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On slice measures of Green currents on $\mathbb{C}\mathbb{P}^2$

Christophe Dupont and Virgile Tapiero

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Abstract

Let f be a holomorphic map of $\mathbb{C}\mathbb{P}^2$ of degree $d \geq 2$, let T be its Green current and $\mu = T \wedge T$ be its equilibrium measure. We give a new proof of a theorem due to Dujardin asserting that $\mu \ll T \wedge \omega_{\mathbb{P}^2}$ implies $\lambda_2 = \frac{1}{2} \text{Log } d$, where $\lambda_1 \geq \lambda_2$ are the Lyapunov exponents of μ . Then, assuming $\mu \ll T \wedge \omega_{\mathbb{P}^2}$, we study slice measures $\nu := T \wedge dd^c|W|^2$, where W is a holomorphic local submersion. We give sufficient conditions on the Radon-Nikodym derivative of μ with respect to the trace measure $T \wedge \omega_{\mathbb{P}^2}$ ensuring $\mu = \nu$. The involved submersion W comes from normal coordinates for the inverse branches of the iterates of f .

Keywords— Holomorphic dynamics, Equilibrium measure, Green current, Lyapunov exponents, normal forms. *MSC 2020* : 32H50, 32U40, 37C40, 37D25

1 Introduction

Let f be a holomorphic map of \mathbb{P}^2 of degree $d \geq 2$. The Green current T and the equilibrium measure μ are invariant objects encoding the dynamical properties of f , we refer to the books [9], [19] by Dinh and Sibony. We recall that $T := \lim_n \frac{1}{d^n} f^{n*} \omega_{\mathbb{P}^2}$, where $\omega_{\mathbb{P}^2}$ is the normalized Fubini-Study $(1, 1)$ -form of \mathbb{P}^2 , this is a closed positive current, with local Hölder potentials. It satisfies $f^*T = dT$ and its auto-intersection $\mu := T \wedge T$ defines an invariant probability measure on \mathbb{P}^2 , which is mixing and satisfies $f^*\mu = d^2\mu$.

Berteloot-Loeb [5] proved that T is a smooth and non degenerate positive $(1, 1)$ -form on a non empty open subset of \mathbb{P}^2 if and only if f is a Lattès map. Berteloot-Dupont [1] established later that Berteloot-Loeb's condition on T characterizes the condition $\mu \ll \text{Leb}_{\mathbb{P}^2}$. Another characterization of Lattès maps involves the Lyapunov exponents $\lambda_1 \geq \lambda_2$ of μ (see Theorem 2.1 for their definitions). Briend-Duval [6] proved that $\lambda_2 \geq \frac{1}{2} \text{Log } d$. The equality $\lambda_1 = \lambda_2 = \frac{1}{2} \text{Log } d$ holds if and only if $\mu \ll \text{Leb}_{\mathbb{P}^2}$, see [12].

It is now natural to characterize the mappings satisfying $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d$. Hopefully, the presence of a minimal Lyapunov exponent should be equivalent to regular properties for T and μ , and perhaps (in some sense) to the existence of a one dimensional Lattès-like factor. The following result, due to Dujardin, nicely fits into this program.

Theorem 1.1 (Dujardin [10, Theorem 3.6]). *If $\mu \ll \sigma_T := T \wedge \omega_{\mathbb{P}^2}$ then $\lambda_2 = \frac{1}{2} \text{Log } d$.*

He also asked the question of the reverse implication, a partial answer is provided in [20]. The measure σ_T is called the trace of T and carries its mass. The proof of Dujardin's theorem is based on a construction of a df -invariant sub-bundle $\mathcal{T} \subset T\mathbb{P}^2$ (called Fatou directions) of rank ≥ 1 that satisfies for σ_T -almost every x and for any $\vec{v} \in \mathcal{T}_x \setminus \{0\}$:

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \text{Log} \|d_x f^n(\vec{v})\| \leq \frac{1}{2} \text{Log } d. \quad (1)$$

Indeed, using $\mu \ll \sigma_T$ and Oseledec's Theorem 2.1, for μ -almost every x , the lim sup in (1) tends to λ_1 or λ_2 , which implies in both cases $\lambda_2 \leq \frac{1}{2} \text{Log } d$. Briend-Duval inequality $\lambda_2 \geq \frac{1}{2} \text{Log } d$ then implies the equality. We note that these arguments only require that $\mu \ll \sigma_T$ on an open subset U charged by μ .

On a chart $U \subset \mathbb{P}^2$ equipped with holomorphic coordinates (Z, W) we have :

$$\sigma_T \asymp (T \wedge dd^c|Z|^2 + T \wedge dd^c|W|^2).$$

By construction $\text{Supp}(\mu) \subset \text{Supp}(\sigma_T)$, hence if $\mu(U) > 0$ then $(T \wedge dd^c|Z|^2)(U) > 0$ or $(T \wedge dd^c|W|^2)(U) > 0$. Actually in this case $T \wedge dd^c|Z|^2$ and $T \wedge dd^c|W|^2$ both charge U . The idea is that if $T \wedge dd^c|Z|^2 \equiv 0$ on U , then the potentials of T would be harmonic on almost every vertical disc contained in U , which implies that μ is null on U , see [13, §3.3].

The measures $T \wedge dd^c|Z|^2$ and $T \wedge dd^c|W|^2$ are called *slices* of T . Dujardin's theorem gives an information on λ_2 when $\mu \ll \sigma_T$, the proof is obtained by applying general results concerning Fatou directions [10]. Our purpose in this article is to analyse more deeply the relations between μ and slices of T when $\mu \ll \sigma_T$.

We first provide another proof of Dujardin's Theorem by using normal forms for the generic inverse branches of f^n and forward recurrent properties. We then provide another proof (of a weaker version of) Theorem 1.1 using backward recurrent properties (we indeed assume that the density of μ with respect to T is bounded). The interest of this second proof is to introduce a decomposition of μ using normal coordinates. This decomposition is the cornerstone of the proof of Theorem 1.2 that we now introduce.

Relations between μ and slices of T can be obtained when f preserves a pencil of lines, given for instance by the meromorphic function $\pi[z : w : t] = [z : w]$. Dupont-Taffin [14] proved that in this situation μ and T are related by the formula :

$$\mu = T \wedge \pi^* \mu_\theta \quad (\text{which implies } \mu \ll \sigma_T), \quad (2)$$

where θ is the rational map satisfying $\pi \circ f = \theta \circ \pi$. We note that Jonsson [15] previously established an analogous formula for polynomial skew products on \mathbb{C}^2 . On another hand, Berteloot-Loeb [4] proved that if θ is a Lattès map, then for every $a \in \mathbb{P}^1$ outside a finite subset, there exists a holomorphic coordinate ζ_a such that $\mu_\theta = dd^c|\zeta_a|^2$ on a neighborhood V_a of a (μ_θ is the equilibrium measure of θ). Combining this result with (2), we obtain on $\pi^{-1}(V_a)$:

$$\mu = T \wedge dd^c|W|^2, \text{ with } W := \zeta_a \circ \pi. \quad (3)$$

This gives an example where a slice of T is equal to μ . Theorem 1.2 shows a similar formula. We assume $\mu \ll \sigma_T$ and denote $\psi \in L^1(\sigma_T)$ the Radon-Nikodym derivative of μ with respect to σ_T . Let $J := \text{Supp}(\mu)$ and $\psi|_J$ be the restriction of ψ on J . Note that the measurable function ψ is defined on \mathbb{P}^2 , it is not unique since one can modify it outside J by still preserving $\mu = \psi \sigma_T$. However, one has $\mu = (\psi \mathbb{1}_J) \sigma_T$ on \mathbb{P}^2 for every version of ψ . We denote $\psi_\Omega := \psi \mathbb{1}_\Omega$ for every borel subset Ω of \mathbb{P}^2 . We also denote by $\hat{x} = (x_n)_{n \in \mathbb{Z}}$ the full orbits of the mapping f , by $\pi_0 : \hat{x} \mapsto x_0$ the projection map and by $\hat{\mu}$ the π_0 -pullback of μ on the set of full orbits, see Section 2.2. In the next statement, the local submersion $W_{\hat{x}} : B(x_0, \eta_\varepsilon(\hat{x})) \rightarrow \mathbb{C}$ is provided by the normal form Theorem 2.2.

Theorem 1.2. *Assume that $\mu = \psi_J \sigma_T$ for some measurable function ψ on \mathbb{P}^2 (in particular $\lambda_2 = \frac{1}{2} \text{Log } d$) and that λ_1 is strictly larger than λ_2 .*

1. If $\psi|_J$ is continuous, then for $\widehat{\mu}$ -almost every \widehat{x} there exist $r_1(\widehat{x}) \leq \eta_\varepsilon(\widehat{x})$ and $C_{\widehat{x}} > 0$ such that :

$$\mu = C_{\widehat{x}} (T \wedge dd^c |W_{\widehat{x}}|^2) |_J \text{ on } B(x_0, r_1(\widehat{x})).$$

2. If ψ is continuous on an open neighborhood V of J and if $\mu = \psi_V \sigma_T$, then there exist $r_2(\widehat{x}) \leq \eta_\varepsilon(\widehat{x})$ and $C_{\widehat{x}} > 0$ such that :

$$\mu = C_{\widehat{x}} T \wedge dd^c |W_{\widehat{x}}|^2 \text{ on } B(x_0, r_2(\widehat{x})).$$

The first Item implies that for $\widehat{\mu}$ -almost every \widehat{x}, \widehat{y} satisfying $\pi_0(\widehat{x}) = \pi_0(\widehat{y})$, the measures $(T \wedge dd^c |W_{\widehat{x}}|^2) |_J$ and $(T \wedge dd^c |W_{\widehat{y}}|^2) |_J$ coincide on a small ball centered at $\pi_0(\widehat{x})$. In particular, the germ of measure $(T \wedge dd^c |W_{\widehat{x}}|^2) |_J$ does not depend on the full orbit \widehat{x} but only on the π_0 -projection of \widehat{x} . A similar remark holds for the second Item.

Let us also observe that the hypothesis $\lambda_1 > \lambda_2$ in Theorem 1.2 is not restrictive. Assume indeed that $\mu = \psi_J \sigma_T$ and that $\lambda_1 = \lambda_2$. Then the Lyapunov exponents are both minimal (equal to $\frac{1}{2} \text{Log } d$) and f is a Lattès mapping on \mathbb{P}^2 . In this case there exists a finite ramified covering $\sigma : \mathbb{C}^2/\Lambda \rightarrow \mathbb{P}^2$ such that σ^*T is equal to a positive definite hermitian form H with constant coefficients (Λ being a cocompact real lattice in \mathbb{C}^2), see [11]. This implies that, around every point p outside an algebraic subset of \mathbb{P}^2 (the critical values of σ), the current T coincides with H in local coordinates given by σ . It follows that, for every local submersion W defined near p , the measure $\mu = T \wedge T$ is equal to $T \wedge dd^c |W|^2$ multiplied by a smooth positive function.

