fix any  $\mathcal{V} \subset \mathcal{U}$  of cardinality  $k := \mathcal{N}(K, \mathcal{U})$  and  $x_n \in K_n$  as above. By compactness, there exists a subsequence of  $(x_n)$  which converges towards some  $x \in X$  and clearly  $x \in K$ . Since  $\mathcal{E}(\mathcal{V})$  is open, its complement is closed. As a consequence,  $x \in X \setminus \mathcal{E}(\mathcal{V})$ . Hence  $K$  is not contained in  $\mathcal{E}(\mathcal{V})$ . Since it is true for every  $\mathcal{V}$ , this means that  $\mathcal{N}(K, \mathcal{U}) > k$  which is a contradiction.  $\square$

**Theorem 2.17.** *Any invertible finitely positively expansive system  $(X, f)$  has zero entropy.*

*Proof.* Let  $N$  be such that the system  $(X, f)$  is positively  $N$ -expansive, and  $\varepsilon > 0$  such that for every  $x$ ,  $|W_\varepsilon(x)| \leq N$ .

(1) **Notation simplification.** In this proof, in order to simplify a bit the formula, let us set

$$\mathcal{U}^n := \bigvee_{i=0}^{n-1} f^{-i}\mathcal{U}$$

for every  $n$  and all cover  $\mathcal{U}$ .

(2) **Claim.** Let us consider any open cover  $\mathcal{U}$  whose every element has diameter smaller than  $\varepsilon/2$ , and fix any integer  $n \geq 1$ . We claim that there exists some  $m_n$  such that for every  $m \geq m_n$ , and every sequence  $u_0, \dots, u_m$  of elements of  $\overline{\mathcal{U}}$ , we have:

$$\mathcal{N}(f^{-n}(u_0) \cap f^{-1-n}(u_1) \cap \dots \cap f^{-m-n}(u_m), \mathcal{U}^n) \leq N.$$

(3) **Proof of the claim.** If the claim were false, there would exist an integer  $n$  and an infinite sequence  $v_0, v_1, \dots \in \overline{\mathcal{U}}$  such that for every  $m_n$  there is  $m > m_n$  such that

$$\mathcal{N}(f^{-n}(v_0) \cap f^{-1-n}(v_1) \cap \dots \cap f^{-m-n}(v_m), \mathcal{U}^n) \geq N + 1.$$

By Lemma 2.16, we have

$$\mathcal{N}(K, \mathcal{U}^n) \geq N + 1,$$

where

$$K = f^{-n} \left( \bigcap_{i \in \mathbb{N}} f^{-i}(v_i) \right).$$

However, by positive  $N$ -expansiveness and because  $f$  is a homeomorphism,  $|K| \leq N$ . Hence  $\mathcal{N}(K, \mathcal{U}^n) \leq N$ , which is a contradiction. This proves the claim.

(4) **Consequences.** We deduce from the above claim that for every  $m \geq m_n$ , and for any element  $u$  of  $\overline{\mathcal{U}^m}$ , there exists a set of  $N$  elements of  $\mathcal{U}^n$  which cover  $f^{-n}(u)$ , and thus there are  $N$  elements of  $\overline{\mathcal{U}^n}$  which also cover  $f^{-n}(u)$ . This implies:

$$\mathcal{N}(\overline{\mathcal{U}^{n+m}}) \leq N \cdot \mathcal{N}(\overline{\mathcal{U}^m}).$$

Since this is satisfied for every  $m \geq m_n$ , for every  $k \geq 1$ , we have:

$$\mathcal{N}(\overline{\mathcal{U}^{m+kn}}) \leq N^k \mathcal{N}(\overline{\mathcal{U}^m}).$$

With an application of the logarithm function:

$$\frac{\log(\mathcal{N}(\overline{\mathcal{U}^{m+kn}}))}{m + kn} \leq \frac{k}{m + kn} \log(N) + \frac{\log(\mathcal{N}(\overline{\mathcal{U}^m}))}{m + kn}.$$

By taking  $k \rightarrow +\infty$ , we obtain:

$$h(\overline{\mathcal{U}}) \leq \frac{\log(N)}{n}.$$

Since  $n$  was taken arbitrary,  $h(\overline{\mathcal{U}}) = 0$ . As a direct consequence of Lemma 2.14, the topological entropy of  $(X, f)$  is equal to zero.  $\square$

In contrast, finitely positively expansive systems which are not homeomorphisms can have arbitrary entropy value: consider for instance one-sided shifts, which are all positively 1-expansive.

**2.5. Shadowing property.** It is known that an invertible dynamical system which is positively expansive is finite. The situation is different for the property of finite positive expansiveness. For instance we will see some counter-examples in Section 4 (in the context of symbolic dynamics). On the other hand, we can recover the same result as for positive expansiveness by adding the constraint of having *shadowing property*.

**Definition 2.18.** For a dynamical system  $(X, f)$  on a metric space  $(X, d)$ , an  $\epsilon$ -**pseudo-orbit** is a sequence  $(x_n)_{n \in \mathbb{N}}$  such that for every  $n \in \mathbb{N}$ ,

$$d(f(x_n), x_{n+1}) < \epsilon.$$

If for  $p, q \in X$  there is  $m > 0$  and a finite sequence  $(x_0, \dots, x_m)$  such that for each  $n = 0, \dots, m-1$

$$d(f(x_n), x_{n+1}) < \epsilon.$$

and  $x_0 = p, x_m = q$  then we say that there is an  $\epsilon$ -**chain** from  $p$  to  $q$ . If for every  $\epsilon > 0$  there is an  $\epsilon$ -chain between any two points  $p, q$  then the system  $(X, f)$  is *chain-transitive*.

**Definition 2.19.** A dynamical system  $(X, f)$  has the **shadowing property** when for every  $\epsilon > 0$ , there exists some  $\delta > 0$  such that every  $\epsilon$ -pseudo-orbit  $(x_n)$  is  $\delta$ -**shadowed** by some  $x \in X$  such that for every  $n \in \mathbb{N}$ ,

$$d(x_n, f^n(x)) < \epsilon.$$

The following theorem is contained in Theorem 3.7 in [LO13]:

**Theorem 2.20.** A dynamical system with the shadowing property has positive entropy if and only if it has a non-equicontinuous point.

Now we can recover [ACCV20, Theorem G], with another proof, based on relations between shadowing property and entropy.

**Theorem 2.21.** Any invertible finitely positively expansive system with the shadowing property is finite.

*Proof.* From Theorem 2.17, such a system must have entropy 0. From Theorem 2.20, this implies that the system is equicontinuous. We can conclude by Corollary 2.10(2).  $\square$

### 3. DYNAMICAL SYSTEMS ON ONE-DIMENSIONAL CONTINUA

It is a well known fact that the interval has no positively expansive dynamical system (e.g. see [ALM00, AH94]). In this section, we prove that it also has no finitely positively expansive one. On the other hand, we know that there are positively expansive systems on the circle ( $z^n$  maps for  $n \notin \{-1, 0, 1\}$ ). This observation naturally leads to the following questions.

**Question 3.1.** Can we find, for every  $n \geq 1$ , a positively  $(n+1)$ -expansive map of the circle which is not positively  $n$ -expansive?

**Question 3.2.** Which one-dimensional continua admit finitely positively expansive systems?

We leave the first question open, but provide some results in the direction of the second one.

**3.1. Systems over the interval.** The aim of this section is to prove Theorem 3.8. For the remainder of this section, we set  $I := [0, 1]$ .

A dynamical system  $(X, f)$  is said to be *mixing* when there exists an integer  $N$  such that for every non-empty open subsets  $u, v$  of  $X$ , and for every  $n \geq N$ ,  $f^n(u) \cap v \neq \emptyset$ .

**Proposition 3.3.** For every sensitive system  $(I, f)$  over the interval, there exists a non-degenerate subinterval  $J \subset I$  and some  $k > 0$  such that  $f_{|J}^k$  is mixing. Moreover, it has positive entropy.

*Proof.* From [Blo82, Theorem 1.3] (see [Rue17, Proposition 2.40] for a proof), every sensitive system of the interval admits a non-degenerate (so-called *periodic*) interval  $J'$  and  $p > 0$  such that  $f_{|\bigsqcup_{0 \leq i < p} f^i(I)}$  is a transitive subsystem. This implies that  $f_{|J'}^p$  is a transitive system over interval  $J$ . Moreover, [Rue17, Propositions 2.16 and 2.17] imply that  $f_{|J}^{2p}$  is mixing, for some non-degenerate interval  $J \subset I$ . By [Blo82, Lemma 1.2] (see [Rue17, Proposition 4.70]), any transitive system of some non-degenerate interval has positive entropy.  $\square$



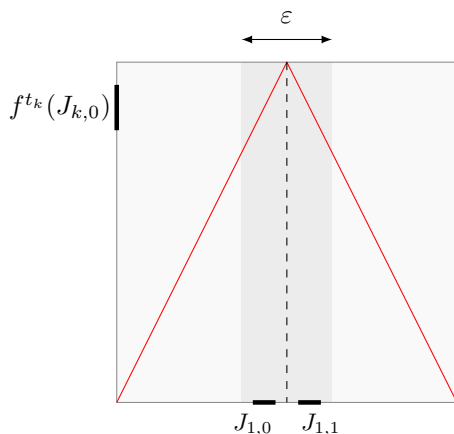


FIGURE 1. Illustration of the intervals  $J_{k,i}$  and  $f^{t_k}(J_{k,0})$  from the proof of Lemma 3.7, where  $f$  is the tent map and  $k = 1$ .

**Remark 3.4.** Let  $(X, f)$  be a dynamical system,  $n \geq 1$ ,  $\varepsilon > 0$ ,  $J \subset X$  with cardinality  $n + 1$ , and  $t$  such that:

$$\forall j \in \llbracket 0, t \rrbracket, \text{diam } f^j(J) < \varepsilon \text{ and } |f^t(J)| = 1.$$

Then  $(X, f)$  is not positively  $n$ -expansive with constant  $\varepsilon$ . Indeed,  $J \subseteq W_\varepsilon(x)$ , for any  $x \in J$ .

**Lemma 3.5.** Let us consider an interval system  $f: I \rightarrow I$ , and  $c$  in the interior of  $f(I)$ . For every  $\varepsilon > 0$ , there exists an interval  $J \subset I$  such that  $\text{diam } J < \varepsilon$  such that  $c$  is in the interior of  $f(J)$ .

*Proof.* Let us assume ad absurdum that there exists  $\varepsilon > 0$  such that for all  $J \subset I$  such that  $\text{diam } J < \varepsilon$ ,  $c$  is not in the interior of  $f(J)$ . This implies that for all  $x \in f^{-1}(c)$ , there exists some  $J \subset I$  such that  $x \in J$  and  $f(J) \leq c$  or  $f(J) \geq c$ . Let us denote by  $\delta^+$  (resp.  $\delta^-$ ) the set of  $x \in f^{-1}(c)$  such that there exists some  $J$  containing  $x$  such that  $f(J) \geq c$  (resp.  $f(J) \leq c$ ). If any of the sets  $\delta^+, \delta^-$  was equal to  $f^{-1}(c)$ , then we would have  $f(I) \leq c$  or  $f(I) \geq c$  which was assumed to be false ( $c$  is in the interior of  $f(I)$ ). Without loss of generality, we can assume that there exists  $x \in \delta^-, x' \in \delta^+$  such that  $x < x'$ . Let us then denote by  $y$  the supremum over the elements of  $\delta^-$  which are smaller than  $x'$ , and  $z$  the infimum over the elements of  $\delta^+$  which are greater than  $y$ . Then else  $y < z$ , and by the intermediate value theorem one can find some  $x'' \in f^{-1}(c)$  such that  $y < x'' < z$ , and thus can't be in  $\delta^-$  or  $\delta^+$ , which is impossible, or  $y = z$  which means that there exists some  $J$  containing  $z$  in its interior such that  $f(J) = \{c\}$ , which contradicts the hypothesis ad absurdum. This concludes the proof.  $\square$

**Remark 3.6.** For every interval continuous map  $f: I \rightarrow I$ , and every interval  $J \subset f(I)$ , there exists a subinterval  $I' \subset I$  such that  $f(I') = J$ .

**Lemma 3.7.** If an interval system  $f: I \rightarrow I$  is mixing, then it is not finitely positively expansive.

*Proof.* Since  $f$  is mixing,  $f$  is not monotone. As a consequence,  $f$  has a local extremum  $c \in \text{int } I$ . Let us fix some  $\varepsilon > 0$  and prove, by induction over  $k \in \mathbb{N}$ , that there exist  $2^k$  disjoint intervals  $J_{k,1}, \dots, J_{k,2^k}$  and an integer  $t_k$  such that:

$$\forall j \in \llbracket 0, t_k \rrbracket, \text{diam} \left( \bigcup_{i=1}^{2^k} f^j(J_{k,i}) \right) < \varepsilon \text{ and } \forall i, j \in \llbracket 1, 2^k \rrbracket, f^{t_k}(J_{k,i}) = f^{t_k}(J_{k,j}).$$

One can find an illustration of these sets for the tent map on Figure 1. The above statement implies that  $f$  is not positively  $(2^k - 1)$ -expansive: indeed,  $J = f^{-t_k}(x) \cap \bigcup_{i=1}^{2^k} f^j(J_{k,i})$ , where  $x \in f^{t_k}(J_{k,0})$  is arbitrary, satisfies the hypothesis of Remark 3.4.

