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# On slice measures of Green currents on $\mathbb{C}\mathbb{P}^2$

Christophe Dupont and Virgile Tapiero

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## Abstract

Let  $f$  be a holomorphic map of  $\mathbb{C}\mathbb{P}^2$  of degree  $d \geq 2$ , let  $T$  be its Green current and  $\mu = T \wedge T$  be its equilibrium measure. We give a new proof of a theorem due to Dujardin asserting that  $\mu \ll T \wedge \omega_{\mathbb{P}^2}$  implies  $\lambda_2 = \frac{1}{2} \text{Log } d$ , where  $\lambda_1 \geq \lambda_2$  are the Lyapunov exponents of  $\mu$ . Then, assuming  $\mu \ll T \wedge \omega_{\mathbb{P}^2}$ , we study slice measures  $\nu := T \wedge dd^c|W|^2$ , where  $W$  is a holomorphic local submersion. We give sufficient conditions on the Radon-Nikodym derivative of  $\mu$  with respect to the trace measure  $T \wedge \omega_{\mathbb{P}^2}$  ensuring  $\mu = \nu$ . The involved submersion  $W$  comes from normal coordinates for the inverse branches of the iterates of  $f$ .

**Keywords**— Holomorphic dynamics, Equilibrium measure, Green current, Lyapunov exponents, normal forms. *MSC 2020* : 32H50, 32U40, 37C40, 37D25

## 1 Introduction

Let  $f$  be a holomorphic map of  $\mathbb{P}^2$  of degree  $d \geq 2$ . The Green current  $T$  and the equilibrium measure  $\mu$  are invariant objects encoding the dynamical properties of  $f$ , we refer to the books [9], [18] by Dinh and Sibony. We recall that  $T := \lim_n \frac{1}{d^n} f^{n*} \omega_{\mathbb{P}^2}$ , where  $\omega_{\mathbb{P}^2}$  is the normalized Fubini-Study  $(1, 1)$ -form of  $\mathbb{P}^2$ , this is a closed positive current, with local Hölder potentials. It satisfies  $f^*T = dT$  and its auto-intersection  $\mu := T \wedge T$  defines an invariant probability measure on  $\mathbb{P}^2$ , which is mixing and satisfies  $f^*\mu = d^2\mu$ .

Berteloot-Loeb [5] proved that  $T$  is a smooth and non degenerate positive  $(1, 1)$ -form on an open subset of  $\mathbb{P}^2$  if and only if  $f$  is a Lattès map. Berteloot-Dupont [1] established later that Berteloot-Loeb's condition on  $T$  characterizes the condition  $\mu \ll \text{Leb}_{\mathbb{P}^2}$ . Another characterization of Lattès maps involves the Lyapunov exponents  $\lambda_1 \geq \lambda_2$  of  $\mu$  (see Theorem 2.1 for their definitions). Briend-Duval [6] proved that  $\lambda_2 \geq \frac{1}{2} \text{Log } d$ . The equality  $\lambda_1 = \lambda_2 = \frac{1}{2} \text{Log } d$  holds if and only if  $\mu \ll \text{Leb}_{\mathbb{P}^2}$ , see [12].

It is now natural to characterize the mappings satisfying  $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d$ . Hopefully, the presence of a minimal Lyapunov exponent should be equivalent to regular properties for  $T$  and  $\mu$ , and perhaps (in some sense) to the existence of a one dimensional Lattès-like factor. The following result, due to Dujardin, nicely fits into this program.

**Theorem 1.1** (Dujardin [10, Theorem 3.6]). *If  $\mu \ll \sigma_T := T \wedge \omega_{\mathbb{P}^2}$  then  $\lambda_2 = \frac{1}{2} \text{Log } d$ .*

He also asked the question of the reverse implication, a partial answer is provided in [19]. The measure  $\sigma_T$  is called the trace of  $T$  and carries its mass. The proof of Dujardin's theorem is based on a construction of a  $df$ -invariant sub-bundle  $\mathcal{T} \subset T\mathbb{P}^2$  (called Fatou directions) of rank  $\geq 1$  that satisfies for  $\sigma_T$ -almost every  $x$  and for any  $\vec{v} \in \mathcal{T}_x \setminus \{0\}$  :

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \text{Log} \|d_x f^n(\vec{v})\| \leq \frac{1}{2} \text{Log } d. \quad (1)$$

Indeed, using  $\mu \ll \sigma_T$  and Oseledec's Theorem 2.1, for  $\mu$ -almost every  $x$ , the lim sup in (1) tends to  $\lambda_1$  or  $\lambda_2$ , which implies in both cases  $\lambda_2 \leq \frac{1}{2} \text{Log } d$ . Briend-Duval inequality  $\lambda_2 \geq \frac{1}{2} \text{Log } d$  then implies the equality. We note that these arguments only require that  $\mu \ll \sigma_T$  on an open sub-set  $U$  charged by  $\mu$ .

On a chart  $U \subset \mathbb{P}^2$  equipped with holomorphic coordinates  $(Z, W)$  we have :

$$\sigma_T \asymp (T \wedge dd^c|Z|^2 + T \wedge dd^c|W|^2).$$

By construction  $\text{Supp}(\mu) \subset \text{Supp}(\sigma_T)$ , hence if  $\mu(U) > 0$  then  $(T \wedge dd^c|Z|^2)(U) > 0$  or  $(T \wedge dd^c|W|^2)(U) > 0$ . Actually in this case  $T \wedge dd^c|Z|^2$  and  $T \wedge dd^c|W|^2$  both charge  $U$ . The idea is that if  $T \wedge dd^c|Z|^2 \equiv 0$  on  $U$ , then the potentials of  $T$  would be harmonic on almost every vertical disc contained in  $U$ , which implies that  $\mu$  is null on  $U$ , see [13, §3.3].

The measures  $T \wedge dd^c|Z|^2$  and  $T \wedge dd^c|W|^2$  are called *slices* of  $T$ . Dujardin's theorem gives an information on  $\lambda_2$  when  $\mu \ll \sigma_T$ , the proof is obtained by applying general results concerning Fatou directions [10]. Our purpose in this article is to analyse more deeply the relations between  $\mu$  and slices of  $T$  when  $\mu \ll \sigma_T$ .

**We first provide another proof of Dujardin's Theorem by using normal forms for the generic inverse branches of  $f^n$  and forward recurrent properties.** We then provide another proof (of a weaker version of) Theorem 1.1 using backward recurrent properties (we indeed assume that the density of  $\mu$  with respect to  $T$  is bounded). The interest of this second proof is to introduce a decomposition of  $\mu$  using normal coordinates. This decomposition is the cornerstone of the proof of Theorem 1.2 that we now introduce.

Relations between  $\mu$  and slices of  $T$  can be obtained when  $f$  preserves a pencil of lines, given for instance by the meromorphic function  $\pi[z : w : t] = [z : w]$ . Dupont-Taffin [14] proved that in this situation  $\mu$  and  $T$  are related by the formula :

$$\mu = T \wedge \pi^* \mu_\theta \quad (\text{which implies } \mu \ll \sigma_T), \quad (2)$$

where  $\theta$  is the rational map satisfying  $\pi \circ f = \theta \circ \pi$ . On another hand, Berteloot-Loeb [4] proved that if  $\theta$  is a Lattès map, then for every  $a \in \mathbb{P}^1$  outside a finite sub-set, there exists a holomorphic coordinate  $\zeta_a$  such that  $\mu_\theta = dd^c|\zeta_a|^2$  on a neighborhood  $V_a$  of  $a$  ( $\mu_\theta$  is the equilibrium measure of  $\theta$ ). Combining this result with (2), we obtain on  $\pi^{-1}(V_a)$  :

$$\mu = T \wedge dd^c|W|^2, \quad \text{with } W := \zeta_a \circ \pi. \quad (3)$$

This gives an example where a slice of  $T$  is equal to  $\mu$ . Theorem 1.2 shows a similar formula. We assume  $\mu \ll \sigma_T$  and denote  $\psi \in L^1(\sigma_T)$  the Radon-Nikodym derivative of  $\mu$  with respect to  $\sigma_T$ . Let  $J := \text{Supp}(\mu)$  and  $\psi|_J$  be the restriction of  $\psi$  on  $J$ . Note that the measurable function  $\psi$  is defined on  $\mathbb{P}^2$ , it is not unique since one can modify it outside  $J$  by still preserving  $\mu = \psi \sigma_T$ . However, one has  $\mu = (\psi \mathbb{1}_J) \sigma_T$  on  $\mathbb{P}^2$  for every version of  $\psi$ . We denote  $\psi_\Omega := \psi \mathbb{1}_\Omega$  for every borel sub-set  $\Omega$  of  $\mathbb{P}^2$ . In our statement, the local submersion  $W_{\hat{x}} : B(x_0, \eta_\varepsilon(\hat{x})) \rightarrow \mathbb{C}$  is provided by the normal form Theorem 2.2.

**Theorem 1.2.** *Assume that  $\mu = \psi_J \sigma_T$  for some measurable function  $\psi$  on  $\mathbb{P}^2$  (in particular  $\lambda_2 = \frac{1}{2} \text{Log } d$ ) and that  $\lambda_1$  is strictly larger than  $\lambda_2$ .*

1. *If  $\psi|_J$  is continuous, then for  $\hat{\mu}$ -almost every  $\hat{x}$  there exist  $r_1(\hat{x}) \leq \eta_\varepsilon(\hat{x})$  and  $C_{\hat{x}} > 0$  such that :*

$$\mu = C_{\hat{x}} (T \wedge dd^c|W_{\hat{x}}|^2)|_J \text{ on } B(x_0, r_1(\hat{x})).$$

2. If  $\psi$  is continuous on an open neighborhood  $V$  of  $J$  and if  $\mu = \psi_V \sigma_T$ , then there exist  $r_2(\hat{x}) \leq \eta_\varepsilon(\hat{x})$  and  $C_{\hat{x}} > 0$  such that :

$$\mu = C_{\hat{x}} T \wedge dd^c |W_{\hat{x}}|^2 \text{ on } B(x_0, r_2(\hat{x})).$$

The first Item implies that for  $\hat{\mu}$ -almost every  $\hat{x}, \hat{y}$  satisfying  $\pi_0(\hat{x}) = \pi_0(\hat{y})$ , the measures  $(T \wedge dd^c |W_{\hat{x}}|^2)|_J$  and  $(T \wedge dd^c |W_{\hat{y}}|^2)|_J$  coincide on a small ball centered at  $\pi_0(\hat{x})$ . In particular, the germ of measure  $(T \wedge dd^c |W_{\hat{x}}|^2)|_J$  does not depend on the full orbit  $\hat{x}$  but only on the  $\pi_0$ -projection of  $\hat{x}$ . A similar remark holds for the second Item.

Let us also observe that the hypothesis  $\lambda_1 > \lambda_2$  in Theorem 1.2 is not restrictive. Assume indeed that  $\mu = \psi_J \sigma_T$  and that  $\lambda_1 = \lambda_2$ . Then the Lyapunov exponents are both minimal (equal to  $\frac{1}{2} \text{Log } d$ ) and  $f$  is a Lattès mapping on  $\mathbb{P}^2$ . In this case there exists a finite ramified covering  $\sigma : \mathbb{C}^2/\Lambda \rightarrow \mathbb{P}^2$  such that  $\sigma^* T$  is equal to a positive definite hermitian form  $H$  with constant coefficients ( $\Lambda$  being a cocompact real lattice in  $\mathbb{C}^2$ ), see [11]. This implies that, around every point  $p$  outside an algebraic sub-set of  $\mathbb{P}^2$  (the critical values of  $\sigma$ ), the current  $T$  coincides with  $H$  in local coordinates given by  $\sigma$ . It follows that, for every local submersion  $W$  defined near  $p$ , the measure  $\mu = T \wedge T$  is equal to  $T \wedge dd^c |W|^2$  multiplied by a smooth positive function.

