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# GEOMETRIC PROPERTIES OF PARTIALLY HYPERBOLIC MEASURES AND APPLICATIONS TO MEASURE RIGIDITY

ALEX ESKIN, RAFAEL POTRIE, AND ZHIYUAN ZHANG

ABSTRACT. We give a geometric characterization of the *quantitative joint non-integrability*, introduced by Katz in [Ka], of strong stable and unstable bundles of partially hyperbolic measures and sets in dimension 3. This is done via the use of higher order templates for the invariant bundles. Using the recent work of Katz, we derive some consequences, including the measure rigidity of  $uu$ -states and the existence of physical measures.

## 1. INTRODUCTION

Let  $f : M \rightarrow M$  be a partially hyperbolic diffeomorphism of a closed 3-manifold: the tangent bundle  $TM = E^u \oplus E^c \oplus E^s$  splits into  $Df$ -invariant one dimensional bundles with the property that there is some integer  $N > 0$  such that for every  $x \in M$ , we have

$$\begin{aligned} \|Df^N|_{E^s(x)}\| &\leq \frac{1}{2} \min\{1, \|Df^N|_{E^c(x)}\|\} \\ &< 2 \max\{1, \|Df^N|_{E^c(x)}\|\} \leq \|Df^N|_{E^u(x)}\|. \end{aligned}$$

Any such diffeomorphism  $f$  admits (uniquely defined)  $f$ -invariant foliations  $W^s$  and  $W^u$  tangent respectively to the bundles  $E^s$  and  $E^u$  (see e.g. [CP]).

Consider a lamination  $\Lambda \subset M$  which is  $f$ -invariant and  $W^u$ -saturated. The geometric properties of its leaves, when projected along stable holonomy, are very relevant to understanding several problems: ergodicity of conservative systems (e.g. [BW]), finiteness of attractors (e.g. [CPS]), mixing properties (e.g. [TZ]), among other properties. More recently, some quantitative measures of joint non-integrability have been used by Katz [Ka] to obtain measure rigidity results based on ideas coming from homogeneous and Teichmüller dynamics [EL, EM] (related progress is that of random dynamical systems [BRH], see also [Ob] for its connection with partially hyperbolic dynamics). In this paper, we intend to look into the notion of quantitative non-joint integrability (QNI) proposed by [Ka]. We consider here exclusively  $C^\infty$  diffeomorphisms, and obtain in this setting equivalent notions that seem more conceptual and easier to verify and work with.

**Definition 1.1.** We say that a compact invariant set  $\Lambda$  of a partially hyperbolic diffeomorphism  $f : M \rightarrow M$  is *jointly integrable up to order  $\ell$*  if

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there is  $\rho > 0$  and a continuous family of  $C^\ell$  smooth surfaces  $\{S_x\}_{x \in \Lambda}$  which verifies that:

- (i)  $W_\rho^u(x) \cup W_\rho^s(x) \subset S_x$ ,
- (ii) for every  $x \in \Lambda$  and  $y \in W_\rho^u(x) \cap \Lambda$  (resp.  $y \in W_\rho^s(x) \cap \Lambda$ ) we have that  $W_{loc}^s(y)$  is tangent to order  $\ell$  to  $S_x$  at  $y$  (resp.  $W_{loc}^u(y)$  is tangent to order  $\ell$  to  $S_x$  at  $y$ ).

Here, when we say that the curve  $\gamma$  is tangent to order  $\ell$  to  $S_x$  we mean that there is a constant  $C > 0$ <sup>1</sup> such that when parametrized by arc-length the distance from a point  $y \in \gamma$  to the surface  $S_x$  is less than  $Ct^\ell$  where  $t$  is the arc-length from  $y$  to  $x$ .

Our main results concern the study of  $uu$ -states of partially hyperbolic systems. By definition, an ergodic  $uu$ -state is an ergodic invariant measure that is absolutely continuous with respect to strong unstable manifolds of the foliation  $W^u$ . These measures always exist (see e.g. [BDV, §11]) and are usually the place to look for *physical measures* (i.e. those for which the statistical basin has positive Lebesgue measure).

The results in this paper are also obtained in the more general setting of partially hyperbolic measures where analogous results hold. While very similar, the proofs require more careful analysis in some parts of the argument. We refer the reader to §2 for precise statements.

Combined with the recent results of [Ka] our results give:

**Theorem 1.2.** *Let  $f : M \rightarrow M$  be a  $C^\infty$  partially hyperbolic diffeomorphism on a closed 3-manifold and let  $\mu$  be a  $uu$ -state with positive center Lyapunov exponent, then, either  $\mu$  is physical, or the support of  $\mu$  is jointly integrable up to order  $\ell$  for every  $\ell > 0$ .*

Note that in [ABV] the physicality of  $uu$ -states is proved under the assumption that *every* such measure has positive center exponents.

In principle similar results may hold in higher dimensions which may be worth investigating. This may involve adapting some definitions to take care of some higher dimensional phenomena that may occur. We decided to restrict to the 3-dimensional case since it already presents some challenges and applications. We note that right now the results in [Ka] require one-dimensional center, but it may be possible to extend this to higher dimensional centers by incorporating the techniques of the work in progress [BEFRH].

We also note that our results require very high regularity to compensate for the fact that we deal with the case where holonomies are not regular (which is the usual case). In some cases, there are reasons that force more regularity of holonomies, even in open sets, and in those cases recently arguments have been made to obtain similar results assuming less regularity of the map, see [ALOS]. Theorem 1.2 will be used in [ACEPWZ] to understand  $uu$ -states of partially hyperbolic Anosov diffeomorphisms in dimension 3 (addressing a conjecture of [GKM]) and will be strengthened to show that if one assumes that the strong unstable foliation of a partially hyperbolic

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<sup>1</sup>Later in the paper we will also work with a measurable version of this, for partially hyperbolic measures, see Definition 2.21.

diffeomorphism of a 3-dimensional manifold fills center unstable disks<sup>2</sup>, then joint integrability up to order  $\ell$  implies actual joint integrability.

The main technical contribution of this paper is to extend the notion of templates introduced in [TZ] to partially hyperbolic dynamics, in particular, dealing with higher order templates to deduce quantitative forms of non-integrability of dynamically defined bundles.

## 2. CONTEXT AND MAIN TECHNICAL RESULT

Throughout this paper we let  $f : M \rightarrow M$  be a  $C^\infty$ -diffeomorphism<sup>3</sup> of a closed 3-manifold  $M$ . We fix a smooth Riemannian metric  $\|\cdot\|_0$  on  $M$ .

**2.1. Partially hyperbolic measures.** An ergodic  $f$ -invariant measure  $\mu$  is *partially hyperbolic* if the following is true:

- $f$  has *simple spectrum*. Namely,  $f$  has three different Lyapunov exponents  $\chi_1 > \chi_2 > \chi_3$ ,
- $\chi_1 > 0 > \chi_3$ .