Let us finally note that the assumption of Item 2 morally replaces the function $\psi_J = \psi \mathbb{1}_J$ of the first Item (which is singular due to the multiplication by $\mathbb{1}_J$) by a continuous function on a neighborhood V of J . More precisely, it requires the existence of an open neighborhood V of J such that $\psi|_V$ is continuous and such that $\text{Supp}(\psi \sigma_T)|_V = J$. This assumption is fulfilled if one assumes that the restriction $\psi|_J$ is continuous and that $\text{Supp } T = J$, by applying Tietze-Urysohn theorem to $\psi|_J$.

Question 1.3. *Is it sufficient to assume $\psi|_J$ continuous to get $\mu = T \wedge dd^c |W_{\widehat{x}}|^2$ locally?*

We remark that the continuity of $\psi|_J$ is satisfied by suspensions of Lattès maps on \mathbb{P}^1 , we provide explicit computations of T, μ and ψ in Section 8.

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2 Classical results

2.1 Lyapunov exponents

Let f be a holomorphic mapping of \mathbb{P}^2 of degree $d \geq 2$. Let T be its Green current and $\mu = T \wedge T$ be its equilibrium measure, it is f -invariant and ergodic. Since T has local continuous psh potentials, μ integrates local psh functions. In particular μ does not charge analytic subsets of \mathbb{P}^2 and $\mu(\mathcal{C}) = 0$, where $\mathcal{C} := \bigcup_{n \in \mathbb{Z}} f^n(\text{Crit } f)$. It also allows to define the Lyapunov exponents $\lambda_1 \geq \lambda_2$:

$$\lambda_1 = \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbb{P}^2} \text{Log} \|df^n\| d\mu \text{ and } \lambda_1 + \lambda_2 = \int_{\mathbb{P}^2} \text{Log} |\det_{\mathbb{C}} df| d\mu.$$

Briend-Duval [6] proved that $\lambda_2 \geq \frac{1}{2} \text{Log } d$. The Oseledec theorem states as follows.

Theorem 2.1 (Oseledec).

1. If $\lambda_1 = \lambda_2 = \lambda$, then there exists an invariant borel subset A_{os} of full μ -measure and disjoint from \mathcal{C} such that for every $x \in A_{os}$:

$$\forall \vec{v} \in T_x \mathbb{P}^2 \setminus \{0\}, \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \text{Log} \|d_x f^n(\vec{v})\| = \lambda.$$

2. If $\lambda_1 > \lambda_2$, then there exists an invariant borel subset A_{os} of full μ -measure and disjoint from \mathcal{C} such that for every $x \in A_{os}$, there exists $v_s(x) \in \mathbb{P}(T_x \mathbb{P}^2)$ satisfying :

$$\forall \vec{v} \in T_x \mathbb{P}^2 \setminus v_s(x), \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \text{Log} \|d_x f^n(\vec{v})\| = \lambda_1.$$

$$\forall \vec{v} \in v_s(x) \setminus \{0\}, \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \text{Log} \|d_x f^n(\vec{v})\| = \lambda_2.$$

Moreover v_s is measurable and satisfies $[d_x f](v_s(x)) = v_s(f(x))$ for every $x \in A_{os}$, where $[d_x f]$ is the projectivization of the tangent map $d_x f$.

2.2 Normal forms for inverse branches

We denote $X := \mathbb{P}^2 \setminus \mathcal{C}$ which is totally invariant by f , and we denote \widehat{X} the set of all sequences $\widehat{x} = (x_n)_{n \in \mathbb{Z}}$ of elements of X that are orbits under the action of f i.e. $x_{n+1} = f(x_n)$. Equipped with the natural projection $\pi_0 : \widehat{X} \rightarrow X$ defined by $\pi_0(\widehat{x}) = x_0$, the dynamical system (X, f, μ) admits a natural extension $(\widehat{X}, \widehat{f}, \widehat{\mu})$, where \widehat{f} is the left shift on \widehat{X} and $\widehat{\mu}$ is the unique probability measure on \widehat{X} invariant by \widehat{f} such that $(\pi_0)_* \widehat{\mu} = \mu$. The measure $\widehat{\mu}$ is ergodic as μ , see [8, Chapter 10].

Given an orbit $\widehat{x} \in \widehat{X}$, there exists a family of inverse branches $(f_{\widehat{x}}^{-n})_{n \in \mathbb{N}}$ such that $f_{\widehat{x}}^{-n}$ is defined on $B(x_0, \eta_\varepsilon(\widehat{x}))$ for every $n \geq 0$, and such that $\text{Lip}(f_{\widehat{x}}^{-n}) \leq \beta_\varepsilon(\widehat{x}) e^{-n(\lambda_2 - \varepsilon)}$. The functions $\widehat{x} \mapsto \eta_\varepsilon(\widehat{x})$ and $\widehat{x} \mapsto \beta_\varepsilon(\widehat{x})$ are ε -tempered, i.e. they satisfy :

$$\forall n \in \mathbb{Z}, \quad e^{-|n|\varepsilon} \varphi(\widehat{x}) \leq \varphi(\widehat{f}^n(\widehat{x})) \leq e^{+|n|\varepsilon} \varphi(\widehat{x}).$$

We refer to Briend-Duval [6] (see also [7, 12]) for details about the construction of the inverse branches. The following result provides normal forms for those mappings, see [2, 3, 16].

Theorem 2.2. *For every $\varepsilon > 0$ small enough with respect to the Lyapunov exponents $\lambda_1 \geq \lambda_2$, there exists an invariant borel subset Λ_ε of full $\widehat{\mu}$ -measure, ε -tempered functions $\rho_\varepsilon : \Lambda_\varepsilon \rightarrow]0, 1]$, $\beta_\varepsilon, L_\varepsilon, M_\varepsilon : \Lambda_\varepsilon \rightarrow [1, +\infty[$ and a function $N : \Lambda_\varepsilon \rightarrow \mathbb{N}$ satisfying the following properties. For every $\widehat{x} \in \Lambda_\varepsilon$, there exist injective holomorphic mappings*

$$\xi_{\widehat{x}} : B(x_0, \eta_\varepsilon(\widehat{x})) \longrightarrow \mathbb{D}^2(\rho_\varepsilon(\widehat{x}))$$

satisfying :

1. $\xi_{\widehat{x}}(x_0) = 0$ and $d_{x_0} \xi_{\widehat{x}}(v_s(x_0))$ is the vertical axis in \mathbb{C}^2 ,
2. $\forall p, q \in B(x_0, \eta_\varepsilon(\widehat{x}))$, $\frac{1}{2} \text{dist}(p, q) \leq \|\xi_{\widehat{x}}(p) - \xi_{\widehat{x}}(q)\| \leq \beta_\varepsilon(\widehat{x}) \text{dist}(p, q)$,
3. the following diagram commutes for every $n \geq N(\widehat{x})$:

$$\begin{array}{ccc} B(x_{-n}, \eta_\varepsilon(\widehat{x}_{-n})) & \xleftarrow{f_{\widehat{x}}^{-n}} & B(x_0, \eta_\varepsilon(\widehat{x})) \\ \xi_{\widehat{x}_{-n}} \downarrow & & \downarrow \xi_{\widehat{x}} \\ \mathbb{D}^2(\rho_\varepsilon(\widehat{x}_{-n})) & \xleftarrow{R_{n, \widehat{x}}} & \mathbb{D}^2(\rho_\varepsilon(\widehat{x})) \end{array} \quad (4)$$

The mappings $R_{n,\hat{x}}$ have the following form depending on (λ_1, λ_2) :

(i) If $\lambda_1 = \lambda_2 = \lambda$, then $R_{n,\hat{x}}$ is a linear and satisfies

$$e^{-n(\lambda+\varepsilon)}\|(z, w)\| \leq \|R_{n,\hat{x}}(z, w)\| \leq e^{-n(\lambda-\varepsilon)}\|(z, w)\|.$$

(ii) If $\lambda_1 = k\lambda_2$ for some $k \geq 2$ (we say that λ_1 and λ_2 are resonant), then $R_{n,\hat{x}}(z, w) = (\alpha_{n,\hat{x}}z, \beta_{n,\hat{x}}w) + (\gamma_{n,\hat{x}}w^k, 0)$.

(iii) If $\lambda_1 \notin \{k\lambda_2, k \geq 1\}$, then $R_{n,\hat{x}}(z, w) = (\alpha_{n,\hat{x}}z, \beta_{n,\hat{x}}w)$.

Moreover, in the cases (ii) and (iii), we have

$$e^{-n(\lambda_1+\varepsilon)} \leq |\alpha_{n,\hat{x}}| \leq e^{-n(\lambda_1-\varepsilon)} \quad , \quad |\gamma_{n,\hat{x}}| \leq M_\varepsilon(\hat{x})e^{-n(\lambda_1-\varepsilon)}$$

and

$$e^{-n(\lambda_2+\varepsilon)} \leq |\beta_{n,\hat{x}}| \leq e^{-n(\lambda_2-\varepsilon)}.$$

Remark 2.3. The diagram (4) commutes for every fixed $n \geq 0$ by reducing the radius of the balls. The property $n \geq N(\hat{x})$ actually ensures that $f_{\hat{x}}^{-n}(B(x_0, \eta_\varepsilon(\hat{x}))) \subset B(x_{-n}, \eta_\varepsilon(\hat{x}_{-n}))$.

We shall need the following control of $(f^n)^*\omega_{\mathbb{P}^2}$ in Section 3.

Lemma 2.4. For every $\hat{x} \in \Lambda_\varepsilon$ and $n \geq 0$, there exists $0 < r_n \leq \eta_\varepsilon(\hat{x})$ such that

$$(f^n)^*\omega_{\mathbb{P}^2} \geq 4\beta_\varepsilon(\hat{x})^{-2}e^{2n(\lambda_2-2\varepsilon)}\omega_{\mathbb{P}^2} \text{ on } B(x_0, r_n).$$

Proof : We assume to simplify that $\lambda_1 > \lambda_2$ and that λ_1, λ_2 are not resonant, the case of equality and the resonant case can be treated similarly (up to modifying the function β_ε). Let $\varepsilon > 0$ be small enough such that $\lambda_1 - \lambda_2 > 2\varepsilon$, ensuring $|\alpha_{n,\hat{x}_n}|^{-1} \geq |\beta_{n,\hat{x}_n}|^{-1}$. According to Remark 2.3, there exists $r_n > 0$ and $0 < r'_n \leq \eta_\varepsilon(\hat{x}_n)$ such that $f_{\hat{x}_n}^{-n} = \xi_{\hat{x}}^{-1} \circ R_{n,\hat{x}_n} \circ \xi_{\hat{x}_n}$ on $B(x_n, r'_n)$ and $B(x_0, r_n) \subset f_{\hat{x}_n}^{-n}(B(x_n, r'_n))$. Since we have on $B(x_0, r_n)$:

$$(f^n)^*\omega_{\mathbb{P}^2} \geq \|(df^n)^{-1}\|^{-2}\omega_{\mathbb{P}^2}$$

and

$$\|(df^n)^{-1}\|^{-2} = \|df_{\hat{x}_n}^{-n}\|^{-2} \geq \|d\xi_{\hat{x}}^{-1}\|^{-2}\|dR_{n,\hat{x}_n}\|^{-2}\|d\xi_{\hat{x}_n}\|^{-2},$$

Theorem 2.2 implies

$$(f^n)^*\omega_{\mathbb{P}^2} \geq 4\|dR_{n,\hat{x}_n}\|^{-2}\beta_\varepsilon(\hat{x}_n)^{-2}\omega_{\mathbb{P}^2} \text{ on } B(x_0, r_n).$$

Since $|\alpha_{n,\hat{x}_n}|^{-1} \geq |\beta_{n,\hat{x}_n}|^{-1}$ and β_ε is ε -tempered, we deduce on $B(x_0, r_n)$:

$$(f^n)^*\omega_{\mathbb{P}^2} \geq 4|\beta_{n,\hat{x}_n}|^{-2}\beta_\varepsilon(\hat{x})^{-2}e^{-2n\varepsilon}\omega_{\mathbb{P}^2}.$$

The conclusion follows from $|\beta_{n,\hat{x}_n}|^{-2} \geq e^{2n(\lambda_2-\varepsilon)}$. \square

We shall also need the following lemma in Section 5.

