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ON PSEUDO-ROTATIONS OF THE ANNULUS WITH GENERIC ROTATION NUMBER

BARNEY BRAMHAM AND ZHIYUAN ZHANG

ABSTRACT. We show that for a Baire generic rotation number $\alpha \in \mathbb{R}/\mathbb{Z}$, the set of area preserving C^∞ -pseudo-rotations of the annulus \mathbb{A} with rotation number α equals the closure of the set of area preserving C^∞ -pseudo-rotations which are smoothly conjugate to the rotation R_α . As a corollary, a C^∞ -generic area preserving pseudo-rotation of the annulus with a Baire generic rotation number α is weakly mixing.

1. INTRODUCTION

In this paper we denote the 2-dimensional annulus by $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times [0, 1]$ equipped with the standard area-form ω . We let $F_{\mathbb{A}}^\infty$ denote the set of ω -preserving C^∞ *pseudo-rotations* of \mathbb{A} (the precise definition will appear in Definition 2.9). Namely, we set

$$F_{\mathbb{A}}^\infty := \{f \in \text{Diff}^\infty(\mathbb{A}, \omega) \mid f \text{ is isotopic to Id and has no periodic points}\}.$$

The study of pseudo-rotations can be essentially traced back to the question of Birkhoff [B41] (see also [H98]) as to whether there are non-trivial analytic diffeomorphisms of the 2-sphere with 2 fixed points (the existence of such diffeomorphisms was recently announced by Berger [B22]). The name ‘‘pseudo-rotation’’ was introduced by Béguin, Crovisier, Le Roux and Patou in [BCLP04]. By a result of Franks [F88], for ω -preserving homeomorphisms of \mathbb{A} , the notion of irrational pseudo-rotations in [BCLP04] coincides with ours. In particular, each $f \in F_{\mathbb{A}}^\infty$ admits a *rotation number* $\rho(f) \in (0, 1)/\mathbb{Q}$ (see Definition 2.10 for the details). For each $\alpha \in (0, 1) \setminus \mathbb{Q}$ we set

$$\begin{aligned} F_{\mathbb{A}}^\infty(\alpha) &:= \{f \in F_{\mathbb{A}}^\infty \mid \rho(f) = \alpha\}, \\ O_{\mathbb{A}}^\infty(\alpha) &:= \{hR_\alpha h^{-1} \mid h \in \text{Diff}^\infty(\mathbb{A}, \omega)\} \end{aligned}$$

where R_α denotes the rotation $(x, y) \mapsto (x + \alpha, y)$. It is clear that $O_{\mathbb{A}}^\infty(\alpha) \subset F_{\mathbb{A}}^\infty(\alpha)$ for any $\alpha \in (0, 1) \setminus \mathbb{Q}$. It is not hard to see that $\overline{F_{\mathbb{A}}^\infty(\alpha)}$ is closed in the C^∞ -topology for each irrational α . In particular, $\overline{O_{\mathbb{A}}^\infty(\alpha)} \subset F_{\mathbb{A}}^\infty(\alpha)$ for any $\alpha \in (0, 1) \setminus \mathbb{Q}$, where the closure is taken in the C^∞ -topology. Our main result is the following.

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THEOREM 1. For a Baire generic $\alpha \in (0, 1) \setminus \mathbb{Q}$, we have

$$F_{\mathbb{A}}^{\infty}(\alpha) = \overline{O_{\mathbb{A}}^{\infty}(\alpha)}$$

where the closure is taken in $\text{Diff}^{\infty}(\mathbb{A})$.

In other words, for a Baire generic α , any pseudo-rotation f with rotation number α is the C^{∞} -limit of a sequence f_k of area preserving diffeomorphisms, which up to a smooth area preserving change of coordinates, is the standard rotation R_{α} . In particular f is approximable by *integrable systems*.

We can see Theorem 1 as a natural analogue of a well-known theorem of Herman in [H79]. Recall that one of the most prominent results in the study of circle diffeomorphisms is the following.

THEOREM 2 (Herman-Yoccoz). For any irrational $\alpha \in \mathbb{R}/\mathbb{Z}$, we denote by $F^{\infty}(\alpha)$ the set of C^{∞} circle diffeomorphisms with rotation number α , and denote by $O^{\infty}(\alpha)$ the set of C^{∞} circle diffeomorphisms which are C^{∞} -conjugate to the standard rotation R_{α} . Then we have

$$F^{\infty}(\alpha) = \begin{cases} \overline{O^{\infty}(\alpha)} = O^{\infty}(\alpha) & \text{if } \alpha \text{ is Diophantine,} \\ \overline{O^{\infty}(\alpha)} \neq O^{\infty}(\alpha) & \text{if } \alpha \text{ is Liouville.} \end{cases}$$

Here in the above the closures are taken under the C^{∞} -topology.

The above result for Diophantine α was conjectured by Arnold, who showed in [A78] that any C^{ω} circle diffeomorphism with a Diophantine rotation number α which is sufficiently close to R_{α} in the C^{ω} -topology, is in fact C^{ω} -conjugate to R_{α} . Arnold's result was then generalised to the C^{∞} -category by Moser in [M66]. This is the beginning of what is now known as the Kolmogorov-Arnold-Moser theory. The global picture was for the first time established by Herman in the seminal paper [H79]. In [H79], Theorem 2 was proved for a subset of α with full Lebesgue measure. Khanin and Sinai gave in [KS89] a different proof of the main result in [H79] building on a renormalization theory for circle diffeomorphisms. The Diophantine part of Theorem 2 was completed by Yoccoz in [Y84]. We also mention Katznelson-Ornstein's papers [KO89a, KO89b] on circle diffeomorphisms with low regularity, and Yoccoz's paper in [EKMY02] on C^{ω} -linearization under the sharp arithmetic condition, i.e., \mathcal{H} -condition. For a recent survey of this development and beyond, we refer the reader to [EFK18].

The Liouville part of Theorem 2 was conjectured by Herman in [H79, Conjecture 7.1]. In fact Herman already showed that Theorem 2 holds for a Baire generic set of α , see [H79, Theorem 7.3]. However, his proof was based on the Diophantine part of Theorem 2 (at least for a full measure set of α), and used certain properties of the function $t \mapsto \rho(R_t f)$ of a circle diffeomorphism f . It is still an open question of Herman whether the

Diophantine rigidity holds within pseudo-rotations. Moreover, it is unclear how to deform a general pseudo-rotation within the set of pseudo-rotations, and change the rotation number. This blocks a direct generalisation of Herman's approach for pseudo-rotations. The full answer to Herman's conjecture was provided by Yoccoz in [Y95b]. Yoccoz showed that any C^∞ circle diffeomorphism with a Liouville rotation number can be C^∞ -approximated by a *quasi-rotation*: this is a class of circle diffeomorphisms which, among other things, admits a renormalization that is a standard rotation. Our proof of Theorem 1 is somewhat similar to the proof of Yoccoz: we also consider certain renormalizations of a pseudo-rotation. However, the type of estimates are very different. We are unable to transfer the strong estimates for circle diffeomorphisms, such as Denjoy's inequality in [Y84], to general pseudo-rotations, due to the possible occurrence of complicated geometry which does not appear in dimension 1. On the other hand, the area-preserving hypothesis provides us with certain strong C^0 -estimates established in [AFLXZ20]. Combining such estimates with a suitable arithmetic condition, we are able to extract some useful information from a sequence of suitably renormalized pseudo-rotations.

We can also compare Theorem 1 with the main result in [B15b], which says that any smooth area preserving pseudo-rotation f on the closed 2-disc, meaning that f fixes the origin and has no periodic points on the annulus complementary to the origin, is the C^0 -limit of smooth periodic disc maps f_k , that each fix the origin. In the latter there are no restrictions on the (irrational) rotation number of f , but in this current paper our integrable approximations are in every way stronger: 1) In [B15b] the sequence of approximations f_k , while C^∞ -smooth, only converge in the C^0 topology to f . 2) In [B15b] the f_k 's are not necessarily area preserving. 3) In [B15b] the approximation maps f_k have rational rotation numbers $p_k/q_k \in \mathbb{Q}$ converging to the rotation number α of f as $k \rightarrow \infty$. One cannot perturb the f_k in [B15b] in an obvious way to make the rotation number equal to α while retaining closeness to f . On the other hand, it is easy to modify the f_k 's in the current article, if one so wishes, to make the rotation number rational and keep the closeness to f . In short, the fact that in this paper we are able to find integrable approximations without altering the rotation number is also a stronger conclusion than in [B15b].

Another motivation behind our result is the important work of Anosov and Katok [AK70] and the extensions in Fayad-Saprykina [FS05], see also Fayad-Katok [FK04], in which, for generic rotation numbers, more precisely all Liouville rotation numbers, examples of pseudo-rotations are constructed which are dynamically interesting, that is, not conjugate to a rotation. These "exotic" pseudo-rotations of Anosov-Katok lie, by construction, in the C^∞ -closure of $\cup_{t \in \mathbb{Q}} O_{\mathbb{A}}^\infty(t)$ rather than the closure of $O_{\mathbb{A}}^\infty(\alpha)$. It however follows from our main result that, for a possibly smaller Baire-generic set of rotation numbers than the Liouville numbers, that the Anosov-Katok

constructions do indeed lie in the closure of $O_{\mathbb{A}}^{\infty}(\alpha)$ with *fixed* rotation number α . Moreover, we have the following interesting corollary.

COROLLARY A. *For a Baire generic $\alpha \in (0, 1) \setminus \mathbb{Q}$, the set of weakly mixing pseudo-rotations in $F_{\mathbb{A}}^{\infty}(\alpha)$ forms a Baire set with empty interior, with respect to the C^{∞} topology.*

Proof. On the one hand, Anosov and Katok, see [AK70], show that for a Baire generic α , weak mixing is a C^{∞} -generic property in $\overline{O_{\mathbb{A}}^{\infty}(\alpha)}$. Thus by Theorem 1 weak mixing is a C^{∞} -generic property in $F_{\mathbb{A}}^{\infty}(\alpha)$.

On the other hand, the second statement follows since elements of $O_{\mathbb{A}}^{\infty}(\alpha)$ are never weak mixing and by Theorem 1 the complement $F_{\mathbb{A}}^{\infty}(\alpha) \setminus O_{\mathbb{A}}^{\infty}(\alpha)$ has empty interior. \square

REMARK 1. *Corollary A is seen to be rather sharp in the following sense:*

- (1) *the genericity of α cannot be improved into any subset of $(0, 1)$ with positive Lebesgue measure. This follows from the KAM result of Fayad-Krikorian [FK09] (attributed by the authors to Herman), that a neighborhood of R_{α} in $F_{\mathbb{A}}^{\infty}(\alpha)$ lies in $O_{\mathbb{A}}^{\infty}(\alpha)$, for any Diophantine α .*
- (2) *Weakly mixing cannot be replaced by mixing. In fact, it follows from the proof of [B15a] and [AFLXZ20] that for a Baire generic α , $F_{\mathbb{A}}^{\infty}(\alpha)$ contains no topologically mixing maps. See also Theorem 3.*

Recently, Avila and Krikorian have announced ¹ an improvement of Theorem 1: for every non-Brjuno α , one has $F_{\mathbb{A}}^{\infty}(\alpha) = \overline{O_{\mathbb{A}}^{\infty}(\alpha)}$. Moreover, they have announced the following result: for every pseudo-rotation f in an open neighborhood of the rigid rotations on \mathbb{D} , there exists a sequence of area-preserving diffeomorphism h_n such that $h_n f h_n^{-1}$ converges to $R_{\rho(f)}$ in the C^{∞} topology. Their method involves delicate estimates on high iterates of the maps, while our method for getting this weaker result relies only on rather soft estimates.

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¹See the minicourse of Krikorian in the program “Renormalization and universality in Conformal Geometry, Dynamics, Random Processes, and Field Theory” in 2020 at Simons Center for Geometry and Physics.

NOTATION

In the rest of this paper, we use the following notation. For a subset $A \subset \mathbb{R}^2$ we denote by $\text{Int}(A)$ the interior of A . For a C^r diffeomorphism $f : U \rightarrow V$ between open subsets $U, V \subset \mathbb{R}^2$ and $r \in \mathbb{N}$, we set $\|D^r f\| = \max_{x \in U, |\alpha|=r} \{\|\partial^\alpha f(x)\|\}$, $\|f\|_{C^r} = \sup_{1 \leq \ell \leq r} \|D^\ell f\|$ and $\|f\|_{\text{Diff}^r(U)} = \|f\|_{C^r(U)} + \|f^{-1}\|_{C^r(V)}$. For C^r diffeomorphisms $f, g : U \rightarrow V$, we denote $d_{\text{Diff}^r(U)}(f, g) = \max(\|f^{-1}g\|_{\text{Diff}^r(U)}, \|g^{-1}f\|_{\text{Diff}^r(U)})$. We abbreviate $\|f\|_{\text{Diff}^r(U)}$, resp. $d_{\text{Diff}^r(U)}$, as $\|f\|_{\text{Diff}^r}$, resp. d_{Diff^r} , when there is no confusion. For two homeomorphisms $f, g : U \rightarrow V$, we denote $\|f - g\| = \sup_{x \in U} d(f(x), g(x))$, where $d(\cdot, \cdot)$ is the euclidean metric.

2. PRELIMINARIES

We denote the boundary components of \mathbb{A} by

$$B_i := \mathbb{R}/\mathbb{Z} \times \{i\} \quad i = 0, 1.$$

The universal covering of \mathbb{A} is $\tilde{\mathbb{A}} = \mathbb{R} \times [0, 1]$. We write $\tilde{B}_i = \mathbb{R} \times \{i\}$ for $i = 0, 1$. We let

$$\pi : \tilde{\mathbb{A}} \rightarrow \mathbb{A}$$

be the natural projection, and let $T : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ be the translation in the first coordinate, i.e., $T(x, y) = (x + 1, y)$. Given a homeomorphism $f : \mathbb{A} \rightarrow \mathbb{A}$ there is a lift $F : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ of f , also a homeomorphism, unique up to composition by a power of T , such that $\pi F = f\pi$. Moreover, any such lift commutes with T .

We have the following lemma, whose proof is elementary and left to the readers.

