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Consistency of eight-dimensional supergravities: Anomalies, Lattices and Counterterms

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ABSTRACT: We reexamine the question of quantum consistency of supergravities in eight dimensions. Theories with both 32 and 16 supercharges suffer from the anomalies under the action of their respective discrete modular groups. In maximal supergravity the anomaly cancellation requires a surprising modification of the Chern-Simons couplings. In minimally supersymmetric theory coupled to Yang-Mills multiples of rank l with the moduli space given by $SO(2, l)/(U(1) \times SO(l))$, the existence of a counterterm together with the requirement that its poles and zeros correspond to the gauge symmetry enhancement imposes nontrivial constraints on the lattice. The counterterms needed for anomaly cancellation for all cases, that are believed to lead to consistent theories of quantum gravity ($l = 2, 10, 18$), are discussed.

Contents

1	Introduction and summary	1
2	Anomaly cancellation in maximal supergravity	6
3	Minimal supergravity and lattices of $(2, l)$ signature	9
3.1	Lattices of $(2, l)$ signature and generalized upper half plane	11
3.2	Action of $O(2, l)$ on generalized upper-half plane	13
3.3	Modular forms on generalized upper-half plane	15
4	Composite $U(1)$ in minimal supergravity	17
4.1	$U(1)$ connection, gauge transformations and gauge fixing	19
4.2	Compensating $U(1)$ transformation	20
5	Constructing the counterterm	23
5.1	Reflective modular forms and reflective lattices	26
5.2	Examples of counterterms	27
6	Anomaly cancellation for $l = 2$	31
6.1	The kinetic term	32
6.2	The counterterm	34
7	Discussion	35
A	Dedekind eta function its multiplier system and theta function	38
B	Orthogonal modular forms	40
B.1	The Weil representation	40
B.2	Vector-valued modular forms	41
B.3	Nearly holomorphic modular forms	42
B.4	Modular forms on generalized upper-half plane	43
B.5	Character of the lattice	46
C	Alternative $l = 2$ counterterm from Hilbert modular forms	47

1 Introduction and summary

Existence of an anomaly cancellation mechanism in (super)gravity theories serves as a good guideline for selecting candidates for theories that can be consistent at the quantum level.

In minimally supersymmetric theories in ten dimensions, existence of Green-Schwarz mechanism reduces the number of possible choices for the gauge groups in the YM sector

to four [1]. From the other side, the existence of an anomaly inflow mechanism to two-dimensional chiral strings coupled to the theory restricts this number to two by ruling out the theories with abelian gauge factors [2, 3]. In six dimensions, there is an infinite number of anomaly-free minimal supergravities [1]. Many, notably infinite families of $(1, 0)$ theories, are ruled out by a closer examination of the inflow mechanism and the anomaly cancellation for two-dimensional $(0, 4)$ strings coupled to the theory [3–5].

The focus of this paper is on eight-dimensional (mostly) minimal supergravities. Classically, the 8D $\mathcal{N} = 1$ supergravity multiplet, made of a graviton, B -field, dilaton, two vector fields as well as spin- $\frac{3}{2}$ and spin- $\frac{1}{2}$ Majorana fermions (gravitino and a dilatino), can be coupled to any number of vector multiplets each comprising a vector field (photon), a gaugino (spin- $\frac{1}{2}$ Majorana fermion) and 2 real scalars [6]. Supposing the number of vector multiplets is l , the $2l$ real scalars contained in the matter sector parametrize the moduli space given by a Kähler manifold

$$\mathcal{M} = \frac{\text{SO}(2, l)}{\text{U}(1) \times \text{SO}(l)}. \quad (1.1)$$

These l vectors together with two vectors in the gravity multiplet form an $(l+2)$ -dimensional representation of $\text{SO}(2, l)$.

The first restriction on admissible values of l once more comes from anomalies - theories with odd numbers of Majorana fermions in 8D and 9D suffer from global anomalies [7], and hence l has to be even [8]. There are further restrictions:

- In theories with 16 supercharges in D dimensions the number of vector multiples consistently coupled to gravity is bound by $26 - D$ in order to assure the unitarity of strings couples to the theory [9]. Hence $l \leq 18$.
- Considering 8D theories on particular backgrounds and using 6D anomaly cancellation it has been argued that in fact the only admissible values of l are $l = 2$, $l = 10$ and $l = 18$ [8].
- The symmetry enhancement (as well as the rank of of the YM algebra coupled to string probes) as predicted by the consistency of the supergravity [10] is an agreement with the landscape of 8D string constructions [11, 12].
- In the formulation of the theory with a four-form potential in the gravity multiplet, constraints on the global structure of the gauge groups can be deduced from the the absence of anomalies between large gauge transformations of B_4 and 1-form symmetries [13, 14]¹.

We would like to reexamine these results from the point of view of 8D anomaly cancellation. Neither the $\mathcal{N} = 1$ theory nor its $\mathcal{N} = 2$ counterpart, where the scalars parametrize the $\text{SL}(2) \times \text{SL}(3)/(\text{U}(1) \times \text{SO}(3))$ coset, suffers from chiral anomalies. However, both

¹Somewhat orthogonal to our discussion, global anomalies and topological analogues of Green-Schwarz mechanism have been discussed in 8D with 16 supercharges [15] and in 10D type IIB theory [16]. In this paper we are mostly concerned with the existence of local counterterms.

theories, with 16 and 32 supercharges, have local anomalies under the composite U(1) in the denominator of the coset.

The moduli space of supergravity theories with extended supersymmetry typically has scalars parametrizing a coset G/H . The numerator of the coset, G , denotes the U-duality group of the theory, and some discrete version of it gives rise to an exact symmetry after quantization. Theory can be formulated in a way that G acts only on bosonic fields. The denominator H , which is the maximal compact subgroup of G , is regarded as a gauge symmetry of the theory. Indeed, the compact part of the Cartan-Maurer form of the coset element transforms as a gauge field under the H transformations. The supersymmetry variations of all fermionic fields, which are inert under G , involve this composite connection corresponding to H . When H contains a U(1) factor, it may couple to fermions in a chiral fashion, a priori giving rise to a composite chiral anomaly [17]. This is exactly what happens in eight dimensions.

The physical content of the theory is usually identified by fixing the gauge, thereby eliminating the redundant bosonic degrees of freedom associated to H . When the local symmetry is gauge fixed, the U-duality becomes non-linearly realized. Moreover, the fermionic fields now transform under G . Part of this transformation may still be realized as a nontrivial phase shift. Therefore, the gauge fixing translates the U(1) anomaly into an anomaly under the surviving discrete part of G , making the theory ill-defined.

The existence of this anomaly implies that the symmetry group (SL(2; \mathbb{R}) or SO(2, l ; \mathbb{R}) in $\mathcal{N} = 2$ and $\mathcal{N} = 1$ theories respectively) may not be continuously maintained in the quantum theory. For the theory to be consistent, a cancellation mechanism should be figured out, in the process deciding to what extend the symmetry survives. The question is if it can be done by the addition of a local counterterm with appropriate modular properties under the transformation of the discrete version of G . Originally such counterterm was discussed in the context of ten-dimensional IIB string theory [18], but the formalism is adapted to 8D theories as well [18, 19]. An (in)ability of finding such a counterterm is the reason why the value of l and the lattice structure of the gauge group in 8D get restricted.²

Let us outline the anomaly cancellation mechanism, up to a point trying to keep the discussion general and applicable to both $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories. Denoting the anomalous composite U(1) connection by Q and its curvature by F^Q , the anomaly is given by the descent formula from the ten-form anomaly polynomial

$$I_{10} = \frac{F^Q}{2\pi} \wedge X_8, \quad (1.2)$$

where $X_8 = X_8(R)$ is an eight-form polynomial in curvature two-form R for the $\mathcal{N} = 2$ theory (whose precise form will be very important in our discussion) and $X_8 = X_8(R, \mathcal{F})$ a polynomial in R and the non-abelian gauge field strength \mathcal{F} for $\mathcal{N} = 1$ case (the exact form of the polynomial in this case on the contrary is not going to play any role in our discussion). The resulting anomalous phase variation in the partition function $\Delta = - \int \Sigma X_8$ can locally

²The construction of the counterterms naturally introduces modular forms. A review of modular forms and their important applications in physics can be found in [20].

be cancelled by adding a term to the action

$$\mathcal{S}_\phi = \int \phi X_8, \quad (1.3)$$

where ϕ is a scalar degree of freedom transforming under $U(1)$: $\phi \rightarrow \phi + \Sigma$. This ϕ can be set to zero by gauge fixing (think of the third scalar in $SL(2; \mathbb{R})$), but since $\delta_M \phi \neq 0$ under the G -valued transformation M , the local counterterm is not G -invariant. As shown explicitly for $SL(2, \mathbb{R})$ in [18] and will be extended to $SO(2, l; \mathbb{R})$ here, one can design a counterterm S such that under the G -valued transformation M

$$\delta_M \mathcal{S} = -\delta_M \mathcal{S}_\phi + \arg \chi(M) \int X_8, \quad (1.4)$$

where $\chi(M)$ is a phase factor and $\delta_M \mathcal{S}_\phi = \int \Sigma X_8$. If this phase factor $\chi(M) \equiv 1$ for any $M \in G$ there will be a complete anomaly cancellation but that does not always happen. Note that in general it should suffice that the partition function is well-defined, and hence δ_M of the entire action integrates to an integer (times 2π).

At this point, the situation becomes drastically different for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases.

- For $\mathcal{N} = 2$, i.e. $G = SL(2, \mathbb{R})$ it is shown in [18] that $\chi(M)$ cannot be equal to 1. However (1.4) is not the only part of the action that is not invariant under δ_M . It might appear somewhat counter-intuitive but the reduction of the higher dimensional Chern-Simons terms also yields a non-invariant term. The result is that there are no particular integrality condition imposed on $\int X_8$ in generic backgrounds. Instead, turning non-trivial four-form fluxes is required.
- For $\mathcal{N} = 1$, there are no extra non-invariant terms. Thus, the value of $\chi(M)$ depends on l and on the details of the lattice of signature $(2, l)$, which will naturally appear during the construction of counterterms. So at the first glance this presents a dilemma: either one should be imposing case-by-case integrality conditions on $\int X_8(R, \mathcal{F})$ or, as we shall argue, opt for a universal consistency condition and require that $\chi(M) = 1$ for every $\mathcal{N} = 1$ theory.

Regardless of philosophy, let us turn to the details of how (1.4) works. The first important point is the precise form of $\delta_M \phi$. For instance, in 10D Type IIB supergravity or in $\mathcal{N} = 2$ theory in 8D, the coset element of $SL(2)/U(1)$ is parametrized by the modular parameter τ and the compensating $U(1)$ transformation under the $SL(2; \mathbb{R})$ takes the form

$$e^{-i\Sigma(M, \tau)} = \left(\frac{c\tau + d}{c\bar{\tau} + d} \right)^{\frac{1}{2}}, \quad M \in \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad M \in SL(2, \mathbb{R}), \quad \tau \in \mathbb{H}. \quad (1.5)$$

The second crucial point is that there exists a function of τ , the Dedekind eta function $\eta(\tau)$, that under $SL(2, \mathbb{Z})$ transformation picks a factor $\sim (c\tau + d)^{1/2}$. As a consequence, a ratio of $\eta(\tau)$ and its complex conjugate can be used in constructing the counterterm [18]. As mentioned there can be a phase factor $\chi(M)$, and the consequences of for $\mathcal{N} = 2$ theory, where it is necessarily nontrivial, will be discussed in section 2.

In this paper we are mainly interested in the eight-dimensional supergravity with 16 supercharges, and the moduli space of the theory is given by \mathcal{M} (1.1). The moduli space \mathcal{M} is a realization of the hermitian symmetric space. Moreover, the tube domain, called the generalized upper-half plane \mathbb{H}_l , can be realized in this space [21, 22]. We find that the generalized upper-half plane \mathbb{H}_l provides the correct framework to describe the gauge transformations. By introducing Calabi-Vesentini coordinates [23], we explicitly compute the U(1) gauge potential and its field strength, and show how the U(1) compensating transformation generalizes equation (1.5). It is formed by the so called automorphy factor $j(M, Z)$ (where M is an $\text{SO}(2, l; \mathbb{R})$ transformation, and Z denotes the coordinates on the generalized upper half plane):

$$e^{-i\Sigma(M, Z)} = \frac{j(M, Z)}{|j(M, Z)|}. \quad (1.6)$$

Equivalently we have $-\Sigma = \arg j(M, Z) = \text{Arg } j(M, Z) + 2k\pi$ for $k \in \mathbb{Z}$ and Arg denotes the principal branch of the argument taking the value from $[-\pi, \pi)$. Finding a function $\Psi(Z)$ such that

$$\Psi(M\langle Z \rangle) = \chi(M)j(M, Z)^r \Psi(Z) \quad (1.7)$$

would allow to construct the counterterm \mathcal{S} as

$$\mathcal{S} = \frac{1}{r} \int \arg \Psi(Z) X_8. \quad (1.8)$$

Indeed such functions, or more precisely meromorphic modular forms on the orthogonal group $\text{O}(2, l)$ of weight r and multiplier system χ , can be found by using the Borcherds products [24]. The original discovery that the automorphic forms on $\text{O}^+(2, s+2)$ ($l = s+2$) can be written as infinite products was made in the context of unimodular lattices. Following the use of theta correspondence, which gave an alternative approach to these results [25], the generalisation of the constructions of modular forms to non-unimodular lattices was provided [26].

The case $l = 2$ case requires special treatment. Strictly speaking, the Borcherds product does not apply and an alternative derivation of the counterterms is needed.

As we shall see, the requirement that the modular form $\Psi(Z)$ has a trivial $\chi(M) \equiv 1$, ensuring the complete anomaly cancellation, is not particularly restrictive. However there is an additional consideration: the local counterterms constructed from the meromorphic $\Psi(Z)$ are not well defined at its zeros or poles. On any Borcherds product these points lie on the so-called rational quadratic divisors (RQD). In fact some of these divisors have physical interpretation and correspond to the symmetry enhancement points in the moduli space [25].³ For these, the theory will continue being consistent even if the counterterm is not well-defined. Moreover the gauge symmetry enhancement should be in agreement with the symmetries of the lattice. We will show that these physical constrains lead to the

³Note that this observation was first made in the context of threshold corrections 4D $\mathcal{N} = 2$ theories, and the functions involved are required to be automorphic. Here we need anomaly cancelling counterterms, which require modular forms of non-trivial weights. We shall return to the comparison of the 4D and 8D in section 7.

requirement that the lattice is *reflective* (defined in section 5). The number of reflective lattices is finite and their rank is bounded by $l = 26$. These bounds are less stringent than those imposed by swampland.

The organisation of this paper is as follows. In section 2, we discuss the anomaly cancellation in eight-dimensional theory with 32 supercharges and the constraints imposed on consistent backgrounds. Other than brief comments in the last section of the paper, this is the only part concerned with the maximally supersymmetric theory. The bulk of the paper is about the theories with 16 supercharges. In section 3 we spell out the anomaly that needs to be cancelled, and introduce the necessary mathematical preliminaries needed for the construction of the counterterms (with further details collected in appendix B). Section 4 is devoted to the derivation of the compensating U(1) transformation (equation 1.6). The construction of counterterms is presented in section 5. In this section we also consider the implications of zeros and poles of the modular forms and the ensuing constraints on admissible lattices. This discussion is suitable only if $l \geq 3$. The $l = 2$ case requires a separate discussion that is presented in section 6. A brief summary and discussion of some open questions are presented in section 7.

2 Anomaly cancellation in maximal supergravity

We first discuss the mechanism in $\mathcal{N} = 2$ and the implications of $\chi(M) \neq 1$. The moduli space of the theory is

$$\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{U}(1)} \times \frac{\mathrm{SL}(3, \mathbb{R})}{\mathrm{SO}(3)}, \quad (2.1)$$

and the first factor (the only one relevant for the anomaly) is parametrized by τ . The theory was originally obtained by reducing the 11D supergravity [27]. The conversion into an SL(2) covariant formalism requires taking $\tau = -2C_{8910} + iV_{T^3}$. Alternatively, the reduction on T^2 of Type IIB theory can be considered, and there τ is identified as the complex structure of the torus.

The counterterm derivation would follow closely the discussion of [18] for 10D type IIB theory, and details of the 8D can be found in [19], so we will be brief.

The field content of this 8D theory is given by a single supermultiplet which contains one graviton, two gravitini (doublet under Spin(3) = SU(2)), six vectors, six dilatini (doublet + quadruplet under Spin(3) = SU(2)), seven real scalars, three two-forms and one three-form. The U(1) charges of the gravitini, of the doublet of dilatini and of the quadruplet of dilatini are respectively (they are all positive chiral): $\frac{1}{2}$, $\frac{3}{2}$ and $-\frac{1}{2}$. Finally, the 4-form field strength can be split in self-dual and anti-self-dual part, carrying charges 1 and -1 respectively under U(1). Hence, the 10-form anomaly polynomial is given by

$$I_{10} = \frac{F^Q}{2\pi} \wedge \left[2 \times \frac{1}{2} I_{3/2}^{d=8} - 4 \times \frac{1}{2} I_{1/2} + 2 \times \frac{3}{2} I_{1/2} + 2 \times I_{\mathrm{SD}} \right]_{8\text{-form}}, \quad (2.2)$$

where, F^Q is the composite field strength built out of τ : $F = dQ = \frac{d\tau \wedge d\bar{\tau}}{4i\tau^2}$. The anomalous

phase variation of the path integral is given by [19, 28]

$$\Delta = -12 \int \Sigma X_8(R) = -12 \int \frac{\Sigma}{192(2\pi)^4} \left(\text{tr} R^4 - \frac{1}{4}(\text{tr} R^2)^2 \right). \quad (2.3)$$

As discussed, gauge fixing translates the U(1) anomaly into an SL(2, Z) anomaly.

Maximal eight-dimensional supergravity has a 1-loop UV divergence which breaks local non-linear supersymmetry. This is in agreement with the local U(1) composite anomaly since the commutator of two local non-linear supersymmetries contains this local U(1) [29].⁴

The form of the the compensating U(1) transformation is given in (1.5), and $-\delta_M \phi = \arg(c\tau + d)$ and in the notation of (1.7), $j(M, \tau) = c\tau + d$. Hence ideally one would require a well-defined modular form $f(\tau)$ satisfying

$$f(M\tau) = (c\tau + d)^r f(\tau), \quad M\tau = \frac{a\tau + b}{c\tau + d}, \quad (2.4)$$

for arbitrary transformation $M \in \text{SL}(2, \mathbb{Z})$ in order to cancel the anomaly. In addition, we also require the counterterm built from $f(\tau)$ to have correct decompactification limit. This condition, as we shall see momentarily, is satisfied if and only if the function f is a cusp form. As already discussed, the counterterm can be constructed from the well-known Dedekind eta function $\eta(\tau)$, which is the weight $\frac{1}{2}$ cusp form. However, the Dedekind eta function has non-trivial multiplier system (see appendix A), which can be given in terms of the standard T and S generators, $\chi_\eta(T) = e^{\frac{\pi i}{12}}$ and $\chi_\eta(S) = e^{-\frac{\pi i}{4}}$.

To recap, a counterterm built solely from $\eta(\tau)$ and curvatures;

$$12 \int \arg(\eta^2(\tau)) X_8, \quad (2.5)$$

will not cancel the SL(2, Z) anomaly unless $\int X_8 \in \mathbb{Z}$. It is not hard to come up with examples of consistent supersymmetry-preserving string backgrounds where such a condition does not hold. We propose that the correct, much milder, requirement is

$$\int \left[X_8 + \frac{1}{2} G \wedge G \right] \in \mathbb{Z}. \quad (2.6)$$

As the first step towards justifying this claim let us recall that at the large volume limit, i.e. when $\text{Im} \tau \rightarrow \infty$, the cusp form $\eta(\tau)$ has limit

$$\lim_{\text{Im} \tau \rightarrow \infty} \arg(\eta^2(\tau)) = \frac{\pi \tau_1}{6} \quad (2.7)$$

Recalling that there is a term $\sim \tau$ among the couplings of the classical 8D supergravity, obtained from the reduction of the 11D Chern-Simons term, the large volume limit of the

⁴We thank Renata Kallosh for bringing this to our attention.

anomalous couplings (2.5) becomes:

$$\lim_{\text{Im } \tau \rightarrow \infty} \mathcal{S} = 2\pi \int \tau_1 \left[X_8 + \frac{1}{2} G \wedge G \right]. \quad (2.8)$$

Further decompactification to nine dimensions yields

$$2\pi \int A_1 \wedge \left[X_8 + \frac{1}{2} G \wedge G \right]. \quad (2.9)$$

For type IIA $A_1 = B_{\mu 9} dx^\mu$, and further lift recovers the full set of 10D Chern-Simons couplings [30, 31]. Note that the two term combination - with an importantly fixed relative coefficient - is needed for the simultaneous cancellation of failure of the diffeomorphism invariance and $C_3 \rightarrow C_3 + d\Lambda_2$ transformation in the presence of fivebranes. For IIB, $A_1 = (\alpha'/R_9^2) g_{\mu 9} dx^\mu$, where R is the radius of the circle, and the coupling is suppressed in the IIB ten-dimensional limit. The origin of the two terms is respectively the winding mode one-loop contribution and the self-duality of the IIB five-form field-strength [32, 33].

One may also note that (2.6) is the condition that appears in the tadpole cancellation in M-theory compactifications to 3D and lower. It would be natural that when considering M-theory on backgrounds that involve a product of 3D and 8D spaces, the integrality condition that is imposed on 8D part involves the same players regardless of 8D space being compact or not (up to possibly boundary modifications).

Should one try to think of (2.8) as a large volume limit not only for the first term, but also the second? Indeed, having $G \wedge G$ multiplied by a nontrivial modular function of τ of weight zero (that does not pick up a $j(M, \tau)^r$ factor) with the same phase factor as $\eta^2(\tau)$ would result in a complete cancellation of the $\text{SL}(2, \mathbb{Z})$ anomaly without imposing any ad hoc conditions on the 8D space-time. Given that the coupling $\tau_1 G \wedge G$ is not $\text{SL}(2)$ invariant (the existing 8D supergravity action constructed from reduction of 11D theory has only explicit $\text{SL}(3)$ symmetry), such a modification would appear not unwelcome. From other side, if the eight-dimensional spacetime has an isometry it should be possible to recast the maximally supersymmetric theory in an explicitly $\text{SL}(5)$ invariant form. Note that upon reduction the four-form G gives rise to a pair of fields with 3-form field strength which together with three 8D 3-forms will form a quintet of $\text{SL}(5)$, and the Chern-Simons coupling becomes part of kinetic term for this quintet. A nontrivial function in τ will obstruct promoting $\text{SL}(2) \times \text{SL}(3)$ to $\text{SL}(5)$. At the same time, existence on an isometry would render the 8-form on 8D manifold $X_8(R)$ trivial, and its integral will vanish. Putting all these considerations together, we arrive a modification

$$\int \tau_1 \frac{1}{2} G \wedge G \quad \mapsto \quad \int [\tau_1 + x(6 \arg(g(\tau)) - \tau_1)] \frac{1}{2} G \wedge G, \quad (2.10)$$

where x can take only two values, $x = 0$ for trivial X_8 and $x = 1$ otherwise.⁵ To obtain

⁵Note that X_8 being nontrivial necessarily means that only a fraction of supersymmetry is preserved.

