

# ON PSEUDO-ROTATIONS OF THE ANNULUS WITH GENERIC ROTATION NUMBER

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ABSTRACT. We show that for a Baire generic rotation number  $\alpha \in \mathbb{R}/\mathbb{Z}$ , the set of area preserving  $C^\infty$ -pseudo-rotations of the annulus  $\mathbb{A}$  with rotation number  $\alpha$  equals the closure of the set of area preserving  $C^\infty$ -pseudo-rotations which are smoothly conjugate to the rotation  $R_\alpha$ . As a corollary, a  $C^\infty$ -generic area preserving pseudo-rotation of the annulus with a Baire generic rotation number  $\alpha$  is weakly mixing.

## 1. INTRODUCTION

In this paper we denote the 2-dimensional annulus by  $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times [0, 1]$  equipped with the standard area-form  $\omega$ . We let  $F_{\mathbb{A}}^\infty$  denote the set of  $\omega$ -preserving  $C^\infty$  *pseudo-rotations* of  $\mathbb{A}$  (the precise definition will appear in Definition 2.9). Namely, we set

$$F_{\mathbb{A}}^\infty := \{f \in \text{Diff}^\infty(\mathbb{A}, \omega) \mid f \text{ is isotopic to Id and has no periodic points}\}.$$

The study of pseudo-rotations can be essentially traced back to the question of Birkhoff [B41] (see also [H98]) as to whether there are non-trivial analytic diffeomorphisms of the 2-sphere with 2 fixed points (the existence of such diffeomorphisms was recently announced by Berger [B22]). The name ‘‘pseudo-rotation’’ was introduced by Béguin, Crovisier, Le Roux and Patou in [BCLP04]. By a result of Franks [F88], for  $\omega$ -preserving homeomorphisms of  $\mathbb{A}$ , the notion of irrational pseudo-rotations in [BCLP04] coincides with ours. In particular, each  $f \in F_{\mathbb{A}}^\infty$  admits a *rotation number*  $\rho(f) \in (0, 1)/\mathbb{Q}$  (see Definition 2.10 for the details). For each  $\alpha \in (0, 1) \setminus \mathbb{Q}$  we set

$$\begin{aligned} F_{\mathbb{A}}^\infty(\alpha) &:= \{f \in F_{\mathbb{A}}^\infty \mid \rho(f) = \alpha\}, \\ O_{\mathbb{A}}^\infty(\alpha) &:= \{hR_\alpha h^{-1} \mid h \in \text{Diff}^\infty(\mathbb{A}, \omega)\} \end{aligned}$$

where  $R_\alpha$  denotes the rotation  $(x, y) \mapsto (x + \alpha, y)$ . It is clear that  $O_{\mathbb{A}}^\infty(\alpha) \subset F_{\mathbb{A}}^\infty(\alpha)$  for any  $\alpha \in (0, 1) \setminus \mathbb{Q}$ . It is not hard to see that  $\overline{F_{\mathbb{A}}^\infty(\alpha)}$  is closed in the  $C^\infty$ -topology for each irrational  $\alpha$ . In particular,  $\overline{O_{\mathbb{A}}^\infty(\alpha)} \subset F_{\mathbb{A}}^\infty(\alpha)$  for any  $\alpha \in (0, 1) \setminus \mathbb{Q}$ , where the closure is taken in the  $C^\infty$ -topology. Our main result is the following.

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THEOREM 1. For a Baire generic  $\alpha \in (0, 1) \setminus \mathbb{Q}$ , we have

$$F_{\mathbb{A}}^{\infty}(\alpha) = \overline{O_{\mathbb{A}}^{\infty}(\alpha)}$$

where the closure is taken in  $\text{Diff}^{\infty}(\mathbb{A})$ .

In other words, for a Baire generic  $\alpha$ , any pseudo-rotation  $f$  with rotation number  $\alpha$  is the  $C^{\infty}$ -limit of a sequence  $f_k$  of area preserving diffeomorphisms, which up to a smooth area preserving change of coordinates, is the standard rotation  $R_{\alpha}$ . In particular  $f$  is approximable by *integrable systems*.

We can see Theorem 1 as a natural analogue of a well-known theorem of Herman in [H79]. Recall that one of the most prominent results in the study of circle diffeomorphisms is the following.

THEOREM 2 (Herman-Yoccoz). For any irrational  $\alpha \in \mathbb{R}/\mathbb{Z}$ , we denote by  $F^{\infty}(\alpha)$  the set of  $C^{\infty}$  circle diffeomorphisms with rotation number  $\alpha$ , and denote by  $O^{\infty}(\alpha)$  the set of  $C^{\infty}$  circle diffeomorphisms which are  $C^{\infty}$ -conjugate to the standard rotation  $R_{\alpha}$ . Then we have

$$F^{\infty}(\alpha) = \begin{cases} \overline{O^{\infty}(\alpha)} = O^{\infty}(\alpha) & \text{if } \alpha \text{ is Diophantine,} \\ \overline{O^{\infty}(\alpha)} \neq O^{\infty}(\alpha) & \text{if } \alpha \text{ is Liouville.} \end{cases}$$

Here in the above the closures are taken under the  $C^{\infty}$ -topology.

The above result for Diophantine  $\alpha$  was conjectured by Arnold, who showed in [A78] that any  $C^{\omega}$  circle diffeomorphism with a Diophantine rotation number  $\alpha$  which is sufficiently close to  $R_{\alpha}$  in the  $C^{\omega}$ -topology, is in fact  $C^{\omega}$ -conjugate to  $R_{\alpha}$ . Arnold's result was then generalised to the  $C^{\infty}$ -category by Moser in [M66]. This is the beginning of what is now known as the Kolmogorov-Arnold-Moser theory. The global picture was for the first time established by Herman in the seminal paper [H79]. In [H79], Theorem 2 was proved for a subset of  $\alpha$  with full Lebesgue measure. Khanin and Sinai gave in [KS89] a different proof of the main result in [H79] building on a renormalization theory for circle diffeomorphisms. The Diophantine part of Theorem 2 was completed by Yoccoz in [Y84]. We also mention Katznelson-Ornstein's papers [KO89a, KO89b] on circle diffeomorphisms with low regularity, and Yoccoz's paper in [EKMY02] on  $C^{\omega}$ -linearization under the sharp arithmetic condition, i.e.,  $\mathcal{H}$ -condition. For a recent survey of this development and beyond, we refer the reader to [EFK18].

The Liouville part of Theorem 2 was conjectured by Herman in [H79, Conjecture 7.1]. In fact Herman already showed that Theorem 2 holds for a Baire generic set of  $\alpha$ , see [H79, Theorem 7.3]. However, his proof was based on the Diophantine part of Theorem 2 (at least for a full measure set of  $\alpha$ ), and used certain properties of the function  $t \mapsto \rho(R_t f)$  of a circle diffeomorphism  $f$ . It is still an open question of Herman whether the

Diophantine rigidity holds within pseudo-rotations. Moreover, it is unclear how to deform a general pseudo-rotation within the set of pseudo-rotations, and change the rotation number. This blocks a direct generalisation of Herman's approach for pseudo-rotations. The full answer to Herman's conjecture was provided by Yoccoz in [Y95b]. Yoccoz showed that any  $C^\infty$  circle diffeomorphism with a Liouville rotation number can be  $C^\infty$ -approximated by a *quasi-rotation*: this is a class of circle diffeomorphisms which, among other things, admits a renormalization that is a standard rotation. Our proof of Theorem 1 is somewhat similar to the proof of Yoccoz: we also consider certain renormalizations of a pseudo-rotation. However, the type of estimates are very different. We are unable to transfer the strong estimates for circle diffeomorphisms, such as Denjoy's inequality in [Y84], to general pseudo-rotations, due to the possible occurrence of complicated geometry which does not appear in dimension 1. On the other hand, the area-preserving hypothesis provides us with certain strong  $C^0$ -estimates established in [AFLXZ20]. Combining such estimates with a suitable arithmetic condition, we are able to extract some useful information from a sequence of suitably renormalized pseudo-rotations.

We can also compare Theorem 1 with the main result in [B15b], which says that any smooth area preserving pseudo-rotation  $f$  on the closed 2-disc, meaning that  $f$  fixes the origin and has no periodic points on the annulus complementary to the origin, is the  $C^0$ -limit of smooth periodic disc maps  $f_k$ , that each fix the origin. In the latter there are no restrictions on the (irrational) rotation number of  $f$ , but in this current paper our integrable approximations are in every way stronger: 1) In [B15b] the sequence of approximations  $f_k$ , while  $C^\infty$ -smooth, only converge in the  $C^0$  topology to  $f$ . 2) In [B15b] the  $f_k$ 's are not necessarily area preserving. 3) In [B15b] the approximation maps  $f_k$  have rational rotation numbers  $p_k/q_k \in \mathbb{Q}$  converging to the rotation number  $\alpha$  of  $f$  as  $k \rightarrow \infty$ . One cannot perturb the  $f_k$  in [B15b] in an obvious way to make the rotation number equal to  $\alpha$  while retaining closeness to  $f$ . On the other hand, it is easy to modify the  $f_k$ 's in the current article, if one so wishes, to make the rotation number rational and keep the closeness to  $f$ . In short, the fact that in this paper we are able to find integrable approximations without altering the rotation number is also a stronger conclusion than in [B15b].

Another motivation behind our result is the important work of Anosov and Katok [AK70] and the extensions in Fayad-Saprykina [FS05], see also Fayad-Katok [FK04], in which, for generic rotation numbers, more precisely all Liouville rotation numbers, examples of pseudo-rotations are constructed which are dynamically interesting, that is, not conjugate to a rotation. These "exotic" pseudo-rotations of Anosov-Katok lie, by construction, in the  $C^\infty$ -closure of  $\cup_{t \in \mathbb{Q}} O_{\mathbb{A}}^\infty(t)$  rather than the closure of  $O_{\mathbb{A}}^\infty(\alpha)$ . It however follows from our main result that, for a possibly smaller Baire-generic set of rotation numbers than the Liouville numbers, that the Anosov-Katok

constructions do indeed lie in the closure of  $O_{\mathbb{A}}^{\infty}(\alpha)$  with *fixed* rotation number  $\alpha$ . Moreover, we have the following interesting corollary.

**COROLLARY A.** *For a Baire generic  $\alpha \in (0, 1) \setminus \mathbb{Q}$ , the set of weakly mixing pseudo-rotations in  $F_{\mathbb{A}}^{\infty}(\alpha)$  forms a Baire set with empty interior, with respect to the  $C^{\infty}$  topology.*

*Proof.* On the one hand, Anosov and Katok, see [AK70], show that for a Baire generic  $\alpha$ , weak mixing is a  $C^{\infty}$ -generic property in  $\overline{O_{\mathbb{A}}^{\infty}(\alpha)}$ . Thus by Theorem 1 weak mixing is a  $C^{\infty}$ -generic property in  $F_{\mathbb{A}}^{\infty}(\alpha)$ .

On the other hand, the second statement follows since elements of  $O_{\mathbb{A}}^{\infty}(\alpha)$  are never weak mixing and by Theorem 1 the complement  $F_{\mathbb{A}}^{\infty}(\alpha) \setminus O_{\mathbb{A}}^{\infty}(\alpha)$  has empty interior.  $\square$

**REMARK 1.** *Corollary A is seen to be rather sharp in the following sense:*

- (1) *the genericity of  $\alpha$  cannot be improved into any subset of  $(0, 1)$  with positive Lebesgue measure. This follows from the KAM result of Fayad-Krikorian [FK09] (attributed by the authors to Herman), that a neighborhood of  $R_{\alpha}$  in  $F_{\mathbb{A}}^{\infty}(\alpha)$  lies in  $O_{\mathbb{A}}^{\infty}(\alpha)$ , for any Diophantine  $\alpha$ .*
- (2) *Weakly mixing cannot be replaced by mixing. In fact, it follows from the proof of [B15a] and [AFLXZ20] that for a Baire generic  $\alpha$ ,  $F_{\mathbb{A}}^{\infty}(\alpha)$  contains no topologically mixing maps. See also Theorem 3.*

Recently, Avila and Krikorian have announced <sup>1</sup> an improvement of Theorem 1: for every non-Brjuno  $\alpha$ , one has  $F_{\mathbb{A}}^{\infty}(\alpha) = \overline{O_{\mathbb{A}}^{\infty}(\alpha)}$ . Moreover, they have announced the following result: for every pseudo-rotation  $f$  in an open neighborhood of the rigid rotations on  $\mathbb{D}$ , there exists a sequence of area-preserving diffeomorphism  $h_n$  such that  $h_n f h_n^{-1}$  converges to  $R_{\rho(f)}$  in the  $C^{\infty}$  topology. Their method involves delicate estimates on high iterates of the maps, while our method for getting this weaker result relies only on rather soft estimates.

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<sup>1</sup>See the minicourse of Krikorian in the program “Renormalization and universality in Conformal Geometry, Dynamics, Random Processes, and Field Theory” in 2020 at Simons Center for Geometry and Physics.

## NOTATION

In the rest of this paper, we use the following notation. For a subset  $A \subset \mathbb{R}^2$  we denote by  $\text{Int}(A)$  the interior of  $A$ . For a  $C^r$  diffeomorphism  $f : U \rightarrow V$  between open subsets  $U, V \subset \mathbb{R}^2$  and  $r \in \mathbb{N}$ , we set  $\|D^r f\| = \max_{x \in U, |\alpha|=r} \{\|\partial^\alpha f(x)\|\}$ ,  $\|f\|_{C^r} = \sup_{1 \leq \ell \leq r} \|D^\ell f\|$  and  $\|f\|_{\text{Diff}^r(U)} = \|f\|_{C^r(U)} + \|f^{-1}\|_{C^r(V)}$ . For  $C^r$  diffeomorphisms  $f, g : U \rightarrow V$ , we denote  $d_{\text{Diff}^r(U)}(f, g) = \max(\|f^{-1}g\|_{\text{Diff}^r(U)}, \|g^{-1}f\|_{\text{Diff}^r(U)})$ . We abbreviate  $\|f\|_{\text{Diff}^r(U)}$ , resp.  $d_{\text{Diff}^r(U)}$ , as  $\|f\|_{\text{Diff}^r}$ , resp.  $d_{\text{Diff}^r}$ , when there is no confusion. For two homeomorphisms  $f, g : U \rightarrow V$ , we denote  $\|f - g\| = \sup_{x \in U} d(f(x), g(x))$ , where  $d(\cdot, \cdot)$  is the euclidean metric.

## 2. PRELIMINARIES

We denote the boundary components of  $\mathbb{A}$  by

$$B_i := \mathbb{R}/\mathbb{Z} \times \{i\} \quad i = 0, 1.$$

The universal covering of  $\mathbb{A}$  is  $\tilde{\mathbb{A}} = \mathbb{R} \times [0, 1]$ . We write  $\tilde{B}_i = \mathbb{R} \times \{i\}$  for  $i = 0, 1$ . We let

$$\pi : \tilde{\mathbb{A}} \rightarrow \mathbb{A}$$

be the natural projection, and let  $T : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$  be the translation in the first coordinate, i.e.,  $T(x, y) = (x + 1, y)$ . Given a homeomorphism  $f : \mathbb{A} \rightarrow \mathbb{A}$  there is a lift  $F : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$  of  $f$ , also a homeomorphism, unique up to composition by a power of  $T$ , such that  $\pi F = f\pi$ . Moreover, any such lift commutes with  $T$ .

We have the following lemma, whose proof is elementary and left to the readers.

