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Adding Semantics to Fuzzy Similarity Measures through the d-Choquet Integral

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Abstract. This paper introduces three classes of similarity measures for fuzzy description profiles, defined through the d-Choquet integral. Such classes of similarity measures are parameterized by the choice of a capacity and a restricted dissimilarity function, and generalize the classical Jaccard index for binary profiles. Semantics is added to such similarity measures on three different levels: (i) how common and different parts of profiles are aggregated (via the choice of the similarity functional form); (ii) how interactions among attributes are weighted (via the choice of the capacity); (iii) how pointwise dissimilarities are evaluated (via the choice of the restricted dissimilarity function).

Keywords: Fuzzy similarity measure · capacity · restricted dissimilarity function · d-Choquet integral.

1 Introduction

The recent trend of *eXplainable AI (XAI)* is based on decision models whose results can be interpreted by human agents, especially when high stake decisions are involved [27]. At the same time, *similarity measures* play a more and more prominent role in machine learning and decision support systems, since they capture the intuitive idea of “proximity”.

As is well-known, the most naive way to model similarity is to map object description profiles to elements of a metric space, and then rely on the underlying distance function. This approach is deeply tied to the nature of the available data and is often inconsistent with human reasoning, as acknowledged by Tversky, in his seminal work [29]. Therefore, during the last years, many similarity measures have been proposed (see, e.g., [23]), mainly focusing on the particular nature of data and on the properties required to a similarity measure [3, 11].

With XAI in view, the concept of similarity demands for a deeper semantics and understanding. In turn, this requires an investigation of the ordering structure induced by a particular similarity measure together with more complex functional forms, able to embody semantic concepts like attribute interactions.

Concerning the first issue, a series of papers (see, e.g., [2, 7–9]) coped with the understanding of the comparative nature of similarity (and dissimilarity) measures on fuzzy description profiles. On the other hand, the issue of modeling interactions has been considered in [1] for binary data, and then generalized in [10] for fuzzy data (see also [28]).

In this paper we extend the three classes of similarity measures introduced in [1, 10] by relying on the notion of *d-Choquet integral* [5]. The goal of our extension is to obtain a three-level semantics ruling: (i) aggregation of common and different parts of profiles; (ii) interactions among attributes; (iii) evaluation of pointwise dissimilarities. Hence, we get three classes of similarity measures parameterized by a *capacity* ν and by a *restricted dissimilarity function* δ .

Since the most difficult part for obtaining an operative similarity measure belonging to such classes is the elicitation of ν , we face the learning of ν , by relying on the *Particle Swarm Optimization (PSO)* technique [21]. We also investigate the tuning of a parametric version of δ . This part of the paper provides some preliminary results inserting in the literature of *similarity learning* (see, e.g., [14, 26]).

Choosing ν , δ and one of the proposed functional forms of similarity measure that maximize accuracy in a classification problem, we can obtain an interpretation in terms of the three levels of semantics recalled above. In particular, the Möbius inverse of the learned ν can be seen as a witness of attribute interactions that can be, in principle, either positive or negative, since ν is a capacity.

The paper is structured as follows. Section 2 recalls the necessary material on the d-Choquet integral. Section 3 introduces the three families of similarity measures based on the d-Choquet integral, and investigates their properties. Section 4 addresses the problem of similarity learning through the PSO technique. Finally, Section 5 collects our conclusions and future perspectives.

2 Preliminaries

Following [5], a function $\delta : [0, 1]^2 \rightarrow [0, 1]$ is called a *restricted dissimilarity function* if it satisfies, for all $x, y, z \in [0, 1]$, the following conditions:

1. $\delta(x, y) = \delta(y, x)$;
2. $\delta(x, y) = 1$ if and only if $\{x, y\} = \{0, 1\}$;
3. $\delta(x, y) = 0$ if and only if $x = y$;
4. if $x \leq y \leq z$, then $\delta(x, y) \leq \delta(x, z)$ and $\delta(y, z) \leq \delta(x, z)$.

The prototypical example of a restricted dissimilarity function is

$$\delta_{1,1}(x, y) = |x - y|, \quad (1)$$

and other functions of this type can be generated via $[0, 1]$ -automorphisms. We recall that a function $\varphi : [0, 1] \rightarrow [0, 1]$ is a $[0, 1]$ -*automorphism* if it is continuous, strictly increasing and such that $\varphi(0) = 0$ and $\varphi(1) = 1$.

Given two (possibly distinct) $[0, 1]$ -automorphisms φ_1, φ_2 , then the function

$$\delta_{\varphi_1, \varphi_2}(x, y) = \varphi_1^{-1}(|\varphi_2(x) - \varphi_2(y)|), \quad (2)$$

is a restricted dissimilarity function [5]. In particular, in what follows we will restrict to the case $\varphi_1(x) = x^q$ and $\varphi_2(x) = x^p$, for $p, q \in (0, +\infty)$, in which case (2) reduces to

$$\delta_{p,q}(x, y) = |x^p - y^p|^{\frac{1}{q}}, \quad (3)$$

that has (1) as particular case for $p = q = 1$. In this work, due to space limitations, we will analyze only the cases $\delta_{p,p}$, $\delta_{1,p}$, $\delta_{p,1}$, parameterized by $p \in (0, +\infty)$.

Let $N = \{1, \dots, n\}$ be endowed with the power set 2^N . As is well-known (see, e.g., [16]), a (*normalized*) *capacity* is a set function $\nu : 2^N \rightarrow [0, 1]$ satisfying:

- (i) $\nu(\emptyset) = 0$ and $\nu(N) = 1$;
- (ii) $\nu(A) \leq \nu(B)$ when $A \subseteq B$, for all $A, B \in 2^N$.

Moreover, every capacity ν is associated with a set function $\mu : 2^N \rightarrow \mathbb{R}$ called *Möbius inverse* such that, for all $A \in 2^N$, it holds that

$$\mu(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \nu(B) \quad \text{and} \quad \nu(A) = \sum_{B \subseteq A} \mu(B).$$

As shown in [6], a function $\mu : 2^N \rightarrow \mathbb{R}$ is the Möbius inverse of a capacity ν , if and only if it satisfies:

- (i) $\mu(\emptyset) = 0$;
- (ii) $\sum_{B \in 2^N} \mu(B) = 1$;
- (iii) $\sum_{\{i\} \subseteq B \subseteq A} \mu(B) \geq 0$, for all $A \in 2^N$ and all $i \in A$.