(2.6) while agreeing with (2.8), we should require that

$$g(M\tau) = \chi_\eta^2(M)g(\tau). \quad (2.11)$$

and has the same large volume limit as $\arg(\eta^2(\tau))$ (2.7).

A natural way for finding $g(\tau)$, is to use the quotient $\eta^2(\tau)$ by the well-defined modular form of weight 1 that does not have an extra phase (and goes to 1 in $\text{Im } \tau \rightarrow \infty$ limit). The theta function $\theta(\tau)$ transforms under the congruence subgroup $\Gamma_0(4)$ as shown in equation (A.12) and has the limit $\theta^2(\tau) \rightarrow 1$ under $\text{Im } \tau \rightarrow \infty$. $\Gamma_0(4)$ has three generators

$$-\mathbb{1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}. \quad (2.12)$$

Only under the $-\mathbb{1}$, $(\frac{-1}{d}) = -1$. In other words, if we consider the physical symmetry $\text{P}\Gamma_0(4) = \Gamma_0(4)/\{\pm\mathbb{1}\}$, $\theta^2(\tau)$ acts as weight 1 modular form with trivial character. With these considerations, we see that restricting the symmetry group to $\Gamma_0(4)$ the function

$$g(\tau) = \frac{\eta^2(\tau)}{\theta^2(\tau)} \quad (2.13)$$

satisfies all necessary requirements. Hence for

$$\mathcal{S}(x=1) = 12 \int \left[\arg(\eta^2(\tau))X_8 + \arg\left(\frac{\eta^2(\tau)}{\theta^2(\tau)}\right) \frac{1}{2}G \wedge G \right], \quad (2.14)$$

under the transformation $M \in \text{P}\Gamma_0(4)$,

$$\delta_M \mathcal{S} + \delta_M \mathcal{S}_\phi = 12 \arg(\chi_\eta^2(M)) \int \left(X_8 + \frac{1}{2}G \wedge G \right). \quad (2.15)$$

The multiplier system (or character) $\chi_\eta^2(M)$ is of order 12 with respect to the group $\text{SL}(2, \mathbb{Z})$ (or $\Gamma_0(4)$), and imposing the integrality condition (2.6) on $[X_8 + \frac{1}{2}G \wedge G]$ (rather than on $X_8(R)$) leaves the partition function invariant and is sufficient for the anomaly cancellation.

While the counterterm (2.14) works only for $M \in \text{P}\Gamma_0(4)$, this does not signify the breaking of actual symmetry of the theory but rather a problem of supergravity action, which we recall once more displays $\text{SL}(2)$ symmetry only at the level of equations of motion.

It would be interesting to verify the coupling (2.14) directly by string theory calculation, which for $x=1$ has to be done in a nontrivial gravitational background.

3 Minimal supergravity and lattices of $(2, l)$ signature

We can now turn to the minimal supergravity in 8D and its possible anomaly counterterms. Generically such theory comprises a single gravity multiplet and l vector multiplets. The field content of $\mathcal{N} = 1, D = 8$ supergravity is given by

$$(e_\mu{}^m, \psi_\mu, \chi, B_{\mu\nu}, A_\mu{}^i, \sigma), \quad i = 1, 2, \quad (3.1)$$

where e_μ^m is the graviton, ψ_μ is the gravitino, χ is dilatino, $B_{\mu\nu}$ is the antisymmetric tensor (background field), σ is the dilaton. Both ψ_μ and χ are pseudo-Majorana spinors. A_μ^i and the scalar σ are real. Coupling l vector multiplets of the form (λ, A_μ, ϕ^i) and combining the field content together we obtain

$$(e_\mu^m, \psi_\mu, \chi, B_{\mu\nu}, A_\mu^I, \phi^\alpha, \sigma), \quad (3.2)$$

where $I = 0, \dots, l+1$, $\alpha = 1, \dots, 2l$. Here we adopt the metric convention

$$\eta_{AB} = \eta_{IJ} = (+1, +1, -1, \dots, -1). \quad (3.3)$$

The $2l$ real scalars ϕ^α parameterize the moduli space

$$\mathcal{M} = \frac{\text{SO}(2, l)}{\text{SO}(2) \times \text{SO}(l)} \cong \frac{\text{SO}(2, l)}{\text{U}(1) \times \text{SO}(l)}. \quad (3.4)$$

The fermions of the theory have chiral couplings to one of the composite U(1) in the denominators of the coset [6]. The U(1) charges of the gravitino (positive chirality), the dilatino (negative chirality) and the gaugini (positive chirality) are all $\frac{1}{2}$.⁶ Hence, the anomaly polynomial is

$$I_{8D} = I_{3/2} - I_{1/2}^{\text{dilatino}} + I_{1/2}^{\text{gaugini}}. \quad (3.5)$$

If the gauge group is given by G ($\text{rank}(G) = l$), the gaugini couple both to G (the fields strength of the gauge field will be denoted by \mathcal{F}) and the composite U(1) (whose field strength is again denoted by $F^Q = dQ$). The resulting polynomial is of the form (1.2) with (see [19] for details)

$$X_8(R, \mathcal{F}) = \frac{1}{32(2\pi)^3} \left[(248 + \dim G) \left[\frac{\text{tr } R^4}{360} + \frac{(\text{tr } R^2)^2}{288} \right] - (\text{tr } R^2)^2 + \frac{1}{6} \text{tr } R^2 \text{Tr } \mathcal{F}^2 + \frac{2}{3} \text{Tr } \mathcal{F}^4 \right], \quad (3.6)$$

and given the variation $\delta Q = d\Sigma$ the anomalous phase is

$$\Delta_G = - \int \Sigma X_8(R, \mathcal{F}), \quad (3.7)$$

The precise form of $X_8(R, \mathcal{F})$ in (3.7) is not important for our discussion. The idea is to constrain the admissible theories and their lattices rather than try to cancel (3.7) by imposing case by case conditions on the integrality properties of $X_8(R, \mathcal{F})$. In addition, the counterterm can be changed by adding massive states and integrating them out. While the role of the massive completions presents interesting questions, here we are concerned by the possibility of writing a counterterm that will lead to an anomaly cancellation for any (discrete) SO(2, l) transformation.

The construction of the counterterm will be following the discussion of the $\mathcal{N} = 2$ case

⁶Notice that the same charge assignment is valid in dual formulation of the theory where the two-form B is replaced by a four-form [34], and the discussion of the counterterms applies to both.

in Sec. 2, but now we will be interested in modular forms for orthogonal group $\text{SO}(2, l)$. We will have to construct the compensating $\text{U}(1)$ with respect to this group (see Sec. 4) and find the modular forms with right properties to serve as counterterms (1.8).

In the rest of this section we will discuss some of the necessary background and set up the notation. In order to make the presentation self-contained, we will include some of the basic definitions. Further details can be found in Appendix B, where the presentation follows closely [21, 22].

3.1 Lattices of $(2, l)$ signature and generalized upper half plane

A typical lattice L in \mathbb{R}^b has the form $L := \left\{ \sum_{i=1}^b a_i v_i \mid a_i \in \mathbb{Z} \right\}$, where $\{v_1, \dots, v_b\}$ is the basis set. Usually the lattice is equipped with a quadratic form $q : L \rightarrow \mathbb{R}$, which defines the norm of the vector x in the lattice as $q(x)$ and naturally induces a symmetric bilinear form $(\cdot, \cdot) : \Lambda \times \Lambda \rightarrow \mathbb{R}$

$$(x, y) := q(x + y) - q(x) - q(y), \quad \text{for } x, y \in L. \quad (3.8)$$

It is easy to see that $q(x) = \frac{1}{2}(x, x)$ since q is a quadratic bilinear form. The lattice is called even if $q(x) \in \mathbb{Z}$ for arbitrary $x \in L$. The dual lattice L' is defined as

$$L' := \{y \in L \otimes \mathbb{Q} \mid (y, x) \in \mathbb{Z} \text{ for } \forall x \in L\}. \quad (3.9)$$

A lattice L is called self-dual or unimodular if it is equal to its dual $L = L'$. The quadratic form q has signature, denoted by (b^+, b^-) and $b^+ + b^- = b$, where b^+ (b^-) denotes the number of the $+$ ($-$) signs. An important theorem states that there are no indefinite even unimodular lattices unless $b^+ - b^- \equiv 0 \pmod{8}$.

Let us denote the lattice and its quadratic form by a pair (L, q) . Suppose (L, q) is a lattice that has a signature $(2, l)$. Consider the Grassmannian of 2-dimensional subspaces of $V = L \otimes \mathbb{R}$ on which the quadratic form is positive definite

$$\text{Gr}_2(V) := \{v \subset V \mid \dim v = 2 \text{ and } q|_v > 0\}, \quad (3.10)$$

where $q|_v > 0$ means that for every element $x \in v$, $q(x) > 0$ ⁷. We define the orthogonal group and the special orthogonal group as

$$\text{O}(V; \mathbb{R}) := \{\sigma \in \text{Aut}(V) \mid \sigma \text{ is an isometry of } V\}, \quad \text{SO}(V; \mathbb{R}) := \{\sigma \in \text{O}(V; \mathbb{R}) \mid \det \sigma = 1\}.$$

Since V is a usual linear space on \mathbb{R}^{l+2} , one can think of these two as matrix groups. If two spaces V_1, V_2 have the same signature, it can be proved that the orthogonal groups are isomorphic, i.e. $\text{O}(V_1; \mathbb{R}) \cong \text{O}(V_2; \mathbb{R})$. Thus we can denote the (special) orthogonal group by using the signature like $\text{O}(2, l; \mathbb{R})$ ($\text{SO}(2, l; \mathbb{R})$). One can prove that $\text{O}(2, l; \mathbb{R})$ acts transitively on $\text{Gr}_2(V)$. If $v_0 \in \text{Gr}_2(V)$ is fixed, the stabilizer K of v_0 is a maximal

⁷We have defined q on the lattice L , i.e. $q(v_i)$ has a clear definition for $1 \leq i \leq l+2$. With the help of the induced bilinear form $(x, y) = q(x+y) - q(x) - q(y)$, we can safely extend the quadratic form to the space $V = L \otimes \mathbb{R}$.

compact subgroup of $O(2, l; \mathbb{R})$ and $K \cong O(2) \times O(l)$. This constructs an isomorphism $\text{Gr}_2(V) \cong O(2, l; \mathbb{R})/K$, which is a realization of the hermitian symmetric space.

To see the complex structure, we consider the complexification $V(\mathbb{C}) = V \otimes \mathbb{C}$ of V . Since V has negative signatures, there exists non-trivial isotropic vector x , which satisfies $q(x) = 0$ and $x \neq 0$. The isotropic subspace (or called the zero quadric) is

$$\mathcal{I} := \{Z_L \in V(\mathbb{C}) \setminus \{0\} \mid (Z_L, Z_L) = 0\} . \quad (3.11)$$

We consider the projective space $P\mathcal{I} := \mathcal{I} / \sim$, where the equivalence relation is $Z_L \sim tZ_L$ for arbitrary $t \in \mathbb{C}, t \neq 0$. The equivalence class can be denoted as $[Z_L]$. Consider the subset

$$\mathcal{K} := \{[Z_L] \in P\mathcal{I} \mid (Z_L, Z_L) = 0, (Z_L, \overline{Z_L}) > 0\} , \quad (3.12)$$

\mathcal{K} is a complex manifold of dimension l consisting of two connected components. The subgroup $O^+(2, l; \mathbb{R})$ of elements whose spinor norm equals the determinant preserves the components of \mathcal{K} , whereas $O(2, l; \mathbb{R}) \setminus O^+(2, l; \mathbb{R})$ interchanges them. We can denote the components \mathcal{K}^+ and \mathcal{K}^- respectively.

For arbitrary $Z_L \in V(\mathbb{C})$, we can write $Z_L = X_L + iY_L, X_L, Y_L \in V$, and construct a map between \mathcal{K}^+ and $\text{Gr}_2(V)$

$$[Z_L] \mapsto v(Z_L) = \{aX_L + bY_L \mid a, b \in \mathbb{R}\} . \quad (3.13)$$

This map is an analytic isomorphism. Thus, the set \mathcal{K}^+ and the map indeed provide a complex structure on $\text{Gr}_2(V)$. We refer the reader to [22] for detailed proof.

As mentioned, for the $\mathcal{N} = 1$ theory with a duality group $SO(2, l; \mathbb{R})$ we shall try to replicate the discussion of $\mathcal{N} = 2$ case with the duality group $SL(2; \mathbb{R})$. The moduli space is much more complicated however, and some generalisations are not straightforward. An important concept, useful for us, is that of the generalized upper-half plane corresponding to the $SO(2, l; \mathbb{R})$ transformation. We will first describe the formal way of constructing the generalized upper-half plane \mathbb{H}_l [22]. A specific method for achieving the generalization [35, 36], which is more appropriate for our discussion, will be discussed later.

Suppose $z \in L$ is a primitive norm zero vector, i.e. $q(z) = 0$ and $\mathbb{Q}z \cap L = \mathbb{Z}z$. Let $z' \in L'$ be another vector which satisfies $(z, z') = 1$. We define the sub-lattice K

$$K := L \cap z^\perp \cap z'^\perp , \quad (3.14)$$

where z^\perp denotes the orthogonal subspace of z , such that that all vectors x in this subspace satisfy $(x, z) = 0$. Then K is Lorentzian, i.e. of the signature is $(1, l - 1)$. The space V can be decomposed into

$$V = \mathbb{R}z \oplus (K \otimes \mathbb{R}) \oplus \mathbb{R}z', \quad V(\mathbb{C}) = \mathbb{C}z \oplus (K \otimes \mathbb{C}) \oplus \mathbb{C}z' . \quad (3.15)$$

For arbitrary vector $Z_L \in V(\mathbb{C})$, there exists a unique combination (a, Z, b) such that $Z_L = az + Z + bz', Z \in K \otimes \mathbb{C}, a, b \in \mathbb{C}$, which means that we can use the combination

(a, Z, b) to represent a vector. We define a set $\tilde{\mathbb{H}}_l$

$$\tilde{\mathbb{H}}_l = \{Z = X + iY \in K \otimes \mathbb{C} \mid X, Y \in K \otimes \mathbb{R}, q(Y) > 0\} . \quad (3.16)$$

Since the lattice K has signature $(1, l-1)$, the set of positive-norm vectors in $K \otimes \mathbb{R}$ splits into two connected components, \mathcal{K}^\pm . We define the map f

$$\begin{aligned} f : \tilde{\mathbb{H}}_n &\longrightarrow \mathcal{K} \\ Z &\longmapsto f(Z) = [(-q(Z) - q(z'), Z, 1)] , \end{aligned} \quad (3.17)$$

where $(-q(Z) - q(z'), Z, 1)$ is an $l+2$ dimensional vector in space $V(\mathbb{C})$. As the references [21, 22] show, f is a biholomorphic map. Under the map f , two connected components of $\tilde{\mathbb{H}}_l$ map into the two connected components \mathcal{K}^\pm of \mathcal{K} separately. We choose \mathbb{H}_l to be the component of $\tilde{\mathbb{H}}_l$ that maps \mathcal{K}^+ . This realization of \mathcal{K}^+ as a tube domain can be viewed as generalized upper-half plane $\mathbb{R}^l + i\Omega^l$, where Ω^l is the positive-norm cone.

3.2 Action of $O(2, l)$ on generalized upper-half plane

To construct the generalized upper-half we will split the lattice (L, q) into $(L, q) = (L_0, q_0) \oplus \Pi_{1,1}$ where (L_0, q_0) is a lattice of signature $(1, l-1)$, equipped with the quadratic form q_0 and $\Pi_{1,1}$ is the unimodular lattice with signature $(1, 1)$ equipped with the quadratic form $q((a, b)) = ab$. The generalized upper-half plane can be defined in the following way

$$\mathbb{H}_l = \{Z = X + iY \in L_0 \otimes \mathbb{C} \mid X, Y \in L_0 \otimes \mathbb{R}, Y \in P\} , \quad (3.18)$$

where P denotes the future light cone of the Minkowski space $L_0 \otimes \mathbb{R}$.

We set $l = s + 2$ and take $s \geq 0$ and even throughout the discussion. Suppose \hat{S} is an $s \times s$ symmetric positive definite real matrix (when $s = 0$, \hat{S} collapses, but the discussion below still applies) and define

$$\begin{aligned} S_0 &= \begin{pmatrix} & & 1 \\ & -\hat{S} & \\ 1 & & \end{pmatrix} \in \text{Sym}(s+2; \mathbb{R}) , \\ S &= \begin{pmatrix} & & & 1 \\ & & & \\ & & S_0 & \\ 1 & & & \end{pmatrix} \in \text{Sym}(s+4; \mathbb{R}) , \end{aligned} \quad (3.19)$$

where $\text{Sym}(s+2; \mathbb{R})$ denotes the set of $(s+2) \times (s+2)$ symmetric real matrix. We then define for $Z_L \in L$ (or $L \otimes \mathbb{C}$), $q(Z_L) = \frac{1}{2} Z_L^T S Z_L$ and for $Z \in L_0$ (or $L_0 \otimes \mathbb{C}$), $q_0(Z) = \frac{1}{2} Z^T S_0 Z$. Here the superscript T means the transpose operation.

In the following we will frequently use the notation $L(N)$ for a positive integer N (Some references use the notation $\sqrt{N}L$). $L(N)$ indicates the lattice with the same basis as L but equipped with the scaled quadratic form NS (or vectors scaled by factor \sqrt{N} equivalently). We consider explicitly only the lattices $L = \Pi_{1,1}(1) \oplus \Pi_{1,1}(1) \oplus \hat{L}$ with lattice \hat{L} equipped

with the quadratic form \hat{S} . The discussion can be extended to $L = \Pi_{1,1}(N_1) \oplus \Pi_{1,1}(N_2) \oplus \hat{L}$ for arbitrary positive integer $N_{1,2}$ and the results still apply.

With the help of these quadratic forms, by setting $z = (1, 0, \dots, 0)^T \in L$ and $z' = (0, \dots, 0, 1)^T \in L'$, the map f (3.17) can be written as

$$\begin{aligned} f : \mathbb{H}_l &\longrightarrow \mathcal{K}^+ \\ Z &\longmapsto [Z_L] = \left[(-q_0(Z), Z, 1)^T \right]. \end{aligned} \quad (3.20)$$

The (special) orthogonal transformation $O(2, l; \mathbb{R})$ ($SO(2, l; \mathbb{R})$)⁸ now is

$$\begin{aligned} O(2, l; \mathbb{R}) &= \{ M \in \text{Mat}(l+4; \mathbb{R}) \mid M^T S M = S \}, \\ SO(2, l; \mathbb{R}) &= \{ M \in \text{SL}(l+4; \mathbb{R}) \mid M^T S M = S \}. \end{aligned} \quad (3.21)$$

Since \mathcal{K}^+ and \mathbb{H}_l are isomorphic, the action of $O^+(2, l; \mathbb{R})$ on the set \mathcal{K}^+ naturally induces the action on the generalized upper-half plane \mathbb{H}_l . If $M \in O^+(2, l; \mathbb{R})$, the action on \mathcal{K}^+ is defined as

$$\begin{aligned} M : \mathcal{K}^+ &\longrightarrow \mathcal{K}^+ \\ [Z_L] &\longmapsto [MZ_L], \end{aligned} \quad (3.22)$$

where MZ_L is the usual linear transformation of a vector in \mathbb{C}^{l+2} . Because the real orthogonal transformation will not change the the norm (Z_L, Z_L) and (Z_L, \bar{Z}_L) , the element $[MZ_L]$ still stays in the set \mathcal{K}^+ . Then we can define the action of M on the generalized upper-half plane \mathbb{H}_l , $Z \mapsto M\langle Z \rangle$, such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{K}^+ & \xrightarrow{[Z_L] \mapsto [MZ_L]} & \mathcal{K}^+ \\ \uparrow f & & \uparrow f \\ \mathbb{H}_l & \xrightarrow{Z \mapsto M\langle Z \rangle} & \mathbb{H}_l \end{array} \quad (3.23)$$

In the equation form we have

$$[Mf(Z)] = [f(M\langle Z \rangle)]. \quad (3.24)$$

For convenience, we decompose the matrix M in the following way

$$M = \begin{pmatrix} \alpha & a^T & \beta \\ b & P & c \\ \gamma & d^T & \delta \end{pmatrix} \in O^+(2, l; \mathbb{R}), \quad \begin{cases} \alpha, \beta, \gamma, \delta \in \mathbb{R}, \\ a, b, c, d \in \mathbb{R}^l, \\ P \in \text{Mat}(l; \mathbb{R}). \end{cases} \quad (3.25)$$

⁸The definition here, while using a different metric, is isomorphic to the definition using $\eta = (+1, +1, -1, \dots, -1)$.

Expanding the equation (3.24), we have

$$\begin{pmatrix} \alpha & a^T & \beta \\ b & P & c \\ \gamma & d^T & \delta \end{pmatrix} \begin{pmatrix} -q_0(Z) \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} -\alpha q_0(Z) + a^T Z + \beta \\ -b q_0(Z) + P Z + c \\ -\gamma q_0(Z) + d^T Z + \delta \end{pmatrix} = j(M, Z) \begin{pmatrix} -q_0(W) \\ W \\ 1 \end{pmatrix}, \quad (3.26)$$

where $W = M\langle Z \rangle$, $j(M, Z) \in \mathbb{C}$. From this equation we extract the definition the action of (special) orthogonal group on the generalized upper-half plane \mathbb{H}_l directly

$$\begin{aligned} W = M\langle Z \rangle &:= (-b q_0(Z) + P Z + c) (-\gamma q_0(Z) + d^T Z + \delta)^{-1}, \\ j(M, Z) &:= -\gamma q_0(Z) + d^T Z + \delta. \end{aligned} \quad (3.27)$$

With such definition, the equality (3.26) clearly holds for last two components. The first components of the vector on two sides of (3.26) are equal due to the norm-zero condition.

3.3 Modular forms on generalized upper-half plane

As proposed in the section 1, a specific function transforming in the particular way under the modular group (1.7) will play a central role in the construction of the counterterm. We turn now to the modular forms on generalized upper-half plane, commonly referred to as orthogonal modular forms. We will restrict for now to $l \geq 3$, where the application of the results of Borcherds [24, 26] apply. The case $l = 2$ will be considered in section 6.