LEMMA 2.1. *Suppose  $f : \mathbb{A} \rightarrow \mathbb{A}$  is a homeomorphism with  $\|f - \text{Id}\| < 1/2$ . Then there is a unique lift  $F : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$  satisfying  $\|F - \text{Id}\| < 1/2$ . This lift satisfies*

$$(2.1) \quad d(F(\tilde{x}), \tilde{x}) = d(f(x), x)$$

for all  $x \in \mathbb{A}$  and all  $\tilde{x} \in \tilde{\mathbb{A}}$  with  $\pi(\tilde{x}) = x$ .

DEFINITION 2.1. We say that  $\gamma \subset \mathbb{A}$  is a *simple regular curve* connecting  $B_0$  and  $B_1$  if  $\gamma = \phi([0, 1])$  where  $\phi : [0, 1] \rightarrow \mathbb{A}$  is an injective continuous map mapping 0, resp. 1, into  $B_0$ , resp.  $B_1$ , that maps  $(0, 1)$  into  $\mathbb{A} \setminus \partial\mathbb{A}$ . We define simple regular curves in  $\tilde{\mathbb{A}}$  connecting  $\tilde{B}_0$  and  $\tilde{B}_1$  in a similar way.

DEFINITION 2.2. For a pair of simple regular curves  $\gamma_1, \gamma_2$  in  $\tilde{\mathbb{A}}$  we will say that  $\gamma_1$  is to the left of  $\gamma_2$ , or  $\gamma_2$  is to the right of  $\gamma_1$ , and write

$$\gamma_1 < \gamma_2$$

if  $\gamma_1 \cap \gamma_2 = \emptyset$  and  $\gamma_1$  lies in the component of  $\tilde{\mathbb{A}} \setminus \gamma_2$  containing points with arbitrarily negative  $\mathbb{R}$ -coordinate. This defines a partial ordering on simple regular curves in  $\tilde{\mathbb{A}}$ .

DEFINITION 2.3. Let  $f$  be a homeomorphism of  $\mathbb{A}$ . A simple regular curve  $\gamma$  in  $\mathbb{A}$  connecting  $B_0$  and  $B_1$ , satisfying

$$\gamma \cap f(\gamma) = \emptyset$$

is called a *Brouwer curve* for  $f$ . A Brouwer curve  $\gamma$  is called *smooth* if  $\gamma$  is a  $C^\infty$ -curve. Brouwer curves for homeomorphisms of  $\tilde{\mathbb{A}}$  are defined in a similar way.

Note that if  $F$  is a lift of a homeomorphism  $f : \mathbb{A} \rightarrow \mathbb{A}$  and  $\tilde{\gamma}$  is a lift of a Brouwer curve  $\gamma \subset \mathbb{A}$  for  $f$ , then

$$T^k F(\tilde{\gamma}) \cap \tilde{\gamma} = \emptyset \quad \forall k \in \mathbb{Z}.$$

DEFINITION 2.4. Let  $f$  be a homeomorphism of  $\mathbb{A}$  and  $Q \geq 2$  an integer. We say that  $\gamma$  is a  *$Q$ -good smooth curve* if  $\gamma$  is a smooth Brouwer curve for each of the maps  $f, f^2, \dots, f^{Q-1}$ .

DEFINITION 2.5. Let  $\gamma_1, \gamma_2$  be two disjoint simple regular curves in  $\mathbb{A}$  connecting  $B_0$  and  $B_1$ . There is a unique closed region  $\mathcal{R}$  in  $\mathbb{A}$  with left boundary  $\gamma_1$  and right boundary  $\gamma_2$ . More precisely, if each  $\gamma_i$  is oriented from  $B_0$  to  $B_1$ , then  $\partial\mathcal{R} \cap (\mathbb{A} \setminus \partial\mathbb{A})$  has orientation agreeing with  $\gamma_2 - \gamma_1$ . We say that  $\mathcal{R}$  is the *region bounded by*  $(\gamma_1, \gamma_2)$ .

DEFINITION 2.6. Let  $\{K_r\}_{r \geq 1}$  be an increasing sequence of positive real numbers. Let  $U, V$  be smooth surfaces, possibly with boundary, and let  $\phi : U \rightarrow V$  be a  $C^\infty$ -diffeomorphism. We say that  $\phi$  is  $\{K_r\}_{r \geq 1}$ -smooth if

$$\|\phi\|_{\text{Diff}^r(U)} < K_r \quad \forall r \in \mathbb{Z}_+.$$

DEFINITION 2.7. Let  $\gamma$  be a Brouwer curve for  $f$ , and let  $\mathcal{R} \subset \mathbb{A}$  be the closed region bounded by  $(\gamma, f(\gamma))$ . We say that an orientation preserving  $C^\infty$ -diffeomorphism

$$\phi : \mathcal{U} \rightarrow \mathcal{R}'$$

from an open neighborhood  $\mathcal{U} \subset \tilde{\mathbb{A}}$  of  $[0, 1]^2$  to an open neighborhood  $\mathcal{R}' \subset \mathbb{A}$  of  $\mathcal{R}$  is an *admissible coordinate* for  $(\mathcal{R}, f)$  if the following hold:

- (1)  $\phi$  has constant Jacobian,
- (2)  $\phi$  satisfies

$$(2.2) \quad \phi(\{0\} \times [0, 1]) = \gamma,$$

$$(2.3) \quad \phi(\{1\} \times [0, 1]) = f(\gamma),$$

$$(2.4) \quad \phi([0, 1] \times \{0\}) \subset B_0,$$

$$(2.5) \quad \phi([0, 1] \times \{1\}) \subset B_1,$$

- (3) there is a neighborhood  $U_L$  of  $\{0\} \times [0, 1]$  in  $\mathcal{U}$ , so that

$$f\phi(x) = \phi T(x) \quad \forall x \in U_L.$$

Without loss of generality we may assume that  $T(U_L) \subset \mathcal{U}$  by choosing  $U_L$  sufficiently small.

We make a similar definition for lifts. Namely, let  $\tilde{\gamma}$  be a lift of a Brouwer curve  $\gamma$  for  $f$ , let  $F$  be an lift of  $f$  such that  $F(\tilde{\gamma})$  is on the right of  $\tilde{\gamma}$ , and let  $\tilde{\mathcal{R}}$  be the region bounded by  $(\tilde{\gamma}, F(\tilde{\gamma}))$ . We say that a  $C^\infty$ -diffeomorphism

$$\phi : \mathcal{U} \rightarrow \mathcal{R}'$$

from an open neighborhood  $\mathcal{U}$  of  $[0, 1]^2$  in  $\tilde{\mathbb{A}}$  to an open neighborhood  $\mathcal{R}'$  of  $\tilde{\mathcal{R}}$  in  $\tilde{\mathbb{A}}$  is an *admissible coordinate* for  $(\tilde{\mathcal{R}}, F)$  if  $\phi$  satisfies the analogous properties above with  $(\mathcal{R}, \mathbb{A}, f)$  replaced by  $(\tilde{\mathcal{R}}, \tilde{\mathbb{A}}, F)$ .

REMARK 2. We notice that by item (3) in Definition 2.7, an admissible coordinate  $\phi : \mathcal{U} \rightarrow \mathcal{R}'$  is determined by its restriction to  $[0, 1]^2$  in the following sense: if  $\phi_i : \mathcal{U}_i \rightarrow \mathcal{R}_i$ ,  $i = 1, 2$  are two admissible coordinates for  $(\mathcal{R}, f)$  such that  $\phi_1|_{[0, 1]^2} = \phi_2|_{[0, 1]^2}$ , then there exists an open neighborhood  $\mathcal{U}_3$  of  $[0, 1]^2$  in  $\tilde{\mathbb{A}}$  such that  $\phi_1|_{\mathcal{U}_3} = \phi_2|_{\mathcal{U}_3}$ . For this reason, we will sometimes identify an admissible coordinate for  $(\mathcal{R}, f)$  or  $(\mathcal{R}, F)$  with a map from  $[0, 1]^2$  to  $\mathcal{R}$ .

DEFINITION 2.8. Let  $r \in \mathbb{N}$  and  $K \in (0, \infty)$ . We say that  $\phi : [0, 1]^2 \rightarrow \mathcal{R}$  is a  $(r, K)$ -admissible coordinate for  $(\mathcal{R}, f)$  if  $\phi$  is an admissible coordinate, in the sense of Definition 2.7, defined on an open neighborhood  $\mathcal{U}$  of  $[0, 1]^2$ , satisfying

$$\|\phi\|_{\text{Diff}^r(\mathcal{U})} < K.$$

Analogously for lifts: We say that  $\phi : \mathcal{U} \rightarrow \tilde{\mathcal{R}}$  is a  $(r, K)$ -admissible coordinate for  $(\tilde{\mathcal{R}}, F)$  if  $\phi$  is an admissible coordinate for  $(\tilde{\mathcal{R}}, F)$ , in the sense of Definition 2.7, defined on an open neighborhood  $\mathcal{U}$  of  $[0, 1]^2$ , satisfying  $\|\phi\|_{\text{Diff}^r(\mathcal{U})} < K$ .

DEFINITION 2.9. A *pseudo-rotation* is a non-wandering homeomorphism  $f : \mathbb{A} \rightarrow \mathbb{A}$  that is isotopic to the identity, maps  $B_0$  (resp.  $B_1$ ) to itself, and has no periodic points.

Recall that a homeomorphism  $f : \mathbb{A} \rightarrow \mathbb{A}$  is said to be *non-wandering* if for every open subset  $U \subset \mathbb{A}$ , there exists an integer  $n > 0$  such that  $f^n(U) \cap U \neq \emptyset$ . In this paper we will only be considering  $C^\infty$ -smooth pseudo-rotations that preserve a smooth area form  $\omega$ . By a slight abuse of notation, we denote by  $\omega$  both the area form on  $\tilde{\mathbb{A}}$  and  $\mathbb{A}$ . If  $f$  is an  $\omega$ -preserving diffeomorphism then so is each lift  $F$  an  $\omega$ -preserving diffeomorphism.

DEFINITION 2.10. Let  $f : \mathbb{A} \rightarrow \mathbb{A}$  be a pseudo-rotation with lift  $F : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ . Denote by  $p_1 : \tilde{\mathbb{A}} \rightarrow \mathbb{R}$  the projection on the first coordinate. We define  $\rho(F) \in \mathbb{R}$  by

$$\rho(F) := \lim_{n \rightarrow \infty} \frac{1}{n} (p_1(F^n(x, y)) - x) \quad x \in \mathbb{R}, y \in [0, 1].$$

It is known that the limit on the right hand side above exists for all  $(x, y)$  and is independent of  $(x, y)$ , see [F88, FH12]. Moreover, we always have  $\rho(F) \notin \mathbb{Q}$ , see [F88]. For any pseudo-rotation  $f$ , there exists a unique lift  $F$

of  $f$  such that  $\rho(F) \in (0, 1) \setminus \mathbb{Q}$ , which we denote by  $\rho(f)$ . We call  $\rho(f)$  the *rotation number* of  $f$ .

Let us recall a few well known facts about the best rational approximations to  $\alpha \in (0, 1) \setminus \mathbb{Q}$  and its continued fraction expansion. Readers can consult [HW, Chapters X, XI] for more details.

First, for  $x \in \mathbb{R}$  we will write

$$\begin{aligned} \lfloor x \rfloor &:= \max\{n \in \mathbb{Z} \mid n \leq x\} \in \mathbb{Z}, \\ q(x) &:= \lfloor 1/x \rfloor \in \mathbb{Z} \end{aligned}$$

for the integer parts of  $x$  and  $1/x$  respectively. Moreover, if  $x \in \mathbb{R} \setminus \mathbb{Q}$  then

$$\|x\|_{\mathbb{R}/\mathbb{Z}} := d(x, \mathbb{Z}) \in (0, 1/2).$$

The Gauss map  $\mathcal{G} : [0, 1) \rightarrow [0, 1)$  is defined by

$$\mathcal{G}(x) = \frac{1}{x} - q(x)$$

on  $(0, 1)$  and  $\mathcal{G}(0) := 0$ . If  $x$  is irrational then so is  $\mathcal{G}(x)$ .

DEFINITION 2.11. For any  $\alpha \in (0, 1) \setminus \mathbb{Q}$ , we define the sequences  $(\alpha_n)_{n \geq 0}$  and  $(\beta_n)_{n \geq 0}$  in  $(0, 1) \setminus \mathbb{Q}$  by

$$\begin{aligned} \alpha_0 &:= \alpha, & \alpha_n &:= \mathcal{G}^n(\alpha_0) & \forall n \geq 1, \\ \beta_n &:= \prod_{i=0}^n \alpha_i & \forall n \geq 0. \end{aligned}$$

Furthermore we define sequences of non-negative integers  $(a_n)_{n \geq 0}$ ,  $(q_n)_{n \geq 0}$  as follows:

$$(2.6) \quad a_0 := 0, \quad a_n := q(\alpha_{n-1}) \quad \forall n \geq 1,$$

$$(2.7) \quad q_0 := 1, \quad q_1 := q(\alpha), \quad q_{n+2} := q_n + q_{n+1}a_{n+2} \quad \forall n \geq 1.$$

We also define  $(p_n)_{n \geq 0}$  by  $p_0 = 0$ , and for  $n \geq 1$ , define  $p_n$  to be the closest integer to  $q_n \alpha$ , which is unique by irrationality of  $\alpha$ . We will use the notation  $\alpha_n(\alpha)$ ,  $q_n(\alpha)$  and  $p_n(\alpha)$  when it is necessary to indicate the dependence of the sequences on  $\alpha$ .

Note that

$$(2.8) \quad \alpha_{n-1}^{-1} = a_n + \alpha_n \quad \forall n \geq 1,$$

since  $\alpha_n = \mathcal{G}(\alpha_{n-1}) = 1/\alpha_{n-1} - q(\alpha_{n-1}) = 1/\alpha_{n-1} - a_n$ . It is also well known that

$$(2.9) \quad p_{n+1}q_n - p_nq_{n+1} = (-1)^n$$

$$(2.10) \quad \beta_n = (-1)^n(q_n \alpha - p_n) > 0 \quad \forall n \geq 0.$$



In particular for each  $n \geq 1$ ,  $\beta_n = |q_n \alpha - p_n| = d(q_n \alpha, \mathbb{Z})$ , and

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n + \alpha_n}}}} \quad \frac{p_n}{q_n} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}}$$

For any  $n \geq 1$ , the integers  $p_n$  and  $q_n$  are relatively prime, and  $p_n/q_n$  is called the  $n$ -th best rational approximation of  $\alpha$ . The following simple inequalities are known, and will be used later:

$$(2.11) \quad \frac{1}{2q_{n+1}} < \frac{1}{q_n + q_{n+1}} < \beta_n < \frac{1}{q_{n+1}},$$

$$(2.12) \quad \alpha_n, q(\alpha_n)^{-1} \in \left( \frac{q_n}{2q_{n+1}}, \frac{2q_n}{q_{n+1}} \right).$$

### 3. EXISTENCE OF A BROUWER CURVE WITH UNIFORM BOUNDS

In order to avoid lengthy computations, in the rest of the paper we will introduce various *increasing functions* to keep track of parameter dependence. We say that a function  $A : U \rightarrow \mathbb{R}$  defined on an open subset  $U \subset \mathbb{R}^n$  is *increasing* if for any  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n) \in U$  with  $x_i \geq y_i$  for all  $1 \leq i \leq n$ , we have  $A(x) \geq A(y)$ . We define *decreasing functions* in a similar way. Typically, all the variables and values of the increasing/decreasing functions that we will consider lie in  $\mathbb{R}_+$ .