LEMMA 2.1. *Suppose $f : \mathbb{A} \rightarrow \mathbb{A}$ is a homeomorphism with $\|f - \text{Id}\| < 1/2$. Then there is a unique lift $F : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ satisfying $\|F - \text{Id}\| < 1/2$. This lift satisfies*

$$(2.1) \quad d(F(\tilde{x}), \tilde{x}) = d(f(x), x)$$

for all $x \in \mathbb{A}$ and all $\tilde{x} \in \tilde{\mathbb{A}}$ with $\pi(\tilde{x}) = x$.

DEFINITION 2.1. We say that $\gamma \subset \mathbb{A}$ is a *simple regular curve* connecting B_0 and B_1 if $\gamma = \phi([0, 1])$ where $\phi : [0, 1] \rightarrow \mathbb{A}$ is an injective continuous map mapping 0, resp. 1, into B_0 , resp. B_1 , that maps $(0, 1)$ into $\mathbb{A} \setminus \partial\mathbb{A}$. We define simple regular curves in $\tilde{\mathbb{A}}$ connecting \tilde{B}_0 and \tilde{B}_1 in a similar way.

DEFINITION 2.2. For a pair of simple regular curves γ_1, γ_2 in $\tilde{\mathbb{A}}$ we will say that γ_1 is to the left of γ_2 , or γ_2 is to the right of γ_1 , and write

$$\gamma_1 < \gamma_2$$

if $\gamma_1 \cap \gamma_2 = \emptyset$ and γ_1 lies in the component of $\tilde{\mathbb{A}} \setminus \gamma_2$ containing points with arbitrarily negative \mathbb{R} -coordinate. This defines a partial ordering on simple regular curves in $\tilde{\mathbb{A}}$.

DEFINITION 2.3. Let f be a homeomorphism of \mathbb{A} . A simple regular curve γ in \mathbb{A} connecting B_0 and B_1 , satisfying

$$\gamma \cap f(\gamma) = \emptyset$$

is called a *Brouwer curve* for f . A Brouwer curve γ is called *smooth* if γ is a C^∞ -curve. Brouwer curves for homeomorphisms of $\tilde{\mathbb{A}}$ are defined in a similar way.

Note that if F is a lift of a homeomorphism $f : \mathbb{A} \rightarrow \mathbb{A}$ and $\tilde{\gamma}$ is a lift of a Brouwer curve $\gamma \subset \mathbb{A}$ for f , then

$$T^k F(\tilde{\gamma}) \cap \tilde{\gamma} = \emptyset \quad \forall k \in \mathbb{Z}.$$

DEFINITION 2.4. Let f be a homeomorphism of \mathbb{A} and $Q \geq 2$ an integer. We say that γ is a *Q -good smooth curve* if γ is a smooth Brouwer curve for each of the maps f, f^2, \dots, f^{Q-1} .

DEFINITION 2.5. Let γ_1, γ_2 be two disjoint simple regular curves in \mathbb{A} connecting B_0 and B_1 . There is a unique closed region \mathcal{R} in \mathbb{A} with left boundary γ_1 and right boundary γ_2 . More precisely, if each γ_i is oriented from B_0 to B_1 , then $\partial\mathcal{R} \cap (\mathbb{A} \setminus \partial\mathbb{A})$ has orientation agreeing with $\gamma_2 - \gamma_1$. We say that \mathcal{R} is the *region bounded by* (γ_1, γ_2) .

DEFINITION 2.6. Let $\{K_r\}_{r \geq 1}$ be an increasing sequence of positive real numbers. Let U, V be smooth surfaces, possibly with boundary, and let $\phi : U \rightarrow V$ be a C^∞ -diffeomorphism. We say that ϕ is $\{K_r\}_{r \geq 1}$ -smooth if

$$\|\phi\|_{\text{Diff}^r(U)} < K_r \quad \forall r \in \mathbb{Z}_+.$$

DEFINITION 2.7. Let γ be a Brouwer curve for f , and let $\mathcal{R} \subset \mathbb{A}$ be the closed region bounded by $(\gamma, f(\gamma))$. We say that an orientation preserving C^∞ -diffeomorphism

$$\phi : \mathcal{U} \rightarrow \mathcal{R}'$$

from an open neighborhood $\mathcal{U} \subset \tilde{\mathbb{A}}$ of $[0, 1]^2$ to an open neighborhood $\mathcal{R}' \subset \mathbb{A}$ of \mathcal{R} is an *admissible coordinate* for (\mathcal{R}, f) if the following hold:

- (1) ϕ has constant Jacobian,
- (2) ϕ satisfies

$$(2.2) \quad \phi(\{0\} \times [0, 1]) = \gamma,$$

$$(2.3) \quad \phi(\{1\} \times [0, 1]) = f(\gamma),$$

$$(2.4) \quad \phi([0, 1] \times \{0\}) \subset B_0,$$

$$(2.5) \quad \phi([0, 1] \times \{1\}) \subset B_1,$$

- (3) there is a neighborhood U_L of $\{0\} \times [0, 1]$ in \mathcal{U} , so that

$$f\phi(x) = \phi T(x) \quad \forall x \in U_L.$$

Without loss of generality we may assume that $T(U_L) \subset \mathcal{U}$ by choosing U_L sufficiently small.

We make a similar definition for lifts. Namely, let $\tilde{\gamma}$ be a lift of a Brouwer curve γ for f , let F be an lift of f such that $F(\tilde{\gamma})$ is on the right of $\tilde{\gamma}$, and let $\tilde{\mathcal{R}}$ be the region bounded by $(\tilde{\gamma}, F(\tilde{\gamma}))$. We say that a C^∞ -diffeomorphism

$$\phi : \mathcal{U} \rightarrow \mathcal{R}'$$

from an open neighborhood \mathcal{U} of $[0, 1]^2$ in $\tilde{\mathbb{A}}$ to an open neighborhood \mathcal{R}' of $\tilde{\mathcal{R}}$ in $\tilde{\mathbb{A}}$ is an *admissible coordinate* for $(\tilde{\mathcal{R}}, F)$ if ϕ satisfies the analogous properties above with $(\mathcal{R}, \mathbb{A}, f)$ replaced by $(\tilde{\mathcal{R}}, \tilde{\mathbb{A}}, F)$.

REMARK 2. We notice that by item (3) in Definition 2.7, an admissible coordinate $\phi : \mathcal{U} \rightarrow \mathcal{R}'$ is determined by its restriction to $[0, 1]^2$ in the following sense: if $\phi_i : \mathcal{U}_i \rightarrow \mathcal{R}_i$, $i = 1, 2$ are two admissible coordinates for (\mathcal{R}, f) such that $\phi_1|_{[0, 1]^2} = \phi_2|_{[0, 1]^2}$, then there exists an open neighborhood \mathcal{U}_3 of $[0, 1]^2$ in $\tilde{\mathbb{A}}$ such that $\phi_1|_{\mathcal{U}_3} = \phi_2|_{\mathcal{U}_3}$. For this reason, we will sometimes identify an admissible coordinate for (\mathcal{R}, f) or (\mathcal{R}, F) with a map from $[0, 1]^2$ to \mathcal{R} .

DEFINITION 2.8. Let $r \in \mathbb{N}$ and $K \in (0, \infty)$. We say that $\phi : [0, 1]^2 \rightarrow \mathcal{R}$ is a (r, K) -admissible coordinate for (\mathcal{R}, f) if ϕ is an admissible coordinate, in the sense of Definition 2.7, defined on an open neighborhood \mathcal{U} of $[0, 1]^2$, satisfying

$$\|\phi\|_{\text{Diff}^r(\mathcal{U})} < K.$$

Analogously for lifts: We say that $\phi : \mathcal{U} \rightarrow \tilde{\mathcal{R}}$ is a (r, K) -admissible coordinate for $(\tilde{\mathcal{R}}, F)$ if ϕ is an admissible coordinate for $(\tilde{\mathcal{R}}, F)$, in the sense of Definition 2.7, defined on an open neighborhood \mathcal{U} of $[0, 1]^2$, satisfying $\|\phi\|_{\text{Diff}^r(\mathcal{U})} < K$.

DEFINITION 2.9. A *pseudo-rotation* is a non-wandering homeomorphism $f : \mathbb{A} \rightarrow \mathbb{A}$ that is isotopic to the identity, maps B_0 (resp. B_1) to itself, and has no periodic points.

Recall that a homeomorphism $f : \mathbb{A} \rightarrow \mathbb{A}$ is said to be *non-wandering* if for every open subset $U \subset \mathbb{A}$, there exists an integer $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. In this paper we will only be considering C^∞ -smooth pseudo-rotations that preserve a smooth area form ω . By a slight abuse of notation, we denote by ω both the area form on $\tilde{\mathbb{A}}$ and \mathbb{A} . If f is an ω -preserving diffeomorphism then so is each lift F an ω -preserving diffeomorphism.

DEFINITION 2.10. Let $f : \mathbb{A} \rightarrow \mathbb{A}$ be a pseudo-rotation with lift $F : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$. Denote by $p_1 : \tilde{\mathbb{A}} \rightarrow \mathbb{R}$ the projection on the first coordinate. We define $\rho(F) \in \mathbb{R}$ by

$$\rho(F) := \lim_{n \rightarrow \infty} \frac{1}{n} (p_1(F^n(x, y)) - x) \quad x \in \mathbb{R}, y \in [0, 1].$$

It is known that the limit on the right hand side above exists for all (x, y) and is independent of (x, y) , see [F88, FH12]. Moreover, we always have $\rho(F) \notin \mathbb{Q}$, see [F88]. For any pseudo-rotation f , there exists a unique lift F

of f such that $\rho(F) \in (0, 1) \setminus \mathbb{Q}$, which we denote by $\rho(f)$. We call $\rho(f)$ the *rotation number* of f .

Let us recall a few well known facts about the best rational approximations to $\alpha \in (0, 1) \setminus \mathbb{Q}$ and its continued fraction expansion. Readers can consult [HW, Chapters X, XI] for more details.

First, for $x \in \mathbb{R}$ we will write

$$\begin{aligned} \lfloor x \rfloor &:= \max\{n \in \mathbb{Z} \mid n \leq x\} \in \mathbb{Z}, \\ q(x) &:= \lfloor 1/x \rfloor \in \mathbb{Z} \end{aligned}$$

for the integer parts of x and $1/x$ respectively. Moreover, if $x \in \mathbb{R} \setminus \mathbb{Q}$ then

$$\|x\|_{\mathbb{R}/\mathbb{Z}} := d(x, \mathbb{Z}) \in (0, 1/2).$$

The Gauss map $\mathcal{G} : [0, 1) \rightarrow [0, 1)$ is defined by

$$\mathcal{G}(x) = \frac{1}{x} - q(x)$$

on $(0, 1)$ and $\mathcal{G}(0) := 0$. If x is irrational then so is $\mathcal{G}(x)$.

DEFINITION 2.11. For any $\alpha \in (0, 1) \setminus \mathbb{Q}$, we define the sequences $(\alpha_n)_{n \geq 0}$ and $(\beta_n)_{n \geq 0}$ in $(0, 1) \setminus \mathbb{Q}$ by

$$\begin{aligned} \alpha_0 &:= \alpha, & \alpha_n &:= \mathcal{G}^n(\alpha_0) & \forall n \geq 1, \\ \beta_n &:= \prod_{i=0}^n \alpha_i & \forall n \geq 0. \end{aligned}$$

Furthermore we define sequences of non-negative integers $(a_n)_{n \geq 0}$, $(q_n)_{n \geq 0}$ as follows:

$$(2.6) \quad a_0 := 0, \quad a_n := q(\alpha_{n-1}) \quad \forall n \geq 1,$$

$$(2.7) \quad q_0 := 1, \quad q_1 := q(\alpha), \quad q_{n+2} := q_n + q_{n+1}a_{n+2} \quad \forall n \geq 1.$$

We also define $(p_n)_{n \geq 0}$ by $p_0 = 0$, and for $n \geq 1$, define p_n to be the closest integer to $q_n \alpha$, which is unique by irrationality of α . We will use the notation $\alpha_n(\alpha)$, $q_n(\alpha)$ and $p_n(\alpha)$ when it is necessary to indicate the dependence of the sequences on α .

Note that

$$(2.8) \quad \alpha_{n-1}^{-1} = a_n + \alpha_n \quad \forall n \geq 1,$$

since $\alpha_n = \mathcal{G}(\alpha_{n-1}) = 1/\alpha_{n-1} - q(\alpha_{n-1}) = 1/\alpha_{n-1} - a_n$. It is also well known that

$$(2.9) \quad p_{n+1}q_n - p_nq_{n+1} = (-1)^n$$

$$(2.10) \quad \beta_n = (-1)^n(q_n \alpha - p_n) > 0 \quad \forall n \geq 0.$$

In particular for each $n \geq 1$, $\beta_n = |q_n \alpha - p_n| = d(q_n \alpha, \mathbb{Z})$, and

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n + \alpha_n}}}} \quad \frac{p_n}{q_n} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}}$$

For any $n \geq 1$, the integers p_n and q_n are relatively prime, and p_n/q_n is called the n -th best rational approximation of α . The following simple inequalities are known, and will be used later:

$$(2.11) \quad \frac{1}{2q_{n+1}} < \frac{1}{q_n + q_{n+1}} < \beta_n < \frac{1}{q_{n+1}},$$

$$(2.12) \quad \alpha_n, q(\alpha_n)^{-1} \in \left(\frac{q_n}{2q_{n+1}}, \frac{2q_n}{q_{n+1}} \right).$$

3. EXISTENCE OF A BROUWER CURVE WITH UNIFORM BOUNDS

In order to avoid lengthy computations, in the rest of the paper we will introduce various *increasing functions* to keep track of parameter dependence. We say that a function $A : U \rightarrow \mathbb{R}$ defined on an open subset $U \subset \mathbb{R}^n$ is *increasing* if for any $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in U$ with $x_i \geq y_i$ for all $1 \leq i \leq n$, we have $A(x) \geq A(y)$. We define *decreasing functions* in a similar way. Typically, all the variables and values of the increasing/decreasing functions that we will consider lie in \mathbb{R}_+ .

The following theorem, essentially proven in [AFLXZ20], allows one to control the C^r -distance of a pseudo-rotation to the identity in terms of its rotation number.