We denote by  $E^1, E^2, E^3$  the Oseledets bundles for  $\mu$  corresponding to  $\chi_1, \chi_2, \chi_3$  respectively, and denote by  $W^1, W^3$  the Pesin laminations associated to  $E^1, E^3$  respectively (see [BP] and also §2.3 for a statement that makes more precise the properties of these manifolds). We fix some  $0 < \epsilon \ll \min_{i \in \{1,2\}} |\chi_i - \chi_{i+1}|$ , and denote by  $\|\cdot\|$  the Lyapunov norm (with parameter  $\epsilon$ ) for  $\mu$  satisfying the following property. For  $\mu$ -almost every  $x \in M$  we have, for  $i \in \{1, 2, 3\}$ :

$$(2.1) \quad \|D_x f|_{E^i(x)}\| = \frac{\|D_x f(v)\|}{\|v\|} \in (e^{\chi_i - \epsilon}, e^{\chi_i + \epsilon}), \quad v \in E^i(x) \setminus \{0\}.$$

If we fix an orientation on each of the (one dimensional) bundles  $E^i$  we get a unit vector  $e^i(x)$  in  $E^i(x)$  for almost every  $x \in M$  and  $i \in \{1, 2, 3\}$ . We define  $\lambda_{i,x} \in \mathbb{R}$  by equation:

$$(2.2) \quad D_x f(e^i(x)) = \lambda_{i,x} e^i(f(x)),$$

and we have that  $\lambda_{i,x} \in \{\pm \|D_x f|_{E^i(x)}\|$ .

The general theory allows us to disintegrate the measure  $\mu$  along the leaves of  $W^1$  and  $W^3$ . We will denote by  $\mu_x^i$  (with  $i \in \{1, 3\}$ ) the conditional measure along the leaves of  $W^i$ ,  $i \in \{1, 3\}$ , (see [BP]).

**Definition 2.1.** An ergodic  $f$ -invariant partially hyperbolic measure  $\mu$  will be called *non-degenerate* if for almost every  $x \in M$  the measures  $\mu_x^1$  and  $\mu_x^3$  are without atoms.

A measure  $\mu$  is called a *uu-state* if for  $\mu$ -a.e.  $x$  the measure  $\mu_x^1$  is absolutely continuous with respect to the length induced by the Riemannian metric on the leaves of  $W^1$ . Note that if  $\chi_2 > 0$  then  $E^1 \oplus E^2$  is  $\mu$ -a.e. tangent to the leaves of the Pesin unstable lamination which we denote by  $W^{12}$ . We denote

<sup>2</sup>More precisely, a minimal subset of the strong unstable foliation verifies that it 'fills center unstable disks' if it contains open sets in some center unstable disk.

<sup>3</sup>All results hold in finite regularity which depends on the properties (Lyapunov exponents) of the measure one looks at as well as some uniform constants of  $f$  around the support of the measure. We will not attempt to estimate the precise regularity since in any case it will be usually very high.

the disintegration of  $\mu$  along  $\mathcal{W}^{12}$  at a  $\mu$ -typical point  $x$  by  $\mu_x^{12}$ . A measure  $\mu$  is said to be *SRB* (Sinai-Ruelle-Bowen) in this context if  $\mu_x^{12}$  is absolutely continuous with respect to the Riemannian volume induced on the leaves of  $\mathcal{W}^{12}$ .

*Remark 2.2.* Note that by Ledrappier-Young's entropy formula in [LY2], any  $uu$ -state which has  $\chi_2 > 0$  must be non-degenerate.

**2.2. Partially hyperbolic sets.** A particularly important case in our discussion is the one where the diffeomorphism  $f : M \rightarrow M$  is *partially hyperbolic*. More generally, we let  $f : M \rightarrow M$  be a smooth diffeomorphism of a closed 3-manifold and  $\Lambda$  be a compact  $f$ -invariant set admitting a partially hyperbolic splitting  $T_\Lambda M = E^u \oplus E^c \oplus E^s$  which is  $Df$ -invariant and verifies that there is an integer  $N > 0$  so that for every  $x \in \Lambda$  we have:

$$\begin{aligned} \|Df^N|_{E^s(x)}\|_0 &< \min\{1, \|Df^N|_{E^c(x)}\|_0\} \\ &< \max\{1, \|Df^N|_{E^c(x)}\|_0\} \leq \|Df^N|_{E^u(x)}\|_0. \end{aligned}$$

We call  $\Lambda$  a (uniformly) *partially hyperbolic set* for  $f$ . We can assume that  $N = 1$  by choosing an appropriate (adapted) metric that we will denote by  $\|\cdot\|$  in analogy with the Lyapunov metric in the case of partially hyperbolic measures. See [CP].

Note that in this case, every ergodic  $f$ -invariant measure supported in  $\Lambda$  is partially hyperbolic.

We note that it is a standard result that every partially hyperbolic diffeomorphism admits at least one ergodic  $uu$ -state, but the existence of SRB measures is not clear in general (see [BDV, Chapter 11]). We state the following for later reference:

**Fact 2.3.** *Let  $f : M \rightarrow M$  be a partially hyperbolic diffeomorphism. Then there exists a partially hyperbolic measure  $\mu$  which is a  $uu$ -state.*

We note that the same holds if there is a partially hyperbolic attractor (i.e. there is an open set  $U$  such that  $f(\overline{U}) \subset U$  and the set  $\Lambda = \bigcap_{n>0} f^n(U)$  is partially hyperbolic).

**2.3. Normal Forms.** We refer the reader to [KK, §3.1] for more details and [KS] for more general results.

**Proposition 2.4.** *Let  $\mu$  be a partially hyperbolic measure. Then for  $i \in \{1, 3\}$  and  $\mu$ -almost every  $x \in M$  there exists  $\Phi_x^i : T_x \mathcal{W}^i(x) \rightarrow \mathcal{W}^i(x)$  a smooth diffeomorphism such that:*

- (i)  $x \mapsto \Phi_x^i$  varies measurably<sup>4</sup>,
- (ii)  $\Phi_x^i(0) = x$  and  $D_0 \Phi_x^i = \text{id}$ ,
- (iii)  $f(\Phi_x^i(t)) = \Phi_{f(x)}^i(\lambda_{i,x} t)$  for every  $t \in \mathbb{R}$ ,
- (iv) if  $y \in \mathcal{W}^i(x)$  then  $(\Phi_y^i)^{-1} \circ \Phi_x^i$  is an affine map.

In the following, for  $\mu$ -a.e.  $x$ , we identify  $T_x \mathcal{W}^i(x)$  with  $\mathbb{R}$  so that 1 corresponds to a vector of norm 1 in  $T_x \mathcal{W}^i(x)$  with respect to the Lyapunov

<sup>4</sup>One can locally trivialize  $E^i(x) \cong T_x \mathcal{W}^i(x)$  using the vector fields in (2.2) as it is a measurable with respect to  $x$  and therefore use a common domain for all maps to make sense of the variation of the maps.

norm  $\|\cdot\|$ . We will fix a collection of maps  $\Phi_x^i$ ,  $i = 1, 3$ , given by Proposition 2.4. We can naturally identify  $\Phi_x^i$  with a diffeomorphism from  $\mathbb{R}$  to  $\mathcal{W}^i(x)$ .