The pivotal observation is that the group $\mathrm{SL}(2; \mathbb{R})$ and $\mathrm{O}(2, l; \mathbb{R})$ form a dual reductive pair. To construct the modular forms of orthogonal group $\mathrm{O}(2, l)$, the modular forms on $\mathrm{SL}(2, \mathbb{Z})$ can be lifted by integrating against the Siegel theta function $\Theta(\tau, Z)$ [25, 26]. A brief review of the relevant background, following the presentation of [21] is given in appendix B.

Suppose $\mathrm{O}(L)$ is the orthogonal group of a even lattice L with signature $(2, l)$ defined by

$$\mathrm{O}(L) := \{M \in \mathrm{O}(2, l; \mathbb{R}) \mid M L = L\}. \quad (3.28)$$

The orthogonal group of the discriminant group $D(L) := L'/L$ can be defined similarly and will be denoted as $\mathrm{O}(L'/L)$. We then denote by $\mathrm{O}_d(L)$ the discriminant kernel of $\mathrm{O}(L)$, which is the subgroup of finite index of $\mathrm{O}(L)$ consisting of all elements which act trivially on the discriminant group L'/L , i.e.

$$\mathrm{O}_d(L) := \mathrm{Ker}(\mathrm{O}(L) \rightarrow \mathrm{O}(L'/L)). \quad (3.29)$$

We define the intersection with $\mathrm{O}^+(V)$, $V = L \otimes \mathbb{R}$ as the modular group

$$\Gamma(L) := \mathrm{O}^+(V) \cap \mathrm{O}_d(L). \quad (3.30)$$

Recalling the definition of $j(M, Z)$, we can rewrite it as $j(M, Z) = (M Z_L, z)$ with $l + 2$ -dimensional vectors $z = (1, 0, \dots, 0)^T$ and $Z_L = (-q_0(Z), Z, 1)^T$. Following Theorem 13.3 in [26] (Theorem B.1), we can lift a nearly holomorphic modular form $f(\tau) = \sum_{\gamma \in L'/L} f_\gamma \mathbf{e}_\gamma$:

$\mathbb{H} \rightarrow \mathbb{C}[L'/L]$ (see Definition B.3) of weight $1 - l/2$ with Fourier expansion

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z} + q(\gamma)} c(\gamma, n) \mathbf{e}_\gamma(n\tau), \quad (3.31)$$

to the meromorphic function $\Psi(Z) : \mathbb{H}_l \rightarrow \mathbb{C}$ with the following transformation property

$$\Psi(M\langle Z \rangle) = \chi(M) j(M, Z)^{c(0,0)/2} \Psi(Z), \quad M \in \Gamma(L). \quad (3.32)$$

$\chi(M)$ is called the multiplier system (see Definition B.4) and $\Psi(Z)$ is a modular form on generalized upper-half plane (also called Borchers product) of weight $c(0,0)/2$ with the multiplier system (or character if the weight is integer) χ and modular group $\Gamma(L)$. This modular group contains some elements that do not preserve orientation. Since our symmetry group is $\mathrm{SO}(2, l; \mathbb{R})$, the modular group we use is actually $\mathrm{S}\Gamma(L) := \Gamma(L) \cap \mathrm{SO}(L)$. We will be interested in the logarithm (the argument) of such modular forms. Hence we also need the information of its poles and zeros, where the argument at these points is not well defined. Remarkably, the positions of zeros and poles are totally determined by the principal part, consisting of all the terms with n (in equation (3.31)) negative

$$\sum_{\beta \in L'/L} \sum_{\substack{n \in \mathbb{Z} + q(\beta) \\ n < 0}} c(\beta, n) \mathbf{e}_\beta(n\tau). \quad (3.33)$$

By Theorem 13.3 in [26] (Theorem B.2), zeros and poles of $\Psi(Z)$ lie in the divisor (Ψ) , which is the linear combinations of rational quadratic divisors $H(\beta, m)$ (Heegner divisors). The rational quadratic divisors $H(\beta, m)$ are unions of orthogonal subspaces H_λ with respect to the vector $\lambda \in \beta + L$, for $\beta \in L'/L$ and rational negative norm m ,

$$H_\lambda = \{[Z_L] \in \mathcal{K}^+ \mid (Z_L, \lambda) = 0\}. \quad (3.34)$$

A rational quadratic divisor $H(\beta, m)$ is defined as

$$H(\beta, m) = \sum_{\substack{\lambda \in \beta + L \\ q(\lambda) = m}} H_\lambda. \quad (3.35)$$

The zeros and poles of $\Psi(Z)$ are contained in the divisor (Ψ) which is given by

$$(\Psi) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z} + q(\beta) \\ m < 0}} c(\beta, m) H(\beta, m). \quad (3.36)$$

These rational quadratic divisors are closely related to the gauge symmetry enhancement [25]. We will return to this in section 5.

4 Composite U(1) in minimal supergravity

We can now turn to computing the compensating U(1) transformation. As the first step a suitable parametrization of the coset space

$$\mathcal{M} = \frac{\text{SO}(2, l)}{\text{SO}(2) \times \text{SO}(l)} \cong \frac{\text{SO}(2, l)}{\text{U}(1) \times \text{SO}(l)} \quad (4.1)$$

is needed. The coset construction in [6] provides a good starting point. At the level of Lie algebra the representative of the coset $\mathfrak{so}(2, l)/(\mathfrak{so}(2) \oplus \mathfrak{so}(l))$ can be written as a matrix

$$\begin{pmatrix} 0_{2 \times 2} & H_{2 \times l} \\ (H^T)_{l \times 2} & 0_{l \times l} \end{pmatrix}, \quad H \in \text{Mat}(2 \times l, \mathbb{R}). \quad (4.2)$$

Here the $2l$ real scalars ϕ^α in $\mathcal{N} = 1$ vector multiplets (see (3.2)) are packaged in H , and $\text{Mat}(2 \times l, \mathbb{R})$ is the set of the real $2 \times l$ matrices. An element $\Lambda \in \mathcal{M}$, $\Lambda^T \eta \Lambda = \eta$, can be represented as

$$\Lambda = \exp \begin{pmatrix} 0 & H \\ H^T & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{1 + qq^T} & q \\ q^T & \sqrt{1 + q^T q} \end{pmatrix}, \quad (4.3)$$

where $q \in M(2 \times l, \mathbb{R})$ is given by

$$q = H \left(\frac{\sinh H^T H}{H^T H} \right)^{\frac{1}{2}}. \quad (4.4)$$

By direct matrix multiplication we can see that

$$\sqrt{1 + qq^T} = \cosh(HH^T)^{1/2}, \quad \sqrt{1 + q^T q} = \cosh(H^T H)^{1/2}. \quad (4.5)$$

The matrices $\sqrt{1 + qq^T}$ and $\sqrt{1 + q^T q}$ satisfy the relation

$$\sqrt{1 + q^T q} = \mathbb{1} + q^T \left(\sqrt{1 + qq^T} - \mathbb{1} \right) (qq^T)^{-1} q, \quad (4.6)$$

which can be checked by squaring the expression on the both sides. The negative power of the matrix in this expression should be considered in the sense of Taylor expansion since the matrix $H^T H$ might not be invertible. Based on this parametrization, we can further simplify the expression by introducing the so called Calabi-Vesentini coordinates [23, 37].

The matrix elements can be labeled by Λ_I^A , with I being the row index and A represents the column index. All the capital Latin indices take integer value from 0 to $l + 1$ ($I, A = 0, 1, \dots, l + 1$), and the metric-preserving property can be written in components as

$$\Lambda_I^A \Lambda_J^B \eta_{AB} = \eta_{IJ}. \quad (4.7)$$

The inverse matrix element of Λ^{-1} is denoted as Λ^I_A and satisfies

$$\Lambda_I^A \Lambda^I_B = \delta^A_B, \quad \Lambda^I_A = \eta^{IJ} \eta_{AB} \Lambda_J^B. \quad (4.8)$$

We can now define

$$\Phi^A = \frac{1}{\sqrt{2}} (\Lambda_0^A + i\Lambda_1^A), \quad (4.9)$$

which can be verified to satisfy

$$\begin{aligned} \bar{\Phi}^A \Phi^B \eta_{AB} &= \frac{1}{2} (\Lambda_0^A - i\Lambda_1^A) (\Lambda_0^B + i\Lambda_1^B) \eta_{AB} = \frac{1}{2} (\eta_{00} + \eta_{11}) = 1, \\ \Phi^A \Phi^B \eta_{AB} &= \frac{1}{2} (\Lambda_0^A + i\Lambda_1^A) (\Lambda_0^B + i\Lambda_1^B) \eta_{AB} = \frac{1}{2} (\eta_{00} - \eta_{11}) = 0. \end{aligned} \quad (4.10)$$

A natural Ansatz for Φ^A satisfying these constraints takes the form

$$\Phi^A = \frac{X^A}{\sqrt{\bar{X}^A X^B \eta_{AB}}}. \quad (4.11)$$

where X^A are components of a $l+2$ dimensional complex vector \vec{X} such that $\vec{X}^T \eta \vec{X} = 0$. In terms of X^A the matrix (4.3) can be written as

$$\Lambda = \frac{1}{\sqrt{2\bar{X}^A X^B \eta_{AB}}} \begin{pmatrix} X^0 + \bar{X}^0 & -i(X^0 - \bar{X}^0) & \dots \\ X^1 + \bar{X}^1 & -i(X^1 - \bar{X}^1) & \dots \\ \vdots & \vdots & * \\ X^{l+1} + \bar{X}^{l+1} & -i(X^{l+1} - \bar{X}^{l+1}) & \dots \end{pmatrix}. \quad (4.12)$$

Notice that $-i(X^0 - \bar{X}^0) = X^1 + \bar{X}^1$, Λ is a symmetric real matrix.

One way to parametrize X^A in terms of l independent complex scalars is

$$X^A = \left(\frac{1+y^2}{2}, \frac{i}{2}(1-y^2), y_i \right), \quad i = 1, \dots, l, \quad (4.13)$$

where y_i is a complex scalar and $y^2 := y_i y_i$. Here, and in the rest of the discussion, a summation over all repeated indices is implied. In addition, y_i should satisfy [23]

$$\bar{X}^A X^B \eta_{AB} > 0 \quad \Rightarrow \quad 1 - 2\bar{y}_i y_i + y^2 \bar{y}^2 > 0, \quad \bar{y}_i y_i < 1, \quad (4.14)$$

which is the bounded choice of the region for y_i , known as Calabi-Vesentini coordinates. In terms of y^i

$$\Lambda = \frac{1}{\sqrt{1 - 2\bar{y}_k y_k + y^2 \bar{y}^2}} \begin{bmatrix} 1 + \frac{1}{2}(y^2 + \bar{y}^2) & -\frac{i}{2}(y^2 - \bar{y}^2) & \dots \\ -\frac{i}{2}(y^2 - \bar{y}^2) & 1 - \frac{1}{2}(y^2 + \bar{y}^2) & \dots \\ y_1 + \bar{y}_1 & -i(y_1 - \bar{y}_1) & \\ \vdots & \vdots & * \\ y_l + \bar{y}_l & -i(y_l - \bar{y}_l) & \end{bmatrix}. \quad (4.15)$$

Under the parametrization (4.12), it is easy to see that \vec{X} is equivalent with $t\vec{X}$ if $t \in \mathbb{R}$. Also, recall that Λ is a coset representative, i.e. $\Lambda \sim \Lambda U$ where U is a $\text{SO}(2) \times \text{SO}(l)$

transformation parametrized by a real θ :

$$\begin{aligned}\Lambda \sim \Lambda U &= \frac{\sqrt{2}}{\sqrt{\vec{X}^\dagger \eta \vec{X}}} \left(\text{Re}(\vec{X}) \quad \text{Im}(\vec{X}) \quad * \right) \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} U_{\text{SO}(l)} \\ &= \frac{\sqrt{2}}{\sqrt{\vec{X}^\dagger \eta \vec{X}}} \left(\text{Re}(\vec{X} e^{-i\theta}) \quad \text{Im}(\vec{X} e^{-i\theta}) \quad * \right),\end{aligned}\quad (4.16)$$

which means that $\vec{X} \sim \vec{X} e^{-i\theta}$. Combined with the scaling transformation, this leads to a conclusion that \vec{X} lives in the projective space and $\vec{X} \sim \alpha \vec{X}$ for an arbitrary non-zero complex number α .

4.1 U(1) connection, gauge transformations and gauge fixing

We are now ready to construct explicitly the composite connection associated with the local U(1) gauge symmetry. It can be expressed in terms of the Maurer-Cartan form [6] as

$$Q = (\Lambda^{-1} d\Lambda)_0^1, \quad Q_\mu = (\Lambda^{-1} \partial_\mu \Lambda)_0^1, \quad (4.17)$$

where d is the exterior derivative defined on the spacetime manifold and ∂_μ is the partial derivative with respect to the μ spacetime coordinates. Using the expression of Λ of q (4.3), we have

$$(\Lambda^{-1} \partial_\mu \Lambda)_{2 \times 2} = \sqrt{1 + qq^T} \partial_\mu \sqrt{1 + qq^T} - q \partial_\mu q^T, \quad (4.18)$$

where the subscript indicates the 2×2 upper left corner of the matrix $\Lambda^{-1} \partial_\mu \Lambda$. Since y_i are unconstrained variables, expressing Q in terms of these avoids ambiguities and we have:

$$Q_\mu = 2i \frac{\bar{y}_i - \bar{y}^2 y_i}{1 - 2\bar{y}_k y_k + y^2 \bar{y}^2} \partial_\mu y_i - \frac{i}{2} \partial_\mu \ln(1 - 2\bar{y}_k y_k + y^2 \bar{y}^2). \quad (4.19)$$

Notice that

$$\frac{1}{2} (1 - 2\bar{y}_k y_k + y^2 \bar{y}^2) = \bar{X}^A X^B \eta_{AB} = \vec{X}^\dagger \eta \vec{X}, \quad (4.20)$$

the denominator is naturally invariant under the transformation $\vec{X} \rightarrow \vec{X} e^{-i\Sigma}$. Besides,

$$\begin{aligned}\vec{X}^\dagger \eta \partial_\mu \vec{X} &= \left(\frac{1 + \bar{y}^2}{2}, -\frac{i}{2} (1 - \bar{y}^2), \bar{y}^\dagger \right) \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} y_i \partial_\mu y_i \\ -i y_i \partial_\mu y_i \\ \partial_\mu \bar{y} \end{pmatrix} \\ &= \bar{y}^2 y_i \partial_\mu y_i - \bar{y}^\dagger \partial_\mu \bar{y}\end{aligned}\quad (4.21)$$

allows to write Q_μ compactly in terms of \vec{X} ,

$$Q_\mu = -i \frac{\vec{X}^\dagger \eta \partial_\mu \vec{X}}{\vec{X}^\dagger \eta \vec{X}} - \frac{i}{2} \partial_\mu \ln(2 \vec{X}^\dagger \eta \vec{X}). \quad (4.22)$$

Under the U(1) gauge transformation $\vec{X} \rightarrow \vec{X}' = \vec{X} e^{-i\Sigma}$ we have

$$\delta Q_\mu = Q'_\mu - Q_\mu = -i \frac{\vec{X}'^\dagger \eta \partial_\mu \vec{X}'}{\vec{X}'^\dagger \eta \vec{X}'} + i \frac{\vec{X}^\dagger \eta \partial_\mu \vec{X}}{\vec{X}^\dagger \eta \vec{X}} = -\partial_\mu \Sigma, \quad (4.23)$$

as expected (Q'_μ here denotes the U(1) transformed connection).

Notice that in the expression for the coset element Λ (4.3) we have chosen the gauge $\phi = 0$, where ϕ represents the variable parametrizing local U(1) gauge symmetry. In order to maintain the gauge ($\phi = 0$), a left action on $\Lambda \in \mathcal{M}$ by an $\text{SO}(2, l)$ transformation should be compensated by a right action of a $\text{SO}(2) \times \text{SO}(l)$ transformation, i.e.

$$\Lambda \rightarrow \Lambda' = R\Lambda U^{-1}, \quad U \in \text{SO}(2) \times \text{SO}(l), \quad (4.24)$$

which leads to

$$\begin{aligned} \frac{1}{\sqrt{\vec{X}^\dagger \eta \vec{X}}} R \left(\text{Re}(\vec{X}) \quad \text{Im}(\vec{X}) \right) &= \frac{1}{\sqrt{\vec{Y}^\dagger \eta \vec{Y}}} \left(\text{Re}(\vec{Y}) \quad \text{Im}(\vec{Y}) \right) \begin{pmatrix} \cos \Sigma & -\sin \Sigma \\ \sin \Sigma & \cos \Sigma \end{pmatrix} \\ &= \frac{1}{\sqrt{\vec{Y}^\dagger \eta \vec{Y}}} \left(\text{Re}(\vec{Y} e^{-i\Sigma}) \quad \text{Im}(\vec{Y} e^{-i\Sigma}) \right), \end{aligned} \quad (4.25)$$

where the complex vector \vec{Y} parametrizes the new coset representative Λ' . The equation is obviously equivalent to

$$\frac{1}{\sqrt{\vec{X}^\dagger \eta \vec{X}}} R \vec{X} = \frac{1}{\sqrt{\vec{Y}^\dagger \eta \vec{Y}}} \vec{Y} e^{-i\Sigma}. \quad (4.26)$$

In the next subsection we will proceed to formally solving this equation and obtaining an analytic expression of $\Sigma = \Sigma(R, \vec{X})$. Before doing so we should recall that in the familiar case of $\text{SL}(2, \mathbb{R})/\text{U}(1)$, the compensating U(1) transformation (the phase factor) is given by (1.5) in terms of the modular variable τ which lives in the complex upper-half plane \mathbb{H} . As we shall see, the relation between the U(1) anomaly and modular variables is universal.

4.2 Compensating U(1) transformation

We start by recalling that the vector \vec{X} , which lives in the projective space, satisfies the constraints

$$\vec{X}^T \eta \vec{X} = 0, \quad \vec{X}^\dagger \eta \vec{X} > 0. \quad (4.27)$$

This matches the condition (3.12) on the generalized upper-half plane for the group $\text{O}^+(2, l; \mathbb{R})$ (see section 3.2). The Calabi-Vesentini coordinates (4.13) are not very convenient for solving the equation (4.26) and determining $\Sigma(R, \vec{X})$. Instead, we should rotate to the reference frame with basis already discussed in section 3, and notably use matrices $M \in \text{O}^+(2, l; \mathbb{R})_S$. The subscript here emphasizes that the orthogonal group is defined with respect to a metric S . This definition applies throughout our discussion, and we shall often omit the subscript.

Since \hat{S} , defining the metric S , introduced in (3.19) is a symmetric positive-definite real matrix, there must exist a orthogonal matrix \hat{P} such that $\hat{P} \hat{S} \hat{P}^T = \hat{V}$, where \hat{V} is

the diagonal matrix with positive diagonal elements. One can define the square root of the inverse $\sqrt{\hat{V}^{-1}}$ such that

$$\mathbb{1} = \hat{Q}\hat{S}\hat{Q}^T, \quad \hat{Q} = \sqrt{\hat{V}^{-1}}\hat{P}. \quad (4.28)$$

There exists a orthogonal matrix U such that $USU^T = \eta$, explicitly we have

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}}J & \frac{1}{\sqrt{2}}\mathbb{1}_2 \\ \hat{Q} & \\ -\frac{1}{\sqrt{2}}J & \frac{1}{\sqrt{2}}\mathbb{1}_2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad UU^T = \begin{pmatrix} \mathbb{1}_2 & \\ & \hat{V} \\ & & \mathbb{1}_2 \end{pmatrix}. \quad (4.29)$$

Inserting U into the equation (4.26) we have

$$\frac{1}{\sqrt{\vec{Z}^\dagger S \vec{Z}}} M \vec{Z} = \frac{1}{\sqrt{\vec{W}^\dagger S \vec{W}}} \vec{W} e^{-i\Sigma}, \quad \vec{Z} = U^T \vec{X}, \quad \vec{W} = U^T \vec{Y} \quad M = U^T R (U^T)^{-1}. \quad (4.30)$$

It is not difficult to verify that $M \in O^+(2, l; \mathbb{R})_S$. To further demonstrate that such a choice of coordinates would be realized as the generalized upper-half plane, we explicitly expand the equation and derive the constraints satisfied by \vec{Z} [38]. After the rotation we have

$$\vec{Z} = U^T \vec{X} = \begin{pmatrix} \frac{i}{2\sqrt{2}}(1-y^2) - \frac{1}{\sqrt{2}}y_{s+2} \\ \frac{1}{2\sqrt{2}}(1+y^2) - \frac{1}{\sqrt{2}}y_{s+1} \\ \hat{Q}^T \vec{y}_s \\ \frac{1}{2\sqrt{2}}(1+y^2) + \frac{1}{\sqrt{2}}y_{s+1} \\ \frac{i}{2\sqrt{2}}(1-y^2) + \frac{1}{\sqrt{2}}y_{s+2} \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{s+2} \\ \beta_{s+3} \end{pmatrix}, \quad (4.31)$$

satisfying the constraint

$$\begin{cases} \vec{Z}^T S \vec{Z} = 0 & \Rightarrow 2\beta_0\beta_1 + 2\beta_{s+2}\beta_{s+3} - \vec{\beta}_s^T \hat{S} \vec{\beta}_s = 0, \\ \vec{Z}^\dagger S_1 \vec{Z} > 0 & \Rightarrow \overline{\beta_0}\beta_1 + \overline{\beta_1}\beta_0 - \vec{\beta}_s^\dagger \hat{S} \vec{\beta}_s + \overline{\beta_{s+2}}\beta_{s+3} + \beta_{s+2}\overline{\beta_{s+3}} > 0, \end{cases} \quad (4.32)$$

where $\vec{\beta}_s$ is the vector with components $(\beta_2, \beta_3, \dots, \beta_{s+1})$. First, let us verify that $\beta_{s+3} \neq 0$. Indeed, assuming $\beta_{s+3} = 0$, would yield

$$\begin{cases} 2\beta_0\beta_1 - \vec{\beta}_s^T \hat{S} \vec{\beta}_s = 0, \\ \overline{\beta_0}\beta_1 + \overline{\beta_1}\beta_0 - \vec{\beta}_s^\dagger \hat{S} \vec{\beta}_s > 0. \end{cases} \quad (4.33)$$

However, by absolute inequality, we have

$$|2\beta_0\beta_1| = |\vec{\beta}_s^T \hat{S} \vec{\beta}_s| \leq \vec{\beta}_s^\dagger \hat{S} \vec{\beta}_s < \overline{\beta_0}\beta_1 + \overline{\beta_1}\beta_0. \quad (4.34)$$