- (1) **Base case.** The case  $k = 0$  is trivial, for any choice of interval  $J_{0,1}$  with length less than  $\varepsilon$ .
- (2) **Induction step.** Let us assume that the assumption has been proved for some  $k \in \mathbb{N}$ . Since  $f$  is mixing and  $c \in \text{int } I$ , [Rue17, Proposition 2.8] gives some  $t > 0$  such that

$$c \in \text{int } f^t(f^{t_k}(J_{k,0})).$$

By uniform continuity of  $f^j$ , there exists  $\delta > 0$  such that for every subinterval  $J \subset I$  with length less than  $\delta$  and every  $j \in \llbracket 0, t \rrbracket$ ,  $\text{diam } f^j(J) < \varepsilon$ . By Lemma 3.5 applied to  $f^t$ , there exists a

subinterval  $J \subset f^{t_k}(J_{k,0})$  with length less than  $\delta$  (hence for every  $j \in \llbracket 0, t \rrbracket$ ,  $\text{diam } f^j(J) < \varepsilon$ ) such that

$$c \in \text{int } f^t(J).$$

Set  $t_{k+1} = t_k + t$ . Since  $c$  is a local extremum, there are disjoint intervals  $I^- \subset f^t(J) \cap [0, c)$  and  $I^+ \subset f^t(J) \cap (c, 0]$  such that  $f(I^-) = f(I^+)$ . For every  $j \in \llbracket 1, 2^k \rrbracket$ , the induction step gives that  $J_{k,j} \cap f^{-t-t_k}(I^-)$  and  $J_{k,j} \cap f^{-t-t_k}(I^+)$  are nontrivial intervals. Hence one can apply Remark 3.6 to the restriction of  $f^{t_{k+1}}$  to these subintervals and to  $J$ : one finds two subintervals  $J_{k+1,2j} \subset J_{k,j} \cap f^{-t_{k+1}}(I^-)$  and  $J_{k+1,2j+1} \subset J_{k,j} \cap f^{-t_{k+1}}(I^+)$  such that  $f^{t_{k+1}}(J_{k+1,2j}) = f(I^-) = f(I^+) = f^{t_{k+1}}(J_{k+1,2j+1})$ . The collection  $(J_{k+1,i})_i$  satisfies the wanted properties.  $\square$

**Theorem 3.8.** *No interval system is finitely positively expansive.*

*Proof.* By Proposition 3.3 and Item 3 of Corollary 2.10, any finitely positively expansive interval system would admit a non-degenerate subinterval  $J$  and  $p > 0$  such that  $f_{|J}^p$  is mixing. From Remark 2.1,  $f_{|J}^p$  would also be a finitely positively expansive system of an interval. This contradicts Lemma 3.7.  $\square$

**3.2. Hawaiian earring.** We present exploratory result towards Question 3.2. What differentiates the interval and the circle regarding finite positive expansiveness is the possibility to have locally invertible expanding continuous maps. The example of **Hawaiian earring** reveals another (topological) property which prevents existence of finitely positively expansive maps. The Hawaiian earring is defined as  $\mathcal{H} := \bigcup_{n \in \mathbb{Z}_+} C_n$ , where:

$$C_n := \left\{ (x, y) \in \mathbb{R}^2 \mid \left( x - \frac{1}{n} \right)^2 + y^2 \leq \frac{1}{n^2} \right\}.$$

This definition is illustrated on Figure 2.

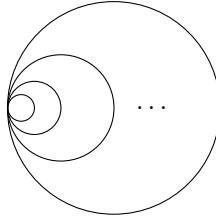


FIGURE 2. Illustration of the Hawaiian earring.

The Hawaiian earring is an example of topological graph. The formal definition of topological graph can be found, for instance, in [Cro78]. What is relevant here is that a **topological graph**  $G$  is a locally connected union of (possibly countably many) topological intervals (i.e. subsets of  $\mathbb{R}^3$  homeomorphic to the interval  $I$ ) which do not intersect in their interiors (i.e.  $e \cap \text{int } e' \neq \emptyset$  for every distinct such intervals  $e, e'$ ). The **degree** of a point  $w \in G$ , denoted by  $\text{deg}_G(w)$ , is the limsup of the number of connected components of  $\mathcal{B}_\varepsilon(w) \setminus \{w\}$ , when  $\varepsilon$  goes to 0. By definition, the degree of any point is in  $\llbracket 1, \infty \rrbracket$ . For instance, every point in the interior of one interval from the definition of the graph has degree 2. We call **vertex** every point which has degree 1 or at least 3. A **subgraph** of  $G$  is simply a closed subset  $G' \subset G$  (in that case it is also a topological graph).

**Lemma 3.9.** *Let  $G$  be a topological graph,  $f : G \rightarrow G$  continuous such that  $(G, f)$  is a positively  $n$ -expansive system, and  $G' \subset G$  a subgraph. Then for every  $w \in G'$ ,  $\text{deg}_{G'}(w) \leq n \text{deg}_{f(G')}(f(w))$ .*

*Proof.* Were it not the case,  $f$  would map strictly more than  $n$  disjoint non-trivial (open) intervals with endpoint  $w$  onto some common non-trivial interval with endpoint  $f(w)$ . Therefore, one could find a set of  $(n+1)$  points arbitrarily close to each other (and to  $w$ ), all mapped to the same point, contradicting Lemma 2.2.  $\square$

**Remark 3.10.** *For any integer  $n$  and every  $k \leq n$ , we can construct a map  $f$  on  $C_{\llbracket 1, n \rrbracket}$  which is positively  $k$ -expansive and not positively  $(k-1)$ -expansive: for instance a map which fixes 0, acts as  $z \mapsto z^2$  on each circle  $C_k, \dots, C_n$ , and maps the circles  $C_1, \dots, C_{k-1}$  onto  $C_k$  in such a way that for every  $i < k$ ,  $f_{|C_i}$  is one-to-one.*

**Proposition 3.11.** *There is no finitely positively expansive map on the Hawaiian earring.*

*Proof.* Let us consider a map  $f$  from the Hawaiian earring to itself and assume ad absurdum that it is finitely positively expansive, and fix  $\varepsilon > 0$  and  $N$  such that for all  $x$ ,  $|W_\varepsilon(x)| \leq N$ .

- (1) **The accumulation point  $\mathbf{0}$  is fixed by  $f$ .** This is a direct application of Remark 3.9 to  $w = \mathbf{0}$ , because  $\deg_{\mathcal{H}}(\mathbf{0}) = \infty$ .
- (2) **There exists a finite  $\Lambda \subset \mathbb{Z}_+$  and some map  $\kappa : \mathbb{Z}_+ \setminus \Lambda \rightarrow \mathbb{Z}_+$  such that for every  $n \in \mathbb{Z}_+ \setminus \Lambda$ ,  $f(C_n)$  is contained in  $C_{\kappa(n)}$ .** Indeed, let  $\Lambda := \{n \in \mathbb{Z}_+ \mid f^{-1}(\mathbf{0}) \cap C_n \setminus \{\mathbf{0}\} \neq \emptyset\} \neq \emptyset$ . By Corollary 2.3 and disjointness of all  $C_n \setminus \{\mathbf{0}\}$ ,  $\Lambda$  is finite. Moreover, for  $n \in \mathbb{Z}_+ \setminus \Lambda$ , the image of  $C_n \setminus \{\mathbf{0}\}$  by  $f$  is a connected set not containing  $\mathbf{0}$ , hence it is contained in some circle  $C_{\kappa(n)}$  (and so is  $C_n$ , since  $f(\mathbf{0}) = \mathbf{0} \in C_{\kappa(n)}$ ).
- (3) **There exists a finite  $\Gamma \supset \Lambda$  such that for every  $n \in \mathbb{Z}_+$ , there exists  $t \in \mathbb{N}$  such that  $\kappa^t(n) \in \Gamma$ .** Indeed, by Point 3 of Corollary 2.10,  $f$  is  $\varepsilon$ -sensitive. There exists  $p$  such that  $\text{diam}(C_{\llbracket p, \infty \rrbracket}) \leq \varepsilon$ . Let  $\Gamma := \llbracket 0, p \rrbracket \cup \Lambda$ . Let  $n \in \mathbb{Z}_+$ ,  $x \in C_n \setminus \{\mathbf{0}\}$ , and  $\delta > 0$  such that  $\mathcal{B}_\delta(x) \subset C_n$ . By  $\varepsilon$ -sensitivity, there exists  $t \in \mathbb{N}$  and  $y \in \mathcal{B}_\delta(x)$  such that  $d(f^t(x), f^t(y)) > \varepsilon$ . This means that at least one of the points  $f^t(x)$  and  $f^t(y)$  is in  $C_{\llbracket 0, p \rrbracket} \subset C_\Gamma$ . Since  $x, y \in C_n$ , we deduce that either  $\kappa^t(n) \in \Gamma$ , or  $\kappa^t(n)$  is undefined. In the latter case, it means that there exists  $t' < t$  such that  $\kappa^{t'}(n) \in \Lambda \subset \Gamma$ .
- (4) **There is a finite  $J \supset \Gamma$  such that  $f(C_J) \subset C_J$ , and for every  $n \in \mathbb{Z}_+$ , there exists  $t \in \mathbb{N}$  such that  $f^t(C_n) \subset C_J$ .** Let us prove this. First, for every  $n \in \mathbb{Z}_+$ , the set  $f(C_n)$  is contained in some  $C_{D_n}$  for some finite set  $D_n \subset \mathbb{Z}_+$  (otherwise the point  $\mathbf{0}$  would have infinitely many preimages, which is not possible). Consequently,  $f(C_\Gamma)$  is contained in some  $C_D$  for some finite set  $D := \bigcup_{n \in \Gamma} D_n \subset \mathbb{Z}_+$ . Since  $D \setminus \Gamma$  does not intersect  $\Lambda$ ,  $\kappa$  is well defined over  $D \setminus \Gamma$ . Furthermore, by the previous point, for every  $n \in D \setminus \Gamma$ , there exists some  $t_n > 0$  such that  $\kappa^{t_n}(n) \in \Gamma$ . Now let

$$J := \Gamma \cup \left\{ \kappa^{t'}(n) \mid n \in D \setminus \Gamma, 0 \leq t' \leq t_n \right\}.$$

Let us prove that  $f(C_J) \subset C_J$ . If  $m \in \Gamma$ , then  $f(C_m) \subset C_D \subset C_{\kappa^0(D \setminus \Gamma) \cup \Gamma} \subset C_J$ . If  $m \in J \setminus \Gamma$ , then there exists  $n \in D \setminus \Gamma$  and  $t' < t_n$  (note that  $t' = t_n$  implies that  $m \in \Gamma$ ) such that  $m = \kappa^{t'}(n)$ ; then  $f(C_m) \subset C_{\kappa(m)} = C_{\kappa^{t'+1}(n)} \subset C_J$ . The other statement is directly derived from the previous point and the fact that  $J \supset \Gamma$ .

- (5) **The system is not positively  $N$ -expansive.** Let  $n > N|J|$ . From the previous point, there exists  $t \in \mathbb{N}$  such that  $f^t(C_{\llbracket 1, n \rrbracket}) \subset C_J$ . Then  $\deg_{C_{\llbracket 1, n \rrbracket}}(\mathbf{0}) = n > N|J| \geq N \deg_{f^t(C_{\llbracket 1, n \rrbracket})}(f^t(\mathbf{0}))$ , which, by Lemma 3.9, gives that  $f^t$  cannot be positively  $N$ -expansive. We conclude by Remark 2.1.  $\square$

This proof can be generalized to graphs with one infinite-degree vertex whose all neighborhoods have complements which are graphs defined as finite unions of intervals.

#### 4. TWO-SIDED SHIFTS

It is well known that  $\mathbb{N}$ -shifts are all positively expansive. We will see in this section that the case of  $\mathbb{Z}$ -shifts is more complex. Furthermore the facts that they are all homeomorphisms, and that  $\mathbb{Z}$ -shifts with the shadowing property are exactly the well-known shifts of finite type make this class a good playground for exploring generalizations of Theorem 2.21: this is the object of study of Section 4.2.

We also found some examples of  $\mathbb{Z}$ -shifts which are finitely positively expansive. As these examples can be obtained using substitutions, we explored the relation between substitutions and this property, and provide some sufficient conditions on the substitutions for the corresponding  $\mathbb{Z}$ -shift to be finitely positively expansive [Section 4.3]. We still leave open the following questions:

**Question 4.1.** *Which  $\mathbb{Z}$ -shifts are finitely positively expansive? Which substitutive /  $S$ -adic ones are?*

**4.1. Preliminary facts.** A standard reference to symbolic dynamics is [LM95]. Below we recall a few standard facts that will be used later. Let  $\mathcal{A}$  be a finite set which we will call **alphabet**. The *shift* action, denoted by  $\sigma$ , is the function from  $\mathcal{A}^{\mathbb{Z}}$  to itself such that for every  $x$  in this set and  $i \in \mathbb{Z}$ ,  $\sigma(x)_i = x_{i+1}$ . A  $\mathbb{Z}$ -shift  $X$  on alphabet  $\mathcal{A}$  is a dynamical system formed by the shift action restricted to a compact subset of  $\mathcal{A}^{\mathbb{Z}}$  which is stable under the shift action. We will denote by  $\mathcal{L}_r(X) := \{x_{\llbracket 0, r \rrbracket} \mid x \in X\}$  its **language** of length  $r \in \mathbb{N}$ .