The following theorem, essentially proven in [AFLXZ20], allows one to control the  $C^r$ -distance of a pseudo-rotation to the identity in terms of its rotation number.

**THEOREM 3.** *There is a sequence of increasing functions*

$$A_r : (0, 1/2) \times \mathbb{R}_+ \rightarrow \mathbb{R}_+, \\ \lim_{t \rightarrow 0} A_r(t, \cdot) \equiv 0$$

for each  $r \in \mathbb{N}$ , so that for any  $\{K_r\}_{r \geq 1}$ -smooth  $\omega$ -preserving pseudo-rotation  $f : \mathbb{A} \rightarrow \mathbb{A}$  there holds

$$\|f - \text{Id}\| < A_0(\|\rho(f)\|_{\mathbb{R}/\mathbb{Z}}, K_1), \\ \|D^r f\| < A_r(\|\rho(f)\|_{\mathbb{R}/\mathbb{Z}}, K_{r+1}), \quad \forall r \geq 1.$$

*Proof.* By [AFLXZ20, Corollary A], we know that

$$\|f - \text{Id}\| < (1 + 2K_1) \|\rho(f)\|_{\mathbb{R}/\mathbb{Z}}^{1/2} =: A_0(\|\rho(f)\|_{\mathbb{R}/\mathbb{Z}}, K_1).$$

By the Hadamard-Kolmogorov convexity theorem, we get corresponding bounds on the “inbetween” derivatives: for any  $r \geq 1$

$$\begin{aligned} \|D^r f\| &\leq C_r \|f - \text{Id}\|_{\text{Diff}^{r+1}}^{\frac{1}{r+1}} \|f\|_{\text{Diff}^{r+1}}^{\frac{r}{r+1}} \\ &\leq C_r \|\rho(f)\|_{\mathbb{R}/\mathbb{Z}}^{\frac{1}{2(r+1)}} (1 + 2K_{r+1}) =: A_r(\|\rho(f)\|_{\mathbb{R}/\mathbb{Z}}, K_{r+1}). \end{aligned}$$

□

By Theorem 3, there is a decreasing function

$$(3.1) \quad \rho_* : \mathbb{R}_+ \rightarrow (0, 1/2)$$

such that for any smooth  $\omega$ -preserving pseudo-rotation  $f$  satisfying

$$(3.2) \quad \|\rho(f)\|_{\mathbb{R}/\mathbb{Z}} < \rho_*(\|Df\|),$$

we have

$$\|f - \text{Id}\| \leq A_0(\|\rho(f)\|_{\mathbb{R}/\mathbb{Z}}, \|Df\|) < A_0(\rho_*(\|Df\|), \|Df\|) < 1/2.$$

Then by Lemma 2.1, there is a unique lift  $F$  of  $f$  satisfying

$$(3.3) \quad \|F - \text{Id}\| = \|f - \text{Id}\| < 1/2.$$

LEMMA 3.1. *There are increasing functions*

$$C', C'' : \mathbb{R}_+ \times (0, 1) \rightarrow \mathbb{R}_+$$

with  $\lim_{t \rightarrow 0} C''(t, \cdot) \equiv 0$  and

$$(3.4) \quad C''(\alpha, K^{-1}) < \rho_*(K^{\alpha-1}) \quad \forall (\alpha, K) \in (0, 1) \times (1, \infty)$$

such that the following holds: if  $\alpha \in (0, 1) \setminus \mathbb{Q}$  satisfies

$$(3.5) \quad \alpha \in (0, \rho_*(K)),$$

$$(3.6) \quad \mathcal{G}(\alpha) < C''(\alpha, K^{-1})$$

for some  $K > 1$ , where  $\rho_*$  is given in (3.1), then every  $C^1$ -smooth  $\omega$ -preserving pseudo-rotation with  $\|Df\| < K$  and  $\rho(f) = \alpha$  satisfies

$$(3.7) \quad \inf_{x \in \mathbb{A}} d(x, f(x)) \geq C'(\alpha, K^{-1}).$$

*Proof.* Fix  $K > 0$  and  $\alpha \in (0, 1) \setminus \mathbb{Q}$  satisfying (3.5) and (3.6). Let  $f$  be a  $C^1$ -smooth  $\omega$ -preserving pseudo-rotation with  $\|Df\| < K$  and  $\rho(f) = \alpha$ . Denote by  $F$  the lift of  $f$  with  $\rho(F) = \rho(f)$ . We abbreviate  $q := q(\alpha) = \lfloor 1/\alpha \rfloor$ .

Choose the function  $C'' : \mathbb{R}_+ \times (0, 1) \rightarrow \mathbb{R}_+$ , with  $\lim_{t \rightarrow 0} C''(t, \cdot) \equiv 0$ , sufficiently small so that for each  $t \in (0, 1)$ , the condition

$$s < C''(t, K^{-1})$$

implies

$$(3.8) \quad A_0(s, K^{t-1}) < 1/2.$$

Without loss of generality we may arrange that  $C''$  is increasing and satisfies (3.4). By Theorem 3, we have

$$\|f^q - \text{Id}\| \leq A_0(\|\rho(f^q)\|_{\mathbb{R}/\mathbb{Z}}, \|Df^q\|) \leq A_0(\|q\rho(f)\|_{\mathbb{R}/\mathbb{Z}}, K^q).$$

Since  $\|q\rho(f)\|_{\mathbb{R}/\mathbb{Z}} = 1 - q\alpha = \alpha\mathcal{G}(\alpha) \leq \mathcal{G}(\alpha)$  and  $K^q \leq K^{\alpha^{-1}}$ , the above gives

$$\|f^q - \text{Id}\| \leq A_0(\mathcal{G}(\alpha), K^{\alpha^{-1}}).$$

Thus, if  $\alpha$  satisfies (3.6), then by (3.8),

$$(3.9) \quad \|f^q - \text{Id}\| < 1/2.$$

The unique lift of  $f^q$  with rotation number in  $(-1/2, 1/2)$  is  $T^{-1}F^q$ , since  $\rho(T^{-1}F^q) = q\rho(F) - 1 = q\alpha - 1 \in (-1/2, 1/2)$ . So by Lemma 2.1

$$\|F^q - T\| = \|T^{-1}F^q - \text{Id}\| = \|f^q - \text{Id}\|.$$

Thus for all  $\tilde{x} \in \tilde{\mathbb{A}}$

$$(3.10) \quad d(F^q(\tilde{x}), \tilde{x}) \geq d(T(\tilde{x}), \tilde{x}) - d(F^q(\tilde{x}), T(\tilde{x})) > 1 - 1/2 = 1/2.$$

On the other hand, for each  $\tilde{x} \in \tilde{\mathbb{A}}$ , there holds

$$d(F^q(\tilde{x}), \tilde{x}) \leq \sum_{j=0}^{q-1} d(F^{j+1}(\tilde{x}), F^j(\tilde{x})) \leq \sum_{j=0}^{q-1} \|DF\|^j d(F(\tilde{x}), \tilde{x}) \leq Cd(F(\tilde{x}), \tilde{x})$$

where  $C = \frac{K^q - 1}{K - 1} \leq \frac{K^{\alpha^{-1}} - 1}{K - 1}$ . Combining the last line with (3.10) and (3.3), we obtain

$$(3.11) \quad \|F(\tilde{x}) - \tilde{x}\| \geq \frac{1}{2C} \quad \forall \tilde{x} \in \tilde{\mathbb{A}}.$$

By Lemma 2.1 and (3.5), we conclude that (3.7) holds if we define  $C'$  by

$$C'(t, K^{-1}) = \frac{1}{2} \frac{K - 1}{K^{t-1} - 1}.$$

□

**DEFINITION 3.1.** For each  $K > 1$  let  $\mathcal{L}(K)$  be the set of  $\alpha \in (0, 1/2) \setminus \mathbb{Q}$  such that

$$(3.12) \quad \alpha \in (0, \rho_*(K)),$$

$$(3.13) \quad \mathcal{G}(\alpha) < C''(\alpha, K^{-1})$$

where  $\rho_*$  is from (3.1) and  $C''$  is from Lemma 3.1.

**LEMMA 3.2.** *There is a sequence of increasing functions*

$$C'_r : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \quad \forall r \geq 1,$$

*such that the following is true. If  $\{K_r\}_{r \geq 1}$  is any sequence in  $(1, \infty)$  and  $\alpha \in \mathcal{L}(K_1)$ , then every  $\{K_r\}_{r \geq 1}$ -smooth  $\omega$ -preserving pseudo-rotation  $f$  with  $\rho(f) = \alpha$  has for each  $r \in \mathbb{N}$  a Brouwer curve  $\gamma = \gamma_r$  for which the closed region in  $\mathbb{A}$  bounded by  $(\gamma, f(\gamma))$  admits a  $(r, C'_r(\alpha^{-1}, K_{r+1}))$ -admissible coordinate.*

*Proof.* Fix a sequence  $(K_r)_{r \geq 1}$  in  $(1, \infty)$ . Fix  $r \geq 1$ . For each  $c > 0$  and  $K \geq 1$  define the following subset of  $\text{Diff}^r(\mathbb{A}, \omega)$ :

$$\mathcal{H}_r(c, K) := \left\{ g \in \text{Diff}^r(\mathbb{A}, \omega) \mid \|g\|_{\text{Diff}^r} \leq K, \quad \inf_{x \in \mathbb{A}} d(x, g(x)) \geq c \right\}.$$

Denote by

$$\overline{\mathcal{H}_{r+1}(c, K)} \subset \text{Diff}^r(\mathbb{A}, \omega)$$

the closure of  $\mathcal{H}_{r+1}(c, K)$  in the  $C^r$ -topology. We observe that  $\overline{\mathcal{H}_{r+1}(c, K)}$  is compact in  $\text{Diff}^r(\mathbb{A})$  and contains only diffeomorphisms without fixed points. Moreover, for  $c' \geq c$  and  $K' \leq K$  we have  $\overline{\mathcal{H}_{r+1}(c', K')} \subset \overline{\mathcal{H}_{r+1}(c, K)}$ .

We now prove a version of the lemma for elements of  $\overline{\mathcal{H}_{r+1}(c, K)}$ , for each fixed  $c > 0, K \geq 1$ . Then we argue that the union of  $\overline{\mathcal{H}_{r+1}(c, K_{r+1})}$  over  $c \in (0, 1]$  contains all pseudo-rotations satisfying the assumptions of the Lemma.

To this end, fix  $c > 0, K \geq 1$  and consider  $g \in \overline{\mathcal{H}_{r+1}(c, K)}$ . Since  $g$  has no fixed points, a strong refinement of Brouwer's plane translation theorem due to Guillou [G94, Théorème 5.1] yields a  $C^0$  Brouwer curve  $\gamma_0$  for  $g$  in the sense of Definition 2.3. Any sufficiently  $C^0$ -close smooth approximation of  $\gamma_0$  that continues to connect the two boundary components yields a smooth Brouwer curve  $\gamma$  for  $g$ . Clearly we can choose such a  $\gamma$  to meet both boundary components of the annulus orthogonally. We can then apply the following lemma.

**LEMMA 3.3.** *Let  $r \geq 1$  and  $\theta \in (0, 1)$ . Suppose  $g \in \text{Diff}^{r, \theta}(\mathbb{A}, \omega)$  has a smooth Brouwer curve  $\gamma$  that meets both boundary components orthogonally. Then there exist  $D = D(g, \gamma, r) > 1$ , and a neighborhood  $\mathcal{V}$  of  $g$  in  $\text{Diff}^{r, \theta}(\mathbb{A}, \omega)$  such that for every  $g' \in \mathcal{V}$ , the region  $(\gamma, g'(\gamma))$  has admissible coordinates  $\phi'$  whose  $C^r$ -norm is bounded by  $D$ .*

*Proof.* Let  $\mathcal{R}$  denote the region  $(\gamma, g(\gamma))$ . We first construct a  $C^\infty$ -diffeomorphism  $\psi_L$  with constant Jacobian  $\omega(\mathcal{R})$  from a neighborhood of  $\{0\} \times [0, 1]$  in  $\tilde{\mathbb{A}}$  onto its image a neighborhood of  $\gamma$  in  $\mathbb{A}$ . Indeed, without loss of generality  $\gamma : [0, 1] \rightarrow \mathbb{A}$  meets the boundary of  $\mathbb{A}$  orthogonally near both end points and is parametrised by arclength so that  $\|\dot{\gamma}\| = L$  is constant,  $L \geq 1$ . Then  $n := -i\dot{\gamma}/L$  is a normal vector field along  $\gamma$  and the map  $\tilde{\mathbb{A}} \rightarrow \mathbb{R}^2, (x, y) \mapsto \gamma(y) + x\omega(\mathcal{R})n(y)/L$  extends  $\gamma$  to a smooth diffeomorphism from a sufficiently small tubular neighborhood of  $\{0\} \times [0, 1]$  in  $\tilde{\mathbb{A}}$  to a neighborhood of the image of  $\gamma$  in  $\mathbb{A}$ , with constant Jacobian  $\omega(\mathcal{R})$  along  $\{0\} \times [0, 1]$  and also near to the boundary of  $\tilde{\mathbb{A}}$ . The Jacobian of this map away from  $\{0\} \times [0, 1]$  depends only on the  $y$  variable and therefore by a further change of coordinates it is easily modified to have constant Jacobian on a whole neighborhood of  $\{0\} \times [0, 1]$  while still mapping  $\{0\} \times [0, 1]$  to  $\gamma$ . The resulting map, which we denote by  $\psi_L$ , clearly has finite  $C^{r, \theta}$ -norm.

Then  $g\psi_L T^{-1}$  is a  $C^{r, \theta}$ -diffeomorphism  $\psi_R$  with constant Jacobian  $\omega(\mathcal{R})$  from a neighborhood of  $\{1\} \times [0, 1]$  in  $\tilde{\mathbb{A}}$  onto its image, a neighborhood

of  $g(\gamma)$  in  $\mathbb{A}$ . We construct the chart  $\phi : [0, 1]^2 \rightarrow \mathbb{A}$  by first specifying its restriction to a neighborhood of  $\{0\} \times [0, 1]$ , resp.  $\{1\} \times [0, 1]$ , to be  $\psi_L$ , resp.  $\psi_R$ . Then we extend it by hand to neighborhoods of  $[0, 1] \times \{0\}$  and  $[0, 1] \times \{1\}$ , also with constant Jacobian  $\omega(\mathcal{R})$ . Finally, using [A10, Corollary 4] (or, in our simple application, using directly [A10, Theorem 3] which follows from [DM90]), we extend the map to all of  $[0, 1]^2$  so as to have constant Jacobian, and the  $C^r$  norm of the resulting map is bounded in terms of the  $C^{r,\theta}$  norm of  $g$ . By construction the conditions in Definition 2.7 hold.

Each  $g$  has a  $C^0$ -small (and hence also  $C^r$ -small) neighborhood in  $\text{Diff}^r(\mathbb{A}, \omega)$  for which the same  $\gamma$  can be applied. The uniform bounds for the  $C^r$ -norm of  $\phi$  is an immediate consequence of the construction. We omit the proof of this latter fact and refer the readers to [A10, DM90] for details.  $\square$

By Lemma 3.3 and by compactness of  $\overline{\mathcal{H}_{r+1}(c, K)} \subset \text{Diff}^{r,1/2}(\mathbb{A}, \omega) \subset \text{Diff}^r(\mathbb{A}, \omega)$ , we find a finite collection of neighborhoods (with respect to the  $C^{r,1/2}$ -topology) as in Lemma 3.3 whose union covers  $\mathcal{H}_{r+1}(c, K)$ , and thus obtain a uniform bound  $E_r(c, K) > 0$  on the  $C^r$ -norm of admissible coordinates that applies to all elements of  $\overline{\mathcal{H}_{r+1}(c, K)}$ . Due to the inclusions  $\overline{\mathcal{H}_{r+1}(c', K')} \subset \overline{\mathcal{H}_{r+1}(c, K)}$  for  $c' > c, K' < K$ , we can assume that  $E_r(c, K)$  is decreasing in  $c$  and increasing in  $K$ .