Lemma 2.5. Assume that $\mu = \psi_J\sigma_T$ for some measurable function ψ on \mathbb{P}^2 . Let $\hat{x} \in \Lambda_\varepsilon$ and $B := B(x_0, \eta_\varepsilon(\hat{x}))$. Let $A_n := \{p \in B \cap J, \psi \circ f_{\hat{x}}^{-n}(p) > 0\}$.

1. For every $n \geq 0$, the measures σ_T and $(f_{\hat{x}}^{-n})^*\sigma_T$ are equivalent on B , and $A_n = A_0$ modulo a borel subset of zero σ_T -measure.
2. If $A \subset B \cap J$ satisfies $\mu(A) = 0$, then $\sigma_T(A \cap A_0) = 0$. In particular, for $a, b \geq 0$,
 - (i) If $\psi \circ f_{\hat{x}}^{-n} \leq b$ μ -a.e. on $B \cap J$, then $\psi \circ f_{\hat{x}}^{-n} \leq b$ σ_T -a.e. on $B \cap J$.

(ii) If $\psi \circ f_{\hat{x}}^{-n} \geq a$ μ -a.e. on $B \cap J$, then $\psi \circ f_{\hat{x}}^{-n} \geq a$ σ_T -a.e. on $B \cap J \cap A_0$.

Proof : Since $f_{\hat{x}}^{-n} : B \rightarrow f_{\hat{x}}^{-n}(B)$ is a biholomorphism (and up to shrinking B), there exists $0 < \alpha < \beta$ such that

$$\alpha \omega_{\mathbb{P}^2} \leq (f_{\hat{x}}^{-n})^* \omega_{\mathbb{P}^2} \leq \beta \omega_{\mathbb{P}^2} \text{ on } B.$$

Using that T is f -invariant, we deduce that σ_T and $(f_{\hat{x}}^{-n})^* \sigma_T$ are equivalent on B . Now for every $p \in B \cap J$ and every r small enough such that $B(p, r) \subset B$,

$$\mu(B(p, r)) = \int_{B(p, r)} \psi_J \sigma_T.$$

But $\mu(B(p, r)) = d^{2n} \mu(f_{\hat{x}}^{-n}(B(p, r)))$ and

$$\mu(f_{\hat{x}}^{-n}(B(p, r))) = \int_{B(p, r)} \psi_J \circ f_{\hat{x}}^{-n} (f_{\hat{x}}^{-n})^* \sigma_T \simeq_{\alpha, \beta} d^{-n} \int_{B(p, r)} \psi_J \circ f_{\hat{x}}^{-n} \sigma_T,$$

where $\simeq_{\alpha, \beta}$ means that the equality holds up to a multiplicative constant between α and β . We deduce by dividing by $\sigma_T(B(p, r))$ (which is positive since $p \in J \subset \text{Supp } T$) :

$$\int_{B(p, r)} \psi_J \sigma_T \simeq_{\alpha, \beta} d^{-n} \int_{B(p, r)} \psi_J \circ f_{\hat{x}}^{-n} \sigma_T.$$

By Lebesgue Theorem [17, Theorem 2.12] (taking limits when r tends to zero), we get for σ_T -a.e. $p \in B \cap J$ that $\psi(p) > 0$ if and only if $\psi \circ f_{\hat{x}}^{-n}(p) > 0$. The second item comes from $0 = \mu(A) = \int_{A \cap A_0} \psi_J \sigma_T$, which gives $\sigma_T(A \cap A_0) = 0$ since ψ_J is positive on $A \cap A_0$. To obtain (i) and (ii), we apply this fact to $\{p \in B \cap J, \psi \circ f_{\hat{x}}^{-n}(p) > b\}$ and to $\{p \in B \cap J, \psi \circ f_{\hat{x}}^{-n}(p) < a\}$ (which both have zero μ -measure by assumption) and the fact that $A_n = A_0$ modulo a borel subset of zero σ_T -measure (provided by the first item). \square

3 A new proof of Dujardin's theorem

We provide another proof of Theorem 1.1. We shall use Theorem 2.2 and forward dynamics. We assume that $\mu \ll \sigma_T = T \wedge \omega_{\mathbb{P}^2}$. This implies that there exists $\psi \in L^1(\sigma_T)$ such that $\mu = \psi_J \sigma_T$. We recall that $\lambda_2 \geq \frac{1}{2} \text{Log } d$ for every endomorphism of \mathbb{P}^2 , it remains to prove the reverse inequality $\lambda_2 \leq \frac{1}{2} \text{Log } d$ in our case of absolute continuity. For every $n \geq 0$, the set

$$E_n := \left\{ x \in \mathbb{P}^2 : \lim_{\rho \rightarrow 0^+} \int_{B(x, \rho)} \psi_J \circ f^n \sigma_T = \psi_J \circ f^n(x) \right\}$$

is a borel subset of full σ_T -measure (and so of full μ -measure) by Lebesgue Theorem. We define the borel set

$$E := \left(\bigcap_{n \in \mathbb{N}} E_n \right) \setminus \left(\bigcup_{n \geq 1} \text{Crit}(f^n) \right),$$

which is also of full μ -measure (μ does not charge analytic subsets). Let us define

$$F := E \cap \left\{ \frac{1}{\tau} \leq \psi_J \leq \tau \right\} \subset J,$$

where $\tau > 0$ is chosen large enough to have $\mu(F) > 0$. According to Poincaré recurrence theorem, for $\hat{\mu}$ -almost every $\hat{x} \in \pi_0^{-1}(F)$, there exists a sub-sequence \hat{x}_{n_k} which satisfies

$\widehat{x}_{n_k} \in \pi_0^{-1}(F)$. Let us fix $\varepsilon > 0$ and apply the normal form Theorem 2.2, it yields a borel subset Λ_ε of full $\widehat{\mu}$ -measure of good orbits. Let us fix $\widehat{x} \in \pi_0^{-1}(F) \cap \Lambda_\varepsilon$ and a sub-sequence $(n_k)_k$ as before. Let us also fix an integer k . Since $x_0 \in E$ avoids the critical set of f^{n_k} , there exists $\rho_k > 0$ small enough such that f^{n_k} is injective on the ball $B(x_0, \rho_k)$. Then we have for any $\rho \in]0, \rho_k]$:

$$\mu(A_\rho) = d^{-2n_k} \mu(f^{n_k}(A_\rho)), \quad (5)$$

where $A_\rho := B(x_0, \rho)$. Up to taking a smaller ρ_k , we can assume that

$$f^{n_k}(A_\rho) \subset B(x_{n_k}, \eta_\varepsilon(\widehat{x}_{n_k})) \text{ and } A_\rho \subset B(x_0, r_{n_k}),$$

where r_{n_k} comes from Lemma 2.4. From $\mu = \psi_J \sigma_T$, Formula (5) yields for every $\rho \in]0, \rho_k]$:

$$\begin{aligned} \int_{A_\rho} \psi_J \sigma_T &= d^{-2n_k} \int_{f^{n_k}(A_\rho)} \psi_J \sigma_T \\ &= d^{-2n_k} \int_{A_\rho} (\psi_J \circ f^{n_k}) d^{n_k} T \wedge (f^{n_k})^* \omega_{\mathbb{P}^2} \end{aligned}$$

where the second equality uses the invariance of the Green current $(f^{n_k})^* T = d^{n_k} T$ and the fact that f^{n_k} is injective on A_ρ . Since $A_\rho \subset B(x_0, r_{n_k})$, we can use the estimate given by Lemma 2.4 with $n = n_k$ to deduce

$$\begin{aligned} \int_{A_\rho} \psi_J \sigma_T &\geq d^{-n_k} \int_{A_\rho} (\psi_J \circ f^{n_k}) 4\beta_\varepsilon(\widehat{x})^{-2} e^{2n_k(\lambda_2 - 2\varepsilon)} T \wedge \omega_{\mathbb{P}^2} \\ &= 4d^{-n_k} \beta_\varepsilon(\widehat{x})^{-2} e^{2n_k(\lambda_2 - 2\varepsilon)} \int_{A_\rho} (\psi_J \circ f^{n_k}) \sigma_T. \end{aligned}$$

We chose $x_0 \in J$, thus $\sigma_T(A_\rho) > 0$ and we have :

$$\int_{A_\rho} \psi_J \sigma_T \geq 4d^{-n_k} \beta_\varepsilon(\widehat{x})^{-2} e^{2n_k(\lambda_2 - 2\varepsilon)} \int_{A_\rho} (\psi_J \circ f^{n_k}) \sigma_T.$$

But we also have $x_0 \in E \subset E_0 \cap E_{n_k}$, thus we can take limits when $\rho \rightarrow 0^+$ to get

$$\psi_J(x_0) \geq 4d^{-n_k} \beta_\varepsilon(\widehat{x})^{-2} e^{2n_k(\lambda_2 - 2\varepsilon)} \times \psi_J(f^{n_k}(x_0)).$$

Since x_0 and $x_{n_k} = f^{n_k}(x_0)$ belongs to $F \subset \{\frac{1}{\tau} \leq \psi_J \leq \tau\}$, we deduce that :

$$\tau \geq \frac{4}{\tau} \beta_\varepsilon(\widehat{x})^{-2} e^{-2n_k(\frac{1}{2} \text{Log } d - \lambda_2 + 2\varepsilon)}.$$

Since this is true for every integer k , we get $\frac{1}{2} \text{Log } d - \lambda_2 + 2\varepsilon \geq 0$. We obtain $\lambda_2 \leq \frac{1}{2} \text{Log } d$ as desired, since $\varepsilon > 0$ is arbitrary small.