In the following, for every finite set  $\mathcal{A}$  and every  $z \in \mathcal{A}^{\mathbb{N}}$ , we will denote by  $\bar{z}$  the element of  $\mathcal{A}^{-\mathbb{N}}$  defined by: for all  $k < 0$ ,  $\bar{z}_k = z_{-k-1}$ . Furthermore, for every  $z' \in \mathcal{A}^{\mathbb{N}}$  and  $z \in \mathcal{A}^{\mathbb{Z}-}$ , we will simply denote by  $zz'$  the element of  $\mathcal{A}^{\mathbb{Z}}$  which coincides with  $z$  on  $\mathbb{Z}-$ , and with  $z'$  on  $\mathbb{N}$ . For every  $a \in \mathcal{A}$ , we

will denote by  $a^\infty$  the infinite word  $z \in \mathcal{A}^\mathbb{N}$  such that for every  $k \in \mathbb{N}$ ,  $z_k = a$ . We also simplify the notation  $\overline{a^\infty}$  into  ${}^\infty a$ .

An asymptotically periodic configuration is one such that there exist  $q \in \mathbb{Z}$  and  $p \in \mathbb{Z}_+$ , respectively called **preperiod** and **asymptotic period** such that for every  $i \geq q$ ,  $x_{i+p} = x_i$ .

A shift  $X$  is said to be of **finite type** when there exists a finite set of words  $\mathcal{F}$  such that  $X = \{x \in \mathcal{A}^\mathbb{Z} \mid \forall i, j \in \mathbb{Z}, x_{[i, j]} \notin \mathcal{F}\}$ . The following is a well known fact by Walters [Wal78]:

**Proposition 4.2.** *Every  $\mathbb{Z}$ -shift has the shadowing property if and only if it is of finite type.*

As a consequence, Theorem 2.21 immediately implies the following.

**Corollary 4.3.** *Any finitely positively expansive  $\mathbb{Z}$ -shift of finite type is finite.*

It is natural to wonder if this result can be extended to sofic shifts, since they are to some extent similar to shifts of finite type. They are usually defined as follows:

**Definition 4.4.** *A  $\mathbb{Z}$ -shift  $X$  is **sofic** when there exists a shift of finite type  $Z$  and a factor map  $\phi : Z \rightarrow X$  (i.e. a shift-commuting continuous surjective map).*

Using a characterization of sofic shifts in terms of predecessor sets, we can indeed generalize the result to sofic shifts. As a matter of fact, we will see that it even holds for shifts with countably many predecessor sets.

**4.2. Predecessor sets.** There is a natural relation between sofic shifts and finite positive expansiveness, as they can be both characterized in terms of predecessor sets.

**Definition 4.5.** *A **predecessor** of infinite word  $z' \in \mathcal{A}^\mathbb{N}$  in  $\mathbb{Z}$ -shift  $X$  is some left-infinite word  $z \in \mathcal{A}^{\mathbb{Z}_-}$  such that  $\exists x \in X, x_{\mathbb{N}} = z'$  and  $x_{\mathbb{Z}_-} = z$ .*

A proof of the following can be derived from [LM95, Section 3.2]:

**Theorem 4.6.** *A  $\mathbb{Z}$ -shift is sofic if and only if it has a finite number of predecessor sets.*

The following is straightforward from the definition of predecessor sets:

**Lemma 4.7.** *A  $\mathbb{Z}$ -shift is positively  $n$ -expansive if and only if all of its predecessor sets have at most  $n$  elements. Equivalently, for every infinite word  $z$  which is the restriction of a configuration of the shift on  $\llbracket 0, \infty \rrbracket$ , and every integer  $\ell$ , the number of predecessors of  $z$  with length  $\ell$  is at most  $n$ .*

**Corollary 4.8.** *Every finitely positively expansive  $\mathbb{Z}$ -shift with countably many predecessor sets is finite.*

*Proof.* It is sufficient to see that these conditions imply that the shift is countable, and to use Item 5 of Corollary 2.10.  $\square$

**Corollary 4.9.** *Every finitely positively expansive sofic  $\mathbb{Z}$ -shift is finite.*

**4.3. S-adic shifts.** In this section, we consider  $\mathbb{Z}$ -shifts which are obtained from substitutions. For a finite alphabet  $\mathcal{A}$ , let us denote by  $\mathcal{A}^+$  the set of non-empty words over  $\mathcal{A}$ . A *substitution* is a function  $\tau : \mathcal{A} \rightarrow \mathcal{B}^+$ , where  $\mathcal{A}$  and  $\mathcal{B}$  are finite sets. In the following, all substitutions are assumed to be non-erasing, meaning none of its images is the empty word. Such a function can naturally be extended into a monoid homomorphism  $\tau^* : \mathcal{A}^* \rightarrow \mathcal{B}^*$ . On the other hand it is more difficult to extend naturally on  $\mathcal{A}^\mathbb{Z}$ . A more natural object to define in this case is the set of configurations in  $\mathcal{B}^\mathbb{Z}$  which can be obtained by application of the substitution on an element of  $\mathcal{A}^\mathbb{Z}$ . We say that such configurations can be *desubstituted*.

**Definition 4.10.** *Let us consider  $\tau : \mathcal{B} \rightarrow \mathcal{A}^+$  a substitution. For every  $x \in \mathcal{B}^\mathbb{Z}$ , we call **desubstitution scheme** of  $x$  for  $\tau$  any increasing sequence  $\mathbf{k} = (k_s)_{s \in \mathbb{Z}}$  in  $\mathbb{Z}$  such that there exists  $y \in \mathcal{A}$ , such that for every  $s \in \mathbb{Z}$  we have:*

$$x_{\llbracket k_s, k_{s+1} \rrbracket} = \tau(y_s).$$

*The sequence  $y$  is called a **desubstitution** of  $x$ . The desubstitution scheme and desubstitution are said to be **standard** when  $\max\{s \in \mathbb{Z} \mid k_s \leq 0\} = 0$ .*

We will denote by  $|w|$  the length of  $w$ , when  $w$  is a word on some alphabet  $\mathcal{A}$ . For two substitutions  $\tau : \mathcal{B} \rightarrow \mathcal{A}^+$  and  $\tau' : \mathcal{C} \rightarrow \mathcal{B}^+$ , the restriction of  $\tau^* \circ \tau'^*$  to  $\mathcal{C}$  is a substitution which is denoted, by abuse of notation, by  $\tau\tau'$ . Furthermore, for any substitution  $\tau : \mathcal{B} \rightarrow \mathcal{A}$ , we will set  $|\tau| := \min_{a \in \mathcal{B}} |\tau(a)|$  and  $\|\tau\| := \max_{a \in \mathcal{B}} |\tau(a)|$ , and call these numbers respectively the **minimal length** and **maximal length** of  $\tau$ .

**Remark 4.11.** *Provided a substitution  $\tau$ , if  $\mathbf{k}$  is a standard desubstitution scheme of  $x$  for  $\tau$ , then for every  $i \in \mathbb{N}$ , we have*

$$i|\tau| - \|\tau\| < i|\tau| + k_0 \leq k_i \leq i\|\tau\| + k_0 \leq i\|\tau\|.$$

**Remark 4.12.** *Let us consider a substitution  $\tau : \mathcal{B} \rightarrow \mathcal{A}^+$ ,  $x \in \mathcal{A}^{\mathbb{Z}}$  and  $i \in \mathbb{Z}$ . Moreover, if  $\mathbf{k}$  is a desubstitution scheme of  $x$  for  $\tau$ , and  $j \in \mathbb{Z}$ , then  $\mathbf{k} - j$  is a desubstitution scheme for  $\sigma^j(x)$ , and if  $\mathbf{k}$  was standard, then  $\sigma^m(\mathbf{k} - j)$  is a standard desubstitution scheme for  $\sigma^j(x)$ , where  $m := \max_{k_i \leq j} i$ . As a direct consequence of Remark 4.11,  $m \in \left[ \left\lfloor \frac{j}{\|\tau\|} \right\rfloor, \left\lceil \frac{j}{\|\tau\|} \right\rceil \right]$ .*

**Definition 4.13.** *Let  $\mathcal{A} = (A_t)_{t \in \mathbb{N}}$  be a sequence of non-empty finite sets and  $\tau = (\tau_t)_{t \in \mathbb{N}}$  be a sequence of substitutions such that for every  $t \in \mathbb{N}$ ,  $\tau_t$  is a function from  $\mathcal{A}_{t+1}$  to  $\mathcal{A}_t^+$ . Such a sequence is called **directive sequence on  $\mathcal{A}$** .*

The number  $\liminf_{t \rightarrow \infty} |\mathcal{A}_t|$  (which can be positive or infinite) is called the (alphabet) **rank** of the directive sequence  $\tau$ . We also set, for every  $t', t \in \mathbb{N}$  such that  $t \leq t'$ ,  $\tau_{[t, t']}] := \tau_t \cdots \tau_{t-1}$ , and  $\sigma^t \tau := (\tau_{t'})_{t' \geq t}$  the shifted directive sequence. For every  $t \in \mathbb{N}$ , we define  $\tau_{[0, t]]}(\mathcal{A}_t^{\mathbb{Z}})$  as the set of elements of  $\mathcal{A}_0^{\mathbb{Z}}$  which admit a desubstitution for  $\tau_{[0, t]]}$ . It is straightforward to see that for every  $t \in \mathbb{N}$ , an element of  $\mathcal{A}_0^{\mathbb{Z}}$  admits a desubstitution for  $\tau_{[0, t+1]]}$  if and only if it admits a desubstitution for  $\tau_{[0, t]]}$  which itself admits a desubstitution for  $\tau_t$ .

**Definition 4.14.** *The **S-adic limit set** associated to a directive sequence is the set defined as follows:*

$$\Omega_{\tau} := \bigcap_{t \in \mathbb{N}} \tau_{[0, t]]}(\mathcal{A}_t^{\mathbb{Z}}).$$

*By compactness,  $\Omega_{\tau}$  is a  $\mathbb{Z}$ -shift.*

**Remark 4.15.** *The term S-adic shift associated with a directive sequence  $\tau$  usually refers to the set of configurations whose patterns appear in at least some  $\tau_{[0, t]]}(a)$ ,  $t \in \mathbb{N}$ ,  $a \in \mathcal{A}_t$ . This set is contained in  $\Omega_{\tau}$ . Hence whenever the shift is finitely positively expansive on  $\Omega_{\tau}$ , it is also on this set.*

**Remark 4.16.** *If there exists  $t \in \mathbb{N}$  such that  $\mathcal{A}_t$  is a singleton  $\{a\}$  (in particular if the rank is 1), then  $\Omega_{\tau}$  is the shift orbit of the periodic configuration  ${}^{\infty}\tau_{[0, t]]}(a)^{\infty}$ . Alphabet rank 1 is thus more constrained than measure-theoretical rank 1, or even topological rank 1.*

When all the substitutions in a directive sequence  $\tau$  are equal to the same substitution  $\tau$ ,  $\Omega_{\tau} := \Omega_{\tau}$  is then the limit set of substitution  $\tau$ . The rank of the directive sequence is equal to the size of the alphabet  $\mathcal{A}_0$  (hence is finite).

S-adic shifts can be much richer than substitution ones, since they include (in some weak sense) all minimal shifts (see [HPS92]). In particular, they cannot all be finitely positively expansive, since some of them have positive entropy. In the present section, we study sufficient conditions for S-adic limit sets (and thus the corresponding S-adic shifts) to have finite positive expansiveness.

**Definition 4.17.** *We say that a directive sequence  $\tau$  is **everywhere-growing** when it satisfies the equality  $\lim_{t \rightarrow \infty} |\tau_{[0, t]]}| = \infty$ . We say that it is **expanding** when each of the substitutions  $\tau_t$ ,  $t \in \mathbb{N}$ , is **expanding**, which means that  $|\tau_t| \geq 2$ .*

**Remark 4.18.** *A directive sequence  $\tau$  is everywhere-growing if and only if for every  $t' \in \mathbb{N}$ ,  $\lim_{t \rightarrow \infty} |\tau_{[t', t]]}| = \infty$ . This is due to subadditivity:  $|\tau_{[0, t]]}| \leq |\tau_{[0, t']}]| |\tau_{[t', t]]}|$ .*

The following is classical and straightforward:

**Remark 4.19.** *A directive sequence is everywhere-growing if and only if it has a telescoping which is expanding, meaning that there exists an increasing sequence  $(t_i)_{i \in \mathbb{N}}$  of non-negative integers such that  $t_0 = 0$  and for every  $i \in \mathbb{N}$ , such that the directive sequence  $(\tau_{[t_i, t_{i+1}]})_{i \in \mathbb{N}}$  is expanding.*

**4.3.1. Recognizability.** A substitution  $\tau : \mathcal{B} \rightarrow \mathcal{A}^+$  is called **injective** if it is one-to-one as a map (not necessarily its extension). For configurations on  $\mathbb{Z}$ , the intuition behind injectivity is formalised as recognizability.

The following definition comes from [BGM22], and is inspired by more classical definitions [Mos96, BSTY19].

**Definition 4.20.** *A substitution  $\tau : \mathcal{B} \rightarrow \mathcal{A}^+$  is said to be **quasi-recognizable** over  $X \subset \mathcal{A}^{\mathbb{Z}}$  if every  $x \in X$  admits at most one standard desubstitution scheme. It is **recognizable** over  $X$  if every  $x \in X$*

admits at most one desubstitution. When the set  $X$  is omitted, it is assumed to be the full shift (or equivalently, the set of desubstitutable configurations). We will say that a directive sequence  $\tau$  is **quasi-recognizable** if every  $\tau_{[0,t[}$  is quasi-recognizable over  $\Omega_\tau$  (in particular if every  $\tau_t$  is quasi-recognizable over  $\Omega_{\sigma^t \tau}$ ).