The above properties imply that $\mu(\{i\}) \geq 0$, for all $i \in N$. Moreover, if $\mu(B) \geq 0$, for all $B \in 2^N$, then the corresponding ν is a *completely monotone capacity* [16].

A capacity ν is then called *k-additive* (with $1 \leq k \leq n$) if $\mu(A) = 0$, for all $A \in 2^N$ with $|A| > k$, and there exists $A \in 2^N$ with $|A| = k$ such that $\mu(A) \neq 0$ [15]. In particular, a 1-additive capacity reduces to a *probability measure*. In what follows, we denote by ν_u the *uniform probability measure* such that $\nu_u(\{i\}) = \frac{1}{n}$, for all $i \in N$, whose Möbius inverse is $\mu_u(\{i\}) = \frac{1}{n}$, for all $i \in N$, and 0 otherwise.

In the context of similarity measures, ν can be seen as a non-additive weighting function related to a set of fuzzy attributes indexed by N . Under this interpretation, the Möbius inverse μ is the actual weight attached to every set of attributes, allowing for modeling (positive or negative) interactions among fuzzy attributes. With this meaning in view, in [1, 10] μ has been called a *significance assessment*.

We recall the notion of d-Choquet integral introduced in [5].

Definition 1. Let $\nu : 2^N \rightarrow [0, 1]$ be a capacity and $\delta : [0, 1]^2 \rightarrow [0, 1]$ be a restricted dissimilarity function. The **d-Choquet integral** with respect to ν and δ is the functional $\mathbb{C}_{\nu, \delta} : [0, 1]^N \rightarrow [0, n]$ defined, for all $X \in [0, 1]^N$, as

$$\mathbb{C}_{\nu, \delta}(X) = \sum_{i=1}^n \delta(X(\sigma(i)), X(\sigma(i-1))) \nu(\{\sigma(i), \dots, \sigma(n)\}),$$

where σ is a permutation of N such that $X(\sigma(1)) \leq \dots \leq X(\sigma(n))$ and $X(\sigma(0)) := 0$. In particular, if $X \in \{0, 1\}^N$, then X reduces to the indicator $\mathbf{1}_A$ of a subset A of N , and so $\mathbb{C}_{\nu, \delta}(\mathbf{1}_A) = \nu(A)$.

Though $\mathbb{C}_{\nu,\delta}(\mathbf{1}_\emptyset) = 0$ and $\mathbb{C}_{\nu,\delta}(\mathbf{1}_N) = 1$, for any choice of ν and δ , we have that for some choices of δ , $\mathbb{C}_{\nu,\delta}$ can take values greater than 1. Nevertheless, taking $\delta_{p,p}$ with $0 < p \leq 1$, $\mathbb{C}_{\nu,\delta_{p,p}}$ ranges in $[0, 1]$ for any choice of ν [5]. In particular, for $p = 1$, $\mathbb{C}_{\nu,\delta_{1,1}}$ reduces to the classical Choquet integral.

The following proposition investigates when $\mathbb{C}_{\nu,\delta}$ is null, assuming a strictly positive ν on $2^N \setminus \{\emptyset\}$, i.e., satisfying the property:

(P) $\nu(A) > 0$, for all $A \in 2^N \setminus \{\emptyset\}$.

Proposition 1. *If ν satisfies (P), then $\mathbb{C}_{\nu,\delta}(X) = 0$ if and only if $X = \mathbf{1}_\emptyset$.*

Proof. By Definition 1 we have that $\mathbb{C}_{\nu,\delta}(X)$ is a weighted sum where all weights $\nu(\{\sigma(i), \dots, \sigma(n)\})$'s are strictly positive and all terms $\delta(X(\sigma(i)), X(\sigma(i-1)))$'s are non-negative. Thus, $\mathbb{C}_{\nu,\delta}(X) = 0$ if and only if $\delta(X(\sigma(i)), X(\sigma(i-1))) = 0$, for $i = 1, \dots, n$. Finally, by property 3 of restricted dissimilarity functions, we get that $\delta(X(\sigma(i)), X(\sigma(i-1))) = 0$ if and only if $X(\sigma(i)) = X(\sigma(i-1))$, for $i = 1, \dots, n$, and since $X(\sigma(0)) := 0$, this is equivalent to $X = \mathbf{1}_\emptyset$. \square

Let us notice that $\mathbb{C}_{\nu,\delta}$ is generally not monotone on $[0, 1]^N$ endowed with the partial order \leq such that $X \leq Y$ if and only if $X(i) \leq Y(i)$, for all $i \in N$, with $X, Y \in [0, 1]^N$. Theorem 4.8 in [5] states that monotonicity of $\mathbb{C}_{\nu,\delta}$ is equivalent to the following condition for δ :

(M) $\delta(0, x_1) + \delta(x_1, x_2) + \dots + \delta(x_{m-1}, x_m) \leq \delta(0, y_1) + \delta(y_1, y_2) + \dots + \delta(y_{m-1}, y_m)$
for all $1 \leq m \leq n$ and $x_1, \dots, x_m, y_1, \dots, y_m \in [0, 1]$ where $x_i \leq x_j$, $y_i \leq y_j$, $x_i \leq y_i$, with $1 \leq i \leq j \leq m$.

The following example, that will be developed in the following section, shows that taking $\delta = \delta_{p,p}$ or $\delta = \delta_{1,p}$, $\mathbb{C}_{\nu,\delta}$ may fail monotonicity, even in the case ν is a probability measure. We point out that the lack of monotonicity of the d-Choquet integral is already discussed in Example 4.13 in [5].

Example 1. Let $N = \{1, 2, 3\}$. Take the uniform probability measure ν_u , and $X, Y, X', Y' \in [0, 1]^N$ such that

$$\begin{array}{c|ccc} N & 1 & 2 & 3 \\ \hline X & 0 & 0.6 & 0.8 \\ Y & 0.2 & 0.8 & 1 \end{array} \quad \text{and} \quad \begin{array}{c|ccc} N & 1 & 2 & 3 \\ \hline X' & 0 & 0.1 & 0.9 \\ Y' & 0.1 & 0.6 & 1 \end{array}$$

For $\delta = \delta_{\frac{1}{2}, \frac{1}{2}}$ we have that $X \leq Y$ but

$$\begin{aligned} \mathbb{C}_{\nu_u, \delta_{\frac{1}{2}, \frac{1}{2}}}(X) &= \frac{0.6 + (\sqrt{0.8} - \sqrt{0.6})^2}{3} \\ &> \frac{0.2 + (\sqrt{0.8} - \sqrt{0.2})^2 + (1 - \sqrt{0.8})^2}{3} = \mathbb{C}_{\nu_u, \delta_{\frac{1}{2}, \frac{1}{2}}}(Y), \end{aligned}$$

while for $\delta = \delta_{1, \frac{1}{2}}$ we have that $X' \leq Y'$ but

$$\mathbb{C}_{\nu_u, \delta_{1, \frac{1}{2}}}(X') = \frac{0.1^2 + 0.8^2}{3} > \frac{0.1^2 + 0.5^2 + 0.4^2}{3} = \mathbb{C}_{\nu_u, \delta_{1, \frac{1}{2}}}(Y').$$

◆

On the other hand, taking $\delta = \delta_{p,1}$, $\mathbb{C}_{\nu,\delta}$ is always monotone (see [5]), as $\delta_{p,1}$ satisfies **(M)**.