4 Decomposition of μ

4.1 Normal coordinates and absolute continuity

According to Theorem 2.2, there exists an invariant subset of full $\widehat{\mu}$ -measure Λ_ε such that for any $\widehat{x} \in \Lambda_\varepsilon$, there exist inverse branches

$$f_{\widehat{x}}^{-n} : B(x_0, \eta_\varepsilon(\widehat{x})) \longrightarrow B(x_{-n}, \eta_\varepsilon(\widehat{x}_{-n})), \quad n \geq N(\widehat{x}),$$

and holomorphic charts

$$\xi_{\hat{x}} = (Z_{\hat{x}}, W_{\hat{x}}) : B(x_0, \eta_\varepsilon(\hat{x})) \longrightarrow \mathbb{D}^2(\rho_\varepsilon(\hat{x})) \quad (6)$$

such that $\frac{1}{2} \leq \|d\xi_{\hat{x}}\| \leq \beta_\varepsilon(\hat{x})$ on $B(x_0, \eta_\varepsilon(\hat{x}))$ and

$$f_{\hat{x}}^{-n} = \xi_{\hat{x}_{-n}}^{-1} \circ R_{n, \hat{x}} \circ \xi_{\hat{x}}$$

for any $n \geq N(\hat{x})$, where $R_{n, \hat{x}}$ is a polynomial map $(\mathbb{C}^2, (Z_{\hat{x}}, W_{\hat{x}})) \rightarrow (\mathbb{C}^2, (Z_{\hat{x}_{-n}}, W_{\hat{x}_{-n}}))$. When $\lambda_1 > \lambda_2$ are not resonant, $R_{n, \hat{x}}$ is a linear diagonal map with eigenvalues $|\alpha_{n, \hat{x}}| \simeq e^{-n(\lambda_1 \pm \varepsilon)}$ and $|\beta_{n, \hat{x}}| \simeq e^{-n(\lambda_2 \pm \varepsilon)}$. Let $(\omega_{\hat{x}, AB})_{A, B \in \{Z, W\}}$ be the smooth functions defined by the formula

$$\omega_{\mathbb{P}^2} = \sum_{A, B \in \{Z, W\}} \omega_{\hat{x}, AB} \frac{i}{2} (dA_{\hat{x}} \wedge d\bar{B}_{\hat{x}}) \quad \text{on } B(x_0, \eta_\varepsilon(\hat{x})). \quad (7)$$

For every A, B in $\{Z, W\}$ and for every $n \geq N(\hat{x})$ we define the following functions :

$$F_{\hat{x}, n}^{AB} := \omega_{\hat{x}_{-n}, AB} \circ f_{\hat{x}}^{-n} : B(x_0, \eta_\varepsilon(\hat{x})) \longrightarrow \mathbb{C}.$$

Proposition 4.1. *Assume that $\mu \ll \sigma_T$ and let ψ_J be the Radon-Nikodym derivative of μ with respect to σ_T . We assume that the Lyapunov exponents $\lambda_1 > \lambda_2$ are not resonant. For every $\hat{x} \in \Lambda_\varepsilon$, for every borel set $U \subset B(x_0, \eta_\varepsilon(\hat{x}))$ and for every $n \geq N(\hat{x})$ we have :*

$$\mu(U) = d^n \int_U (\psi_J \circ f_{\hat{x}}^{-n}) \sum_{A, B \in \{Z, W\}} F_{\hat{x}, n}^{AB} \alpha_{n, \hat{x}}^A \overline{\alpha_{n, \hat{x}}^B} T \wedge \frac{i}{2} dA_{\hat{x}} \wedge d\bar{B}_{\hat{x}},$$

where $\alpha_{n, \hat{x}}^Z := \alpha_{n, \hat{x}}$ and $\alpha_{n, \hat{x}}^W := \beta_{n, \hat{x}}$.

Proof : From the equalities $\mu(U) = d^{2n} \mu(f_{\hat{x}}^{-n}(U))$ and $\mu = \psi_J(T \wedge \omega_{\mathbb{P}^2})$, we obtain

$$\mu(U) = d^{2n} \int_U (\psi_J \circ f_{\hat{x}}^{-n}) (f_{\hat{x}}^{-n})^* T \wedge (f_{\hat{x}}^{-n})^* \omega_{\mathbb{P}^2}.$$

By invariance of the Green current, we get $(f_{\hat{x}}^{-n})^* T = d^{-n} T$ on $B(x_0, \eta_\varepsilon(\hat{x}))$ and

$$\mu(U) = d^n \int_U (\psi_J \circ f_{\hat{x}}^{-n}) T \wedge (f_{\hat{x}}^{-n})^* \omega_{\mathbb{P}^2}.$$

We conclude by pulling back Formula (7) (\hat{x} being replaced by \hat{x}_{-n}) by $f_{\hat{x}}^{-n}$ and using the fact that $R_{n, \hat{x}}$ is linear and diagonal with eigenvalues $\alpha_{n, \hat{x}}$ and $\beta_{n, \hat{x}}$. \square

Definition 4.2. *Assume that $\mu \ll \sigma_T$ and let $\psi_J = \psi \mathbb{1}_J$ be the Radon-Nikodym derivative of μ with respect to σ_T . We assume that the Lyapunov exponents $\lambda_1 > \lambda_2$ are not resonant. For every $U \subset B(x_0, \eta_\varepsilon(\hat{x}))$ and $n \geq 0$, we have according to Proposition 4.1 :*

$$\mu(U) = I_n(U \cap J) + J_n(U \cap J),$$

where for every $D \subset B(x_0, \eta_\varepsilon(\hat{x}))$:

$$I_n(D) := d^n |\beta_{n, \hat{x}}|^2 \int_D (\psi \circ f_{\hat{x}}^{-n}) F_{\hat{x}, n}^{WW} T \wedge \frac{i}{2} (dW_{\hat{x}} \wedge d\bar{W}_{\hat{x}})$$

and

$$J_n(D) := d^n \sum_{(A, B) \neq (W, W)} \alpha_{n, \hat{x}}^A \overline{\alpha_{n, \hat{x}}^B} \int_D (\psi \circ f_{\hat{x}}^{-n}) F_{\hat{x}, n}^{AB} T \wedge \frac{i}{2} (dA_{\hat{x}} \wedge d\bar{B}_{\hat{x}})$$

where $\alpha_{n, \hat{x}}^Z := \alpha_{n, \hat{x}}$ and $\alpha_{n, \hat{x}}^W := \beta_{n, \hat{x}}$.

Remark 4.3. *We observe that if $\mu = \psi_V \sigma_T$ as in the second item of Theorem 1.2, then*

$$\mu(U) = I_n(U \cap V) + J_n(U \cap V)$$

for every $U \subset B(x_0, \eta_\varepsilon(\hat{x}))$.

4.2 Recurrence in \mathcal{A}_τ

Definition 4.4. For $\tau > 0$ let $\mathcal{A}_\tau \subset \Lambda_\varepsilon$ be the subset consisting of elements \hat{y} satisfying

1. $|\omega_{\hat{y}, AB}| \leq \tau$,
2. $\frac{1}{\tau} \leq \omega_{\hat{y}, AA} \leq \tau$,
3. $S_{AB}(\hat{y}) := \sup_{B(y_0, \eta_\varepsilon(\hat{y}))} \|d\omega_{\hat{y}, AB}\| \leq \tau$,
4. $\psi_J \circ \pi_0(\hat{y}) \geq \frac{1}{\tau}$,

the two first items holding on $B(y_0, \eta_\varepsilon(\hat{y}))$, the three first ones for every $A, B \in \{Z, W\}$.

Since the distortion of the coordinates $\xi_{\hat{x}} = (Z_{\hat{x}}, W_{\hat{x}})$ is controlled by Theorem 2.2, there exists τ large enough such that \mathcal{A}_τ has positive $\hat{\mu}$ -measure. By Poincaré recurrence theorem, for $\hat{\mu}$ -almost every \hat{x} , there exists $(n_k)_{k \in \mathbb{N}}$ such that $n_k \geq N(\hat{x})$ and

$$\hat{x}_{-n_k} \in \mathcal{A}_\tau.$$

We have specified $n_k \geq N(\hat{x})$ to have $f_{\hat{x}}^{-n_k}(B(x_0, \eta_\varepsilon(\hat{x}))) \subset B(x_{-n_k}, \eta_\varepsilon(\hat{x}_{-n_k}))$. In this way the functions $F_{\hat{x}, n_k}^{AB}$ are well defined on $B(x_0, \eta_\varepsilon(\hat{x}))$. By definition of $F_{\hat{x}, n}^{AB}$, we get from the first item of Definition 4.4 :

$$|F_{\hat{x}, n_k}^{AB}| \leq \tau \text{ on } B(x_0, \eta_\varepsilon(\hat{x})). \quad (8)$$

Moreover, using the third item of Definition 4.4 and $\text{Lip}(f_{\hat{x}}^{-n_k}) \leq \beta_\varepsilon(\hat{x})e^{-n_k(\lambda_2 - \varepsilon)}$ (given by Theorem 2.2), we get

$$\|dF_{\hat{x}, n_k}^{AB}\| \leq \tau \beta_\varepsilon(\hat{x})e^{-n_k(\lambda_2 - \varepsilon)} \text{ on } B(x_0, \eta_\varepsilon(\hat{x})).$$

We deduce that

$$F_{\hat{x}, n_k}^{AB} \text{ uniformly converges on } B(x_0, \eta_\varepsilon(\hat{x})) \text{ to a constant } F_{\hat{x}}^{AB} \text{ up to a subsequence} \quad (9)$$

and that (according to the second item of Definition 4.4) :

$$F_{\hat{x}, n_k}^{AA} \geq \frac{1}{\tau} \text{ on } B(x_0, \eta_\varepsilon(\hat{x})). \quad (10)$$

4.3 A weaker version of Dujardin's theorem

We give another proof of Theorem 1.1 assuming that the Radon-Nikodym derivative $\psi_J \in L^\infty(\sigma_T)$. Our motivation is to use backward iterates, namely the inverse branches $f_{\hat{x}}^{-n_k}$, in order to prepare the proof of Theorem 1.2 in Section 6.1. As in Section 3, it suffices to prove that $\lambda_2 \leq \frac{1}{2} \text{Log } d$.

For sake of simplicity, we assume that the Lyapunov exponents $\lambda_1 > \lambda_2$ are not resonant. This implies that the polynomial map $R_{n, \hat{x}}$ appearing in the diagram (4) is linear and diagonal. The proof can be adapted in the resonant case, see Section 7.

We apply the normal form Theorem 2.2 with $\varepsilon > 0$, it yields a borel subset Λ_ε of full $\hat{\mu}$ -measure of good orbits. We use the set \mathcal{A}_τ defined in Section 4.2. Let us fix a $\hat{\mu}$ -generic element $\hat{x} \in \Lambda_\varepsilon$, in particular $x_0 \in J$. Let $(n_k)_{k \in \mathbb{N}}$ be such that $\hat{x}_{-n_k} \in \mathcal{A}_\tau$.

1) Construction of a set of continuity C .