**Definition 4.21.** A **right-quasi-recognizability radius** of a substitution  $\tau : \mathcal{B} \rightarrow \mathcal{A}^+$  over a set  $Z \subset (\mathcal{A}^\mathbb{Z})^2$  is a natural number  $R \in \mathbb{N}$  such that for every pair  $(x, x') \in Z$  with standard desubstitution schemes  $\mathbf{k}$  and  $\mathbf{k}'$  and  $x_{\mathbb{N}} = x'_{\mathbb{N}}$ , there exist  $j, j' \in \llbracket 0, R \rrbracket$  such that  $\mathbf{k}_{\llbracket j, \infty \llbracket} = \mathbf{k}'_{\llbracket j', \infty \llbracket}$  and  $x_{\llbracket k_j, \infty \llbracket} = x'_{\llbracket k'_{j'}, \infty \llbracket}$ .

We will say that  $R$  is a right-quasi-recognizability radius over  $X \subset \mathcal{A}^\mathbb{Z}$  when it is for  $Z = X \times X$ .

Let us notice that if  $R$  is a right-quasi-recognizability radius over some set  $Z$ , then any  $R' > R$  is also a quasi-recognizability radius over  $Z$ .

Note that the last condition  $x_{\llbracket k_j, \infty \llbracket} = x'_{\llbracket k'_{j'}, \infty \llbracket}$  derives straightforwardly from  $x_{\mathbb{N}} = x'_{\mathbb{N}}$ , except when  $j = j' = 0$  (because then  $k_j$  may be negative). Informally,  $R$  is a right-quasi-recognizability radius if one can desubstitute uniquely any right-half configuration, up to  $R$  ‘pre-image letters’. The following remark will be useful in short time.

**Remark 4.22.** Let us consider fixed some right-quasi-recognizability radius  $R$  on  $Z$  for  $\tau$ . Consider some  $z \in \mathcal{A}^\mathbb{N}$  such that there is  $x \in Z$  with  $x_{\mathbb{N}} = z$ , and  $x$  and  $\mathbf{k}$  be such that  $k_R$  is minimal among all choices of such  $x$  and their standard desubstitution schemes  $\mathbf{k}$  (this exists since  $k_R \geq R|\tau| - \|\tau\|$ , from Remark 4.11). Then for every  $x' \in Z$  and every desubstitution scheme  $\mathbf{k}'$ , there exists  $j' \in \llbracket 0, R \rrbracket$  such that  $\mathbf{k}_{\llbracket R, \infty \llbracket} = \mathbf{k}'_{\llbracket j', \infty \llbracket}$  and  $x_{\llbracket k_R, \infty \llbracket} = x'_{\llbracket k'_{j'}, \infty \llbracket}$ . Besides, Remark 4.11 again gives:

$$j' \geq \frac{k'_{j'}}{\|\tau\|} = \frac{k_R}{\|\tau\|} > R \frac{|\tau|}{\|\tau\|} - 1,$$

so that in the end  $j' \in \left[ \left\lceil R \frac{|\tau|}{\|\tau\|} \right\rceil, R \right]$ . Note that if  $R(1 - \frac{|\tau|}{\|\tau\|}) < 1$ , then  $j' = R$  (this happens when  $R \in \{0, 1\}$  for instance).

**Proposition 4.23.** A substitution  $\tau$  is quasi-recognizable over  $X$  if and only if it admits a right-quasi-recognizability radius over  $X$ .

*Proof.* ( $\Leftarrow$ ) Roughly speaking, the strategy for proving this is, for every  $x \in X$ , to use the definition of right-quasi-recognizability radius on elements of  $\mathcal{O}(x)$ . Let us get more precise. Let us assume that  $\tau$  admits a right-quasi-recognizability radius  $R$  over  $X$ , and consider  $z \in X$  which can be desubstituted, and  $\mathbf{k}, \mathbf{k}'$  two standard desubstitution schemes for  $z$ . Let us fix  $i \in \llbracket R, \infty \llbracket$ , and set  $y := \sigma^{k-i}(x)$ , which admits both  $(k_{n-i} - k_{-i})_{n \in \mathbb{Z}}$  and  $(k'_{n-i'} - k_{-i})_{n \in \mathbb{Z}}$  as standard desubstitution schemes, for some  $i'$ . Applying the definition of right-quasi-recognizability radius to  $x = x' = y$ , there exist  $j, j' \in \llbracket 0, R \rrbracket$  such that  $\mathbf{k}_{\llbracket j-i, \infty \llbracket} - k_{-i} = \mathbf{k}'_{\llbracket j'-i', \infty \llbracket} - k_{-i}$ . Since  $j-i \leq R-i \leq 0$  and  $\mathbf{k}$  is standard,  $j-i$  is the number of nonpositive elements in  $\mathbf{k}_{\llbracket j-i, \infty \llbracket}$ . Since  $\mathbf{k}_{\llbracket j-i, \infty \llbracket} = \mathbf{k}'_{\llbracket j'-i', \infty \llbracket}$  and  $\mathbf{k}'$  is standard, this number is also  $j' - i'$ . Thus  $j - i = j' - i'$ . We have proven that for arbitrarily large  $i$ ,  $\mathbf{k}_{\llbracket j-i, \infty \llbracket} = \mathbf{k}'_{\llbracket j'-i', \infty \llbracket}$ . The fact that for any choice of  $j$  as above,  $k_{j-i}$  becomes closer and closer to  $-\infty$  when  $i$  goes to  $\infty$  allows us to conclude that  $\mathbf{k} = \mathbf{k}'$ . We just proved that every  $x$  admits at most one desubstitution scheme.

( $\Rightarrow$ ) Conversely, assume that  $\tau$  is quasi-recognizable. By a classical compactness argument, there exists a **quasi-recognizability radius**  $r \in \mathbb{N}$  such that for every standard desubstitution schemes  $\mathbf{k}, \mathbf{k}'$  of two configurations  $x, x' \in X$  such that  $x_{\llbracket -r, r \llbracket} = x'_{\llbracket -r, r \llbracket}$ , one has  $k_0(x) = k_0(x')$ . Let us set  $R := \left\lceil \frac{r-1+\|\tau\|}{|\tau|} \right\rceil$  and fix two configurations  $x, x'$  such that  $x_{\mathbb{N}} = x'_{\mathbb{N}}$ , with desubstitution schemes  $\mathbf{k}, \mathbf{k}'$ . Let  $i \in \mathbb{Z}$  be such that  $k_i \geq r$ . One can apply the definition of quasi-recognizability radius to  $\sigma^{k_i}(x)$  and  $\sigma^{k_i}(x')$  with desubstitution schemes  $\sigma^i(\mathbf{k}) - k_i$  and  $\sigma^i(\mathbf{k}') - k_i$ , for  $m := \max_{k'_i \leq k_i} j$  (see Remark 4.12): we get  $0 = k_i - k_i = k_m - k_i$ , which means that  $k_i \in \{k'_m \mid m \in \mathbb{Z}\}$ . Hence  $\{k_i \mid i \in \mathbb{Z}\} \cap \llbracket r, \infty \llbracket \subset \{k'_i \mid i \in \mathbb{Z}\}$ . By symmetry of the condition, and the fact that  $k_i$  and  $k'_i$  are strictly increasing, we get that  $\mathbf{k}_{\llbracket j, \infty \llbracket} = \mathbf{k}'_{\llbracket j', \infty \llbracket}$ , where  $j := \min_{k_i \geq r} i$  and  $j' := \min_{k'_i \geq r} i$ . Since  $k_j, k'_{j'} \geq r \geq 0 \geq k_0$ , we have  $j, j' \geq 0$ . Moreover, Remark 4.11 gives that  $k_R, k'_R \geq r$ , so  $j, j' \leq R$ .  $\square$

Our definition of quasi-recognizability radius may appear less natural than a ‘symmetric’ version of the quasi-recognizability radius. However it is justified by the following lemma.

Let us remind that for any  $\mathbb{Z}$ -shift  $X$  on some alphabet  $\mathcal{A}$ , and  $z' \in \mathcal{A}^{\mathbb{N}}$ , we denote by  $\mathcal{P}_X(z')$  the set of left-infinite words  $z \in \mathcal{A}^{\mathbb{Z}^-}$  such that  $zz' \in X$ . For every  $\ell \geq 1$ , we also denote by  $\mathcal{P}_X^\ell(z')$  the set of length- $\ell$  words  $w$  on  $\mathcal{A}$  which are suffix of a word  $z \in \mathcal{A}^{\mathbb{Z}^-}$  such that  $zz' \in X$ .

**Lemma 4.24.** *Let us fix some  $\mathbb{Z}$ -shift  $Y \subset \mathcal{B}^{\mathbb{Z}}$ , and a substitution  $\tau : \mathcal{B} \rightarrow \mathcal{A}^+$  which admits a right-quasirecognizability radius  $R$  over  $\{x\} \times X$ , where  $X$  is the set of configurations which admit a desubstitution in  $Y$ , and  $x \in X$ . Then for every  $\ell \geq 1$ ,*

$$\left| \mathcal{P}_X^{\ell|\tau|}(x_{\mathbb{N}}) \right| \leq \left| \mathcal{P}_X^h(x_{\mathbb{N}}) \right| \leq \sum_{j'=\lceil R \frac{|\tau|}{\|\tau\|} \rceil}^R |\mathcal{B}|^{j'+\ell},$$

where  $h := \min_{u \in \mathcal{L}_\ell(Y)} |\tau(u)|$ .

*Proof.* Let  $\mathbf{k}$  be a standard desubstitution scheme for  $x$ . Should we swap  $x$  for some asymptotic configuration  $x'$ , we can assume without loss of generality that  $k_R$  is the minimal possible  $k_R$  over all possible standard desubstitution schemes of all possible  $x'$  such that  $x'_{\mathbb{N}} = x_{\mathbb{N}}$  (see Remark 4.22). Let  $\ell \geq 1$ . Let us consider any map  $\phi : \mathcal{P}_X^h(x_{\mathbb{N}}) \rightarrow \bigcup_{j'=\lceil R \frac{|\tau|}{\|\tau\|} \rceil}^R \mathcal{B}^{j'+\ell}$  such that for any  $w \in \mathcal{P}_X^h(x_{\mathbb{N}})$ ,  $\phi(w) = y'_{[-\ell, j']}$ , where  $y'$  is some standard desubstitution of a configuration  $x'$  such that  $x'_{[-h, \infty]} = wx_{\mathbb{N}}$ ,  $\mathbf{k}'$  the corresponding desubstitution scheme, and  $j' \in \left[ \left\lceil R \frac{|\tau|}{\|\tau\|} \right\rceil, R \right]$  is obtained from Remark 4.22. Beware that this map is not constructively defined; we now simply prove the injectivity of such a map, which will yield our cardinality inequality. For every configuration  $x'$  such that  $x'_{[-h, \infty]} = wx_{\mathbb{N}}$  and every standard desubstitution scheme  $\mathbf{k}'$  and corresponding desubstitution  $y'$ , one has

$$\tau\phi(w) = \tau(y'_{[-\ell, j']}) = x'_{\llbracket k'_{-\ell}, k'_{j'} \rrbracket} = x'_{\llbracket k'_{-\ell}, k_R \rrbracket},$$

and  $k'_{-\ell} \leq k'_0 - |\tau\phi(w)| \leq -h$ , by definition of  $h$  and because  $k'_0 \leq 0$  since the desubstitution scheme is standard. Hence, if  $R > 0$ , then  $w = x'_{[-h, 0]} = \tau\phi(w)_{|\tau\phi(w)|-k_R+[-h, 0]}$ . If  $R = 0$  and  $h \geq -k_0$ , then  $w = x'_{[-h, 0]} = \tau\phi(w)_{|\tau\phi(w)|+[-h-k_0, 0]} x_{\llbracket k_0, 0 \rrbracket}$ . If  $R = 0$  and  $h \leq -k_0$ , then  $w = x_{[-h, 0]}$  (there is actually just one such predecessor). In all three cases, we have proven that  $\phi$  is injective, which straightforwardly gives the second inequality. The first one comes from noticing that  $h \geq \ell |\tau|$ .  $\square$

**4.3.2. Right-quasi-recognizability radius and finite positive expansiveness.** Using the notion of right-quasi-recognizability radius, we can provide upper bounds on the number of predecessors of an element of  $\mathcal{A}^{\mathbb{N}}$ , and then deduce finitely positively expansiveness for the S-adic limit set under certain conditions. We formulate them in this section.