Since both sides of the equation have positive signs, we can square without changing the direction of the inequality:

$$4\overline{\beta_0}\beta_0\overline{\beta_1}\beta_1 < (\overline{\beta_0}\beta_1)^2 + (\overline{\beta_1}\beta_0)^2 + 2\overline{\beta_0}\beta_0\overline{\beta_1}\beta_1 \iff (\text{Im } \overline{\beta_0}\beta_1)^2 < 0, \quad (4.35)$$

leading to contradiction. Thus $\beta_{s+3} \neq 0$ and we can safely normalize the vector \vec{Z} by dividing the final component,

$$\vec{Z} = \alpha(z) \begin{pmatrix} -q_0(Z) \\ Z \\ 1 \end{pmatrix} = \alpha(Z)Z_L, \quad \alpha(Z) = \beta_{s+3}, \quad Z \in \mathbb{C}^{s+2}, \quad Z_j = \frac{\beta_j}{\beta_{s+3}}, \quad j = 1, \dots, s+2. \quad (4.36)$$

Here $q_0(Z) = Z^T S_0 Z / 2$ as defined in section 3.2. With the definition of the quadratic form $q(Z_L) = \frac{1}{2} Z_L^T S Z_L$ and $(A, B) = q(A+B) - q(A) - q(B)$, we can rewrite the constraints (4.32) of Z_L as

$$(Z_L, Z_L) = 0, \quad (Z_L, \bar{Z}_L) > 0, \quad (4.37)$$

so we conclude that $Z_L \in \mathcal{K}$ defined by equation (3.12). Without loss of generality we assume that $Z_L \in \mathcal{K}^+$, and can check that the only constraint on the range of Z is given by $q_0(\text{Im}(Z)) > 0$. If we assume $\text{Im}(Z)$ lives in the future light cone of the Minkowski space, Z indeed lives in the generalized upper-half plane \mathbb{H}_l . With these settings we can rewrite the equation (4.30) as

$$\frac{e^{i\hat{\phi}(Z)}}{\sqrt{Z_L^\dagger S Z_L}} M Z_L = \frac{e^{i\hat{\phi}(W)}}{\sqrt{W_L^\dagger S W_L}} W_L e^{-i\Sigma}, \quad (4.38)$$

where $Z_L = (-q_0(Z), Z, 1)^T$, $W_L = (-q_0(W), W, 1)^T$, and

$$e^{i\hat{\phi}(Z)} = \frac{\alpha(Z)}{|\alpha(Z)|}, \quad e^{i\hat{\phi}(W)} = \frac{\alpha(W)}{|\alpha(W)|}. \quad (4.39)$$

Recalling the discussion of the action of the orthogonal group on generalized upper-half plane (equation (3.25) and (3.26)), we conclude that $W = M(z)$ and

$$e^{-i\Sigma(M,Z)} = e^{i\hat{\phi}(Z) - i\hat{\phi}(W)} \frac{\sqrt{W_L^\dagger S W_L}}{\sqrt{Z_L^\dagger S Z_L}} (-\gamma q_0(Z) + d^T Z + \delta). \quad (4.40)$$

Recall that

$$M Z_L = \begin{pmatrix} -\alpha q_0(Z) + a^T Z + \beta \\ -b q_0(Z) + P Z + c \\ -\gamma q_0(Z) + d^T Z + \delta \end{pmatrix} = (-\gamma q_0(Z) + d^T Z + \delta) W_L, \quad (4.41)$$

with the property that the real orthogonal transformation doesn't change the norm, i.e. $\sqrt{Z_L^\dagger S Z_L} = \sqrt{(M Z_L)^\dagger S (M Z_L)}$, we conclude that

$$e^{-i\Sigma(M,Z)} = e^{i\hat{\phi}(Z) - i\hat{\phi}(W)} \frac{-\gamma q_0(Z) + d^T Z + \delta}{|-\gamma q_0(Z) + d^T Z + \delta|} = e^{i\hat{\phi}(Z) - i\hat{\phi}(W)} \frac{j(M, Z)}{|j(M, Z)|}. \quad (4.42)$$

By choosing the specific gauge, the compensating U(1) transformation is given by

$$e^{-i\Sigma(M,Z)} = \frac{j(M,Z)}{|j(M,Z)|}. \quad (4.43)$$

This is the direct generalization of the compensating U(1) transformation for $\text{SL}(2, \mathbb{R})/\text{U}(1)$ given in (1.5) to the generalized upper-half plane \mathbb{H}_l .

5 Constructing the counterterm

In this section we shall examine the anomaly cancellation for $l \geq 3$, while leaving the treatment of $l = 2$ to section 6. As already discussed in the beginning of section 3, eight-dimensional $\mathcal{N} = 1$ theories suffer from a composite U(1) anomaly. The anomalous phase raised in the local U(1) gauge transformation ($\phi \rightarrow \phi + \Sigma$) is

$$\Delta_G = - \int \Sigma X_8(R, \mathcal{F}). \quad (5.1)$$

A direct way to cancel the anomalous phase is to add the local counterterm

$$\mathcal{S}_\phi = \int \phi X_8(R, \mathcal{F}), \quad (5.2)$$

where ϕ parametrizes the local U(1) gauge symmetry. When we apply the U(1) gauge transformation $\phi \rightarrow \phi + \Sigma$, $\delta\mathcal{S}_\phi$ can cancel the anomalous phase above. But the drawback is that the local counterterm is not invariant under $\text{SO}(2, l; \mathbb{R})$ symmetry transformations, as shown in the equation (4.43)

$$\delta_M \phi = - \arg(j(M, Z)). \quad (5.3)$$

Here δ_M indicates an $\text{SO}(2, l; \mathbb{R})$ gauge transformation with respect to the element M . Since the compensating U(1) transformation is the argument of the automorphy factor, it is natural to construct the counterterm by using modular forms on generalized upper-half plane and it is of the form

$$\mathcal{S} = \frac{1}{r} \int \arg(\Psi(Z)) X_8(R, \mathcal{F}), \quad (5.4)$$

where $\Psi(Z)$ satisfies the modular property (1.7). We have already seen that the continuous symmetry group should be discretized since no suitable functions that can maintain the continuous symmetry. As mentioned in section 3.3 the analogue of $\text{SL}(2; \mathbb{Z})$ in $\mathcal{N} = 2$ case is the discrete modular group $\text{S}\Gamma(L)$ with respect to the lattice L of signature $(2, l)$. Such a discrete lattice L will be the root lattice of the gauge group G (or contain the sublattices which are the root lattices of the gauge group G). Hence the anomaly cancellation may lead to nontrivial restrictions on the lattices L (on the gauge groups G).

As discussed in section 3, the Borcherds products provide necessary tools for constructing $\Psi(Z)$ with requisite properties to cancel the anomaly (5.1). As long as a nearly

holomorphic modular form of weight $1 - l/2$ with respect to the lattice L can be found, one can obtain the modular form $\Psi(Z)$ on generalized upper-half plane of weight $r = c(0, 0)/2$. However the counterterms needs to satisfy some natural physical conditions leading to constraints that will be outlined bellow.⁹

- **The character of the lattice L (the modular group $\text{S}\Gamma(L)$)**

Since the modular form $\Psi(Z)$ satisfies the modular property (3.32), where the weight $r = c(0, 0)/2$, the counterterm is transformed under the $\text{S}\Gamma(L)$ transformation as

$$\delta_M \mathcal{S} = -\delta_M \mathcal{S}_\phi + \arg \chi(M) \int X_8(R, \mathcal{F}). \quad (5.5)$$

In order to completely cancel the anomaly without imposing extra conditions on the background manifold, such as integrality of $\int X_8(R, \mathcal{F})$, $\chi(M) \equiv 1$ for arbitrary $M \in \text{S}\Gamma(L)$ is required. To the best of our knowledge, the necessary and sufficient condition for the character to be trivial is not known.¹⁰ A sufficient condition is known (Theorem B.4). Moreover, it cannot be weakened too much (see the counter example (Example 1.4) in [39]). More details are in the appendix B. Notably any lattice that contains A_2 sublattice has $\chi(M) \equiv 1$.

- **Rational quadratic divisor (RQD)**

The counterterm is obviously ill-defined at the zeroes or poles of $\Psi(Z)$. Fortunately, through Borchers product (Theorem B.2) we know that all the zeroes and poles lie in the rational quadratic divisors (Definition B.6). To circumvent this issue, one could have required the Borchers product $\Psi(Z)$ to be well-defined and have no zeroes on the entire generalized upper-half plane, which is equivalent to requiring $c(\beta, m) \equiv 0$ if $m < 0$ for all $m \in \mathbb{Z} + q(\beta)$ and $\beta \in L'/L$. This would in turn mean that the principal of the nearly holomorphic modular form $f(\tau)$ has zero principal part, so $f(\tau)$ is actually a holomorphic modular form of $\text{SL}(2, \mathbb{Z})$. However, no nonzero holomorphic modular form of non positive weight ($1 - l/2 \leq 0$ for $l \geq 2$) exists and thus the counterterm will always have ill-defined points in moduli space.

As originally explained in the context of 4D $\mathcal{N} = 2$ theories [25], these points in moduli space, corresponding to symmetry enhancements, are contained in the rational quadratic divisors. This is the set of the orthogonal subspaces determined by the negative-norm vectors $\ell \in L'$, such that the reflections orthogonal to them are symmetries of the lattice. Viewing a general even lattice L as the momentum lattice, it would exclude some rational quadratic divisors. The reason is that ℓ might not be in L . This means that in the general expansion of the divisors in terms of RQDs (3.36), we should take $c(\beta, m) = 0$ if $\beta \neq 0$ ($\beta \in L'/L$).¹¹ By using shorthand notation $H(m) = \frac{1}{2}H(0, m)$ (having suppressed the

⁹Here we recall another time that throughout this discussion we have taken the lattice L to be even.

¹⁰In general $\chi(M)$ is called the multiplier system and is different from character if the weight of the Borchers product is not integral. Through suitable normalization we can always obtain the Borchers product of integral weights so we will not consider the cases of rational weights.

¹¹This is automatic for all unimodular lattices.

vector index), the divisor of modular form $\Psi(Z)$ can be written as

$$(\Psi) = \sum_{m \in \mathbb{Z}, m < 0} c(m)H(m). \quad (5.6)$$

Since the Borchers product comes from the lifting of the nearly holomorphic modular form $f(\tau)$, such $f(\tau)$ exists if and only if it satisfies the Theorem B.3, i.e. the coefficients in the principal part (3.33) need to satisfy

$$\sum_{m \in \mathbb{Z}, m < 0} c(m)a(-m) = 0. \quad (5.7)$$

Here $a(-m) = a_{0,-m}$ is the functional that maps the cusp form $g \in S_{\kappa,L}$ into its $(0, -m)$ Fourier coefficient and $S_{\kappa,L}$ is the space of the cusp forms of weight $\kappa = 1 + l/2$ for the dual Weil representation (more details can be found in the discussion of Theorem B.3).

The simplest solution to this condition is when the cusp form space $S_{\kappa,L}$ is trivial (there exists no nonzero cusp form of weight $1 + l/2$ of dual Weil representation). Lattices with such property exist and are called simple lattices. As shown in [40], there are only 15 simple even lattices of signature $(2, l)$, $l \geq 4$, of square free level¹² up to isomorphisms (see the Theorem 2 in [40]). For signature $(2, 18)$, only the even unimodular lattice $\Pi_{2,18} \cong \Pi_{1,1} \oplus \Pi_{1,1} \oplus E_8(-1) \oplus E_8(-1)$ is simple, and for $(2, 10)$ only the even unimodular $\Pi_{2,10} \cong \Pi_{1,1} \oplus \Pi_{1,1} \oplus E_8(-1)$ and $\Pi_{1,1} \oplus \Pi_{1,1}(2) \oplus E_8(-1)$ are simple.

- **Reflective lattices**

Requiring that the $(2, l)$ lattice L is reflective is sufficient for finding a solution of (5.7). This condition will be discussed in detail in the next subsection. At this point, we only mention that the reflective symmetries of the lattice are directly linked to the enhancement of the gauge symmetry. There is a finite number of such lattices, and their rank is bounded by $l = 26$. There is a complete classification of reflective lattices of prime level. All these lattices are of even rank and hence should be considered. A complete classification for any level is available for a particular subclass, the 2-reflective lattices that have norm -2 roots.¹³ Here we find lattices of odd rank, which should be discarded due to the global anomalies.

Recall that in ten dimensions, anomaly cancellation allows for not only rank 16 theories (with a unimodular lattice $E_8 \oplus E_8$), but also of theories with gauge group $E_8 \times U(1)^{248}$ and $U(1)^{496}$. Simple reduction of these theories would produce 8D theories with $l = 258$ and $l = 498$ respectively. The fact that the condition of reflectivity bounds the rank of the lattice to be equal or less than 24 tells us that for these no suitable 8D counterpart can be

¹²The level of the lattice L is a positive integer p such that $p = \min\{n \in \mathbb{N} | nq(\gamma) \in \mathbb{Z} \text{ for all } \gamma \in L'\}$.

¹³A bibliographical note: In [41] all strongly reflective modular forms of singular weight on lattices of prime level were classified. A proof that there are only finitely many even lattices with $l \geq 7$ which admit 2-reflective modular forms and the highest rank such lattice is the even unimodular lattice $\Pi_{2,26}$ is given by [42]. These were subsequently classified in [43]. In [44], all possible reflective lattices of prime level were classified. In our discussion of reflective modular forms (see subsection 5.1), we will adopt the conventions of [44].

found (even if they admit 10D Green-Schwartz term). So it seems these theories can be ruled out purely based on anomaly cancellation, and without swampland considerations.

- **Counterterms as obstructions to ten-dimensional lifts**

Given the form of a reflective lattice (5.10) it is natural to ask about possible decompactifications to ten dimensions. If such decompactification is possible, i.e. a good “large volume limit” exists, the 8d theory can be considered consistent only if a lifting to the ten-dimensional $E_8 \oplus E_8$ lattice exists.¹⁴

We have not done an exhaustive check on which reflective lattices can or cannot be lifted to an $E_8 \oplus E_8$ lattice in 10D. Any lattice with $l > 18$ clearly does not have such lifting. The rank 8 self-dual lattice $\Pi_{1,1} \oplus \Pi_{1,1} \oplus E_8(-1)$ also does not have such lifting. For such lattices the anomaly cancellation can be validated only if they are “intrinsically eight-dimensional”, i.e. if their counterterm obstructs the decompactification to 10D. As we shall see in section 6, we find such example in $l = 2$ case, where the two complex scalars parametrizing the coset cannot be identified with the moduli of a two-torus.¹⁵

5.1 Reflective modular forms and reflective lattices

Let L be an even lattice of signature $(2, l)$ and its dual is L' . The level of L is the smallest positive integer N such that $N(x, x) \in 2\mathbb{Z}$ for all $x \in L'$. The discriminant of L denoted L'/L can be decomposed by Jordan components and we denote this decomposition by D_L . The genus of L is the set of lattices which have the same signature and the same discriminant form (up to isomorphism) as L . A holomorphic modular form for the modular group $\Gamma(L)$ is called reflective if its zeroes are contained in the union of rational quadratic divisors ℓ^\perp associated to roots of L , namely the reflection

$$\sigma_\ell : \alpha \mapsto \alpha - \frac{2(\alpha, \ell)}{(\ell, \ell)} \ell, \quad \alpha \in L \quad (5.8)$$

belongs to $O^+(L)$. A lattice is called reflective if it has a reflective modular form. A modular form is called symmetric if it is modular for $O^+(L)$ and it is known that L is reflective if and only if L has symmetric reflective modular form. Recall the definition of the modular group $\Gamma(L)$

$$\Gamma(L) = O^+(L \otimes \mathbb{R}) \cap \text{Ker} (O(L) \rightarrow O(L'/L)) , \quad (5.9)$$

$\Gamma(L) = O^+(L)$ ($S\Gamma(L) = \Gamma(L) \cap SO(L)$) is the largest modular symmetry group. It is reasonable to require the local counterterms are constructed by symmetric modular form since we want to maintain the discrete symmetry maximally.

¹⁴For $L \cong \Pi_{1,1} \oplus \Pi_{1,1} \oplus \sum_i \hat{L}_i$, there always exists a straightforward lift in ten dimensions with $L(10D) = \sum_i \hat{L}_i$. Other than for the Narain lattice, all these lifts can be discarded. The CHL lattice $\Pi_{1,1} \oplus \Pi_{1,1} \oplus D_8(-1) \cong \Pi_{1,1} \oplus \Pi_{1,1}(2) \oplus E_8(-1)$ also allows for a lift to $E_8 \oplus E_8$. Very loosely, all lattices that would allow to have central charge $c_L = 18$ can be potentially liftable to ten dimensions

¹⁵Similarly, the function F_1^{het} that appears in the one-loop gravitational couplings in the $\mathcal{N} = 2$ heterotic compactification with two vector multiplets also does not allow such identification [45].

We consider the lattices of the same genus $\Pi_{2,l}(p^{\epsilon_p l_p})$, where $l \geq 3$,¹⁶ p is a prime number, $\epsilon_p = -$ or $+$, $1 \leq l_p \leq l/2 + 1$ and ϵ_p is completely determined by l, p and l_p . If two lattices of signature $(2, l)$ and prime level p have the same determinant then they are isomorphic. We refer the readers to [41] for more details. Let L be such a lattice. By [46], L can be represented as¹⁷

$$\Pi_{1,1} \oplus \Pi_{1,1}(p) \oplus \hat{L}(-1) \quad \text{or} \quad \Pi_{1,1} \oplus \Pi_{1,1} \oplus \hat{L}(-1), \quad (5.10)$$

where $\Pi_{1,1}$ is a hyperbolic plane as we defined above and \hat{L} is a positive definite lattice. A primitive vector $v \in L$ is reflective if and only if $(v, v) = -2$ or $(v, v) = -2p$ and $v/p \in L'$. By [47] and Eichler criterion (see e.g. [48]) all the vectors of norm -2 in L are in the same $O^+(L)$ -orbit, and all reflective vectors of norm $-2p$ in L are also in the same $O^+(L)$ -orbit. Therefore, for a symmetry reflective modular form, all 2-reflective divisors (the rational quadratic divisors defined by the vector v of norm -2) have the same multiplicity, which is denoted by c_1 . All $2p$ -reflective divisors (the rational quadratic divisors defined by the vector v of norm $-2p$ and $v/p \in L'$) have the same multiplicity denoted by c_p . A symmetric reflective modular form is called 2-reflective (resp. $2p$ -reflective) if $c_p = 0$ (resp. $c_1 = 0$). A lattice L is called 2-reflective (resp. $2p$ -reflective) if it has a 2-reflective (resp. $2p$ -reflective) modular form.

The positions of the zeroes and poles of the modular form $\Psi(Z)$, where the counterterm (5.4) is ill-defined, should be interpreted as the symmetry enhancement points. These points corresponds to the rational quadratic divisors, which are defined as the orthogonal subspace with respect to some negative norm vectors (roots of the lattices). The symmetry is enhanced due to the reflective symmetry of the lattice. Requiring that $\Psi(Z)$ is symmetric reflective modular form, and thus the corresponding lattice L should be reflective, ensures that the theory is well-defined and anomaly-free throughout the moduli space. This is a strong constraint for the lattice. As shown in [44], only 55 possible types of reflective lattices of genus $\Pi_{2,l}(p^{\epsilon_p l_p})$ with $1 \leq l_p \leq 1 + l/2$ exist for prime level $p > 1$. And only three even unimodular lattices ($p = 1$) $\Pi_{2,10}$, $\Pi_{2,18}$ and $\Pi_{2,26}$ are reflective. Among these lattices, only those with trivial character (a big majority) can provide suitable counterterms (5.4) and hence lead to theories that are anomaly-free. Further restrictions may be imposed by the consistency of the large volume limits. Since only the Narain and the CHL lattice pass the tests of full quantum consistency, all other lattices which lead to anomaly-free theories constitute the finite swampland of the eight-dimensional minimal supergravity.

5.2 Examples of counterterms

Before turning to specific examples of counterterms, we point out that if we choose the divisor of $\Psi(Z)$ to be a linear combination of some rational quadratic divisors, $\Psi(Z)$ must be the same function up to normalization as we construct from Borchers product. More precisely (see Theorem 1.2 in [49]), assuming that $L \cong \Pi_{1,1} \oplus \Pi_{1,1}(N) \oplus \hat{L}(-1)$ for some

¹⁶As we shall see in section 6, the condition of reflectivity is required for the $l = 2$ case as well.

¹⁷For some lattices, such as CHL, both representations are possible. Of course the last factor in two different ways of representing the lattice will also be different.

positive integer N and $l \geq 3$, every meromorphic modular form $F(Z)$ with respect to $\Gamma(L)$ whose divisor is a linear combination of special divisors $H(\beta, m)$ is (up to a non-zero constant factor) the Borcherds product $\Psi(Z)$ of some $f \in M_{1-l/2}^!$.

We can now discuss examples, which include two fully consistent 8D $\mathcal{N} = 1$ supergravities with $l = 18$ and $l = 10$.

- **Signature** (2, 18)

As already mentioned reflectivity imposes an upper bound $l = 26$ on the rank of the gauge group. Moreover, for $l > 18$ there are very few reflective lattices with l even: the self-dual lattice $\Pi_{2,26}$ and two lattices at level 2 and 3, $\Pi_{2,22}(2)$ and $\Pi_{2,20}(3)$ respectively. If these allow a decompactification limit, they can be ruled out.

The $l = 18$ case, in addition to the self-dual (Narain) lattice, includes five different level 2 reflective lattices. These five will necessarily have enhancement points corresponding to norm -4 root vectors. Their modular forms cannot be decomposed into products of 2-reflective and 4-reflective forms, and they have no string theory realization.

For the theory obtained via compactification of 10D heterotic string on a two-torus [50], the momentum lattice structure is given by the Narain lattice

$$L = \Pi_{1,1} \oplus \Pi_{1,1} \oplus E_8(-1) \oplus E_8(-1), \quad (5.11)$$

while the symmetry enhancement appears when $p^2 = -2$.¹⁸ The symmetry enhancement points are given by the rational quadratic divisor

$$H(-1) = \bigcup_{(v,v)=-2, v \in L} v^\perp, \quad (5.12)$$

requiring that the lattice admits a 2-reflective modular form (the lattice L is 2-reflective). This is the case for the even unimodular lattice $\Pi_{2,18}$.