*Remark 2.5.* The sign of the values of  $\lambda_{i,x}$  depends on the chosen orientations of the bundles  $E^i$ . It is sometimes impossible to find a continuous orientation of the bundles, so it cannot be made so that the values are always positive even after taking iterates or finite covering. For the purposes of this paper, this is not an issue, so we will sometimes assume that the value of  $\lambda_{i,x}$  is always positive to simplify the exposition when it is possible to treat the general case in a similar way.

We denote  $W_r^i(x) = \Phi_x^i((-r, r))$ ,  $i \in \{1, 3\}$ . We denote by  $W_{loc}^i(x)$  a neighborhood of  $x$  in  $\mathcal{W}^i(x)$  whose size may vary from line to line. Since we will use some dynamically defined scales, we introduce the following notation for each integer  $k > 0$  and each  $\rho > 0$ :

$$(2.3) \quad W_\rho^{1,k}(x) := f^{-k}(W_\rho^1(f^k(x))), \quad W_\rho^{3,k}(x) := f^k(W_\rho^3(f^{-k}(x))).$$

We will use the following notation for  $i \in \{1, 3\}$ :

$$(2.4) \quad \hat{\mu}_x^i = [(\Phi_x^i)^{-1}]_* \mu_x^i.$$

The above conditions determine  $\hat{\mu}_x^i$  as a Radon measure on  $\mathbb{R}$  up to a multiple. Given  $i \in \{1, 3\}$ , we have  $\hat{\mu}_{f(x)}^i = cf_* \hat{\mu}_x^i$  for some  $c > 0$ . In the following we normalise  $\hat{\mu}_x^i$  so that its restriction to  $(-1, 1)$  is a probability measure.

With a slight abuse of notation, we use  $\hat{\mu}_x^i$  to denote the probability measure restricted to  $(-1, 1)$ .

The following is an alternative way to characterize  $uu$ -states.

**Proposition 2.6.** *The measure  $\mu$  is an  $uu$ -state if and only if the measures  $\hat{\mu}_x^1$  defined in (2.4) are Lebesgue.*

See [BRH, §6.5] for a proof based on the rigidity result of Ledrappier-Young [LY].

In a similar way as in §2.3 one can find continuous<sup>5</sup> normal form coordinates in the uniform partially hyperbolic setting in dimension 3 (see [KK]):

**Proposition 2.7.** *Let  $f : M \rightarrow M$  be a smooth diffeomorphism of a 3-manifold  $M$  and  $\Lambda$  a compact  $f$ -invariant and partially hyperbolic set. For every  $x \in \Lambda$  there exists  $\Phi_x^i : T_x \mathcal{W}^i(x) \rightarrow \mathcal{W}^i(x)$  a smooth diffeomorphism such that:*

- (i)  $x \mapsto \Phi_x^i$  varies continuously,
- (ii)  $\Phi_x^i(0) = x$  and  $D_0 \Phi_x^i = \text{id}$ ,
- (iii)  $f(\Phi_x^i(t)) = \Phi_{f(x)}^i(\lambda_{i,x} t)$  for every  $t \in \mathbb{R}$ ,
- (iv) if  $y \in \mathcal{W}^i(x)$  then  $(\Phi_y^i)^{-1} \circ \Phi_x^i$  is an affine map.

Remark 2.5 applies to this proposition too.

**2.4. Quantitative non-integrability.** Recently, in [Ka] the author proposed a geometric condition on  $uu$ -states that allows one to apply the scheme introduced in [EM, EL]. Let us recall the following crucial definition in [Ka] (although this notion is only defined for  $uu$ -states in [Ka], it can be stated for partially hyperbolic measure considered here):

<sup>5</sup>Note that here the one-dimensionality of the bundle is crucial for this.

**Definition 2.8.** A partially hyperbolic measure  $\mu$  has the *quantitative non-integrability* property (QNI) if:

- there is  $\alpha > 0$  and,
- for every  $\varepsilon > 0$  a subset  $\mathcal{P} \subset M$  of measure  $\mu(\mathcal{P}) > 1 - \varepsilon$  and,
- for every  $\nu > 0$  constants  $C := C(\nu, \varepsilon) > 0$  and  $k_0 := k_0(\nu, \varepsilon) > 0$  such that:

if an integer  $k \geq k_0$  and  $x \in \mathcal{P}$  satisfy  $f^k(x), f^{-k}(x) \in \mathcal{P}$  then

- there is a subset  $S_x \subset W_1^{3,k}(x)$ <sup>6</sup> with  $\mu_x^3(S_x) > (1 - \nu)\mu_x^3(W_1^{3,k}(x))$  satisfying the following property:
- for every  $y \in S_x$  there exists  $U_y \subset W_1^{1,k}(x)$  with  $\mu_x^1(U_y) > (1 - \nu)\mu_x^1(W_1^{1,k}(x))$  so that if  $z \in U_y$  then

$$(2.5) \quad d(W_1^1(y), W_1^3(z)) > Ce^{-\alpha k}.$$

We do not assume that the measure is a  $uu$ -state because this allows us to define the notion in a more general setting; and even though our main application is for  $uu$ -states we wish to make the arguments symmetric:

**Proposition 2.9.** *A partially hyperbolic measure  $\mu$  has QNI for  $f$  if and only if it has QNI for  $f^{-1}$ .*

The proof is a simple Fubini argument that we postpone to Appendix A. In Appendix A we also discuss this definition as well as other formulations and compare them with the ones in the work of Katz [Ka].

*Remark 2.10.* The main difference between our definition and that of [Ka] is the notion of local stable and local unstable manifolds. For notational simplicity (helped by the fact that we are working with one dimensional stable and unstable strong manifolds) we chosen to use local manifolds to be of a certain length with respect to the normal form coordinates. In [Ka] the local stable and unstable manifolds are considered with respect to a measurable partition of the stable/unstable measurable (Pesin) lamination; this is more natural and it extends better to higher dimensions. We could have chosen to use this formalism, but some arguments where we reduce to cocycles defined on fixed intervals would be more cumbersome to write. We explain the equivalence of the definitions in more detail in Appendix A.

**2.5. Cocycle normal forms and good charts.** We will consider good coordinate charts that incorporate the normal coordinates as in [TZ, §4].