THEOREM 3. *There is a sequence of increasing functions*

$$A_r : (0, 1/2) \times \mathbb{R}_+ \rightarrow \mathbb{R}_+, \\ \lim_{t \rightarrow 0} A_r(t, \cdot) \equiv 0$$

for each $r \in \mathbb{N}$, so that for any $\{K_r\}_{r \geq 1}$ -smooth ω -preserving pseudo-rotation $f : \mathbb{A} \rightarrow \mathbb{A}$ there holds

$$\|f - \text{Id}\| < A_0(\|\rho(f)\|_{\mathbb{R}/\mathbb{Z}}, K_1), \\ \|D^r f\| < A_r(\|\rho(f)\|_{\mathbb{R}/\mathbb{Z}}, K_{r+1}), \quad \forall r \geq 1.$$

Proof. By [AFLXZ20, Corollary A], we know that

$$\|f - \text{Id}\| < (1 + 2K_1) \|\rho(f)\|_{\mathbb{R}/\mathbb{Z}}^{1/2} =: A_0(\|\rho(f)\|_{\mathbb{R}/\mathbb{Z}}, K_1).$$

By the Hadamard-Kolmogorov convexity theorem, we get corresponding bounds on the “inbetween” derivatives: for any $r \geq 1$

$$\begin{aligned} \|D^r f\| &\leq C_r \|f - \text{Id}\|_{\text{Diff}^{r+1}}^{\frac{1}{r+1}} \|f\|_{\text{Diff}^{r+1}}^{\frac{r}{r+1}} \\ &\leq C_r \|\rho(f)\|_{\mathbb{R}/\mathbb{Z}}^{\frac{1}{2(r+1)}} (1 + 2K_{r+1}) =: A_r(\|\rho(f)\|_{\mathbb{R}/\mathbb{Z}}, K_{r+1}). \end{aligned}$$

□

By Theorem 3, there is a decreasing function

$$(3.1) \quad \rho_* : \mathbb{R}_+ \rightarrow (0, 1/2)$$

such that for any smooth ω -preserving pseudo-rotation f satisfying

$$(3.2) \quad \|\rho(f)\|_{\mathbb{R}/\mathbb{Z}} < \rho_*(\|Df\|),$$

we have

$$\|f - \text{Id}\| \leq A_0(\|\rho(f)\|_{\mathbb{R}/\mathbb{Z}}, \|Df\|) < A_0(\rho_*(\|Df\|), \|Df\|) < 1/2.$$

Then by Lemma 2.1, there is a unique lift F of f satisfying

$$(3.3) \quad \|F - \text{Id}\| = \|f - \text{Id}\| < 1/2.$$

LEMMA 3.1. *There are increasing functions*

$$C', C'' : \mathbb{R}_+ \times (0, 1) \rightarrow \mathbb{R}_+$$

with $\lim_{t \rightarrow 0} C''(t, \cdot) \equiv 0$ and

$$(3.4) \quad C''(\alpha, K^{-1}) < \rho_*(K^{\alpha-1}) \quad \forall (\alpha, K) \in (0, 1) \times (1, \infty)$$

such that the following holds: if $\alpha \in (0, 1) \setminus \mathbb{Q}$ satisfies

$$(3.5) \quad \alpha \in (0, \rho_*(K)),$$

$$(3.6) \quad \mathcal{G}(\alpha) < C''(\alpha, K^{-1})$$

for some $K > 1$, where ρ_* is given in (3.1), then every C^1 -smooth ω -preserving pseudo-rotation with $\|Df\| < K$ and $\rho(f) = \alpha$ satisfies

$$(3.7) \quad \inf_{x \in \mathbb{A}} d(x, f(x)) \geq C'(\alpha, K^{-1}).$$

Proof. Fix $K > 0$ and $\alpha \in (0, 1) \setminus \mathbb{Q}$ satisfying (3.5) and (3.6). Let f be a C^1 -smooth ω -preserving pseudo-rotation with $\|Df\| < K$ and $\rho(f) = \alpha$. Denote by F the lift of f with $\rho(F) = \rho(f)$. We abbreviate $q := q(\alpha) = \lfloor 1/\alpha \rfloor$.

Choose the function $C'' : \mathbb{R}_+ \times (0, 1) \rightarrow \mathbb{R}_+$, with $\lim_{t \rightarrow 0} C''(t, \cdot) \equiv 0$, sufficiently small so that for each $t \in (0, 1)$, the condition

$$s < C''(t, K^{-1})$$

implies

$$(3.8) \quad A_0(s, K^{t-1}) < 1/2.$$

Without loss of generality we may arrange that C'' is increasing and satisfies (3.4). By Theorem 3, we have

$$\|f^q - \text{Id}\| \leq A_0(\|\rho(f^q)\|_{\mathbb{R}/\mathbb{Z}}, \|Df^q\|) \leq A_0(\|q\rho(f)\|_{\mathbb{R}/\mathbb{Z}}, K^q).$$

Since $\|q\rho(f)\|_{\mathbb{R}/\mathbb{Z}} = 1 - q\alpha = \alpha\mathcal{G}(\alpha) \leq \mathcal{G}(\alpha)$ and $K^q \leq K^{\alpha^{-1}}$, the above gives

$$\|f^q - \text{Id}\| \leq A_0(\mathcal{G}(\alpha), K^{\alpha^{-1}}).$$

Thus, if α satisfies (3.6), then by (3.8),

$$(3.9) \quad \|f^q - \text{Id}\| < 1/2.$$

The unique lift of f^q with rotation number in $(-1/2, 1/2)$ is $T^{-1}F^q$, since $\rho(T^{-1}F^q) = q\rho(F) - 1 = q\alpha - 1 \in (-1/2, 1/2)$. So by Lemma 2.1

$$\|F^q - T\| = \|T^{-1}F^q - \text{Id}\| = \|f^q - \text{Id}\|.$$

Thus for all $\tilde{x} \in \tilde{\mathbb{A}}$

$$(3.10) \quad d(F^q(\tilde{x}), \tilde{x}) \geq d(T(\tilde{x}), \tilde{x}) - d(F^q(\tilde{x}), T(\tilde{x})) > 1 - 1/2 = 1/2.$$

On the other hand, for each $\tilde{x} \in \tilde{\mathbb{A}}$, there holds

$$d(F^q(\tilde{x}), \tilde{x}) \leq \sum_{j=0}^{q-1} d(F^{j+1}(\tilde{x}), F^j(\tilde{x})) \leq \sum_{j=0}^{q-1} \|DF\|^j d(F(\tilde{x}), \tilde{x}) \leq Cd(F(\tilde{x}), \tilde{x})$$

where $C = \frac{K^q - 1}{K - 1} \leq \frac{K^{\alpha^{-1}} - 1}{K - 1}$. Combining the last line with (3.10) and (3.3), we obtain

$$(3.11) \quad \|F(\tilde{x}) - \tilde{x}\| \geq \frac{1}{2C} \quad \forall \tilde{x} \in \tilde{\mathbb{A}}.$$

By Lemma 2.1 and (3.5), we conclude that (3.7) holds if we define C' by

$$C'(t, K^{-1}) = \frac{1}{2} \frac{K - 1}{K^{t-1} - 1}.$$

□

DEFINITION 3.1. For each $K > 1$ let $\mathcal{L}(K)$ be the set of $\alpha \in (0, 1/2) \setminus \mathbb{Q}$ such that

$$(3.12) \quad \alpha \in (0, \rho_*(K)),$$

$$(3.13) \quad \mathcal{G}(\alpha) < C''(\alpha, K^{-1})$$

where ρ_* is from (3.1) and C'' is from Lemma 3.1.

LEMMA 3.2. *There is a sequence of increasing functions*

$$C'_r : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \quad \forall r \geq 1,$$

such that the following is true. If $\{K_r\}_{r \geq 1}$ is any sequence in $(1, \infty)$ and $\alpha \in \mathcal{L}(K_1)$, then every $\{K_r\}_{r \geq 1}$ -smooth ω -preserving pseudo-rotation f with $\rho(f) = \alpha$ has for each $r \in \mathbb{N}$ a Brouwer curve $\gamma = \gamma_r$ for which the closed region in \mathbb{A} bounded by $(\gamma, f(\gamma))$ admits a $(r, C'_r(\alpha^{-1}, K_{r+1}))$ -admissible coordinate.

Proof. Fix a sequence $(K_r)_{r \geq 1}$ in $(1, \infty)$. Fix $r \geq 1$. For each $c > 0$ and $K \geq 1$ define the following subset of $\text{Diff}^r(\mathbb{A}, \omega)$:

$$\mathcal{H}_r(c, K) := \left\{ g \in \text{Diff}^r(\mathbb{A}, \omega) \mid \|g\|_{\text{Diff}^r} \leq K, \quad \inf_{x \in \mathbb{A}} d(x, g(x)) \geq c \right\}.$$

Denote by

$$\overline{\mathcal{H}_{r+1}(c, K)} \subset \text{Diff}^r(\mathbb{A}, \omega)$$

the closure of $\mathcal{H}_{r+1}(c, K)$ in the C^r -topology. We observe that $\overline{\mathcal{H}_{r+1}(c, K)}$ is compact in $\text{Diff}^r(\mathbb{A})$ and contains only diffeomorphisms without fixed points. Moreover, for $c' \geq c$ and $K' \leq K$ we have $\overline{\mathcal{H}_{r+1}(c', K')} \subset \overline{\mathcal{H}_{r+1}(c, K)}$.

We now prove a version of the lemma for elements of $\overline{\mathcal{H}_{r+1}(c, K)}$, for each fixed $c > 0, K \geq 1$. Then we argue that the union of $\overline{\mathcal{H}_{r+1}(c, K_{r+1})}$ over $c \in (0, 1]$ contains all pseudo-rotations satisfying the assumptions of the Lemma.

To this end, fix $c > 0, K \geq 1$ and consider $g \in \overline{\mathcal{H}_{r+1}(c, K)}$. Since g has no fixed points, a strong refinement of Brouwer's plane translation theorem due to Guillou [G94, Théorème 5.1] yields a C^0 Brouwer curve γ_0 for g in the sense of Definition 2.3. Any sufficiently C^0 -close smooth approximation of γ_0 that continues to connect the two boundary components yields a smooth Brouwer curve γ for g . Clearly we can choose such a γ to meet both boundary components of the annulus orthogonally. We can then apply the following lemma.

LEMMA 3.3. *Let $r \geq 1$ and $\theta \in (0, 1)$. Suppose $g \in \text{Diff}^{r, \theta}(\mathbb{A}, \omega)$ has a smooth Brouwer curve γ that meets both boundary components orthogonally. Then there exist $D = D(g, \gamma, r) > 1$, and a neighborhood \mathcal{V} of g in $\text{Diff}^{r, \theta}(\mathbb{A}, \omega)$ such that for every $g' \in \mathcal{V}$, the region $(\gamma, g'(\gamma))$ has admissible coordinates ϕ' whose C^r -norm is bounded by D .*

Proof. Let \mathcal{R} denote the region $(\gamma, g(\gamma))$. We first construct a C^∞ -diffeomorphism ψ_L with constant Jacobian $\omega(\mathcal{R})$ from a neighborhood of $\{0\} \times [0, 1]$ in $\tilde{\mathbb{A}}$ onto its image a neighborhood of γ in \mathbb{A} . Indeed, without loss of generality $\gamma : [0, 1] \rightarrow \mathbb{A}$ meets the boundary of \mathbb{A} orthogonally near both end points and is parametrised by arclength so that $\|\dot{\gamma}\| = L$ is constant, $L \geq 1$. Then $n := -i\dot{\gamma}/L$ is a normal vector field along γ and the map $\tilde{\mathbb{A}} \rightarrow \mathbb{R}^2, (x, y) \mapsto \gamma(y) + x\omega(\mathcal{R})n(y)/L$ extends γ to a smooth diffeomorphism from a sufficiently small tubular neighborhood of $\{0\} \times [0, 1]$ in $\tilde{\mathbb{A}}$ to a neighborhood of the image of γ in \mathbb{A} , with constant Jacobian $\omega(\mathcal{R})$ along $\{0\} \times [0, 1]$ and also near to the boundary of $\tilde{\mathbb{A}}$. The Jacobian of this map away from $\{0\} \times [0, 1]$ depends only on the y variable and therefore by a further change of coordinates it is easily modified to have constant Jacobian on a whole neighborhood of $\{0\} \times [0, 1]$ while still mapping $\{0\} \times [0, 1]$ to γ . The resulting map, which we denote by ψ_L , clearly has finite $C^{r, \theta}$ -norm.

Then $g\psi_L T^{-1}$ is a $C^{r, \theta}$ -diffeomorphism ψ_R with constant Jacobian $\omega(\mathcal{R})$ from a neighborhood of $\{1\} \times [0, 1]$ in $\tilde{\mathbb{A}}$ onto its image, a neighborhood

of $g(\gamma)$ in \mathbb{A} . We construct the chart $\phi : [0, 1]^2 \rightarrow \mathbb{A}$ by first specifying its restriction to a neighborhood of $\{0\} \times [0, 1]$, resp. $\{1\} \times [0, 1]$, to be ψ_L , resp. ψ_R . Then we extend it by hand to neighborhoods of $[0, 1] \times \{0\}$ and $[0, 1] \times \{1\}$, also with constant Jacobian $\omega(\mathcal{R})$. Finally, using [A10, Corollary 4] (or, in our simple application, using directly [A10, Theorem 3] which follows from [DM90]), we extend the map to all of $[0, 1]^2$ so as to have constant Jacobian, and the C^r norm of the resulting map is bounded in terms of the $C^{r,\theta}$ norm of g . By construction the conditions in Definition 2.7 hold.

Each g has a C^0 -small (and hence also C^r -small) neighborhood in $\text{Diff}^r(\mathbb{A}, \omega)$ for which the same γ can be applied. The uniform bounds for the C^r -norm of ϕ is an immediate consequence of the construction. We omit the proof of this latter fact and refer the readers to [A10, DM90] for details. \square

By Lemma 3.3 and by compactness of $\overline{\mathcal{H}_{r+1}(c, K)} \subset \text{Diff}^{r,1/2}(\mathbb{A}, \omega) \subset \text{Diff}^r(\mathbb{A}, \omega)$, we find a finite collection of neighborhoods (with respect to the $C^{r,1/2}$ -topology) as in Lemma 3.3 whose union covers $\mathcal{H}_{r+1}(c, K)$, and thus obtain a uniform bound $E_r(c, K) > 0$ on the C^r -norm of admissible coordinates that applies to all elements of $\overline{\mathcal{H}_{r+1}(c, K)}$. Due to the inclusions $\overline{\mathcal{H}_{r+1}(c', K')} \subset \overline{\mathcal{H}_{r+1}(c, K)}$ for $c' > c, K' < K$, we can assume that $E_r(c, K)$ is decreasing in c and increasing in K .