Let us finally note that the assumption of Item 2 morally replaces the function  $\psi_J = \psi \mathbb{1}_J$  of the first Item (which is singular due to the multiplication by  $\mathbb{1}_J$ ) by a continuous function on a neighborhood  $V$  of  $J$ . More precisely, it requires the existence of an open neighborhood  $V$  of  $J$  such that  $\psi|_V$  is continuous and such that  $\text{Supp}(\psi \sigma_T)|_V = J$ . This assumption is fulfilled if one assumes that the restriction  $\psi|_J$  is continuous and that  $\text{Supp} T = J$ , by applying Tietze-Urysohn theorem to  $\psi|_J$ .

**Question 1.3.** *Is it sufficient to assume  $\psi|_J$  continuous to get  $\mu = T \wedge dd^c |W_{\hat{x}}|^2$  locally ?*

We remark that the continuity of  $\psi|_J$  is satisfied by suspensions of Lattès maps on  $\mathbb{P}^1$ , we provide explicit computations of  $T, \mu$  and  $\psi$  in Section 8.

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## 2 Classical results

### 2.1 Lyapunov exponents

Let  $f$  be a holomorphic mapping of  $\mathbb{P}^2$  of degree  $d \geq 2$ . Let  $T$  be its Green current and  $\mu = T \wedge T$  be its equilibrium measure, it is  $f$ -invariant and ergodic. Since  $T$  has local continuous  $psh$  potentials,  $\mu$  integrates local  $psh$  functions. In particular  $\mu$  does not charge analytic sub-sets of  $\mathbb{P}^2$  and  $\mu(\mathcal{C}) = 0$ , where  $\mathcal{C} := \bigcup_{n \in \mathbb{Z}} f^n(\text{Crit } f)$ . It also allows to define the Lyapunov exponents  $\lambda_1 \geq \lambda_2$  :

$$\lambda_1 = \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbb{P}^2} \text{Log} \|df^n\| d\mu \text{ and } \lambda_1 + \lambda_2 = \int_{\mathbb{P}^2} \text{Log} |\det_{\mathbb{C}} df| d\mu.$$

Briend-Duval [6] proved that  $\lambda_2 \geq \frac{1}{2} \text{Log } d$ . The Oseledec theorem states as follows.

**Theorem 2.1** (Oseledec).

1. If  $\lambda_1 = \lambda_2 = \lambda$ , then there exists an invariant borel subset  $A_{os}$  of full  $\mu$ -measure and disjoint from  $\mathcal{C}$  such that for every  $x \in A_{os}$  :

$$\forall \vec{v} \in T_x \mathbb{P}^2 \setminus \{0\}, \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \text{Log} \|d_x f^n(\vec{v})\| = \lambda.$$

2. If  $\lambda_1 > \lambda_2$ , then there exists an invariant borel subset  $A_{os}$  of full  $\mu$ -measure and disjoint from  $\mathcal{C}$  such that for every  $x \in A_{os}$ , there exists  $v_s(x) \in \mathbb{P}(T_x \mathbb{P}^2)$  satisfying :

$$\forall \vec{v} \in T_x \mathbb{P}^2 \setminus v_s(x), \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \text{Log} \|d_x f^n(\vec{v})\| = \lambda_1.$$

$$\forall \vec{v} \in v_s(x) \setminus \{0\}, \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \text{Log} \|d_x f^n(\vec{v})\| = \lambda_2.$$

Moreover  $v_s$  is measurable and satisfies  $[d_x f](v_s(x)) = v_s(f(x))$  for every  $x \in A_{os}$ , where  $[d_x f]$  is the projectivization of the tangent map  $d_x f$ .

## 2.2 Normal forms for inverse branches

We denote  $X := \mathbb{P}^2 \setminus \mathcal{C}$  which is totally invariant by  $f$ , and we denote  $\widehat{X}$  the set of all sequences  $\widehat{x} = (x_n)_{n \in \mathbb{Z}}$  of  $X$  that are orbits under the action of  $f$  i.e.  $x_{n+1} = f(x_n)$ . Equipped with the natural projection  $\pi_0 : \widehat{X} \rightarrow X$  defined by  $\pi_0(\widehat{x}) = x_0$ , the dynamical system  $(X, f, \mu)$  admits a natural extension  $(\widehat{X}, \widehat{f}, \widehat{\mu})$ , where  $\widehat{f}$  is the left shift on  $\widehat{X}$  and  $\widehat{\mu}$  is the unique probability measure on  $\widehat{X}$  invariant by  $\widehat{f}$  such that  $(\pi_0)_* \widehat{\mu} = \mu$ . The measure  $\widehat{\mu}$  is ergodic as  $\mu$ , see [8, Chapter 10].

Given an orbit  $\widehat{x} \in \widehat{X}$ , there exists a family of inverse branches  $(f_{\widehat{x}}^{-n})_{n \in \mathbb{N}}$  such that  $f_{\widehat{x}}^{-n}$  is defined on  $B(x_0, \eta_\varepsilon(\widehat{x}))$  for every  $n \geq 0$ , and such that  $\text{Lip}(f_{\widehat{x}}^{-n}) \leq \beta_\varepsilon(\widehat{x}) e^{-n(\lambda_2 - \varepsilon)}$ . The functions  $\widehat{x} \mapsto \eta_\varepsilon(\widehat{x})$  and  $\widehat{x} \mapsto \beta_\varepsilon(\widehat{x})$  are  $\varepsilon$ -tempered, they satisfy :

$$\forall n \in \mathbb{Z}, \quad e^{-|n|\varepsilon} \varphi(\widehat{x}) \leq \varphi(\widehat{f}^n(\widehat{x})) \leq e^{+|n|\varepsilon} \varphi(\widehat{x}).$$

We refer to Briend-Duval [6] (see also [7, 12]) for details about the construction of the inverse branches. The following result provides normal forms for those mappings, see [2, 3, 15].

**Theorem 2.2.** *For every  $\varepsilon > 0$  small enough with respect to the Lyapunov exponents  $\lambda_1 \geq \lambda_2$ , there exists an invariant borel sub-set  $\Lambda_\varepsilon$  of full  $\widehat{\mu}$ -measure,  $\varepsilon$ -tempered functions  $\rho_\varepsilon : \Lambda_\varepsilon \rightarrow ]0, 1]$ ,  $\beta_\varepsilon, L_\varepsilon, M_\varepsilon : \Lambda_\varepsilon \rightarrow [1, +\infty[$  and a function  $N : \Lambda_\varepsilon \rightarrow \mathbb{N}$  satisfying the following properties. For every  $\widehat{x} \in \Lambda_\varepsilon$ , there exist injective holomorphic mappings*

$$\xi_{\widehat{x}} : B(x_0, \eta_\varepsilon(\widehat{x})) \longrightarrow \mathbb{D}^2(\rho_\varepsilon(\widehat{x}))$$

satisfying :

1.  $\xi_{\widehat{x}}(x_0) = 0$  and  $d_{x_0} \xi_{\widehat{x}}(v_s(x_0))$  is the vertical axis in  $\mathbb{C}^2$ ,
2.  $\forall p, q \in B(x_0, \eta_\varepsilon(\widehat{x}))$ ,  $\frac{1}{2} \text{dist}(p, q) \leq \|\xi_{\widehat{x}}(p) - \xi_{\widehat{x}}(q)\| \leq \beta_\varepsilon(\widehat{x}) \text{dist}(p, q)$ ,
3. the following diagram commutes for every  $n \geq N(\widehat{x})$  :

$$\begin{array}{ccc} B(x_{-n}, \eta_\varepsilon(\widehat{x}_{-n})) & \xleftarrow{f_{\widehat{x}}^{-n}} & B(x_0, \eta_\varepsilon(\widehat{x})) \\ \xi_{\widehat{x}_{-n}} \downarrow & & \downarrow \xi_{\widehat{x}} \\ \mathbb{D}^2(\rho_\varepsilon(\widehat{x}_{-n})) & \xleftarrow{R_{n, \widehat{x}}} & \mathbb{D}^2(\rho_\varepsilon(\widehat{x})) \end{array} \quad (4)$$

The mappings  $R_{n, \widehat{x}}$  have the following form depending on  $(\lambda_1, \lambda_2)$  :

(i) If  $\lambda_1 = \lambda_2 = \lambda$ , then  $R_{n,\hat{x}}$  is a linear and satisfies

$$e^{-n(\lambda+\varepsilon)}\|(z, w)\| \leq \|R_{n,\hat{x}}(z, w)\| \leq e^{-n(\lambda-\varepsilon)}\|(z, w)\|.$$

(ii) If  $\lambda_1 = k\lambda_2$  for some  $k \geq 2$  (we say that  $\lambda_1$  and  $\lambda_2$  are resonant), then  $R_{n,\hat{x}}(z, w) = (\alpha_{n,\hat{x}}z, \beta_{n,\hat{x}}w) + (\gamma_{n,\hat{x}}w^k, 0)$ .

(iii) If  $\lambda_1 \notin \{k\lambda_2, k \geq 1\}$ , then  $R_{n,\hat{x}}(z, w) = (\alpha_{n,\hat{x}}z, \beta_{n,\hat{x}}w)$ .

Moreover, in the cases (ii) and (iii), we have

$$e^{-n(\lambda_1+\varepsilon)} \leq |\alpha_{n,\hat{x}}| \leq e^{-n(\lambda_1-\varepsilon)}, \quad |\gamma_{n,\hat{x}}| \leq M_\varepsilon(\hat{x})e^{-n(\lambda_1-\varepsilon)}$$

and

$$e^{-n(\lambda_2+\varepsilon)} \leq |\beta_{n,\hat{x}}| \leq e^{-n(\lambda_2-\varepsilon)}.$$

**Remark 2.3.** The diagram (4) commutes for every fixed  $n \geq 0$  by reducing the radius of the balls. The property  $n \geq N(\hat{x})$  actually ensures that  $f_{\hat{x}}^{-n}(B(x_0, \eta_\varepsilon(\hat{x}))) \subset B(x_{-n}, \eta_\varepsilon(\hat{x}_{-n}))$ .

We shall need the following control of  $(f^n)^*\omega_{\mathbb{P}^2}$  in Section 3.