**Definition 2.11** (0-good unstable charts). Let  $\mu$  be a partially hyperbolic measure. A measurable collection of smooth diffeomorphisms  $\{\iota_x : (-\|Df\|, \|Df\|)^3 \rightarrow M\}_{x \in M}$  is a family of *unstable charts* if it verifies that for  $\mu$ -almost every  $x \in M$  we have that  $\iota_x(t_1, 0, 0) = \Phi_x^1(t_1)$ ,  $\iota_x(0, 0, t_3) = \Phi_x^3(t_3)$  for  $t_1, t_3 \in (-1, 1)$ ,  $\partial_2 \iota_x(0, 0, 0)$  is a unit vector in  $E^2(x)$ . Moreover, if we write  $F_x := \iota_{f(x)}^{-1} \circ f \circ \iota_x = (F_{x,1}, F_{x,2}, F_{x,3})$  then the map  $F_x : (-\|Df\|, \|Df\|)^3 \rightarrow \mathbb{R}^3$  verifies that:

- (i)  $\partial_2 F_{x,2}(t, 0, 0) = \lambda_{2,x}$  for all  $t \in (-1, 1)$ ,
- (ii)  $\partial_3 F_{x,3}(t, 0, 0) = \lambda_{3,x}$  for all  $t \in (-1, 1)$ ,

<sup>6</sup>Recall notation (2.3).

(iii)  $\partial_2 F_{x,3}(t, 0, 0) = 0$  for all  $t \in (-1, 1)$ .

We say that a family of unstable charts is *0-good* if for some constant  $d$  (independent of  $x$ ) we have that

$$(2.6) \quad \partial_3 F_{x,2}(t, 0, 0) \text{ is a polynomial of degree } \leq d \text{ in } t \in (-1, 1).$$

Note that in [TZ, §4] similar charts are constructed for Anosov flows. It is not hard to adapt the argument to our case. We will prove the following in §3.

**Proposition 2.12.** *For every partially hyperbolic measure  $\mu$  there is a family of 0-good unstable charts.*

Given a family of unstable charts, for  $\mu$ -a.e.  $x$ , the map  $F_x$  satisfies that for every  $t_1 \in (-1, 1)$  we have that  $F_x(t_1, 0, 0) = (\lambda_{1,x}t_1, 0, 0)$  and

$$(2.7) \quad \begin{pmatrix} \frac{\partial F_{x,2}}{\partial x_2} & \frac{\partial F_{x,2}}{\partial x_3} \\ \frac{\partial F_{x,3}}{\partial x_2} & \frac{\partial F_{x,3}}{\partial x_3} \end{pmatrix} (t_1, 0, 0) = \begin{pmatrix} \lambda_{2,x} & r_x(t_1) \\ 0 & \lambda_{3,x} \end{pmatrix}.$$

We think of this as a linear cocycle over the unstable manifolds. This means that for each point of the unstable manifold (seen in normal coordinates) we have a linear transformation between linear spaces above each point, and as we iterate these get composed with the corresponding linear transformations depending on the base dynamics. See §3 for more details.

In equation (2.7) the function  $r_x$  is a smooth function. It follows from the general theory of *cocycle normal forms*, developed in [BEFRH], that one can change coordinates in order to make  $r_x$  a polynomial of degree depending only on the values of the functions  $\lambda_{2,x}$  and  $\lambda_{3,x}$  (see Proposition 3.3 below). This is how Proposition 2.12 is proven.

Note that the strong stable bundle along the strong unstable manifold can be modeled as a section of this cocycle (cf. §3.1), what will be referred to as a *template*. Since the cocycle is 2 dimensional and can be taken smoothly into an upper triangular form (cf. equation (2.7)) we can think of this template, under the normal form coordinates, as a function on the strong unstable manifold. Therefore this reduction allows one to distinguish between the case where such template is a polynomial or not. This is a reformulation of one of the main observations from [Ts, TZ] (see [Ts, Remark 1.2]). We will show that whenever the template is not a polynomial, then the QNI condition is verified. Else, one can continue doing this for higher order  $\ell + 1$ -good charts of the stable manifolds along a strong unstable manifold, see Theorem 2.24. When we look at  $\ell$ -order jets, we obtain in this way a two-dimensional cocycle over the unstable manifolds, whose diagonal entries are  $\lambda_{2,x}$  and  $\lambda_{3,x}^\ell$ . We refer the reader to §3 for more details on the particular case of cocycle normal forms we will use and to [BEFRH, Appendix A] for a more detailed account.

We introduce the following notion.

**Definition 2.13** ( *$\ell$ -good unstable charts*). Let  $\{\iota_x\}_{x \in M}$  be a family of unstable charts for a partially hyperbolic measure  $\mu$ . Let  $\ell$  be a positive integer. We say that the family is  *$\ell$ -good* if for  $\mu$ -almost every  $x \in M$  there is a unique<sup>7</sup>

<sup>7</sup>It is unique almost everywhere and up to zero measure.



collection of measurable functions  $\mathcal{T}_x^\ell : (-1, 1) \rightarrow \mathbb{R}$ ,  $a_x, b_x : (-1, 1)^2 \rightarrow \mathbb{R}$  such that for  $\hat{\mu}_x^1$ -almost every  $t \in (-1, 1)$  we have that:

$$(2.8) \quad \iota_x^{-1}(W_{loc}^3(\Phi_x^1(t))) = \{(t + a_x(t, s)s, \mathcal{T}_x^\ell(t)s^{\ell+1} + b_x(t, s)s^{\ell+2}, s) : s \in (-1, 1)\}$$

and for some constant  $d := d(\ell, f, \mu)$  (independent of  $x$ ) we have that

$$(2.9) \quad \partial_3^{\ell+1} F_{x,2}(t, 0, 0) \text{ is a polynomial of degree } \leq d \text{ in } t \in (-1, 1).$$

In this case, we call  $\mathcal{T}_x^\ell$  in equation (2.8) a *stable template of  $\ell$ -jets* at  $x$ .

One can define in a similar way  $\ell$ -good stable charts for  $\mu$ .

*Remark 2.14.* We point out again the fact that the relevant conditions about  $\ell$ -good unstable charts at a point  $x$  all concern information that can be read in arbitrarily small neighborhoods of  $W_1^1(x)$  and therefore to analyze the existence of such charts it is enough to understand the associated linear cocycles along the unstable manifold. This will be expanded in §3.

We note that since the leaves of the invariant laminations are smooth, the functions  $a_x(t, s)$  and  $b_x(t, s)$  are smooth in  $s$  for  $\hat{\mu}_x^1$ -almost every  $t \in (-1, 1)$ . In particular, there is a (measurable) function  $c_x : (-1, 1) \rightarrow \mathbb{R}_+$  such that for  $\hat{\mu}_x^1$ -almost every  $t \in (-1, 1)$  and for any  $|s| < 1$  we have

$$(2.10) \quad |a_x(t, s)|, |b_x(t, s)| < c_x(t).$$

Moreover, we have the following, which will be used in Section 7.