**Lemma 4.25.** *Let  $\tau$  be an everywhere-growing directive sequence such that there exist  $\alpha \geq 1, R \in \mathbb{N}$  and  $x \in \Omega_\tau$  such that, for infinitely many  $t \in \mathbb{N}$ , we have that  $|\mathcal{A}_{t+1}| \leq \alpha$  and  $R$  is a right-quasi-recognizability radius for  $\tau_{\llbracket 0, t \rrbracket}$  over  $\{x\} \times \Omega_\tau$ . Then  $|\mathcal{P}_{\Omega_\tau}(x_{\mathbb{N}})| \leq \sum_{j'=\lceil R \frac{|\tau|}{\|\tau\|} \rceil}^R \alpha^{j'+1}$ .*

*Proof.* Since  $\tau$  is everywhere-growing, for every  $m \geq 1$ , there exists  $t \in \mathbb{N}$  such that  $|\tau_{\llbracket 0, t \rrbracket}| \geq m$ . By assumption, there exists  $t \geq t'$  such that  $|\mathcal{A}_{t+1}| \leq \alpha$  and  $R$  is a right-quasi-recognizability radius for  $\tau_{\llbracket 0, t \rrbracket}$  over  $\{x\} \times \Omega_\tau$ . If  $Y = \Omega_{\sigma^t \tau}$ , then the set of configurations which admit a desubstitution in  $Y$  is  $X = \Omega_\tau$ . An application of Lemma 4.24 for substitution  $\tau_{\llbracket 0, t \rrbracket}$ , configuration  $x$  and integer  $\ell = 1$  then gives

$$\left| \mathcal{P}_{\Omega_\tau}^m(x_{\mathbb{N}}) \right| \leq \left| \mathcal{P}_{\Omega_\tau}^1{}^{\tau_{\llbracket 0, t \rrbracket}}(x_{\mathbb{N}}) \right| \leq \sum_{j'=\lceil R \frac{|\tau|}{\|\tau\|} \rceil}^R |\mathcal{A}_{t+1}|^{j'+1} \leq \sum_{j'=\lceil R \frac{|\tau|}{\|\tau\|} \rceil}^R \alpha^{j'+1}.$$

Since this is true for every  $m \geq 1$ , compactness gives that  $|\mathcal{P}_{\Omega_\tau}(x_{\mathbb{N}})| \leq \sum_{j'=\lceil R \frac{|\tau|}{\|\tau\|} \rceil}^R \alpha^{j'+1}$ .  $\square$

**Theorem 4.26.** *Let  $\tau$  be an everywhere-growing directive sequence with finite rank  $\alpha$  such that there exists some  $R \in \mathbb{N}$  which is a right-quasi-recognizability radius  $R$  for  $\tau_{\llbracket 0, t \rrbracket}$  over  $\Omega_\tau$ , for every  $t \in \mathbb{N}$ . Then the S-adic limit set  $\Omega_\tau$  is positively  $\alpha^{R+1}$ -expansive.*

*Proof.* This is a direct application of Lemma 4.25 using that when  $\alpha \geq 2$ , the sum  $\sum_{j'=\lceil R \frac{|\tau|}{\|\tau\|} \rceil}^R \alpha^{j'+1}$  is bounded by  $\alpha^{R+1}$ . When  $\alpha = 1$ , Remark 4.16 gives that the shift is finite, hence positively expansive.  $\square$

**Remark 4.27.** *The hypothesis in Theorem 4.26 can be slightly relaxed (for readability we left the theorem written as it is). For instance, the condition on the right-quasi-recognizability radius can be replaced by the existence of a common right-quasi-recognizability radius  $R$  for  $\tau_{[0,t]}$ , for infinitely many  $t \in \mathbb{N}$  such that  $|\mathcal{A}_{t+1}| \leq \alpha$ .*

**Remark 4.28.** *Notice also that the quasi-recognizability assumption involves the shift  $\Omega_\tau$ . The condition might be hard to check directly, because the set is hard to describe; however, it is implied by any similar condition where  $\Omega_\tau$  is replaced by any shift  $X$  which includes it (for instance the full shift), for the reason that if  $R$  is a right-quasi-recognizability radius for  $\tau$  on  $X$ , then it is also on  $\Omega_\tau$ . As a matter of fact, many substitutions are actually recognizable on the full shift.*

4.3.3. *Suffix codes and uniform substitutions.*

**Definition 4.29.** *A **suffix code** is a set of words none of which is a strict suffix of any other.*

**Lemma 4.30.** *Let  $\tau : \mathcal{B} \rightarrow \mathcal{A}^*$  be a quasi-recognizable substitution such that  $\tau(\mathcal{B})$  is a suffix code. Then 1 is a right-quasi-recognizability radius for  $\tau$ .*

*Proof.* From Proposition 4.23,  $\tau$  has a right-quasi-recognizability radius  $R$ . We can assume that it is the minimal one and assume ad absurdum that  $R > 1$ . Let us consider two configurations  $x, x' \in \mathcal{A}^{\mathbb{Z}}$  which admit standard desubstitution schemes  $\mathbf{k}$  and  $\mathbf{k}'$  respectively, and desubstitutions  $y, y'$ , and satisfy  $x_{\mathbb{N}} = x'_{\mathbb{N}}$ . By definition there exist some  $j, j' \in [0, R]$  such that  $k_{[j, \infty[} = k'_{[j', \infty[}$ . Assume that  $x, x'$  witness that  $R$  is minimal, meaning that  $j = R$  and  $k_{R-1} \neq k'_{j'-1}$  or  $x_{[k_{R-1}, k_R[} \neq x'_{[k'_{j'-1}, k'_{j'}[}$ . Since  $R > 1$ , we know that  $k_{R-1} \geq 0$ , so that  $x_{[k_{R-1}, k_R[} = x'_{[k_{R-1}, k_{j'}[}$ . Having  $k_{R-1} = k'_{j'-1}$  would contradict the previous sentence. On the other hand, having  $k_{R-1} < k'_{j'-1}$  would give that the  $x_{[k_{R-1}, k_R[}$  admits  $x'_{[k'_{j'-1}, k'_{j'}[}$  as a strict suffix. Since they are both images by  $\tau$  of a letter in  $\mathcal{B}$ , this contradicts the hypothesis. Symmetrically, having  $k_{R-1} > k'_{j'-1}$  would give that the  $x_{[k_{R-1}, k_R[}$  is a strict suffix of  $x'_{[k'_{j'-1}, k'_{j'}[}$ . This also contradicts the hypothesis. Hence  $R \leq 1$ .  $\square$

**Corollary 4.31.** *For every directive sequence  $\tau$  which is everywhere-growing, quasi-recognizable, with finite rank  $\alpha$ , and such that  $\tau_t(\mathcal{A}_{t+1})$  is a suffix code for every  $t \in \mathbb{N}$ ,  $\Omega_\tau$  is positively  $\alpha^2$ -expansive.*

*Proof.* This is direct from Theorem 4.26 and Lemma 4.30.  $\square$

The suffix code condition is actually a natural generalization of uniform substitutions, for which we can thus derive a similar statement.

**Definition 4.32.** *A substitution  $\tau$  is **uniform** if  $|\tau| = \|\tau\|$ , that is, all the images  $\tau(a)$  have the same length. A directive sequence  $\tau$  is **uniform** if for every  $t$ ,  $\tau_t$  is uniform. In that case, notice that for every  $t$ ,  $\tau_{[0,t]}$  is also uniform.*

**Corollary 4.33.** *For any directive sequence  $\tau$  which is everywhere-growing, quasi-recognizable on the full shift, uniform, with finite rank  $\alpha$ ,  $\Omega_\tau$  is positively  $\alpha^2$ -expansive.*

*Proof.* This derives directly from Corollary 4.31 and the fact that every set of words with uniform length is clearly a suffix code.  $\square$

**Example 4.34.** *The substitutive shift corresponding to*

$$\tau : \begin{cases} a & \mapsto abc \\ b & \mapsto bbc \\ c & \mapsto aba \end{cases},$$

*is positively 9-expansive. This upper bound does not seem tight, and we suspect that a more careful study (of how letters split up between the ones with common suffix and the others) would allow to prove that the hypotheses of Corollary 4.31 imply  $\Omega_\tau$  are positively  $(\lfloor \alpha/2 \rfloor + 1)(\lceil \alpha/2 \rceil + 1)$ -expansive when  $\alpha$  is odd.*

**Example 4.35.** *On the contrary, the substitution*

$$\tau : \begin{cases} a & \mapsto aa \\ b & \mapsto bb \end{cases}$$

*is a counter-example to Corollary 4.33 when dropping the hypothesis of quasi-recognizability over the full shift. Indeed,  $\Omega_\tau$  contains non-periodic asymptotically periodic configurations, for instance  ${}^\infty ab^\infty$ , which prevents any finite positive expansiveness, by Corollary 2.7.*



4.3.4. *Right-marked, return, Tœplitz substitutions.* In some cases, one can be a little more precise about the finite positive expansiveness degree.

**Definition 4.36.** We say that a substitution  $\tau : \mathcal{B} \rightarrow \mathcal{A}^+$  is ***q-right-recoverable*** if it is injective and  $\{\tau(a)_{\llbracket q, |\tau(a)| \rrbracket} \mid a \in \mathcal{B}\}$  is a suffix code, where  $q \in \llbracket 1, |\tau| \rrbracket$ . When there exists  $q \in \llbracket 1, |\tau| \rrbracket$  such that  $\tau$  is *q-right-recoverable*, we simply say that it is *right-recoverable*.

**Remark 4.37.** Notice that any *q-right-recoverable* substitution, with  $q \geq 1$ , is *k-right-recoverable* for every  $k \leq q$ .

**Lemma 4.38.** For every *q-right-recoverable* substitution  $\tau : \mathcal{B} \rightarrow \mathcal{A}^*$ , the generated monoid homomorphism  $\tau^* : \mathcal{B}^* \rightarrow \mathcal{A}^*$  is injective.

*Proof.* Let us consider two words  $u = u_0 \dots u_m, v = v_0 \dots v_n \in \mathcal{A}^*$  such that  $\tau^*(u) = \tau^*(v)$ . As a consequence of this equality, one of the words  $\tau(u_m)_{\llbracket q, |\tau(u_m)| \rrbracket}$  and  $\tau(v_n)_{\llbracket q, |\tau(v_n)| \rrbracket}$  is suffix of the other. Since the set  $\{\tau(a)_{\llbracket q, |\tau(a)| \rrbracket} \mid a \in \mathcal{B}\}$  is a suffix code, this implies that  $\tau(u_m)_{\llbracket q, |\tau(u_m)| \rrbracket} = \tau(v_n)_{\llbracket q, |\tau(v_n)| \rrbracket}$  and thus that  $\tau(u_m)$  and  $\tau(v_n)$  have the same length. Using  $\tau^*(u) = \tau^*(v)$  again, we have that  $\tau(u_m) = \tau(v_n)$ . Since  $\tau$  is injective, we have  $u_m = v_n$ . Finally, we have that  $\tau^*$  has the same images on  $u_0 \dots u_{m-1}$  and  $v_0 \dots v_{n-1}$ . We conclude by a recursion argument.  $\square$

**Definition 4.39.** We call ***extension*** of a suffix code  $W$  any set  $W'$  such that there exists a surjective function  $\phi : W \rightarrow W'$  such that for all  $w \in W$ ,  $w$  is a suffix of  $\phi(w)$ . Since  $W$  is a suffix code, such a function is also injective.

**Lemma 4.40.** Every extension of a suffix code is a suffix code.

*Proof.* Let us consider an extension  $W'$  of a suffix code  $W$ , and set  $\phi : W \rightarrow W'$  surjective such that for all  $w, w'$  is a suffix of  $\phi(w)$ . We have to prove that if two words  $u', v'$  in  $W'$  are such that  $u'$  is a suffix of  $v'$ , then  $u'$  and  $v'$  are equal. Let us fix two such words  $u', v'$ . We denote by  $u, v$  their respective preimages by  $\phi$ . As a consequence, one of the two words  $u, v$  is suffix of the other. Therefore  $u = v$ . Since  $\phi$  is also injective,  $u' = v'$ .  $\square$

**Lemma 4.41.** For every suffix code  $W$  over alphabet  $\mathcal{B}$ , and every substitution  $\tau : \mathcal{B} \rightarrow \mathcal{A}^*$  such that  $\tau^*$  is injective, the set  $\tau^*(W)$  is a suffix code.

*Proof.* It is sufficient to see that for every words  $u, v$  such that  $u$  is suffix of  $v$ ,  $(\tau^*)^{-1}(u)$  is a suffix of  $(\tau^*)^{-1}(v)$ .  $\square$

**Lemma 4.42.** If  $\tau : \mathcal{B} \rightarrow \mathcal{A}^+$  is *right-recoverable* and  $\tau' : \mathcal{C} \rightarrow \mathcal{B}^+$  is *q-right-recoverable*, then  $\tau\tau'$  is *q| $\tau$ |-right-recoverable*.

*Proof.* In order to prove that  $\tau\tau'$  is *q| $\tau$ |-right-recoverable*, we need to prove that  $q|\tau| \in \llbracket 1, |\tau\tau'| \rrbracket$ , that  $W' := \{\tau\tau'(c)_{\llbracket q|\tau|, |\tau\tau'(c)| \rrbracket} \mid c \in \mathcal{C}\}$  is a suffix code, and that  $\tau\tau'$  is injective. The first point follows from the fact that  $\tau'$  is *q-right-recoverable* and thus  $q \in \llbracket 1, |\tau'| \rrbracket$ , and that  $|\tau||\tau'| \leq |\tau\tau'|$ . The third point follows from the fact that  $\tau^*$  is injective (Lemma 4.38) and that  $\tau'$  is injective. We are left to prove the second point. Since  $\tau'$  is *q-right-recoverable*,  $W := \{\tau'(c)_{\llbracket q, |\tau'(c)| \rrbracket} \mid c \in \mathcal{C}\}$  is a suffix code. By Lemma 4.41,  $\tau^*(W)$  is a suffix code, and by Lemma 4.40,  $W'$  is a suffix code, as an extension of  $\tau^*(W)$ .  $\square$

**Lemma 4.43.** Let us fix an integer  $q \geq 1$  and  $\tau : \mathcal{B} \rightarrow \mathcal{A}^+$  be a *quasi-recognizable q-right-recoverable* substitution, and  $x \in X := \tau(\mathcal{B}^{\mathbb{Z}})$ . Then

$$|\mathcal{P}_X^q(x_{\mathbb{N}})| \leq |\mathcal{B}|.$$

*Proof.* Right-recoverability implies the hypothesis of Lemma 4.30, so that 1 is a right-quasi-recognizability radius. Let  $\mathbf{k}$  be the unique standard desubstitution scheme for  $x$ , and  $y$  the unique desubstitution corresponding to this desubstitution scheme (it is unique because  $\tau$  is injective). We distinguish two cases.