**Lemma 2.4.** For every  $\hat{x} \in \Lambda_\varepsilon$  and  $n \geq 0$ , there exists  $0 < r_n \leq \eta_\varepsilon(\hat{x})$  such that

$$(f^n)^*\omega_{\mathbb{P}^2} \geq \beta_\varepsilon(\hat{x})^{-2}e^{2n(\lambda_2-2\varepsilon)}\omega_{\mathbb{P}^2} \text{ on } B(x_0, r_n).$$

**Proof :** We assume to simplify that  $\lambda_1 > \lambda_2$  and that  $\lambda_1, \lambda_2$  are not resonant, the case of equality and the resonant case can be treated similarly (up to modify the function  $\beta_\varepsilon$ ). Let  $\varepsilon > 0$  be small enough such that  $\lambda_1 - \lambda_2 > 2\varepsilon$ , ensuring  $|\alpha_{n,\hat{x}_n}|^{-1} \geq |\beta_{n,\hat{x}_n}|^{-1}$ . According to Remark 2.3, there exists  $r_n > 0$  and  $0 < r'_n \leq \eta_\varepsilon(\hat{x}_n)$  such that  $f_{\hat{x}_n}^{-n} = \xi_{\hat{x}}^{-1} \circ R_{n,\hat{x}_n} \circ \xi_{\hat{x}_n}$  on  $B(x_n, r'_n)$  and  $B(x_0, r_n) \subset f_{\hat{x}_n}^{-n}(B(x_n, r'_n))$ . Since we have on  $B(x_0, r_n)$  :

$$(f^n)^*\omega_{\mathbb{P}^2} \geq \|(df^n)^{-1}\|^{-2}\omega_{\mathbb{P}^2}$$

and

$$\|(df^n)^{-1}\|^{-2} = \|df_{\hat{x}_n}^{-n}\|^{-2} \geq \|d\xi_{\hat{x}}^{-1}\|^{-2}\|dR_{n,\hat{x}_n}\|^{-2}\|d\xi_{\hat{x}_n}\|^{-2},$$

Theorem 2.2 implies

$$(f^n)^*\omega_{\mathbb{P}^2} \geq 4\|dR_{n,\hat{x}_n}\|^{-2}\beta_\varepsilon(\hat{x}_n)^{-2}\omega_{\mathbb{P}^2} \text{ on } B(x_0, r_n).$$

Since  $|\alpha_{n,\hat{x}_n}|^{-1} \geq |\beta_{n,\hat{x}_n}|^{-1}$  and  $\beta_\varepsilon$  is  $\varepsilon$ -tempered, we deduce on  $B(x_0, r_n)$  :

$$(f^n)^*\omega_{\mathbb{P}^2} \geq 4|\beta_{n,\hat{x}_n}|^{-2}\beta_\varepsilon(\hat{x})^{-2}e^{-2n\varepsilon}\omega_{\mathbb{P}^2}.$$

The conclusion follows from  $|\beta_{n,\hat{x}_n}|^{-2} \geq e^{2n(\lambda_2-\varepsilon)}$ .  $\square$

We shall also need the following lemma in Section 5.

**Lemma 2.5.** Assume that  $\mu = \psi_J\sigma_T$  for some measurable function  $\psi$  on  $\mathbb{P}^2$ . Let  $\hat{x} \in \Lambda_\varepsilon$  and  $B := B(x_0, \eta_\varepsilon(\hat{x}))$ . Let  $A_n := \{p \in B \cap J, \psi \circ f_{\hat{x}}^{-n}(p) > 0\}$ .

1. For every  $n \geq 0$ , the measures  $\sigma_T$  and  $(f_{\hat{x}}^{-n})^*\sigma_T$  are equivalent on  $B$ , and  $A_n = A_0$  modulo a borel sub-set of zero  $\sigma_T$ -measure.
2. If  $A \subset B \cap J$  satisfies  $\mu(A) = 0$ , then  $\sigma_T(A \cap A_0) = 0$ . In particular, for  $a, b \geq 0$ ,
  - (i) If  $\psi \circ f_{\hat{x}}^{-n} \leq b$   $\mu$ -a.e. on  $B \cap J$ , then  $\psi \circ f_{\hat{x}}^{-n} \leq b$   $\sigma_T$ -a.e. on  $B \cap J$ .
  - (ii) If  $\psi \circ f_{\hat{x}}^{-n} \geq a$   $\mu$ -a.e. on  $B \cap J$ , then  $\psi \circ f_{\hat{x}}^{-n} \geq a$   $\sigma_T$ -a.e. on  $B \cap J \cap A_0$ .

**Proof :** Since  $f_{\hat{x}}^{-n} : B \rightarrow f_{\hat{x}}^{-n}(B)$  is a biholomorphism (and up to shrink  $B$ ), there exists  $0 < \alpha < \beta$  such that

$$\alpha \omega_{\mathbb{P}^2} \leq (f_{\hat{x}}^{-n})^* \omega_{\mathbb{P}^2} \leq \beta \omega_{\mathbb{P}^2} \text{ on } B.$$

Using that  $T$  is  $f$ -invariant, we deduce that  $\sigma_T$  and  $(f_{\hat{x}}^{-n})^* \sigma_T$  are equivalent on  $B$ . Now for every  $p \in B \cap J$  and every  $r$  small enough such that  $B(p, r) \subset B$ ,

$$\mu(B(p, r)) = \int_{B(p, r)} \psi_J \sigma_T.$$

But  $\mu(B(p, r)) = d^{2n} \mu(f_{\hat{x}}^{-n}(B(p, r)))$  and

$$\mu(f_{\hat{x}}^{-n}(B(p, r))) = \int_{B(p, r)} \psi_J \circ f_{\hat{x}}^{-n} (f_{\hat{x}}^{-n})^* \sigma_T \simeq_{\alpha, \beta} d^{-n} \int_{B(p, r)} \psi_J \circ f_{\hat{x}}^{-n} \sigma_T,$$

where  $\simeq_{\alpha\beta}$  means that the equality holds up to a multiplicative constant between  $\alpha$  and  $\beta$ . We deduce by dividing by  $\sigma_T(B(p, r))$  (which is positive since  $p \in J \subset \text{Supp } T$ ) :

$$\int_{B(p, r)} \psi_J \sigma_T \simeq_{\alpha, \beta} d^{-n} \int_{B(p, r)} \psi_J \circ f_{\hat{x}}^{-n} \sigma_T.$$

By Lebesgue Theorem [16, Theorem 2.12] (taking limits when  $r$  tends to zero), we get for  $\sigma_T$ -a.e.  $p \in B \cap J$  that  $\psi(p) > 0$  if and only if  $\psi \circ f_{\hat{x}}^{-n}(p) > 0$ . The second item comes from  $0 = \mu(A) = \int_{A \cap A_0} \psi_J \sigma_T$ , which gives  $\sigma_T(A \cap A_0) = 0$  since  $\psi_J$  is positive on  $A \cap A_0$ . To obtain (i) and (ii), we apply this fact to  $\{p \in B \cap J, \psi \circ f_{\hat{x}}^{-n}(p) > b\}$  and to  $\{p \in B \cap J, \psi \circ f_{\hat{x}}^{-n}(p) < a\}$  (which both have zero  $\mu$ -measure by assumption) and the fact that  $A_n = A_0$  modulo a borel sub-set of zero  $\sigma_T$ -measure (provided by the first item).  $\square$

### 3 A new proof of Dujardin's theorem

We provide another proof of Theorem 1.1. We shall use Theorem 2.2 and forward dynamics. Let  $\psi \in L^1(\sigma_T)$  be such that  $\mu = \psi_J \sigma_T$ . We recall that  $\lambda_2 \geq \frac{1}{2} \text{Log } d$  for every endomorphism of  $\mathbb{P}^2$ , it remains to prove the reverse inequality  $\lambda_2 \leq \frac{1}{2} \text{Log } d$  in our case of absolute continuity. For every  $n \geq 0$ , the set

$$E_n := \left\{ x \in \mathbb{P}^2 : \lim_{\rho \rightarrow 0^+} \int_{B(x, \rho)} \psi_J \circ f^n \sigma_T = \psi_J \circ f^n(x) \right\}$$

is a borel sub-set of full  $\sigma_T$ -measure (and so of full  $\mu$ -measure) by the Lebesgue Theorem. We define the borel set

$$E := \left( \bigcap_{n \in \mathbb{N}} E_n \right) \setminus \left( \bigcup_{n \geq 1} \text{Crit}(f^n) \right),$$

which is also of full  $\mu$ -measure ( $\mu$  does not charge analytic sub-sets). Let us define

$$F := E \cap \left\{ \frac{1}{\tau} \leq \psi_J \leq \tau \right\} \subset J,$$

where  $\tau > 0$  is chosen large enough to have  $\mu(F) > 0$ . According to Poincaré recurrence theorem, for  $\hat{\mu}$ -almost every  $\hat{x} \in \pi_0^{-1}(F)$ , there exists a sub-sequence  $\hat{x}_{n_k}$  which satisfies  $\hat{x}_{n_k} \in \pi_0^{-1}(F)$ . Let us fix  $\varepsilon > 0$  and apply the normal form Theorem 2.2, it yields a borel



sub-set  $\Lambda_\varepsilon$  of full  $\widehat{\mu}$ -measure of good orbits. Let us fix  $\widehat{x} \in \pi_0^{-1}(F) \cap \Lambda_\varepsilon$  and a sub-sequence  $(n_k)_k$  as before. Let us also fix an integer  $k$ . Since  $x_0 \in E$  avoids the critical set of  $f^{n_k}$ , there exists  $\rho_k > 0$  small enough such that  $f^{n_k}$  is injective on the ball  $B(x_0, \rho_k)$ . Then we have for any  $\rho \in ]0, \rho_k]$  :

$$\mu(A_\rho) = d^{-2n_k} \mu(f^{n_k}(A_\rho)), \quad (5)$$

where  $A_\rho := B(x_0, \rho)$ . Up to take a smaller  $\rho_k$ , we can assume that

$$f^{n_k}(A_{\rho_k}) \subset B(x_{n_k}, \eta_\varepsilon(\widehat{x}_{n_k})) \text{ and } A_\rho \subset B(x_0, r_{n_k}),$$

where  $r_{n_k}$  comes from Lemma 2.4. From  $\mu = \psi_J \sigma_T$ , Formula (5) yields for every  $\rho \in ]0, \rho_k]$  :

$$\begin{aligned} \int_{A_\rho} \psi_J \sigma_T &= d^{-2n_k} \int_{f^{n_k}(A_\rho)} \psi_J \sigma_T \\ &= d^{-2n_k} \int_{A_\rho} (\psi_J \circ f^{n_k}) d^{n_k} T \wedge (f^{n_k})^* \omega_{\mathbb{P}^2} \end{aligned}$$

where the second equality uses the invariance of the Green current  $(f^{n_k})^* T = d^{n_k} T$  and the fact that  $f^{n_k}$  is injective on  $A_\rho$ . Since  $A_\rho \subset B(x_0, r_{n_k})$ , we can use the estimate given by Lemma 2.4 with  $n = n_k$  to deduce

$$\begin{aligned} \int_{A_\rho} \psi_J \sigma_T &\geq d^{-n_k} \int_{A_\rho} (\psi_J \circ f^{n_k}) \beta_\varepsilon(\widehat{x})^{-2} e^{2n_k(\lambda_2 - 2\varepsilon)} T \wedge \omega_{\mathbb{P}^2} \\ &= d^{-n_k} \beta_\varepsilon(\widehat{x})^{-2} e^{2n_k(\lambda_2 - 2\varepsilon)} \int_{A_\rho} (\psi_J \circ f^{n_k}) \sigma_T. \end{aligned}$$

We chose  $x_0 \in J$ , thus  $\sigma_T(A_\rho) > 0$  and we have :

$$\int_{A_\rho} \psi_J \sigma_T \geq d^{-n_k} \beta_\varepsilon(\widehat{x})^{-2} e^{2n_k(\lambda_2 - 2\varepsilon)} \int_{A_\rho} (\psi_J \circ f^{n_k}) \sigma_T.$$

But we also have  $x_0 \in E \subset E_0 \cap E_{n_k}$ , thus we can take limits when  $\rho \rightarrow 0^+$  to get

$$\psi_J(x_0) \geq d^{-n_k} \beta_\varepsilon(\widehat{x})^{-2} e^{2n_k(\lambda_2 - 2\varepsilon)} \times \psi_J(f^{n_k}(x_0)).$$

Since  $x_0$  and  $x_{n_k} = f^{n_k}(x_0)$  belongs to  $F \subset \{\frac{1}{\tau} \leq \psi_J \leq \tau\}$ , we deduce that :

$$\tau \geq \frac{1}{\tau} \beta_\varepsilon(\widehat{x})^{-2} e^{-2n_k(\frac{1}{2} \text{Log } d - \lambda_2 + 2\varepsilon)}.$$

Since this is true for every integer  $k$ , we get  $\frac{1}{2} \text{Log } d - \lambda_2 + 2\varepsilon \geq 0$ . We obtain  $\lambda_2 \leq \frac{1}{2} \text{Log } d$  as desired, since  $\varepsilon > 0$  is arbitrary small.