3 Fuzzy d-Choquet Similarity Measures

We assume that every object is described by a set of attributes indexed by the finite set $N = \{1, \dots, n\}$, and that each one can be present with a different degree of membership: any object description is thus regarded as a fuzzy subset of N [30]. In order to avoid cumbersome notation, every fuzzy subset X of N is identified with its *membership function*, so, we simply denote it as a function $X : N \rightarrow [0, 1]$. Denote by $\mathcal{F} = [0, 1]^N$ the set of all possible fuzzy object descriptions and by $\mathcal{C} = \{0, 1\}^N$ the subset of crisp object descriptions.

We consider a t-norm T together with its dual t-conorm S and the complement $(\cdot)^c = 1 - (\cdot)$ to perform fuzzy set-theoretic operations. As usual (see [22]), we denote the main t-norms and t-conorms, for every $x, y \in [0, 1]$, as

$$\begin{aligned} T_M(x, y) &= \min\{x, y\}, & S_M(x, y) &= \max\{x, y\}, \\ T_P(x, y) &= x \cdot y, & S_P(x, y) &= x + y - x \cdot y, \\ T_L(x, y) &= \max\{x + y - 1, 0\}, & S_L(x, y) &= \min\{x + y, 1\}. \end{aligned}$$

For every $X, Y \in \mathcal{F}$, we define $X \cap Y = T(X, Y)$, $X \setminus Y = T(X, Y^c)$, $Y \setminus X = T(Y, X^c)$, $X \Delta Y = S(X \setminus Y, Y \setminus X)$ and $X \cup Y = S(X, Y)$, where all operations are intended pointwise on the elements of N . All t-norms and t-conorms extend uniquely to k -ary operations, for $k \geq 2$, due to associativity [22], and so do the corresponding fuzzy set-theoretic operations.

Different definitions of similarities have been given for fuzzy subsets [2, 12, 13] essentially based on the “common” and the “different” parts of the compared fuzzy subsets.

We introduce three classes of similarity measures $\mathbf{S}_i^{\nu, \delta} : \mathcal{F}^2 \rightarrow [0, +\infty)$, for $i = 1, 2, 3$, each parameterized by a capacity ν and by a restricted dissimilarity function δ , defined, for every $X, Y \in \mathcal{F}$, as:

$$\mathbf{S}_1^{\nu, \delta}(X, Y) = \frac{\mathbb{C}_{\nu, \delta}(X \cap Y)}{\mathbb{C}_{\nu, \delta}(X \setminus Y) + \mathbb{C}_{\nu, \delta}(Y \setminus X) + \mathbb{C}_{\nu, \delta}(X \cap Y)}, \quad (4)$$

$$\mathbf{S}_2^{\nu, \delta}(X, Y) = \frac{\mathbb{C}_{\nu, \delta}(X \cap Y)}{\mathbb{C}_{\nu, \delta}(X \Delta Y) + \mathbb{C}_{\nu, \delta}(X \cap Y)}, \quad (5)$$

$$\mathbf{S}_3^{\nu, \delta}(X, Y) = \frac{\mathbb{C}_{\nu, \delta}(X \cap Y)}{\mathbb{C}_{\nu, \delta}(X \cup Y)}. \quad (6)$$

Taking $\nu = \nu_u$ and $\delta = \delta_{1,1}$, the restrictions of $\mathbf{S}_i^{\nu, \delta}(X, Y)$ on \mathcal{C} , for $i = 1, 2, 3$, reduce to the classical *Jaccard's index* [18]. More generally, for a probability measure ν and $\delta = \delta_{1,1}$ we get a weighted version of the Jaccard's index [1].

The similarity measures $\mathbf{S}_i^{\nu, \delta}(X, Y)$, for $i = 1, 2, 3$, embody three levels of semantics:

- (i) The choice of the functional form $\mathbf{S}_i^{\nu, \delta}$ implies how common and different parts of fuzzy profiles are aggregated: in the particular case $\mathbf{S}_1^{\nu, \delta}$, we get a symmetric fuzzy version of Tversky's contrast model [29].

- (ii) The choice of the capacity ν expresses how interactions among attributes are weighted: the corresponding Möbius inverse μ acts as a significance assessment that allows for positive or negative interactions.
- (iii) The choice of the restricted dissimilarity function δ encodes how point-wise dissimilarities are evaluated: choosing one of the parametric forms $\delta_{p,p}, \delta_{1,p}, \delta_{p,1}$, a tuning on sample similarity comparisons can be performed.

The following proposition shows that, assuming a capacity ν which satisfies **(P)** and $T = T_M$, the ratios in (4)–(6) are always well-defined, except for the case $X = Y = \mathbf{1}_\emptyset$. In this limit case, we set $\mathbf{S}_i^{\nu,\delta}(\mathbf{1}_\emptyset, \mathbf{1}_\emptyset) := 1$, for $i = 1, 2, 3$.

Proposition 2. *Let ν satisfying **(P)**, δ an arbitrary restricted dissimilarity function, and $T = T_M$. Then, the denominator of $\mathbf{S}_i^{\nu,\delta}$, for $i = 1, 2, 3$, is 0 if and only if $X = Y = \mathbf{1}_\emptyset$.*

Proof. If $X = Y = \mathbf{1}_\emptyset$, then we immediately get that all denominators are 0. We prove the converse implication for each similarity measure.