**Lemma 2.15.** *For every  $\varepsilon > 0$ , there exist a constant  $C > 1$  and a subset  $\mathcal{P} \subset M$  such that  $\mu(\mathcal{P}) > 1 - \varepsilon$ , and for every  $x \in \mathcal{P}$  and every  $r \in (0, C^{-1})$ , the set of  $t \in (-r, r)$  satisfying  $|c_x(t)| < C$  has  $\hat{\mu}_x^1$ -measure at least  $(1 - \varepsilon)\hat{\mu}_x^1((-r, r))$ .*

*Proof.* Fix an arbitrary  $\varepsilon > 0$ . By Lusin's lemma, we can find a compact subset  $\mathcal{Q} \subset M$  such that  $\mu(\mathcal{Q}) > 1 - \varepsilon^2$ , and  $W_{loc}^1(x)$ ,  $W_{loc}^3(x)$  as well as the chart  $\iota_x$  are continuous restricted to  $x \in \mathcal{Q}$ . Then by definition, for any  $x \in \mathcal{Q}$  and  $t \in (-1, 1)$  such that  $\Phi_x^1(t) \in \mathcal{Q}$ , we see that  $|c_x(t)|$  can be chosen uniformly bounded from above. Then the lemma follows immediately from Proposition B.1.  $\square$

*Remark 2.16.* Note that the stable templates depend on the charts. In [TZ] the stable templates at  $x$  are taken to be the family of all possible  $\mathcal{T}_x^0$  as we change the underlying 0-good unstable charts. We emphasize that we usually expect to have non-smooth  $\mathcal{T}_x^\ell$ . Indeed, one of the main points here is that if  $\mathcal{T}_x^\ell$  is smooth in some regions, then one can produce a higher order approximation.

*Remark 2.17.* The existence of  $\ell$ -good charts implies that the *stable Hopf brush* at a point  $x$ , by which we mean  $\mathcal{H}_x^s = \bigcup_{t \in (-1, 1)} W_1^3(\Phi_x^1(t))$ , is more regular than expected: it can be approximated to order  $\ell$  by the stable templates of  $\ell$ -jet. One has a similar approximation for the *unstable Hopf brush* defined by  $\mathcal{H}_x^u = \bigcup_{t \in (-1, 1)} W_1^1(\Phi_x^3(t))$ . Note that the regularity of  $\mathcal{H}_x^s$  and  $\mathcal{H}_x^u$  may be pretty bad, but the templates used to approximate these sets to high order may have good regularity.

One useful consequence of (2.9) is the following simple computation:

*Remark 2.18.* Note that the condition (2.8) together with the properties of unstable normal coordinate charts imply that:

$$(2.11) \quad \frac{\lambda_{2,x}}{\lambda_{3,x}^{\ell+1}} \mathcal{J}_x^\ell(t) + \frac{1}{\lambda_{3,x}^{\ell+1}} \frac{\partial_3^{\ell+1} F_{x,2}(t, 0, 0)}{(\ell+1)!} = \mathcal{J}_{f(x)}^\ell(\lambda_{1,x} t).$$

If (2.9) is verified, we know that  $\frac{1}{(\ell+1)! \lambda_{3,x}^{\ell+1}} \partial_3^{\ell+1} F_{x,2}(t, 0, 0)$  is a polynomial in  $t$  which depends only on the coordinates we have chosen. Consequently, the property that  $\mathcal{J}_x^\ell$  is a polynomial of degree  $\leq d$  is independent of the choice of the  $\ell$ -good chart. See Proposition 3.13 for more details.

Before we state the main inductive step for proving Theorem 1.2, we recall the notion about Whitney smoothness.

For a function  $\varphi : (-1, 1) \rightarrow \mathbb{R}$  and  $K \subset (-1, 1)$  a compact set, we say that  $\varphi$  is  $C^r$  in the sense of Whitney on  $K$  if there exists a  $C^r$  function  $\tilde{\varphi}$  on an open neighborhood of  $K$  such that  $\tilde{\varphi}|_K = \varphi$ . Another equivalent condition (cf. Whitney's extension theorem, see [W]) is given by the existence of continuous functions  $a_i : K \rightarrow \mathbb{R}$ ,  $1 \leq i \leq r$ , satisfying a family of compatibility conditions (see [W]). In particular, for any  $t, s \in K$  we have

$$(2.12) \quad |\varphi(s) - (\varphi(t) + a_1(t)(s-t) + \dots + a_r(t)(s-t)^r)| = o(|s-t|^r).$$

We say that  $\varphi$  is smooth in the sense of Whitney on  $K$  if it is  $C^r$  in the sense of Whitney on  $K$  for every integer  $r > 0$ .

We will prove in §4 the following proposition.

**Proposition 2.19** (Dichotomy). *Let  $\mu$  be a partially hyperbolic measure with  $\ell$ -good unstable charts. Then either there are  $(\ell+1)$ -good unstable charts, or, for almost every  $x \in M$  we have that  $\mathcal{J}_x^\ell$  as defined in (2.8) is not Whitney smooth restricted to any subset of  $\mathcal{W}^1(x)$  with positive  $\mu_x^1$ -measure (in particular, it is not a polynomial of degree  $\leq d$ ).*

We can see from the above proposition that the smoothness of  $\mathcal{J}_x^\ell$  (an intrinsic property about  $(f, \mu)$ ) can be expressed naturally using normal coordinates (see Proposition 2.6 for another application of such an idea).

It says that the assertion that the  $\ell$ -order jets of the strong stable lamination are smooth along the strong unstable direction is equivalent to the a priori stronger condition that they are polynomial in the normal coordinates.

**2.6. Compatible charts.** Note that for the time one map of the geodesic flow on a constant negatively curved surface, the volume measure verifies that for every  $\ell > 0$  the map admits  $\ell$ -good stable and unstable charts, and, at the same time, verifies a strong form of quantitative non-integrability due to the contact structure. Thus, to be able to extract more information from the existence of  $\ell$ -good stable and unstable charts, we will show that there is some compatibility between these charts assuming that the QNI condition is not verified.

**Definition 2.20** (Compatible charts). For a partially hyperbolic measure  $\mu$  we say that it admits  $\ell$ -compatible good charts if there exist  $L := L(\ell, f) \geq \ell$ ,  $L$ -good stable charts  $\{i_x\}_{x \in M}$  and  $L$ -good unstable charts  $\{i'_x\}_{x \in M}$  such that for  $\mu$  almost every  $x \in M$  we have that: for all  $(t_1, t_3)$  close to  $(0, 0)$ ,

$$(2.13) \quad (i'_x)^{-1} \circ i_x(t_1, 0, t_3) = (s_1, O((|s_1| + |s_3|)^\ell), s_3).$$

We say that  $\mu$  admits *compatible good charts* if it admits  $\ell$ -compatible good charts for every  $\ell > 0$ .

The existence of compatible good charts implies that the measure is *jointly integrable up to order  $l$* , a notion defined below:

**Definition 2.21.** We say that a partially hyperbolic measure  $\mu$  is *jointly integrable up to order  $l$*  if there is a measurable family of  $C^l$  smooth surfaces (with boundaries)  $\{S_x\}_{x \in M}$  in  $M$  such that for  $\mu$  almost every  $x \in M$ , there is  $\rho_x > 0$  such that :

- (i)  $W_{\rho_x}^1(x) \cup W_{\rho_x}^3(x) \subset S_x$ , and,
- (ii) for  $\mu_x^1$  almost every  $y \in W_{\rho_x}^1(x)$  (resp.  $\mu_x^3$  almost every  $y \in W_{\rho_x}^3(x)$ ) we have that  $W_1^3(y)$  is tangent to  $S_x$  to order  $l$  at  $y$  (resp.  $W_1^1(y)$  is tangent to  $S_x$  to order  $l$  at  $y$ ).