## 4 Decomposition of $\mu$

### 4.1 Normal coordinates and absolute continuity

According to Theorem 2.2, there exists an invariant sub-set of full  $\widehat{\mu}$ -measure  $\Lambda_\varepsilon$  such that for any  $\widehat{x} \in \Lambda_\varepsilon$ , there exist inverse branches

$$f_{\widehat{x}}^{-n} : B(x_0, \eta_\varepsilon(\widehat{x})) \longrightarrow B(x_{-n}, \eta_\varepsilon(\widehat{x}_{-n})), \quad n \geq N(\widehat{x}),$$

and holomorphic charts

$$\xi_{\widehat{x}} = (Z_{\widehat{x}}, W_{\widehat{x}}) : B(x_0, \eta_\varepsilon(\widehat{x})) \longrightarrow \mathbb{D}^2(\rho_\varepsilon(\widehat{x})) \quad (6)$$



such that  $\frac{1}{2} \leq \|d\xi_{\hat{x}}\| \leq \beta_\varepsilon(\hat{x})$  on  $B(x_0, \eta_\varepsilon(\hat{x}))$  and

$$f_{\hat{x}}^{-n} = \xi_{\hat{x}_{-n}}^{-1} \circ R_{n, \hat{x}} \circ \xi_{\hat{x}}$$

for any  $n \geq N(\hat{x})$ , where  $R_{n, \hat{x}}$  is a polynomial map  $(\mathbb{C}^2, (Z_{\hat{x}}, W_{\hat{x}})) \rightarrow (\mathbb{C}^2, (Z_{\hat{x}_{-n}}, W_{\hat{x}_{-n}}))$ . When  $\lambda_1, \lambda_2$  are not resonant,  $R_{n, \hat{x}}$  is a linear diagonal map with eigenvalues  $|\alpha_{n, \hat{x}}| \simeq e^{-n(\lambda_1 \pm \varepsilon)}$  and  $|\beta_{n, \hat{x}}| \simeq e^{-n(\lambda_2 \pm \varepsilon)}$ . Let  $(\omega_{\hat{x}, AB})_{A, B \in \{Z, W\}}$  be the smooth functions defined by the formula

$$\omega_{\mathbb{P}^2} = \sum_{A, B \in \{Z, W\}} \omega_{\hat{x}, AB} \frac{i}{2} (dA_{\hat{x}} \wedge d\overline{B}_{\hat{x}}) \text{ on } B(x_0, \eta_\varepsilon(\hat{x})). \quad (7)$$

For every  $A, B$  in  $\{Z, W\}$  and for every  $n \geq N(\hat{x})$  we define the following functions :

$$F_{\hat{x}, n}^{AB} := \omega_{\hat{x}_{-n}, AB} \circ f_{\hat{x}}^{-n} : B(x_0, \eta_\varepsilon(\hat{x})) \longrightarrow \mathbb{C}.$$

**Proposition 4.1.** *Assume that  $\mu \ll \sigma_T$  and let  $\psi_J$  be the Radon-Nikodym derivative of  $\mu$  with respect to  $\sigma_T$ . We assume that the Lyapunov exponents  $\lambda_1 > \lambda_2$  are not resonant. For every  $\hat{x} \in \Lambda_\varepsilon$ , for every borel set  $U \subset B(x_0, \eta_\varepsilon(\hat{x}))$  and for every  $n \geq N(\hat{x})$  we have :*

$$\mu(U) = d^n \int_U (\psi_J \circ f_{\hat{x}}^{-n}) \sum_{A, B \in \{Z, W\}} F_{\hat{x}, n}^{AB} \alpha_{n, \hat{x}}^A \overline{\alpha_{n, \hat{x}}^B} T \wedge \frac{i}{2} dA_{\hat{x}} \wedge d\overline{B}_{\hat{x}},$$

where  $\alpha_{n, \hat{x}}^Z := \alpha_{n, \hat{x}}$  and  $\alpha_{n, \hat{x}}^W := \beta_{n, \hat{x}}$ .

**Proof :** From the equalities  $\mu(U) = d^{2n} \mu(f_{\hat{x}}^{-n}(U))$  and  $\mu = \psi_J(T \wedge \omega_{\mathbb{P}^2})$ , we obtain

$$\mu(U) = d^{2n} \int_U (\psi_J \circ f_{\hat{x}}^{-n}) (f_{\hat{x}}^{-n})^* T \wedge (f_{\hat{x}}^{-n})^* \omega_{\mathbb{P}^2}.$$

By invariance of the Green current, we get  $(f_{\hat{x}}^{-n})^* T = d^{-n} T$  on  $B(x_0, \eta_\varepsilon(\hat{x}))$  and

$$\mu(U) = d^n \int_U (\psi_J \circ f_{\hat{x}}^{-n}) T \wedge (f_{\hat{x}}^{-n})^* \omega_{\mathbb{P}^2}.$$

We conclude by pulling back Formula (7) ( $\hat{x}$  being replaced by  $\hat{x}_{-n}$ ) by  $f_{\hat{x}}^{-n}$  and using the fact that  $R_{n, \hat{x}}$  is linear and diagonal with eigenvalues  $\alpha_{n, \hat{x}}$  and  $\beta_{n, \hat{x}}$ .  $\square$

**Definition 4.2.** *Assume that  $\mu \ll \sigma_T$  and let  $\psi_J = \psi \mathbb{1}_J$  be the Radon-Nikodym derivative of  $\mu$  with respect to  $\sigma_T$ . We assume that the Lyapunov exponents  $\lambda_1 > \lambda_2$  are not resonant. For every  $U \subset B(x_0, \eta_\varepsilon(\hat{x}))$  and  $n \geq 0$ , we have according to Proposition 4.1 :*

$$\mu(U) = I_n(U \cap J) + J_n(U \cap J),$$

where for every  $D \subset B(x_0, \eta_\varepsilon(\hat{x}))$  :

$$I_n(D) := d^n |\beta_{n, \hat{x}}|^2 \int_D (\psi \circ f_{\hat{x}}^{-n}) F_{\hat{x}, n}^{WW} T \wedge \frac{i}{2} (dW_{\hat{x}} \wedge d\overline{W}_{\hat{x}})$$

and

$$J_n(D) := d^n \sum_{(A, B) \neq (W, W)} \alpha_{n, \hat{x}}^A \overline{\alpha_{n, \hat{x}}^B} \int_D (\psi \circ f_{\hat{x}}^{-n}) F_{\hat{x}, n}^{AB} T \wedge \frac{i}{2} (dA_{\hat{x}} \wedge d\overline{B}_{\hat{x}})$$

where  $\alpha_{n, \hat{x}}^Z := \alpha_{n, \hat{x}}$  and  $\alpha_{n, \hat{x}}^W := \beta_{n, \hat{x}}$ .

**Remark 4.3.** *We observe that if  $\mu = \psi_V \sigma_T$  as in the second item of Theorem 1.2, then*

$$\mu(U) = I_n(U \cap V) + J_n(U \cap V)$$

for every  $U \subset B(x_0, \eta_\varepsilon(\hat{x}))$ .

## 4.2 Recurrence in $\mathcal{A}_\tau$

**Definition 4.4.** For  $\tau > 0$  let  $\mathcal{A}_\tau \subset \Lambda_\varepsilon$  be the subset consisting of elements  $\hat{y}$  satisfying

1.  $|\omega_{\hat{y},AB}| \leq \tau$ ,
2.  $\frac{1}{\tau} \leq \omega_{\hat{y},AA} \leq \tau$ ,
3.  $S_{AB}(\hat{y}) := \sup_{B(y_0, \eta_\varepsilon(\hat{y}))} \|d\omega_{\hat{y},AB}\| \leq \tau$ ,
4.  $\psi_J \circ \pi_0(\hat{y}) \geq \frac{1}{\tau}$ ,

the two first items holding on  $B(y_0, \eta_\varepsilon(\hat{y}))$ , the three first ones for every  $A, B \in \{Z, W\}$ .

Since the distortion of the coordinates  $\xi_{\hat{x}} = (Z_{\hat{x}}, W_{\hat{x}})$  is controlled by Theorem 2.2, there exists  $\tau$  large enough such that  $\mathcal{A}_\tau$  has positive  $\hat{\mu}$ -measure. By Poincaré recurrence theorem, for  $\hat{\mu}$ -almost every  $\hat{x}$ , there exists  $(n_k)_{k \in \mathbb{N}}$  such that  $n_k \geq N(\hat{x})$  and

$$\hat{x}_{-n_k} \in \mathcal{A}_\tau.$$

We have specified  $n_k \geq N(\hat{x})$  to have  $f_{\hat{x}}^{-n_k}(B(x_0, \eta_\varepsilon(\hat{x})) \subset B(x_{-n_k}, \eta_\varepsilon(\hat{x}_{-n_k}))$ . In this way the functions  $F_{\hat{x}, n_k}^{AB}$  are well defined on  $B(x_0, \eta_\varepsilon(\hat{x}))$ . By definition of  $F_{\hat{x}, n}^{AB}$ , we get from the first item of Definition 4.4 :

$$|F_{\hat{x}, n_k}^{AB}| \leq \tau \text{ on } B(x_0, \eta_\varepsilon(\hat{x})). \quad (8)$$

Moreover, using the third item of Definition 4.4 and  $\text{Lip}(f_{\hat{x}}^{-n_k}) \leq \beta_\varepsilon(\hat{x})e^{-n_k(\lambda_2 - \varepsilon)}$  (given by Theorem 2.2), we get

$$\|dF_{\hat{x}, n_k}^{AB}\| \leq \tau \beta_\varepsilon(\hat{x})e^{-n_k(\lambda_2 - \varepsilon)} \text{ on } B(x_0, \eta_\varepsilon(\hat{x})).$$

We deduce that

$$F_{\hat{x}, n_k}^{AB} \text{ uniformly converges on } B(x_0, \eta_\varepsilon(\hat{x})) \text{ to a constant } F_{\hat{x}}^{AB} \text{ up to a subsequence} \quad (9)$$

and that (according to the second item of Definition 4.4) :

$$F_{\hat{x}, n_k}^{AA} \geq \frac{1}{\tau} \text{ on } B(x_0, \eta_\varepsilon(\hat{x})). \quad (10)$$

## 4.3 A weaker version of Dujardin's theorem

We give another proof of Theorem 1.1 assuming that the Radon-Nikodym derivative  $\psi_J \in L^\infty(\sigma_T)$ . Our motivation is to use backward iterates, namely the inverse branches  $f_{\hat{x}}^{-n_k}$ , in order to prepare the proof of Theorem 1.2 in Section 6.1. As in Section 3, it suffices to prove that  $\lambda_2 \leq \frac{1}{2} \text{Log } d$ .