Now, suppose f is as in Lemma 3.2. That is, f is a $(K_r)_{r \geq 1}$ -smooth ω -preserving pseudo-rotation with rotation number $\rho(f) = \alpha \in \mathcal{L}(K_1)$. By Lemma 3.1,

$$f \in \mathcal{H}_{r+1}\left(C'(\alpha, K_1^{-1}), K_{r+1}\right).$$

Thus there exists a region $(\gamma, f(\gamma))$ having admissible coordinates whose C^r -norm is bounded by $E_r(C'(\alpha, K_1^{-1}), K_{r+1})$, for some smooth Brouwer curve γ . Hence Lemma 3.2 holds with $C'_r(\alpha^{-1}, K_{r+1}) := E_r(C'(\alpha, K_1^{-1}), K_{r+1})$. Evidently C'_r is an increasing function, since C' is increasing and $E_r(c, K)$ is decreasing in c and increasing in K . \square

It will be useful to fix the following notation:

DEFINITION 3.2. For each integer $r \geq 1$ and each $K > 1$ let $\mathcal{L}_r(K)$ be the set of $\alpha \in \mathcal{L}(K)$ such that

$$(3.14) \quad A_0(\alpha \mathcal{G}(\alpha), K^{q(\alpha)}) < K^{-q(\alpha)} C'_r(\alpha^{-1}, K)^{-1}$$

where $C'_r : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, $r \geq 1$ are the increasing functions produced by Lemma 3.2.

We will see later that each $\mathcal{L}_r(K)$ is non-empty for each integer $r \geq 1$ and each $K > 1$. Moreover, by (3.13) and (3.14) we have

$$(3.15) \quad \mathcal{L}_r(K') \subset \mathcal{L}_r(K) \quad \forall K' > K.$$

Now we can strengthen Lemma 3.2. Indeed, we show that if more restrictions are placed on the rotation number of a ω -preserving pseudo-rotation

f , then the produced Brouwer curve is actually Q -good, for some large Q depending on the rotation number of f :

PROPOSITION 3.1. *Let f be a $\{K_r\}_{r \geq 1}$ -smooth ω -preserving pseudo-rotation. If $\rho(f) \in \mathcal{L}_r(K_{r+1})$ for some integer $r \geq 1$, then the Brouwer curve γ for f produced by Lemma 3.2 (corresponding to r) is $q(\rho(f))$ -good.*

Note that the condition $\rho(f) \in \mathcal{L}_r(K_{r+1})$ in this Proposition includes the condition on $\rho(f)$ used in Lemma 3.2, since $\mathcal{L}_r(K) \subset \mathcal{L}(K)$ by definition.

Proof. We denote by d_H the Hausdorff distance on $\tilde{\mathbb{A}}$, i.e. for any two subsets A, B of $\tilde{\mathbb{A}}$,

$$d_H(A, B) = \sup_{x \in A, y \in B} \max(d(x, B), d(y, A)).$$

Set $\alpha = \rho(f) \in \mathcal{L}_r(K_{r+1})$ for some $r \geq 1$. Let γ be the Brouwer curve given by Lemma 3.2 for this value of r . Let $\mathcal{R} \subset \mathbb{A}$ be the region bounded by $(\gamma, f(\gamma))$ and let $\psi : \mathcal{R} \rightarrow [0, 1] \times [0, 1]$ be the inverse of some $(r, C'_r(\alpha^{-1}, K_{r+1}))$ -admissible coordinates produced by Lemma 3.2. Applying the intermediate value theorem to ψ yields

$$(3.16) \quad d_H(\gamma, f(\gamma)) > C'_r(\alpha^{-1}, K_{r+1})^{-1}.$$

Fix a lift $\tilde{\gamma} \subset \tilde{\mathbb{A}}$ of γ , and let F be the lift of f for which $\rho(F) = \alpha$. Since $\tilde{\gamma} \cap F(\tilde{\gamma}) = \emptyset$ it follows from $\alpha > 0$ and the order of boundary points that $\tilde{\gamma} < F(\tilde{\gamma})$. By Definition 2.7, (3.16) and $K_{r+1} \geq K_1$, we can see that

$$(3.17) \quad \begin{aligned} d_H(F^{q(\alpha)-1}(\tilde{\gamma}), F^{q(\alpha)}(\tilde{\gamma})) &> K_1^{-q(\alpha)} d_H(\tilde{\gamma}, F(\tilde{\gamma})) \\ &\geq K_{r+1}^{-q(\alpha)} C'_r(\alpha^{-1}, K_{r+1})^{-1}. \end{aligned}$$

Here we implicitly used that the analogue of (3.16) holds for the lifts, since $d_H(\tilde{\gamma}, F(\tilde{\gamma})) \geq d_H(\gamma, f(\gamma))$ holds - in fact for all choice of lifts F and $\tilde{\gamma}$. Moreover, since F is injective, for each $i \in \mathbb{N}$ we have $F^i(\tilde{\gamma}) \cap F^{i+1}(\tilde{\gamma}) = \emptyset$ and so from the order of boundary points we have for all $i \in \mathbb{N}$,

$$(3.18) \quad \tilde{\gamma} < F(\tilde{\gamma}) < F^2(\tilde{\gamma}) < \dots < F^i(\tilde{\gamma}).$$

Thus it suffices to show that

$$(3.19) \quad F^{q(\alpha)-1}(\tilde{\gamma}) < T\tilde{\gamma}$$

and it will follow that the iterates $F(\tilde{\gamma}), \dots, F^{q(\alpha)-1}(\tilde{\gamma})$ all lie strictly in the region between $\tilde{\gamma}$ and $T(\tilde{\gamma})$ in $\tilde{\mathbb{A}}$ and therefore that the iterates $f(\gamma), \dots, f^{q(\alpha)-1}(\gamma)$ are all disjoint from γ as required.

To show (3.19) we argue by contradiction. Assuming (3.19) is not true, we have

$$(3.20) \quad F^{q(\alpha)-1}(\tilde{\gamma}) \cap T\tilde{\gamma} \neq \emptyset$$

because of the order of boundary points. We will show that this means that $F^{q(\alpha)-1}(\tilde{\gamma})$ and $F^{q(\alpha)}(\tilde{\gamma})$ pass somewhere very close to each other, because $F^{q(\alpha)}(\tilde{\gamma})$ is close to $T\tilde{\gamma}$. This will contradict (3.17).

First, notice that by $\alpha \in \mathcal{L}_r(K_{r+1})$, (3.4) and that ρ_* is decreasing, we have

$$\begin{aligned} \|\rho(f^{q(\alpha)})\|_{\mathbb{R}/\mathbb{Z}} &= |q(\alpha)\alpha - 1| = \alpha\mathcal{G}(\alpha) < \mathcal{G}(\alpha) < C''(\alpha, K_{r+1}^{-1}) \\ &< \rho_*(K_{r+1}^{\alpha^{-1}}) \leq \rho_*(\|Df\|^{q(\alpha)}) \leq \rho_*(\|Df^{q(\alpha)}\|). \end{aligned}$$

Then by (3.3), we have

$$d_H(F^{q(\alpha)}(\tilde{\gamma}), T\tilde{\gamma}) \leq \|F^{q(\alpha)} - T\| = \|T^{-1}F^{q(\alpha)} - \text{Id}\| = \|f^{q(\alpha)} - \text{Id}\|.$$

The last equality can be justified by Lemma 2.1, just as in the proof of Lemma 3.1. Thus by Theorem 3 and $K_{r+1} \geq K_1$, we have

$$\begin{aligned} d_H(F^{q(\alpha)}(\tilde{\gamma}), T\tilde{\gamma}) &\leq A_0(\|\rho(f^{q(\alpha)})\|_{\mathbb{R}/\mathbb{Z}}, K_1^{q(\alpha)}) \\ (3.21) \quad &\leq A_0(\alpha\mathcal{G}(\alpha), K_{r+1}^{q(\alpha)}). \end{aligned}$$

Then along with (3.17), (3.14), and the hypothesis $\alpha \in \mathcal{L}_r(K_{r+1})$, we have

$$\begin{aligned} d_H(F^{q(\alpha)^{-1}}(\tilde{\gamma}), T\tilde{\gamma}) &\geq d_H(F^{q(\alpha)}(\tilde{\gamma}), F^{q(\alpha)^{-1}}(\tilde{\gamma})) - d_H(F^{q(\alpha)}(\tilde{\gamma}), T\tilde{\gamma}) \\ &\geq K_{r+1}^{-q(\alpha)} C'_r(\alpha^{-1}, K_{r+1})^{-1} - A_0(\alpha\mathcal{G}(\alpha), K_{r+1}^{q(\alpha)}) > 0. \end{aligned}$$

However this contradicts (3.20). Thus we have (3.19). \square

4. SMOOTH DOMAIN BOUNDED BY GOOD CURVES

4.1. Renormalization of pseudo-rotations. Given a C^∞ pseudo-rotation f on \mathbb{A} and an integer $n \geq 1$, we denote by F_n the unique lift of f^n to $\tilde{\mathbb{A}}$ such that $\rho(F_n) \in (0, 1)$. Given a smooth Brouwer curve γ_n for f^n , we let $\tilde{\gamma}_n$ be an arbitrary lift of γ_n to $\tilde{\mathbb{A}}$. We let Ω_n be the unique closed region in $\tilde{\mathbb{A}}$ bounded by $(\tilde{\gamma}_n, F_n(\tilde{\gamma}_n))$.

Given a C^∞ admissible coordinate $H : [0, 1]^2 \rightarrow \Omega_n$ for (Ω_n, F_n) (see Remark 2), we can uniquely extend H to a C^∞ diffeomorphism of $\tilde{\mathbb{A}}$ with constant Jacobian, denoted again by H , satisfying

$$(4.1) \quad HT = F_n H.$$

We notice that although the C^r norms of F_n and T are uniformly bounded throughout $\tilde{\mathbb{A}}$, the C^r norm of H need not be uniformly bounded. However it is clear from (4.1) that for any integer $L > 0$, the norm $\|D^r H\|_{[-L, L] \times [0, 1]}$ is bounded in terms of L and $\|F_n\|_{\text{Diff}^r(\tilde{\mathbb{A}})}$.

Denote $\alpha = \rho(f) \in (0, 1) \setminus \mathbb{Q}$. We have $\rho(F_1) = \alpha$. We abbreviate

$$F^{a,b} := T^b F_1^a \quad \forall a, b \in \mathbb{Z}.$$

By our previous definition, we have $F_n = F^{n, -\lfloor n\alpha \rfloor}$. Denote by $J : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ the orientation reversing diffeomorphism

$$J(x, y) = (-x, y).$$

Notice that by (4.1) we have

$$(4.2) \quad JH^{-1}F_n HJ = JTJ = T^{-1}.$$

We set

$$(4.3) \quad F_H^{a,b} := JH^{-1}F^{a,b}HJ \quad \forall a, b \in \mathbb{Z}.$$

Since $F^{a,b}$ is ω -preserving and commutes with F_n , and since H has constant Jacobian, we deduce from (4.2) that $F_H^{a,b}$ is also ω -preserving, and commutes with T . Consequently $F_H^{a,b}$ descends to an ω -preserving C^∞ diffeomorphism $f_H^{a,b} : \mathbb{A} \rightarrow \mathbb{A}$.

We have the following lemma.

LEMMA 4.1. *There is a sequence of increasing functions $\{E_r : \mathbb{N}^2 \times \mathbb{R}_+^2 \rightarrow \mathbb{R}_+\}_{r \geq 1}$ such that the following is true. Let f be a $\{K_r\}_{r \geq 1}$ -smooth ω -preserving pseudo-rotation with $\rho(f) = \alpha$; let $n \neq 0$ be an integer; let Ω_n be the region bounded by $(\tilde{\gamma}_n, F_n(\tilde{\gamma}_n))$ where $\tilde{\gamma}_n$ is some lift of a Brouwer curve for f^n . Assume that (Ω_n, F_n) admits a (r, L_r) -admissible coordinate H . Then for any $a, b \in \mathbb{Z}$ with $a \lfloor n\alpha \rfloor + bn \neq 0$, $f_H^{a,b}$ is a pseudo-rotation in $\text{Diff}^\infty(\mathbb{A}, \omega)$ with*

$$\rho(f_H^{a,b}) = \left\{ \frac{-a\alpha - b}{\lfloor n\alpha \rfloor} \right\}.$$

Moreover, we have

$$\|f_H^{a,b}\|_{\text{Diff}^r} \leq E_r(|a|, |b|, K_r, L_r).$$

Proof. We first show that $f_H^{a,b}$ has no periodic points. Assume to the contrary there are integers $p \in \mathbb{Z}$, $q > 0$ and some $z \in \tilde{\mathbb{A}}$ such that

$$(F_H^{a,b})^q(z) = T^p(z).$$

Then by (4.2) and (4.3) we have

$$(4.4) \quad z = (F_H^{a,b})^q T^{-p}(z) = JH^{-1}F_1^{(qa+pn)}T^{(qb-p\lfloor n\alpha \rfloor)}HJ(z).$$

However, (4.4) and the condition $a \lfloor n\alpha \rfloor + bn \neq 0$ implies that $HJ(z)$ descends to a periodic point for f , which contradicts the hypothesis that f is a pseudo-rotation. We conclude that $f_H^{a,b}$ is a pseudo-rotation in $\text{Diff}^\infty(\mathbb{A}, \omega)$.

To compute $\rho(f_H^{a,b})$, it suffices to study the trajectory of an arbitrary $z \in \tilde{\mathbb{A}}$ under the iterates of $(F_H^{a,b})^q T^{-p}$ using the second equality in (4.4); for example a point on the boundary. This standard argument is left to the readers.

For the C^r -norm of $f_H^{a,b}$, it suffices to control the C^r -norm of $F_H^{a,b}$ restricted to $(-1, 2) \times [0, 1]$. By (4.3), $\|D^r F_H^{a,b}\|_{(-1,2) \times [0,1]}$ depends only on the C^r -norms of $F^{a,b}$, and the C^r -norm of J and H on

$$F_H^{a,b}((-1, 2) \times [0, 1]) \cup (-1, 2) \times [0, 1].$$

This yields a bound depending only on $r, |a|, |b|, K_r, L_r$ as required. \square

4.2. Finding a good curve. In the following statement we consider $n \in \mathbb{N}$ even, so that $q_n \alpha - p_n > 0$.