It is natural to compare the above definition with Definition 1.1.

**Proposition 2.22.** *Let  $\mu$  be a partially hyperbolic measure with compatible good charts. Then  $\mu$  is jointly integrable up to order  $l$  for every  $l > 0$ .*

*Proof.* We fix an arbitrary integer  $l > 0$ , and let  $\ell = 10l$ . We let  $L > \ell$  be a large integer, let  $\{\iota_x\}_{x \in M}$  be a collection of  $L$ -good stable charts, and let  $\{\iota'_x\}_{x \in M}$  be a collection of  $L$ -good unstable charts such that (2.13) is satisfied for  $\ell$ . Let  $x$  be a  $\mu$ -typical point such that  $\iota_x$  and  $\iota'_x$  are defined at  $x$ , and (2.13) holds.

For all  $(t_1, t_3)$  close to  $(0, 0)$ , we write

$$(\iota'_x)^{-1} \circ \iota_x(t_1, 0, t_3) = (h_1(t_1, t_3), h_2(t_1, t_3), h_3(t_1, t_3)).$$

Here  $h_1, h_2, h_3$  are smooth functions, and we have

$$(2.14) \quad h_1(t_1, 0) = t_1, \quad h_3(0, t_3) = t_3, \quad |h_2(t_1, t_3)| \leq C_x(|t_1|^\ell + |t_3|^\ell)$$

for some  $C_x > 1$ .

Let  $\rho_x > 0$  be a small constant to be determined, and we set  $E = (-\rho_x, \rho_x) \times \{0\} \cup \{0\} \times (-\rho_x, \rho_x)$ . For any integers  $i, j \geq 0$  such that  $i + j \leq l$ , we define

$$\varphi_{i,j}(t_1, t_3) = \begin{cases} \partial_{t_1}^i \partial_{t_3}^j h_2(0, t_3), & t_1 = 0, t_3 \in (-\rho_x, \rho_x) \setminus \{0\}, \\ 0, & t_3 = 0, t_1 \in (-\rho_x, \rho_x). \end{cases}$$

By (2.14), the above formula gives a collection of continuous functions on  $E$ . Moreover, by  $2l \leq \ell$  and Taylor's expansion of  $h_2$  at the origin, we see that  $|\varphi_{i,j}(t_1, t_3)| \leq C'_x(|t_1| + |t_3|)^l$  for the above  $i, j$  and some  $C'_x > 0$ . Then we have the following compatibility conditions: for any  $i, j$  as above, for any  $(t_1, t_3)$  and  $(s_1, s_3)$  on  $E$ , we have

$$\begin{aligned} \varphi_{i,j}(t_1, t_3) &= \sum_{i'+j' \leq l-i-j} \frac{\varphi_{i+i', j+j'}(s_1, s_3)}{(i')!(j')!} (t_1 - s_1)^{i'} (t_3 - s_3)^{j'} \\ &\quad + o((|s_1 - t_1| + |s_3 - t_3|)^{l-i-j}). \end{aligned}$$

Here the implicit constant depends only on  $l$  and  $x$ . Then by Whitney's extension theorem, there exists a  $C^l$  function  $\varphi$  defined in a neighborhood of  $(0, 0)$  such that  $\partial_{t_1}^i \partial_{t_3}^j \varphi|_E = \varphi_{i,j}$  for any integers  $i, j \geq 0$  such that  $i + j \leq l$ .

We define  $H(t_1, t_2, t_3) = (t_1, t_2 + \varphi(t_1, t_3), t_3)$ , and define a map  $\iota_x^b : (-\rho_x, \rho_x)^3 \rightarrow M$  by  $\iota_x^b = \iota'_x \circ H$ . We denote  $T_0 = \{(t_1, t_2, t_3) : t_2 = 0\}$  and define

$$S_x = (\iota_x^b)^{-1}(T_0), \quad S_x^s = (\iota_x)^{-1}(T_0), \quad S_x^u = (\iota'_x)^{-1}(T_0).$$

By construction, we see that  $S_x$  and  $S_x^s$  are tangent to order  $l$  along  $W_{loc}^s(x)$ , and  $S_x$  and  $S_x^u$  are tangent to order  $l$  along  $W_{loc}^u(x)$ . Then it is straightforward to see that  $\mu$  is jointly integrable up to order  $l$ . This concludes the proof.  $\square$

**2.7. Main technical statement.** We have the following dichotomy, which proposes a more geometric way to deal with the QNI condition (at least when the diffeomorphism is sufficiently smooth).

**Theorem 2.23.** *Let  $\mu$  be a partially hyperbolic measure for a  $C^\infty$  smooth diffeomorphism  $f$  of a closed 3-manifold. Then,  $\mu$  has the QNI property if and only if it does not admit compatible good charts (cf. Definition 2.20).*

It is easy to check that if  $\mu$  admits compatible good charts then it cannot verify QNI, so the main point of the Theorem is to establish that if  $\mu$  does not admit compatible good charts, then it has to have the QNI property. We divide the proof into two natural steps:

**Theorem 2.24.** *Let  $\mu$  be a partially hyperbolic measure and assume that it does not admit  $\ell$ -good unstable charts for some integer  $\ell > 0$ , then  $\mu$  verifies the QNI property.*

The proof of this theorem will be given in §6. The symmetric statement holds for the existence of  $\ell$ -good stable charts (cf. Proposition 2.9).

The second part is to show that the  $\ell$ -good charts must be compatible unless QNI holds:

**Theorem 2.25.** *Let  $\ell \geq 1$  and let  $\mu$  be a partially hyperbolic measure. Then there is an integer  $L > 0$  such that if  $\mu$  admits  $L$ -good stable charts and  $L$ -good unstable charts and  $\mu$  does not have QNI then there is a family of  $\ell$ -compatible good charts.*

This will be shown in §7. In §2.9 we discuss and prove some uniform versions of these results when the diffeomorphism is (uniformly) partially hyperbolic.

**2.8. Applications.** We restate here a consequence of the main result of [Ka].

**Theorem 2.26** (Katz [Ka]). *Assume that  $\mu$  is an ergodic partially hyperbolic measure with  $\chi_2 > 0$  which is a  $uu$ -state and verifies the QNI property. Then,  $\mu$  is SRB.*

It is worth pointing out that in [Ka] the flow case is treated. Note that for diffeomorphisms one can take the suspension flow and the results from [Ka] will apply and thus give the statement we just quoted.