Let $\delta > 0$ be such that $\mu(B(x_0, \eta_\varepsilon(\hat{x}))) > 2\delta$. By Lusin's theorem there exists a compact set $C_0 \subset B(x_0, \eta_\varepsilon(\hat{x}))$ such that $\mu(B(x_0, \eta_\varepsilon(\hat{x})) \setminus C_0) < \delta$ and ψ_J is continuous on C_0 . Using

again Lusin's theorem we construct by induction a sequence $(C_n)_n$ of compact subsets of $B(x_0, \eta_\varepsilon(\hat{x}))$ such that

$$C_{n+1} \subset C_n \text{ and } \mu(C_n \setminus C_{n+1}) < 2^{-(n+1)}\delta$$

and

$$\psi_J \circ f_{\hat{x}}^{-(n+1)} \in C^0(C_{n+1}).$$

Then $\mu(C_n) > \mu(C_{n-1}) - 2^{-n}\delta > \dots > \mu(C_0) - \delta > \mu(B(x_0, \eta_\varepsilon(\hat{x}))) - 2\delta > 0$. Thus

$$C := \bigcap_{n \in \mathbb{N}} C_n \subset B(x_0, \eta_\varepsilon(\hat{x}))$$

satisfies $\mu(C) > 0$. Let μ_C be the measure on $B(x_0, \eta_\varepsilon(\hat{x}))$ defined by

$$\mu_C(\cdot) := \mu(\cdot \cap C).$$

Its support $\text{supp } \mu_C$ is included in $C \cap J$ because C is a closed subset of $B(x_0, \eta_\varepsilon(\hat{x}))$.

2) Construction of x'_0 in $\text{supp } \mu_C$, with boundedness properties.

Let G be a borel subset of full σ_T -measure (and so of full μ -measure) satisfying

$$\forall p \in G, 0 \leq \psi_J(p) \leq \|\psi_J\|_{L^\infty(\sigma_T)},$$

and let G_k be the borel set

$$G_k := f_{\hat{x}}^{-n_k}(B(x_0, \eta_\varepsilon(\hat{x}))) \cap G.$$

Let us verify that $\mu(f^{n_k}(G_k)) = \mu(B(x_0, \eta_\varepsilon(\hat{x})))$. Since $\mu(G) = 1$, we have $\mu(G_k) = \mu(f_{\hat{x}}^{-n_k}(B(x_0, \eta_\varepsilon(\hat{x})))) = d^{-2n_k} \mu(B(x_0, \eta_\varepsilon(\hat{x})))$. But f^{n_k} is injective on $f_{\hat{x}}^{-n_k}(B(x_0, \eta_\varepsilon(\hat{x})))$, which contains G_k , hence $\mu(f^{n_k}(G_k)) = d^{2n_k} \mu(G_k)$, completing our verification.

Therefore

$$\mu\left(\bigcap_{k \in \mathbb{N}} f^{n_k}(G_k)\right) = \mu(B(x_0, \eta_\varepsilon(\hat{x}))),$$

which is positive since $x_0 \in J$. Let

$$x'_0 \in \bigcap_{k \in \mathbb{N}} f^{n_k}(G_k) \cap \left\{ \frac{1}{\tau'} \leq \psi_J \right\} \cap \text{supp } \mu_C, \quad (11)$$

where $\tau' > 1$ is large enough so that the set defined in (11) is not empty. Using the fact that $f_{\hat{x}}^{-n_k} \circ (f^{n_k})|_{G_k} = \text{Id}_{G_k}$, we observe that

$$\forall k \in \mathbb{N}, f_{\hat{x}}^{-n_k}(x'_0) \in G_k \subset G. \quad (12)$$

3) Conclusion.

For a given k , let $\rho_k > 0$ be small enough so that for any $0 < \rho < \rho_k$:

$$A_\rho := (B(x'_0, \rho) \cap C) \subset B(x_0, \eta_\varepsilon(\hat{x})).$$

Since $\|d\xi_{\hat{x}}\| \leq \beta_\varepsilon(\hat{x})$ we can assume that $\sum_{A,B} |T \wedge \frac{i}{2} dA_{\hat{x}} \wedge d\bar{B}_{\hat{x}}| \leq \beta_\varepsilon(\hat{x})^2 \sigma_T$ on $B(x_0, \eta_\varepsilon(\hat{x}))$. According to Proposition 4.1, $|F_{\hat{x}, n_k}^{AB}| \leq \tau$ (see Equation (8)) and $|\alpha_{n, \hat{x}}| \leq |\beta_{n, \hat{x}}|$, we deduce

$$\int_{A_\rho} \psi_J d\sigma_T = \mu(A_\rho) \leq d^{n_k} |\beta_{n_k, \hat{x}}|^2 \tau \beta_\varepsilon(\hat{x})^2 \int_{A_\rho} (\psi_J \circ f_{\hat{x}}^{-n_k}) d\sigma_T.$$

According to Equation (11), we have $x'_0 \in \text{supp } \mu_C$, thus $\sigma_T(A_\rho) > 0$ and

$$\int_{A_\rho} \psi_J \, d\sigma_T \leq d^{n_k} |\beta_{n_k, \hat{x}}|^2 \tau \beta_\varepsilon(\hat{x})^2 \int_{A_\rho} (\psi_J \circ f_{\hat{x}}^{-n_k}) \, d\sigma_T.$$

Equation (11) also gives $\frac{1}{\tau'} \leq \psi_J(x'_0)$ and $x'_0 \in \text{supp } \mu_C \subset C \subset C_0 \cap C_{n_k}$. We moreover have $A_\rho \subset C$ by definition. Hence, if ρ tends to 0, we get by continuity of ψ_J (resp. $\psi_J \circ f_{\hat{x}}^{-n_k}$) on C_0 (resp. on C_{n_k}) :

$$\frac{1}{\tau'} \leq \psi_J(x'_0) \leq d^{n_k} |\beta_{n_k, \hat{x}}|^2 \tau \beta_\varepsilon(\hat{x})^2 (\psi_J \circ f_{\hat{x}}^{-n_k})(x'_0) \leq d^{n_k} |\beta_{n_k, \hat{x}}|^2 \tau \beta_\varepsilon(\hat{x})^2 \|\psi_J\|_{L^\infty(\sigma_T)},$$

the last inequality coming from the choice of x'_0 , see Equation (12). We obtain for every k :

$$(\tau' \tau \beta_\varepsilon(\hat{x})^2 \|\psi_J\|_{L^\infty(\sigma_T)})^{-1} \leq d^{n_k} |\beta_{n_k, \hat{x}}|^2 \leq e^{2n_k(\frac{1}{2} \text{Log } d - \lambda_2 + \varepsilon)}.$$

Hence $\lambda_2 \leq \frac{1}{2} \text{Log } d + \varepsilon$, which gives $\lambda_2 \leq \frac{1}{2} \text{Log } d$ when ε tends to zero, as desired. \square

5 Estimates on I_n and J_n in the decomposition of μ

5.1 Study of I_n and J_n

We recall that I_n and J_n were introduced in Definition 4.2.

Proposition 5.1. *Assume that $\mu \ll \sigma_T$ and let $\psi \mathbb{1}_J$ be the Radon-Nikodym derivative of μ with respect to σ_T . Assume that $\lambda_1 > \lambda_2$ are not resonant and let $\varepsilon > 0$ be small enough such that*

$$\lambda_1 + \lambda_2 - 2\varepsilon > \text{Log } d \quad , \quad \lambda_1 > \frac{1}{2} \text{Log } d + \varepsilon.$$

Let \hat{x} be a $\hat{\mu}$ -generic orbit and let $(n_k)_k$ be a sub-sequence such that $\hat{x}_{-n_k} \in \mathcal{A}_T$. Let

$$\sigma_{\hat{x}} := T \wedge dd^c |W_{\hat{x}}|^2.$$

Let $B := B(x_0, \eta_\varepsilon(\hat{x}))$, $A_0 := \{p \in B \cap J, \psi(p) > 0\}$ and $U \subset B$ be a borel set.

1. If $\psi \circ f_{\hat{x}}^{-n_k} \leq b \mu - a.e.$ on B for every $k \geq 0$, then there exists $C_{\hat{x}} > 0$ which does not depend on $U \subset B$ such that

$$J_{n_k}(U \cap J) \leq C_{\hat{x}} e^{-n_k(\lambda_1 + \lambda_2 - \text{Log } d)} e^{2n_k \varepsilon} \xrightarrow[k \rightarrow \infty]{} 0.$$

2. If $\psi \circ f_{\hat{x}}^{-n_k} \leq b \mu - a.e.$ on B for every $k \geq 0$ and if $\mu(U) > 0$, then

$$\sigma_{\hat{x}}(U \cap J \cap A_0) > 0.$$

3. If $0 < a \leq \psi \circ f_{\hat{x}}^{-n_k} \leq b \mu - a.e.$ on B for every $k \geq 0$, then

$$d^{n_k} |\beta_{n_k, \hat{x}}|^2 a \tau^{-1} \sigma_{\hat{x}}(U \cap J \cap A_0) \leq I_{n_k}(U \cap J) \leq d^{n_k} |\beta_{n_k, \hat{x}}|^2 b \tau \sigma_{\hat{x}}(U \cap J).$$

In particular $(d^{n_k} |\beta_{n_k, \hat{x}}|^2)_k$ converges, up to a sub-sequence, to some $u > 0$.

Remark 5.2. *We assume in Proposition 5.1 that λ_1, λ_2 are not resonant. The same statement actually holds in the resonant case, this is explained in Section 7.*

Proof of Proposition 5.1 : Let us prove the first item. By Definition 4.2,

$$J_{n_k}(U \cap J) = \sum_{(A,B) \neq (W,W)} d^{n_k} \alpha_{n_k, \hat{x}}^A \overline{\alpha_{n_k, \hat{x}}^B} \int_{U \cap J} (\psi \circ f_{\hat{x}}^{-n_k}) F_{\hat{x}, n_k}^{AB} T \wedge dA_{\hat{x}} \wedge d\overline{B}_{\hat{x}}.$$

Let us introduce the measures (which are $\ll \sigma_T$) :

$$\sigma_{\hat{x}} := T \wedge dd^c |W_{\hat{x}}|^2, \quad \lambda_{\hat{x}} := T \wedge dd^c |Z_{\hat{x}}|^2.$$

Using $|\alpha_{n, \hat{x}}| \leq |\beta_{n, \hat{x}}|$, the (assumed) upper bound on $\psi \circ f_{\hat{x}}^{-n_k}$, the upper bound on $F_{\hat{x}, n_k}^{AB}$ provided by (8), the item 2.(i) of Lemma 2.5 and Cauchy-Schwarz inequality, we get

$$|J_{n_k}(U \cap J)| \leq d^{n_k} |\alpha_{n_k, \hat{x}} \beta_{n_k, \hat{x}}| b \tau (\sigma_{\hat{x}} + \lambda_{\hat{x}})(U \cap J).$$

The first item then follows from $|\alpha_{n, \hat{x}}| \leq e^{-n(\lambda_1 - \varepsilon)}$ and $|\beta_{n, \hat{x}}| \leq e^{-n(\lambda_2 - \varepsilon)}$.