- (1) **First case.** Assume that there exists  $x' \in X$  such that  $x'_{\mathbb{N}} = x_{\mathbb{N}}$  having a standard desubstitution scheme  $\mathbf{k}'$  such that  $k'_0 > -q$ . Because 1 is a right-quasi-recognizability radius, there are  $j, j' \in \{0, 1\}$  such that  $\mathbf{k}_{\llbracket j, \infty \rrbracket} = \mathbf{k}'_{\llbracket j, \infty \rrbracket}$ . Since  $k_0, k'_0 \leq 0 < k_1, k'_1$ , we can deduce that  $k'_1 = k_1$ . If we denote by  $y'$  the desubstitution corresponding to  $\mathbf{k}'$ , then  $\tau(y'_0) = x'_{\llbracket k'_0, k'_1 \rrbracket} = x'_{\llbracket k_0, k_1 \rrbracket}$  and  $\tau(y_0) = x_{\llbracket k_0, k_1 \rrbracket}$ . Hence  $\tau(y'_0)_{\llbracket q, |\tau(y'_0)| \rrbracket} = x_{\llbracket k'_0+q, k_1 \rrbracket}$  and  $\tau(y_0)_{\llbracket q, |\tau(y_0)| \rrbracket} = x_{\llbracket k_0+q, k_1 \rrbracket}$ , so that one is suffix of the other. By recoverability,  $\tau(y'_0) = \tau(y_0)$ . Since  $\tau$  is injective,  $y'_0 = y_0$ . Hence 0 is actually also a radius over  $\{x\} \times X$ . We conclude by using Lemma 4.25.

(2) **Second case.** On the other hand, suppose that for every  $x' \in X$  such that  $x'_\mathbb{N} = x_\mathbb{N}$  and standard desubstitution scheme  $\mathbf{k}'$ , one has  $k'_0 \leq -q$ . Let us consider any map  $\phi : \mathcal{P}_X^q(x_\mathbb{N}) \rightarrow \mathcal{B}$  such that for any  $w \in \mathcal{P}_X^q(x_\mathbb{N})$ ,  $\phi(w) = y'_0$  for some desubstitution  $y'$  of a configuration  $x'$  such that  $x'_{[-q, \infty[} = wx_\mathbb{N}$ . For every configuration  $x'$  such that  $x'_{[-q, \infty[} = wx_\mathbb{N}$  and every standard desubstitution scheme  $\mathbf{k}'$  and desubstitution  $y'$ , one has  $k'_0 \leq -q$ , by assumption. Since  $\tau(y'_0) = x'_{[k'_0, k_1]}$ , one gets  $w = \tau(\phi(w))_{|\tau(\phi(w))| - k_1 + [-q, 0]}$ , which implies that  $\phi$  is injective. This directly gives the inequality.  $\square$

**Corollary 4.44.** *Let  $\tau$  be a quasi-recognizable everywhere-growing directive sequence with finite rank  $\alpha$  such that for every  $t' \in \mathbb{N}$  and every sufficiently large  $t > t'$ ,  $\tau_{[t', t]}$  is right-recoverable. Then  $\Omega_\tau$  is positively  $\alpha$ -expansive.*

Because of injectivity, such quasi-recognizable directive sequences will actually be eventually recognizable in the sense of [BSTY19].

*Proof.* By the everywhere-growing property, for every  $m \geq 1$ , there exists  $t' \in \mathbb{N}$  such that  $|\tau_{[0, t']}| \geq m$ . By assumption, there exists  $t \geq t'$  such that  $\mathcal{A}_{t+1} \leq \alpha$ , and  $\tau_{[t', t]}$  is 1-right-recoverable. By Lemma 4.42, this implies that  $\tau_{[0, t]} = \tau_{[0, t']}\tau_{[t', t]}$  is  $|\tau_{[0, t']}|$ -right-recoverable, hence  $m$ -right-recoverable. By Lemma 4.43, we get that for every  $x \in \Omega_\tau$ ,  $|\mathcal{P}_X^m(x_\mathbb{N})| \leq |\mathcal{A}_t| \leq \alpha$ . Since this is true for every  $m$ , we conclude by compactness that  $|\mathcal{P}_X(x_\mathbb{N})| \leq \alpha$ .  $\square$

**Definition 4.45.** *We will say that a substitution  $\tau$  is **right-marked** when  $a \mapsto \tau(a)_{|\tau(a)|-1}$  is one-to-one, and that a directive sequence  $\tau$  is **right-marked** when for every  $t \in \mathbb{N}$ ,  $\tau_t$  is right-marked. In that case, notice that for every  $t \in \mathbb{N}$ ,  $\tau_{[0, t]}$  is also right-marked.*

**Remark 4.46.** *It is clear that if a substitution is right-marked and expanding, then it is  $(|\tau| - 1)$ -right-recoverable. Moreover, the composition of two right-marked substitutions is clearly still right-marked. By telescoping, one can then see that any quasi-recognizable everywhere-growing right-marked directive sequence satisfies the hypothesis of Corollary 4.44.*

**Remark 4.47.** *When a directive sequence  $\tau$  is right-marked,  $t \mapsto |\mathcal{A}_t|$  is non-increasing, and thus eventually reaches the rank of  $\tau$ .*

**Corollary 4.48.** *For every everywhere-growing quasi-recognizable right-marked directive sequence  $\tau$  with finite rank  $\alpha$ ,  $\Omega_\tau$  is positively  $\alpha$ -expansive, and not positively  $(\alpha - 1)$ -expansive.*

*Proof.* The fact that it is  $\alpha$ -expansive comes directly from Remark 4.46 and Corollary 4.44.

As Remark 4.47 states, there is  $t' \in \mathbb{N}$  such that for every  $t \geq t'$ ,  $|\mathcal{A}_t| = \alpha$ . For any  $t \geq t'$ , let us fix a one-sided sequence  $z^t \in \mathcal{A}_t^\mathbb{N}$ , and for each  $a \in \mathcal{A}_t$ , consider some configuration  $x^{t,a}$  such that  $x_{-1}^{t,a} = a$  and  $x_\mathbb{N}^{t,a} = z^t$ . Since  $\tau_{[0, t]}$  is right-marked, all the configurations with standard desubstitution scheme  $\mathbf{k}$  for  $\tau_{[0, t]}$  such that  $k_0 = 0$  and with desubstitution  $x^{a,t}$  differ at position  $-1$ , but coincide over  $\mathbb{N}$ . By compactness, there exist  $r$  configurations in  $\Omega_\tau$  which differ at position  $-1$  but coincide over  $\mathbb{N}$ . This proves that  $\Omega_\tau$  is not positively  $(\alpha - 1)$ -expansive.  $\square$

**Example 4.49.** *There are right-marked substitutions which are not quasi-recognizable, for instance:*

$$\tau : \begin{cases} a \mapsto aa \\ b \mapsto bb \end{cases} \quad \text{and} \quad \tau' : \begin{cases} a \mapsto aa \\ b \mapsto ab \end{cases} .$$

*The corresponding substitution limit sets are not finitely positively expansive, because they admit some non-periodic asymptotically periodic configurations, like  ${}^\infty ab^\infty$  for the first one and  ${}^\infty aba^\infty$  for the second. As a consequence, it is not possible to drop the quasi-recognizability hypothesis in Corollary 4.48 (and Corollary 4.44).*

Here are some examples of applications of Corollary 4.48.

**Example 4.50.** *The Fibonacci ( $\tau(a) = ab$  and  $\tau(b) = a$ ) and Thue-Morse ( $\tau(a) = ab$  and  $\tau(b) = ba$ ) substitutions are clearly right-marked and everywhere-growing and they are known to be quasi-recognizable [Mos96]. Hence the two corresponding  $S$ -adic limit sets, and thus the two corresponding substitutive shifts are both positively 2-expansive.*

Another very classical class of substitutions connected to Corollary 4.44 are return substitutions.

**Definition 4.51.**  $\tau : \mathcal{B} \rightarrow \mathcal{A}^*$  is a **return substitution** with respect to some word  $w \in \mathcal{A}^+$  if it is injective and  $w$  appears exactly twice in each  $\tau(a)w$ , for  $a \in \mathcal{B}$ , once as a prefix (and once as a suffix). A substitution  $\tau : \mathcal{B} \rightarrow \mathcal{A}^*$  is **left-proper** if for every  $a, b \in \mathcal{B}$ ,  $\tau(a)_0 = \tau(b)_0$ . A word  $w \in \mathcal{A}^*$  is **nonoverlapping** if  $\mathcal{A}^\ell w \cap w \mathcal{A}^\ell = \emptyset$  for every  $\ell \in \llbracket 1, |w| \rrbracket$ .

**Remark 4.52.** Every return substitution is left-proper. Every injective left-proper substitution is 1-right-recoverable. Every return substitution with respect to some word of length at least 2 is expanding.

**Proposition 4.53.** Let  $\tau : \mathcal{B} \rightarrow \mathcal{A}^*$  be a return substitution with respect to some nonoverlapping word  $w$ . Then  $\tau$  is recognizable.

*Proof.* Since  $\tau$  is injective, it is enough to prove quasi-recognizability. Suppose that there exists  $x \in \mathcal{A}^\mathbb{Z}$  with two standard desubstitution schemes  $\mathbf{k}$  and  $\mathbf{k}'$ . Since the images of letters all start with  $w$ , we get for every  $i, j \in \mathbb{Z}$ ,  $x_{\llbracket k_i, k_i + |w| \rrbracket} = w = x_{\llbracket k'_j, k'_j + |w| \rrbracket}$ . Since  $w$  is nonoverlapping, one can deduce that  $k'_j \notin \llbracket k_i, k_i + |w| \rrbracket$ . Suppose that there exists  $i \in \mathbb{Z}$  such that  $k_i \notin \{k'_j \mid j \in \mathbb{Z}\}$ , and let  $j$  be maximal such that  $k'_j < k_i$ . This maximality and the previous sentence give that  $k'_{j+1} \geq k_i + |w|$ . There exists  $a \in \mathcal{B}$  such that  $\tau(a) = x_{\llbracket k'_j, k'_{j+1} \rrbracket}$ , but  $\tau(a)w$  contains  $x_{\llbracket k_i, k_i + |w| \rrbracket} = w$  as a strict factor, which contradicts the definition of return substitution. We have proved that  $\{k_i \mid i \in \mathbb{Z}\} \subset \{k'_j \mid j \in \mathbb{Z}\}$ . By symmetry we have equality. Since the two desubstitution schemes are standard, they are equal.  $\square$

**Corollary 4.54.** Let  $\tau$  be an everywhere-growing directive sequence of return substitutions with respect to nonoverlapping words, with finite rank  $\alpha$ . Then  $\Omega_\tau$  is positively  $\alpha$ -expansive.

*Proof.* Proposition 4.53 gives that  $\tau$  is quasi-recognizable. Remark 4.52 implies that all return substitutions are 1-recoverable. Corollary 4.44 can then be applied.  $\square$

If one drops the injectivity assumption from the definition of return substitution, one can prove quasi-recognizability (instead of recognizability), and still obtain positive  $\alpha^2$ -expansiveness, thanks to the suffix code property and Corollary 4.31.

Example 4.50 can be generalized to the class of Arnoux-Rauzy shifts (sometimes called episturmian, with some variants), which include all sturmian shifts. One way to define them (see [And21]) is as S-adic  $\mathbb{Z}$ -shifts corresponding to aperiodic directive sequences of (finitely many) substitutions of the form:

$$\tau_i : \begin{array}{l} a_j \mapsto a_i a_j \\ a_i \mapsto a_i \end{array},$$

where  $i \in \llbracket 0, \alpha \rrbracket$  and  $\alpha \geq 2$  is the alphabet size.

**Corollary 4.55.** Every Arnoux-Rauzy shift over  $\alpha$  letters is positively  $\alpha$ -expansive (and not positively  $(\alpha - 1)$ -expansive).

*Proof.* This directly comes from Corollary 4.48, the fact that substitution  $\tau_i$  is a right-marked return substitutions with respect to nonoverlapping word  $a_i$ , which implies recognizability by Proposition 4.53, and that the directive sequences involving infinitely many of at least two distinct substitutions are everywhere-growing.  $\square$

Corollary 4.54 can be applied to much more general class of subshifts than Arnoux-Rauzy shift. By classical results [HPS92], every minimal  $\mathbb{Z}$ -shift can be expressed as an S-adic  $\mathbb{Z}$ -shift corresponding to an everywhere-growing directive sequence of return substitutions with respect to words of length 1 (which are clearly nonoverlapping). Unfortunately, this classical construction often yields an infinite-rank directive sequence (even in cases when the shift could also correspond to a finite-rank one), which prevents application of our results. Indeed, (infinite-rank) minimal shifts may for instance have positive entropy and thus cannot be finitely positively expansive, by Theorem 2.17. Yet, our corollary applies to many minimal  $\mathbb{Z}$ -shifts, including examples presented earlier in this text. In particular, there are some criteria for minimal  $\mathbb{Z}$ -shifts to have finite rank, such as having non-superlinear complexity [DDMP20, Theorem 5.5]. Interested readers are referred to [DDMP20] and references therein for more details. In what follows we will provide an example of technique of this kind for Toeplitz shifts.