Therefore, our main technical statement has the following consequence:

**Corollary 2.27.** *Assume that an ergodic partially hyperbolic measure  $\mu$  is a  $uu$ -state with  $\chi_2 > 0$ , then either  $\mu$  is SRB or  $\mu$  is jointly integrable up to order  $\ell$  for every  $\ell > 0$ .*

**2.9. Uniform versions of the results.** We come back to the context of the introduction.

*Remark 2.28.* We will use normal form coordinates for points in  $\Lambda$  as in §2.3. In the setting of one dimensional stable and unstable manifolds we are working on, it turns out that one can choose the normal form coordinates to vary continuously on the point  $x \in \Lambda$ . This will be relevant for our statements, and in higher dimensions presents a challenge to generalize our results. See [KS] and references therein.

The results announced in the introduction are not a direct consequence of their measurable counterparts stated in the previous subsections due to the fact that the estimates are measurable functions instead of continuous ones. In order to obtain the continuous version, it is just necessary to check that the arguments in the measurable case do give uniform estimates when necessary since there is a continuous invariant splitting to start with. We will explain this in §8 (pointing out how the arguments simplify in some places for the uniform case). Here we will provide the corresponding definitions and main statements for the convenience of the reader.

We will consider a continuous orientation on  $E^i$  up to finite cover<sup>8</sup> and the induced unit vector fields  $e^i(x)$ . For  $x \in \Lambda$  and  $i \in \{1, 2, 3\}$ , we define  $\lambda_x^i \in \pm \|D_x f|_{E^i(x)}\|$  by equation:

$$(2.15) \quad D_x f e^i(x) = \lambda_x^i e^i(f(x)),$$

where  $E^1 = E^u$ ,  $E^2 = E^c$  and  $E^3 = E^s$ . We know by the choice of the Riemannian metric that these are continuous functions which verify that  $|\lambda_x^1| > |\lambda_x^2| > |\lambda_x^3|$  as well as  $|\lambda_x^1| > 1 > |\lambda_x^3|$ . We consider the laminations  $\mathcal{W}^1, \mathcal{W}^3$  tangent respectively to  $E^1$  and  $E^3$  given by the stable manifold theorem with their corresponding normal form coordinates (cf. Remark 2.28 or Proposition 2.7 below).

The notion of quantitative non-joint integrability which one obtains in the uniform case is also a bit stronger due to the uniform assumptions.

We have the following result:

**Theorem 2.29.** *Let  $\Lambda$  be a partially hyperbolic set of a smooth diffeomorphism  $f$ . Then, if there is a fully supported non-degenerate measure  $\mu$  on  $\Lambda$  which does not have the QNI property, then the set  $\Lambda$  is jointly integrable up to order  $\ell$  for every  $\ell > 0$  (cf. Definition 1.1).*

One consequence of this theorem is that having one measure without QNI forces every measure with the same support to have this property:

**Definition 2.30.** We say that a partially hyperbolic set  $\Lambda$  has *topological QNI* if every measure which is fully supported on  $\Lambda$  has QNI.

Theorem 2.29 implies that either  $\Lambda$  has topological QNI, or every fully supported measure is degenerate, or  $\Lambda$  is jointly integrable up to order  $\ell$  for

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<sup>8</sup>Note that if  $\Lambda \subset M$  is not everything, there may not be a finite cover of  $M$  that orients the bundles (e.g. the Plykin attractor), however, we are only interested on the dynamics in a neighborhood of  $\Lambda$  and one can always find a finite cover of such neighborhood with the desired properties. Note that this is just a notational issue, to avoid having to add  $\pm$  signs in each equation.

every  $\ell$ . The second case happens for instance when the set  $\Lambda$  is contained in a normally hyperbolic surface tangent to  $E^u \oplus E^c$ .

*Remark 2.31.* We note that it has been proved in [BC] that if  $\Lambda$  has no *strong connections* (i.e. for every  $x \in \Lambda$  we have that  $W_{loc}^s(x) \cap \Lambda = \{x\}$ ) then  $\Lambda$  is contained in a locally invariant surface, that is, there is a compact surface with boundary  $S$  containing  $\Lambda$  in its interior and an open neighborhood  $U$  of  $\Lambda$  in  $S$  such that  $f(U) \subset S$ . In this case, every fully supported measure in  $\Lambda$  is degenerate.

### 3. EXISTENCE OF NORMAL COORDINATE CHARTS AND COCYCLE NORMAL FORMS

In this section we prove Proposition 2.12 and Proposition 3.9. We restate some results which are done in more generality in [BEFRH, Appendix A] but in a somewhat different form.

**3.1. Cocycle normal forms.** Let  $f : M \rightarrow M$  be a smooth diffeomorphism preserving a partially hyperbolic measure  $\mu$ . We let  $\mathcal{E} \rightarrow M$  be a (measurable) vector bundle over  $(M, \mu)$  and  $A : \mathcal{E} \rightarrow \mathcal{E}$  a vector bundle automorphism that lifts  $(f, \mu)$  (i.e. for  $\mu$ -a.e.  $x \in M$  we have that  $A_x := A|_{\mathcal{E}_x} : \mathcal{E}_x \rightarrow \mathcal{E}_{f(x)}$  is a linear automorphism).

We will be concerned only with two dimensional vector bundles (i.e.  $\dim(\mathcal{E}_x) = 2$  for  $\mu$ -a.e.  $x \in M$ ). We refer to [BEFRH, Appendix A] for more general results.

We say that a bundle  $\mathcal{E}$  is *smooth along unstable manifolds* if for  $\mu$ -a.e.  $x$ , the restriction of  $\mathcal{E}$  to  $W^1(x)$  is smooth. In this case, a *smooth trivialization along unstable manifolds* of  $\mathcal{E}$  is a family of pairs  $\mathcal{S} = \{\mathcal{S}_x = (\xi_x, \xi_x^\perp)\}_{x \in M}$  such that for  $\mu$ -a.e.  $x \in M$ ,  $\xi_x, \xi_x^\perp : (-\|Df\|, \|Df\|) \rightarrow \mathcal{E}$  are smooth maps such that  $\xi_x(t), \xi_x^\perp(t)$  are linearly independent vectors in  $\mathcal{E}_{\Phi_x^1(t)}$ .

*Remark 3.1.* We note that for a partially hyperbolic measure  $\mu$  almost every point has a well defined strong unstable manifold, however, not every point in this manifold is generic with respect to  $\mu$ .

We start by presenting an example which corresponds to the first step of our induction.

*Example 3.2.* Consider the two dimensional vector bundle  $\mathcal{E} \rightarrow M$  defined for  $\mu$ -a.e.  $x \in M$  as the quotient  $\mathcal{E}_x = T_x M / E^1(x)$ . We fix a non-degenerate inner product on  $\mathcal{E}_x$  on each  $x$  which we denote by  $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ . Clearly, since the cocycle  $Df$  preserves  $E^1$ , it induces a vector bundle automorphism  $A$  given by:

$$A[v] = [D_x f(v)] = D_x f(v) + E^1(f(x)) \in T_{f(x)} M / E^1(f(x))$$

where  $v \in T_x M$  and  $[v] \in T_x M / E^1(x)$  denotes  $v + E^1(x)$  the representative of  $v$  in the quotient.