We show the second item. By using Item 1 of Lemma 2.5 (for the last equality), we get

$$\mu(U) = \mu(U \cap J) = d^{2n} \mu(f_{\hat{x}}^{-n}(U \cap J)) = d^{2n} \int_{U \cap J \cap A_0} (\psi \circ f_{\hat{x}}^{-n}) (f_{\hat{x}}^{-n})^*(T \wedge \omega_{\mathbb{P}^2}).$$

Cauchy-Schwarz inequality implies (up to multiplication by a constant) :

$$T \wedge \omega_{\mathbb{P}^2} \leq \omega_{\hat{x}_{-n}, WW} \sigma_{\hat{x}_{-n}} + \omega_{\hat{x}_{-n}, ZZ} \lambda_{\hat{x}_{-n}} \text{ on } B(\hat{x}_{-n}, \eta_\varepsilon(\hat{x}_{-n})).$$

We now fix $n = n_k$. Using the (assumed) upper bound on $\psi \circ f_{\hat{x}}^{-n_k}$ on B combined with item 2.(i) of Lemma 2.5, and the upper bounds on $F_{\hat{x}, n_k}^{WW} = \omega_{\hat{x}_{-n}, WW} \circ f_{\hat{x}}^{-n_k}$, $F_{\hat{x}, n_k}^{ZZ} = \omega_{\hat{x}_{-n}, ZZ} \circ f_{\hat{x}}^{-n_k}$ provided by Equation (8), we get

$$\mu(U) \leq b d^{2n_k} \tau \left(\int_{U \cap J \cap A_0} (f_{\hat{x}}^{-n_k})^*(\sigma_{\hat{x}_{-n_k}}) + \int_{U \cap J \cap A_0} (f_{\hat{x}}^{-n_k})^*(\lambda_{\hat{x}_{-n_k}}) \right).$$

We deduce

$$\mu(U) \leq b d^{2n_k} \tau \left(\int_{U \cap J \cap A_0} d^{-n_k} |\beta_{n_k, \hat{x}}|^2 \sigma_{\hat{x}} + \int_{U \cap J \cap A_0} d^{-n_k} |\alpha_{n_k, \hat{x}}|^2 \lambda_{\hat{x}} \right),$$

hence

$$0 < \mu(U) \leq b d^{n_k} \tau (|\beta_{n_k, \hat{x}}|^2 \sigma_{\hat{x}}(U \cap J \cap A_0) + |\alpha_{n_k, \hat{x}}|^2 \lambda_{\hat{x}}(U \cap J \cap A_0)).$$

To conclude we observe that

$$d^{n_k} |\alpha_{n_k, \hat{x}}|^2 \leq e^{n_k(\text{Log } d - 2\lambda_1 + 2\varepsilon)},$$

which tends to 0 when k tends to infinity. Hence $\sigma_{\hat{x}}(U \cap J \cap A_0) > 0$.

Let us prove the third item. According to item 2.(ii) of Lemma 2.5, we have

$$\psi \circ f_{\hat{x}}^{-n_k} \geq a, \quad \sigma_{\hat{x}} - a.e. \text{ on } B \cap J \cap A_0.$$

Hence, according to Definition 4.2, we get

$$I_{n_k}(U \cap J \cap A_0) \geq d^{n_k} |\beta_{n_k, \hat{x}}|^2 a \int_{U \cap J \cap A_0} F_{\hat{x}, n_k}^{WW} \sigma_{\hat{x}} \geq d^{n_k} |\beta_{n_k, \hat{x}}|^2 a \frac{1}{\tau} \sigma_{\hat{x}}(U \cap J \cap A_0),$$

where the last inequality comes from Equation (10). Similarly, by using item 2.(i) of Lemma 2.5 and Equation (8), we get

$$I_{n_k}(U \cap J) \leq d^{n_k} |\beta_{n_k, \hat{x}}|^2 b \tau \sigma_{\hat{x}}(U \cap J)$$

This proves the two stated inequalities on $I_{n_k}(U \cap J)$. Let us verify the last fact. Since $0 < \mu(B) = I_{n_k}(B \cap J) + J_{n_k}(B \cap J)$ and since $J_{n_k}(B)$ tends to 0 by the first item, we get $\frac{1}{2}\mu(B) \leq I_{n_k}(B \cap J) \leq 2\mu(B)$ for k large enough, which implies

$$0 < \frac{1}{2b\tau} \frac{\mu(B)}{\sigma_{\hat{x}}(B \cap J)} \leq d^{n_k} |\beta_{n_k, \hat{x}}|^2 \leq \frac{2\tau}{a} \frac{\mu(B)}{\sigma_{\hat{x}}(B \cap J \cap A_0)} < +\infty,$$

since $\sigma_{\hat{x}}(B \cap J \cap A_0) > 0$ by the second item. \square

5.2 Application : an equivalence between μ and a slice of T

We prove that μ can be approximated by a slice of T under absolute continuity assumptions. The proof is a direct application of the decomposition of μ fixed in Definition 4.2 and of Proposition 5.1.

Proposition 5.3. *Assume that $\mu \ll \sigma_T$ and $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d$. Let $\psi \mathbb{1}_J$ be the Radon-Nikodym derivative of μ with respect to σ_T . Let \hat{x} be a $\hat{\mu}$ -generic element and let $(n_k)_k$ be a sub-sequence such that $\hat{x}_{-n_k} \in \mathcal{A}_\tau$. Let $B := B(x_0, \eta_\varepsilon(\hat{x}))$ and $A_0 := \{p \in B \cap J, \psi(p) > 0\}$. If*

$$0 < a \leq \psi \circ f_{\hat{x}}^{-n_k} \leq b \mu - \text{a.e. on } B \text{ for every } k \geq 0,$$

then there exists $C_{\hat{x}} > 0$ such that on B :

$$C_{\hat{x}}^{-1}(T \wedge dd^c |W_{\hat{x}}|^2)|_{J \cap A_0} \leq \mu \leq C_{\hat{x}}(T \wedge dd^c |W_{\hat{x}}|^2)|_J.$$

Proof : Let $U \subset B$ be a borel set and let $I_k := I_{n_k}(U \cap J)$, $J_k := J_{n_k}(U \cap J)$. According to Definition 4.2, we have $\mu(U) = I_k + J_k$. The first item of Proposition 5.1 yields $J_k \rightarrow 0$. Using the third item of Proposition 5.1 (which in particular provides $\lim_k d^{n_k} |\beta_{n_k, \hat{x}}|^2 = u > 0$), Equation (8) and item 2.(i) of Lemma 2.5, we get

$$\mu(U) \leq u b \tau \sigma_{\hat{x}}(U \cap J).$$

Similarly, by using Equation (10) and item 2.(ii) of Lemma 2.5, we get

$$\mu(U) \geq \frac{u a}{\tau} \sigma_{\hat{x}}(U \cap J \cap A_0).$$

We conclude by setting $C_{\hat{x}} := \max\{\tau/ua, ub\tau\}$. \square

6 Proof of Theorem 1.2

6.1 Proof of the first item

Lemma 6.1. *Assume that $\mu \ll \sigma_T$ and $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d$. Let $\psi \mathbb{1}_J$ be the Radon-Nikodym derivative of μ with respect to σ_T , and assume that $\psi|_J$ is continuous. Let \hat{x} be a $\hat{\mu}$ -generic element and let $(n_k)_k$ be a sub-sequence such that $\hat{x}_{-n_k} \in \mathcal{A}_\tau$ with $n_k \geq N(\hat{x})$. There exists $r_1(\hat{x}) \leq \eta_\varepsilon(\hat{x})$ and $x_{-\infty} \in J$ such that, up to a sub-sequence :*

1. $\lim_k d^{n_k} |\beta_{n_k, \hat{x}}|^2 = u > 0$ and $\lim_k J_{n_k}(U \cap J) = 0$ for any borel set $U \subset B(x_0, r_1(\hat{x}))$,
2. $\psi \circ f_{\hat{x}}^{-n_k}$ uniformly converges to $\psi(x_{-\infty})$ on $B(x_0, r_1(\hat{x})) \cap J$.

Proof : To prove the first item, in view of Proposition 5.1, it suffices to verify

$$\frac{1}{2\tau} \leq \psi \circ f_{\hat{x}}^{-n_k}$$

on some $B(x_0, r_1(\hat{x})) \cap J$. By uniform continuity, there exists $\delta(\tau) > 0$ such that

$$\forall a, b \in J, \text{dist}(a, b) \leq \delta(\tau) \implies |\psi(a) - \psi(b)| \leq \frac{1}{2\tau}.$$

Defining $r_1(\hat{x}) := \min \left\{ \eta_\varepsilon(\hat{x}), \frac{\delta(\tau)}{\beta_\varepsilon(\hat{x})} \right\}$, we have for every $p \in B(x_0, r_1(\hat{x}))$:

$$\text{dist}(f_{\hat{x}}^{-n_k}(p), f_{\hat{x}}^{-n_k}(x_0)) \leq \text{Lip}(f_{\hat{x}}^{-n_k})r_1(\hat{x}) \leq \beta_\varepsilon(\hat{x})r_1(\hat{x}) \leq \delta(\tau).$$

Using $\psi(x_{-n_k}) \geq \frac{1}{\tau}$ (coming from $\hat{x}_{-n_k} \in \mathcal{A}_\tau$), we get $\psi \circ f_{\hat{x}}^{-n_k} \geq \frac{1}{2\tau}$ on $B(x_0, r_1(\hat{x})) \cap J$.

For the second item, the sequence $(x_{-n_k})_k \in J$ converges by compactness to some $x_{-\infty} \in J$. Since $\text{Lip} f_{\hat{x}}^{-n_k}$ tends to 0, $(f_{\hat{x}}^{-n_k})_k$ converges uniformly on $B(x_0, r_1(\hat{x}))$ to the constant mapping $x_{-\infty}$. The conclusion follows from the continuity of ψ on J . \square

We now prove the first item of Theorem 1.2. Let $U \subset B(x_0, r_1(\hat{x}))$ be a borel set. We recall that $\sigma_{\hat{x}} = T \wedge dd^c |W_{\hat{x}}|^2$ and that by Definition 4.2 :

$$\mu(U) = \mu(U \cap J) = d^{n_k} |\beta_{n_k, \hat{x}}|^2 \int_{U \cap J} (\psi \circ f_{\hat{x}}^{-n_k}) F_{\hat{x}, n_k}^{WW} \sigma_{\hat{x}} + J_{n_k}(U \cap J).$$

According to Equation (9), $F_{\hat{x}, n_k}^{WW}$ uniformly converges on $B(x_0, r_1(\hat{x}))$ to some constant $F_{\hat{x}}$, up to a sub-sequence. We conclude by using the two items of Lemma 6.1, which yield $\mu(U) = C_{\hat{x}} \sigma_{\hat{x}}(U \cap J)$, where $C_{\hat{x}} := u F_{\hat{x}} \psi(x_{-\infty})$. Note that $C_{\hat{x}}$ does not depend on U and is positive by taking $U = B(x_0, r_1(\hat{x}))$.