Now, suppose  $f$  is as in Lemma 3.2. That is,  $f$  is a  $(K_r)_{r \geq 1}$ -smooth  $\omega$ -preserving pseudo-rotation with rotation number  $\rho(f) = \alpha \in \mathcal{L}(K_1)$ . By Lemma 3.1,

$$f \in \mathcal{H}_{r+1}\left(C'(\alpha, K_1^{-1}), K_{r+1}\right).$$

Thus there exists a region  $(\gamma, f(\gamma))$  having admissible coordinates whose  $C^r$ -norm is bounded by  $E_r(C'(\alpha, K_1^{-1}), K_{r+1})$ , for some smooth Brouwer curve  $\gamma$ . Hence Lemma 3.2 holds with  $C'_r(\alpha^{-1}, K_{r+1}) := E_r(C'(\alpha, K_1^{-1}), K_{r+1})$ . Evidently  $C'_r$  is an increasing function, since  $C'$  is increasing and  $E_r(c, K)$  is decreasing in  $c$  and increasing in  $K$ .  $\square$

It will be useful to fix the following notation:

**DEFINITION 3.2.** For each integer  $r \geq 1$  and each  $K > 1$  let  $\mathcal{L}_r(K)$  be the set of  $\alpha \in \mathcal{L}(K)$  such that

$$(3.14) \quad A_0(\alpha \mathcal{G}(\alpha), K^{q(\alpha)}) < K^{-q(\alpha)} C'_r(\alpha^{-1}, K)^{-1}$$

where  $C'_r : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ ,  $r \geq 1$  are the increasing functions produced by Lemma 3.2.

We will see later that each  $\mathcal{L}_r(K)$  is non-empty for each integer  $r \geq 1$  and each  $K > 1$ . Moreover, by (3.13) and (3.14) we have

$$(3.15) \quad \mathcal{L}_r(K') \subset \mathcal{L}_r(K) \quad \forall K' > K.$$

Now we can strengthen Lemma 3.2. Indeed, we show that if more restrictions are placed on the rotation number of a  $\omega$ -preserving pseudo-rotation

$f$ , then the produced Brouwer curve is actually  $Q$ -good, for some large  $Q$  depending on the rotation number of  $f$ :

**PROPOSITION 3.1.** *Let  $f$  be a  $\{K_r\}_{r \geq 1}$ -smooth  $\omega$ -preserving pseudo-rotation. If  $\rho(f) \in \mathcal{L}_r(K_{r+1})$  for some integer  $r \geq 1$ , then the Brouwer curve  $\gamma$  for  $f$  produced by Lemma 3.2 (corresponding to  $r$ ) is  $q(\rho(f))$ -good.*

Note that the condition  $\rho(f) \in \mathcal{L}_r(K_{r+1})$  in this Proposition includes the condition on  $\rho(f)$  used in Lemma 3.2, since  $\mathcal{L}_r(K) \subset \mathcal{L}(K)$  by definition.

*Proof.* We denote by  $d_H$  the Hausdorff distance on  $\tilde{\mathbb{A}}$ , i.e. for any two subsets  $A, B$  of  $\tilde{\mathbb{A}}$ ,

$$d_H(A, B) = \sup_{x \in A, y \in B} \max(d(x, B), d(y, A)).$$

Set  $\alpha = \rho(f) \in \mathcal{L}_r(K_{r+1})$  for some  $r \geq 1$ . Let  $\gamma$  be the Brouwer curve given by Lemma 3.2 for this value of  $r$ . Let  $\mathcal{R} \subset \mathbb{A}$  be the region bounded by  $(\gamma, f(\gamma))$  and let  $\psi : \mathcal{R} \rightarrow [0, 1] \times [0, 1]$  be the inverse of some  $(r, C'_r(\alpha^{-1}, K_{r+1}))$ -admissible coordinates produced by Lemma 3.2. Applying the intermediate value theorem to  $\psi$  yields

$$(3.16) \quad d_H(\gamma, f(\gamma)) > C'_r(\alpha^{-1}, K_{r+1})^{-1}.$$

Fix a lift  $\tilde{\gamma} \subset \tilde{\mathbb{A}}$  of  $\gamma$ , and let  $F$  be the lift of  $f$  for which  $\rho(F) = \alpha$ . Since  $\tilde{\gamma} \cap F(\tilde{\gamma}) = \emptyset$  it follows from  $\alpha > 0$  and the order of boundary points that  $\tilde{\gamma} < F(\tilde{\gamma})$ . By Definition 2.7, (3.16) and  $K_{r+1} \geq K_1$ , we can see that

$$(3.17) \quad \begin{aligned} d_H(F^{q(\alpha)-1}(\tilde{\gamma}), F^{q(\alpha)}(\tilde{\gamma})) &> K_1^{-q(\alpha)} d_H(\tilde{\gamma}, F(\tilde{\gamma})) \\ &\geq K_{r+1}^{-q(\alpha)} C'_r(\alpha^{-1}, K_{r+1})^{-1}. \end{aligned}$$

Here we implicitly used that the analogue of (3.16) holds for the lifts, since  $d_H(\tilde{\gamma}, F(\tilde{\gamma})) \geq d_H(\gamma, f(\gamma))$  holds - in fact for all choice of lifts  $F$  and  $\tilde{\gamma}$ . Moreover, since  $F$  is injective, for each  $i \in \mathbb{N}$  we have  $F^i(\tilde{\gamma}) \cap F^{i+1}(\tilde{\gamma}) = \emptyset$  and so from the order of boundary points we have for all  $i \in \mathbb{N}$ ,

$$(3.18) \quad \tilde{\gamma} < F(\tilde{\gamma}) < F^2(\tilde{\gamma}) < \dots < F^i(\tilde{\gamma}).$$

Thus it suffices to show that

$$(3.19) \quad F^{q(\alpha)-1}(\tilde{\gamma}) < T\tilde{\gamma}$$

and it will follow that the iterates  $F(\tilde{\gamma}), \dots, F^{q(\alpha)-1}(\tilde{\gamma})$  all lie strictly in the region between  $\tilde{\gamma}$  and  $T(\tilde{\gamma})$  in  $\tilde{\mathbb{A}}$  and therefore that the iterates  $f(\gamma), \dots, f^{q(\alpha)-1}(\gamma)$  are all disjoint from  $\gamma$  as required.

To show (3.19) we argue by contradiction. Assuming (3.19) is not true, we have

$$(3.20) \quad F^{q(\alpha)-1}(\tilde{\gamma}) \cap T\tilde{\gamma} \neq \emptyset$$

because of the order of boundary points. We will show that this means that  $F^{q(\alpha)-1}(\tilde{\gamma})$  and  $F^{q(\alpha)}(\tilde{\gamma})$  pass somewhere very close to each other, because  $F^{q(\alpha)}(\tilde{\gamma})$  is close to  $T\tilde{\gamma}$ . This will contradict (3.17).

First, notice that by  $\alpha \in \mathcal{L}_r(K_{r+1})$ , (3.4) and that  $\rho_*$  is decreasing, we have

$$\begin{aligned} \|\rho(f^{q(\alpha)})\|_{\mathbb{R}/\mathbb{Z}} &= |q(\alpha)\alpha - 1| = \alpha\mathcal{G}(\alpha) < \mathcal{G}(\alpha) < C''(\alpha, K_{r+1}^{-1}) \\ &< \rho_*(K_{r+1}^{\alpha^{-1}}) \leq \rho_*(\|Df\|^{q(\alpha)}) \leq \rho_*(\|Df^{q(\alpha)}\|). \end{aligned}$$

Then by (3.3), we have

$$d_H(F^{q(\alpha)}(\tilde{\gamma}), T\tilde{\gamma}) \leq \|F^{q(\alpha)} - T\| = \|T^{-1}F^{q(\alpha)} - \text{Id}\| = \|f^{q(\alpha)} - \text{Id}\|.$$

The last equality can be justified by Lemma 2.1, just as in the proof of Lemma 3.1. Thus by Theorem 3 and  $K_{r+1} \geq K_1$ , we have

$$\begin{aligned} d_H(F^{q(\alpha)}(\tilde{\gamma}), T\tilde{\gamma}) &\leq A_0(\|\rho(f^{q(\alpha)})\|_{\mathbb{R}/\mathbb{Z}}, K_1^{q(\alpha)}) \\ (3.21) \quad &\leq A_0(\alpha\mathcal{G}(\alpha), K_{r+1}^{q(\alpha)}). \end{aligned}$$

Then along with (3.17), (3.14), and the hypothesis  $\alpha \in \mathcal{L}_r(K_{r+1})$ , we have

$$\begin{aligned} d_H(F^{q(\alpha)^{-1}}(\tilde{\gamma}), T\tilde{\gamma}) &\geq d_H(F^{q(\alpha)}(\tilde{\gamma}), F^{q(\alpha)^{-1}}(\tilde{\gamma})) - d_H(F^{q(\alpha)}(\tilde{\gamma}), T\tilde{\gamma}) \\ &\geq K_{r+1}^{-q(\alpha)} C'_r(\alpha^{-1}, K_{r+1})^{-1} - A_0(\alpha\mathcal{G}(\alpha), K_{r+1}^{q(\alpha)}) > 0. \end{aligned}$$

However this contradicts (3.20). Thus we have (3.19).  $\square$

#### 4. SMOOTH DOMAIN BOUNDED BY GOOD CURVES

**4.1. Renormalization of pseudo-rotations.** Given a  $C^\infty$  pseudo-rotation  $f$  on  $\mathbb{A}$  and an integer  $n \geq 1$ , we denote by  $F_n$  the unique lift of  $f^n$  to  $\tilde{\mathbb{A}}$  such that  $\rho(F_n) \in (0, 1)$ . Given a smooth Brouwer curve  $\gamma_n$  for  $f^n$ , we let  $\tilde{\gamma}_n$  be an arbitrary lift of  $\gamma_n$  to  $\tilde{\mathbb{A}}$ . We let  $\Omega_n$  be the unique closed region in  $\tilde{\mathbb{A}}$  bounded by  $(\tilde{\gamma}_n, F_n(\tilde{\gamma}_n))$ .

Given a  $C^\infty$  admissible coordinate  $H : [0, 1]^2 \rightarrow \Omega_n$  for  $(\Omega_n, F_n)$  (see Remark 2), we can uniquely extend  $H$  to a  $C^\infty$  diffeomorphism of  $\tilde{\mathbb{A}}$  with constant Jacobian, denoted again by  $H$ , satisfying

$$(4.1) \quad HT = F_n H.$$

We notice that although the  $C^r$  norms of  $F_n$  and  $T$  are uniformly bounded throughout  $\tilde{\mathbb{A}}$ , the  $C^r$  norm of  $H$  need not be uniformly bounded. However it is clear from (4.1) that for any integer  $L > 0$ , the norm  $\|D^r H\|_{[-L, L] \times [0, 1]}$  is bounded in terms of  $L$  and  $\|F_n\|_{\text{Diff}^r(\tilde{\mathbb{A}})}$ .

Denote  $\alpha = \rho(f) \in (0, 1) \setminus \mathbb{Q}$ . We have  $\rho(F_1) = \alpha$ . We abbreviate

$$F^{a,b} := T^b F_1^a \quad \forall a, b \in \mathbb{Z}.$$

By our previous definition, we have  $F_n = F^{n, -\lfloor n\alpha \rfloor}$ . Denote by  $J : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$  the orientation reversing diffeomorphism

$$J(x, y) = (-x, y).$$

Notice that by (4.1) we have

$$(4.2) \quad JH^{-1}F_n HJ = JTJ = T^{-1}.$$

We set

$$(4.3) \quad F_H^{a,b} := JH^{-1}F^{a,b}HJ \quad \forall a, b \in \mathbb{Z}.$$

Since  $F^{a,b}$  is  $\omega$ -preserving and commutes with  $F_n$ , and since  $H$  has constant Jacobian, we deduce from (4.2) that  $F_H^{a,b}$  is also  $\omega$ -preserving, and commutes with  $T$ . Consequently  $F_H^{a,b}$  descends to an  $\omega$ -preserving  $C^\infty$  diffeomorphism  $f_H^{a,b} : \mathbb{A} \rightarrow \mathbb{A}$ .

We have the following lemma.

LEMMA 4.1. *There is a sequence of increasing functions  $\{E_r : \mathbb{N}^2 \times \mathbb{R}_+^2 \rightarrow \mathbb{R}_+\}_{r \geq 1}$  such that the following is true. Let  $f$  be a  $\{K_r\}_{r \geq 1}$ -smooth  $\omega$ -preserving pseudo-rotation with  $\rho(f) = \alpha$ ; let  $n \neq 0$  be an integer; let  $\Omega_n$  be the region bounded by  $(\tilde{\gamma}_n, F_n(\tilde{\gamma}_n))$  where  $\tilde{\gamma}_n$  is some lift of a Brouwer curve for  $f^n$ . Assume that  $(\Omega_n, F_n)$  admits a  $(r, L_r)$ -admissible coordinate  $H$ . Then for any  $a, b \in \mathbb{Z}$  with  $a \lfloor n\alpha \rfloor + bn \neq 0$ ,  $f_H^{a,b}$  is a pseudo-rotation in  $\text{Diff}^\infty(\mathbb{A}, \omega)$  with*

$$\rho(f_H^{a,b}) = \left\{ \frac{-a\alpha - b}{\lfloor n\alpha \rfloor} \right\}.$$

Moreover, we have

$$\|f_H^{a,b}\|_{\text{Diff}^r} \leq E_r(|a|, |b|, K_r, L_r).$$

*Proof.* We first show that  $f_H^{a,b}$  has no periodic points. Assume to the contrary there are integers  $p \in \mathbb{Z}$ ,  $q > 0$  and some  $z \in \tilde{\mathbb{A}}$  such that

$$(F_H^{a,b})^q(z) = T^p(z).$$

Then by (4.2) and (4.3) we have

$$(4.4) \quad z = (F_H^{a,b})^q T^{-p}(z) = JH^{-1}F_1^{(qa+pn)}T^{(qb-p\lfloor n\alpha \rfloor)}HJ(z).$$

However, (4.4) and the condition  $a \lfloor n\alpha \rfloor + bn \neq 0$  implies that  $HJ(z)$  descends to a periodic point for  $f$ , which contradicts the hypothesis that  $f$  is a pseudo-rotation. We conclude that  $f_H^{a,b}$  is a pseudo-rotation in  $\text{Diff}^\infty(\mathbb{A}, \omega)$ .

To compute  $\rho(f_H^{a,b})$ , it suffices to study the trajectory of an arbitrary  $z \in \tilde{\mathbb{A}}$  under the iterates of  $(F_H^{a,b})^q T^{-p}$  using the second equality in (4.4); for example a point on the boundary. This standard argument is left to the readers.