**Definition 4.56.** A substitution is said to be a **Toeplitz substitution** if it is left-proper, uniform, injective and expanding. A **Toeplitz limit set** is an the limit set of a directive sequence of Toeplitz substitutions.

Note that the expanding assumption is automatic whenever the alphabet is non-trivial. An important remark is that the composition of finitely many Toeplitz substitutions is also a Toeplitz substitution.

Tœplitz shifts are more commonly defined through their quasi-periodic structure, but the definitions are equivalent (see [Kü03, Prop 4.70] for instance). Let us prove a preliminary result. A directive sequence is said to be **weakly primitive** if:

$$\forall t \in \mathbb{N}, \exists t' \geq t, \forall a \in \mathcal{A}_{t'}, \forall b \in \mathcal{A}_t, b \text{ appears in } \tau_{\llbracket t, t' \rrbracket}(a).$$

**Proposition 4.57.**

- (1) Every Tœplitz limit set  $\Omega_\tau$  is aperiodic, unless for all sufficiently large  $t \in \mathbb{N}$ , there exists  $a_t \in \mathcal{A}_t$  such that  $\tau_t(a_{t+1}) = a_t^{|\tau_t|}$ .
- (2) Every weakly primitive Tœplitz limit set is minimal (in particular it is equal to the classical Tœplitz shift).

*Proof.*

- (1) Let  $p \geq 1$  be the minimal possible period for all possible configurations  $x \in \Omega_\tau$ . Up to a shift, one can assume that  $x = \tau_0(y)$  for some  $y \in \Omega_{\sigma\tau}$ .
  - If  $p$  is prime with  $|\tau_0|$ , then for every  $i \in \mathbb{Z}$ , there exists  $k, \ell \in \mathbb{Z}$  such that  $i + kp = \ell |\tau_0|$ . We obtain that  $x_i = x_{i+kp} = x_{\ell|\tau_0|}$  is the first symbol  $a_0$  of the common prefix, i.e.  $x = {}^\infty a_0^\infty$ , and  $p = 1$  by minimality. By injectivity, one gets that  $y = {}^\infty a_1^\infty$  for some symbol  $a_1 \in \mathcal{A}_1$ , and iteratively a sequence of symbols  $a_t \in \mathcal{A}_t$  such that  $\tau_t(a_{t+1}) = a_t^{|\tau_t|}$ .
  - Otherwise, the least common multiple  $p \wedge |\tau_0|$  is also a period for  $x$ , and by injectivity of  $\tau_0$ ,  $p \wedge |\tau_0| / |\tau_0|$  is a period for  $y$ . Hence  $\Omega_{\sigma\tau}$  admits a smaller period than  $p$ , and the other item will be eventually reached.
- (2) Let us prove that every  $u \in \mathcal{L}(\Omega_\tau)$  appears in every configuration  $x \in \Omega_\tau$ . Since  $\tau$  is expanding, there exists  $t \in \mathbb{N}$  such that  $|u| \leq |\tau_{\llbracket 0, t \rrbracket}|$ . By definition,  $u$  appears in some  $\tau_{\llbracket 0, t \rrbracket}(v)$ , and by the previous inequality,  $v$  can be assumed to have length 1 or 2. If  $|v| = 1$ , primitivity gives some  $t' \geq t$  such that  $v$  appears in  $\tau_{\llbracket t, t' \rrbracket}(a)$  for every letter  $a \in \mathcal{A}_{t'}$ . In particular,  $v$  appears in every word of length  $2|\tau_{\llbracket t, t' \rrbracket}| - 1$  in  $\Omega_{\sigma^t\tau}$ , hence  $u$  appears in every word of length  $2|\tau_{\llbracket 0, t' \rrbracket}|$  in  $\Omega_\tau$ . If, on the other hand,  $|v| = 2$ , then either  $v$  appears in some  $\tau_t(a)$  for some  $a \in \mathcal{A}_t$  or  $v = \tau_t(a)_{|\tau_t|-1} \tau_t(b)_0$ . In both cases, we can apply the previous argument to  $a$  instead of  $w$  and  $t + 1$  instead of  $t$ , so that, for some  $t' \geq t + 1$ ,  $a$  appears in every word of length  $2|\tau_{\llbracket t+1, t' \rrbracket}| - 1$  in  $\Omega_{\sigma^{t+1}\tau}$ . In the first case, we get that  $u$  appears in every word of length  $2|\tau_{\llbracket 0, t' \rrbracket}|$  in  $\Omega_\tau$ . In the second case, note that  $\tau_t(b)_0$  is the common first letter for every  $\tau_t(c)$ , where  $c \in \mathcal{A}_{t+1}$ . Hence  $\tau_t(a)\tau_t(b)_0$  appears in every word of length  $2|\tau_{\llbracket t, t' \rrbracket}| + 1$  in  $\Omega_{\sigma^t\tau}$ , and we obtain that  $u$  appears in every word of length  $2|\tau_{\llbracket 0, t' \rrbracket}| + |\tau_{\llbracket 0, t \rrbracket}|$  in  $\Omega_{\sigma^t\tau}$ .  $\square$

**Corollary 4.58.** Every recognizable limit set of Tœplitz directive sequences with finite rank  $\alpha \geq 1$  is positively  $\alpha$ -expansive.

*Proof.* The statement directly derives from Corollary 4.44 and (the second statement of) Remark 4.52.  $\square$

There are criteria for recognizability of Tœplitz  $\mathbb{Z}$ -shifts, like the following one. For every finite word  $u$ , we call **period** of  $u$  any  $p \geq 1$  such that for every  $i < |u|$  with  $i + p < |u|$ , we have  $u_i = u_{i+p}$ .

For a left-proper substitution, we call *maximal common prefix* of this substitution the maximal non-empty word which is prefix of all of its images.

**Proposition 4.59.** Assume that  $\tau : \mathcal{B} \rightarrow \mathcal{A}^*$  is a Tœplitz substitution whose maximal common prefix  $u$  has smallest period  $p$ . Then  $\tau$  is recognizable over the set  $\tau(X)$ , where  $X$  is the set of  $x \in \mathcal{A}^\mathbb{Z}$  satisfying the three following properties: there exists  $i \in \mathbb{Z}$  such that  $\tau(x_i)$  does not start by  $u_{\llbracket u-p, |u| \rrbracket}$ ; there exists  $i \in \mathbb{Z}$  such that  $\tau(x_i)$  does not end by  $u_{\llbracket 0, p \rrbracket}$ ; and there exists  $i \in \mathbb{Z}$  such that  $\tau(x_i)$  does not contain an occurrence of  $u$  disjoint from its prefix  $u$ .

In particular, if  $u$  is nonoverlapping and has length  $|u| \geq |\tau|/2$ , then  $\tau$  is recognizable over the set of aperiodic configurations.

Before proving the proposition, let us state some consequence.

**Corollary 4.60.** Let  $\tau$  be a directive sequence of Tœplitz substitutions with rank  $\alpha \geq 1$ , such that for every  $t \in \mathbb{N}$ , the maximal common prefix of  $\tau_t$  is nonoverlapping and has length at least  $|\tau|/2$ . Then  $\Omega_\tau$  is positively  $\alpha$ -expansive.

*Proof.* The fact that the maximal common prefix is nonoverlapping makes it impossible to have that for all  $t \in \mathbb{N}$  sufficiently large, there exists  $a_t \in \mathcal{A}_t$  such that  $\tau_t(a_{t+1}) = a_t^{|\tau_t|}$ . Indeed, under this hypothesis for all  $t$  the maximal common prefix of  $\tau_t$  would be of the form  $a_t^{p_t}$ , where  $p_t$  is an integer, which implies that it is overlapping. By the first point of Proposition 4.57,  $\Omega_\tau$  is aperiodic. Hence Proposition 4.59 gives that  $\tau$  is recognizable, and we can conclude by Corollary 4.58.  $\square$

*Proof of Proposition 4.59.* In order to prove that  $\tau$  is recognizable, since  $\tau$  is assumed to be injective, it is enough to prove quasi-recognizability.

Let  $y$  be a configuration of  $\tau(X)$  which has  $x$  as desubstitution such that  $x$  satisfies the properties in the statement. By shifting, we can assume that  $y = \tau(x)$ , so that the corresponding standard desubstitution scheme is  $(i|\tau|)_{i \in \mathbb{Z}}$ . Assume that it admits a distinct standard desubstitution scheme  $\mathbf{k} = (k_0 + i|\tau|)_{i \in \mathbb{Z}}$ , where  $k_0 < 0$ . We conclude according to the value of  $k_0$ :

- (1) If  $-|\mathbf{u}| \leq \mathbf{k}_0 < 0$ , then  $y_{\llbracket k_0, |\mathbf{u}| \rrbracket} \in u\mathcal{A}^{-k_0} \cap \mathcal{A}^{-k_0}u$ , which means that  $-k_0$  is a period for  $u$ , hence a multiple of  $p$ . Moreover, for every  $i \in \mathbb{Z}$ ,  $\tau(x_i)$  ends with

$$\tau(x_i)_{\llbracket |\tau|+k_0, |\tau| \rrbracket} = y_{\llbracket (i+1)|\tau|+k_0, (i+1)|\tau| \rrbracket} = y_{\llbracket k_{i+1}, k_{i+1}-k_0 \rrbracket} = u_{\llbracket 0, -k_0 \rrbracket}.$$

Since  $-k_0$  is a multiple of  $p$ , and  $p$  is a period of  $u$ ,  $u_{\llbracket 0, -k_0 \rrbracket}$  ends with  $u_{\llbracket 0, p \rrbracket}$ . This implies that for all  $i$ ,  $\tau(x_i)$  ends with  $u_{\llbracket 0, p \rrbracket}$ , which contradicts the second assumption in the statement.

- (2) If  $|\mathbf{u}| - |\tau| \leq \mathbf{k}_0 < -|\mathbf{u}|$ . For every  $i \in \mathbb{Z}$ ,  $\tau(x_i)$  contains the word

$$\tau(x_i)_{\llbracket |\tau|+k_0, |\tau|+k_0+|\mathbf{u}| \rrbracket} = y_{\llbracket (i+1)|\tau|+k_0, (i+1)|\tau|+k_0+|\mathbf{u}| \rrbracket} = y_{\llbracket k_{i+1}, k_{i+1}+|\mathbf{u}| \rrbracket} = u.$$

Since  $|\tau| + k_0 \geq |\mathbf{u}|$  this other occurrence of  $u$  is disjoint from the prefix  $u$ , which contradicts the third assumption.

- (3) If  $-|\tau| < \mathbf{k}_0 < |\mathbf{u}| - |\tau|$ . Then  $0 < k_1 < |\mathbf{u}|$ , and thus  $\tau(x)_{\llbracket 0, k_1+|\mathbf{u}| \rrbracket} \in u\mathcal{A}^{k_1} \cap \mathcal{A}^{k_1}u$ , which means that  $k_1$  is a period for  $u$ , hence a multiple of  $p$ . For every  $i \in \mathbb{Z}$ ,  $\tau(x_i)_{\llbracket |\mathbf{u}|, |\tau| \rrbracket}$  starts with

$$\tau(x_i)_{\llbracket |\mathbf{u}|, |\mathbf{u}|+k_1 \rrbracket} = y_{\llbracket i|\tau|+|\mathbf{u}|, i|\tau|+|\mathbf{u}|+k_1 \rrbracket} = y_{\llbracket k_{i+1}-k_1+|\mathbf{u}|, k_{i+1}+|\mathbf{u}| \rrbracket} = u_{\llbracket |\mathbf{u}|-k_1, |\mathbf{u}| \rrbracket},$$

which contradicts the first assumption, since  $k_1$  is a period.

Let us now prove the second part of the statement. Assuming that  $u$  is nonoverlapping, then its smallest period is  $|u|$ , so that the first and second hypotheses are actually implied by the third one. It is thus sufficient to prove it and then apply the first part of the statement. When  $|u| > |\tau|/2$  it is straightforward. When  $|u| = |\tau|/2$  on the other hand, for all  $y$  aperiodic and  $x$  a desubstitution of  $y$ , then  $x$  is aperiodic as well and thus there exists  $i \in \mathbb{Z}$  such that  $\tau(x_i) \neq uu$ . This  $i$  satisfies the third hypothesis.  $\square$

The following extends the result of C. Morales on existence for every  $n \geq 2$  of systems which are positively  $n$ -expansive and not  $(n-1)$ -positively expansive, providing examples within minimal dynamical systems.

**Proposition 4.61.** *For every  $n \geq 2$ , there exists a minimal substitutive Tœplitz shift over alphabet  $\llbracket 0, n \rrbracket$  which is positively  $n$ -expansive and not positively  $(n-1)$ -expansive.*

*Proof.* Let us consider substitution  $\tau$  over  $\llbracket 0, n \rrbracket$  defined by  $\tau: k \mapsto 0 \dots (n-1)k$  for each symbol  $k$ . This Tœplitz substitution satisfies all the hypotheses from Propositions 4.57 and 4.59, hence it is recognizable, everywhere-growing and  $\Omega_\tau$  is minimal. Moreover, the substitution is also right-marked, hence Corollary 4.48 implies that it is positively  $n$ -expansive and not  $(n-1)$ -expansive.  $\square$

4.3.5. *Right-quasi-recognizability radius growth.* Let us derive some additional applications from Theorem 4.26.