PROPOSITION 4.1. *There is a sequence of increasing functions $\{G_r : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+\}_{r \geq 1}$ such that the following is true.*

Let f be a pseudo-rotation in $\text{Diff}^\infty(\mathbb{A}, \omega)$, let F be a lift with $\rho(F) = \rho(f) = \alpha \in (0, 1) \setminus \mathbb{Q}$, and let $n \geq 2$ be an even integer. Suppose $\gamma \subset \mathbb{A}$ is a smooth Brouwer curve for f^{q_n} with a lift $\gamma^\dagger \subset \tilde{\mathbb{A}}$ such that the closed region Ω in $\tilde{\mathbb{A}}$ bounded by $(\gamma^\dagger, F^{q_n, -p_n}(\gamma^\dagger))$ admits an (r, K_r) -admissible coordinate

$$H : [0, 1]^2 \rightarrow \Omega$$

for some $r \geq 1$. Suppose further that $\gamma_n \subset \mathbb{A}$ is an a_{n+2} -good smooth curve for $f_H^{q_{n+1}, -p_{n+1}}$,² for which the closed region $\Omega_n \subset \mathbb{A}$ bounded by $(\gamma_n, f_H^{q_{n+1}, -p_{n+1}}(\gamma_n))$ admits an (r, M_r) -admissible coordinate. Then there exists a q_{n+2} -good smooth curve $\hat{\gamma} \subset \mathbb{A}$ for f , for which the closed region $\Omega_ \subset \mathbb{A}$ bounded by $(f^{q_{n+1}}(\hat{\gamma}), \hat{\gamma})$ admits a $(r, G_r(K_r, M_r))$ -admissible coordinate for $f^{q_{n+1}}$.*

Proof. We set

$$S_0 = F^{q_n, -p_n}, \quad S = F^{q_{n+1}, -p_{n+1}}.$$

As explained in Section 4.1, we extend H to a C^∞ -diffeomorphism of $\tilde{\mathbb{A}}$ with constant Jacobian by the formula

$$HT = S_0H.$$

By definition, we know that

- (1) $JH^{-1}S_0HJ = JTJ = T^{-1}$ on $\tilde{\mathbb{A}}$;
- (2) $\tilde{S} := JH^{-1}SHJ$ commutes with T on $\tilde{\mathbb{A}}$, and descends to $f_H^{q_{n+1}, -p_{n+1}}$.

Moreover, we have $\rho(\tilde{S}) = \rho(f_H^{q_{n+1}, -p_{n+1}}) = \alpha_{n+1} \in (0, \frac{1}{2})$.

Let $\tilde{\gamma}$ be an arbitrary lift of γ_n . As γ_n is a simple regular curve and is disjoint from $f_H^{q_{n+1}, -p_{n+1}}(\gamma_n)$, we know by item (1), (2) above that $\tilde{\gamma}$ is also a simple regular curve, and

$$(4.5) \quad T^{-1}(\tilde{\gamma}) \cap \tilde{\gamma} = \tilde{S}(\tilde{\gamma}) \cap \tilde{\gamma} = \emptyset.$$

We let $\gamma' = HJ(\tilde{\gamma})$. Then γ' is also a simple regular curve, and by (4.5) and item (1), (2) above, we have

$$(4.6) \quad S_0(\gamma') \cap \gamma' = S(\gamma') \cap \gamma' = \emptyset.$$

Moreover, we can see that

$$(4.7) \quad S(\gamma') < \gamma' < S_0(\gamma')$$

by considering their boundary points on B_0 and B_1 . We let $\hat{\gamma} = \pi(\gamma')$ (recall that π is the canonical projection from $\tilde{\mathbb{A}}$ to \mathbb{A}).

LEMMA 4.2. *The curve $\hat{\gamma}$ is a simple regular curve connecting B_0 and B_1 . Namely, it has no self-intersection.*

²Recall that by Lemma 4.1 we have $\rho(f_H^{q_{n+1}, -p_{n+1}}) = \alpha_{n+1}$ and $a_{n+2} = q(\alpha_{n+1})$.

Proof. Since $\hat{\gamma} = \pi(\gamma')$, and since γ' is a simple regular curve in $\tilde{\mathbb{A}}$ connecting B_0 to B_1 , it suffices to show that γ' is disjoint from all its translates $T^k(\gamma')$ for $k \in \mathbb{Z} \setminus \{0\}$. Since γ' connects the two boundary components it is enough to show γ' is disjoint from $T(\gamma')$.

By definition and by (2.9), we have $T = S_0^{q_{n+1}}S^{-q_n}$. Then by (4.7), we obtain that $\gamma' < T(\gamma')$. This completes the proof. \square

By item (2) above, the region $\tilde{\Omega}_n$ bounded by $(\tilde{\gamma}, \tilde{S}(\tilde{\gamma}))$ is a lift of Ω_n . Moreover, the (r, M_r) -admissible coordinate for Ω_n lifts to a (r, M_r) -admissible coordinate for $\tilde{\Omega}_n$. We notice that the push forward $HJ(\tilde{\Omega}_n)$ is the region in $\tilde{\mathbb{A}}$ between $S(\gamma')$ and γ' . Moreover, from the proof of Lemma 4.2, we see that

$$T^{-1}(\gamma') < S(\gamma') < \gamma'.$$

Thus the map π induces a diffeomorphism from $HJ(\tilde{\Omega}_n)$ to the region in \mathbb{A} between $f^{q_{n+1}}(\hat{\gamma})$ and $\hat{\gamma}$, that is, to the region Ω_* . We conclude that the (r, M_r) -admissible coordinate for $\tilde{\Omega}_n$, after composing with HJ and projecting, yield $(r, G_r(K_r, M_r))$ -admissible coordinates for Ω_* , for some functions G_r as in the proposition.

It remains to show that $\hat{\gamma}$ is a q_{n+2} -good curve. We divide the proof into two cases.

Case I: Assume that there are integers p and $0 < k < q_{n+2}$ such that $k\alpha + p < 0$ and

$$(4.8) \quad F^{k,p}(\gamma') \cap \gamma' \neq \emptyset.$$

We assume further that for any integers p' and $0 < k' < q_{n+2}$ such that $k'\alpha + p' < 0$ and $F^{k',p'}(\gamma') \cap \gamma' \neq \emptyset$, we have

$$k'\alpha + p' \leq k\alpha + p.$$

- First, we observe that either $q_{n+2} > k \geq q_{n+2} - q_n$ or

$$-\{q_n\alpha\} = p_n - q_n\alpha < k\alpha + p < 0.$$

Indeed, if $k < q_{n+2} - q_n$ and $p_n - q_n\alpha > k\alpha + p$, then we have $k + q_n < q_{n+2}$ and

$$k\alpha + p < (k + q_n)\alpha + p - p_n < 0.$$

In particular, the endpoints of $S_0 F^{k,p}(\gamma')$ are on the left hand side of those of γ' . Then by (4.8) and $S_0 F^{k,p}(\gamma') > F^{k,p}(\gamma')$, we have that

$$F^{k+q_n, -p_n+p}(\gamma') \cap \gamma' \neq \emptyset.$$

This contradicts the choice of k .

- If $q_{n+2} > k \geq q_{n+2} - q_n$, then by (4.6), (4.8) and (2.7), we know that $k > q_{n+1}$. We notice that

$$k\alpha + p < (k - q_{n+1})\alpha + p + p_{n+1} < 0.$$

In particular, the endpoints of $S^{-1}F^{k,p}(\gamma')$ are on the left hand side of those of γ' . Then by (4.8) and $S^{-1}F^{k,p}(\gamma') > F^{k,p}(\gamma')$, we have that

$$F^{k-q_{n+1}, p+p_{n+1}}(\gamma') \cap \gamma' \neq \emptyset.$$

This contradicts the choice of k .

- If $-\{q_n\alpha\} < k\alpha + p < 0$, then we have

$$k \in \{q_{n+1}, \dots, a_{n+2}q_{n+1}\}.$$

If $k = iq_{n+1}$ with $i > 1$, then we have

$$-1 < i(q_{n+1}\alpha - p_{n+1}) < (i-1)(q_{n+1}\alpha - p_{n+1}) = (k - q_{n+1})\alpha - (i-1)p_{n+1} < 0.$$

Then we must have $p = -ip_{n+1}$, and the endpoints of $S^{-1}F^{k,p}(\gamma')$ are on the left hand side of those of γ' . Then by (4.8) and $S^{-1}F^{k,p}(\gamma') > F^{k,p}(\gamma')$, we have that

$$F^{k-q_{n+1}, -(i-1)p_{n+1}}(\gamma') \cap \gamma' \neq \emptyset.$$

If $k = q_{n+1}$, we would have

$$F^{q_{n+1}}(\gamma') \cap (\gamma' + \mathbb{Z}) = \emptyset.$$

Both cases contradict the choice of k .

Case II: Assume that there are integers p and $0 < k < q_{n+2}$ such that $k\alpha + p > 0$ and

$$(4.9) \quad F^{k,p}(\gamma') \cap \gamma' \neq \emptyset.$$

We assume further that for any integers p' and $0 < k' < q_{n+2}$ such that $k'\alpha + p' > 0$ and $F^{k',p'}(\gamma') \cap \gamma' \neq \emptyset$, we have

$$k'\alpha + p' \geq k\alpha + p.$$

- First, we observe that either $q_n \geq k > 0$ or

$$\{q_n\alpha\} = q_n\alpha - p_n > k\alpha + p > 0.$$

Indeed, if $k > q_n$ and $q_n\alpha - p_n < k\alpha + p$, then we have $q_{n+2} > k - q_n > 0$ and

$$k\alpha + p > (k - q_n)\alpha + p + p_n > 0.$$

In particular, the endpoints of $S_0^{-1}F^{k,p}(\gamma')$ are on the right hand side of those of γ' . Then by (4.9) and $F^{k-q_n, p_n+p}(\gamma') = S_0^{-1}F^{k,p}(\gamma') < F^{k,p}(\gamma')$, we have that

$$F^{k-q_n, p_n+p}(\gamma') \cap \gamma' \neq \emptyset.$$

This contradicts the choice of k .

- If $q_n \geq k > 0$, then by (4.6) and (4.9), we have $q_n > k$ and consequently $k + q_{n+1} < q_{n+2}$. Moreover we notice that

$$k\alpha + p > (k + q_{n+1})\alpha + p - p_{n+1} > 0.$$

In particular, the endpoints of $SF^{k,p}(\gamma')$ are on the right hand side of those of γ' . By (4.9) and $SF^{k,p}(\gamma') < F^{k,p}(\gamma')$, we have that

$$F^{k+q_n, p-p_n}(\gamma') \cap \gamma' \neq \emptyset.$$

This contradicts the choice of k .

• If $\{q_n \alpha\} > k\alpha + p > 0$ (and $0 < k < q_{n+2}$), then it is straightforward to verify that

$$(k, p) \in \{(q_n + iq_{n+1}, p_n + ip_{n+1}) \mid 0 < i < a_{n+2}\}.$$

By the hypothesis that γ_n is a a_{n+2} -good curve, we have

$$T^{-1}\tilde{S}^i(\tilde{\gamma}) \cap \tilde{\gamma} = \emptyset \quad \forall 1 \leq i < a_{n+2}.$$

By $T^{-1} = JH^{-1}S_0HJ$, $\tilde{S} = JH^{-1}SHJ$ and $\gamma' = HJ(\tilde{\gamma})$, we obtain

$$F^{q_n+iq_{n+1}, -p_n-ip_{n+1}}(\gamma') \cap \gamma' = S_0S^i(\gamma') \cap \gamma' = \emptyset$$

for all integer $0 < i < a_{n+2}$. This again gives a contradiction.

In summary, we see that for all integer $0 < i < q_{n+2}$,

$$F^i(\gamma') \cap (\gamma' + \mathbb{Z}) = \emptyset.$$

Hence $\hat{\gamma}$ is q_{n+2} -good. This completes the proof of Proposition 4.1. \square

We can now prove the following:

COROLLARY B. *For each integer $r \geq 1$, there exist increasing functions $P_r : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ and $W_r : \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that the following is true. Suppose that there is an odd integer $n \geq 3$ such that*

$$(4.10) \quad q_n > P_r(K_{r+2}, q_{n-1}), \quad q_{n+1} > P_r(K_{r+2}, q_n) \quad \text{and} \quad q_{n+2} > P_r(K_{r+2}, q_{n+1})$$

where $\{q_k\}_{k \geq 0}$ is the sequence of denominators associated to some $\alpha \in (0, 1) \setminus \mathbb{Q}$. Then for any $\{K_k\}_{k \geq 1}$ -smooth pseudo-rotation $f \in \text{Diff}^\infty(\mathbb{A}, \omega)$ with $\rho(f) = \alpha$ has a q_{n+1} -good smooth curve γ such that the closed region in \mathbb{A} bounded by $(f^{q_n}(\gamma), \gamma)$ admits a $(r, W_r(q_{n+1}, K_{r+2}))$ -admissible coordinate for f^{-q_n} .