For the  $C^r$ -norm of  $f_H^{a,b}$ , it suffices to control the  $C^r$ -norm of  $F_H^{a,b}$  restricted to  $(-1, 2) \times [0, 1]$ . By (4.3),  $\|D^r F_H^{a,b}\|_{(-1,2) \times [0,1]}$  depends only on the  $C^r$ -norms of  $F^{a,b}$ , and the  $C^r$ -norm of  $J$  and  $H$  on

$$F_H^{a,b}((-1, 2) \times [0, 1]) \cup (-1, 2) \times [0, 1].$$

This yields a bound depending only on  $r, |a|, |b|, K_r, L_r$  as required.  $\square$



**4.2. Finding a good curve.** In the following statement we consider  $n \in \mathbb{N}$  even, so that  $q_n \alpha - p_n > 0$ .

**PROPOSITION 4.1.** *There is a sequence of increasing functions  $\{G_r : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+\}_{r \geq 1}$  such that the following is true.*

*Let  $f$  be a pseudo-rotation in  $\text{Diff}^\infty(\mathbb{A}, \omega)$ , let  $F$  be a lift with  $\rho(F) = \rho(f) = \alpha \in (0, 1) \setminus \mathbb{Q}$ , and let  $n \geq 2$  be an even integer. Suppose  $\gamma \subset \mathbb{A}$  is a smooth Brouwer curve for  $f^{q_n}$  with a lift  $\gamma^\dagger \subset \tilde{\mathbb{A}}$  such that the closed region  $\Omega$  in  $\tilde{\mathbb{A}}$  bounded by  $(\gamma^\dagger, F^{q_n, -p_n}(\gamma^\dagger))$  admits an  $(r, K_r)$ -admissible coordinate*

$$H : [0, 1]^2 \rightarrow \Omega$$

*for some  $r \geq 1$ . Suppose further that  $\gamma_n \subset \mathbb{A}$  is an  $a_{n+2}$ -good smooth curve for  $f_H^{q_{n+1}, -p_{n+1}}$ ,<sup>2</sup> for which the closed region  $\Omega_n \subset \mathbb{A}$  bounded by  $(\gamma_n, f_H^{q_{n+1}, -p_{n+1}}(\gamma_n))$  admits an  $(r, M_r)$ -admissible coordinate. Then there exists a  $q_{n+2}$ -good smooth curve  $\hat{\gamma} \subset \mathbb{A}$  for  $f$ , for which the closed region  $\Omega_* \subset \mathbb{A}$  bounded by  $(f^{q_{n+1}}(\hat{\gamma}), \hat{\gamma})$  admits a  $(r, G_r(K_r, M_r))$ -admissible coordinate for  $f^{q_{n+1}}$ .*

*Proof.* We set

$$S_0 = F^{q_n, -p_n}, \quad S = F^{q_{n+1}, -p_{n+1}}.$$

As explained in Section 4.1, we extend  $H$  to a  $C^\infty$ -diffeomorphism of  $\tilde{\mathbb{A}}$  with constant Jacobian by the formula

$$HT = S_0H.$$

By definition, we know that

- (1)  $JH^{-1}S_0HJ = JTJ = T^{-1}$  on  $\tilde{\mathbb{A}}$ ;
- (2)  $\tilde{S} := JH^{-1}SHJ$  commutes with  $T$  on  $\tilde{\mathbb{A}}$ , and descends to  $f_H^{q_{n+1}, -p_{n+1}}$ .

Moreover, we have  $\rho(\tilde{S}) = \rho(f_H^{q_{n+1}, -p_{n+1}}) = \alpha_{n+1} \in (0, \frac{1}{2})$ .

Let  $\tilde{\gamma}$  be an arbitrary lift of  $\gamma_n$ . As  $\gamma_n$  is a simple regular curve and is disjoint from  $f_H^{q_{n+1}, -p_{n+1}}(\gamma_n)$ , we know by item (1), (2) above that  $\tilde{\gamma}$  is also a simple regular curve, and

$$(4.5) \quad T^{-1}(\tilde{\gamma}) \cap \tilde{\gamma} = \tilde{S}(\tilde{\gamma}) \cap \tilde{\gamma} = \emptyset.$$

We let  $\gamma' = HJ(\tilde{\gamma})$ . Then  $\gamma'$  is also a simple regular curve, and by (4.5) and item (1), (2) above, we have

$$(4.6) \quad S_0(\gamma') \cap \gamma' = S(\gamma') \cap \gamma' = \emptyset.$$

Moreover, we can see that

$$(4.7) \quad S(\gamma') < \gamma' < S_0(\gamma')$$

by considering their boundary points on  $B_0$  and  $B_1$ . We let  $\hat{\gamma} = \pi(\gamma')$  (recall that  $\pi$  is the canonical projection from  $\tilde{\mathbb{A}}$  to  $\mathbb{A}$ ).

**LEMMA 4.2.** *The curve  $\hat{\gamma}$  is a simple regular curve connecting  $B_0$  and  $B_1$ . Namely, it has no self-intersection.*

<sup>2</sup>Recall that by Lemma 4.1 we have  $\rho(f_H^{q_{n+1}, -p_{n+1}}) = \alpha_{n+1}$  and  $a_{n+2} = q(\alpha_{n+1})$ .

*Proof.* Since  $\hat{\gamma} = \pi(\gamma')$ , and since  $\gamma'$  is a simple regular curve in  $\tilde{\mathbb{A}}$  connecting  $B_0$  to  $B_1$ , it suffices to show that  $\gamma'$  is disjoint from all its translates  $T^k(\gamma')$  for  $k \in \mathbb{Z} \setminus \{0\}$ . Since  $\gamma'$  connects the two boundary components it is enough to show  $\gamma'$  is disjoint from  $T(\gamma')$ .

By definition and by (2.9), we have  $T = S_0^{q_{n+1}}S^{-q_n}$ . Then by (4.7), we obtain that  $\gamma' < T(\gamma')$ . This completes the proof.  $\square$

By item (2) above, the region  $\tilde{\Omega}_n$  bounded by  $(\tilde{\gamma}, \tilde{S}(\tilde{\gamma}))$  is a lift of  $\Omega_n$ . Moreover, the  $(r, M_r)$ -admissible coordinate for  $\Omega_n$  lifts to a  $(r, M_r)$ -admissible coordinate for  $\tilde{\Omega}_n$ . We notice that the push forward  $HJ(\tilde{\Omega}_n)$  is the region in  $\tilde{\mathbb{A}}$  between  $S(\gamma')$  and  $\gamma'$ . Moreover, from the proof of Lemma 4.2, we see that

$$T^{-1}(\gamma') < S(\gamma') < \gamma'.$$

Thus the map  $\pi$  induces a diffeomorphism from  $HJ(\tilde{\Omega}_n)$  to the region in  $\mathbb{A}$  between  $f^{q_{n+1}}(\hat{\gamma})$  and  $\hat{\gamma}$ , that is, to the region  $\Omega_*$ . We conclude that the  $(r, M_r)$ -admissible coordinate for  $\tilde{\Omega}_n$ , after composing with  $HJ$  and projecting, yield  $(r, G_r(K_r, M_r))$ -admissible coordinates for  $\Omega_*$ , for some functions  $G_r$  as in the proposition.

It remains to show that  $\hat{\gamma}$  is a  $q_{n+2}$ -good curve. We divide the proof into two cases.

Case I: Assume that there are integers  $p$  and  $0 < k < q_{n+2}$  such that  $k\alpha + p < 0$  and

$$(4.8) \quad F^{k,p}(\gamma') \cap \gamma' \neq \emptyset.$$

We assume further that for any integers  $p'$  and  $0 < k' < q_{n+2}$  such that  $k'\alpha + p' < 0$  and  $F^{k',p'}(\gamma') \cap \gamma' \neq \emptyset$ , we have

$$k'\alpha + p' \leq k\alpha + p.$$

- First, we observe that either  $q_{n+2} > k \geq q_{n+2} - q_n$  or

$$-\{q_n\alpha\} = p_n - q_n\alpha < k\alpha + p < 0.$$

Indeed, if  $k < q_{n+2} - q_n$  and  $p_n - q_n\alpha > k\alpha + p$ , then we have  $k + q_n < q_{n+2}$  and

$$k\alpha + p < (k + q_n)\alpha + p - p_n < 0.$$

In particular, the endpoints of  $S_0 F^{k,p}(\gamma')$  are on the left hand side of those of  $\gamma'$ . Then by (4.8) and  $S_0 F^{k,p}(\gamma') > F^{k,p}(\gamma')$ , we have that

$$F^{k+q_n, -p_n+p}(\gamma') \cap \gamma' \neq \emptyset.$$

This contradicts the choice of  $k$ .

- If  $q_{n+2} > k \geq q_{n+2} - q_n$ , then by (4.6), (4.8) and (2.7), we know that  $k > q_{n+1}$ . We notice that

$$k\alpha + p < (k - q_{n+1})\alpha + p + p_{n+1} < 0.$$

In particular, the endpoints of  $S^{-1}F^{k,p}(\gamma')$  are on the left hand side of those of  $\gamma'$ . Then by (4.8) and  $S^{-1}F^{k,p}(\gamma') > F^{k,p}(\gamma')$ , we have that

$$F^{k-q_{n+1}, p+p_{n+1}}(\gamma') \cap \gamma' \neq \emptyset.$$

This contradicts the choice of  $k$ .

- If  $-\{q_n\alpha\} < k\alpha + p < 0$ , then we have

$$k \in \{q_{n+1}, \dots, a_{n+2}q_{n+1}\}.$$

If  $k = iq_{n+1}$  with  $i > 1$ , then we have

$$-1 < i(q_{n+1}\alpha - p_{n+1}) < (i-1)(q_{n+1}\alpha - p_{n+1}) = (k - q_{n+1})\alpha - (i-1)p_{n+1} < 0.$$

Then we must have  $p = -ip_{n+1}$ , and the endpoints of  $S^{-1}F^{k,p}(\gamma')$  are on the left hand side of those of  $\gamma'$ . Then by (4.8) and  $S^{-1}F^{k,p}(\gamma') > F^{k,p}(\gamma')$ , we have that

$$F^{k-q_{n+1}, -(i-1)p_{n+1}}(\gamma') \cap \gamma' \neq \emptyset.$$

If  $k = q_{n+1}$ , we would have

$$F^{q_{n+1}}(\gamma') \cap (\gamma' + \mathbb{Z}) = \emptyset.$$

Both cases contradict the choice of  $k$ .

Case II: Assume that there are integers  $p$  and  $0 < k < q_{n+2}$  such that  $k\alpha + p > 0$  and

$$(4.9) \quad F^{k,p}(\gamma') \cap \gamma' \neq \emptyset.$$

We assume further that for any integers  $p'$  and  $0 < k' < q_{n+2}$  such that  $k'\alpha + p' > 0$  and  $F^{k',p'}(\gamma') \cap \gamma' \neq \emptyset$ , we have

$$k'\alpha + p' \geq k\alpha + p.$$

- First, we observe that either  $q_n \geq k > 0$  or

$$\{q_n\alpha\} = q_n\alpha - p_n > k\alpha + p > 0.$$

Indeed, if  $k > q_n$  and  $q_n\alpha - p_n < k\alpha + p$ , then we have  $q_{n+2} > k - q_n > 0$  and

$$k\alpha + p > (k - q_n)\alpha + p + p_n > 0.$$

In particular, the endpoints of  $S_0^{-1}F^{k,p}(\gamma')$  are on the right hand side of those of  $\gamma'$ . Then by (4.9) and  $F^{k-q_n, p_n+p}(\gamma') = S_0^{-1}F^{k,p}(\gamma') < F^{k,p}(\gamma')$ , we have that

$$F^{k-q_n, p_n+p}(\gamma') \cap \gamma' \neq \emptyset.$$

This contradicts the choice of  $k$ .

- If  $q_n \geq k > 0$ , then by (4.6) and (4.9), we have  $q_n > k$  and consequently  $k + q_{n+1} < q_{n+2}$ . Moreover we notice that

$$k\alpha + p > (k + q_{n+1})\alpha + p - p_{n+1} > 0.$$

In particular, the endpoints of  $SF^{k,p}(\gamma')$  are on the right hand side of those of  $\gamma'$ . By (4.9) and  $SF^{k,p}(\gamma') < F^{k,p}(\gamma')$ , we have that

$$F^{k+q_n, p-p_n}(\gamma') \cap \gamma' \neq \emptyset.$$

This contradicts the choice of  $k$ .

• If  $\{q_n \alpha\} > k\alpha + p > 0$  (and  $0 < k < q_{n+2}$ ), then it is straightforward to verify that

$$(k, p) \in \{(q_n + iq_{n+1}, p_n + ip_{n+1}) \mid 0 < i < a_{n+2}\}.$$

By the hypothesis that  $\gamma_n$  is a  $a_{n+2}$ -good curve, we have

$$T^{-1}\tilde{S}^i(\tilde{\gamma}) \cap \tilde{\gamma} = \emptyset \quad \forall 1 \leq i < a_{n+2}.$$

By  $T^{-1} = JH^{-1}S_0HJ$ ,  $\tilde{S} = JH^{-1}SHJ$  and  $\gamma' = HJ(\tilde{\gamma})$ , we obtain

$$F^{q_n+iq_{n+1}, -p_n-ip_{n+1}}(\gamma') \cap \gamma' = S_0S^i(\gamma') \cap \gamma' = \emptyset$$

for all integer  $0 < i < a_{n+2}$ . This again gives a contradiction.

In summary, we see that for all integer  $0 < i < q_{n+2}$ ,

$$F^i(\gamma') \cap (\gamma' + \mathbb{Z}) = \emptyset.$$

Hence  $\hat{\gamma}$  is  $q_{n+2}$ -good. This completes the proof of Proposition 4.1.  $\square$

We can now prove the following:

**COROLLARY B.** *For each integer  $r \geq 1$ , there exist increasing functions  $P_r : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  and  $W_r : \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that the following is true. Suppose that there is an odd integer  $n \geq 3$  such that*

$$(4.10) \quad q_n > P_r(K_{r+2}, q_{n-1}), \quad q_{n+1} > P_r(K_{r+2}, q_n) \quad \text{and} \quad q_{n+2} > P_r(K_{r+2}, q_{n+1})$$

where  $\{q_k\}_{k \geq 0}$  is the sequence of denominators associated to some  $\alpha \in (0, 1) \setminus \mathbb{Q}$ . Then for any  $\{K_k\}_{k \geq 1}$ -smooth pseudo-rotation  $f \in \text{Diff}^\infty(\mathbb{A}, \omega)$  with  $\rho(f) = \alpha$  has a  $q_{n+1}$ -good smooth curve  $\gamma$  such that the closed region in  $\mathbb{A}$  bounded by  $(f^{q_n}(\gamma), \gamma)$  admits a  $(r, W_r(q_{n+1}, K_{r+2}))$ -admissible coordinate for  $f^{-q_n}$ .