A weight 132 modular form $\Psi_{(2,18)}(Z)$ can be obtained by applying the Borcherds product to the nearly holomorphic modular form [54]

$$f(\tau) = \frac{1728E_4}{E_4^3 - E_6^2}(\tau) = \frac{1}{q} + 264 + 8244q + 139520q^2 + \dots, \quad (5.13)$$

where $q = e^{2\pi i\tau}$ and E_4, E_6 are Eisenstein series with the constant term normalized to 1

$$\begin{aligned} E_4 &= 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} = 1 + 240q + 2160q^2 + \dots, \\ E_6 &= 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} = 1 - 504q - 6632q^2 + \dots. \end{aligned} \quad (5.14)$$

¹⁸The full list of the allowed enhancements with the corresponding gauge algebras is worked out in [11, 51] with the help of the results of elliptic $K3$ fibrations [52, 53].

From Theorem B.2 we know that the modular form $\Psi_{(2,18)}(Z)$ is holomorphic (all the coefficient in the principal part is positive) and only has zeroes at the rational quadratic divisor $H(-1)$. Moreover, since the lattice L is even unimodular now, the character for the group $\text{SO}^+(L)$ must be trivial.

- **Signature** (2, 10)

For $l = 10$, if the level of the lattice is prime, there are ten types of reflective lattices.

The simplest of these is the self-dual lattice $L = \Pi_{1,1} \oplus \Pi_{1,1} \oplus E_8$, which is 2-reflective. Requiring that the zeroes of $\Psi(Z)$ are contained in the rational quadratic divisors defined by $(v, v) = -2$, we should look for a nearly holomorphic modular form of weight -4 as an input into the Borcherds product. Such a function exists

$$f(\tau) = \frac{1}{q} + 504 + 16404q + \dots, \quad q = e^{2\pi i\tau}, \quad (5.15)$$

and the corresponding Borcherds product $\Psi(Z)$ is of weight 252. Due to the unimodularity, the character for this lattice is, as required, trivial. Comparison of the possible gauge symmetry enhancements allowed by this lattice to [10] would exclude this lattice.

Since all other lattices are at level $p > 1$, the enhancement points will correspond not only to short roots (vectors with norm -2) as for even unimodular lattices. Indeed, for reflective lattices roots are not only vectors with norm -2 but also vectors v with norm $-2p$ satisfying $(v, u) = 0 \pmod{2}$ for all vectors $u \in L$. In fact, the last condition is equivalent to saying that v/p is in the dual lattice L' .

The most interesting class is for $p = 2$. It contains three lattices, all of which have reflective vectors of norm -2 and -4 ($p = 2$). The CHL lattice [55] of the form

$$L = \Pi_{1,1} \oplus \Pi_{1,1}(2) \oplus E_8(-1) \cong \Pi_{1,1} \oplus \Pi_{1,1} \oplus D_8(-1) \quad (5.16)$$

is among these three. The full list of enhancements and the allowed gauge algebras in the 8D CHL theories [56, 57] is worked out in [12, 58]. By the Theorem 4.1 and Theorem 4.2 in [43], L admits a 2-reflective modular form Ψ_1 of weight 124 and a 4-reflective modular form Ψ_2 of weight 4. These two modular forms are both holomorphic and only have zeros respectively on

$$(\Psi_1) = H(-1) = \bigcup_{(v,v)=-2, v \in L} v^\perp \quad \text{and} \quad (\Psi_2) = \bigcup_{(v,v)=-4, v/2 \in L'} v^\perp \quad (5.17)$$

The lattice satisfies the condition in Theorem B.4 thus the character of the modular group is trivial. Hence an anomaly-cancelling counterterm can be constructed by direct multiplication of these functions $\Psi_{(2,10)}(Z) = \Psi_1(Z)\Psi_2(Z)$. These modular forms are closely related to many interesting results about Enriques surfaces [59–62]. We define the lattice $L_E = \Pi_{1,1} \oplus \Pi_{1,1}(2) \oplus E_8(-2)$. Notice that

$$L'_E(2) \cong \Pi_{1,1} \oplus \Pi_{1,1}(2) \oplus E_8(-1) \cong L, \quad (5.18)$$

the orthogonal group has the following relation

$$\mathrm{O}^+(L_E) \cong \mathrm{O}^+(L'_E) \cong \mathrm{O}^+(L'_E(2)) \cong \mathrm{O}^+(L). \quad (5.19)$$

Hence the (reflective) modular forms with respect to the group $\mathrm{O}^+(L)$ correspond to that of the group $\mathrm{O}^+(L_E)$. Note that the reflective vectors defined above are in the lattice L . For a modular form with respect to the group $\mathrm{O}^+(L_E)$ (lattice L_E), we should check the relation of the reflective vectors and RQDs to those for the lattice L . Under the transformation $L'_E \rightarrow L'_E(2)$ (one can think that each vector is scaled by $\sqrt{2}$), the 2 reflective vectors of L_E and 4 reflective vectors ($v_E \in L_E, v_E/2 \in L'_E$) transform to the 4-reflective vectors and the 2-reflective vectors of L respectively. This correspondence can be summarized as follows:

$$\begin{aligned} \Psi_1(Z) &: 2\text{-reflective for lattice } L \longrightarrow 4\text{-reflective for lattice } L_E, \\ \Psi_2(Z) &: 4\text{-reflective for lattice } L \longrightarrow 2\text{-reflective for lattice } L_E. \end{aligned}$$

$\Psi_2(Z)$ is called Borchers-Enriques modular form Φ_4 , first found in [59] and reconstructed as an example in [26] (see Example 13.7). The lattice $L'_E/L_E \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ so we can use two bits to label the element in L'_E/L_E . There exists a nearly holomorphic modular form of weight -4 , written as $f(\tau) = \sum_{\gamma} \mathbf{e}_{\gamma} f_{\gamma}$ and the components f_{γ} are

$$\begin{aligned} f_{00}(\tau) &= -f_{10}(\tau) = -f_{01}(\tau) = 8\eta^8(2\tau)/\eta^{16}(\tau) = 8 + 128q + 1152q^2 + \dots, \\ f_{11}(\tau) &= 8\eta^8(2\tau)/\eta^{16}(\tau) + \eta^8(\tau/2)/\eta^{16}(\tau) = q^{-1/2} + 36q^{-1/2} + 402q^{3/2} + \dots. \end{aligned} \quad (5.20)$$

By Borchers product (Theorem B.1), the weight of $\Psi_2(Z)$ is 4 ($c(0,0) = 8$) and it is holomorphic (the coefficient in the principal part is positive). There is only one term in the principal part ($q^{-1/2}$) so the RQD is exactly the set of 2-reflective vectors $v_E^2 = -2$. Hence Ψ_2 is a 4-reflective modular form of weight 4 with respect to the lattice L as we expected. Another equivalent way to construct $\Psi_2(Z)$ is to use Jacobi lifting [60]. Construction of the weight 124 2-reflective modular form $\Psi_1(Z)$ is more complicated, and we refer to Lemma 5.4 in [62] for more detailed explanations.

For the two other lattices in this class ($p = 2$), D_8 factor of the CHL lattice in (5.16) is replaced by respectively $D_4 \oplus D_4$ and D'_8 .¹⁹ The counterterms can again be obtained by a direct multiplication of two different modular forms.²⁰ In these cases, the 2-reflective modular forms are of weight 60 and 28 respectively, and the 4-reflective modular forms are of weight 12 and 28 respectively.

¹⁹Notice that further reduction of the CHL strings to seven and six dimensions yields $\Pi_{1,1} \oplus \Pi_{1,1} \oplus \Pi_{1,1} \oplus D_4 \oplus D_4(-1)$ and $\Pi_{1,1} \oplus \Pi_{1,1} \oplus \Pi_{1,1} \oplus \Pi_{1,1} \oplus D'_8(-2)$ lattices respectively [55].

²⁰For any other prime p the lattices of signature $(2, 10)$ do not admit modular forms that can be decomposed into a product of 2-reflective and $2p$ -reflective modular forms (Theorem 4.3 in [43]). Note that there are four $l = 10$ lattices which are 2-reflective and have non-prime level p [44]. Only the CHL lattice yields the gauge symmetry enhancement consistent with the swampland considerations.

6 Anomaly cancellation for $l = 2$

In the previous discussion we mainly focused on the case $l \geq 3$. This restriction allowed us to obtain the modular forms, suitable for building counterterms, using the lattice decomposition as $L = \Pi_{1,1} \oplus \Pi_{1,1} \oplus \hat{L}(-1)$ and the Theorem B.1 to ensure the triviality of the character. The case $l = 2$, where the Borcherds product does not apply generally, needs special considerations.

For $l = 2$ case. we can take advantage of the two to one group homomorphism from $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ to $\mathrm{SO}(2, 2; \mathbb{R})$, and consider²¹

$$\mathcal{M}_{l=2} = \frac{\mathrm{SO}(2, 2)}{\mathrm{SO}(2) \times \mathrm{SO}(2)} \cong \frac{\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})}{\mathrm{U}(1) \times \mathrm{U}(1)}, \quad (6.1)$$

which can be parametrized by a pair of complex scalars Z_1 and Z_2 with modular-invariant kinetic terms²²

$$\mathcal{L}_{\text{scalars}} = \frac{1}{2} \left(\frac{\partial_\mu Z_1 \partial^\mu \bar{Z}_1}{|\mathrm{Im} Z_1|^2} + \frac{\partial_\mu Z_2 \partial^\mu \bar{Z}_2}{|\mathrm{Im} Z_2|^2} \right). \quad (6.2)$$

We can once more use the canonical way to obtain the generalized upper-half plane developed in section 3.2. When $l = 2$, the matrix \hat{S} collapses. The discretized structure has not yet emerged so we can arbitrarily choose a quadratic form of signature $(2, 2)$ since every positive definite symmetric bilinear form of the same signature is equivalent in the vector space $V = L \otimes \mathbb{R}$. A convenient choice is

$$S_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} & 1 \\ S_0 & \\ 1 & \end{pmatrix}, \quad (6.3)$$

with the quadratic forms q_0 and q defined with respect to S_0 and S respectively. Recalling the definition of \mathbb{H}_l (3.18), with $l = 2$ we have

$$\mathbb{H}_{l=2} = \{Z = X + iY \in L_0 \otimes \mathbb{C} | X, Y \in L_0 \otimes \mathbb{R}, Y \in P\}, \quad (6.4)$$

where P denotes the future light cone of the Minkowski space $L_0 \otimes \mathbb{R}$ with signature $(1, 1)$. Denoting $Z = (Z_1, Z_2)^T$, we have $q_0(Y) = \mathrm{Im}(Z_1) \mathrm{Im}(Z_2) > 0$. P then picks the connected component that $\mathrm{Im}(Z_1) > 0$ and $\mathrm{Im}(Z_2) > 0$, i.e. the generalized upper-half plane is exactly the direct product of the usual upper-half planes, $\mathbb{H}_{l=2} \cong \mathbb{H} \times \mathbb{H}$.

²¹Notice that only one $\mathrm{SO}(2)$ factor is anomalous, and this identification requires rotation of the two factors in the denominator

²²Notice that in (6.1) only one $\mathrm{SO}(2)$ factor is anomalous, and this identification requires rotation of the two factors in the denominator. As we shall see the moduli Z_1 and Z_2 cannot be identified as moduli of a two-torus.

6.1 The kinetic term

We will start the discussion of the kinetic terms (6.2) recalling the relation between (Z_1, Z_2) and (y_1, y_2) (4.31) and (4.36),

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \frac{1+y^2-2y_1}{i(1-y^2)+2y_2} \\ \frac{1+y^2+2y_1}{i(1-y^2)+2y_2} \end{pmatrix} = \begin{pmatrix} i \frac{y_1-iy_2-1}{y_1-iy_2+1} \\ i \frac{y_1+iy_2+1}{y_1+iy_2-1} \end{pmatrix}. \quad (6.5)$$

A specific form of Λ should be considered first. A benefit in $l = 2$ case is that q in the element Λ (4.3) is a square matrix. We further assume that q is invertible, i.e.²³

$$\det q = \frac{4}{1 - 2\bar{y}_k y_k + y^2 \bar{y}^2} [\operatorname{Re}(y_1) \operatorname{Im}(y_2) - \operatorname{Re}(y_2) \operatorname{Im}(y_1)] \neq 0. \quad (6.6)$$

This is equivalent to requiring that $y_2 \neq ay_1$ for any real number a . Using the formula (4.6) to express the block $\sqrt{1 + q^T q}$ in terms of y_i and the invertibility of q , we have

$$\begin{aligned} \sqrt{1 + q^T q} &= \mathbb{1} + q^T \left(\sqrt{1 + qq^T} - \mathbb{1} \right) (qq^T)^{-1} q \\ &= \mathbb{1} + q^T \left(\sqrt{1 + qq^T} - \mathbb{1} \right) (q^T)^{-1} = q^T \sqrt{1 + qq^T} (q^T)^{-1}, \end{aligned} \quad (6.7)$$

where

$$(q^T)^{-1} = \frac{\sqrt{1 - 2\bar{y}_k y_k + y^2 \bar{y}^2}}{2 [\operatorname{Re}(y_1) \operatorname{Im}(y_2) - \operatorname{Im}(y_1) \operatorname{Re}(y_2)]} \begin{pmatrix} \operatorname{Im}(y_2) & -\operatorname{Im}(y_1) \\ -\operatorname{Re}(y_2) & \operatorname{Re}(y_1) \end{pmatrix}. \quad (6.8)$$

Direct manipulations yield

$$\sqrt{1 + q^T q} = \mathfrak{S} \begin{pmatrix} 1 + y_1 \bar{y}_1 - y_2 \bar{y}_2 & \bar{y}_1 y_2 + y_1 \bar{y}_2 \\ \bar{y}_1 y_2 + y_1 \bar{y}_2 & 1 + y_2 \bar{y}_2 - y_1 \bar{y}_1 \end{pmatrix}, \quad \mathfrak{S} = \frac{1}{\sqrt{1 - 2\bar{y}_k y_k + y^2 \bar{y}^2}}, \quad (6.9)$$

and expression for Λ in terms y_i :

$$\Lambda(y_1, y_2) = \mathfrak{S} \begin{bmatrix} 1 + \frac{1}{2}(y^2 + \bar{y}^2) & -\frac{i}{2}(y^2 - \bar{y}^2) & y_1 + \bar{y}_1 & y_2 + \bar{y}_2 \\ -\frac{i}{2}(y^2 - \bar{y}^2) & 1 - \frac{1}{2}(y^2 + \bar{y}^2) & -i(y_1 - \bar{y}_1) & -i(y_2 - \bar{y}_2) \\ y_1 + \bar{y}_1 & -i(y_1 - \bar{y}_1) & 1 + y_1 \bar{y}_1 - y_2 \bar{y}_2 & \bar{y}_1 y_2 + y_1 \bar{y}_2 \\ y_2 + \bar{y}_2 & -i(y_2 - \bar{y}_2) & \bar{y}_1 y_2 + y_1 \bar{y}_2 & 1 + y_2 \bar{y}_2 - y_1 \bar{y}_1 \end{bmatrix}. \quad (6.10)$$

The Maurer-Cartan form has decomposition

$$\Lambda^{-1} \partial_\mu \Lambda = \tilde{Q}_\mu + \tilde{P}_\mu, \quad \tilde{Q}_\mu \in \mathfrak{so}(2) \oplus \mathfrak{so}(l), \quad \tilde{P}_\mu \in \mathfrak{p}, \quad (6.11)$$

²³Later we will see that the determinant of q in the denominator cancels out. Hence the result derived under this assumption can be analytically continued to the set of points in the domain where $\det q = 0$.

where \mathfrak{p} is the complement of $\mathfrak{so}(2) \oplus \mathfrak{so}(l)$ ($\mathfrak{g} = \mathfrak{so}(2, l) = (\mathfrak{so}(2) \oplus \mathfrak{so}(l)) \perp \mathfrak{p}$). We can directly obtain \tilde{P}_μ through block decomposition (following [6]).

$$\Lambda^{-1} \partial_\mu \Lambda = \begin{pmatrix} Q_\mu^{\text{SO}(2)} & P_\mu \\ P_\mu^T & Q_\mu^{\text{SO}(l)} \end{pmatrix} = \begin{pmatrix} Q_\mu^{\text{SO}(2)} & \\ & Q_\mu^{\text{SO}(l)} \end{pmatrix} + \begin{pmatrix} & P_\mu \\ P_\mu^T & \end{pmatrix}, \quad (6.12)$$

which leads to the scalar Lagrangian

$$\mathcal{L}_{\text{scalars}} = \frac{1}{2} \text{Tr} \left(\tilde{P}_\mu \tilde{P}^\mu \right) = \text{Tr} \left(P_\mu^T P^\mu \right), \quad (6.13)$$

with the trace taken over the matrix with Λ . For $l = 2$ we can use the explicit form of Λ (equation 6.10) to compute the P_μ block by the formula:

$$\begin{aligned} P_\mu &= \sqrt{1 + qq^T} \partial_\mu q - q \partial_\mu \sqrt{1 + q^T q} \\ &= \mathfrak{S}^2 \begin{bmatrix} \mathcal{Y}_1 (\partial_\mu y_1 + \partial_\mu \bar{y}_1) + \mathcal{Y}_2 (\partial_\mu y_2 - \partial_\mu \bar{y}_2) & \mathcal{Y}_1 (\partial_\mu y_2 + \partial_\mu \bar{y}_2) - \mathcal{Y}_2 (\partial_\mu y_1 - \partial_\mu \bar{y}_1) \\ i\mathcal{Y}_1 (-\partial_\mu y_1 + \partial_\mu \bar{y}_1) - i\mathcal{Y}_2 (\partial_\mu y_2 + \partial_\mu \bar{y}_2) & i\mathcal{Y}_1 (-\partial_\mu y_2 + \partial_\mu \bar{y}_2) + i\mathcal{Y}_2 (\partial_\mu y_1 + \partial_\mu \bar{y}_1) \end{bmatrix}, \end{aligned} \quad (6.14)$$

where $\mathcal{Y}_1 = 1 - y_k \bar{y}_k$ and $\mathcal{Y}_2 = y_1 \bar{y}_2 - \bar{y}_1 y_2$. The Lagrangian is given by

$$\text{Tr} \left(P_\mu^T P^\mu \right) = \frac{4(\mathcal{Y}_1^2 - \mathcal{Y}_2^2) (\partial_\mu y_1 \partial^\mu \bar{y}_1 + \partial_\mu y_2 \partial^\mu \bar{y}_2) + 8\mathcal{Y}_2 \mathcal{Y}_1 (\partial_\mu \bar{y}_1 \partial^\mu y_2 - \partial_\mu y_1 \partial^\mu \bar{y}_2)}{(\mathcal{Y}_1^2 + \mathcal{Y}_2^2)^2}. \quad (6.15)$$

Here we have used that

$$\mathfrak{S}^4 = \frac{1}{(1 - 2\bar{y}_k y_k + y^2 \bar{y}^2)^2} = \frac{1}{(\mathcal{Y}_1^2 + \mathcal{Y}_2^2)^2}. \quad (6.16)$$

Regrouping two complex scalars y_1, y_2 into

$$\xi = y_1 + iy_2, \quad \varphi = y_1 - iy_2, \quad (6.17)$$

the kinetic term further simplifies to

$$\text{Tr} \left(P_\mu^T P^\mu \right) = \frac{2\partial_\mu \xi \partial^\mu \bar{\xi}}{(1 - \xi \bar{\xi})^2} + \frac{2\partial_\mu \varphi \partial^\mu \bar{\varphi}}{(1 - \varphi \bar{\varphi})^2}. \quad (6.18)$$

Recall that the range of y_i is constrained (see (4.14))

$$1 - 2\bar{y}_i y_i + y^2 \bar{y}^2 > 0, \quad \bar{y}_i y_i < 1. \quad (6.19)$$

This translates into requirements that $|\xi| < 1$ and $|\varphi| < 1$. Thus the two separate terms in (6.18) are naturally given by two Poincaré metric tensors on a unit disk. Finally, recalling the equation (6.5)

$$\xi = \frac{Z_2 + i}{Z_2 - i}, \quad \varphi = \frac{Z_1 + i}{i - Z_1}, \quad (6.20)$$

we arrive at the canonical kinetic term

$$\mathcal{L}_{\text{scalar}} = \text{Tr} (P_\mu^T P^\mu) = \frac{1}{2} \left(\frac{\partial_\mu Z_1 \partial^\mu \bar{Z}_1}{|\text{Im} Z_1|^2} + \frac{\partial_\mu Z_2 \partial^\mu \bar{Z}_2}{|\text{Im} Z_2|^2} \right). \quad (6.21)$$

It is worth noting that the Z_2 symmetry exchanging $Z_1 \leftrightarrow Z_2$ is not present in the 8 dimensional supergravity. Such a symmetry would be complemented by the matrix

$$\mathcal{R} = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}, \quad (6.22)$$

which has determinant -1 , and thus it is not in $\text{SO}(2, 2; \mathbb{R})$.

6.2 The counterterm

The next step towards constructing a counterterm is to calculate the explicit form of the compensating $U(1)$ transformation (4.43), namely $\arg [j(M, Z_1, Z_2)]$ for $M \in \text{SO}(2, 2; \mathbb{R})$ and $Z_1, Z_2 \in \mathbb{H}$. The fact that the generalized upper-half plane is isomorphic to the direct product of the usual complex upper-half plane suggests it should be described in terms of the automorphy factor of $\text{SL}(2, \mathbb{Z})$. The action of $\text{SO}(2, 2; \mathbb{R})$ on the generalized upper-half plane is (see (3.27))

$$\begin{aligned} W = M\langle Z \rangle &:= (-bq_0(Z) + PZ + c) (-\gamma q_0(Z) + d^T Z + \delta)^{-1}, \\ j(M, Z) &:= -\gamma q_0(Z) + d^T Z + \delta, \end{aligned} \quad (6.23)$$

for $M \in \text{SO}(2, 2; \mathbb{R})$ decomposed in the form (3.25). From the other side, for two $\text{SL}(2; \mathbb{R})$ matrices,

$$A = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \quad \alpha_1 \delta_1 - \beta_1 \gamma_1 = 1, \quad B = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}, \quad \alpha_2 \delta_2 - \beta_2 \gamma_2 = 1, \quad (6.24)$$

we can define the map from $\text{SL}(2; \mathbb{R}) \times \text{SL}(2; \mathbb{R})$ to $\text{SO}^+(2, 2; \mathbb{R})$ by [36, 63]

$$\Omega(A, B) = \begin{pmatrix} \alpha_1 F B F & \beta_1 F B \\ \gamma_1 B F & \delta_1 B \end{pmatrix}, \quad F = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (6.25)$$

It is easy to verify that this is a surjective group homomorphism. Moreover, the action of $\Omega(A, B)$ is

$$\Omega(A, B) \left\langle \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \right\rangle = \begin{pmatrix} \frac{\alpha_1 Z_1 + \beta_1}{\gamma_1 Z_1 + \delta_1} \\ \frac{\alpha_2 Z_2 + \beta_2}{\gamma_2 Z_2 + \delta_2} \end{pmatrix}, \quad j(M, Z) = (\gamma_1 Z_1 + \delta_1)(\gamma_2 Z_2 + \delta_2). \quad (6.26)$$

By Theorem 3 in [36], the modular group $\Gamma = \text{SO}^+(2, 2; \mathbb{Z})$ is formed by the element $\Omega(A, B)$ where $A, B \in \text{SL}(2, \mathbb{Z})$. The factorization of the automorphy factor $j(M, Z)$ allows

to express the counterterm in terms of $\mathrm{SL}(2, \mathbb{Z})$ modular forms. Notice that two factors $(\gamma_1 Z_1 + \delta_1)$ and $(\gamma_2 Z_2 + \delta_2)$ appear symmetrically, the counterterm must be of the form

$$\mathcal{S} = \frac{1}{r} \int \arg(\Psi_1(Z_1)\Psi_2(Z_2)) X_8, \quad (6.27)$$

where $\Psi_{1,2}$ are the $\mathrm{SL}(2, \mathbb{Z})$ modular forms of the same non-trivial weight r . Demanding once more that the zeros and poles of the function $\Psi_1(Z_1)\Psi_1(Z_2)$ correspond to the symmetry enhancement points in the moduli space leads to

$$\Psi_{1,2}(Z_{1,2}) = E_4(Z_{1,2}), \quad (6.28)$$

where E_4 is the weight 4 Eisenstein series defined in equation (5.14). E_4 has only one simple zero at i within the fundamental domain, thus at these points (its modular image under $\mathrm{SL}(2, \mathbb{Z})$) the symmetry is enhanced. The maximal symmetry enhancement $\mathrm{SU}(2) \times \mathrm{SU}(2)$ appears when $Z_1 = Z_2 = i$.