(Measure $\mathbf{S}_1^{\nu,\delta}$). By Proposition 1, $\mathbb{C}_{\nu,\delta}(X \setminus Y) + \mathbb{C}_{\nu,\delta}(Y \setminus X) + \mathbb{C}_{\nu,\delta}(X \cap Y) = 0$ if and only if $X \setminus Y = Y \setminus X = X \cap Y = \mathbf{1}_\emptyset$. This is equivalent, for all $i \in N$, to $T_M(X(i), 1 - Y(i)) = T_M(Y(i), 1 - X(i)) = T_M(X(i), Y(i)) = 0$, that implies $X(i) = Y(i) = 0$.

(Measure $\mathbf{S}_2^{\nu,\delta}$). By Proposition 1, $\mathbb{C}_{\nu,\delta}(X \Delta Y) + \mathbb{C}_{\nu,\delta}(X \cap Y) = 0$ if and only if $X \Delta Y = X \cap Y = \mathbf{1}_\emptyset$. This is equivalent, for all $i \in N$, to $S_M(T_M(X(i), 1 - Y(i)), T_M(Y(i), 1 - X(i))) = T_M(X(i), Y(i)) = 0$, that implies $X(i) = Y(i) = 0$.

(Measure $\mathbf{S}_3^{\nu,\delta}$). By Proposition 1, $\mathbb{C}_{\nu,\delta}(X \cup Y) = 0$ if and only if $X \cup Y = \mathbf{1}_\emptyset$. This is equivalent, for all $i \in N$, to $S_M(X(i), Y(i)) = 0$, that implies $X(i) = Y(i) = 0$. \square

In light of Proposition 2, we will assume that ν satisfies **(P)** throughout the paper. In turn, this implies that the Möbius inverse of ν is such that $\mu(\{i\}) > 0$, for all $i \in N$, which has a semantic interpretation. Indeed, this last requirement can be justified by interpreting μ as a significance assessment: all attributes included in a description profile should be “significant”, i.e., μ should attach to them a positive weight.

Proposition 3. *Let ν satisfying **(P)**, $\delta \in \{\delta_{p,p}, \delta_{1,p}, \delta_{p,1}\}$, and $T = T_M$. Then, the following properties hold for all $X, Y \in \mathcal{F}$:*

- (i) $\mathbf{S}_i^{\nu,\delta}(X, Y) \leq 1$, for $i = 1, 2$;
- (ii) $\mathbf{S}_3^{\nu,\delta}(X, Y) \leq 1$, if $\delta = \delta_{p,1}$;
- (iii) $\mathbf{S}_3^{\nu,\delta}(X, X) = 1$;
- (iv) $\mathbf{S}_i^{\nu,\delta}(X, Y) = 0$ if and only if $X \cap Y = \mathbf{1}_\emptyset \neq X \cup Y$, for $i = 1, 2, 3$;
- (v) $\mathbf{S}_i^{\nu,\delta}(X, Y) = \mathbf{S}_i^{\nu,\delta}(Y, X)$, for $i = 1, 2, 3$.

Proof. The proof immediately follows by (4)–(6), and Propositions 1 and 2. \square

The following example shows that $\mathbf{S}_3^{\nu,\delta}(X, Y)$ can take values greater than 1 for $\delta = \delta_{p,p}$ or $\delta = \delta_{1,p}$.

Example 2. Let N, X, Y, X', Y', ν_u , and $\delta_{\frac{1}{2}, \frac{1}{2}}, \delta_{1, \frac{1}{2}}$ be as in Example 1. Since $X \leq Y$ and $X' \leq Y'$, taking $T = T_M$, we get that $X \cap Y = X$, $X \cup Y = Y$, $X' \cap Y' = X'$, and $X' \cup Y' = Y'$ thus

$$\begin{aligned} \mathbf{S}_3^{\nu, \delta_{\frac{1}{2}, \frac{1}{2}}}(X, Y) &= \frac{0.6 + (\sqrt{0.8} - \sqrt{0.6})^2}{0.2 + (\sqrt{0.8} - \sqrt{0.2})^2 + (1 - \sqrt{0.8})^2} > 1, \\ \mathbf{S}_3^{\nu, \delta_{1, \frac{1}{2}}}(X', Y') &= \frac{0.1^2 + 0.8^2}{0.1^2 + 0.5^2 + 0.4^2} > 1. \end{aligned}$$

◆

We notice that, the restrictions of $\mathbf{S}_i^{\nu, \delta}$ to \mathcal{C} , for $i = 1, 2, 3$, coincide with the similarity measures defined in [1], for any choice of δ and a completely monotone ν . On the other hand, if we take $\delta = \delta_{1,1}$, then $\mathbf{S}_i^{\nu, \delta_{1,1}}$, for $i = 1, 2, 3$, coincide with the similarity measures defined in [10]. This implies that, in general, $\mathbf{S}_i^{\nu, \delta}(X, X) < 1$, for $i = 1, 2$. In the particular case $\delta = \delta_{1,1}$ and ν is a probability measure, $\mathbf{S}_3^{\nu, \delta}$ is a special case of the similarity measure introduced in [28].

As a by-product, taking $\delta = \delta_{1,1}$, by [10] we derive that $\mathbf{S}_i^{\nu, \delta}$, for $i = 1, 2, 3$, do not generally satisfy T' -transitivity, where T' is a t-norm possibly different from the t-norm T used in the fuzzy set-theoretic operations, i.e., the property:

$$(\mathbf{T}) \quad \mathbf{S}_i^{\nu, \delta}(X, Z) \geq T'(\mathbf{S}_i^{\nu, \delta}(X, Y), \mathbf{S}_i^{\nu, \delta}(Y, Z)), \text{ for all } X, Y, Z \in \mathcal{F}.$$

We notice that for $\delta = \delta_{p,p}$ or $\delta = \delta_{1,p}$ property **(T)** does not make sense for $\mathbf{S}_3^{\nu, \delta}$, since it may take values greater than 1.

In the case $\delta = \delta_{1,1}$ and ν is a probability measure, in [10, 28] it is shown that $\mathbf{S}_3^{\nu, \delta}$ is T_L -transitive. The following proposition shows that T_L -transitivity holds also when $\delta = \delta_{p,1}$

Proposition 4. *If ν satisfies **(P)** and is additive, $T = T_M$ and $\delta = \delta_{p,1}$, then the similarity measure $\mathbf{S}_3^{\nu, \delta}$ satisfies **(T)** with $T' = T_L$.*

Proof. The proof is an immediate modification of the proof of Proposition 1 in [10] (see also [28]). We first notice that $\mathbb{C}_{\nu, \delta_{p,1}}$ is monotone and, for all $X \in \mathcal{F}$, denoting by X^p the element of \mathcal{F} such that $X^p(i) = (X(i))^p$, for all $i \in N$, it holds that $\mathbb{C}_{\nu, \delta_{p,1}}(X) = \sum_{i=1}^n (X(i))^p \nu(\{i\}) = \mathbb{C}_{\nu, \delta_{1,1}}(X^p)$. Moreover, since $\varphi_2(x) = x^p$ is strictly increasing, and \cap and \cup refer to T_M and S_M , respectively, it holds that $X^p \cap Y^p = (X \cap Y)^p$ and $X^p \cup Y^p = (X \cup Y)^p$.