Proof. Fix $r \geq 1$. Let $\{L_k : \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+\}_{k \geq 1}$ be a sequence of increasing functions independent of f such that for each integer $m \geq 0$, f^m is $\{L_k(m, K_k)\}_{k \geq 1}$ -smooth. Let us abbreviate

$$L'_k := L_k(q_{n-1}, K_k) \quad \forall k \geq 1.$$

For each $n \geq 2$ we have $\rho(f^{q_{n-1}}) = \beta_{n-1} \in (0, 1/2)$. If (4.10) holds and P_r is chosen appropriately, then

$$(4.11) \quad \beta_{n-1} \in \mathcal{L}(L'_1).$$

This allows us to apply Lemma 3.2 to $f^{q_{n-1}}$. By Lemma 3.2, there is a smooth Brouwer curve γ for $f^{q_{n-1}}$ for which the closed region $\mathcal{R}'_{n-1} \subset \mathbb{A}$ bounded

by $(\gamma, f^{q_{n-1}}(\gamma))$ admits a $(r+1, Y_{r+1})$ -admissible coordinate $H' : [0, 1]^2 \rightarrow \mathcal{R}'_{n-1}$ where

$$(4.12) \quad Y_{r+1} := C'_{r+1}(\beta_{n-1}^{-1}, L'_{r+2}).$$

We lift H' to an admissible coordinate $H : [0, 1]^2 \rightarrow \mathcal{R}_{n-1}$ where \mathcal{R}_{n-1} is a lift of \mathcal{R}'_{n-1} in $\tilde{\mathbb{A}}$. More precisely, \mathcal{R}_{n-1} is a connected component of $\pi^{-1}(\mathcal{R}'_{n-1})$. As in Section 4.1, we extend H to a diffeomorphism of $\tilde{\mathbb{A}}$ satisfying

$$HT = F^{q_{n-1}, -p_{n-1}}H$$

and set $f_n := f_H^{q_n, -p_n} : \mathbb{A} \rightarrow \mathbb{A}$. That is, f_n is the projection of $F_H^{q_n, -p_n} := (HJ)^{-1}F^{q_n, -p_n}(HJ) : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$. By Lemma 4.1, f_n is a pseudo-rotation in $\text{Diff}^\infty(\mathbb{A}, \omega)$ with $\rho(f_n) = \beta_n / \beta_{n-1} = \alpha_n$ such that

$$(4.13) \quad \|f_n\|_{\text{Diff}^{r+1}} \leq E_{r+1}(q_n, p_n, K_{r+1}, Y_{r+1}).$$

By (4.12), and using that

$$K_{r+1} \leq K_{r+2}, \quad \max(p_n, \beta_{n-1}^{-1}) \leq 2q_n$$

we can rewrite (4.13) as

$$(4.14) \quad \|f_n\|_{\text{Diff}^{r+1}} < V_{r+1}(q_n, K_{r+2})$$

where $V_{r+1} : \mathbb{N} \times \mathbb{R}_+ \rightarrow (1, \infty)$ is an increasing function independent of f .

Now we choose $P_r : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ to be any increasing function that is sufficiently large that the following conditions are fulfilled:

$$(4.15) \quad P_r(D, y) > 2y\rho_*(V_{r+1}(y, D))^{-1}$$

$$(4.16) \quad \frac{2y}{P_r(D, y)} < C'' \left(\frac{1}{2y}, V_{r+1}(y, D)^{-1} \right),$$

for all $D, y > 0$, where ρ_* is the decreasing function introduced in (3.1). We moreover choose P_r sufficiently large that whenever $x > P_r(D, y)$ there holds

$$(4.17) \quad A_0 \left(\frac{2y}{x}, V_{r+1}(y, D)^{2y} \right) < V_{r+1}(y, D)^{-2y} C'_r(2y, V_{r+1}(y, D))^{-1}$$

for all $D, y > 0$. This we can arrange because $\lim_{t \rightarrow 0} A_0(t, \cdot) = 0$.

Assume now that (4.10) holds for a fixed odd integer $n \geq 3$. By (4.15) and then the second inequality in (4.10), we have

$$2q_n < \rho_*(V_{r+1}(q_n, K_{r+2}))P_r(K_{r+2}, q_n) < \rho_*(V_{r+1}(q_n, K_{r+2}))q_{n+1}$$

and therefore by (2.12)

$$(4.18) \quad \alpha_n < \frac{2q_n}{q_{n+1}} < \rho_*(V_{r+1}(q_n, K_{r+2})).$$

Also, by (2.12), the third inequality in (4.10), and (4.16), we obtain

$$\begin{aligned}
(4.19) \quad \mathcal{G}(\alpha_n) = \alpha_{n+1} &< \frac{2q_{n+1}}{q_{n+2}} < \frac{2q_{n+1}}{P_r(K_{r+2}, q_{n+1})} \\
&< C'' \left(\frac{1}{2q_{n+1}}, V_{r+1}(q_{n+1}, K_{r+2})^{-1} \right) \\
(4.20) \quad &< C''(\alpha_n, V_{r+1}(q_n, K_{r+2})^{-1})
\end{aligned}$$

where the last inequality uses the monotonicity of C'' . We claim that

$$(4.21) \quad \rho(f_n) = \alpha_n \in \mathcal{L}_r(V_{r+1}(q_n, K_{r+2}))$$

where \mathcal{L}_r is as in Definition 3.2. Equivalently, we show that

$$(4.22) \quad A_0 \left(\alpha_n \mathcal{G}(\alpha_n), V_{r+1}(q_n, K_{r+2})^{q(\alpha_n)} \right) < V_{r+1}(q_n, K_{r+2})^{-q(\alpha_n)} C'_r \left(\alpha_n^{-1}, V_{r+1}(q_n, K_{r+2}) \right)^{-1}.$$

Using (2.12) we have $\alpha_n \mathcal{G}(\alpha_n) = \alpha_n \alpha_{n+1} < 4q_n/q_{n+2} \leq 2q_{n+1}/q_{n+2}$ and $q(\alpha_n) \leq 2q_{n+1}$ and $\alpha_n^{-1} < 2q_{n+1}/q_n \leq 2q_{n+1}$. Therefore by the monotonicity of A_0 , and by (4.17) together with the third inequality in (4.10), we obtain

$$\begin{aligned}
A_0 \left(\alpha_n \mathcal{G}(\alpha_n), V_{r+1}(q_n, K_{r+2})^{q(\alpha_n)} \right) &< A_0 \left(\frac{2q_{n+1}}{q_{n+2}}, V_{r+1}(q_{n+1}, K_{r+2})^{2q_{n+1}} \right) \\
&< V_{r+1}(q_{n+1}, K_{r+2})^{-2q_{n+1}} C'_r(2q_{n+1}, V_{r+1}(q_{n+1}, K_{r+2}))^{-1}.
\end{aligned}$$

Now inequality (4.22) follows from the monotonicity of V_{r+1} and C'_r , and because $\alpha_n^{-1} < 2q_{n+1}/q_n \leq 2q_{n+1}$ from (2.12). This proves (4.21).

Combining (4.21) with (4.14) we see that f_n satisfies the hypotheses of Proposition 3.1, and therefore f_n has a $q(\rho(f_n))$ -good Brouwer curve γ_n say. By (2.8) $q(\rho(f_n)) = q(\alpha_n) = a_{n+1}$. Thus γ_n is an a_{n+1} -good Brouwer curve for f_n . Moreover, by Lemma 3.2, the region in \mathbb{A} bounded by $(\gamma_n, f_n(\gamma_n))$ admits an $(r, C'_r(\alpha_n^{-1}, \hat{K}_{r+1}))$ -admissible coordinate, provided $\hat{K}_{r+1} \geq \|f_n\|_{\text{Diff}^{r+1}}$ and $\rho(f_n) = \alpha_n \in \mathcal{L}(\hat{K})$ where $\hat{K} \geq \|f_n\|_{\text{Diff}^1}$. By (4.21) and (4.14) we may take $\hat{K}_{r+1} = \hat{K} = V_{r+1}(q_n, K_{r+2})$ and conclude that the region in \mathbb{A} bounded by $(\gamma_n, f_n(\gamma_n))$ admits a (r, U_r) -admissible coordinate where

$$U_r := C'_r(\alpha_n^{-1}, V_{r+1}(q_n, K_{r+2})).$$

We define W_r by

$$W_r(q, K) := G_r(C'_r(2q, L_{r+1}(q, K)), C'_r(2q, V_{r+1}(q, K))).$$

By (2.11) and (2.12), we have $\beta_{n-1}^{-1}, \alpha_n^{-1} < 2q_{n+1}$. Then by $K_{r+2} \geq K_{r+1}$, we have

$$\begin{aligned}
&C'_r(2q_{n+1}, L_{r+1}(q_{n+1}, K_{r+2})) > Y_r \\
\text{and } &C'_r(2q_{n+1}, V_{r+1}(q_{n+1}, K_{r+2})) > U_r.
\end{aligned}$$

Thus we have

$$W_r(q_{n+1}, K_{r+2}) \geq G_r(Y_r, U_r).$$

Then by Proposition 4.1, there is a q_{n+1} -good smooth curve of f , denoted by $\hat{\gamma}$, such that the region in \mathbb{A} bounded by $(f^{q_n}(\hat{\gamma}), \hat{\gamma})$ admits a $(r, W_r(q_{n+1}, K_{r+2}))$ -admissible coordinate for f^{-q_n} . \square

5. CONSTRUCTION OF APPROXIMANTS

This section is mostly occupied by the proof of the following theorem, from which the main result of this paper, Theorem 1, will then follow easily.

THEOREM 4. *For each $(r_0, M, \epsilon) \in \mathbb{Z}_{\geq 2} \times \mathbb{N} \times (0, 1]$, there is an increasing function*

$$P = P_{r_0, \epsilon, M} : \mathbb{N} \rightarrow \mathbb{R}_+$$

so that for any pseudo-rotation $f \in \text{Diff}^\infty(\mathbb{A}, \omega)$ with

$$(5.1) \quad \|f\|_{\text{Diff}^{r_0+2}(\mathbb{A})} < M,$$

whose rotation number $\rho(f) = \alpha \in (0, 1) \setminus \mathbb{Q}$ satisfies the property that there exists an odd integer $n \geq 3$ for which

$$(5.2) \quad q_n > P(q_{n-1}), \quad q_{n+1} > P(q_n) \quad \text{and} \quad q_{n+2} > P(q_{n+1}),$$

then there exists $h_0 \in \text{Diff}^\infty(\mathbb{A}, \omega)$ with

$$(5.3) \quad d_{\text{Diff}^{r_0-1}(\mathbb{A})}(h_0 R_\alpha h_0^{-1}, f) < \epsilon.$$

Proof. Fix some (r_0, M, ϵ) in $\mathbb{Z}_{\geq 2} \times \mathbb{N} \times (0, 1]$. Let $f \in \text{Diff}^\infty(\mathbb{A}, \omega)$ denote a pseudo-rotation and set $\alpha = \rho(f)$.

It will be convenient to use the following notation: for an increasing function $S : \mathbb{N} \rightarrow \mathbb{N}$ we define

$$(5.4) \quad \mathcal{C}(S) := \left\{ \theta \in (0, 1) \setminus \mathbb{Q} \mid \exists n \in \mathbb{N} \text{ odd, so that} \right. \\ \left. q_n(\theta) > S(q_{n-1}(\theta)), q_{n+1}(\theta) > S(q_n(\theta)), q_{n+2}(\theta) > S(q_{n+1}(\theta)) \right\}$$

where $\{p_n(\theta)/q_n(\theta)\}_{n \geq 0}$ is the sequence of continued fractions of θ introduced in Section 2. Our successive restrictions on the rotation number α will take the form:

$$(5.5) \quad \alpha \in \mathcal{C}(S_i)$$

for a finite collection of functions S_1, S_2, \dots to be determined. We will set

$$P := \max_i S_i$$

and then when $\alpha \in \mathcal{C}(P)$, all conditions in (5.5) will be met.

For our first condition on the rotation number, set

$$(5.6) \quad S_1 := P_{r_0}(M, \cdot)$$

where $P_{r_0} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is defined in Corollary B. From now on we assume

$$\alpha \in \mathcal{C}(S_1).$$

Fix any odd $n \geq 3$ for which

$$(5.7) \quad q_n > S_1(q_{n-1}), \quad q_{n+1} > S_1(q_n) \quad \text{and} \quad q_{n+2} > S_1(q_{n+1}).$$

Then by Corollary B, f has a q_{n+1} -good Brouwer curve $\gamma \subset \mathbb{A}$ for which the closed region

$$\mathcal{D}_0 \subset \mathbb{A}$$

bounded by $(f^{q_n}(\gamma), \gamma)$ admits a (r_0, \hat{W}_{r_0}) -admissible coordinate for f^{-q_n} , where

$$\hat{W}_{r_0} := W_{r_0}(q_{n+1}, K_{r_0+2}).$$

Recall that γ is q_{n+1} -good for f means that the curves $f^i(\gamma)$, $0 \leq i \leq q_{n+1} - 1$ are mutually disjoint. Then it is clear that the curves

$$(5.8) \quad f^{q_n}(\gamma), \dots, f^{q_n+q_{n+1}-1}(\gamma)$$

are mutually disjoint as well. In particular $f^{q_{n+1}}(\gamma)$ and $f^{q_n}(\gamma)$ are disjoint and so we can consider the closed region

$$\mathcal{D} \subset \mathbb{A}$$

bounded by $(f^{q_n}(\gamma), f^{q_{n+1}}(\gamma))$.

CLAIM 1. *The annulus \mathbb{A} is covered by the regions*

$$\mathcal{D}, f(\mathcal{D}), \dots, f^{q_n-1}(\mathcal{D}), f^{q_n}(\mathcal{D}_0), \dots, f^{q_{n+1}-1}(\mathcal{D}_0)$$

with mutually disjoint interiors. Moreover, the intersection of any two such neighboring regions equals one of the curves in (5.8).

Proof. The restriction of f to the boundary circle B_0 is an orientation preserving homeomorphism with rotation number α . It is then a well-known fact that for any fixed $x_0 \in B_0$ the intervals

$$\begin{aligned} & [f^{q_n+i}(x_0), f^{q_{n+1}+i}(x_0)], \quad 0 \leq i \leq q_n - 1, \\ \text{and} \quad & [f^{q_n+j}(x_0), f^j(x_0)], \quad q_n \leq j \leq q_{n+1} - 1 \end{aligned}$$

together form a covering for B_0 with mutually disjoint interiors. In particular we may take $x_0 := \gamma \cap B_0$. An analogous statement holds for the other boundary component B_1 . The claim then follows from the disjointness of the curves in (5.8) and our definition of simple regular curve. \square

Fix a lift $\tilde{\gamma}$ of γ to $\tilde{\mathbb{A}}$. Let $F : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ be the unique lift of f satisfying $\rho(F) = \rho(f) \in (0, 1)$. Recall the notation from section 4.1 that $F^{a,b} = T^b F^a$ for all $a, b \in \mathbb{Z}$. We define $\tilde{\mathcal{D}}_0 \subset \tilde{\mathbb{A}}$ to be the region bounded by

$$(F^{q_n, -p_n}(\tilde{\gamma}), \tilde{\gamma})$$

and $\tilde{\mathcal{D}} \subset \tilde{\mathbb{A}}$ to be the region bounded by

$$(F^{q_n, -p_n}(\tilde{\gamma}), F^{q_{n+1}, -p_{n+1}}(\tilde{\gamma})).$$

We claim that $T^{-p_n} F^{q_n}(\tilde{\gamma})$ is contained in the fundamental domain of π bounded by $T^{-1}\tilde{\gamma}$ and $\tilde{\gamma}$. Indeed, this follows from comparing the order of

their endpoints, and by the disjointness of their projections $f^{q_n}(\gamma)$ and γ . Thus π restricts to a diffeomorphism from $\tilde{\mathcal{D}}_0$ to \mathcal{D}_0 . Similarly, π restricts to a diffeomorphism from $\tilde{\mathcal{D}}$ to \mathcal{D} as well.