For sake of simplicity, we assume that the Lyapunov exponents  $\lambda_1 > \lambda_2$  are not resonant. This implies that the polynomial map  $R_{n, \hat{x}}$  appearing in the diagram (4) is linear and diagonal. The proof can be adapted in the resonant case, see Section 7.

We apply the normal form Theorem 2.2 with  $\varepsilon > 0$ , it yields a borel sub-set  $\Lambda_\varepsilon$  of full  $\hat{\mu}$ -measure of good orbits. We use the set  $\mathcal{A}_\tau$  defined in Section 4.2. Let us fix a  $\hat{\mu}$ -generic element  $\hat{x} \in \Lambda_\varepsilon$ , in particular  $x_0 \in J$ . Let  $(n_k)_{k \in \mathbb{N}}$  be such that  $\hat{x}_{-n_k} \in \mathcal{A}_\tau$ .

Let  $\delta > 0$  be such that  $\mu(B(x_0, \eta_\varepsilon(\hat{x}))) > 2\delta$ . By Lusin's theorem there exists a compact set  $C_0 \subset B(x_0, \eta_\varepsilon(\hat{x}))$  such that  $\mu(B(x_0, \eta_\varepsilon(\hat{x})) \setminus C_0) < \delta$  and  $\psi_J$  is continuous on  $C_0$ . Using again Lusin's theorem we construct by induction a sequence  $(C_n)_n$  of compact sub-sets of  $B(x_0, \eta_\varepsilon(\hat{x}))$  such that

$$C_{n+1} \subset C_n \text{ and } \mu(C_n \setminus C_{n+1}) < 2^{-(n+1)} \delta$$

and

$$\psi_J \circ f_{\widehat{x}}^{-(n+1)} \in C^0(C_{n+1}).$$

Then  $\mu(C_n) > \mu(C_{n-1}) - 2^{-n}\delta > \dots > \mu(C_0) - \delta > \mu(B(x_0, \eta_\varepsilon(\widehat{x}))) - 2\delta > 0$ . Thus

$$C := \bigcap_{n \in \mathbb{N}} C_n$$

satisfies  $\mu(C) > 0$ . Let  $G$  be a borel sub-set of full  $\sigma_T$ -measure (and so of full  $\mu$ -measure) such that :

$$\forall p \in G, 0 \leq \psi_J(p) \leq \|\psi_J\|_{L^\infty(\sigma_T)}.$$

Let  $G_k$  be the Borel set :

$$G_k := f_{\widehat{x}}^{-n_k}(B(x_0, \eta_\varepsilon(\widehat{x}))) \cap G.$$

Because  $f_{\widehat{x}}^{n_k}$  is injective on  $f_{\widehat{x}}^{-n_k}(B(x_0, \eta_\varepsilon(\widehat{x})))$ , which contains  $G_k$ , one has  $\mu(f_{\widehat{x}}^{n_k}(G_k)) = d^{2n_k} \mu(G_k) = \mu(B(x_0, \eta_\varepsilon(\widehat{x})))$ , which is positive since  $x_0 \in J$ . Thus  $\mu(\bigcap_{k \in \mathbb{N}} f_{\widehat{x}}^{n_k}(G_k)) = \mu(B(x_0, \eta_\varepsilon(\widehat{x}))) > 0$ . Let

$$x'_0 \in \bigcap_{k \in \mathbb{N}} f_{\widehat{x}}^{n_k}(G_k) \cap \left\{ \frac{1}{\tau'} \leq \psi_J \right\} \cap C \cap J, \quad (11)$$

where  $\tau' > 1$  is large enough so that the set defined in (11) is not empty. Observe that  $f_{\widehat{x}}^{-n_k}(x'_0) \in G$  for every  $k \in \mathbb{N}$ . For a given  $k$ , let  $\rho_k > 0$  be small enough so that for any  $0 < \rho < \rho_k$  :

$$A_\rho := (B(x'_0, \rho) \cap C) \subset B(x_0, \eta_\varepsilon(\widehat{x})).$$

Since  $\|d\xi_{\widehat{x}}\| \leq \beta_\varepsilon(\widehat{x})$  we can assume that  $\sum_{A,B} |T \wedge \frac{i}{2} dA_{\widehat{x}} \wedge d\overline{B}_{\widehat{x}}| \leq \beta_\varepsilon(\widehat{x})^2 \sigma_T$  on  $B(x_0, \eta_\varepsilon(\widehat{x}))$ . According to Proposition 4.1,  $|F_{\widehat{x}, n_k}^{AB}| \leq \tau$  (Equation (8)) and  $|\alpha_{n, \widehat{x}}| \leq |\beta_{n, \widehat{x}}|$ , we deduce that :

$$\int_{A_\rho} \psi_J d\sigma_T = \mu(A_\rho) \leq d^{n_k} |\beta_{n_k, \widehat{x}}|^2 \tau \beta_\varepsilon(\widehat{x})^2 \int_{A_\rho} (\psi_J \circ f_{\widehat{x}}^{-n_k}) d\sigma_T.$$

According to Equation (11) we have  $x'_0 \in J$ , thus  $\sigma_T(A_\rho) > 0$  and

$$\int_{A_\rho} \psi_J d\sigma_T \leq d^{n_k} |\beta_{n_k, \widehat{x}}|^2 \tau \beta_\varepsilon(\widehat{x})^2 \int_{A_\rho} (\psi_J \circ f_{\widehat{x}}^{-n_k}) d\sigma_T.$$

Equation (11) also gives  $\frac{1}{\tau'} \leq \psi_J(x'_0)$  and  $x'_0 \in C \subset C_0 \cap C_{n_k}$ . We also recall that  $A_\rho \subset C$ . Hence, if  $\rho$  tends to 0, we get by continuity of  $\psi_J$  (resp.  $\psi_J \circ f_{\widehat{x}}^{-n_k}$ ) on  $C_0$  (resp.  $C_{n_k}$ ) :

$$\frac{1}{\tau'} \leq \psi_J(x'_0) \leq d^{n_k} |\beta_{n_k, \widehat{x}}|^2 \tau \beta_\varepsilon(\widehat{x})^2 (\psi_J \circ f_{\widehat{x}}^{-n_k})(x'_0) \leq d^{n_k} |\beta_{n_k, \widehat{x}}|^2 \tau \beta_\varepsilon(\widehat{x})^2 \|\psi_J\|_{L^\infty(\sigma_T)},$$

the last inequality coming from  $f_{\widehat{x}}^{-n_k}(x'_0) \in G$ . We finally get for every  $k$  :

$$(\tau' \tau \beta_\varepsilon(\widehat{x})^2 \|\psi_J\|_{L^\infty(\sigma_T)})^{-1} \leq d^{n_k} |\beta_{n_k, \widehat{x}}|^2 \leq e^{2n_k (\frac{1}{2} \text{Log } d - \lambda_2 + \varepsilon)}.$$

Hence  $\lambda_2 \leq \frac{1}{2} \text{Log } d + \varepsilon$ , which gives  $\lambda_2 \leq \frac{1}{2} \text{Log } d$  when  $\varepsilon$  tends to zero, as desired.  $\square$

## 5 Estimates on $I_n$ and $J_n$ in the decomposition of $\mu$

### 5.1 Study of $I_n$ and $J_n$

**Proposition 5.1.** *Assume that  $\mu \ll \sigma_T$  and let  $\psi \mathbb{1}_J$  be the Radon-Nikodym derivative of  $\mu$  with respect to  $\sigma_T$ . Assume that  $\lambda_1 > \lambda_2$  and let  $\varepsilon > 0$  small enough such that  $\lambda_1 + \lambda_2 - 2\varepsilon > \text{Log } d$  and  $\lambda_1 > \frac{1}{2} \text{Log } d + \varepsilon$ . Let  $\hat{x}$  be a  $\hat{\mu}$ -generic orbit and let  $(n_k)_k$  be a sub-sequence such that  $\hat{x}_{-n_k} \in \mathcal{A}_\tau$ . Let  $\sigma_{\hat{x}} := T \wedge dd^c |W_{\hat{x}}|^2$ . Let  $B := B(x_0, \eta_\varepsilon(\hat{x}))$  and  $A_0 := \{p \in B \cap J, \psi(p) > 0\}$ . Let  $U \subset B$  be a borel set.*

1. *If  $\psi \circ f_{\hat{x}}^{-n_k} \leq b \mu - a.e.$  on  $B$  for every  $k \geq 0$ , then*

$$J_{n_k}(U \cap J) \leq C_{\hat{x}} e^{-n_k(\lambda_1 + \lambda_2 - \text{Log } d)} e^{2n_k \varepsilon} \xrightarrow[k \rightarrow \infty]{} 0.$$

2. *If  $\psi \circ f_{\hat{x}}^{-n_k} \leq b \mu - a.e.$  on  $B$  and if  $\mu(U) > 0$ , then  $\sigma_{\hat{x}}(U \cap J \cap A_0) > 0$ .*

3. *If  $0 < a \leq \psi \circ f_{\hat{x}}^{-n_k} \leq b \mu - a.e.$  on  $B$ , then*

$$d^{n_k} |\beta_{n_k, \hat{x}}|^2 a \tau^{-1} \sigma_{\hat{x}}(U \cap J \cap A_0) \leq I_{n_k}(U \cap J) \leq d^{n_k} |\beta_{n_k, \hat{x}}|^2 b \tau \sigma_{\hat{x}}(U \cap J).$$

*In particular  $(d^{n_k} |\beta_{n_k, \hat{x}}|^2)_k$  converges, up to a sub-sequence, to some  $u > 0$ .*

**Proof:** Let us prove the first item. We assume that  $\lambda_1, \lambda_2$  are not resonant, we refer to Section 7 to fix the resonant case. By Definition 4.2,

$$J_{n_k}(U \cap J) = \sum_{(A,B) \neq (W,W)} d^{n_k} \alpha_{n_k, \hat{x}}^A \overline{\alpha_{n_k, \hat{x}}^B} \int_{U \cap J} (\psi \circ f_{\hat{x}}^{-n_k}) F_{\hat{x}, n_k}^{AB} T \wedge dA_{\hat{x}} \wedge d\overline{B}_{\hat{x}}.$$

Using  $|\alpha_{n, \hat{x}}| \leq |\beta_{n, \hat{x}}|$ , the (assumed) upper bound on  $\psi \circ f_{\hat{x}}^{-n_k}$ , the upper bound on  $F_{\hat{x}, n_k}^{AB}$  provided by (8), the item (i) of Lemma 2.5 and Cauchy-Schwarz inequality, we get with  $\sigma_{\hat{x}} := T \wedge dd^c |W_{\hat{x}}|^2$  and  $\lambda_{\hat{x}} := T \wedge dd^c |Z_{\hat{x}}|^2$  (which are  $\ll \sigma_T$ ):

$$|J_{n_k}(U \cap J)| \leq d^{n_k} |\alpha_{n_k, \hat{x}} \beta_{n_k, \hat{x}}| b \tau (\sigma_{\hat{x}} + \lambda_{\hat{x}})(U \cap J).$$

The first item then follows from  $|\alpha_{n, \hat{x}}| \leq e^{-n(\lambda_1 - \varepsilon)}$  and  $|\beta_{n, \hat{x}}| \leq e^{-n(\lambda_2 - \varepsilon)}$ .