For all $X, Y, Z \in \mathcal{F}$, it is sufficient to show that

$$\mathbf{S}_3^{\nu, \delta_{p,1}}(X, Z) + 1 \geq \mathbf{S}_3^{\nu, \delta_{p,1}}(X, Y) + \mathbf{S}_3^{\nu, \delta_{p,1}}(Y, Z).$$

Setting $c = \mathbb{C}_{\nu, \delta_{p,1}}(X \cup Y \cup Z) - \mathbb{C}_{\nu, \delta_{p,1}}(X \cup Y)$ and $c' = \mathbb{C}_{\nu, \delta_{p,1}}(X \cup Y \cup Z) - \mathbb{C}_{\nu, \delta_{p,1}}(Y \cup Z)$ we get that

$$\mathbf{S}_3^{\nu, \delta_{p,1}}(X, Y) \leq \frac{\mathbb{C}_{\nu, \delta_{p,1}}(X \cap Y) + c}{\mathbb{C}_{\nu, \delta_{p,1}}(X \cup Y) + c} \quad \text{and} \quad \mathbf{S}_3^{\nu, \delta_{p,1}}(Y, Z) \leq \frac{\mathbb{C}_{\nu, \delta_{p,1}}(Y \cap Z) + c'}{\mathbb{C}_{\nu, \delta_{p,1}}(Y \cup Z) + c'}.$$

Therefore, we obtain

$$\begin{aligned}
\mathbf{S}_3^{\nu, \delta_{p,1}}(X, Y) + \mathbf{S}_3^{\nu, \delta_{p,1}}(Y, Z) &\leq \frac{\mathbb{C}_{\nu, \delta_{p,1}}(X \cap Y) + c}{\mathbb{C}_{\nu, \delta_{p,1}}(X \cup Y) + c} + \frac{\mathbb{C}_{\nu, \delta_{p,1}}(Y \cap Z) + c'}{\mathbb{C}_{\nu, \delta_{p,1}}(Y \cup Z) + c'} \\
&= \frac{\mathbb{C}_{\nu, \delta_{p,1}}(X \cap Y) + c + \mathbb{C}_{\nu, \delta_{p,1}}(Y \cap Z) + c'}{\mathbb{C}_{\nu, \delta_{p,1}}(X \cup Y \cup Z)} \\
&\leq \frac{\mathbb{C}_{\nu, \delta_{p,1}}(X \cap Y) + c + \mathbb{C}_{\nu, \delta_{p,1}}(Y \cap Z) + c'}{\mathbb{C}_{\nu, \delta_{p,1}}(X \cup Z)} \\
&\leq \frac{\mathbb{C}_{\nu, \delta_{p,1}}(X \cup Z) + \mathbb{C}_{\nu, \delta_{p,1}}(X \cap Z)}{\mathbb{C}_{\nu, \delta_{p,1}}(X \cup Z)} \\
&= 1 + \mathbf{S}_3^{\nu, \delta_{p,1}}(X, Z),
\end{aligned}$$

where the last inequality follows since, for all $i \in N$, we have

$$\begin{aligned}
&T_M(X^p(i), Y^p(i)) - S_M(X^p(i), Y^p(i)) \\
&\quad + T_M(Y^p(i), Z^p(i)) - S_M(Y^p(i), Z^p(i)) \\
&\quad + 2S_M(X^p(i), Y^p(i), Z^p(i)) \leq S_M(X^p(i), Z^p(i)) + T_M(X^p(i), Z^p(i)),
\end{aligned}$$

that holds for all the possible orderings of $X^p(i), Y^p(i), Z^p(i)$. Indeed, if $X^p(i) \geq Z^p(i) \geq Y^p(i)$, then we get $X^p(i) + Y^p(i) - (Z^p(i) - Y^p(i)) \leq X^p(i) + Z^p(i)$ and if $Z^p(i) \geq X^p(i) \geq Y^p(i)$, then we get $Y^p(i) - (X^p(i) - Y^p(i)) + Z^p(i) \leq X^p(i) + Z^p(i)$. While, in all the remaining cases we get $X^p(i) + Z^p(i) \leq X^p(i) + Z^p(i)$. \square

The study of similarity measures appears to be of particular importance since it helps to improve predictions by providing a transparent understanding of the reasoning behind a forecast and helps to make interpretable decisions and implement XAI.

4 Similarity Learning

The three similarity measures $\mathbf{S}_i^{\nu, \delta}$, for $i = 1, 2, 3$, essentially rely on the choice of ν and $\delta \in \{\delta_{p,p}, \delta_{p,1}, \delta_{1,p}\}$, for a suitable $p \in (0, +\infty)$. Surely, the most difficult part in getting an operative $\mathbf{S}_i^{\nu, \delta}$ is the elicitation of ν , due to its exponential size.

From a XAI point of view, learning ν is important since its Möbius inverse μ singles out the interactions between attributes which is, according to each choice of δ and functional form $\mathbf{S}_i^{\nu, \delta}$, tied to the choice of p . In the more general case of a capacity ν , negative interactions between groups of attributes are possible but the learning task is complicated by the set of constraints (i)–(iii) in Section 2, that restrict the feasible μ 's.

By learning the combination of δ , ν and $\mathbf{S}_i^{\delta, \nu}$ that maximizes accuracy in a classification problem, we get a model that gives us three levels of explanations: the chosen $\mathbf{S}_i^{\delta, \nu}$ tells us how common and different parts of profiles are

aggregated; the Möbius inverse μ of ν singles out interactions on the groups of attributes where it is different from zero; δ tells us how pointwise dissimilarities are evaluated and to which degree p . For a fixed functional form $\mathbf{S}_i^{\delta, \nu}$, the large number of parameters naturally raises the problem of identifiability, that has been recently addressed in learning Choquet functionals [4, 20]. The issue of identifiability is particularly relevant if the Möbius inverse μ of ν is taken as an indicator of interactions, therefore, a thorough investigation is planned for future research.