The following q_{n+1} regions have disjoint interiors:

$$(5.9) \quad \tilde{\mathcal{B}}_i := \begin{cases} F^i(\tilde{\mathcal{D}}) & 0 \leq i \leq q_n - 1, \\ F^i(\tilde{\mathcal{D}}_0) & q_n \leq i \leq q_{n+1} - 1. \end{cases}$$

By Claim 1, $\tilde{\mathbb{A}}$ is covered by the union of the following regions with mutually disjoint interiors:

$$(5.10) \quad T^{-k}\tilde{\mathcal{B}}_j, \quad k \in \mathbb{Z}, \quad 0 \leq j \leq q_{n+1} - 1.$$

The intersection of any two neighboring regions in (5.10) therefore equals to one of the following curves

$$(5.11) \quad T^{-k}F^j(\tilde{\gamma}), \quad k \in \mathbb{Z}, \quad q_n \leq j \leq q_n + q_{n+1} - 1.$$

We denote by Γ the union of the curves in (5.11). The region

$$(5.12) \quad U_0 := \bigcup_{i=0}^{q_{n+1}-1} \text{int}(\tilde{\mathcal{B}}_i)$$

satisfies that:

- (1) $T^j(U_0) \cap T^k(U_0) = \emptyset$ for any $j \neq k \in \mathbb{Z}$;
- (2) $\cup_{k \in \mathbb{Z}} T^k(U_0) = \tilde{\mathbb{A}} \setminus \Gamma$.

By Corollary B, there is an admissible coordinate ϕ (see Definition 2.7) from a neighborhood of $[0, 1]^2$ in $\tilde{\mathbb{A}}$ to a neighborhood of \mathcal{D}_0 , satisfying (2.2)-(2.5) for $(f^{-q_n}, f^{q_n}(\gamma))$ in place of (f, γ) , and

$$\begin{aligned} \|\phi\|_{\text{Diff}^0} &< \hat{W}_{r_0}, \\ \phi \circ T(x) &= f^{-q_n} \circ \phi(x) \quad \forall x \in V \end{aligned}$$

where V is some neighborhood of $\{0\} \times [0, 1]$ in $\tilde{\mathbb{A}}$. As we have seen in Remark 2, there is no loss of information by regarding ϕ as a mapping from $[0, 1]^2$ to \mathcal{D}_0 . Since the restriction $\pi : \tilde{\mathcal{D}}_0 \rightarrow \mathcal{D}_0$ is a diffeomorphism we may set

$$\psi := \pi^{-1} \circ \phi$$

which therefore gives us a C^∞ -smooth diffeomorphism that extends to a C^∞ -smooth diffeomorphism with constant Jacobian from a neighborhood of $[0, 1]^2$ in $\tilde{\mathbb{A}}$ to a neighborhood of $\tilde{\mathcal{D}}_0$ satisfying

$$(5.13) \quad \|\psi\|_{\text{Diff}^0} < \hat{W}_{r_0},$$

$$(5.14) \quad \psi \circ T(x) = F^{-q_n, p_n} \circ \psi(x) \quad \forall x \in V.$$

There is a unique extension to a C^∞ -diffeomorphism with constant Jacobian to the whole strip $\tilde{\mathbb{A}}$, which we still denote by

$$\psi : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$$

and which satisfies

$$(5.15) \quad \psi \circ T = F^{-q_n p_n} \circ \psi.$$

To summarise thus far, we have proven the following:

LEMMA 5.1. *For each $(r_0, M) \in \mathbb{Z}_{\geq 2} \times \mathbb{N}$ there is an increasing function*

$$S_1 = S_1(r_0, M) : \mathbb{N} \rightarrow \mathbb{R}_+$$

so that for any pseudo-rotation $f \in \text{Diff}^\infty(\mathbb{A}, \omega)$ satisfying (5.1) whose rotation number $\rho(f) = \alpha$ satisfies the property that there exists an odd integer $n \geq 3$ for which

$$q_n > S_1(q_{n-1}), \quad q_{n+1} > S_1(q_n) \quad \text{and} \quad q_{n+2} > S_1(q_{n+1}),$$

there exists a q_{n+1} -good Brouwer curve $\gamma \subset \mathbb{A}$, so that for any lift $\tilde{\gamma} \in \tilde{\mathbb{A}}$, there exists a C^∞ -diffeomorphism,

$$\psi : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$$

with constant Jacobian which satisfies

$$(5.16) \quad \|\psi\|_{\text{Diff}^0([0,1]^2)} < \hat{W}_{r_0} := W_{r_0}(q_{n+1}, M)$$

where W_{r_0} is the function produced by Corollary B, and

$$(5.17) \quad \psi \circ T = F^{-q_n p_n} \circ \psi$$

on the whole of $\tilde{\mathbb{A}}$, where F is the unique lift of f with rotation number in $(0, 1)$. Moreover ψ maps $[0, 1]^2$ onto the region $\tilde{\mathcal{D}}_0$ bounded by $(\tilde{F}^{q_n p_n}(\tilde{\gamma}), \tilde{\gamma})$.

To continue our proof of Theorem 4 we require the following:

LEMMA 5.2. *For each $(r_0, M) \in \mathbb{Z}_{\geq 2} \times \mathbb{N}$, there is an increasing function*

$$Q_{r_0, M} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \quad \text{with} \quad \lim_{t \rightarrow 0} Q_{r_0, M}(t, \cdot) \equiv 0,$$

and there is an increasing function

$$S_2 = S_2(r_0, M) : \mathbb{N} \rightarrow \mathbb{R}_+$$

with $S_2 \geq S_1$, so that for any pseudo-rotation $f \in \text{Diff}^\infty(\mathbb{A}, \omega)$ satisfying (5.1) whose rotation number $\rho(f) = \alpha$ satisfies the property that there exists an odd integer $n \geq 3$ for which

$$q_n > S_1(q_{n-1}), \quad q_{n+1} > S_1(q_n) \quad \text{and} \quad q_{n+2} > S_1(q_{n+1}),$$

then the following holds for the unique lift $F : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ of f with $\rho(F) \in (0, 1)$: let γ be a q_{n+1} -good Brouwer curve for f given by Lemma 5.1³, and let $\tilde{\gamma}$ be any lift of γ . Let $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}_0$ be the regions in $\tilde{\mathbb{A}}$ bounded by

$$(F^{q_n p_n}(\tilde{\gamma}), F^{q_{n+1} p_{n+1}}(\tilde{\gamma})), \quad (F^{q_n p_n}(\tilde{\gamma}), \tilde{\gamma})$$

³Lemma 5.1 is applicable since by hypothesis $S_2 \geq S_1$.

respectively. Then there are open neighborhoods $\hat{\mathcal{D}}$ of $\tilde{\mathcal{D}}$ and $\hat{\mathcal{D}}_0$ of $\tilde{\mathcal{D}}_0$; a C^∞ -diffeomorphism

$$h : \hat{\mathcal{D}} \rightarrow \hat{\mathcal{D}}_0;$$

and neighborhoods U_L of $F^{q_n, -p_n}(\tilde{\gamma})$ and U_R of $F^{q_{n+1}, -p_{n+1}}(\tilde{\gamma})$, so that

$$(5.18) \quad h|_{U_L} = \text{Id}, \quad h|_{U_R} = F^{-q_{n+1}, p_{n+1}},$$

$$(5.19) \quad \|h - \text{Id}\|_{\text{Diff}^{r_0}(\hat{\mathcal{D}}, \hat{\mathcal{D}}_0)} < Q_{r_0, M}(\beta_{n+1}, q_{n+1}).$$

Proof. By Lemma 5.1 there is a diffeomorphism $\psi : \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}$ satisfying (5.16) and which maps $[0, 1]^2$ onto $\tilde{\mathcal{D}}_0$. Set

$$\gamma' := F^{q_{n+1}, -p_{n+1}}(\tilde{\gamma}).$$

By letting q_{n+2} be sufficiently large depending on q_{n+1} and M , we have

$$A_0(\beta_{n+1}, M^{q_{n+1}}) < 1/2$$

where A_0 is produced by Theorem 3. Then

$$\|f^{q_{n+1}} - \text{Id}\| \leq A_0(\|\rho(f^{q_{n+1}})\|_{\mathbb{R}/\mathbb{Z}}, M^{q_{n+1}}) = A_0(\beta_{n+1}, M^{q_{n+1}}) < 1/2.$$

Using Theorem 3, Lemma 2.1 and (2.11), we obtain

$$(5.20) \quad d_H(\tilde{\gamma}, \gamma') = d_H(f^{q_{n+1}}(\gamma), \gamma) < A_0(\beta_{n+1}, M^{q_{n+1}}).$$

Notice that by (5.16) and (5.17) we also have that $\|\psi\|_{\text{Diff}^{r_0}([0, 2] \times [0, 1])}$ is bounded in terms of q_{n+1}, M, r_0 . Then by (5.20) and by letting β_{n+1} be sufficiently large depending on q_{n+1}, M, r_0 , we have

$$\gamma' \subset \psi([1/2, 3/2] \times [0, 1]).$$

We define

$$\gamma'' = \psi^{-1}(\gamma').$$

Then

$$\{1/2\} \times [0, 1] \subset \gamma'' \subset \{3/2\} \times [0, 1]$$

if β_{n+1} is sufficiently large depending on q_{n+1}, M, r_0 . Denote by \mathcal{U}' the region in $\tilde{\mathcal{A}}$ bounded by $\{0\} \times [0, 1]$ and γ'' . It remains to construct a C^∞ -diffeomorphism

$$\varphi : \mathcal{U}' \rightarrow [0, 1]^2$$

such that φ equals Id near $\{0\} \times [0, 1]$; equals $\psi^{-1}F^{-q_{n+1}, p_{n+1}}\psi$ near γ'' ; and tends to Id in the C^{r_0} -topology as β_{n+1} tends to 0 for each fixed q_{n+1}, M, r_0 . Indeed, after the above φ is constructed, we can define $h = \varphi\psi^{-1}$.

By Theorem 3, for given q_{n+1}, M, r_0 , we see that $F^{-q_{n+1}, p_{n+1}}$ tends to Id in the C^{r_0} -topology as q_{n+2} tends to infinity. Thus it suffices to construct φ by smooth interpolation. \square

Continuing the proof of Theorem 4, we first construct a periodic approximation of f as follows. We define G on U_0 (see (5.12)) by

$$(5.21) \quad G|_U = \begin{cases} F & U = \cup_{i=0}^{q_n-2} \text{int}(\tilde{\mathcal{B}}_i), \\ F^{q_n} h F^{1-q_n} & U = \text{int}(\tilde{\mathcal{B}}_{q_n-1}), \\ F & U = \cup_{i=q_n}^{q_{n+1}-2} \text{int}(\tilde{\mathcal{B}}_i), \\ T^{p_{n+1}} h^{-1} F^{1-q_{n+1}} & U = \text{int}(\tilde{\mathcal{B}}_{q_{n+1}-1}). \end{cases}$$

By property (1),(2) below (5.12), we may extend G to a self-map of $\tilde{\mathbb{A}} \setminus \Gamma$, still denoted by G , satisfying

$$(5.22) \quad GT = TG.$$

Moreover, by (5.21) the following identity holds on $\tilde{\mathbb{A}} \setminus \Gamma$:

$$(5.23) \quad G^{q_{n+1}} = T^{p_{n+1}}.$$

By construction, for any two curves γ_a, γ_b in (5.11), there are integers i, j such that

$$(5.24) \quad \gamma_b = T^i G^j(\gamma_a).$$

By (5.21) in the definition of G , we also have that

$$(5.25) \quad \tilde{\mathcal{B}}_i = G^i(\tilde{\mathcal{B}}_0), \quad 0 \leq i \leq q_{n+1} - 1.$$

We denote

$$(5.26) \quad \tilde{\mathcal{B}}_{q_{n+1}} := G^{q_{n+1}}(\tilde{\mathcal{B}}_0) = T^{p_{n+1}}(\tilde{\mathcal{B}}_0).$$

Now we set

$$\phi := \psi^{-1} h$$

where ψ is in Lemma 5.1 and h is in Lemma 5.2.

LEMMA 5.3. *There is a neighborhood $V' \subset \tilde{\mathbb{A}}$ of $F^{q_{n+1}, -p_{n+1}}(\tilde{\gamma})$ so that G^{q_n} extends to a C^∞ map on V' , and*

$$(5.27) \quad T\phi(x) = \phi T^{-p_n} G^{q_n}(x), \quad x \in V'.$$

Proof. To see this, we first notice that

$$F^{q_{n+1}, -p_{n+1}}(\tilde{\gamma}) = \tilde{\mathcal{B}}_0 \cap T^{p_n - p_{n+1}}(\tilde{\mathcal{B}}_{q_{n+1} - q_n}),$$

and $\tilde{\mathcal{B}}_0 \cup T^{p_n - p_{n+1}}(\tilde{\mathcal{B}}_{q_{n+1} - q_n})$ is a neighborhood of $F^{q_{n+1}, -p_{n+1}}(\tilde{\gamma})$ in $\tilde{\mathbb{A}}$. Let U_L, U_R be given by Lemma 5.2. The set V' defined by

$$V' = F^{q_{n+1} - q_n, -p_{n+1} + p_n}(U_L) \cap U_R \cap (\tilde{\mathcal{B}}_0 \cup T^{p_n - p_{n+1}}(\tilde{\mathcal{B}}_{q_{n+1} - q_n}))$$

is a neighborhood of $F^{q_{n+1}, -p_{n+1}}(\tilde{\gamma})$.