6.2 Proof of the second item

The following lemma is the counterpart of Lemma 6.1, the function ψ being now continuous on a neighborhood V of J . The two items hold without any restriction to J . The proof is similar using Remark 4.3 (decomposition of μ on V), the uniform continuity of ψ on V , and by introducing a radius $r_2(\hat{x})$ satisfying $B(x_0, r_2(\hat{x})) \subset V$. Note that $f_{\hat{x}}^{-n_k}(B(x_0, r_2(\hat{x}))) \subset V$ for k large enough.

Lemma 6.2. *Assume that $\mu \ll \sigma_T$ and $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d$. We assume that there exists a neighborhood V of J and a function $\psi \in L^1(\sigma_T)$ such that ψ is continuous on V and satisfies $\mu = \psi_V \sigma_T$ (we recall that $\psi_V = \psi \mathbb{1}_V$). Let \hat{x} be a $\hat{\mu}$ -generic element and let $(n_k)_k$ be a sub-sequence such that $\hat{x}_{-n_k} \in \mathcal{A}_\tau$ with $n \geq N(\hat{x})$. There exists $r_2(\hat{x}) \leq \eta_\varepsilon(\hat{x})$ (small enough to have $B(x_0, r_2(\hat{x})) \subset V$) and $x_{-\infty} \in J$ such that, up to a sub-sequence :*

1. $\lim_k d^{n_k} |\beta_{n_k, \hat{x}}|^2 = u > 0$ and $\lim_k J_{n_k}(U) = 0$ for any borel set $U \subset B(x_0, r_2(\hat{x}))$,
2. $\psi \circ f_{\hat{x}}^{-n_k}$ uniformly converges to $\psi(x_{-\infty})$ on $B(x_0, r_2(\hat{x}))$.

The second item of Theorem 1.2 follows as in the end of Section 6.1, by using

$$\mu(U) = d^{n_k} |\beta_{n_k, \hat{x}}|^2 \int_U (\psi \circ f_{\hat{x}}^{-n_k}) F_{\hat{x}, n_k}^{WW} \sigma_{\hat{x}} + J_{n_k}(U)$$

for every $U \subset B(x_0, r_2(\hat{x})) \subset V$, this formula is provided by Definition 4.2 and Remark 4.3.

7 About the resonant case

We assume that λ_1, λ_2 are resonant : $\lambda_1 = q\lambda_2$ for some $q \geq 2$. We assume that $\mu \ll \sigma_T$, let ψ denote the Radon-Nikodym derivative of μ with respect to σ_T . Let us first explain how Proposition 4.1 is affected. We shall compute the corresponding $J_n(D)$ appearing in Definition 4.2. According to Theorem 2.2, the map $R_{n,\hat{x}}$ is equal to :

$$R_{n,\hat{x}}(z, w) = (\alpha_{n,\hat{x}}z, \beta_{n,\hat{x}}w) + (\gamma_{n,\hat{x}}w^q, 0),$$

where $|\gamma_{n,\hat{x}}| \leq M_\varepsilon(\hat{x})e^{-n(\lambda_1-\varepsilon)}$. Let $U \subset B(x_0, \eta_\varepsilon(\hat{x}))$ be a borel set. Denoting $\psi_n := \psi \circ f_{\hat{x}}^{-n}$, we have $\mu(U) = I_n(U \cap J) + J_n(U \cap J)$, where $I_n(D)$ has the same expression as in the non resonant case :

$$I_n(D) = d^n |\beta_{n,\hat{x}}|^2 \int_D \psi_n F_{\hat{x},n}^{WW} T \wedge \frac{i}{2} (dW_{\hat{x}} \wedge d\bar{W}_{\hat{x}}).$$

In the resonant case, the formula giving $J_n(D)$ is modified as follows :

$$\begin{aligned} J_n(D) &= d^n |\alpha_{n,\hat{x}}|^2 \int_D \psi_n F_{\hat{x},n}^{ZZ} T \wedge \frac{i}{2} (dZ_{\hat{x}} \wedge d\bar{Z}_{\hat{x}}) \\ &\quad + 2d^n \int_D \psi_n \operatorname{Re} \left[\left((qF_{\hat{x},n}^{ZZ} \alpha_{n,\hat{x}} \bar{\gamma}_{n,\hat{x}} \bar{W}_{\hat{x}}^{q-1} + F_{\hat{x},n}^{ZW} \alpha_{n,\hat{x}} \bar{\beta}_{n,\hat{x}}) T \wedge \frac{i}{2} (dZ_{\hat{x}} \wedge d\bar{W}_{\hat{x}}) \right) \right] \\ &\quad + d^n \int_D \psi_n q^2 F_{\hat{x},n}^{ZZ} |\gamma_{n,\hat{x}}|^2 |W_{\hat{x}}|^{2(q-1)} T \wedge \frac{i}{2} (dW_{\hat{x}} \wedge d\bar{W}_{\hat{x}}) \\ &\quad + d^n \int_D 2\operatorname{Re} \left(qF_{\hat{x},n}^{ZW} \gamma_{n,\hat{x}} \bar{\beta}_{n,\hat{x}} W_{\hat{x}}^{q-1} \right) T \wedge \frac{i}{2} (dW_{\hat{x}} \wedge d\bar{W}_{\hat{x}}). \end{aligned}$$

Let us verify that the first item of Proposition 5.1 remains valid in the resonant case. Let \hat{x} be a $\hat{\mu}$ -generic element and let $(n_k)_k$ be a sub-sequence such that $\hat{x}_{-n_k} \in \mathcal{A}_\tau$. Assume

$$\psi_n = \psi \circ f_{\hat{x}}^{-n_k} \leq b, \quad \mu - a.e. \text{ on } B$$

and let us verify that there exists $C'(\hat{x})$ such that for every borel set $U \subset B$:

$$J_{n_k}(U) \leq C'(\hat{x}) e^{-n_k(\lambda_1 + \lambda_2 - \operatorname{Log} d)} e^{2n_k \varepsilon}.$$

We use the formula on $J_n(D)$ given above and Cauchy-Schwarz inequality. We also use $|F_{\hat{x},n}^{AB}| \leq \tau$ given by Equation (8), item (i) of Lemma 2.5 and $|W_{\hat{x}}| \leq \rho_\varepsilon := \rho_\varepsilon(\hat{x})$ on B provided by Equation (6). If $\sigma_{\hat{x}} := T \wedge dd^c |W_{\hat{x}}|^2$, $\lambda_{\hat{x}} := T \wedge dd^c |Z_{\hat{x}}|^2$, we indeed obtain

$$\begin{aligned} J_{n_k}(U) &\leq d^{n_k} |\alpha_{n_k,\hat{x}}|^2 b\tau \lambda_{\hat{x}}(U) \\ &\quad + 2d^{n_k} |\alpha_{n_k,\hat{x}}| b\tau \left(q|\gamma_{n_k,\hat{x}}| \rho_\varepsilon^{q-1} + |\beta_{n_k,\hat{x}}| \right) (\lambda_{\hat{x}} + \sigma_{\hat{x}})(U) \\ &\quad + qd^{n_k} |\gamma_{n_k,\hat{x}}| b\tau \left(q|\gamma_{n_k,\hat{x}}| \rho_\varepsilon^{2(q-1)} + 2|\beta_{n_k,\hat{x}}| \rho_\varepsilon^{q-1} \right) \sigma_{\hat{x}}(U). \end{aligned}$$

The existence of $C'(\hat{x})$ then comes from the upper bounds $|\alpha_{n,\hat{x}}| \leq e^{-n(\lambda_1-\varepsilon)}$, $|\beta_{n,\hat{x}}| \leq e^{-n(\lambda_2-\varepsilon)}$, $|\gamma_{n,\hat{x}}| \leq M_\varepsilon(\hat{x})e^{-n(\lambda_1-\varepsilon)}$ and from $\lambda_1 > \lambda_2$.

8 Suspensions of one-dimensional Lattès maps

Let $\theta = [P : Q]$ be Lattès map on \mathbb{P}^1 of degree $d \geq 2$ and let $f := [P : Q : t^d]$ be its suspension on \mathbb{P}^2 . These maps were studied by Berteloot-Loeb [4]. We identify the first affine chart with \mathbb{C}^2 . Let

$$G_\theta := \lim_{n \rightarrow +\infty} \frac{1}{d^n} \operatorname{Log} \|(P^n, Q^n)\| : \mathbb{C}^2 \rightarrow \mathbb{R} \cup \{-\infty\}$$

be the Green function of the polynomial mapping (P, Q) and let $G := \max\{G_\theta, 0\}$. The attracting basin A of $(0, 0)$ is bounded and equal to $\{G_\theta < 0\}$. The Green current of f satisfies $T = dd^c G$ on \mathbb{C}^2 , the support of μ coincides with the boundary ∂A of A .

By [4, Proposition 3.1] (see also [11] in higher dimensions), for every p outside a finite number of circles drawn on ∂A , there exists a biholomorphism $\mathbf{p} : (\mathbb{D}^2, 0) \longrightarrow (\mathbb{D}^2, p)$ such that

$$G_0(z, w) := G_\theta \circ \mathbf{p}(z, w) = \operatorname{Re}(z) + |w|^2 \text{ on } (\mathbb{D}^2, 0).$$

We denote

$$T_0 := \mathbf{p}^* T = dd^c(\max\{G_0, 0\}) \text{ and } \mu_0 := \mathbf{p}^* \mu.$$

The closed positive current T_0 has the following matrix representation

$$T_0 = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix},$$

where $T_{11} := T_0 \wedge dd^c |w|^2$ and $T_{22} := T \wedge dd^c |z|^2$. We describe below T_0 and μ_0 on $(\mathbb{D}^2, 0)$. We assume to simplify that the germ $(\mathbb{D}^2, 0)$ is the open set $D :=]-1, 1[\times \mathbb{D}$. Let

$$\Omega := \{\operatorname{Re}(z) + |w|^2 > 0\} \cap D \text{ and } M_0 := \{\operatorname{Re}(z) + |w|^2 = 0\} \cap D.$$

Observe that M_0 is the image (intersected with D) of the euclidian 3-sphere of \mathbb{C}^2 by the classical Cayley transformation, see [18, Chapter 2.3]. Let us parametrize M_0 by

$$\Phi :]-1, 0[\times]-1, 1[\times]0, 2\pi[\rightarrow M_0, \quad (u, v, \theta) \mapsto (u + iv, \sqrt{-u}e^{i\theta}) \quad (13)$$

and let $\operatorname{Leb}_{M_0} := \Phi_* \operatorname{Leb}_{\mathbb{R}^3}$. Let also ω_0 be the standard hermitian $(1, 1)$ -form of \mathbb{C}^2 .