In this section we address the problem of learning the capacity ν and tuning the parameter $p \in (0, +\infty)$ for each $\mathbf{S}_i^{\nu, \delta}$, by relying on a set of labeled fuzzy description profiles. Due to the identifiability issue, we focus on the learning of the significance assessment μ corresponding to ν , by restricting to the case of a k -additive and completely monotone ν that satisfies **(P)**, and taking $T = T_M$. All the learning and calibration procedure is carried out in `Python 3.10`.

Due to space limitations, and since our aim is only to highlight the whole process, we refer to the `Iris` dataset, which is available in the `Kaggle` platform [19]. The dataset has been pre-processed, by normalizing attribute ranges in $[0, 1]$. The processed dataset has 4 attributes and a class label taking 3 possible values, with 150 rows.

Proceeding in analogy to [1], we perform a learning task executing a stratified 4-fold cross validation that splits the dataset in 4 balanced parts, namely $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$. For $h = 1, 2, 3, 4$, \mathcal{T}_h is taken as test set, while the union of the remaining three parts $\mathcal{D}_h = \bigcup_{k \neq h} \mathcal{T}_k$ is taken as training set. For $h = 1, 2, 3, 4$, we have that $\mathcal{D}_h = \{(X_1, y_1), \dots, (X_{N_h}, y_{N_h})\}$, where $X_j \in \mathcal{F} = [0, 1]^4$ is a fuzzy description profile, while y_j is the corresponding class.

For a fixed similarity $\mathbf{S}_i^{\nu, \delta}$, for $i = 1, 2, 3$, where $\delta \in \{\delta_{p,p}, \delta_{p,1}, \delta_{1,p}\}$, we define a *Nearest-Neighborhood (NN) classifier*: each fuzzy description profile $X_j \in \mathcal{D}_h$ is assigned to the class y_j^* solving the problem

$$(X_j^*, y_j^*) = \underset{(X_m, y_m) \in \mathcal{D}_h \setminus \{(X_j, y_j)\}}{\arg \max} \mathbf{S}_i^{\nu, \delta}(X_j, X_m).$$

Our aim is to find the significance assessment μ that maximises the *Leave-One-Out (LOO) objective function*

$$N_{LOO}(\mu) = |\{y_j : y_j = y_j^*, (X_j, y_j) \in \mathcal{D}_h\}|,$$

which counts the number of correctly classified instances.

The maximization of $N_{LOO}(\mu)$ in the space of non-negative Möbius inverses gives rise to a continuous optimization problem with a non-continuous objective function, thus classical optimization techniques cannot be used. Here, in analogy to [1], we adopt the *Particle Swarm Optimization (PSO)* technique, which is a stochastic incomplete method operating on a fixed number of candidate μ 's [21]. For the PSO implementation we refer to the `PySwarms` library [24] version 1.3.0. Since the search space is very large, we restrict to at most k -additive Möbius inverses, for $k = 1, 2$, and the optimization is carried on for 20 epochs. We further consider an initial set of 20 particles built as a 1-additive neighborhood of μ_u ,

obtained perturbing $\mu_u(\{i\}) = \frac{1}{n}$ with $(-1)^{i-1} \cdot \epsilon_i$, where $\epsilon_i \sim \mathbf{Unif}(0, \frac{1}{n})$, for all $i \in N$.

Once the optimal μ_h^* for the training set \mathcal{D}_h has been selected, accuracy is measured by computing $N_{LOO}(\mu_h^*)$ on \mathcal{T}_h and passing to percentages. We finally compute the average accuracy in the 4-folds, by referring to the four learned $\mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*$.

To justify the choice of ν_u as a reference, Figures 1a, 1c, and 1e show the average accuracy of a NN classifier, performed on the four folds $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$, using $\mathbf{S}_i^{\nu_u, \delta}$, for $i = 1, 2, 3$ and $\delta \in \{\delta_{p,p}, \delta_{p,1}, \delta_{1,p}\}$. To favor a comparison, we also report results for the *Euclidean* and the *cosine similarity measures*:

$$\mathbf{S}_E(X, Y) = 1 - \frac{1}{n} \sum_{i=1}^n (X(i) - Y(i))^2,$$

$$\mathbf{S}_C(X, Y) = \frac{\sum_{i=1}^n X(i)Y(i)}{\sqrt{\sum_{i=1}^n X(i)^2} \sqrt{\sum_{i=1}^n Y(i)^2}}.$$

We have that $\mathbf{S}_i^{\nu_u, \delta_{1,1}}$, for $i = 2, 3$, behaves better than \mathbf{S}_C , and better than \mathbf{S}_E , for $i = 3$, while $\mathbf{S}_1^{\nu_u, \delta}$ is always below \mathbf{S}_C , for all δ 's. The best performance is achieved by $\mathbf{S}_3^{\nu_u, \delta_{p,1}}$, for $p \geq 1$, which always dominates all other similarity measures (see Figure 1c), resulting in an average accuracy of more than 95%. In view of XAI, this suggests that the *Iris* dataset does not show strong interactions among the attributes that further seem to be equally significant, when the similarity is of the form $\mathbf{S}_3^{\nu_u, \delta_{p,1}}$, i.e., when the d-Choquet integral of the fuzzy union of the two compared profiles is taken in the denominator. In a sense, this also partially justifies the good behavior of \mathbf{S}_E in this dataset due to its metric properties, in which uniform weighting and no interactions are considered.

Results obtained with $\mathbf{S}_i^{\nu_u, \delta}$, for $i = 1, 2, 3$ and $\delta \in \{\delta_{p,p}, \delta_{p,1}, \delta_{1,p}\}$, serve as a benchmark, since the PSO learning procedure starts with ν_u in the initial set of particles. Figures 1b, 1d and 1f show the mean accuracy of the capacities obtained through PSO: solid lines refer to 1-additive capacities and dashed lines to at most 2-additive capacities. For a sake of robustness, due to the stochastic nature of PSO, Figures 1b, 1d and 1f report average values on 4 runs. The most evident effect of the learning procedure is for $\delta_{p,p}$ and $\delta_{1,p}$, while for $\delta_{p,1}$ we have a light improvement for $\mathbf{S}_1^{\nu_u, \delta_{p,1}}$ and for $\mathbf{S}_2^{\nu_u, \delta_{p,1}}$, while $\mathbf{S}_3^{\nu_u, \delta_{p,1}}$ shows some slight worsening for some values of p .