We show the second item. By using Item 1 of Lemma 2.5 (for the last equality), we get

$$\mu(U) = \mu(U \cap J) = d^{2n} \mu(f_{\hat{x}}^{-n}(U \cap J)) = d^{2n} \int_{U \cap J \cap A_0} (\psi \circ f_{\hat{x}}^{-n}) (f_{\hat{x}}^{-n})^*(T \wedge \omega_{\mathbb{P}^2}).$$

Cauchy-Schwarz inequality implies (up to multiplication by a constant) :

$$T \wedge \omega_{\mathbb{P}^2} \leq \omega_{\hat{x}_{-n}, WW} \sigma_{\hat{x}_{-n}} + \omega_{\hat{x}_{-n}, ZZ} \lambda_{\hat{x}_{-n}} \text{ on } B(\hat{x}_{-n}, \eta_\varepsilon(\hat{x}_{-n})).$$

We now fix  $n = n_k$ . Using the (assumed) upper bound on  $\psi \circ f_{\hat{x}}^{-n_k}$  on  $B$  combined with item (i) of Lemma 2.5, and the upper bounds on  $F_{\hat{x}, n_k}^{WW} = \omega_{\hat{x}_{-n}, WW} \circ f_{\hat{x}}^{-n_k}$ ,  $F_{\hat{x}, n_k}^{ZZ} = \omega_{\hat{x}_{-n}, ZZ} \circ f_{\hat{x}}^{-n_k}$  provided by Equation (8), we get

$$\mu(U) \leq b d^{2n_k} \tau \left( \int_{U \cap J \cap A_0} (f_{\hat{x}}^{-n_k})^*(\sigma_{\hat{x}_{-n_k}}) + \int_{U \cap J \cap A_0} (f_{\hat{x}}^{-n_k})^*(\lambda_{\hat{x}_{-n_k}}) \right).$$

We deduce

$$\mu(U) \leq b d^{2n_k} \tau \left( \int_{U \cap J \cap A_0} d^{-n_k} |\beta_{n_k, \hat{x}}|^2 \sigma_{\hat{x}} + \int_{U \cap J \cap A_0} d^{-n_k} |\alpha_{n_k, \hat{x}}|^2 \lambda_{\hat{x}} \right),$$

hence

$$0 < \mu(U) \leq b d^{n_k} \tau (|\beta_{n_k, \hat{x}}|^2 \sigma_{\hat{x}}(U \cap J \cap A_0) + |\alpha_{n_k, \hat{x}}|^2 \lambda_{\hat{x}}(U \cap J \cap A_0)).$$

To conclude we observe that

$$d^{n_k} |\alpha_{n_k, \hat{x}}|^2 \leq e^{n_k(\text{Log } d - 2\lambda_1 + 2\varepsilon)},$$

which tends to 0 when  $k$  tends to infinity. Hence  $\sigma_{\hat{x}}(U \cap J \cap A_0) > 0$ .

Let us prove the third item. According to item (ii) of Lemma 2.5, we have

$$\psi \circ f_{\hat{x}}^{-n_k} \geq a, \quad \sigma_{\hat{x}} - a.e. \text{ on } B \cap J \cap A_0.$$

Hence, according to Definition 4.2, we get

$$I_{n_k}(U \cap J \cap A_0) \geq d^{n_k} |\beta_{n_k, \hat{x}}|^2 a \int_{U \cap J \cap A_0} F_{\hat{x}, n}^{WW} \sigma_{\hat{x}} \geq d^{n_k} |\beta_{n_k, \hat{x}}|^2 a \frac{1}{\tau} \sigma_{\hat{x}}(U \cap J \cap A_0),$$

where the last inequality comes from Equation (10). Similarly, by using item (i) of Lemma 2.5 and Equation (8), we get

$$I_{n_k}(U \cap J) \leq d^{n_k} |\beta_{n_k, \hat{x}}|^2 b \tau \sigma_{\hat{x}}(U \cap J)$$

This proves the two stated inequalities on  $I_{n_k}(U \cap J)$ . Let us verify the last fact. Since  $0 < \mu(B) = I_{n_k}(B \cap J) + J_{n_k}(B \cap J)$  and since  $J_{n_k}(B)$  tends to 0 by the first item, we get  $\frac{1}{2}\mu(B) \leq I_{n_k}(B \cap J) \leq 2\mu(B)$  for  $k$  large enough, which implies

$$0 < \frac{1}{2b\tau} \frac{\mu(B)}{\sigma_{\hat{x}}(B \cap J)} \leq d^{n_k} |\beta_{n_k, \hat{x}}|^2 \leq \frac{2\tau}{a} \frac{\mu(B)}{\sigma_{\hat{x}}(B \cap J \cap A_0)} < +\infty,$$

since  $\sigma_{\hat{x}}(B \cap J \cap A_0) > 0$  by the second item.  $\square$

## 5.2 Application : an equivalence between $\mu$ and a slice of $T$

We prove that  $\mu$  can be approximated by a slice of  $T$  under absolute continuity assumptions. The proof is a direct application of the decomposition of  $\mu$  fixed in Definition 4.2 and of Proposition 5.1.

**Proposition 5.2.** *Assume that  $\mu \ll \sigma_T$  and  $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d$ . Let  $\psi \mathbb{1}_J$  be the Radon-Nikodym derivative of  $\mu$  with respect to  $\sigma_T$ . Let  $\hat{x}$  be a  $\hat{\mu}$ -generic element and let  $(n_k)_k$  be a sub-sequence such that  $\hat{x}_{-n_k} \in \mathcal{A}_\tau$ . Let  $B := B(x_0, \eta_\varepsilon(\hat{x}))$  and  $A_0 := \{p \in B \cap J, \psi(p) > 0\}$ . If*

$$0 < a \leq \psi \circ f_{\hat{x}}^{-n_k} \leq b \mu - a.e. \text{ on } B \text{ for every } k \geq 0,$$

then there exists  $C_{\hat{x}} > 0$  such that on  $B$  :

$$C_{\hat{x}}^{-1} (T \wedge dd^c |W_{\hat{x}}|^2)|_{J \cap A_0} \leq \mu \leq C_{\hat{x}} (T \wedge dd^c |W_{\hat{x}}|^2)|_J.$$

**Proof :** Let  $U \subset B$  be a borel set and let  $I_k := I_{n_k}(U \cap J)$ ,  $J_k := J_{n_k}(U \cap J)$ . According to Definition 4.2, we have  $\mu(U) = I_k + J_k$ . The first item of Proposition 5.1 yields  $J_k \rightarrow 0$ . Using the third item of Proposition 5.1 (which in particular provides  $\lim_k d^{n_k} |\beta_{n_k, \hat{x}}|^2 = u > 0$ ), Equation (8) and item (i) of Lemma 2.5, we get

$$\mu(U) \leq u b \tau \sigma_{\hat{x}}(U \cap J).$$

Similarly, by using Equation (10) and item (ii) of Lemma 2.5, we get

$$\mu(U) \geq \frac{u a}{\tau} \sigma_{\hat{x}}(U \cap J \cap A_0).$$

We conclude by setting  $C_{\hat{x}} := \max\{\tau/ua, ub\tau\}$ .  $\square$

## 6 Proof of Theorem 1.2

### 6.1 Proof of the first item

**Lemma 6.1.** *Assume that  $\mu \ll \sigma_T$  and  $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d$ . Let  $\psi \mathbb{1}_J$  be the Radon-Nikodym derivative of  $\mu$  with respect to  $\sigma_T$ , and assume that  $\psi|_J$  is continuous. Let  $\hat{x}$  be a  $\hat{\mu}$ -generic element and let  $(n_k)_k$  be a sub-sequence such that  $\hat{x}_{-n_k} \in \mathcal{A}_\tau$  with and  $n_k \geq N(\hat{x})$ . There exists  $r_1(\hat{x}) \leq \eta_\varepsilon(\hat{x})$  and  $x_{-\infty} \in J$  such that, up to a sub-sequence :*

1.  $\lim_k d^{n_k} |\beta_{n_k, \hat{x}}|^2 = u > 0$  and  $\lim_k J_{n_k}(U \cap J) = 0$  for any borel set  $U \subset B(x_0, r_1(\hat{x}))$ ,
2.  $\psi \circ f_{\hat{x}}^{-n_k}$  uniformly converges to  $\psi(x_{-\infty})$  on  $B(x_0, r_1(\hat{x})) \cap J$ .

**Proof :** To prove the first item of the Lemma, in view of Proposition 5.1, it suffices to verify

$$\frac{1}{2\tau} \leq \psi \circ f_{\hat{x}}^{-n_k}$$

on some  $B(x_0, r_1(\hat{x})) \cap J$ . By uniform continuity, there exists  $\delta(\tau) > 0$  such that

$$\forall a, b \in J, \text{dist}(a, b) \leq \delta(\tau) \implies |\psi(a) - \psi(b)| \leq \frac{1}{2\tau}.$$

Defining  $r_1(\hat{x}) := \min \left\{ \eta_\varepsilon(\hat{x}), \frac{\delta(\tau)}{\beta_\varepsilon(\hat{x})} \right\}$ , we have for every  $p \in B(x_0, r_1(\hat{x}))$  :

$$\text{dist}(f_{\hat{x}}^{-n_k}(p), f_{\hat{x}}^{-n_k}(x_0)) \leq \text{Lip}(f_{\hat{x}}^{-n_k}) r_1(\hat{x}) \leq \beta_\varepsilon(\hat{x}) r_1(\hat{x}) \leq \delta(\tau).$$

Using  $\psi(x_{-n_k}) \geq \frac{1}{\tau}$  (coming from  $\hat{x}_{-n_k} \in \mathcal{A}_\tau$ ), we get  $\psi \circ f_{\hat{x}}^{-n_k} \geq \frac{1}{2\tau}$  on  $B(x_0, r_1(\hat{x})) \cap J$ .

For the second item, the sequence  $(x_{-n_k})_k \in J$  converges by compactness to some  $x_{-\infty} \in J$ . Since  $\text{Lip } f_{\hat{x}}^{-n_k}$  tends to 0,  $(f_{\hat{x}}^{-n_k})_k$  converges uniformly on  $B(x_0, r_1(\hat{x}))$  to the constant mapping  $x_{-\infty}$ . The conclusion follows from the continuity of  $\psi$  on  $J$ .  $\square$

We now prove the first item of Theorem 1.2. Let  $U \subset B(x_0, r_1(\hat{x}))$  be a borel set. We recall that  $\sigma_{\hat{x}} = T \wedge dd^c |W_{\hat{x}}|^2$  and that by Definition 4.2 :

$$\mu(U) = \mu(U \cap J) = d^{n_k} |\beta_{n_k, \hat{x}}|^2 \int_{U \cap J} (\psi \circ f_{\hat{x}}^{-n_k}) F_{\hat{x}, n_k}^{WW} \sigma_{\hat{x}} + J_{n_k}(U \cap J).$$

According to Equation (9),  $F_{\hat{x}, n_k}^{WW}$  uniformly converges on  $B(x_0, r_1(\hat{x}))$  to some constant  $F_{\hat{x}}$ , up to a sub-sequence. We conclude by using the two items of Lemma 6.1, which yield  $\mu(U) = C_{\hat{x}} \sigma_{\hat{x}}(U \cap J)$ , where  $C_{\hat{x}} := u F_{\hat{x}} \psi(x_{-\infty})$ . Note that  $C_{\hat{x}}$  does not depend on  $U$  and is positive by taking  $U = B(x_0, r_1(\hat{x}))$ .