Few comments are in order. The choice (6.28) reflects the knowledge of the moduli spaces of $l = 2$ theories, which notably do not have $\mathrm{SU}(3)$ enhancement points.

It is worth noting that the Eisenstein function $E_4 \rightarrow 1$ in (6.28) for large $\mathrm{Im}(Z_1)$ or $\mathrm{Im}(Z_2)$ and we do not have suitable ten-dimensional decompactification limit. Hence Z_1 and Z_2 cannot be identified as moduli of a two-torus. This is indeed the case for the known $l = 2$ 8D theories, none of which comes from compactifications of 10D heterotic string [64–67]. At least for the theory obtained via perturbative IIB construction [64] may hope to compute this counterterm (6.27) explicitly.

Unlike the cases with $l \geq 3$, this construction is not tied to any particular lattice structure and should apply to both known $l = 2$ theories.

An alternative construction using Hilbert modular forms may be considered. In fact it leads to an anomaly-cancelling counterterm for $l = 2$ case. There exists a function $f \in A_0^+(5, \chi_5)$ (see Appendix C)

$$f(\tau) = q^{-1} + 5 + 11q - 54q^4 + O(q^5), \quad q = e^{2\pi i\tau}, \quad (6.29)$$

yielding the Hilbert modular form $\Psi(Z)$ of weight 5 with trivial multiplier system. However this function may allow symmetry enhancements, such as $\mathrm{SU}(3)$, which should not appear in the 8D (2, 2) theories [10], and hence is not physical. We discuss the details of this construction outside of the main text in Appendix C.

7 Discussion

The moduli space of the eight-dimensional minimal supergravities coupled to l Yang-Mills multiplets is given by

$$\mathcal{M} = \frac{\mathrm{SO}(2, l)}{\mathrm{U}(1) \times \mathrm{SO}(l)}.$$

The composite $\mathrm{U}(1)$ connection, under which the fermions of the theory are chirally charged, is anomalous. The gauge fixing translates this anomaly into an anomaly under the discrete

part of the coset denominator, which can be shown to coincide with the discrete modular group of the corresponding lattice. The consistency of the theory requires a suitable counterterm to cancel this discrete anomaly.

The counterterms can be constructed with the use of the Borcherds product of the modular forms on the orthogonal group, $\Psi(Z)$:

$$\mathcal{S} = \frac{1}{r} \int \arg(\Psi(Z)) X_8(R, \mathcal{F}), \quad (7.1)$$

where $X_8(R, \mathcal{F})$ is the anomaly polynomial and r is the weight of the modular form satisfying some conditions required by the anomaly cancellation. These conditions can be summarized as

- The character for the modular group $\Gamma(L)$ (or the lattice L) must be trivial.
- The zeros and poles of $\Psi(Z)$ lie on the rational quadratic divisors. If these points can be interpreted as the symmetry enhancement points, it requires $\Psi(Z)$ should be reflective modular form and L is the reflective lattice.

For the $l = 2$ case, the homomorphism between $SL(2; \mathbb{R}) \times SL(2; \mathbb{R})$ to $SO^+(2, 2; \mathbb{R})$ can be used in order to construct the local counterterm from the usual $SL(2; \mathbb{Z})$ modular forms. An alternative way to cancel the anomaly by using Hilbert modular forms at the cost of shrinking the symmetry exists. However it would allow for enhanced gauge symmetry that is not consistent with the string-theoretic constructions.

We will conclude by outlining some open questions and directions for further research.

Relation to the Swampland It is not surprising that we find a larger set of theories with a mechanism for anomaly cancellation than what is allowed by swampland considerations. It is however curious, that there are finite number of admissible lattices and they are bounded by 26. In fact, the only two lattices for $l > 2$ that are believed to lead to consistent theories of quantum gravity [66] are even more special and admit 2-reflective modular forms. It would be of great interest to find out if there exist physical requirements that lead to further constraints on the lattice structure.

Notice that we always assume the lattice to be even. This condition enters crucially in the construction of the modular forms on the orthogonal groups, and it is hard to see how a counterterm can be constructed otherwise. We do not know a more direct supergravity (swampland?) argument in support of this condition that arises very naturally in string theory.

Counterterms and massive sectors In our $\mathcal{N} = 1$ discussion the precise form of the anomaly polynomial played no role. In fact (3.7) is computed only by knowing the massless spectrum. On the other hand, the string amplitudes receive contributions from massive states. For a very recent interesting discussion of importance of these see [67]. At the supergravity level one could generate corrections to the counterterm to (3.7) by adding massive states and integrating them out. It is hard to believe that the choices of massive sector are arbitrary, and as discussed in [19] one expects that reduction on \mathbb{P}^1 to

six-dimensional $(1, 0)$ would impose strong constraints on the possible massive sectors. The question of whether and when a theory admits different consistent massive completions is certainly of great interest.

K3 reductions and 4D physics It is also of interest to explore the implications of the 8D counterterms discussed here for compactifications, particularly 4D couplings. There are very direct parallels between 8D maximally and minimally supersymmetric theories and 4D $\mathcal{N} = 4$ and $\mathcal{N} = 2$ respectively.

4D $\mathcal{N} = 4$ supergravity (coupled to YM) also has composite anomalies, recently discussed in e.g. [68–70]. The moduli space is given by $\text{SL}(2)/\text{U}(1) \times \text{SO}(6, n_V)/\text{SO}(6) \times \text{SO}(n_V)$. As in maximally supersymmetric 8D theory and consistently with the supersymmetry algebra, the $\text{U}(1)$ composite anomaly is also an anomaly of a nonlinear local supersymmetry [70]. Putting the maximally supersymmetric theory 8D on $K3$ yields 4D theory coupled to 22 vector multiplets.²⁴ It is not hard to see that the $\text{SL}(2)/\text{U}(1)$ factor directly descends from 8D. It can be checked the $K3$ reduction of the counterterm (2.14) in the large volume limit agrees with the one computed in [69] for $n_V = 22$. The reduction closely follows that of type IIA Chern-Simons couplings to six dimensions [30, 31]. In fact a generic 4D $\mathcal{N} = 2$ supergravity coupled to arbitrary number of vectors, provided $n_V \geq 2$ can be seen as coming from a torus reduction of 6D $(1, 1)$ theory, with a relation between the 6D CS couplings and 4D counterterms identical to that between their 10D and 8D counterparts as discussed in section 2.

The $K3$ reduction of 8D theory with 16 supercharges to a 4D $\mathcal{N} = 2$ theory parallels the reduction of 10D heterotic strings on $K3$. There, a separate integration of the Bianchi identity (with the constraints that the instanton numbers should sum up to 24) and of the Green-Schwarz term yield two different four-forms that agree with those obtained in the factorised anomaly polynomial in the resulting 6D $(1, 0)$ theory (see e.g. [72]). So one could wonder about similar reduction of the counterterm in 8D.

Choosing an instanton in group $H \subset G$ ($\text{rank}(G) = l$) breaks the gauge group to G_0 stabilised by H in G . The Bianchi identity can be written in general as (following the notation of [10])

$$dH_3 = \kappa \text{tr} R^2 + \ell \cdot \text{tr} \mathcal{F}^2$$

where κ can take values 1 or 0 (only for $l = 2$), and ℓ is the level of the current algebra (for a product gauge group, summation over different gauge factors is implied), and hence $\ell \cdot c_2(H) = 24\kappa$.²⁵ Denoting $\text{rank}(H) = h$,

$$\text{SO}(2, l) \longrightarrow \text{SO}(2, l - h).$$

But in 4D, $n_V = l - h + 1$, and the extra multiplet comprises one of the vectors in 8D gravity multiplet, and the dilaton-axion. Notice that while in 8D the counterterm must have nontrivial modular properties, the 4D threshold corrections $\sim \text{tr} R^2$ involve automorphic

²⁴For other constructions of 4D $\mathcal{N} = 4$ theories from Type II strings see e.g. [71]

²⁵For $\kappa = 0$, there cannot be nontrivial gauge configurations over $K3$. The reduction yields a 4D $\mathcal{N} = 2$ theory with three vector multiplets and 20 neutral hypermultiplets.

functions on $\text{SO}(2, n_V)$. The addition of the extra scalar (“conformal compensator” in vector moduli space) should be responsible for this change. It would be of some interest to understand how this works in more detail.

It has been argued that the $K3$ reduction of $\mathcal{N} = 1$ theories in 8D provides a good framework for studying 4D $\mathcal{N} = 2$ compactifications since it encompasses not only the $K3 \times T^2$ but also the heterotic flux backgrounds [73]. Considering the space of all 8D $\text{SO}(2, l)$ for $l = 2, 10, 18$ would enlarge this space and hopefully cover all $\mathcal{N} = 2$ theories of heterotic type, i.e. those for which the dilaton is in the vector multiplets. This raises an interesting possibility that all threshold corrections in these theories would in some way be governed and be derivable from the special $\text{SO}(2, l)$ modular forms from which the counterterms (7.1) are built.

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A Dedekind eta function its multiplier system and theta function

In this appendix we collect some relevant facts about the Dedekind eta function $\eta(\tau)$ and the theta function $\theta(\tau)$, used in section 2. Under the modular transformation, both will pick a square root of $c\tau + d$, and the branch of the square root needs to be specified. In the main text we have already defined the argument of $z \in \mathbb{C}$ as $\text{Arg } z \in [-\pi, \pi)$. Thus the square root for $z \in \mathbb{C}$ is

$$z^{\frac{1}{2}} = \sqrt{|z|} e^{\frac{i}{2} \text{Arg } z}, \quad (\text{A.1})$$

and this convention will be used throughout the discussion.

The Dedekind eta function can be written in the form of infinite products,

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \quad (\text{A.2})$$

Since two $\text{SL}(2, \mathbb{Z})$ -matrices generate the whole group, its modular properties can be captured by

$$\eta(T\tau) = \eta(\tau + 1) = e^{\frac{\pi i}{12}} \eta(\tau), \quad \eta(S\tau) = \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau), \quad (\text{A.3})$$

where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (\text{A.4})$$

More generally, the modular properties of $\eta(\tau)$ under $\text{SL}(2, \mathbb{Z})$ can be written as [74]

$$\eta(M\tau) = \chi_\eta(M)(c\tau + d)^{1/2}\eta(\tau) \quad (\text{A.5})$$

with a nontrivial multiplier system $\chi_\eta(M)$. Let c and d be integers such that $\gcd(c, d) = 1$, d is odd and $c \neq 0$. Let $\text{sgn}(x) = \frac{x}{|x|}$ be the sign of a real number $x \neq 0$. Then

$$\left(\frac{c}{d}\right)^* = \left(\frac{c}{|d|}\right), \quad \text{and} \quad \left(\frac{c}{d}\right)_* = \left(\frac{c}{|d|}\right) \cdot (-1)^{\frac{1}{4}(\text{sgn}(c)-1)(\text{sgn}(d)-1)}, \quad (\text{A.6})$$

where $\left(\frac{c}{d}\right)$ is the Legendre symbol and we set

$$\left(\frac{0}{1}\right)^* = \left(\frac{0}{-1}\right)^* = 1, \quad \left(\frac{0}{1}\right)_* = 1, \quad \left(\frac{0}{-1}\right)_* = -1. \quad (\text{A.7})$$

For arbitrary element in $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2; \mathbb{Z})$, the multiplier system of the Dedekind eta function is given by

$$\begin{aligned} \chi_\eta(M) &= \left(\frac{d}{c}\right)^* q\left(\frac{1}{24}[(a+d)c - bd(c^2 - 1) - 3c]\right) && \text{if } c \text{ is odd,} \\ \chi_\eta(M) &= \left(\frac{c}{d}\right)_* q\left(\frac{1}{24}[(a+d)c - bd(c^2 - 1) + 3d - 3 - 3cd]\right) && \text{if } c \text{ is even,} \end{aligned} \quad (\text{A.8})$$

where $q(z) = e^{2\pi iz}$. It should be noted that $\chi_\eta(M)$ cannot form the homomorphism from $\text{SL}(2; \mathbb{Z})$ to $\text{U}(1)$. Since the $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ transformation satisfy $S^2 = -\mathbb{1}$, we have

$$\begin{aligned} \eta(\tau) &= \eta((-1)\tau) = \chi_\eta(-1)(-1)^{1/2}\eta(\tau) = \chi_\eta(-1)(-i)\eta(\tau) \\ &\Rightarrow \chi_\eta(-1) = i \neq \chi_\eta(S)^2 = -i, \end{aligned} \quad (\text{A.9})$$

In the main text, the congruence subgroup $\Gamma_0(4)$ of $\text{SL}(2; \mathbb{Z})$ was introduced:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\} \quad (\text{A.10})$$

for any positive integer N . Within the congruence subgroup $\Gamma_0(4)$ the weight 1/2 modular form is well-defined [75, 76]. For element $M \in \Gamma_0(4)$, the transformation of the square of the theta function, given as

$$\theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q^2 + 2q^4 + \dots, \quad (\text{A.11})$$

takes the form

$$\theta^2(M\tau) = \left(\frac{-1}{d}\right) (c\tau + d)\theta^2(\tau), \quad (\text{A.12})$$

where $\left(\frac{-1}{d}\right)$ denotes the Legendre symbol, $\left(\frac{-1}{d}\right) = (-1)^{\frac{d-1}{2}}$.

B Orthogonal modular forms

Some necessary properties of orthogonal modular forms were reviewed in subsection 3.3. In order to make the paper more self-contained, more background material is collected in this Appendix. Definitions and theorems are given without proofs. Our presentation follows closely [21], which can be consulted for detailed explanations.

Throughout this section, as in the main text, we denote by L an even lattice of signature $(2, l)$ and assume $l \geq 3$.

B.1 The Weil representation

We denote the complex upper-half plane $\mathbb{H} = \{\tau \in \mathbb{C}; \text{Im } \tau > 0\}$. τ is the standard variable on \mathbb{H} and we use x and y for its real and imaginary parts respectively ($\tau = x + iy$). For $z \in \mathbb{C}$ we define $e(z) = e^{2\pi iz}$ and denote by $\sqrt{z} = z^{1/2}$ the principal branch of the square root. For arbitrary $b \in \mathbb{C}$, we define $z^b = e^{b \text{Ln } z}$ where $\text{Ln } z$ denotes the principal branch of the logarithm. We denote by $\text{Mp}(2; \mathbb{R})$ the metaplectic group, i.e. the double cover of group $\text{SL}(2; \mathbb{R})$, realized by the two choices of holomorphic square roots of $\tau \rightarrow c\tau + d$ for arbitrary element $M \in \text{SL}(2; \mathbb{R})$,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}, \quad \det M = ad - bc = 1. \quad (\text{B.1})$$

Any element in $\text{Mp}(2; \mathbb{R})$ can be written as $(M, \phi(\tau))$ where $M \in \text{SL}(2; \mathbb{R})$ and $\phi(\tau)^2 = c\tau + d$. The multiplication in the group $\text{Mp}(2; \mathbb{R})$ is defined as

$$(M_1, \phi_1(\tau))(M_2, \phi_2(\tau)) = (M_1 M_2, \phi_1(M_2 \tau) \phi_2(\tau)), \quad (\text{B.2})$$

where $M\tau = (a\tau + b)/(c\tau + d)$ denotes the usual action of $\text{SL}(2; \mathbb{R})$. By fixing the choice $\phi(\tau) = \sqrt{c\tau + d}$, we actually define a locally isomorphic embedding $\text{SL}(2; \mathbb{R}) \hookrightarrow \text{Mp}(2; \mathbb{R})$

$$M \mapsto \widetilde{M} = \left(M, \sqrt{c\tau + d} \right). \quad (\text{B.3})$$

$\text{Mp}(2; \mathbb{Z})$ is generated by two elements T, S

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right). \quad (\text{B.4})$$

One has the relation $S^2 = (ST)^3 = Z$, where

$$Z = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right) \quad (\text{B.5})$$

is the standard generator of the center of $\text{Mp}(2; \mathbb{Z})$. For convenience we define $\Gamma_1 = \text{SL}(2; \mathbb{Z})$,

$$\begin{aligned}\Gamma_\infty &= \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}; n \in \mathbb{Z} \right\} \leq \Gamma_1, \\ \tilde{\Gamma}_\infty = \langle T \rangle &= \left\{ \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, 1 \right); n \in \mathbb{Z} \right\},\end{aligned}\tag{B.6}$$

where $\langle T \rangle$ denotes the group generated by T .

Suppose L is an even lattice equipped with a symmetric \mathbb{Z} -valued bilinear form (z_1, z_2) for $z_1, z_2 \in L$ and the associated quadratic form $q(z) = (z, z)/2$ is integer for arbitrary $z \in L$. We denote by L' the dual lattice. The quotient L'/L is a finite Abelian group, the so-called discriminant group. Since the quadratic form can be extended to the dual lattice, we can define the quadratic form on L'/L , which takes values in \mathbb{Q}/\mathbb{Z} . There is a unitary representation ϱ of $\text{Mp}(2; \mathbb{Z})$ on the algebra $\mathbb{C}[L'/L]$. If we denote the standard basis of $\mathbb{C}[L'/L]$ by $\{\mathbf{e}_\gamma | \gamma \in L'/L\}$, then ϱ can be defined by the action of the generators $S, T \in \text{Mp}(2; \mathbb{Z})$ as follows

$$\begin{aligned}\varrho(T)\mathbf{e}_\gamma &= e(q(\gamma))\mathbf{e}_\gamma, \\ \varrho(S)\mathbf{e}_\gamma &= \frac{\sqrt{i}^{b^- - b^+}}{\sqrt{|L'/L|}} \sum_{\delta \in L'/L} e(-(\gamma, \delta))\mathbf{e}_\delta.\end{aligned}\tag{B.7}$$

This is the so-called Weil representation. Based on the relation $S^2 = Z$, we have

$$\begin{aligned}\varrho(Z)\mathbf{e}_\gamma &= \frac{i^{b^- - b^+}}{|L'/L|} \sum_{\delta, \lambda \in L'/L} e(-(\gamma, \delta))e(-(\delta, \lambda))\mathbf{e}_\lambda \\ &= i^{b^- - b^+} \mathbf{e}_{-\gamma}.\end{aligned}\tag{B.8}$$

We denote by $\langle \cdot, \cdot \rangle$ the standard product of $\mathbb{C}[L'/L]$, i.e.

$$\left\langle \sum_{\gamma \in L'/L} \lambda_\gamma \mathbf{e}_\gamma, \sum_{\gamma \in L'/L} \mu_\gamma \mathbf{e}_\gamma \right\rangle = \sum_{\gamma \in L'/L} \lambda_\gamma \bar{\mu}_\gamma.\tag{B.9}$$

For $\gamma, \delta \in L'/L$, we can define the representation matrix element $\varrho_{\gamma\delta}(M, \phi) = \langle \varrho(M, \phi)\mathbf{e}_\delta, \mathbf{e}_\gamma \rangle$.

B.2 Vector-valued modular forms

Definition B.1 (*Petersson slash operator*) Let $\kappa \in \frac{1}{2}\mathbb{Z}$ and f be a $\mathbb{C}[L'/L]$ -valued function on \mathbb{H} . For $(M, \phi) \in \text{Mp}(2; \mathbb{Z})$ we define the Petersson slash operator $|_\kappa(M, \phi)$ by

$$(f|_\kappa(M, \phi))(\tau) = \phi(\tau)^{-2\kappa} \varrho(M, \phi)^{-1} f(M\tau).\tag{B.10}$$

We denote by ϱ^* the dual representation of ϱ . If we think of $\varrho(M, \phi)$ as a matrix with entries in \mathbb{C} , then $\varrho^*(M, \phi)$ is simply the complex conjugate of $\varrho(M, \phi)$. The "dual operation" of

$\text{Mp}(2; \mathbb{Z})$ on functions $f : \mathbb{H} \rightarrow \mathbb{C}[L'/L]$ is given by

$$(f|_{\kappa}^*(M, \phi))(\tau) = \phi(\tau)^{-2\kappa} \varrho^*(M, \phi)^{-1} f(M\tau). \quad (\text{B.11})$$

If we assume that the function $f : \mathbb{H} \rightarrow \mathbb{C}[L'/L]$ is a holomorphic function which is invariant under the $|_{\kappa}^*$ operation of $T \in \text{Mp}(2; \mathbb{Z})$. Since f can be expanded by the basis \mathbf{e}_{γ} of L'/L , we have $f = \sum_{\gamma} f_{\gamma} \mathbf{e}_{\gamma}$. The invariance is satisfied if and only if

$$\begin{aligned} f_{\gamma}(\tau) &= f_{\gamma}|_{\kappa}^* T(\tau) = e^*(q(\gamma))^{-1} f_{\gamma}(\tau + 1) \\ \Leftrightarrow e(q(\gamma)\tau) f_{\gamma}(\tau) &= e(q(\gamma)(\tau + 1)) f_{\gamma}(\tau + 1), \end{aligned} \quad (\text{B.12})$$

which means the invariance of f under T implies that the function $e(q(\gamma)\tau) f_{\gamma}(\tau)$ is periodic with period 1. We can directly Fourier expand f by

$$f(\tau) e(q(\gamma)\tau) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z}} c(\gamma, n) e(n\tau) \mathbf{e}_{\gamma}. \quad (\text{B.13})$$

To have a compact expression, we define $\mathbf{e}_{\gamma}(n\tau) = e(n\tau) \mathbf{e}_{\gamma}$ and write

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z} - q(\gamma)} c(\gamma, n) \mathbf{e}_{\gamma}(n\tau), \quad (\text{B.14})$$

with Fourier coefficients

$$c(\gamma, n) = \int_0^1 \langle f(\tau), \mathbf{e}_{\gamma}(n\bar{\tau}) \rangle dx. \quad (\text{B.15})$$

Definition B.2 (*holomorphic modular form of dual Weil representation*) Let $\kappa \in \frac{1}{2}\mathbb{Z}$. A function $f : \mathbb{H} \rightarrow \mathbb{C}[L]$ is called a modular form of weight κ with respect to ϱ^* and $\text{Mp}(2; \mathbb{Z})$ if

- i) $f|_{\kappa}^*(M, \phi) = f$ for all $(M, \phi) \in \text{Mp}(2; \mathbb{Z})$,
- ii) f is holomorphic on \mathbb{H} ,
- iii) f is holomorphic at the cusp ∞ . If $c(\gamma, 0) \equiv 0$, f is called a cusp form.