By (5.21), (5.25) and (5.22), we have

$$(5.28) \quad G^{q_n}|_{\text{Int}(\tilde{\mathcal{B}}_0)} = F^{q_n} h,$$

$$(5.29) \quad G^{q_n}|_{T^{p_n - p_{n+1}}(\text{Int}(\tilde{\mathcal{B}}_{q_{n+1} - q_n}))} = T^{p_n} h^{-1} F^{q_n - q_{n+1}} T^{p_{n+1} - p_n}.$$

By Lemma 5.2, we see that

$$\text{RHS of (5.28)}|_{V'} = \text{RHS of (5.29)}|_{V'} = T^{p_{n+1}}F^{q_n - q_{n+1}}.$$

Then it is clear that G^{q_n} extends to a C^∞ map on V' . Again by Lemma 5.2, we have for any $x \in V'$

$$\begin{aligned} \phi T^{-p_n} G^{q_n}(x) &= \psi^{-1} h T^{p_{n+1} - p_n} F^{q_n - q_{n+1}}(x) = \psi^{-1} T^{p_{n+1} - p_n} F^{q_n - q_{n+1}}(x), \\ T\phi(x) &= T\psi^{-1} h(x) = T\psi^{-1} T^{-p_{n+1}} F^{q_{n+1}}(x). \end{aligned}$$

Thus (5.27) follows from (5.17). \square

We have the following corollary.

COROLLARY C. *The map G on $\tilde{\mathbb{A}} \setminus \Gamma$ extends to an element in $\text{Diff}^\infty(\tilde{\mathbb{A}})$.*

Proof. By (5.23), clearly $G^{q_{n+1}}$ extends to a smooth map $T^{p_{n+1}}$ on $\tilde{\mathbb{A}}$. By Lemma 5.3, (5.22) and (5.24), we see that G^{q_n} also extends to a map in $\text{Diff}^\infty(\tilde{\mathbb{A}})$. To conclude the proof it suffices to notice that

$$G = (G^{q_n})^{-p_{n+1}} (G^{q_{n+1}})^{p_n}.$$

\square

By a slight abuse of notation, we again write $G \in \text{Diff}^\infty(\tilde{\mathbb{A}})$ for the extension provided in Corollary C. Clearly (5.22) continues to hold. Consequently G descends to a periodic diffeomorphism $g \in \text{Diff}^\infty(\mathbb{A})$ satisfying

$$g^{q_{n+1}} = \text{Id}.$$

We define map $\tilde{H} : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ by

$$(5.30) \quad \tilde{H}(x) := T^{-kq_{n+1} + jp_{n+1}} \phi G^{-j} T^k(x), \quad x \in T^{-k} \tilde{\mathcal{B}}_j$$

for all $k \in \mathbb{Z}$ and all $0 \leq j \leq q_{n+1} - 1$. Then by (5.23), (5.25), (5.26) and (5.30) we have

$$(5.31) \quad \tilde{H}T = T^{q_{n+1}}\tilde{H}, \quad \tilde{H}G = T^{p_{n+1}}\tilde{H}.$$

We set $H_1 := D_{q_{n+1}}\tilde{H}$, where

$$D_\lambda(x, y) = (\lambda^{-1}x, y), \quad \lambda \in \mathbb{R} \setminus \{0\}.$$

Notice that for every $\lambda \in \mathbb{R} \setminus \{0\}$ and every $p \in \mathbb{R}$ we have

$$D_\lambda T_p D_\lambda^{-1} = T_{\frac{p}{\lambda}}$$

where for each $c \in \mathbb{R}$, $T_c : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ denotes the map $T_c(x, y) = (x + c, y)$. Then by (5.31), we see that $H_1 : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ satisfies

$$(5.32) \quad TH_1 = H_1T, \quad H_1G = T_{p_{n+1}/q_{n+1}}H_1.$$

We now show that $\tilde{H} \in \text{Diff}^\infty(\tilde{\mathbb{A}})$. By construction, \tilde{H} is C^∞ -smooth in the interior of each $T^{-k}\tilde{\mathcal{B}}_j$, $k \in \mathbb{Z}$, $0 \leq j \leq q_{n+1} - 1$. Thus it remains to show that \tilde{H} is C^∞ in a neighborhood of each of the curves in (5.11).

By (5.9), we can see that the set

$$(5.33) \quad \tilde{\mathcal{B}}_* := \tilde{\mathcal{B}}_0 \cup T^{-p_n} \tilde{\mathcal{B}}_{q_n}$$

is a neighborhood of $T^{-p_n} F^{q_n}(\tilde{\gamma})$. By (5.24), we deduce that every curve in (5.11) has a neighborhood of the form $T^i G^j(\tilde{\mathcal{B}}_*)$. By (5.31), it remains to verify that \tilde{H} is C^∞ over $\tilde{\mathcal{B}}_*$. This follows from a similar argument as in the proof of Lemma 5.3 using (5.27) and (5.30). Consequently \tilde{H} and H_1 are C^∞ -smooth diffeomorphisms of $\tilde{\mathbb{A}}$. Then by (5.32) H_1 descends to a C^∞ -diffeomorphism $h_1 : \mathbb{A} \rightarrow \mathbb{A}$ satisfying

$$(5.34) \quad g = h_1^{-1} R_{p_{n+1}/q_{n+1}} h_1.$$

Moreover, we have the claim that:

$\|H_1\|_{\text{Diff}^0(\tilde{\mathbb{A}})}$, and hence $\|h_1\|_{\text{Diff}^0(\mathbb{A})}$ as well, are bounded in terms of q_{n+1}, M, r_0 .

Indeed, by definition it is clear that $\|\phi\|_{\text{Diff}^0}$ is bounded in terms of q_{n+1}, M, r_0 ; and for each integer $0 \leq j \leq q_{n+1} - 1$, $\|G\|_{\text{Diff}^0(\tilde{\mathcal{B}}_j, \tilde{\mathcal{B}}_{j+1})}$ is bounded in terms of q_{n+1}, M, r_0 . Our claim follows immediately from (5.30) and the fact that H_1 commutes with T .

Recall that r_0, M and ϵ are fixed. In the following we will show that:

LEMMA 5.4. *For sufficiently fast growing P depending on r_0, M and ϵ the following holds. Assume n satisfies the conditions in Lemma 5.1 and Lemma 5.2; and moreover (5.2) holds for n . Then we have:*

- (1) $d_{\text{Diff}^0(\mathbb{A})}(f, g) < \epsilon/4$;
- (2) $d_{\text{Diff}^0(\mathbb{A})}(h_1^{-1} R_\alpha h_1, g) < \epsilon/4$;
- (3) there is $h_2 \in \text{Diff}^\infty(\mathbb{A})$ such that $(h_2 h_1)_* \omega = \omega$ and

$$d_{\text{Diff}^{r_0-1}(\mathbb{A})}(h_1^{-1} h_2^{-1} R_\alpha h_2 h_1, h_1^{-1} R_\alpha h_1) < \epsilon/4.$$

Proof. *Proof of (1):* By (5.21), there is $D > 0$ depending only on r_0 such that

$$d_{\text{Diff}^0(\mathbb{A})}(f, g) \leq D \max(d_{\text{Diff}^0(\mathbb{A})}(f^{q_n} h f^{-q_n}, \text{Id}), d_{\text{Diff}^0(\mathbb{A})}(h^{-1} f^{-q_{n+1}}, \text{Id})).$$

By Theorem 3, Lemma 5.2, and by letting q_{n+2} be sufficiently large depending on $\epsilon, r_0, M, q_{n+1}$, we have

$$d_{\text{Diff}^0(\mathbb{A})}(f^{q_n} h f^{-q_n}, \text{Id}), \quad d_{\text{Diff}^0(\mathbb{A})}(h^{-1} f^{-q_{n+1}}, \text{Id}) < (4D)^{-1} \epsilon.$$

Thus we have

$$d_{\text{Diff}^0(\mathbb{A})}(f, g) < \epsilon/4.$$

Proof of (2): Since $\|h_1\|_{\text{Diff}^0(\mathbb{A})}$ admits an upper bound depending only on q_{n+1}, M and r_0 , by letting q_{n+2} be sufficiently large depending on $\epsilon, r_0, M, q_{n+1}$, we have

$$d_{\text{Diff}^0(\mathbb{A})}(h_1^{-1} R_\alpha h_1, g) < \epsilon/4.$$

Proof of (3): Set

$$\lambda_0 = \det(H_1) = \frac{1}{q_{n+1}} \det(\tilde{H}) \in C^\infty(\tilde{\mathbb{A}}).$$

Recall that both ψ and F have constant Jacobians, hence by (5.30), on each $\tilde{\mathcal{B}}_j$ we have

$$\log \lambda_0 = \log \det h \circ (G^{-j}) + \log \det(G^{-j}F^j) \circ F^j + c_1$$

for some constant c_1 . On $T^{-k}\tilde{\mathcal{B}}_j$ we have a similar formula. Thus

$$\sup_{1 \leq r \leq r_0} \|D^r \log \lambda_0\| \leq (1 + d_{\text{Diff}^{r_0}(\mathbb{A})}(f, \text{Id}))^{Cq_{n+1}r_0} (d_{\text{Diff}^{r_0}(\mathbb{A})}(h, \text{Id}) + d_{\text{Diff}^{r_0}(\mathbb{A})}(g, f)).$$

By Lemma 5.2 and the argument above for bounding $d_{\text{Diff}^{r_0}(\mathbb{A})}(g, f)$, we see that $\sup_{1 \leq r \leq r_0} \|D^r \log \lambda_0\|$ can be made arbitrarily small by making q_{n+2} sufficiently large while keeping q_{n+1} , M and r_0 fixed. By (5.32), we have

$$\int_{[0,1]^2} \lambda_0 d\omega = 1.$$

Then it is direct to see, for some absolute constant $C > 0$, that

$$\|\lambda_0 - 1\| < C \|D \log \lambda_0\|$$

given that $\|D \log \lambda_0\| < 1$. Consequently, for any $\delta > 0$, we have $\|\lambda_0 - 1\|_{C^{r_0}} < \delta$ if q_{n+2} is sufficiently large depending on δ , q_{n+1} , M and r_0 .

By Dacorogna-Moser's theorem ([DM90, Theorem 1]), there exists $h_2 \in \text{Diff}^\infty(\mathbb{A})$ such that

$$(h_2)_*(\lambda_0 \omega) = \omega.$$

Moreover, by [DM90, Theorem 2 and Lemma 3], we can choose h_2 with $d_{\text{Diff}^{r_0-1}(\mathbb{A})}(h_2, \text{Id})$ arbitrarily small provided $\|\lambda_0 - 1\|_{C^{r_0}}$ is sufficiently small.

In summary, if q_{n+2} is sufficiently large depending on ϵ , q_{n+1} , M and r_0 , we can choose $h_2 \in \text{Diff}^\infty(\mathbb{A})$ sufficiently close to Id in $\text{Diff}^{r_0}(\mathbb{A})$ so that

$$d_{\text{Diff}^{r_0-1}(\mathbb{A})}(h_1^{-1}h_2^{-1}R_\alpha h_2 h_1, h_1^{-1}R_\alpha h_1) < \epsilon/4.$$

□

We set $h_0 = h_2 h_1$. By Lemma 5.4(3) we have $h_0 \in \text{Diff}^\infty(\mathbb{A}, \omega)$. Then by Lemma 5.4, we have

$$d_{\text{Diff}^{r_0}(\mathbb{A})}(h_0^{-1}R_\alpha h_0, f) < \epsilon$$

This completes the proof of Theorem 4. □

We can now prove the main result with the aid of the following lemma whose straightforward proof is omitted.

LEMMA 5.5. *For any function $P : \mathbb{N} \rightarrow \mathbb{N}$, the set*

$$\mathcal{C} := \{\alpha \in (0, 1) \setminus \mathbb{Q} \mid \exists n \geq 3 \text{ odd such that } q_n(\alpha) > P(q_{n-1}(\alpha)), \\ q_{n+1}(\alpha) > P(q_n(\alpha)), q_{n+2}(\alpha) > P(q_{n+1}(\alpha))\}$$

is open and dense in $(0, 1) \setminus \mathbb{Q}$.

Proof of Theorem 1. We have seen in the introduction that $\overline{O_{\mathbb{A}}^{\infty}(\alpha)} \subset F_{\mathbb{A}}^{\infty}(\alpha)$ for any $\alpha \in (0, 1)/\mathbb{Q}$. It remains to show that for a Baire generic $\alpha \in (0, 1)/\mathbb{Q}$

$$(5.35) \quad F_{\mathbb{A}}^{\infty}(\alpha) \subset \overline{O_{\mathbb{A}}^{\infty}(\alpha)}$$

with the closure taken in the C^{∞} -topology. For each tuple (r_0, M, k) in $\mathbb{Z}_{\geq 1}^3$, with $r_0 \geq 2$, let $P_{r_0, M, k^{-1}}$ be the function produced by Theorem 4. Therefore by Lemma 5.5 the set

$$\mathcal{A}(r_0, M, k) := \left\{ \alpha \in (0, 1) \setminus \mathbb{Q} \mid \text{there is an odd } n \geq 3 \text{ such that} \right. \\ \left. q_n(\alpha) > P_{r_0, M, k^{-1}}(q_{n-1}(\alpha)), \quad q_{n+1}(\alpha) > P_{r_0, M, k^{-1}}(q_n(\alpha)) \right. \\ \left. \text{and } q_{n+2}(\alpha) > P_{r_0, M, k^{-1}}(q_{n+1}(\alpha)) \right\}$$

is open and dense in $(0, 1) \setminus \mathbb{Q}$. Thus the countable intersection

$$\mathcal{A} := \bigcap_{r \geq 2} \bigcap_{k \geq 1} \bigcap_{M \geq 2} \mathcal{A}(r, M, k)$$

is a residual subset of $(0, 1) \setminus \mathbb{Q}$. Fix $\alpha \in \mathcal{A}$ and suppose $f \in F_{\mathbb{A}}^{\infty}(\alpha)$. Then for any $\epsilon > 0$ and $r \in \mathbb{N}$ with $r \geq 2$ choose $M \in \mathbb{N}$ so that $\|f\|_{\text{Diff}^{r+2}(\mathbb{A})} < M$, then by $\alpha \in \mathcal{A}(r, M, \lceil \epsilon \rceil^{-1})$ and by Theorem 4 there exists $h \in \text{Diff}^{\infty}(\mathbb{A}, \omega)$ so that

$$d_{C^{r-1}}(f, hR_{\alpha}h^{-1}) < \epsilon.$$

This gives $F_{\mathbb{A}}^{\infty}(\alpha) \subset \overline{O_{\mathbb{A}}^{\infty}(\alpha)}$ with the closure in the C^{r-1} -topology. Since r is arbitrary we easily conclude (5.35) holds with closure in the C^{∞} -topology. \square

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