### 6.2 Proof of the second item

The following lemma is the counterpart of Lemma 6.1, the function  $\psi$  being now continuous on a neighborhood  $V$  of  $J$ . The two items hold without any restriction to  $J$ . The proof is similar using Remark 4.3 (decomposition of  $\mu$  on  $V$ ), the uniform continuity of  $\psi$  on  $V$ , and by introducing a radius  $r_2(\hat{x})$  satisfying  $B(x_0, r_2(\hat{x})) \subset V$ . Note that  $f_{\hat{x}}^{-n_k}(B(x_0, r_2(\hat{x}))) \subset V$  for  $k$  large enough.

**Lemma 6.2.** *Assume that  $\mu \ll \sigma_T$  and  $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d$ . We assume that there exists a neighborhood  $V$  of  $J$  and a function  $\psi \in L^1(\sigma_T)$  such that  $\psi$  is continuous on  $V$  and satisfies  $\mu = \psi_V \sigma_T$  (we recall that  $\psi_V = \psi \mathbb{1}_V$ ). Let  $\hat{x}$  be a  $\hat{\mu}$ -generic element and let  $(n_k)_k$  be a sub-sequence such that  $\hat{x}_{-n_k} \in \mathcal{A}_\tau$  with  $n \geq N(\hat{x})$ . There exists  $r_2(\hat{x}) \leq \eta_\varepsilon(\hat{x})$  (small enough to have  $B(x_0, r_2(\hat{x})) \subset V$ ) and  $x_{-\infty} \in J$  such that, up to a sub-sequence :*

1.  $\lim_k d^{n_k} |\beta_{n_k, \hat{x}}|^2 = u > 0$  and  $\lim_k J_{n_k}(U) = 0$  for any borel set  $U \subset B(x_0, r_2(\hat{x}))$ ,
2.  $\psi \circ f_{\hat{x}}^{-n_k}$  uniformly converges to  $\psi(x_{-\infty})$  on  $B(x_0, r_2(\hat{x}))$ .

The second item of Theorem 1.2 follows as in the end of Section 6.1, by using

$$\mu(U) = d^{n_k} |\beta_{n_k, \hat{x}}|^2 \int_U (\psi \circ f_{\hat{x}}^{-n_k}) F_{\hat{x}, n_k}^{WW} \sigma_{\hat{x}} + J_{n_k}(U)$$

for every  $U \subset B(x_0, r_2(\hat{x})) \subset V$ , this formula is provided by Definition 4.2 and Remark 4.3.

## 7 About the resonant case

We assume that  $\lambda_1, \lambda_2$  are resonant :  $\lambda_1 = q\lambda_2$  for some  $q \geq 2$ . We assume that  $\mu \ll \sigma_T$ , let  $\psi$  denote the Radon-Nikodym derivative of  $\mu$  with respect to  $\sigma_T$ . Let us explain how Proposition 4.1 is affected. We shall compute the corresponding  $J_n(D)$  appearing in Definition 4.2. According to Theorem 2.2, the map  $R_{n, \hat{x}}$  is equal to :

$$R_{n, \hat{x}}(z, w) = (\alpha_{n, \hat{x}} z, \beta_{n, \hat{x}} w) + (\gamma_{n, \hat{x}} w^q, 0),$$

where  $|\gamma_{n, \hat{x}}| \leq M_\varepsilon(\hat{x}) e^{-n(\lambda_1 - \varepsilon)}$ . Let  $U \subset B(x_0, \eta_\varepsilon(\hat{x}))$  be a borel set. Denoting  $\psi_n := \psi \circ f_{\hat{x}}^{-n}$ , we have  $\mu(U) = I_n(U \cap J) + J_n(U \cap J)$ , where  $I_n(D)$  has the same expression as in the non resonant case :

$$I_n(D) = d^n |\beta_{n, \hat{x}}|^2 \int_D \psi_n F_{\hat{x}, n}^{WW} T \wedge \frac{i}{2} (dW_{\hat{x}} \wedge d\overline{W}_{\hat{x}}).$$

In the resonant case, the formula giving  $J_n(D)$  is modified as follows :

$$\begin{aligned} J_n(D) &= d^n |\alpha_{n, \hat{x}}|^2 \int_D \psi_n F_{\hat{x}, n}^{ZZ} T \wedge \frac{i}{2} (dZ_{\hat{x}} \wedge d\overline{Z}_{\hat{x}}) \\ &\quad + 2d^n \int_D \psi_n \operatorname{Re} \left[ \left( (q F_{\hat{x}, n}^{ZZ} \alpha_{n, \hat{x}} \overline{\gamma}_{n, \hat{x}} \overline{W}_{\hat{x}}^{q-1} + F_{\hat{x}, n}^{ZW} \alpha_{n, \hat{x}} \overline{\beta}_{n, \hat{x}}) T \wedge \frac{i}{2} (dZ_{\hat{x}} \wedge d\overline{W}_{\hat{x}}) \right) \right] \\ &\quad + d^n \int_D \psi_n \left[ q^2 F_{\hat{x}, n}^{ZZ} |\gamma_{n, \hat{x}}|^2 |W_{\hat{x}}|^{2(q-1)} + 2 \operatorname{Re} \left( q F_{\hat{x}, n}^{ZW} \gamma_{n, \hat{x}} \overline{\beta}_{n, \hat{x}} W_{\hat{x}}^{q-1} \right) \right] T \wedge \frac{i}{2} (dW_{\hat{x}} \wedge d\overline{W}_{\hat{x}}). \end{aligned}$$

Let us verify that the first item of Proposition 5.1 remains valid in the resonant case. Let  $\hat{x}$  be a  $\hat{\mu}$ -generic element and let  $(n_k)_k$  be a sub-sequence such that  $\hat{x}_{-n_k} \in \mathcal{A}_T$ . Assume

$$\psi_n = \psi \circ f_{\hat{x}}^{-n_k} \leq b, \quad \mu - a.e. \text{ on } B$$

and let us verify that there exists  $C'(\hat{x})$  such that for every borel set  $U \subset B$  :

$$J_{n_k}(U) \leq C'(\hat{x}) e^{-n_k(\lambda_1 + \lambda_2 - \operatorname{Log} d)} e^{2n_k \varepsilon}.$$

We use the formula on  $J_n(D)$  given above and Cauchy-Schwarz inequality. We also use  $|F_{\hat{x}, n}^{AB}| \leq \tau$  given by Equation (8), item (i) of Lemma 2.5 and  $|W_{\hat{x}}| \leq \rho_\varepsilon := \rho_\varepsilon(\hat{x})$  on  $B$  provided by Equation (6). If  $\sigma_{\hat{x}} := T \wedge dd^c |W_{\hat{x}}|^2$ ,  $\lambda_{\hat{x}} := T \wedge dd^c |Z_{\hat{x}}|^2$ , we indeed obtain

$$\begin{aligned} J_{n_k}(U) &\leq d^{n_k} |\alpha_{n_k, \hat{x}}|^2 b \tau \lambda_{\hat{x}}(U) \\ &\quad + 2d^{n_k} |\alpha_{n_k, \hat{x}}| b \tau \left( q |\gamma_{n_k, \hat{x}}| \rho_\varepsilon^{q-1} + |\beta_{n_k, \hat{x}}| \right) (\lambda_{\hat{x}} + \sigma_{\hat{x}})(U) \\ &\quad + q d^{n_k} |\gamma_{n_k, \hat{x}}| b \tau \left( q |\gamma_{n_k, \hat{x}}| \rho_\varepsilon^{2(q-1)} + 2 |\beta_{n_k, \hat{x}}| \rho_\varepsilon^{q-1} \right) \sigma_{\hat{x}}(U). \end{aligned}$$

The existence of  $C'(\hat{x})$  then comes from the upper bounds  $|\alpha_{n, \hat{x}}| \leq e^{-n(\lambda_1 - \varepsilon)}$ ,  $|\beta_{n, \hat{x}}| \leq e^{-n(\lambda_2 - \varepsilon)}$ ,  $|\gamma_{n, \hat{x}}| \leq M_\varepsilon(\hat{x}) e^{-n(\lambda_1 - \varepsilon)}$  and from  $\lambda_1 > \lambda_2$ .



## 8 Suspensions of one-dimensional Lattès maps

Let  $\theta = [P : Q]$  be Lattès map on  $\mathbb{P}^1$  of degree  $d \geq 2$  and let  $f := [P : Q : t^d]$  be its suspension on  $\mathbb{P}^2$ . These maps were studied by Berteloot-Loeb [4]. We identify the first affine chart with  $\mathbb{C}^2$ . Let

$$G_\theta := \lim_{n \rightarrow +\infty} \frac{1}{d^n} \text{Log} \|(P^n, Q^n)\| : \mathbb{C}^2 \rightarrow \mathbb{R} \cup \{-\infty\}$$

be the Green function of the polynomial mapping  $(P, Q)$  and let  $G := \max\{G_\theta, 0\}$ . The attracting basin  $A$  of  $(0, 0)$  is bounded and equal to  $\{G = 0\}$ . The Green current of  $f$  satisfies  $T = dd^c G$  on  $\mathbb{C}^2$ , the support of  $\mu$  coincides with the boundary  $\partial A$  of  $A$ .

By [4, Proposition 3.1] (see also [11] in higher dimensions), for every  $p$  outside a finite number of circles drawn on  $\partial A$ , there exists a biholomorphism  $\mathbf{p} : (\mathbb{D}^2, 0) \rightarrow (\mathbb{D}^2, p)$  such that

$$G_0(z, w) := G_\theta \circ \mathbf{p}(z, w) = \text{Re}(z) + |w|^2 \text{ on } (\mathbb{D}^2, 0).$$

We denote

$$T_0 := \mathbf{p}^* T = dd^c(\max\{G_0, 0\}) \text{ and } \mu_0 := \mathbf{p}^* \mu.$$

The closed positive current  $T_0$  has the following matrix representation

$$T_0 = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix},$$

where  $T_{11} := T_0 \wedge dd^c |w|^2$  and  $T_{22} := T \wedge dd^c |z|^2$ . We describe below  $T_0$  and  $\mu_0$  on  $(\mathbb{D}^2, 0)$ . We assume to simplify that the germ  $(\mathbb{D}^2, 0)$  is the open set  $D := ]-1, 1[ \times \mathbb{D}$ . Let

$$\Omega := \{\text{Re}(z) + |w|^2 > 0\} \cap D \text{ and } M_0 := \{\text{Re}(z) + |w|^2 = 0\} \cap D.$$

Observe that  $M_0$  is the image (intersected with  $D$ ) of the euclidian 3-sphere of  $\mathbb{C}^2$  by the classical Cayley transformation, see [17, Chapter 2.3]. Let us parametrize  $M_0$  by

$$\Phi : ]-1, 0[ \times ]-1, 1[ \times ]0, 2\pi[ \rightarrow M_0, \quad (u, v, \theta) \mapsto (u + iv, \sqrt{-ue^{i\theta}}) \quad (12)$$

and let  $\text{Leb}_{M_0} := \Phi_* \text{Leb}_{\mathbb{R}^3}$ . Let also  $\omega_0$  be the standard hermitian  $(1, 1)$ -form of  $\mathbb{C}^2$ .