The condition (iii) requires f has a Fourier expansion of the form

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} - q(\gamma) \\ n \geq 0}} c(\gamma, n) \mathbf{e}_{\gamma}(n\tau). \quad (\text{B.16})$$

The \mathbb{C} -vector space of modular forms of weight κ with respect to ϱ^* and $\text{Mp}(2; \mathbb{Z})$ is denoted by $M_{\kappa, L}$ and the subspace of cusp forms is denoted by $S_{\kappa, L}$. Similar to the usual complex valued modular form of $\text{SL}(2; \mathbb{Z})$, the linear space $M_{\kappa, L}$ is finite dimensional.

B.3 Nearly holomorphic modular forms

Definition B.3 (*nearly holomorphic modular form*) A function $f : \mathbb{H} \rightarrow \mathbb{C}[L]$ is called a nearly holomorphic modular form of weight k (with respect to ϱ and $\text{Mp}(2; \mathbb{Z})$), if

- i) $f|_k(M, \phi) = f$ for all $(M, \phi) \in Mp(2; \mathbb{Z})$,
- ii) f is holomorphic on \mathbb{H} ,
- iii) f has a pole in ∞ , i.e. f has a Fourier expansion of the form

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} + q(\gamma) \\ n \gg -\infty}} c(\gamma, n) \mathbf{e}_\gamma(n\tau). \quad (\text{B.17})$$

The space of these nearly holomorphic modular forms is denoted by $M_{k,L}^!$. The summation $n \gg -\infty$ indicates that there exists a finite negative number n_0 such that all $n \geq n_0$. This condition implies that the pole at the cusp (∞) has finite order. The Fourier polynomial

$$\sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} + q(\gamma) \\ n < 0}} c(\gamma, n) \mathbf{e}_\gamma(n\tau) \quad (\text{B.18})$$

is called the principal part of f .

As shown in [21], the space of nearly holomorphic modular form is generated by the Poincaré series, thus is finite dimensional. The principal part should satisfy the Theorem B.3.

B.4 Modular forms on generalized upper-half plane

The orthogonal modular forms and Borcherds product were introduced in the main text. Recall the definition of $j(M, Z)$ in (3.27)). More generally we can rewrite it as

$$j(M, Z) = (MZ_L, z), \quad (\text{B.19})$$

where $z = (1, 0, \dots, 0)^T$ is the $l + 2$ vector and $Z_L = (-q_0(Z), Z, 1)^T$. Suppose L is an even lattice of signature $(2, l)$ and $V = L \otimes \mathbb{R}$. The function $j(M, Z)$ on $O^+(V) \times \mathbb{H}_l$ is an automorphy factor for $O^+(V)$, i.e. it satisfies the cocycle relation

$$j(M_1 M_2, Z) = j(M_1, M_2 \langle Z \rangle) j(M_2, Z). \quad (\text{B.20})$$

For an arbitrary $a \in \mathbb{C}$, we have already specified $\text{Arg } a \in [-\pi, \pi)$, which is the principal value of argument of a . We denote by Ln the logarithm of the principal branch, which is defined as $\text{Ln } a = \ln |a| + i \text{Arg } a$. For an arbitrary $a, b \in \mathbb{C}$, we define $a^b = e^{b \text{Ln } a}$. Let $r \in \mathbb{Q}$, if $M \in O^+(V)$ and $Z \in \mathbb{H}_l$, then $j(M, Z)^r = e^{r \text{Ln } j(M, Z)}$. There exists a map w_r from $O^+(V) \times O^+(V)$ to the set of roots of unity (of order bounded by the denominator of r) such that

$$j(M_1 M_2, Z)^r = w_r(M_1, M_2) j(M_1, M_2 \langle Z \rangle)^r j(M_2, Z)^r. \quad (\text{B.21})$$

Definition B.4 (multiplier system) Let $\Gamma \leq O^+(V)$ be a subgroup and $r \in \mathbb{Q}$ as above. By a multiplier system of weight r we mean a map

$$\chi : \Gamma \longrightarrow S^1 = \{t \in \mathbb{C} \mid |t| = 1\} \quad (\text{B.22})$$

satisfying

$$\chi(M_1 M_2) = w_r(M_1, M_2) \chi(M_1) \chi(M_2), \quad M_1, M_2 \in \Gamma. \quad (\text{B.23})$$

If $r \in \mathbb{Z}$, then χ is actually a character of Γ , then $\chi(M)j(M, Z)^r$ is a cocycle of Γ .

Definition B.5 (modular form on generalized upper-half plane) Let $\Gamma \leq \Gamma(L)$ be a subgroup of finite index and χ a multiplier system for Γ of weight $r \in \mathbb{Q}$. A meromorphic function F on \mathbb{H}_l is called a meromorphic modular form of weight r and multiplier system χ with respect to Γ , if

$$\Psi(M\langle Z \rangle) = \chi(M)j(M, Z)^r \Psi(Z) \quad (\text{B.24})$$

for all $M \in \Gamma$. If Ψ is even holomorphic on \mathbb{H}_l then it is called a holomorphic modular form.

The Borchers product can lift a nearly holomorphic modular form $f(\tau) = \sum_{\gamma \in L'/L} f_\gamma \mathbf{e}_\gamma : \mathbb{H} \rightarrow \mathbb{C}[L'/L]$ (see Definition B.3) of weight $1 - l/2$ with Fourier expansion

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z} + q(\gamma)} c(\gamma, n) \mathbf{e}_\gamma(n\tau), \quad (\text{B.25})$$

to the meromorphic function $\Psi(Z) : \mathbb{H} \rightarrow \mathbb{C}$ of weight $c(0, 0)/2$. The precise theorem is stated as follows.

Theorem B.1 (Theorem 13.3 (1) in [26] or Theorem 3.22 (i) in [21]) Let L be an even lattice of signature $(2, l)$ with $l \geq 3$, and $z \in L$ a primitive isotropic vector. Let $z' \in L'$ and $K = L \cap z^\perp \cap z'^\perp$. Moreover, assume that K also contains an isotropic vector. Let f be a nearly holomorphic modular form of weight $k = 1 - l/2$ whose Fourier coefficients $c(\gamma, n)$ are integral for $n < 0$. Then

$$\Psi(Z) = \prod_{\beta \in L'/L} \prod_{\substack{m \in \mathbb{Z} + q(\beta) \\ m < 0}} \Psi_{\beta, m}(Z)^{c(\beta, m)/2} \quad (\text{B.26})$$

is a meromorphic function on \mathbb{H}_l of (rational) weight $c(0, 0)/2$ for the modular group $\Gamma(L)$ with some multiplier systems χ of finite order. If $c(0, 0) \in 2\mathbb{Z}$, then χ is the character of group $\Gamma(L)$.

For functions $\Psi_{\beta, m}(Z)$ see Definition 3.14 in [21].

We can turn to the zeros and poles of $\Psi(Z)$. A nowhere-vanishing holomorphic modular forms $\Psi(Z)$ obtained through Borchers product cannot exist since there is no input nonzero holomorphic modular form $f(\tau)$ of negative weight $1 - l/2$. Before determining the positions of poles and zeroes, it is necessary to explain the concept of rational quadratic divisors (Heegner divisors).

Let $z \in L$ be a primitive norm 0 vector, $z' \in L'$ with $(z, z') = 1$. Let N be unique positive integer with such that $(z, L) = N\mathbb{Z}$. Then we have $z/N \in L'$. Denote by K the lattice

$$K = L \cap z^\perp \cap z'^\perp. \quad (\text{B.27})$$

K has signature $(b^+ - 1, b^- - 1) = (1, l - 1)$. For an arbitrary vector $n \in V = L \otimes \mathbb{R}$, n_K denotes the orthogonal projection n to $K \otimes \mathbb{R}$ and

$$n_K = n - (n, z)z' + (n, z)(z', z')z - (n, z')z. \quad (\text{B.28})$$

If $n \in L'$, then n_K lies in the dual lattice K' of K . Let $\zeta \in L$ be a lattice vector with $(\zeta, z) = N$. Let $n \in L$, then the vector

$$\tilde{n} = n - (n, z/N)\zeta - (n, z')z + (n, z/N)(\zeta, z')z \quad (\text{B.29})$$

lies in L and easy to verify that $\tilde{n} \perp z$ and $\tilde{n} \perp z'$. Hence $\tilde{n} \in K$ and each element $n \in L$ can be uniquely decomposed in this way, or equivalently, $L = K \oplus \mathbb{Z}\zeta \oplus \mathbb{Z}z$. Now let $\lambda \in L'$ be a vector of negative norm, i.e. $q(\lambda) < 0$. Then the orthogonal complement $\lambda^\perp \subset L \otimes \mathbb{R}$ is a rational quadratic space of type $(2, l - 1)$. With these settings we can define the rational quadratic divisors

Definition B.6 (*rational quadratic divisor or Heegner divisor*) Let $\lambda \in L'$ be a vector of negative norm m , we set

$$H_\lambda = \{[Z_L] \in \mathcal{K}^+ \mid (Z_L, \lambda) = 0\}. \quad (\text{B.30})$$

Moreover, due to the decomposition $Z_L = (-q(Z) - q(z'))z + Z + z'$ (recall the equation (3.17)) and $\lambda = bz + \lambda_K + az'$, expanding the inner product (Z_L, λ) yields

$$H_\lambda \cong \{Z \in \mathbb{H}_l \mid aq(Z) - (Z, \lambda_K) - aq(z') - b = 0\} \quad (\text{B.31})$$

in coordinates on \mathbb{H}_l . This set defines a prime divisor on \mathbb{H}_l . Suppose $\beta \in L'/L$ and m is a negative rational number; the sum

$$H(\beta, m) = \sum_{\substack{\lambda \in \beta + L \\ q(\lambda) = m}} H_\lambda \quad (\text{B.32})$$

is called the rational quadratic divisor (or Heegner divisor) of discriminant (β, m) , which is a $\Gamma(L)$ -invariant divisor on \mathbb{H}_l . When $\beta = 0$, we usually denote $H(m) = \frac{1}{2}H(0, m)$.

This definition is suitable for lattices of signature $(2, l)$ with arbitrary Gram matrix. If we specify the Gram matrix of $L = \Pi_{1,1} \oplus L_0$ as defined in the equation (3.19) and the vector z, z' , equivalently we have

$$H_\lambda = \{Z \in \mathbb{H}_l \mid aq_0(Z) - (Z, \lambda_K)_0 - b = 0\}, \quad (\text{B.33})$$

where the subscript emphasizes that the inner product is associated with the quadratic form S_0 . With this definition we can describe the position of the zeros and poles by the following theorem.

Theorem B.2 (*Theorem 13.3 (2) in [26] or Theorem 3.22 (ii) in [21]*) The zeros and poles of $\Psi(Z)$ lies on the divisor of $\Psi(Z)$ on \mathbb{H}_l , which is the linear combinations of Heegner

divisors determined by the principal part of the nearly holomorphic modular form f

$$(\Psi) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z} + q(\beta) \\ m < 0}} c(\beta, m) H(\beta, m). \quad (\text{B.34})$$

The multiplicities of $H(\beta, m)$ are 2, if $2\beta = 0$ in L'/L , and 1, if $2\beta \neq 0$ in L'/L .

As we saw from the above theorems, the properties of the Borcherds product $\Psi(Z)$ are completely captured by the nearly holomorphic modular form $f(\tau)$, in particular by the principal part of $f(\tau)$:

$$\sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} + q(\gamma) \\ n < 0}} c(\gamma, n) \mathfrak{e}_\gamma(n\tau). \quad (\text{B.35})$$

Pairing the form $f(\tau)$ with a vector valued cusp form of weight $1 + l/2$ for the dual Weil representation (see Definition B.2) gives a meromorphic elliptic modular form of weight 2 for $\text{SL}(2; \mathbb{Z})$, hence its constant term must vanish by the residue theorem (no nonzero $\text{SL}(2; \mathbb{Z})$ modular form of weight 2) and this gives the conditions on the principal part on f , stated as the following theorem. By setting $\kappa = 1 + l/2$ and denoting the space of the vector valued modular cusp form of weight κ with respect to lattice L as $S_{\kappa, L}$, we have

Theorem B.3 (Theorem 1.17 in [21]) *There exists a nearly holomorphic modular form $f \in M_{k, L}^!$ with prescribed principal part*

$$\sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z} + q_1(\beta) \\ m < 0}} c(\beta, m) \mathfrak{e}_\beta(m\tau) \quad (\text{B.36})$$

($c(\beta, m) \in \mathbb{C}$ with $c(\beta, m) = c(-\beta, m)$), if and only if the functional

$$\sum_{\beta \in L'/L} \sum_{\substack{m \in \mathbb{Z} + q_1(\beta) \\ m < 0}} c(\beta, m) a_{\beta, -m}, \quad (\text{B.37})$$

equals zero in $S_{\kappa, L}^*$. For $\gamma \in D(L)$ and $n \in \mathbb{Z} - q(\gamma)$ with $n > 0$, $a_{\gamma, n} : S_{\kappa, L} \rightarrow \mathbb{C}$ denote the functional in the dual space $S_{\kappa, L}^*$ of $S_{\kappa, L}$ which maps a cusp form f to its (γ, n) -th Fourier coefficient $a_{\gamma, n}(f)$.

Obviously this imposes non-trivial condition on the principal part of the nearly holomorphic modular form f .

B.5 Character of the lattice

If the weight of the modular form is integer, which is the case of interest, the multiplier system is actually the character of the modular group $\Gamma(L)$, or the character of the lattice L . This forms a homomorphism from the modular group to $\text{U}(1)$. From the well-known Pontryagin duality, the abelianisation $G^{ab} = G/[G, G]$ of the group G is isomorphic to the character group $\text{Hom}(G, \mathbb{C}^\times)$. Thus to obtain the character we need to consider the

abelianisation of the modular group $\Gamma(L)$. As discussed in section 5, the anomaly cancellation imposes the triviality of the character for the admissible lattices. To the best of our knowledge, the sufficient and necessary conditions for a lattice of signature $(2, l)$ to have trivial characters are not known. A sufficient condition is known:

Theorem B.4 (Theorem 1.7 in [39]) *Let L be an even integral lattice containing at least two hyperbolic planes $(\Pi_{1,1})$, such that $\text{rank}_3(L) \geq 5^{26}$ and $\text{rank}_2(L) \geq 6$, then the $\Gamma(L)^{ab} \cong \mathbb{Z}/2\mathbb{Z}$ and $S\Gamma(L)^{ab}$ is trivial, where $S\Gamma(L)$ is the modular group intersect with the special orthogonal group of lattice L , i.e. $S\Gamma(L) := \Gamma(L) \cap SO(L)$.*

An immediate corollary is that if $L = \Pi_{1,1} \oplus \Pi_{1,1} \oplus \hat{L}$ and \hat{L} contains a sublattice isomorphic to A_2 , it satisfies the so-called Kneser conditions [39, 77] and the character for group $S\Gamma(L)$ is trivial. Notably, if the lattice $L = \Pi_{1,1} \oplus \Pi_{1,1} \oplus \hat{L}$ is an even unimodular lattice of rank at least 6, we have the same conclusion that the $\Gamma(L)^{ab} \cong \mathbb{Z}/2\mathbb{Z}$ and $S\Gamma(L)^{ab}$ is trivial.

C Alternative $l = 2$ counterterm from Hilbert modular forms

In this appendix, we will provide a brief overview of the alternative construction for the case $l = 2$ mentioned at the end of section 6, which follows a similar path to the procedure for $l \geq 3$ cases. For this we need to introduce Hilbert modular forms, following closely to these good references [22, 78, 79].

Let $\mathbb{K} = \mathbb{Q}(\sqrt{p})$, $p \in \mathbb{N}$, $p > 1$ squarefree, be a real quadratic number field with the ring of integers and discriminant

$$\mathcal{O}_{\mathbb{K}} = \mathbb{Z} + \mathbb{Z}\omega_{\mathbb{K}}, \quad \omega_{\mathbb{K}} = \begin{cases} (1 + \sqrt{p})/2, \\ \sqrt{p}, \end{cases} \quad d_{\mathbb{K}} = \begin{cases} p, & p \equiv 1 \pmod{4}, \\ 4p, & \text{else.} \end{cases} \quad (\text{C.1})$$

The non-trivial automorphism $\mathbb{K} \rightarrow \mathbb{K}$ is given by

$$\alpha = \alpha_0 + \alpha_1\sqrt{p} \mapsto \alpha^* = \alpha_0 - \alpha_1\sqrt{p}, \quad \alpha_0, \alpha_1 \in \mathbb{Q}. \quad (\text{C.2})$$

The Hilbert modular group is given by $\Gamma_{\mathbb{K}} = \text{SL}(2; \mathcal{O}_{\mathbb{K}})$. With respect to this group, we can define the Hilbert modular form

Definition C.1 (Hilbert modular form) *Let $\mu : \text{SL}(2; \mathcal{O}_{\mathbb{K}}) \rightarrow \mathbb{C}$ be a map of finite order (multiplier system). A Hilbert (Blumenthal) modular form for \mathbb{K} of weight $r = (r_1, r_2) \in \mathbb{Q}^2$ with multiplier system μ is a holomorphic function $f : \mathbb{H}^2 \rightarrow \mathbb{C}$ with the properties*

i) $f(M\tau) = \mu(M)(c\tau + d)^{r_1}(c^\tau_2 + d^*)^{r_2}f(\tau)$ for all $\tau \in \mathbb{H}^2$, $M \in \text{SL}(2; \mathcal{O}_{\mathbb{K}})$, where*

$$M\tau := \left(\frac{a\tau_1 + b}{c\tau_1 + d}, \frac{a^*\tau_2 + b^*}{c^*\tau_2 + d^*} \right), \quad \tau = (\tau_1, \tau_2), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (\text{C.3})$$

²⁶For any prime p the p -rank of L , denoted by $\text{rank}_p(L)$, is the maximal rank of the sublattices M such that $\det(M)$ is coprime to p .

ii) f is regular at cusps of $SL(2; \mathcal{O}_{\mathbb{K}})$.

If f vanishes at all cusps, we call f a cusp form. If f has homogeneous weight $r = (k, k) \in \mathbb{Q}^2$ we will also say that f has weight $k \in \mathbb{Q}$.

If we want to use such Hilbert modular forms to cancel the anomaly, several things need to be clarified. First, the action of Hilbert modular group $\Gamma_{\mathbb{K}}$ on the generalized upper-half plane \mathbb{H}_2 is obtained through the homomorphism Ω in (6.25). For arbitrary $M \in SL(2; \mathbb{K})$ (or $\Gamma_{\mathbb{K}}$),

$$\Omega(M, M^*)\langle Z \rangle = \left(\frac{\alpha Z_1 + \beta}{\gamma Z_1 + \delta}, \frac{\alpha^* Z_2 + \beta^*}{\gamma^* Z_2 + \delta^*} \right)^T, \quad M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad (\text{C.4})$$

$$j(M, Z) = (\gamma Z_1 + \delta)(\gamma^* Z_2 + \delta^*).$$

In other words, the symmetry group now is the Hilbert modular group $\Gamma_{\mathbb{K}}$, different with the previous $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$. Theorem 2 of [63] proves the group $\Gamma_{\mathbb{K}}$ is actually isomorphic to the discriminant kernel of the orthogonal group, so the symmetry further shrinks to $\Gamma_{\mathbb{K}}$.

Furthermore, non-trivial lattice structure emerges. The modular group $\Gamma_{\mathbb{K}}$ is the spin group $\text{Spin}(L)$ of lattice L (section 2.7 of Chapter 2 in [22]), where L can be written as $L = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathcal{O}_{\mathbb{K}}$ with quadratic form $q((a, \nu, b)) = ab - \nu\nu^*$ for $a, b \in \mathbb{Z}$ and $\nu \in \mathcal{O}_{\mathbb{K}}$. For example, suppose $p \equiv 1 \pmod{4}$, and we write the lattice explicitly in terms of matrices

$$L = \left\{ \begin{pmatrix} a & \nu \\ \nu^* & b \end{pmatrix} \mid a, b \in \mathbb{Z}, \nu \in \mathcal{O}_{\mathbb{K}} \right\}, \quad q(X) = \det(X) \text{ for } X \in L. \quad (\text{C.5})$$

The basis of this lattice is easily written in terms of the matrices

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & \frac{1+\sqrt{p}}{2} \\ \frac{1-\sqrt{p}}{2} & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (\text{C.6})$$

One can easily obtain the Gram matrix in terms of these basis, i.e. $S_{ij} = (e_i, e_j) = q(e_i + e_j) - q(e_i) - q(e_j)$, then (we use the symbol S as defined in the previous sections)

$$S = \begin{pmatrix} & & & 1 \\ & -2 & -1 & \\ & -1 & \frac{p-1}{2} & \\ 1 & & & \end{pmatrix}. \quad (\text{C.7})$$

Though S contains a $\Pi_{1,1}$ as usual, the rest presents a non-trivial lattice structure, not realized by known string-theoretic constructions.

The next step is to construct the suitable Hilbert modular forms of non-trivial weight. The conditions in section 5 still need to be satisfied, namely:

- The character μ of the group $\Gamma_{\mathbb{K}}$ (of the lattice L) must be trivial.
- The zeros and poles of the Hilbert modular form correspond to the symmetry enhancement point of the theory.