*Proof.* Fix  $r \geq 1$ . Let  $\{L_k : \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+\}_{k \geq 1}$  be a sequence of increasing functions independent of  $f$  such that for each integer  $m \geq 0$ ,  $f^m$  is  $\{L_k(m, K_k)\}_{k \geq 1}$ -smooth. Let us abbreviate

$$L'_k := L_k(q_{n-1}, K_k) \quad \forall k \geq 1.$$

For each  $n \geq 2$  we have  $\rho(f^{q_{n-1}}) = \beta_{n-1} \in (0, 1/2)$ . If (4.10) holds and  $P_r$  is chosen appropriately, then

$$(4.11) \quad \beta_{n-1} \in \mathcal{L}(L'_1).$$

This allows us to apply Lemma 3.2 to  $f^{q_{n-1}}$ . By Lemma 3.2, there is a smooth Brouwer curve  $\gamma$  for  $f^{q_{n-1}}$  for which the closed region  $\mathcal{R}'_{n-1} \subset \mathbb{A}$  bounded

by  $(\gamma, f^{q_{n-1}}(\gamma))$  admits a  $(r+1, Y_{r+1})$ -admissible coordinate  $H' : [0, 1]^2 \rightarrow \mathcal{R}'_{n-1}$  where

$$(4.12) \quad Y_{r+1} := C'_{r+1}(\beta_{n-1}^{-1}, L'_{r+2}).$$

We lift  $H'$  to an admissible coordinate  $H : [0, 1]^2 \rightarrow \mathcal{R}_{n-1}$  where  $\mathcal{R}_{n-1}$  is a lift of  $\mathcal{R}'_{n-1}$  in  $\tilde{\mathbb{A}}$ . More precisely,  $\mathcal{R}_{n-1}$  is a connected component of  $\pi^{-1}(\mathcal{R}'_{n-1})$ . As in Section 4.1, we extend  $H$  to a diffeomorphism of  $\tilde{\mathbb{A}}$  satisfying

$$HT = F^{q_{n-1}, -p_{n-1}}H$$

and set  $f_n := f_H^{q_n, -p_n} : \mathbb{A} \rightarrow \mathbb{A}$ . That is,  $f_n$  is the projection of  $F_H^{q_n, -p_n} := (HJ)^{-1}F^{q_n, -p_n}(HJ) : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ . By Lemma 4.1,  $f_n$  is a pseudo-rotation in  $\text{Diff}^\infty(\mathbb{A}, \omega)$  with  $\rho(f_n) = \beta_n / \beta_{n-1} = \alpha_n$  such that

$$(4.13) \quad \|f_n\|_{\text{Diff}^{r+1}} \leq E_{r+1}(q_n, p_n, K_{r+1}, Y_{r+1}).$$

By (4.12), and using that

$$K_{r+1} \leq K_{r+2}, \quad \max(p_n, \beta_{n-1}^{-1}) \leq 2q_n$$

we can rewrite (4.13) as

$$(4.14) \quad \|f_n\|_{\text{Diff}^{r+1}} < V_{r+1}(q_n, K_{r+2})$$

where  $V_{r+1} : \mathbb{N} \times \mathbb{R}_+ \rightarrow (1, \infty)$  is an increasing function independent of  $f$ .

Now we choose  $P_r : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  to be any increasing function that is sufficiently large that the following conditions are fulfilled:

$$(4.15) \quad P_r(D, y) > 2y\rho_*(V_{r+1}(y, D))^{-1}$$

$$(4.16) \quad \frac{2y}{P_r(D, y)} < C'' \left( \frac{1}{2y}, V_{r+1}(y, D)^{-1} \right),$$

for all  $D, y > 0$ , where  $\rho_*$  is the decreasing function introduced in (3.1). We moreover choose  $P_r$  sufficiently large that whenever  $x > P_r(D, y)$  there holds

$$(4.17) \quad A_0 \left( \frac{2y}{x}, V_{r+1}(y, D)^{2y} \right) < V_{r+1}(y, D)^{-2y} C'_r(2y, V_{r+1}(y, D))^{-1}$$

for all  $D, y > 0$ . This we can arrange because  $\lim_{t \rightarrow 0} A_0(t, \cdot) = 0$ .

Assume now that (4.10) holds for a fixed odd integer  $n \geq 3$ . By (4.15) and then the second inequality in (4.10), we have

$$2q_n < \rho_*(V_{r+1}(q_n, K_{r+2}))P_r(K_{r+2}, q_n) < \rho_*(V_{r+1}(q_n, K_{r+2}))q_{n+1}$$

and therefore by (2.12)

$$(4.18) \quad \alpha_n < \frac{2q_n}{q_{n+1}} < \rho_*(V_{r+1}(q_n, K_{r+2})).$$

Also, by (2.12), the third inequality in (4.10), and (4.16), we obtain

$$\begin{aligned}
(4.19) \quad \mathcal{G}(\alpha_n) = \alpha_{n+1} &< \frac{2q_{n+1}}{q_{n+2}} < \frac{2q_{n+1}}{P_r(K_{r+2}, q_{n+1})} \\
&< C'' \left( \frac{1}{2q_{n+1}}, V_{r+1}(q_{n+1}, K_{r+2})^{-1} \right) \\
(4.20) \quad &< C''(\alpha_n, V_{r+1}(q_n, K_{r+2})^{-1})
\end{aligned}$$

where the last inequality uses the monotonicity of  $C''$ . We claim that

$$(4.21) \quad \rho(f_n) = \alpha_n \in \mathcal{L}_r(V_{r+1}(q_n, K_{r+2}))$$

where  $\mathcal{L}_r$  is as in Definition 3.2. Equivalently, we show that

$$(4.22) \quad A_0 \left( \alpha_n \mathcal{G}(\alpha_n), V_{r+1}(q_n, K_{r+2})^{q(\alpha_n)} \right) < V_{r+1}(q_n, K_{r+2})^{-q(\alpha_n)} C'_r \left( \alpha_n^{-1}, V_{r+1}(q_n, K_{r+2}) \right)^{-1}.$$

Using (2.12) we have  $\alpha_n \mathcal{G}(\alpha_n) = \alpha_n \alpha_{n+1} < 4q_n/q_{n+2} \leq 2q_{n+1}/q_{n+2}$  and  $q(\alpha_n) \leq 2q_{n+1}$  and  $\alpha_n^{-1} < 2q_{n+1}/q_n \leq 2q_{n+1}$ . Therefore by the monotonicity of  $A_0$ , and by (4.17) together with the third inequality in (4.10), we obtain

$$\begin{aligned}
A_0 \left( \alpha_n \mathcal{G}(\alpha_n), V_{r+1}(q_n, K_{r+2})^{q(\alpha_n)} \right) &< A_0 \left( \frac{2q_{n+1}}{q_{n+2}}, V_{r+1}(q_{n+1}, K_{r+2})^{2q_{n+1}} \right) \\
&< V_{r+1}(q_{n+1}, K_{r+2})^{-2q_{n+1}} C'_r(2q_{n+1}, V_{r+1}(q_{n+1}, K_{r+2}))^{-1}.
\end{aligned}$$

Now inequality (4.22) follows from the monotonicity of  $V_{r+1}$  and  $C'_r$ , and because  $\alpha_n^{-1} < 2q_{n+1}/q_n \leq 2q_{n+1}$  from (2.12). This proves (4.21).

Combining (4.21) with (4.14) we see that  $f_n$  satisfies the hypotheses of Proposition 3.1, and therefore  $f_n$  has a  $q(\rho(f_n))$ -good Brouwer curve  $\gamma_n$  say. By (2.8)  $q(\rho(f_n)) = q(\alpha_n) = a_{n+1}$ . Thus  $\gamma_n$  is an  $a_{n+1}$ -good Brouwer curve for  $f_n$ . Moreover, by Lemma 3.2, the region in  $\mathbb{A}$  bounded by  $(\gamma_n, f_n(\gamma_n))$  admits an  $(r, C'_r(\alpha_n^{-1}, \hat{K}_{r+1}))$ -admissible coordinate, provided  $\hat{K}_{r+1} \geq \|f_n\|_{\text{Diff}^{r+1}}$  and  $\rho(f_n) = \alpha_n \in \mathcal{L}(\hat{K})$  where  $\hat{K} \geq \|f_n\|_{\text{Diff}^1}$ . By (4.21) and (4.14) we may take  $\hat{K}_{r+1} = \hat{K} = V_{r+1}(q_n, K_{r+2})$  and conclude that the region in  $\mathbb{A}$  bounded by  $(\gamma_n, f_n(\gamma_n))$  admits a  $(r, U_r)$ -admissible coordinate where

$$U_r := C'_r(\alpha_n^{-1}, V_{r+1}(q_n, K_{r+2})).$$

We define  $W_r$  by

$$W_r(q, K) := G_r(C'_r(2q, L_{r+1}(q, K)), C'_r(2q, V_{r+1}(q, K))).$$

By (2.11) and (2.12), we have  $\beta_{n-1}^{-1}, \alpha_n^{-1} < 2q_{n+1}$ . Then by  $K_{r+2} \geq K_{r+1}$ , we have

$$\begin{aligned}
&C'_r(2q_{n+1}, L_{r+1}(q_{n+1}, K_{r+2})) > Y_r \\
\text{and } &C'_r(2q_{n+1}, V_{r+1}(q_{n+1}, K_{r+2})) > U_r.
\end{aligned}$$

Thus we have

$$W_r(q_{n+1}, K_{r+2}) \geq G_r(Y_r, U_r).$$

Then by Proposition 4.1, there is a  $q_{n+1}$ -good smooth curve of  $f$ , denoted by  $\hat{\gamma}$ , such that the region in  $\mathbb{A}$  bounded by  $(f^{q_n}(\hat{\gamma}), \hat{\gamma})$  admits a  $(r, W_r(q_{n+1}, K_{r+2}))$ -admissible coordinate for  $f^{-q_n}$ .  $\square$

## 5. CONSTRUCTION OF APPROXIMANTS

This section is mostly occupied by the proof of the following theorem, from which the main result of this paper, Theorem 1, will then follow easily.

**THEOREM 4.** *For each  $(r_0, M, \epsilon) \in \mathbb{Z}_{\geq 2} \times \mathbb{N} \times (0, 1]$ , there is an increasing function*

$$P = P_{r_0, \epsilon, M} : \mathbb{N} \rightarrow \mathbb{R}_+$$

so that for any pseudo-rotation  $f \in \text{Diff}^\infty(\mathbb{A}, \omega)$  with

$$(5.1) \quad \|f\|_{\text{Diff}^{r_0+2}(\mathbb{A})} < M,$$

whose rotation number  $\rho(f) = \alpha \in (0, 1) \setminus \mathbb{Q}$  satisfies the property that there exists an odd integer  $n \geq 3$  for which

$$(5.2) \quad q_n > P(q_{n-1}), \quad q_{n+1} > P(q_n) \quad \text{and} \quad q_{n+2} > P(q_{n+1}),$$

then there exists  $h_0 \in \text{Diff}^\infty(\mathbb{A}, \omega)$  with

$$(5.3) \quad d_{\text{Diff}^{r_0-1}(\mathbb{A})}(h_0 R_\alpha h_0^{-1}, f) < \epsilon.$$

*Proof.* Fix some  $(r_0, M, \epsilon)$  in  $\mathbb{Z}_{\geq 2} \times \mathbb{N} \times (0, 1]$ . Let  $f \in \text{Diff}^\infty(\mathbb{A}, \omega)$  denote a pseudo-rotation and set  $\alpha = \rho(f)$ .

It will be convenient to use the following notation: for an increasing function  $S : \mathbb{N} \rightarrow \mathbb{N}$  we define

$$(5.4) \quad \mathcal{C}(S) := \left\{ \theta \in (0, 1) \setminus \mathbb{Q} \mid \exists n \in \mathbb{N} \text{ odd, so that} \right. \\ \left. q_n(\theta) > S(q_{n-1}(\theta)), q_{n+1}(\theta) > S(q_n(\theta)), q_{n+2}(\theta) > S(q_{n+1}(\theta)) \right\}$$

where  $\{p_n(\theta)/q_n(\theta)\}_{n \geq 0}$  is the sequence of continued fractions of  $\theta$  introduced in Section 2. Our successive restrictions on the rotation number  $\alpha$  will take the form:

$$(5.5) \quad \alpha \in \mathcal{C}(S_i)$$

for a finite collection of functions  $S_1, S_2, \dots$  to be determined. We will set

$$P := \max_i S_i$$

and then when  $\alpha \in \mathcal{C}(P)$ , all conditions in (5.5) will be met.

For our first condition on the rotation number, set

$$(5.6) \quad S_1 := P_{r_0}(M, \cdot)$$

where  $P_{r_0} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  is defined in Corollary B. From now on we assume

$$\alpha \in \mathcal{C}(S_1).$$

Fix any odd  $n \geq 3$  for which

$$(5.7) \quad q_n > S_1(q_{n-1}), \quad q_{n+1} > S_1(q_n) \quad \text{and} \quad q_{n+2} > S_1(q_{n+1}).$$

Then by Corollary B,  $f$  has a  $q_{n+1}$ -good Brouwer curve  $\gamma \subset \mathbb{A}$  for which the closed region

$$\mathcal{D}_0 \subset \mathbb{A}$$

bounded by  $(f^{q_n}(\gamma), \gamma)$  admits a  $(r_0, \hat{W}_{r_0})$ -admissible coordinate for  $f^{-q_n}$ , where

$$\hat{W}_{r_0} := W_{r_0}(q_{n+1}, K_{r_0+2}).$$

Recall that  $\gamma$  is  $q_{n+1}$ -good for  $f$  means that the curves  $f^i(\gamma)$ ,  $0 \leq i \leq q_{n+1} - 1$  are mutually disjoint. Then it is clear that the curves

$$(5.8) \quad f^{q_n}(\gamma), \dots, f^{q_n+q_{n+1}-1}(\gamma)$$

are mutually disjoint as well. In particular  $f^{q_{n+1}}(\gamma)$  and  $f^{q_n}(\gamma)$  are disjoint and so we can consider the closed region

$$\mathcal{D} \subset \mathbb{A}$$

bounded by  $(f^{q_n}(\gamma), f^{q_{n+1}}(\gamma))$ .

CLAIM 1. *The annulus  $\mathbb{A}$  is covered by the regions*

$$\mathcal{D}, f(\mathcal{D}), \dots, f^{q_n-1}(\mathcal{D}), f^{q_n}(\mathcal{D}_0), \dots, f^{q_{n+1}-1}(\mathcal{D}_0)$$

*with mutually disjoint interiors. Moreover, the intersection of any two such neighboring regions equals one of the curves in (5.8).*

*Proof.* The restriction of  $f$  to the boundary circle  $B_0$  is an orientation preserving homeomorphism with rotation number  $\alpha$ . It is then a well-known fact that for any fixed  $x_0 \in B_0$  the intervals

$$\begin{aligned} & [f^{q_n+i}(x_0), f^{q_{n+1}+i}(x_0)], \quad 0 \leq i \leq q_n - 1, \\ \text{and} \quad & [f^{q_n+j}(x_0), f^j(x_0)], \quad q_n \leq j \leq q_{n+1} - 1 \end{aligned}$$

together form a covering for  $B_0$  with mutually disjoint interiors. In particular we may take  $x_0 := \gamma \cap B_0$ . An analogous statement holds for the other boundary component  $B_1$ . The claim then follows from the disjointness of the curves in (5.8) and our definition of simple regular curve.  $\square$

Fix a lift  $\tilde{\gamma}$  of  $\gamma$  to  $\tilde{\mathbb{A}}$ . Let  $F : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$  be the unique lift of  $f$  satisfying  $\rho(F) = \rho(f) \in (0, 1)$ . Recall the notation from section 4.1 that  $F^{a,b} = T^b F^a$  for all  $a, b \in \mathbb{Z}$ . We define  $\tilde{\mathcal{D}}_0 \subset \tilde{\mathbb{A}}$  to be the region bounded by

$$(F^{q_n, -p_n}(\tilde{\gamma}), \tilde{\gamma})$$

and  $\tilde{\mathcal{D}} \subset \tilde{\mathbb{A}}$  to be the region bounded by

$$(F^{q_n, -p_n}(\tilde{\gamma}), F^{q_{n+1}, -p_{n+1}}(\tilde{\gamma})).$$

We claim that  $T^{-p_n} F^{q_n}(\tilde{\gamma})$  is contained in the fundamental domain of  $\pi$  bounded by  $T^{-1}\tilde{\gamma}$  and  $\tilde{\gamma}$ . Indeed, this follows from comparing the order of



their endpoints, and by the disjointness of their projections  $f^{q_n}(\gamma)$  and  $\gamma$ . Thus  $\pi$  restricts to a diffeomorphism from  $\tilde{\mathcal{D}}_0$  to  $\mathcal{D}_0$ . Similarly,  $\pi$  restricts to a diffeomorphism from  $\tilde{\mathcal{D}}$  to  $\mathcal{D}$  as well.