Proposition 8.1. *With the preceding notations,*

$$T_0 = \begin{bmatrix} \frac{1}{8} \operatorname{Leb}_{M_0} & 0 \\ 0 & \operatorname{Leb}_\Omega + \frac{|w|^2}{2} \operatorname{Leb}_{M_0} \end{bmatrix},$$

and

$$\mu_0 = \frac{1}{8} \operatorname{Leb}_{M_0} = T_0 \wedge dd^c |w|^2.$$

In particular,

$$\mu_0 = \psi_0(T_0 \wedge \omega_0), \text{ where } \psi_0(z, w) := \frac{1}{1 + 4|w|^2} \mathbb{1}_{M_0}.$$

The remainder of this Section is devoted to the proof of Proposition 8.1.

Computation of T_{11} .— Let φ be a test function with compact support in D . We denote $\varphi_{z\bar{z}} := \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \varphi$, similarly for $\varphi_{w\bar{w}}$. By definition we have :

$$\langle T_{11}, \varphi \rangle = \int_\Omega G_0(z, w) \varphi_{z\bar{z}}(z, w) \, d\operatorname{Leb}(z, w).$$

Let Δ_w denote the subset $] -1, 1[\times] -1, 1[\times \{w\}$ intersected with Ω , so that

$$\langle T_{11}, \varphi \rangle = \int_{\mathbb{D}} \left[\int_{\Delta_w} G_0(z, w) \varphi_{z\bar{z}}(z, w) \, d\operatorname{Leb}(z) \right] d\operatorname{Leb}(w) =: \int_{\mathbb{D}} I_w(\varphi) \, d\operatorname{Leb}(w). \quad (14)$$

We write $z = u + iv$, hence $\varphi_{z\bar{z}} = \frac{1}{4} \Delta_{u,v} \varphi$. Observe also that G_0 only depends on (u, w) . The number $4I_w(\varphi)$ is then equal to

$$\int_{-1}^1 \left[\int_{-|w|^2}^1 G_0(u, w) \frac{\partial^2 \varphi}{\partial u^2}(u + iv, w) \, du \right] dv + \int_{-|w|^2}^1 G_0(u, w) \left[\int_{-1}^1 \frac{\partial^2 \varphi}{\partial v^2}(u + iv, w) \, dv \right] du.$$

The second term vanishes since $\text{Supp}(\varphi(\cdot, w)) \subset]-1, 1[\times]-1, 1[$. Integrating by parts and using $\partial_u G_0 = 1$, $\partial_v G_0 = 0$ for the first one, we get

$$4I_w(\varphi) = - \int_{-1}^1 \left[\int_{-|w|^2}^1 \frac{\partial \varphi}{\partial u}(u + iv, w) du \right] dv .$$

Using again $\text{Supp}(\varphi(\cdot, w)) \subset]-1, 1[\times]-1, 1[$, we get

$$I_w(\varphi) = \frac{1}{4} \int_{-1}^1 \varphi(-|w|^2 + iv, w) dv .$$

We therefore obtain by (14) :

$$\langle T_{11}, \varphi \rangle = \frac{1}{4} \int_{\mathbb{D}} \int_{-1}^1 \varphi(-|w|^2 + iv, w) dv d\text{Leb}(w)$$

Observe that $(w, v) \mapsto (-|w|^2 + iv, w)$ is another parametrization of M_0 different from Φ defined in (13). Let us write this integral with polar coordinates :

$$\langle T_{11}, \varphi \rangle = \frac{1}{8} \int_{-1}^1 \int_0^{2\pi} \int_0^1 \varphi(-\rho^2 + iv, \rho e^{i\theta}) 2\rho d\rho d\theta dv .$$

Applying the change of variable $u = -\rho^2$ one gets back to the parametrization Φ :

$$\langle T_{11}, \varphi \rangle = \frac{1}{8} \int_{]-1, 0] \times]-1, 1[\times]0, 2\pi[} \varphi(u + iv, \sqrt{-u}e^{i\theta}) d\text{Leb}(u, v, \theta) = \frac{1}{8} \int_{M_0} \varphi d\text{Leb}_{M_0} .$$

Computation of T_{22} . - Let Δ_z be the vertical disc $\{z\} \times \mathbb{D}$ intersected with Ω . One has :

$$\langle T_{22}, \varphi \rangle = \int_{]-1, 1[\times]-1, 1[} \left[\int_{\Delta_z} G_0(z, w) \varphi_{w\bar{w}}(z, w) d\text{Leb}(w) \right] d\text{Leb}(z) .$$

Writing $z = u + iv$, we get

$$\langle T_{22}, \varphi \rangle = \int_{-1}^1 \left[\int_{-1}^0 J_{u,v}(\varphi) du \right] dv + \int_{-1}^1 \left[\int_0^1 J_{u,v}(\varphi) du \right] dv ,$$

where

$$J_{u,v}(\varphi) := \int_{\mathbb{D} \setminus \mathbb{D}(\sqrt{-u})} G_0(u + iv, w) \varphi_{w\bar{w}}(u + iv, w) d\text{Leb}(w) ,$$

with the convention $\mathbb{D}(\sqrt{-u}) = \emptyset$ if $0 \leq u \leq 1$.

Lemma 8.2.

1. $J_{u,v}(\varphi) = \int_{\mathbb{D}} \varphi(u + iv, w) d\text{Leb}(w)$ for $0 \leq u \leq 1$.
2. $J_{u,v}(\varphi) = \int_{\mathbb{D} \setminus \mathbb{D}(\sqrt{-u})} \varphi(z, w) d\text{Leb}(w) - \frac{u}{2} \int_0^{2\pi} \varphi(z, \sqrt{-u}e^{i\theta}) d\theta$ for $-1 \leq u \leq 0$.

Proof : We write $w = s + it$ and use $\varphi_{w\bar{w}} = \frac{1}{4} \Delta_{s,t} \varphi$. Since φ has compact support in D , one gets by Green-Stokes formula :

$$4J_{u,v}(\varphi) := \int_{\mathbb{D}} G_0(u + iv, w) \Delta_{s,t} \varphi d\text{Leb}(w) = \int_{\mathbb{D}} \Delta_{s,t} G_0(u + iv, w) \varphi d\text{Leb}(w)$$

The first item follows since $\partial_s^2 G_0 = \partial_t^2 G_0 = 2$. In the case $-1 \leq u \leq 0$, we have :

$$J_{u,v}(\varphi) = \int_{\mathbb{D} \setminus \mathbb{D}(\sqrt{-u})} G_0(u + iv, w) \varphi_{w\bar{w}}(u + iv, w) d\text{Leb}(w) . \quad (15)$$

Let us denote $z = u + iv$ and $w = s + it \in \mathbb{D} \setminus \mathbb{D}(\sqrt{-u})$. Using $\text{Supp}(\varphi(z, \cdot)) \subset \mathbb{D}$ and $G_0(z, \cdot)|_{\partial\mathbb{D}(\sqrt{-u})} = 0$, Green-Stokes formula gives :

$$\begin{aligned} & \int_{\mathbb{D} \setminus \mathbb{D}(\sqrt{-u})} G_0(z, s + it) \Delta_{s,t} \varphi(z, s + it) - \Delta_{s,t} G_0(z, s + it) \varphi(z, s + it) \, d\text{Leb}(s, t) \\ &= \int_{\partial\mathbb{D}(\sqrt{-u})} \varphi(z, s + it) (\nabla_{(s,t)} G_0(z, s + it)) \cdot (s, t) \, d\sigma(s, t), \end{aligned}$$

where σ is the Lebesgue measure on the circle $\partial\mathbb{D}(\sqrt{-u})$. Since $\Delta_{s,t} G_0(z, s + it) = 4$ and $(\nabla_{(s,t)} G_0(z, s + it)) \cdot (s, t) = 2|w|^2 = -2u$, we infer

$$4 \int_{\mathbb{D} \setminus \mathbb{D}(\sqrt{-u})} \varphi(z, w) - G_0(z, w) \varphi_{w\bar{w}}(z, w) \, d\text{Leb}(w) = \int_{\partial\mathbb{D}(\sqrt{-u})} 2u \varphi(z, w) \, d\sigma(w).$$

The second item follows by combining this formula with (15). \square

Let us now return to the computation of $\langle T_{22}, \varphi \rangle$. Let $\Omega^+ := \Omega \cap \{u > 0\}$ and $\Omega^- := \Omega \cap \{u < 0\}$. According to Lemma 8.2, we get

$$\int_{-1}^1 \int_0^1 J_{u,v}(\varphi) \, du \, dv = \int_{\Omega^+} \varphi(z, w) \, d\text{Leb}(z, w),$$

and that $\int_{-1}^1 \int_{-1}^0 J_{u,v}(\varphi) \, du \, dv$ is equal to

$$\int_{\Omega^-} \varphi(z, w) \, d\text{Leb}(z, w) + \int_{-1}^1 \left[\int_{-1}^0 \left[\int_0^{2\pi} -\frac{u}{2} \varphi(u + iv, \sqrt{-u} e^{-i\theta}) \, d\theta \right] du \right] dv.$$

If $h(z, w) := \frac{|w|^2}{2}$ the second integral is equal to :

$$\int_{]-1,0[\times]-1,1[\times]0,2\pi[} (h\varphi) \circ \Phi(u, v, \theta) \, d(u, v, \theta) = \int_{M_0} h\varphi \, d\text{Leb}_{M_0}.$$

We deduce as desired

$$\langle T_{22}, \varphi \rangle = \int_{-1}^1 \int_{-1}^1 J_{u,v}(\varphi) \, du \, dv = \int_{\Omega} \varphi(z, w) \, d\text{Leb}(z, w) + \int_{M_0} h\varphi \, d\text{Leb}_{M_0}.$$

Computation of μ_0 .— Writing $G_0^+ := \max\{G_0, 0\}$, we have $T_0 = dd^c G_0^+$ and $\mu_0 = dd^c(G_0^+ dd^c G_0^+)$. Hence

$$\begin{aligned} \langle \mu_0, \varphi \rangle &= \langle T_0, G_0^+ dd^c \varphi \rangle = \langle T_{11} dd^c |z|^2 + T_{22} dd^c |w|^2, G_0^+ dd^c \varphi \rangle \\ &= \frac{1}{8} \int_{M_0} G_0 \varphi_{w\bar{w}} \, d\text{Leb}_{M_0} + \int_{\Omega} G_0 \varphi_{z\bar{z}} \, d\text{Leb} + \int_{M_0} G_0 \varphi_{z\bar{z}} h \, d\text{Leb}_{M_0}. \end{aligned}$$

Since $G_0 = 0$ on M_0 , we get $\langle \mu_0, \varphi \rangle = \int_{\Omega} G_0 \varphi_{z\bar{z}} \, d\text{Leb}_{\Omega} = \langle T_{11}, \varphi \rangle$ as desired (observe that this is the Lebesgue part on Ω in T_{22} which provides $\mu = T_{11}$). The formula $\mu_0 = \frac{1}{1+8h} \mathbb{1}_{M_0} T_0 \wedge \omega_0$ immediately follows.

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