The first condition can be assured by an appropriate choice of the value of p . However, identifying the rational quadratic divisor in this case is not straightforward. Necessary details can be found in references [22, 78, 79]. It can be verified (see section 1.3 in [79]) that the generalized upper-half plane \mathbb{H}_2 (isomorphic to \mathcal{K}^+ defined in (3.12)) is exactly of the form of

$$\mathbb{H}_2 = \left\{ \delta \begin{pmatrix} Z_1 Z_2 & Z_1 \\ Z_2 & 1 \end{pmatrix} \mid \text{Im}(Z_1) > 0, \text{Im}(Z_2) > 0, \delta \in \mathbb{C}^* \right\}. \quad (\text{C.8})$$

Since we are working within the projective space, usually the factor δ is ignored. It is also obvious that we can use (Z_1, Z_2) to label the element in \mathbb{H}_2 . In the previous sections we used rational quadratic divisors to describe the position of zeroes and poles of the Borchers product. The corresponding object here is called Hirzebruch-Zagier divisor [80].

Definition C.2 (*Hirzebruch-Zagier divisor*) For $(a, h, b) \in L$ and $Z \in \mathbb{H} \times \mathbb{H}$, we have the innerproduct

$$\left(\begin{pmatrix} Z_1 Z_2 & Z_1 \\ Z_2 & 1 \end{pmatrix}, \begin{pmatrix} a & h \\ h^* & b \end{pmatrix} \right) = bZ_1 Z_2 - h^* Z_1 - hZ_2 + a. \quad (\text{C.9})$$

The zero locus of the right hand side defines an analytic divisor on $\mathbb{H} \times \mathbb{H}$. For a positive number m , in the space $\mathbb{H} \times \mathbb{H}$ we define the set

$$T(m) = \bigcup_{\substack{(a,b,h) \in L'/\{\pm 1\} \\ q(a,b,h) = ab - h^*h = -m/p}} \{(Z_1, Z_2) \in \mathbb{H}^2 \mid aZ_1 Z_2 + hZ_1 + h^* Z_2 + b = 0\}. \quad (\text{C.10})$$

$T(m)$ is called Hirzebruch-Zagier divisor of discriminant m .

Before we extend the Borchers product (Theorem B.1) to this case, we set up some basic notations. In section 2 the congruence subgroup $\Gamma_0(p)$ is defined. Corresponding modular forms may be defined as well.

Definition C.3 (*Modular forms for congruence subgroups*) Let μ be an abelian character $\Gamma_0(p) \rightarrow \mathbb{C}^*$ and $k \in \mathbb{N}_0$ a non negative integer. A holomorphic map $f : \mathbb{H} \rightarrow \mathbb{C}$ with the transformation law

$$f(M\tau) = \mu(M)(c\tau + d)^k f(\tau) \quad \text{for all } M \in \Gamma_0(p), \quad (\text{C.11})$$

for which $f(\infty) := \lim_{\text{Im}(z) \rightarrow \infty} f(z)$ and $f(0) := \lim_{z \rightarrow 0} z^k f(z)$ exist in $\mathbb{C} \cup \{\infty\}$ (it has finite order at the infinity) is called nearly holomorphic modular form for $\Gamma_0(p)$ of weight k with character μ . If $f(\infty)$ and $f(0)$ are complex numbers, then f is called a holomorphic modular form for $\Gamma_0(p)$ of weight k with character μ . If $f(\infty) = f(0) = 0$, then f is called a cusp form.

We define the spaces

$$\begin{aligned}
A_k(p, \mu) & \text{ nearly holomorphic modular forms for } \Gamma_0(p) \text{ of weight } k \text{ with character } \mu, \\
M_k(p, \mu) & \text{ holomorphic modular forms for } \Gamma_0(p) \text{ of weight } k \text{ with character } \mu, \\
S_k(p, \mu) & = \{f \in M_k(p, \mu) \mid f \text{ cusp form}\}, \\
A_k^\pm(p, \chi_p) & = \left\{ f(z) = \sum_{n \in \mathbb{Z}} a(n) e^{2\pi i n z} \in A_k(p, \chi_p) \mid a(n) = 0 \text{ for } \chi_p(n) = \mp 1 \right\}, \\
S_k^\pm(p, \chi_p) & = A_k^\pm(p, \chi_p) \cap S_k(p, \chi_p).
\end{aligned}$$

One can show that $A_k(p, \chi) = A_k^+(p, \chi_p) \oplus A_k^-(p, \chi_p)$. If $f = \sum_{n \in \mathbb{Z}} a(n) q^n$ is a modular form in $A_k^\epsilon(p, \chi_p)$, then we call $\sum_{n < 0} a(n) q^n$ the principal part of f (at $\tau \rightarrow \infty$). For all integers n we define

$$s(n) = 1 + \sum_{j=0}^{p-1} \frac{e^{2\pi i n j / p}}{p} = 2 - \left(\frac{n}{p}\right)^2 = \begin{cases} 2, & \text{if } n \equiv 0 \pmod{p} \\ 1, & \text{if } n \not\equiv 0 \pmod{p} \end{cases}. \quad (\text{C.12})$$

Similar to the Theorem B.3, the principal part of f has non trivial restrictions (Theorem 6 in [78]). There exists a nearly holomorphic modular form $f \in A_k^+(p, \chi_p)$ with prescribed principal part $\sum_{n < 0} a(n) q^n$ (where $a(0) = 0$ if $\chi_p(n) = -1$), if and only if

$$\sum_{n < 0} s(n) a(n) b(-n) = 0 \quad (\text{C.13})$$

for every cusp form $g = \sum_{m > 0} b(m) q^m$ in $S_\kappa^+(p, \chi_p)$, where $\kappa = 2 - k$. The case $k = 0$ and $\kappa = 2$ is of particular interest for us. For prime number $p \equiv 1 \pmod{4}$, the dimension of $S_2(p, \chi_p)$ is $2 \left\lfloor \frac{p-5}{24} \right\rfloor$. Thus the space is empty for $p = 5$ case. Hence for $p = 5$, for all $m \in \mathbb{N}$ there is a nearly holomorphic modular form $f \in A_0^+(p, \chi_p)$ with prescribed principal part $\sum_{n < 0} a(n) q^n$ if and only if $a(n) = 0$ for all $n \in \mathbb{N}$ with $\chi_p(n) = -1$. Such nearly holomorphic modular form is unique [79], and up to the normalization is the function f given in (6.29).

Theorem C.1 (*Borchers product for Hilbert modular forms, Theorem 9 in [78]*) *Let $f = \sum_{n \in \mathbb{Z}} a(n) q^n \in A_0^+(p, \chi_p)$ and assume that $s(n) a(n) \in \mathbb{Z}$ for all $n < 0$. Then there is a meromorphic function Ψ on $\mathbb{H} \times \mathbb{H}$ with the following properties:*

i) Ψ is a meromorphic modular form for Γ_κ (the Hilbert modular group defined in section 6) with some multiplier system of finite order. The weight of Ψ is equal to the constant coefficient $a(0)$ of f .

ii) The divisor of Ψ is determined by the principal part of f . It equals

$$\sum_{n < 0} s(n) a(n) T(-n). \quad (\text{C.14})$$

We can now verify the two required properties. The first is satisfied for $p = 5$, since the multiplier system of Hilbert modular form for $\Gamma_{\mathbb{K}}$ is trivial (Corollary 5.2.1 in [79]). The second condition necessitates a confirmation that the positions of poles and zeros of the Hilbert modular form correspond to the symmetry enhancements. The explicit relationship between these points and the Hirzebruch-Zagier divisor is not yet known. Naively, requiring that the symmetry enhancement appears at the diagonal set $(Z_1, Z_2) = \{(\tau, \tau) | \tau \in \mathbb{H}\}$ similarly to the choices in [25], corresponds to the Hirzebruch-Zagier divisor $T(1)$. There exists a weight 0 modular form $f \in A_0^+(5, \chi_5)$

$$f(\tau) = q^{-1} + 5 + 11q - 54q^4 + O(q^5), q = e^{2\pi i \tau}, \quad (\text{C.15})$$

that has only one term (q^{-1}) in the principal part. By Theorem C.1, we arrive at a holomorphic Hilbert modular form of weight 5 vanishing at $T(1)$. However, as pointed out in the main text, unexpected symmetry enhancements appear. For example, the point $Z_1 = Z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ may lead to $SU(3)$ symmetry, which does not appear in 8D (2,2) theories. Therefore, Hilbert modular forms do not appear in the counterterms for the $\mathcal{N} = 1$ theories with $l = 2$.

References

- [1] W. Taylor, *TASI Lectures on Supergravity and String Vacua in Various Dimensions*, [arXiv:1104.2051](#).
- [2] A. Adams, O. DeWolfe, and W. Taylor, *String universality in ten dimensions*, Phys. Rev. Lett. **105** (2010) 071601, [[arXiv:1006.1352](#)].
- [3] H.-C. Kim, G. Shiu, and C. Vafa, *Branes and the Swampland*, Phys. Rev. D **100** (2019), no. 6 066006, [[arXiv:1905.08261](#)].
- [4] S.-J. Lee and T. Weigand, *Swampland bounds on the abelian gauge sector*, Phys. Rev. D **100** (Jul, 2019) 026015.
- [5] P. Cheng, R. Minasian, and S. Theisen, *Anomalies as obstructions: from dimensional lifts to swampland*, JHEP **01** (2022) 068, [[arXiv:2106.14912](#)].
- [6] A. Salam and E. Sezgin, *$d = 8$ supergravity: Matter couplings, gauging and Minkowski compactification*, Physics Letters B **154** (Apr., 1985) 37–42.
- [7] L. Alvarez-Gaume and E. Witten, *Gravitational Anomalies*, Nucl. Phys. B **234** (1984) 269.
- [8] M. Montero and C. Vafa, *Cobordism conjecture, anomalies, and the string lamppost principle*, Journal of High Energy Physics **2021** (Jan, 2021) 63.
- [9] H.-C. Kim, H.-C. Tarazi, and C. Vafa, *Four-dimensional $\mathcal{N} = 4$ SYM theory and the swampland*, Phys. Rev. D **102** (2020), no. 2 026003, [[arXiv:1912.06144](#)].
- [10] Y. Hamada and C. Vafa, *8d supergravity, reconstruction of internal geometry and the swampland*, Journal of High Energy Physics **2021** (Jun, 2021) 178.
- [11] A. Font, B. Fraiman, M. Graña, C. A. Núñez, and H. P. De Freitas, *Exploring the landscape of heterotic strings on T^d* , JHEP **2020** (2020) 194, [[arXiv:2007.10358](#)].
- [12] A. Font, B. Fraiman, M. Graña, C. A. Núñez, and H. Parra De Freitas, *Exploring the landscape of CHL strings on T^d* , JHEP **2021** (2021) 095, [[arXiv:2104.07131](#)].

- [13] M. Cvetič, M. Dierigl, L. Lin, and H. Y. Zhang, *String Universality and Non-Simply-Connected Gauge Groups in 8d*, Phys. Rev. Lett. **1 25** (2020), no. 21 211602, [[arXiv:2008.10605](#)].
- [14] M. Cvetič, M. Dierigl, L. Lin, and H. Y. Zhang, *Gauge group topology of 8D Chaudhuri-Hockney-Lykken vacua*, Phys. Rev. D **1 04** (2021), no. 8 086018, [[arXiv:2107.04031](#)].
- [15] I. n. García-Etxebarria, H. Hayashi, K. Ohmori, Y. Tachikawa, and K. Yonekura, *8d gauge anomalies and the topological Green-Schwarz mechanism*, JHEP **1 1** (2017) 177, [[arXiv:1710.04218](#)].
- [16] A. Debray, M. Dierigl, J. J. Heckman, and M. Montero, *The anomaly that was not meant IIB*, [[arXiv:2107.14227](#)].
- [17] N. Marcus, *Composite Anomalies in Supergravity*, Phys. Lett. B **1 57** (1985) 383–388.
- [18] M. R. Gaberdiel and M. B. Green, *An $SL(2, z)$ anomaly in IIB supergravity and its f-theory interpretation*, Journal of High Energy Physics **1 998** (nov, 1998) 026–026.
- [19] R. Minasian, S. Sasmal, and R. Savelli, *Discrete anomalies in supergravity and consistency of string backgrounds*, Journal of High Energy Physics **2 017** (Feb, 2017) 25.
- [20] E. D’Hoker and J. Kaidi, *Lectures on modular forms and strings*, 2022.
- [21] J. H. Bruinier, Borcherds products on $O(2, 1)$ and Chern classes of Heegner divisors. Springer Science & Business Media, 2002.
- [22] J. H. Bruinier, G. Van der Geer, G. Harder, and D. Zagier, The 1-2-3 of modular forms: lectures at a summer school in Nordfjordeid, Norway. Springer Science & Business Media, 2008.
- [23] E. Calabi and E. Vesentini, *On compact, locally symmetric kahler manifolds*, Annals of Mathematics **7 1** (1960), no. 3 472–507.
- [24] R. E. Borcherds, *Automorphic forms on $o(s+2, 2, r)$ and infinite products*, Inventiones mathematicae **1 20** (Dec, 1995) 161–213.
- [25] J. A. Harvey and G. Moore, *Algebras, bps states, and strings*, Nuclear Physics B **4 63** (1996), no. 2 315–368.
- [26] R. E. Borcherds, *Automorphic forms with singularities on grassmannians*, Inventiones mathematicae **1 32** (May, 1998) 491–562.
- [27] A. Salam and E. Sezgin, *$d = 8$ SUPERGRAVITY*, Nucl. Phys. B **2 58** (1985) 284.
- [28] G. Bossard and V. Verschinin, *Minimal unitary representations from supersymmetry*, JHEP **1 0** (2014) 008, [[arXiv:1406.5527](#)].
- [29] R. Kallosh, *Is $d=4$ maximal supergravity special?*, 2023.
- [30] C. Vafa and E. Witten, *A One loop test of string duality*, Nucl. Phys. B **4 47** (1995) 261–270, [[hep-th/9505053](#)].
- [31] M. J. Duff, J. T. Liu, and R. Minasian, *Eleven-dimensional origin of string-string duality: A One loop test*, Nucl. Phys. B **4 52** (1995) 261–282, [[hep-th/9506126](#)].
- [32] I. Antoniadis, S. Ferrara, R. Minasian, and K. S. Narain, *R^{**4} couplings in M and type II theories on Calabi-Yau spaces*, Nucl. Phys. B **5 07** (1997) 571–588, [[hep-th/9707013](#)].
- [33] J. T. Liu and R. Minasian, *Computing $1/N^2$ corrections in AdS/CFT*, [[arXiv:1010.6074](#)].

- [34] M. Awada and P. K. Townsend, $d = 8$ MAXWELL-EINSTEIN SUPERGRAVITY, Phys. Lett. B **1** 56 (1985) 51–54.
- [35] B. Williams, *Higher pullbacks of modular forms on orthogonal groups*, Forum Mathematicum **3** 3 (mar, 2021) 631–652.
- [36] A. Krieg, *Integral orthogonal groups*, in Dynamical Systems, Number Theory and Applications: A Festschrift in Honor of Armin Leutbecher’s 80th Birthday, pp. 177–195. World Scientific, 2016.
- [37] T. Kecker, The Quaternion-Kähler Manifold $SO(4, 20)/(SO(4) \times SO(20))$ from the c-map and as Moduli Space of K3 Surfaces. PhD thesis, Universität Hamburg, 2009.
- [38] R. Gilmore, Lie groups, Lie algebras, and some of their applications. Courier Corporation, 2012.
- [39] V. Gritsenko, K. Hulek, and G. Sankaran, *Abelianisation of orthogonal groups and the fundamental group of modular varieties*, Journal of Algebra **3** 22 (2009), no. 2 463–478.
- [40] M. Dittmann, H. Hagemeyer, and M. Schwagenscheidt, *Automorphic products of singular weight for simple lattices*, Mathematische Zeitschrift **2** 79 (Feb, 2015) 585–603.
- [41] N. R. Scheithauer, *On the classification of automorphic products and generalized kac–moody algebras*, Inventiones mathematicae **1** 64 (Jun, 2006) 641–678.
- [42] S. Ma, *Finiteness of 2-reflective lattices of signature $(2, n)$* , American Journal of Mathematics **1** 39 (2017), no. 2 513–524.
- [43] H. Wang, *The classification of 2-reflective modular forms*, arXiv preprint arXiv:1906.10459 (2019).
- [44] H. Wang, *Reflective modular forms on lattices of prime level*, Transactions of the American Mathematical Society **3** 75 (2022), no. 05 3451–3468.
- [45] V. Kaplunovsky, J. Louis, and S. Theisen, *Aspects of duality in $N=2$ string vacua*, Phys. Lett. B **3** 57 (1995) 71–75, [[hep-th/9506110](#)].
- [46] V. V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Mathematics of the USSR-Izvestiya **1** 4 (feb, 1980) 103.
- [47] N. R. Scheithauer, *Some constructions of modular forms for the Weil representation of $SL_2(\mathbb{Z})$* , Nagoya Mathematical Journal **2** 20 (2015), no. none 1 – 43.
- [48] V. A. Gritsenko, *Reflective modular forms and their applications*, Usp.Mat.Nauk **7** 3 (2018), no. 5 53–122.
- [49] J. H. Bruinier, *On the converse theorem for borchers products*, Journal of Algebra **3** 97 (2014) 315–342.
- [50] K. S. Narain, M. H. Sarmadi, and E. Witten, *A Note on Toroidal Compactification of Heterotic String Theory*, Nucl. Phys. B **2** 79 (1987) 369–379.
- [51] M. Cvetič, M. Dierigl, L. Lin, and H. Y. Zhang, *All eight- and nine-dimensional string vacua from junctions*, Phys. Rev. D **1** 06 (Jul, 2022) 026007.
- [52] I. Shimada and D.-Q. Zhang, *Classification of extremal elliptic k3 surfaces and fundamental groups of open k3 surfaces*, Nagoya Mathematical Journal **1** 61 (2001) 23–54.
- [53] I. Shimada, *On elliptic k3 surfaces*, arXiv preprint math/0505140 (2005).

- [54] A. Nagano and K. Ueda, *The ring of modular forms for the even unimodular lattice of signature (2,18)*, Hiroshima Mathematical Journal **5** 2 (2022), no. 1 43 – 51.
- [55] A. Mikhailov, *Momentum lattice for chl string*, Nuclear Physics B **5** 34 (1998), no. 3 612–652.
- [56] S. Chaudhuri, G. Hockney, and J. D. Lykken, *Maximally supersymmetric string theories in $D < 10$* , Phys. Rev. Lett. **7** 5 (1995) 2264–2267, [[hep-th/9505054](#)].
- [57] S. Chaudhuri and J. Polchinski, *Moduli space of CHL strings*, Phys. Rev. D **5** 2 (1995) 7168–7173, [[hep-th/9506048](#)].
- [58] M. Cvetič, M. Dierigl, L. Lin, and H. Y. Zhang, *Gauge group topology of 8d chaudhuri-hockney-lykken vacua*, Phys. Rev. D **1** 04 (Oct, 2021) 086018.
- [59] R. E. Borcherds, *The moduli space of enriques surfaces and the fake monster lie superalgebra*, Topology **3** 5 (1996), no. 3 699–710.
- [60] V. Gritsenko, *Reflective modular forms in algebraic geometry*, 2010.
- [61] S. Kondō, *The moduli space of enriques surfaces and borcherds products*, Journal of Algebraic Geometry **1** 1 (2002), no. 4 601–627.
- [62] V. Gritsenko and K. Hulek, *Moduli of polarized enriques surfaces*, 2015.
- [63] A. Hauße-Waschbüsch and A. Krieg, *The hilbert modular group and orthogonal groups*, Research in Number Theory **8** (Jul, 2022) 47.
- [64] A. Dabholkar and J. Park, *Strings on orientifolds*, Nucl. Phys. B **4** 77 (1996) 701–714, [[hep-th/9604178](#)].
- [65] O. Aharony, Z. Komargodski, and A. Patir, *The Moduli space and M(atrrix) theory of 9d $N=1$ backgrounds of M/string theory*, JHEP **0** 5 (2007) 073, [[hep-th/0702195](#)].
- [66] H. Parra De Freitas, *New supersymmetric string moduli spaces from frozen singularities*, Journal of High Energy Physics **2** 023 (Jan, 2023) 170.
- [67] M. Montero and H. P. de Freitas, *New Supersymmetric String Theories from Discrete Theta Angles*, [arXiv:2209.03361](#).
- [68] P. Tourkine and P. Vanhove, *An R^4 non-renormalisation theorem in $\mathcal{N} = 4$ supergravity*, Class. Quant. Grav. **2** 9 (2012) 115006, [[arXiv:1202.3692](#)].
- [69] J. J. M. Carrasco, R. Kallosh, R. Roiban, and A. A. Tseytlin, *On the $U(1)$ duality anomaly and the S -matrix of $N=4$ supergravity*, JHEP **0** 7 (2013) 029, [[arXiv:1303.6219](#)].
- [70] R. Kallosh and Y. Yamada, *Deformation of $d=4$, $n > 4$ supergravities breaks nonlinear local supersymmetry*, 2023.
- [71] D. Persson and R. Volpato, *Fricke S -duality in CHL models*, JHEP **1** 2 (2015) 156, [[arXiv:1504.07260](#)].
- [72] M. J. Duff, R. Minasian, and E. Witten, *Evidence for heterotic / heterotic duality*, Nucl. Phys. B **4** 65 (1996) 413–438, [[hep-th/9601036](#)].
- [73] I. V. Melnikov, R. Minasian, and S. Theisen, *Heterotic flux backgrounds and their IIA duals*, JHEP **0** 7 (2014) 023, [[arXiv:1206.1417](#)].
- [74] G. Köhler, *Dedekind’s Eta Function and Modular Forms*, pp. 3–30. Springer Berlin Heidelberg, Berlin, Heidelberg, 2011.

- [75] G. Shimura, *On modular forms of half integral weight*, Annals of Mathematics **9** 7 (1973), no. 3 440–481.
- [76] J.-P. Serre and H. M. Stark, *Modular forms of weight $1/2$* , in Modular Functions of One Variable VI (J.-P. Serre and D. B. Zagier, eds.), (Berlin, Heidelberg), pp. 27–67, Springer Berlin Heidelberg, 1977.
- [77] M. Kneser, *Erzeugung ganzzahliger orthogonaler gruppen durch spiegelungen*, Mathematische Annalen **2** 55 (May, 1984) 453–462.
- [78] J. H. Bruinier and M. Bundschuh, *On borchers products associated with lattices of prime discriminant*, The Ramanujan Journal **7** (Mar, 2003) 49–61.
- [79] S. Mayer, Hilbert modular forms for the fields $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{17})$. PhD thesis, RWTH, Aachen, 2007. Aachen, Techn. Hochsch., Diss., 2007.
- [80] F. Hirzebruch and D. Zagier, *Intersection numbers of curves on hilbert modular surfaces and modular forms of nebentypus*, Inventiones Mathematicae **3** 6 (Dec, 1976) 57–113.