It is important to notice that PSO is an incomplete stochastic method thus, though ν_u is in the initial set of particles, the procedure could converge to sub-optimal solutions in the training test \mathcal{D}_h , that behave worse than ν_u on the test set \mathcal{T}_h . We also notice that, since we start from an initial population of 1-additive capacities, the optimal capacity in the at most 2-additive case could still be 1-additive, due to the slow rate of convergence of PSO and the large search space.

The analysis carried out in Figure 1 can be considered as a preliminary task used to choose the most suitable δ , p and $\mathbf{S}_i^{\nu_u, \delta}$. In view of this, for the *Iris*

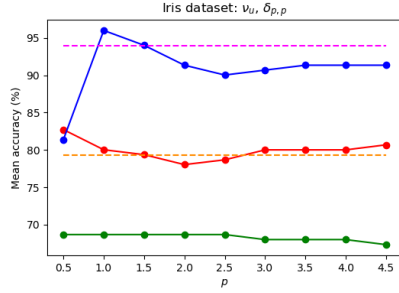
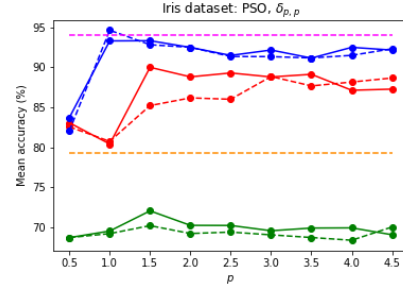
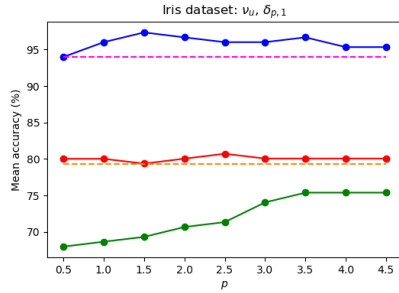
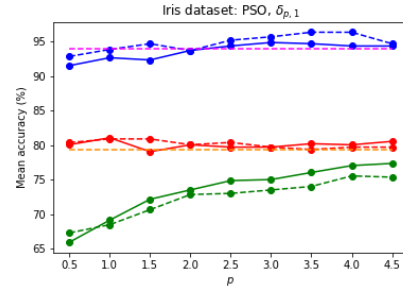
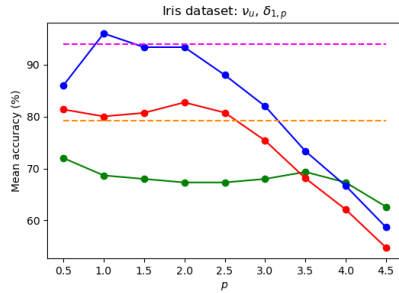
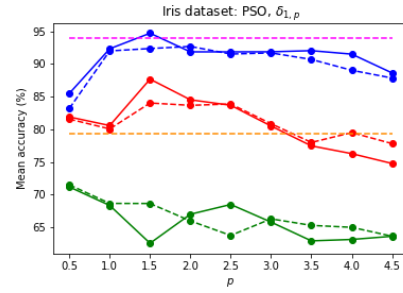
(a) Iris dataset: ν_u and $\delta_{p,p}$ (b) Iris dataset: PSO and $\delta_{p,p}$ (c) Iris dataset: ν_u and $\delta_{p,1}$ (d) Iris dataset: PSO and $\delta_{p,1}$ (e) Iris dataset: ν_u and $\delta_{1,p}$ (f) Iris dataset: PSO and $\delta_{1,p}$

Fig. 1: Mean accuracy (%) seen as a function of p , for $\delta \in \{\delta_{p,p}, \delta_{p,1}, \delta_{1,p}\}$: $\mathbf{S}_1^{\nu,\delta}$ in green; $\mathbf{S}_2^{\nu,\delta}$ in red; $\mathbf{S}_3^{\nu,\delta}$ in blue; \mathbf{S}_E in magenta; \mathbf{S}_C in orange. Capacity $\nu = \nu_u$ in (a), (c), (e); average values on 4 runs for the capacity found through PSO in (b), (d), (f): 1-additive in solid line; at most 2-additive in dashed line.

dataset, an optimal choice is $\delta = \delta_{p,1}$, $p = 3.5$ and $\mathbf{S}_3^{\nu,\delta}$. With this particular choice we can perform the PSO technique for a larger number of epochs or particles, so as to achieve better optimization results. Next, for the sake of interpretability, once $\mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*$ have been found for $\mathbf{S}_3^{\nu,\delta_{3,1}}$, we can look for the (not necessarily unique) μ_h^* that maximizes the average accuracy over all the test sets \mathcal{T}_k 's. Table 1 reports the optimal μ_1^* found in fold 1, still working with 20 particles in the at most 2-additive case, but considering 100 epochs. Such μ_1^* turns out to maximize the average accuracy over all the test sets \mathcal{T}_h 's, reaching 95.95%, so it has a good behavior on the whole dataset.

Attributes	{1}	{2}	{3}	{4}	{1, 2}	{1, 3}	{1, 4}	{2, 3}	{2, 4}	{3, 4}
μ_1^*	0.021	0.106	0.165	0.106	0.034	0.065	0.034	0.256	0.198	0.015

Table 1: Optimal Möbius inverse μ_1^* maximizing the average accuracy over all the test sets (3 decimals rounding).

5 Conclusion

We introduced three classes of similarity measures for fuzzy description profiles, based on the d-Choquet integral, the latter extending the Choquet integral by means of a dissimilarity function. The proposed similarities are parameterized by a capacity ν and a restricted dissimilarity function δ , conveying semantics on three different levels. In order to get an operative similarity measure belonging to one of such classes, the choice of ν and δ can be faced as a similarity learning problem. Due to the exponential size of ν , restrictions on its representation need to be considered, while a parametric version of δ translates in an ensuing tuning problem.

Here, we formulated the learning and tuning tasks relying on the PSO technique, and showed some preliminary results on a reference dataset. Our experimental analysis revealed the slow convergence rate of PSO joined by a very large search space. Future research will be devoted to a systematic experimental study involving several real and artificial datasets. Still in the experimental setting, the comparison with other incomplete stochastic methods, such as *Differential Evolution (DE)* [25], should be carried out so as evaluate rate of convergence and quality of the found solutions.

Concerning the learning of the classical Choquet integral, recent works developed deep learning techniques [4] and dedicated optimization techniques to face sparsity [17]. Though the quoted results are not directly applicable to the present learning task, their adaptation seems an interesting line of future research.