The following  $q_{n+1}$  regions have disjoint interiors:

$$(5.9) \quad \tilde{\mathcal{B}}_i := \begin{cases} F^i(\tilde{\mathcal{D}}) & 0 \leq i \leq q_n - 1, \\ F^i(\tilde{\mathcal{D}}_0) & q_n \leq i \leq q_{n+1} - 1. \end{cases}$$

By Claim 1,  $\tilde{\mathbb{A}}$  is covered by the union of the following regions with mutually disjoint interiors:

$$(5.10) \quad T^{-k}\tilde{\mathcal{B}}_j, \quad k \in \mathbb{Z}, \quad 0 \leq j \leq q_{n+1} - 1.$$

The intersection of any two neighboring regions in (5.10) therefore equals to one of the following curves

$$(5.11) \quad T^{-k}F^j(\tilde{\gamma}), \quad k \in \mathbb{Z}, \quad q_n \leq j \leq q_n + q_{n+1} - 1.$$

We denote by  $\Gamma$  the union of the curves in (5.11). The region

$$(5.12) \quad U_0 := \bigcup_{i=0}^{q_{n+1}-1} \text{int}(\tilde{\mathcal{B}}_i)$$

satisfies that:

- (1)  $T^j(U_0) \cap T^k(U_0) = \emptyset$  for any  $j \neq k \in \mathbb{Z}$ ;
- (2)  $\cup_{k \in \mathbb{Z}} T^k(U_0) = \tilde{\mathbb{A}} \setminus \Gamma$ .

By Corollary B, there is an admissible coordinate  $\phi$  (see Definition 2.7) from a neighborhood of  $[0, 1]^2$  in  $\tilde{\mathbb{A}}$  to a neighborhood of  $\mathcal{D}_0$ , satisfying (2.2)-(2.5) for  $(f^{-q_n}, f^{q_n}(\gamma))$  in place of  $(f, \gamma)$ , and

$$\begin{aligned} \|\phi\|_{\text{Diff}^0} &< \hat{W}_{r_0}, \\ \phi \circ T(x) &= f^{-q_n} \circ \phi(x) \quad \forall x \in V \end{aligned}$$

where  $V$  is some neighborhood of  $\{0\} \times [0, 1]$  in  $\tilde{\mathbb{A}}$ . As we have seen in Remark 2, there is no loss of information by regarding  $\phi$  as a mapping from  $[0, 1]^2$  to  $\mathcal{D}_0$ . Since the restriction  $\pi : \tilde{\mathcal{D}}_0 \rightarrow \mathcal{D}_0$  is a diffeomorphism we may set

$$\psi := \pi^{-1} \circ \phi$$

which therefore gives us a  $C^\infty$ -smooth diffeomorphism that extends to a  $C^\infty$ -smooth diffeomorphism with constant Jacobian from a neighborhood of  $[0, 1]^2$  in  $\tilde{\mathbb{A}}$  to a neighborhood of  $\tilde{\mathcal{D}}_0$  satisfying

$$(5.13) \quad \|\psi\|_{\text{Diff}^0} < \hat{W}_{r_0},$$

$$(5.14) \quad \psi \circ T(x) = F^{-q_n, p_n} \circ \psi(x) \quad \forall x \in V.$$

There is a unique extension to a  $C^\infty$ -diffeomorphism with constant Jacobian to the whole strip  $\tilde{\mathbb{A}}$ , which we still denote by

$$\psi : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$$

and which satisfies

$$(5.15) \quad \psi \circ T = F^{-q_n p_n} \circ \psi.$$

To summarise thus far, we have proven the following:

LEMMA 5.1. *For each  $(r_0, M) \in \mathbb{Z}_{\geq 2} \times \mathbb{N}$  there is an increasing function*

$$S_1 = S_1(r_0, M) : \mathbb{N} \rightarrow \mathbb{R}_+$$

*so that for any pseudo-rotation  $f \in \text{Diff}^\infty(\mathbb{A}, \omega)$  satisfying (5.1) whose rotation number  $\rho(f) = \alpha$  satisfies the property that there exists an odd integer  $n \geq 3$  for which*

$$q_n > S_1(q_{n-1}), \quad q_{n+1} > S_1(q_n) \quad \text{and} \quad q_{n+2} > S_1(q_{n+1}),$$

*there exists a  $q_{n+1}$ -good Brouwer curve  $\gamma \subset \mathbb{A}$ , so that for any lift  $\tilde{\gamma} \in \tilde{\mathbb{A}}$ , there exists a  $C^\infty$ -diffeomorphism,*

$$\psi : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$$

*with constant Jacobian which satisfies*

$$(5.16) \quad \|\psi\|_{\text{Diff}^0([0,1]^2)} < \hat{W}_{r_0} := W_{r_0}(q_{n+1}, M)$$

*where  $W_{r_0}$  is the function produced by Corollary B, and*

$$(5.17) \quad \psi \circ T = F^{-q_n p_n} \circ \psi$$

*on the whole of  $\tilde{\mathbb{A}}$ , where  $F$  is the unique lift of  $f$  with rotation number in  $(0, 1)$ . Moreover  $\psi$  maps  $[0, 1]^2$  onto the region  $\tilde{\mathcal{D}}_0$  bounded by  $(\tilde{F}^{q_n p_n}(\tilde{\gamma}), \tilde{\gamma})$ .*

To continue our proof of Theorem 4 we require the following:

LEMMA 5.2. *For each  $(r_0, M) \in \mathbb{Z}_{\geq 2} \times \mathbb{N}$ , there is an increasing function*

$$Q_{r_0, M} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \quad \text{with} \quad \lim_{t \rightarrow 0} Q_{r_0, M}(t, \cdot) \equiv 0,$$

*and there is an increasing function*

$$S_2 = S_2(r_0, M) : \mathbb{N} \rightarrow \mathbb{R}_+$$

*with  $S_2 \geq S_1$ , so that for any pseudo-rotation  $f \in \text{Diff}^\infty(\mathbb{A}, \omega)$  satisfying (5.1) whose rotation number  $\rho(f) = \alpha$  satisfies the property that there exists an odd integer  $n \geq 3$  for which*

$$q_n > S_1(q_{n-1}), \quad q_{n+1} > S_1(q_n) \quad \text{and} \quad q_{n+2} > S_1(q_{n+1}),$$

*then the following holds for the unique lift  $F : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$  of  $f$  with  $\rho(F) \in (0, 1)$ : let  $\gamma$  be a  $q_{n+1}$ -good Brouwer curve for  $f$  given by Lemma 5.1<sup>3</sup>, and let  $\tilde{\gamma}$  be any lift of  $\gamma$ . Let  $\tilde{\mathcal{D}}$  and  $\tilde{\mathcal{D}}_0$  be the regions in  $\tilde{\mathbb{A}}$  bounded by*

$$(F^{q_n p_n}(\tilde{\gamma}), F^{q_{n+1} p_{n+1}}(\tilde{\gamma})), \quad (F^{q_n p_n}(\tilde{\gamma}), \tilde{\gamma})$$

<sup>3</sup>Lemma 5.1 is applicable since by hypothesis  $S_2 \geq S_1$ .

respectively. Then there are open neighborhoods  $\hat{\mathcal{D}}$  of  $\tilde{\mathcal{D}}$  and  $\hat{\mathcal{D}}_0$  of  $\tilde{\mathcal{D}}_0$ ; a  $C^\infty$ -diffeomorphism

$$h : \hat{\mathcal{D}} \rightarrow \hat{\mathcal{D}}_0;$$

and neighborhoods  $U_L$  of  $F^{q_n, -p_n}(\tilde{\gamma})$  and  $U_R$  of  $F^{q_{n+1}, -p_{n+1}}(\tilde{\gamma})$ , so that

$$(5.18) \quad h|_{U_L} = \text{Id}, \quad h|_{U_R} = F^{-q_{n+1}, p_{n+1}},$$

$$(5.19) \quad \|h - \text{Id}\|_{\text{Diff}^{r_0}(\hat{\mathcal{D}}, \hat{\mathcal{D}}_0)} < Q_{r_0, M}(\beta_{n+1}, q_{n+1}).$$

*Proof.* By Lemma 5.1 there is a diffeomorphism  $\psi : \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}$  satisfying (5.16) and which maps  $[0, 1]^2$  onto  $\tilde{\mathcal{D}}_0$ . Set

$$\gamma' := F^{q_{n+1}, -p_{n+1}}(\tilde{\gamma}).$$

By letting  $q_{n+2}$  be sufficiently large depending on  $q_{n+1}$  and  $M$ , we have

$$A_0(\beta_{n+1}, M^{q_{n+1}}) < 1/2$$

where  $A_0$  is produced by Theorem 3. Then

$$\|f^{q_{n+1}} - \text{Id}\| \leq A_0(\|\rho(f^{q_{n+1}})\|_{\mathbb{R}/\mathbb{Z}}, M^{q_{n+1}}) = A_0(\beta_{n+1}, M^{q_{n+1}}) < 1/2.$$

Using Theorem 3, Lemma 2.1 and (2.11), we obtain

$$(5.20) \quad d_H(\tilde{\gamma}, \gamma') = d_H(f^{q_{n+1}}(\tilde{\gamma}), \tilde{\gamma}) < A_0(\beta_{n+1}, M^{q_{n+1}}).$$

Notice that by (5.16) and (5.17) we also have that  $\|\psi\|_{\text{Diff}^{r_0}([0, 2] \times [0, 1])}$  is bounded in terms of  $q_{n+1}, M, r_0$ . Then by (5.20) and by letting  $\beta_{n+1}$  be sufficiently large depending on  $q_{n+1}, M, r_0$ , we have

$$\gamma' \subset \psi([1/2, 3/2] \times [0, 1]).$$

We define

$$\gamma'' = \psi^{-1}(\gamma').$$

Then

$$\{1/2\} \times [0, 1] \subset \gamma'' \subset \{3/2\} \times [0, 1]$$

if  $\beta_{n+1}$  is sufficiently large depending on  $q_{n+1}, M, r_0$ . Denote by  $\mathcal{U}'$  the region in  $\tilde{\mathcal{A}}$  bounded by  $\{0\} \times [0, 1]$  and  $\gamma''$ . It remains to construct a  $C^\infty$ -diffeomorphism

$$\varphi : \mathcal{U}' \rightarrow [0, 1]^2$$

such that  $\varphi$  equals  $\text{Id}$  near  $\{0\} \times [0, 1]$ ; equals  $\psi^{-1}F^{-q_{n+1}, p_{n+1}}\psi$  near  $\gamma''$ ; and tends to  $\text{Id}$  in the  $C^{r_0}$ -topology as  $\beta_{n+1}$  tends to 0 for each fixed  $q_{n+1}, M, r_0$ . Indeed, after the above  $\varphi$  is constructed, we can define  $h = \varphi\psi\psi^{-1}$ .

By Theorem 3, for given  $q_{n+1}, M, r_0$ , we see that  $F^{-q_{n+1}, p_{n+1}}$  tends to  $\text{Id}$  in the  $C^{r_0}$ -topology as  $q_{n+2}$  tends to infinity. Thus it suffices to construct  $\varphi$  by smooth interpolation.  $\square$

Continuing the proof of Theorem 4, we first construct a periodic approximation of  $f$  as follows. We define  $G$  on  $U_0$  (see (5.12)) by

$$(5.21) \quad G|_U = \begin{cases} F & U = \cup_{i=0}^{q_n-2} \text{int}(\tilde{\mathcal{B}}_i), \\ F^{q_n} h F^{1-q_n} & U = \text{int}(\tilde{\mathcal{B}}_{q_n-1}), \\ F & U = \cup_{i=q_n}^{q_{n+1}-2} \text{int}(\tilde{\mathcal{B}}_i), \\ T^{p_{n+1}} h^{-1} F^{1-q_{n+1}} & U = \text{int}(\tilde{\mathcal{B}}_{q_{n+1}-1}). \end{cases}$$

By property (1),(2) below (5.12), we may extend  $G$  to a self-map of  $\tilde{\mathbb{A}} \setminus \Gamma$ , still denoted by  $G$ , satisfying

$$(5.22) \quad GT = TG.$$

Moreover, by (5.21) the following identity holds on  $\tilde{\mathbb{A}} \setminus \Gamma$ :

$$(5.23) \quad G^{q_{n+1}} = T^{p_{n+1}}.$$

By construction, for any two curves  $\gamma_a, \gamma_b$  in (5.11), there are integers  $i, j$  such that

$$(5.24) \quad \gamma_b = T^i G^j(\gamma_a).$$

By (5.21) in the definition of  $G$ , we also have that

$$(5.25) \quad \tilde{\mathcal{B}}_i = G^i(\tilde{\mathcal{B}}_0), \quad 0 \leq i \leq q_{n+1} - 1.$$

We denote

$$(5.26) \quad \tilde{\mathcal{B}}_{q_{n+1}} := G^{q_{n+1}}(\tilde{\mathcal{B}}_0) = T^{p_{n+1}}(\tilde{\mathcal{B}}_0).$$

Now we set

$$\phi := \psi^{-1}h$$

where  $\psi$  is in Lemma 5.1 and  $h$  is in Lemma 5.2.

LEMMA 5.3. *There is a neighborhood  $V' \subset \tilde{\mathbb{A}}$  of  $F^{q_{n+1}, -p_{n+1}}(\tilde{\gamma})$  so that  $G^{q_n}$  extends to a  $C^\infty$  map on  $V'$ , and*

$$(5.27) \quad T\phi(x) = \phi T^{-p_n} G^{q_n}(x), \quad x \in V'.$$

*Proof.* To see this, we first notice that

$$F^{q_{n+1}, -p_{n+1}}(\tilde{\gamma}) = \tilde{\mathcal{B}}_0 \cap T^{p_n - p_{n+1}}(\tilde{\mathcal{B}}_{q_{n+1} - q_n}),$$

and  $\tilde{\mathcal{B}}_0 \cup T^{p_n - p_{n+1}}(\tilde{\mathcal{B}}_{q_{n+1} - q_n})$  is a neighborhood of  $F^{q_{n+1}, -p_{n+1}}(\tilde{\gamma})$  in  $\tilde{\mathbb{A}}$ . Let  $U_L, U_R$  be given by Lemma 5.2. The set  $V'$  defined by

$$V' = F^{q_{n+1} - q_n, -p_{n+1} + p_n}(U_L) \cap U_R \cap (\tilde{\mathcal{B}}_0 \cup T^{p_n - p_{n+1}}(\tilde{\mathcal{B}}_{q_{n+1} - q_n}))$$

is a neighborhood of  $F^{q_{n+1}, -p_{n+1}}(\tilde{\gamma})$ .