**Proposition 8.1.** *With the preceding notations,*

$$T_0 = \begin{bmatrix} \frac{1}{8} \text{Leb}_{M_0} & 0 \\ 0 & \text{Leb}_\Omega + \frac{|w|^2}{2} \text{Leb}_{M_0} \end{bmatrix},$$

and

$$\mu_0 = \frac{1}{8} \text{Leb}_{M_0} = T_0 \wedge dd^c |w|^2.$$

In particular,

$$\mu_0 = \psi_0(T_0 \wedge \omega_0), \text{ where } \psi_0(z, w) := \frac{1}{1 + 4|w|^2} \mathbb{1}_{M_0}.$$

The remainder of this Section is devoted to the proof of Proposition 8.1.

*Computation of  $T_{11}$ .*— Let  $\varphi$  be a test function with compact support in  $D$ . We denote  $\varphi_{z\bar{z}} := \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \varphi$ , similarly for  $\varphi_{w\bar{w}}$ . By definition we have :

$$\langle T_{11}, \varphi \rangle = \int_{\Omega} G_0(z, w) \varphi_{z\bar{z}}(z, w) \, d\text{Leb}(z, w).$$

Let  $\Delta_w$  denote the subset  $] - 1, 1[\times] - 1, 1[\times\{w\}$  intersected with  $\Omega$ , so that

$$\langle T_{11}, \varphi \rangle = \int_{\mathbb{D}} \left[ \int_{\Delta_w} G_0(z, w) \varphi_{z\bar{z}}(z, w) \, d\text{Leb}(z) \right] d\text{Leb}(w) =: \int_{\mathbb{D}} I_w(\varphi) \, d\text{Leb}(w). \quad (13)$$

We write  $z = u + iv$ , hence  $\varphi_{z\bar{z}} = \frac{1}{4}\Delta_{u,v}\varphi$ . Observe also that  $G_0$  only depends on  $(u, w)$ . The number  $4I_w(\varphi)$  is then equal to

$$\int_{-1}^1 \left[ \int_{-|w|^2}^1 G_0(u, w) \frac{\partial^2 \varphi}{\partial u^2}(u + iv, w) \, du \right] dv + \int_{-|w|^2}^1 G_0(u, w) \left[ \int_{-1}^1 \frac{\partial^2 \varphi}{\partial v^2}(u + iv, w) \, dv \right] du.$$

The second term vanishes since  $\text{Supp}(\varphi(\cdot, w)) \subset ] - 1, 1[\times] - 1, 1[$ . Integrating by parts and using  $\partial_u G_0 = 1$ ,  $\partial_v G_0 = 0$  for the first one, we get

$$4I_w(\varphi) = - \int_{-1}^1 \left[ \int_{-|w|^2}^1 \frac{\partial \varphi}{\partial u}(u + iv, w) \, du \right] dv.$$

Using again  $\text{Supp}(\varphi(\cdot, w)) \subset ] - 1, 1[\times] - 1, 1[$ , we get

$$I_w(\varphi) = \frac{1}{4} \int_{-1}^1 \varphi(-|w|^2 + iv, w) \, dv.$$

We therefore obtain by (13) :

$$\langle T_{11}, \varphi \rangle = \frac{1}{4} \int_{\mathbb{D}} \int_{-1}^1 \varphi(-|w|^2 + iv, w) \, dv \, d\text{Leb}(w)$$

Observe that  $(w, v) \mapsto (-|w|^2 + iv, w)$  is another parametrization of  $M_0$  different from  $\Phi$  defined in (12). Let us write this integral with polar coordinates :

$$\langle T_{11}, \varphi \rangle = \frac{1}{8} \int_{-1}^1 \int_0^{2\pi} \int_0^1 \varphi(-\rho^2 + iv, \rho e^{i\theta}) \, 2\rho d\rho \, d\theta \, dv.$$

Applying the change of variable  $u = -\rho^2$  one gets back to the parametrization  $\Phi$  :

$$\langle T_{11}, \varphi \rangle = \frac{1}{8} \int_{]-1, 0[\times] - 1, 1[\times] 0, 2\pi[} \varphi(u + iv, \sqrt{-u}e^{i\theta}) \, d\text{Leb}(u, v, \theta) = \frac{1}{8} \int_{M_0} \varphi \, d\text{Leb}_{M_0}.$$

*Computation of  $T_{22}$ .* - Let  $\Delta_z$  be the vertical disc  $\{z\} \times \mathbb{D}$  intersected with  $\Omega$ . One has :

$$\langle T_{22}, \varphi \rangle = \int_{]-1, 1[\times] - 1, 1[} \left[ \int_{\Delta_z} G_0(z, w) \varphi_{w\bar{w}}(z, w) \, d\text{Leb}(w) \right] d\text{Leb}(z).$$

Writing  $z = u + iv$ , we get

$$\langle T_{22}, \varphi \rangle = \int_{-1}^1 \left[ \int_{-1}^0 J_{u,v}(\varphi) \, du \right] dv + \int_{-1}^1 \left[ \int_0^1 J_{u,v}(\varphi) \, du \right] dv,$$

where

$$J_{u,v}(\varphi) := \int_{\mathbb{D} \setminus \mathbb{D}(\sqrt{-u})} G_0(u + iv, w) \varphi_{w\bar{w}}(u + iv, w) \, d\text{Leb}(w),$$

with the convention  $\mathbb{D}(\sqrt{-u}) = \emptyset$  if  $0 \leq u \leq 1$ .

**Lemma 8.2.**

1.  $J_{u,v}(\varphi) = \int_{\mathbb{D}} \varphi(u + iv, w) \, d\text{Leb}(w)$  for  $0 \leq u \leq 1$ .
2.  $J_{u,v}(\varphi) = \int_{\mathbb{D} \setminus \mathbb{D}(\sqrt{-u})} \varphi(z, w) \, d\text{Leb}(w) - \frac{u}{2} \int_0^{2\pi} \varphi(z, \sqrt{-u}e^{i\theta}) \, d\theta$  for  $-1 \leq u \leq 0$ .

**Proof :** We write  $w = s + it$  and use  $\varphi_{w\bar{w}} = \frac{1}{4}\Delta_{s,t}\varphi$ . Since  $\varphi$  has compact support in  $D$ , one gets by Green-Stokes formula :

$$4J_{u,v}(\varphi) := \int_{\mathbb{D}} G_0(u + iv, w)\Delta_{s,t}\varphi \, d\text{Leb}(w) = \int_{\mathbb{D}} \Delta_{s,t}G_0(u + iv, w)\varphi \, d\text{Leb}(w)$$

The first item follows since  $\partial_s^2 G_0 = \partial_t^2 G_0 = 2$ . In the case  $-1 \leq u \leq 0$ , we have :

$$J_{u,v}(\varphi) = \int_{\mathbb{D} \setminus \mathbb{D}(\sqrt{-u})} G_0(u + iv, w)\varphi_{w\bar{w}}(u + iv, w) \, d\text{Leb}(w). \quad (14)$$

Let us denote  $z = u + iv$  and  $w = s + it \in \mathbb{D} \setminus \mathbb{D}(\sqrt{-u})$ . Using  $\text{Supp}(\varphi(z, \cdot)) \subset \mathbb{D}$  and  $G_0(z, \cdot)|_{\partial\mathbb{D}(\sqrt{-u})} = 0$ , Green-Stokes formula gives :

$$\begin{aligned} & \int_{\mathbb{D} \setminus \mathbb{D}(\sqrt{-u})} G_0(z, s + it)\Delta_{s,t}\varphi(z, s + it) - \Delta_{s,t}G_0(z, s + it)\varphi(z, s + it) \, d\text{Leb}(s, t) \\ &= \int_{\partial\mathbb{D}(\sqrt{-u})} \varphi(z, s + it) (\nabla_{(s,t)}G_0(z, s + it)) \cdot (s, t) \, d\sigma(s, t), \end{aligned}$$

where  $\sigma$  is the Lebesgue measure on the circle  $\partial\mathbb{D}(\sqrt{-u})$ . Since  $\Delta_{s,t}G_0(z, s + it) = 4$  and  $(\nabla_{(s,t)}G_0(z, s + it)) \cdot (s, t) = 2|w|^2 = -2u$ , we infer

$$4 \int_{\mathbb{D} \setminus \mathbb{D}(\sqrt{-u})} \varphi(z, w) - G_0(z, w)\varphi_{w\bar{w}}(z, w) \, d\text{Leb}(w) = \int_{\partial\mathbb{D}(\sqrt{-u})} 2u\varphi(z, w) \, d\sigma(w).$$

The second item follows by combining this formula with (14).  $\square$

Let us now return to the computation of  $\langle T_{22}, \varphi \rangle$ . Let  $\Omega^+ := \Omega \cap \{u > 0\}$  and  $\Omega^- := \Omega \cap \{u < 0\}$ . According to Lemma 8.2, we get

$$\int_{-1}^1 \int_0^1 J_{u,v}(\varphi) \, du \, dv = \int_{\Omega^+} \varphi(z, w) \, d\text{Leb}(z, w),$$

and that  $\int_{-1}^1 \int_{-1}^0 J_{u,v}(\varphi) \, du \, dv$  is equal to

$$\int_{\Omega^-} \varphi(z, w) \, d\text{Leb}(z, w) + \int_{-1}^1 \left[ \int_{-1}^0 \left[ \int_0^{2\pi} -\frac{u}{2}\varphi(u + iv, \sqrt{-u}e^{-i\theta}) \, d\theta \right] du \right] dv.$$

If  $h(z, w) := \frac{|w|^2}{2}$  the second integral is equal to :

$$\int_{]-1, 0[ \times ]-1, 1[ \times ]0, 2\pi[} (h\varphi) \circ \Phi(u, v, \theta) \, d(u, v, \theta) = \int_{M_0} h\varphi \, d\text{Leb}_{M_0}.$$

We deduce as desired

$$\langle T_{22}, \varphi \rangle = \int_{-1}^1 \int_{-1}^1 J_{u,v}(\varphi) \, du \, dv = \int_{\Omega} \varphi(z, w) \, d\text{Leb}(z, w) + \int_{M_0} h\varphi \, d\text{Leb}_{M_0}.$$

Computation of  $\mu_0$ .— Writing  $G_0^+ := \max\{G_0, 0\}$ , we have  $T_0 = dd^c G_0^+$  and  $\mu_0 = dd^c(G_0^+ dd^c G_0^+)$ . Hence

$$\begin{aligned} \langle \mu_0, \varphi \rangle &= \langle T_0, G_0^+ dd^c \varphi \rangle = \langle T_{11} dd^c |z|^2 + T_{22} dd^c |w|^2, G_0^+ dd^c \varphi \rangle \\ &= \frac{1}{8} \int_{M_0} G_0 \varphi_{w\bar{w}} \, d\text{Leb}_{M_0} + \int_{\Omega} G_0 \varphi_{z\bar{z}} \, d\text{Leb} + \int_{M_0} G_0 \varphi_{z\bar{z}} h \, d\text{Leb}_{M_0}. \end{aligned}$$

Since  $G_0 = 0$  on  $M_0$ , we get  $\langle \mu_0, \varphi \rangle = \int_{\Omega} G_0 \varphi_{z\bar{z}} \, d\text{Leb}_{\Omega} = \langle T_{11}, \varphi \rangle$  as desired (observe that this is the Lebesgue part on  $\Omega$  in  $T_{22}$  which provides  $\mu = T_{11}$ ). The formula  $\mu_0 = \frac{1}{1+8h} \mathbb{1}_{M_0} T_0 \wedge \omega_0$  immediately follows.

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