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References

1. Baiocchi, M., Coletti, G., Petturiti, D.: Weighted Attribute Combinations Based Similarity Measures. In: Greco, S., Bouchon-Meunier, B., Coletti, G., Fedrizzi, M., Matarazzo, B., Yager, R. (eds.) *Advances in Computational Intelligence*. pp. 211–220. Springer Berlin Heidelberg, Berlin, Heidelberg (2012)
2. Bouchon-Meunier, B., Coletti, G., Lesot, M.J., Rifqi, M.: Towards a Conscious Choice of a Fuzzy Similarity Measure: A Qualitative Point of View. In: Hüllermeier, E., Kruse, R., Hoffmann, F. (eds.) *Computational Intelligence for Knowledge-Based Systems Design*. pp. 1–10. Springer Berlin Heidelberg, Berlin, Heidelberg (2010)
3. Bouchon-Meunier, B., Rifqi, M., Bothorel, S.: Towards general measures of comparison of objects. *Fuzzy Sets and Systems* **84**(2), 143–153 (1996)
4. Bresson, R.: Neural learning and validation of hierarchical multi-criteria decision aiding models with interacting criteria. Ph.D. thesis, Université Paris-Saclay (2022)
5. Bustince, H., Mesiar, R., Fernandez, J., Galar, M., Paternain, D., Altalhi, A., Dimuro, G., Bedregal, B., Takáč, Z.: d-Choquet integrals: Choquet integrals based on dissimilarities. *Fuzzy Sets and Systems* **414**, 1–27 (2021)
6. Chateaufneuf, A., Jaffray, J.Y.: Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. *Mathematical Social Sciences* **17**(3), 263–283 (1989)
7. Coletti, G., Bouchon-Meunier, B.: A study of similarity measures through the paradigm of measurement theory: the classic case. *Soft Computing* **23**(16), 6827–6845 (2019)
8. Coletti, G., Bouchon-Meunier, B.: A study of similarity measures through the paradigm of measurement theory: the fuzzy case. *Soft Computing* **24**(15), 11223–11250 (2020)
9. Coletti, G., Petturiti, D., Bouchon-Meunier, B.: A Measurement Theory Characterization of a Class of Dissimilarity Measures for Fuzzy Description Profiles. In: Lesot, M.J., Vieira, S., Reformat, M., Carvalho, J., Wilbik, A., Bouchon-Meunier, B., Yager, R. (eds.) *Information Processing and Management of Uncertainty in Knowledge-Based Systems*. pp. 258–268. Springer International Publishing, Cham (2020)
10. Coletti, G., Petturiti, D., Vantaggi, B.: Fuzzy Weighted Attribute Combinations Based Similarity Measures. In: Antonucci, A., Cholvy, L., Papini, D. (eds.) *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*. pp. 364–374. Springer International Publishing, Cham (2017)
11. Couso, I., Garrido, L., Sánchez, L.: Similarity and dissimilarity measures between fuzzy sets: A formal relational study. *Information Sciences* **229**, 122–141 (2013)
12. De Baets, B., De Meyer, H.: Transitivity-preserving fuzzification schemes for cardinality-based similarity measures. *European Journal of Operational Research* **160**(3), 726–740 (2005)
13. De Baets, B., Janssens, S., De Meyer, H.: On the transitivity of a parametric family of cardinality-based similarity measures. *International Journal of Approximate Reasoning* **50**(1), 104–116 (2009)
14. Garcia, N., Vogiatzis, G.: Learning non-metric visual similarity for image retrieval. *Image and Vision Computing* **82**, 18–25 (2019)
15. Grabisch, M.: k-order additive fuzzy measures. In: *Proceedings of the 6th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU)*, Granada, Spain. pp. 1345–1350 (1996)

16. Grabisch, M.: *Set Functions, Games and Capacities in Decision Making*. Springer (2016)
17. Herin, M., Perny, P., Sokolovska, N.: Learning preference models with sparse interactions of criteria. In: *IJCAI 2023 - The 32nd International Joint Conference On Artificial Intelligence*, Macao, China (2023)
18. Jaccard, P.: Nouvelles recherches sur la distribution florale. *Bulletin de la Société Vaudoise des Sciences Naturelles* **44**, 223–270 (1908)
19. Kaggle: <https://www.kaggle.com>
20. Kaldjob, P.K., Mayag, B., Bouyssou, D.: Study of the Instability of the Sign of the Nonadditivity Index in a Choquet Integral Model. In: Ciucci, D., Couso, I., Medina, J., Ślęzak, D., Petturiti, D., Bouchon-Meunier, B., Yager, R. (eds.) *Information Processing and Management of Uncertainty in Knowledge-Based Systems*. pp. 197–209. Springer International Publishing, Cham (2022)
21. Kennedy, J., Eberhart, R.: *Swarm Intelligence*. Morgan Kaufmann (2001)
22. Klement, E., Mesiar, R., Pap, E.: *Triangular Norms*, Trends in Logic, vol. 8. Kluwer Academic Publishers, Dordrecht/Boston/London (2000)
23. Lesot, M.J., Rifqi, M., Benhadda, H.: Similarity Measures for Binary and Numerical Data: A Survey. *International Journal of Knowledge Engineering and Soft Data Paradigms* **1**(1), 63–84 (2009)
24. Miranda, L.J.V.: PySwarms, a research-toolkit for Particle Swarm Optimization in Python. *Journal of Open Source Software* **3** (2018)
25. Price, K., Storn, R., Lampinen, J.: *Differential Evolution: A Practical Approach to Global Optimization*. Springer (2005)
26. Rahnama, J., Hüllermeier, E.: Learning Tversky Similarity. In: Lesot, M.J., Vieira, S., Reformat, M., Carvalho, J., Wilbik, A., Bouchon-Meunier, B., Yager, R. (eds.) *Information Processing and Management of Uncertainty in Knowledge-Based Systems*. pp. 269–280. Springer International Publishing, Cham (2020)
27. Rudin, C.: Stop explaining black box machine learning models for high stakes decisions and use interpretable models instead. *Nature Machine Intelligence* **1**(5), 206–215 (2019)
28. Scozzafava, R., Vantaggi, B.: Fuzzy inclusion and similarity through coherent conditional probability. *Fuzzy Sets and Systems* **160**(3), 292–305 (2009)
29. Tversky, A.: Features of similarity. *Psychological Review* **84**(4), 327–352 (1977)
30. Zadeh, L.: Fuzzy sets. *Information and Control* **8**(3), 338–353 (1965)