When two substitutions  $\tau$  and  $\tau'$  can be composed, it is clear that  $|\tau\tau'| \leq |\tau||\tau'|$ . It is natural to wonder how composition acts on the right-quasi-recognizability radius that we have defined above.

**Lemma 4.62.** *Let  $\tilde{\tau}: \mathcal{B} \mapsto \mathcal{C}$  be a substitution with right-quasi-recognizability radius  $\tilde{R}$  over  $X$ , and  $\tau: \mathcal{A} \rightarrow \mathcal{B}$  be an injective substitution with right-quasi-recognizability radius  $R$  over  $\tilde{\tau}(X)$ . Then  $\left\lceil \frac{R}{|\tau|} + \tilde{R} \right\rceil$  is a right-quasi-recognizability radius for  $\tau\tilde{\tau}$ .*

*Proof.* Let  $x, x'$  be two configurations with  $x_{\mathbb{N}} = x'_{\mathbb{N}}$ . Any standard desubstitution scheme of  $x$  for  $\tau\tilde{\tau}$  can be written as  $(k'_{\tilde{k}_i})_{i \in \mathbb{Z}}$ , where  $\mathbf{k}$  is a standard desubstitution scheme of  $x$  for  $\tau$ ,  $y$  is a corresponding desubstitution, and  $\tilde{\mathbf{k}}$  is a standard desubstitution scheme of  $y$  for  $\tilde{\tau}$ . Similarly, any desubstitution scheme of  $x'$  for  $\tau\tilde{\tau}$  can be written as  $(k'_{\tilde{k}'_i})_{i \in \mathbb{Z}}$ . Since  $R$  is a right-quasi-recognizability radius for  $\tau$ , there

exist  $j, j' \leq R$  such that  $k_{\lfloor j, \infty \rfloor} = k'_{\lfloor j', \infty \rfloor}$ . By injectivity of  $\tau$ , one has  $\sigma^j(y)_{\mathbb{N}} = \sigma^{j'}(y)_{\mathbb{N}}$ . Following Remark 4.12, let us define  $l := \max_{\tilde{k}_i \leq j} i \leq \left\lfloor \frac{j}{|\tau|} \right\rfloor$ , and  $l' := \max_{\tilde{k}'_i \leq j'} i \leq \left\lfloor \frac{j'}{|\tau|} \right\rfloor$ , which make  $\sigma^l \tilde{k} - j$  a standard desubstitution scheme for  $\sigma^j(y)$  and  $\sigma^{l'} \tilde{k}' - j'$  a standard desubstitution scheme for  $\sigma^{j'}(y')$ . Since  $\tilde{R}$  is a right-quasi-recognizability radius for  $\tilde{\tau}$ , there exist  $\tilde{j}, \tilde{j}' \leq \tilde{R}$  such that  $\tilde{k}_{\lfloor l+\tilde{j}, \infty \rfloor} - j = \tilde{k}'_{\lfloor l'+\tilde{j}', \infty \rfloor} - j'$ . For  $n \in \mathbb{N}$ , we get that

$$k_{\tilde{k}_{l+\tilde{j}+n}} = k_{\tilde{k}'_{l'+\tilde{j}'+n} - j' + j} = k'_{\tilde{k}'_{l'+\tilde{j}'+n}}.$$

Since  $l+\tilde{j}$  and  $l'+\tilde{j}'$  are smaller than  $\left\lfloor \frac{R}{|\tau|} \right\rfloor + \tilde{R}$ . Moreover,  $x_{\lfloor k_j, \infty \rfloor} = x'_{\lfloor k'_{j'}, \infty \rfloor}$ ,  $l+\tilde{j} \geq j$ , and  $l'+\tilde{j}' \geq j'$ , so that we also have  $x_{\lfloor k_{\tilde{k}_{l+\tilde{j}}}, \infty \rfloor} = x'_{\lfloor k'_{\tilde{k}'_{l'+\tilde{j}'}}}, \infty \rfloor}$ . We thus have proved that  $\left\lfloor \frac{R}{|\tau|} \right\rfloor + \tilde{R}$  is a right-recognizability radius for  $\tau \tilde{\tau}$ .  $\square$

**Corollary 4.63.** *Let  $\tau$  be an everywhere-growing directive sequence with finite rank  $\alpha \geq 1$  such that there exists  $m \geq 1$  and  $(R_i)_{i \in \mathbb{N}}$  such that for every  $i \in \mathbb{N}$ ,  $R_i$  is a right-quasirecognizable radius for  $\tau_i$  over the shift  $\Omega_{\sigma^i(\tau)}$  with:*

$$\sum_{i < t} R_i |\tau_{\llbracket 0, i \rrbracket}| \leq |\tau_{\llbracket 0, t \rrbracket}| m.$$

*Then  $\Omega_{\tau}$  is positively  $\alpha^{m+2}$ -expansive.*

*Proof.* By applying Lemma 4.62 inductively, for every  $t$ ,  $\left\lfloor \sum_{i < t} \frac{R_i}{|\tau_{\llbracket i, t \rrbracket}|} \right\rfloor$  is a right-quasi-recognizability radius for  $\tau_{\llbracket 0, t \rrbracket}$ . Since for every  $i, t$ ,  $|\tau_{\llbracket 0, t \rrbracket}| \leq |\tau_{\llbracket 0, i \rrbracket}| |\tau_{\llbracket i, t \rrbracket}|$ , we have:

$$\left\lfloor \sum_{i < t} \frac{R_i}{|\tau_{\llbracket i, t \rrbracket}|} \right\rfloor \leq \left\lfloor \frac{1}{|\tau_{\llbracket 0, t \rrbracket}|} \sum_{i < t} R_i |\tau_{\llbracket 0, i \rrbracket}| \right\rfloor \leq m.$$

Thus  $m$  is a right-quasi-recognizability radius for  $\tau_{\llbracket 0, t \rrbracket}$  for every  $t$ . The statement follows from Theorem 4.26.  $\square$

**Corollary 4.64.** *Let  $\tau$  be a directive sequence and  $\rho > 1$  such that  $|\tau_{\llbracket 0, t \rrbracket}| \sim \rho^t$  and there exists  $R$  which is a right-quasi-recognizability radius of  $\tau_t$  over  $\Omega_{\sigma^t(\tau)}$ , for every  $t \in \mathbb{N}$ . Then  $\Omega_{\tau}$  is finitely positively expansive.*

*Proof.* From the hypotheses, we have that there exists  $\beta > 0$  such that

$$\sum_{i < t} R \frac{|\tau_{\llbracket 0, i \rrbracket}|}{|\tau_{\llbracket 0, t \rrbracket}|} \leq \sum_{i < t} R \beta \rho^{i-t} = \frac{\beta R (1 - \rho^{-t})}{\rho - 1} < \frac{\beta R}{\rho - 1}.$$

We conclude by Corollary 4.63.  $\square$

The second hypothesis of Corollary 4.64 is probably difficult to check because it involves both the different substitutions and the different partial limit sets  $\Omega_{\sigma^t \tau}$ . It is quite natural though (and sometimes even assumed in the definition of S-adic) to consider only finitely many substitutions.

A directive sequence is **finitary** if it involves only finitely many distinct substitutions.

**Corollary 4.65.** *Let  $\tau$  be a finitary directive sequence of substitutions such that  $|\tau_{\llbracket 0, t \rrbracket}|$  grows exponentially. If either all of these substitutions are quasi-recognizable, or  $\Omega_{\tau}$  is aperiodic, then  $\Omega_{\tau}$  is finitely positively expansive.*

*Proof.* The first statement comes from Corollary 4.64 and the existence of a common right-quasi-recognizability to any finite set of quasi-recognizable substitutions, by Proposition 4.23 and taking the maximum. The second statement comes additionally from the fact that all substitutions are recognizable over aperiodic configurations, by [BSTY19, BPR22].  $\square$

**Remark 4.66.** *The assumptions of the last corollaries are quite natural, because many substitutions are recognizable (see [Mos96, BSTY19]), and frequently assumed properties of finitary directive sequences, imply exponential growth. For instance, **strong primitivity**:*

$$\exists j \in \mathbb{N}, \forall t \in \mathbb{N}, \forall a \in \mathcal{A}_{t+j}, \forall b \in \mathcal{A}_t, b \text{ appears in } \tau_{\llbracket t, t+j \rrbracket}(a).$$

**Remark 4.67.** *It is to be noted that every everywhere-growing  $S$ -adic shift is eventually recognizable, in the sense that  $\sigma^t\tau$  is recognizable for some  $t \in \mathbb{N}$  [BSTY19, Theorem 5.2]. From this it can be deduced that they are all factors of recognizable  $S$ -adic shifts [DDMP20]. But the transient part (or the factor map) could in theory yield infinitely many left tails for the same configuration right half. In the case of substitutive shifts, though, it is rather easy to derive finite positive expansiveness.*

**Corollary 4.68.** *Let  $\tau$  be a preperiodic everywhere-growing directive sequence such that the shift  $\Omega_\tau$  is aperiodic. Then  $\Omega_\tau$  is finitely positively expansive.*

*Proof.* It is rather straightforward (see for instance [PF02]) that, if a preperiodic directive sequence  $\tau$  is everywhere-growing, then  $|\tau_{[0,t]}| \sim \rho^t$  for some  $\rho > 1$ . It is also known from [BSTY19, Theorem 5.3] that it is recognizable provided that  $\Omega_\tau$  is aperiodic; there is a bounded radius thanks to Lemma 4.62 and the finite number of distinct limit sets  $\Omega_{\sigma^t\tau}$  (thanks to preperiodicity of the sequence  $\tau$ ). The statement follows from Corollary 4.64.  $\square$

**Remark 4.69.** *Most of our results assume non-erasingness and, in this subsection, injectivity. However one can inductively modify the alphabets so as to get these properties. Indeed, if  $\tau_0(a)$  is the empty word, then one can consider  $\mathcal{A}'_1 := \mathcal{A}_1 \setminus \{a\}$ , over which the behavior of  $\tau$  completely describes the configuration image set, and adapt  $\tau_1$  (removing occurrences of  $a$ ). Similarly, if  $\tau_0(a) = \tau_0(b)$ , then one can consider  $\mathcal{A}'_1 := \mathcal{A}_1 \setminus \{b\}$ , and adapt  $\tau_1$  (replacing occurrences of  $b$  by  $a$ ). Doing this inductively for every  $t \in \mathbb{N}$ , one gets a directive sequence with the same limit set and a non-greater rank (since alphabets are not increased in the process). This may change ultimate periodicity or computability aspects, but this is not a problem for our results.*

## 5. COMMENTS

Several of our results use the hypothesis of finite rank and one may wonder if ultimately this hypothesis is sufficient to have finite positive expansiveness. In the general context of zero-dimensional dynamical systems this is not the case, because any odometer has finite topological rank but is not finitely positively expansive. One possible question would be: which constraints, together with finite topological rank, imply finite positive expansiveness.

How complex can finitely positively expansive dynamical systems get? We already know that the entropy is zero when the function is a homeomorphism. In particular, the entropy of finitely positively expansive  $\mathbb{Z}$ -shifts is always zero. However for the examples considered, we observed that the complexity function is always bounded from above by a linear function  $n \mapsto an + b$ . Is it always the case? A possible direction towards this question would be to consider other topological invariants such as entropy dimension.

**5.1. Higher-dimensional shifts.** Let us briefly sketch some comments and questions about higher-dimensional shifts. For more formal terminology about directional dynamics, one can refer to [GK23].

The subaction of a  $\mathbb{Z}^d$ -shift  $X$  in some direction  $\theta$  is positively  $n$ -expansive if every sufficiently wide half-infinite stripe in direction  $\theta$  admits at most  $n$  valid extensions (this definition remains relevant for irrational directions). It is known, in this setting, that every  $\mathbb{Z}^d$ -shift which is expansive in every direction, or positively expansive in one direction, is finite (see for instance [BL97, GK23]).

Nevertheless, if  $X$  is a shift of finite type (that is, has the  $d$ -dimensional shadowing property), its subactions need not have the shadowing property, so that Theorem 2.21 does not apply. This naturally leads to the following question: Does there exist  $n \in \mathbb{N}$  and an infinite  $\mathbb{Z}^2$ -shift of finite type which is  $n$ -expansive in every direction, or positively  $n$ -expansive in some direction? The corresponding question is already answered for sofic  $\mathbb{Z}^2$ -shifts, as seen in the following example, contradicting a naive generalization of Corollary 4.9.

**Example 5.1.** *There exists an infinite sofic  $\mathbb{Z}^2$ -shift which is positively expansive in every direction, except 0 and  $\pi$ , in which it is positively 2-expansive. Indeed, consider any infinite effective positively 2-expansive  $\mathbb{Z}$ -shift  $X$  (for instance the Thue-Morse shift). Then the shift of all those configurations such that all lines are equal, and every line is in  $X$  is sofic, thanks to [DRS10, AS13].*

A natural related directional property is the following (note that each half-plane corresponds to a positive cone for some total group order of  $\mathbb{Z}^2$ , than can generalize what  $\mathbb{N}$  is for  $\mathbb{Z}$ ): a  $\mathbb{Z}^d$ -shift is weakly positively  $n$ -expansive in direction  $\theta$  if every half-plane in direction  $\theta$  admits at most  $n$  valid extensions.

One can now ask, given a shift (of finite type), what are the constraints on possible maps  $\theta \mapsto n$ , where  $n$  is the minimal integer such that the shift is positively  $n$ -expansive (resp. weakly) in direction  $\theta$ .

The minimal substitutive subshifts from [Gä12, Oll08] are good candidates to be looked into for these questions.

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