By (5.21), (5.25) and (5.22), we have

$$(5.28) \quad G^{q_n}|_{\text{Int}(\tilde{\mathcal{B}}_0)} = F^{q_n}h,$$

$$(5.29) \quad G^{q_n}|_{T^{p_n - p_{n+1}}(\text{Int}(\tilde{\mathcal{B}}_{q_{n+1} - q_n}))} = T^{p_n}h^{-1}F^{q_n - q_{n+1}}T^{p_{n+1} - p_n}.$$

By Lemma 5.2, we see that

$$\text{RHS of (5.28)}|_{V'} = \text{RHS of (5.29)}|_{V'} = T^{p_{n+1}}F^{q_n - q_{n+1}}.$$

Then it is clear that  $G^{q_n}$  extends to a  $C^\infty$  map on  $V'$ . Again by Lemma 5.2, we have for any  $x \in V'$

$$\begin{aligned} \phi T^{-p_n} G^{q_n}(x) &= \psi^{-1} h T^{p_{n+1} - p_n} F^{q_n - q_{n+1}}(x) = \psi^{-1} T^{p_{n+1} - p_n} F^{q_n - q_{n+1}}(x), \\ T\phi(x) &= T\psi^{-1} h(x) = T\psi^{-1} T^{-p_{n+1}} F^{q_{n+1}}(x). \end{aligned}$$

Thus (5.27) follows from (5.17).  $\square$

We have the following corollary.

COROLLARY C. *The map  $G$  on  $\tilde{\mathbb{A}} \setminus \Gamma$  extends to an element in  $\text{Diff}^\infty(\tilde{\mathbb{A}})$ .*

*Proof.* By (5.23), clearly  $G^{q_{n+1}}$  extends to a smooth map  $T^{p_{n+1}}$  on  $\tilde{\mathbb{A}}$ . By Lemma 5.3, (5.22) and (5.24), we see that  $G^{q_n}$  also extends to a map in  $\text{Diff}^\infty(\tilde{\mathbb{A}})$ . To conclude the proof it suffices to notice that

$$G = (G^{q_n})^{-p_{n+1}} (G^{q_{n+1}})^{p_n}.$$

$\square$

By a slight abuse of notation, we again write  $G \in \text{Diff}^\infty(\tilde{\mathbb{A}})$  for the extension provided in Corollary C. Clearly (5.22) continues to hold. Consequently  $G$  descends to a periodic diffeomorphism  $g \in \text{Diff}^\infty(\mathbb{A})$  satisfying

$$g^{q_{n+1}} = \text{Id}.$$

We define map  $\tilde{H} : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$  by

$$(5.30) \quad \tilde{H}(x) := T^{-kq_{n+1} + jp_{n+1}} \phi G^{-j} T^k(x), \quad x \in T^{-k} \tilde{\mathcal{B}}_j$$

for all  $k \in \mathbb{Z}$  and all  $0 \leq j \leq q_{n+1} - 1$ . Then by (5.23), (5.25), (5.26) and (5.30) we have

$$(5.31) \quad \tilde{H}T = T^{q_{n+1}}\tilde{H}, \quad \tilde{H}G = T^{p_{n+1}}\tilde{H}.$$

We set  $H_1 := D_{q_{n+1}}\tilde{H}$ , where

$$D_\lambda(x, y) = (\lambda^{-1}x, y), \quad \lambda \in \mathbb{R} \setminus \{0\}.$$

Notice that for every  $\lambda \in \mathbb{R} \setminus \{0\}$  and every  $p \in \mathbb{R}$  we have

$$D_\lambda T_p D_\lambda^{-1} = T_{\frac{p}{\lambda}}$$

where for each  $c \in \mathbb{R}$ ,  $T_c : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$  denotes the map  $T_c(x, y) = (x + c, y)$ . Then by (5.31), we see that  $H_1 : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$  satisfies

$$(5.32) \quad TH_1 = H_1T, \quad H_1G = T_{p_{n+1}/q_{n+1}}H_1.$$

We now show that  $\tilde{H} \in \text{Diff}^\infty(\tilde{\mathbb{A}})$ . By construction,  $\tilde{H}$  is  $C^\infty$ -smooth in the interior of each  $T^{-k}\tilde{\mathcal{B}}_j$ ,  $k \in \mathbb{Z}$ ,  $0 \leq j \leq q_{n+1} - 1$ . Thus it remains to show that  $\tilde{H}$  is  $C^\infty$  in a neighborhood of each of the curves in (5.11).

By (5.9), we can see that the set

$$(5.33) \quad \tilde{\mathcal{B}}_* := \tilde{\mathcal{B}}_0 \cup T^{-p_n} \tilde{\mathcal{B}}_{q_n}$$

is a neighborhood of  $T^{-p_n} F^{q_n}(\tilde{\gamma})$ . By (5.24), we deduce that every curve in (5.11) has a neighborhood of the form  $T^i G^j(\tilde{\mathcal{B}}_*)$ . By (5.31), it remains to verify that  $\tilde{H}$  is  $C^\infty$  over  $\tilde{\mathcal{B}}_*$ . This follows from a similar argument as in the proof of Lemma 5.3 using (5.27) and (5.30). Consequently  $\tilde{H}$  and  $H_1$  are  $C^\infty$ -smooth diffeomorphisms of  $\tilde{\mathbb{A}}$ . Then by (5.32)  $H_1$  descends to a  $C^\infty$ -diffeomorphism  $h_1 : \mathbb{A} \rightarrow \mathbb{A}$  satisfying

$$(5.34) \quad g = h_1^{-1} R_{p_{n+1}/q_{n+1}} h_1.$$

Moreover, we have the claim that:

$\|H_1\|_{\text{Diff}^0(\tilde{\mathbb{A}})}$ , and hence  $\|h_1\|_{\text{Diff}^0(\mathbb{A})}$  as well, are bounded in terms of  $q_{n+1}, M, r_0$ .

Indeed, by definition it is clear that  $\|\phi\|_{\text{Diff}^0}$  is bounded in terms of  $q_{n+1}, M, r_0$ ; and for each integer  $0 \leq j \leq q_{n+1} - 1$ ,  $\|G\|_{\text{Diff}^0(\tilde{\mathcal{B}}_j, \tilde{\mathcal{B}}_{j+1})}$  is bounded in terms of  $q_{n+1}, M, r_0$ . Our claim follows immediately from (5.30) and the fact that  $H_1$  commutes with  $T$ .

Recall that  $r_0, M$  and  $\epsilon$  are fixed. In the following we will show that:

LEMMA 5.4. *For sufficiently fast growing  $P$  depending on  $r_0, M$  and  $\epsilon$  the following holds. Assume  $n$  satisfies the conditions in Lemma 5.1 and Lemma 5.2; and moreover (5.2) holds for  $n$ . Then we have:*

- (1)  $d_{\text{Diff}^0(\mathbb{A})}(f, g) < \epsilon/4$ ;
- (2)  $d_{\text{Diff}^0(\mathbb{A})}(h_1^{-1} R_\alpha h_1, g) < \epsilon/4$ ;
- (3) there is  $h_2 \in \text{Diff}^\infty(\mathbb{A})$  such that  $(h_2 h_1)_* \omega = \omega$  and

$$d_{\text{Diff}^{r_0-1}(\mathbb{A})}(h_1^{-1} h_2^{-1} R_\alpha h_2 h_1, h_1^{-1} R_\alpha h_1) < \epsilon/4.$$

*Proof.* *Proof of (1):* By (5.21), there is  $D > 0$  depending only on  $r_0$  such that

$$d_{\text{Diff}^0(\mathbb{A})}(f, g) \leq D \max(d_{\text{Diff}^0(\mathbb{A})}(f^{q_n} h f^{-q_n}, \text{Id}), d_{\text{Diff}^0(\mathbb{A})}(h^{-1} f^{-q_{n+1}}, \text{Id})).$$

By Theorem 3, Lemma 5.2, and by letting  $q_{n+2}$  be sufficiently large depending on  $\epsilon, r_0, M, q_{n+1}$ , we have

$$d_{\text{Diff}^0(\mathbb{A})}(f^{q_n} h f^{-q_n}, \text{Id}), \quad d_{\text{Diff}^0(\mathbb{A})}(h^{-1} f^{-q_{n+1}}, \text{Id}) < (4D)^{-1} \epsilon.$$

Thus we have

$$d_{\text{Diff}^0(\mathbb{A})}(f, g) < \epsilon/4.$$

*Proof of (2):* Since  $\|h_1\|_{\text{Diff}^0(\mathbb{A})}$  admits an upper bound depending only on  $q_{n+1}, M$  and  $r_0$ , by letting  $q_{n+2}$  be sufficiently large depending on  $\epsilon, r_0, M, q_{n+1}$ , we have

$$d_{\text{Diff}^0(\mathbb{A})}(h_1^{-1} R_\alpha h_1, g) < \epsilon/4.$$

*Proof of (3):* Set

$$\lambda_0 = \det(H_1) = \frac{1}{q_{n+1}} \det(\tilde{H}) \in C^\infty(\tilde{\mathbb{A}}).$$

Recall that both  $\psi$  and  $F$  have constant Jacobians, hence by (5.30), on each  $\tilde{\mathcal{B}}_j$  we have

$$\log \lambda_0 = \log \det h \circ (G^{-j}) + \log \det(G^{-j}F^j) \circ F^j + c_1$$

for some constant  $c_1$ . On  $T^{-k}\tilde{\mathcal{B}}_j$  we have a similar formula. Thus

$$\sup_{1 \leq r \leq r_0} \|D^r \log \lambda_0\| \leq (1 + d_{\text{Diff}^{r_0}(\mathbb{A})}(f, \text{Id}))^{Cq_{n+1}r_0} (d_{\text{Diff}^{r_0}(\mathbb{A})}(h, \text{Id}) + d_{\text{Diff}^{r_0}(\mathbb{A})}(g, f)).$$

By Lemma 5.2 and the argument above for bounding  $d_{\text{Diff}^{r_0}(\mathbb{A})}(g, f)$ , we see that  $\sup_{1 \leq r \leq r_0} \|D^r \log \lambda_0\|$  can be made arbitrarily small by making  $q_{n+2}$  sufficiently large while keeping  $q_{n+1}$ ,  $M$  and  $r_0$  fixed. By (5.32), we have

$$\int_{[0,1]^2} \lambda_0 d\omega = 1.$$

Then it is direct to see, for some absolute constant  $C > 0$ , that

$$\|\lambda_0 - 1\| < C \|D \log \lambda_0\|$$

given that  $\|D \log \lambda_0\| < 1$ . Consequently, for any  $\delta > 0$ , we have  $\|\lambda_0 - 1\|_{C^{r_0}} < \delta$  if  $q_{n+2}$  is sufficiently large depending on  $\delta$ ,  $q_{n+1}$ ,  $M$  and  $r_0$ .

By Dacorogna-Moser's theorem ([DM90, Theorem 1]), there exists  $h_2 \in \text{Diff}^\infty(\mathbb{A})$  such that

$$(h_2)_*(\lambda_0 \omega) = \omega.$$

Moreover, by [DM90, Theorem 2 and Lemma 3], we can choose  $h_2$  with  $d_{\text{Diff}^{r_0-1}(\mathbb{A})}(h_2, \text{Id})$  arbitrarily small provided  $\|\lambda_0 - 1\|_{C^{r_0}}$  is sufficiently small.

In summary, if  $q_{n+2}$  is sufficiently large depending on  $\epsilon$ ,  $q_{n+1}$ ,  $M$  and  $r_0$ , we can choose  $h_2 \in \text{Diff}^\infty(\mathbb{A})$  sufficiently close to  $\text{Id}$  in  $\text{Diff}^{r_0}(\mathbb{A})$  so that

$$d_{\text{Diff}^{r_0-1}(\mathbb{A})}(h_1^{-1}h_2^{-1}R_\alpha h_2 h_1, h_1^{-1}R_\alpha h_1) < \epsilon/4.$$

□

We set  $h_0 = h_2 h_1$ . By Lemma 5.4(3) we have  $h_0 \in \text{Diff}^\infty(\mathbb{A}, \omega)$ . Then by Lemma 5.4, we have

$$d_{\text{Diff}^{r_0}(\mathbb{A})}(h_0^{-1}R_\alpha h_0, f) < \epsilon$$

This completes the proof of Theorem 4. □

We can now prove the main result with the aid of the following lemma whose straightforward proof is omitted.

LEMMA 5.5. *For any function  $P : \mathbb{N} \rightarrow \mathbb{N}$ , the set*

$$\mathcal{C} := \{\alpha \in (0, 1) \setminus \mathbb{Q} \mid \exists n \geq 3 \text{ odd such that } q_n(\alpha) > P(q_{n-1}(\alpha)), \\ q_{n+1}(\alpha) > P(q_n(\alpha)), q_{n+2}(\alpha) > P(q_{n+1}(\alpha))\}$$

*is open and dense in  $(0, 1) \setminus \mathbb{Q}$ .*

*Proof of Theorem 1.* We have seen in the introduction that  $\overline{O_{\mathbb{A}}^{\infty}(\alpha)} \subset F_{\mathbb{A}}^{\infty}(\alpha)$  for any  $\alpha \in (0, 1)/\mathbb{Q}$ . It remains to show that for a Baire generic  $\alpha \in (0, 1)/\mathbb{Q}$

$$(5.35) \quad F_{\mathbb{A}}^{\infty}(\alpha) \subset \overline{O_{\mathbb{A}}^{\infty}(\alpha)}$$

with the closure taken in the  $C^{\infty}$ -topology. For each tuple  $(r_0, M, k)$  in  $\mathbb{Z}_{\geq 1}^3$ , with  $r_0 \geq 2$ , let  $P_{r_0, M, k^{-1}}$  be the function produced by Theorem 4. Therefore by Lemma 5.5 the set

$$\mathcal{A}(r_0, M, k) := \left\{ \alpha \in (0, 1) \setminus \mathbb{Q} \mid \text{there is an odd } n \geq 3 \text{ such that} \right. \\ \left. q_n(\alpha) > P_{r_0, M, k^{-1}}(q_{n-1}(\alpha)), \quad q_{n+1}(\alpha) > P_{r_0, M, k^{-1}}(q_n(\alpha)) \right. \\ \left. \text{and } q_{n+2}(\alpha) > P_{r_0, M, k^{-1}}(q_{n+1}(\alpha)) \right\}$$

is open and dense in  $(0, 1) \setminus \mathbb{Q}$ . Thus the countable intersection

$$\mathcal{A} := \bigcap_{r \geq 2} \bigcap_{k \geq 1} \bigcap_{M \geq 2} \mathcal{A}(r, M, k)$$

is a residual subset of  $(0, 1) \setminus \mathbb{Q}$ . Fix  $\alpha \in \mathcal{A}$  and suppose  $f \in F_{\mathbb{A}}^{\infty}(\alpha)$ . Then for any  $\epsilon > 0$  and  $r \in \mathbb{N}$  with  $r \geq 2$  choose  $M \in \mathbb{N}$  so that  $\|f\|_{\text{Diff}^{r+2}(\mathbb{A})} < M$ , then by  $\alpha \in \mathcal{A}(r, M, \lceil \epsilon \rceil^{-1})$  and by Theorem 4 there exists  $h \in \text{Diff}^{\infty}(\mathbb{A}, \omega)$  so that

$$d_{C^{r-1}}(f, hR_{\alpha}h^{-1}) < \epsilon.$$

This gives  $F_{\mathbb{A}}^{\infty}(\alpha) \subset \overline{O_{\mathbb{A}}^{\infty}(\alpha)}$  with the closure in the  $C^{r-1}$ -topology. Since  $r$  is arbitrary we easily conclude (5.35) holds with closure in the  $C^{\infty}$ -topology.  $\square$

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