We note that for  $\mu$ -a.e.  $x \in M$ , the restriction  $\mathcal{E}_x$  of the bundle  $\mathcal{E}$  to  $W_1^1(x)$  is a smooth vector bundle because the local unstable manifold is smooth.

We choose a trivialization of  $\mathcal{E}$  as follows. We choose a smooth map  $\xi_x : (-\|Df\|, \|Df\|) \rightarrow \mathcal{E}$  such that

$$\Phi_x^1(t) \mapsto \xi_x(t) \in (E^1(\Phi_x^1(t)) \oplus E^2(\Phi_x^1(t))) / E^1(\Phi_x^1(t))$$

is a section of the bundle  $\mathcal{E} \rightarrow M$ . The existence of such smooth map  $\xi_x$  is guaranteed by the fact that the weak-unstable bundle  $E^1 \oplus E^2$  of the Oseledets decomposition is smooth along strong unstable manifolds (see Proposition C.1). We let  $\xi_x^\perp : (-\|Df\|, \|Df\|) \rightarrow \mathcal{E}$  be a smooth map such that  $\xi_x^\perp(t)$  is a unit vector in  $\mathcal{E}_x$ , and  $\langle \xi_x^\perp(t), \xi_x(t) \rangle_{\mathcal{E}} = 0$ .

In this way, if we set  $\mathcal{S}_x = (\xi_x, \xi_x^\perp)$ , then we can write the matrix corresponds to the action of  $Df$  from  $\mathcal{E}_{\Phi_x^1(t)}$  to  $\mathcal{E}_{\Phi_{f(x)}^1(\lambda_{1,x}t)}$  in the basis  $\mathcal{S}_x(t) = (\xi_x(t), \xi_x^\perp(t))$  and  $\mathcal{S}_{f(x)}(\lambda_{1,x}t) = (\xi_{f(x)}(\lambda_{1,x}t), \xi_{f(x)}^\perp(\lambda_{1,x}t))$ , for  $t \in (-1, 1)$ , as

$$A^{\mathcal{S}}(t) := \begin{pmatrix} \alpha_x(t) & r_x(t) \\ 0 & \beta_x(t) \end{pmatrix}$$

where  $\alpha_x, r_x, \beta_x$  are smooth functions. ◇

We can write the vector bundle automorphism  $A$  in a smooth trivialization as a measurable function

$$A^{\mathcal{S}} : M \times (-1, 1) \rightarrow \mathrm{GL}_2(\mathbb{R})$$

such that  $A^{\mathcal{S}}(x, t)$  denotes the matrix associated to  $A_{\Phi_x^1(t)}$  from the ordered basis  $(\xi_x(t), \xi_x^\perp(t))$  to the ordered basis  $(\xi_{f(x)}(\lambda_{1,x}t), \xi_{f(x)}^\perp(\lambda_{1,x}t))$ .

We say that the vector bundle automorphism  $A$  is *smooth along unstable manifolds* if there is a smooth trivialization of  $\mathcal{E}$  such that for  $\mu$ -a.e.  $x \in M$  the entries of  $A^{\mathcal{S}}(x, \cdot)$  are smooth functions of  $t$ . Note that if there is one such trivialization, the same holds for all smooth trivializations.

The following is the main result from normal forms for cocycles which are smooth along unstable manifolds that we will need. We refer the reader to Appendix C for more discussion on the notions of smoothness along strong unstables and exponents of cocycles.

**Proposition 3.3.** *Let  $\mu$  be a partially hyperbolic measure of a diffeomorphism  $f : M \rightarrow M$ , let  $\mathcal{E} \rightarrow M$  be a  $\mu$ -measurable two dimensional vector bundle which is smooth along unstable manifolds, and let  $A$  be a linear cocycle over  $(f, \mu)$  which is smooth along unstable manifolds and has exponents  $\alpha > \beta$ . Then there exists  $\mathcal{S} = \{\mathcal{S}_x = (\xi_x, \xi_x^\perp)\}_{x \in M}$ , a smooth trivialization along unstable manifolds of  $\mathcal{E}$ , satisfying that for  $\mu$ -a.e.  $x \in M$  we have that for  $t \in (-1, 1)$*

$$(3.1) \quad A^{\mathcal{S}}(x, t) = \begin{pmatrix} \alpha_x & p_x(t) \\ 0 & \beta_x \end{pmatrix}$$

where  $\alpha_x$  and  $\beta_x$  depend measurably on  $x$  such that  $\int \log \alpha_x d\mu(x) = \alpha$  and  $\int \log \beta_x d\mu(x) = \beta$  and  $p_x : (-1, 1) \rightarrow \mathbb{R}$  is a polynomial of degree at most  $d$  which depends only on  $\chi_1, \alpha, \beta$ .

*Proof.* Being smooth along unstable manifolds, we can define a measurable non-degenerate inner product  $\|\cdot\|_{\mathcal{E}}$  on the fibers of  $\mathcal{E}$  which is smooth along unstable manifolds.

By Proposition C.1, there exists a family of smooth trivializations  $\mathcal{S} = \{\mathcal{S}_{0,x} = (\xi_{0,x}, \xi_{0,x}^\perp)\}_{x \in M}$  such that the cocycle  $A^{\mathcal{S}_0}$  is upper triangular along

unstable manifolds. That is, there are smooth functions  $\alpha_x, \beta_x, r_x : (-1, 1) \rightarrow \mathbb{R}$  such that:

$$(3.2) \quad A^{S_0}(x, t) = \begin{pmatrix} \alpha_x(t) & r_x^0(t) \\ 0 & \beta_x(t) \end{pmatrix}.$$

We remark that  $\xi_{0,x}(t)$  belongs to the Oseledets bundle associated to the exponent  $\alpha$  in the fiber  $\mathcal{E}_{\Phi_x^1(t)}$  (whenever it is defined<sup>9</sup>).

Note that to get the smoothness of the trivialization we are considering the orthogonal bundle with respect to the smooth Riemannian metric rather than the Lyapunov metric (which we use to compute contraction and expansion). This is no problem since we only use this to produce a reference smooth bundle that we will later change to make our cocycle be in normal form.

We now make the diagonals to be constant.

**Claim 3.4.** *There exists a unique measurable family of smooth functions  $\{a_x : (-1, 1) \rightarrow \mathbb{R}_{>0}\}_{x \in M}$  so that  $a_x(0) = 1$  and  $A(x, t)\xi_{1,x}(t) = \alpha_x(0)\xi_{1,x}(\lambda_{1,x}t)$  where  $\xi_{1,x}(t) = a_x(t)\xi_{0,x}(t)$ . Similarly, there is a family of smooth functions  $\{a_x^\perp : (-1, 1) \rightarrow \mathbb{R}_{>0}\}_{x \in M}$  so that  $\xi_{1,x}^\perp(t) = a_x^\perp(t)\xi_{0,x}^\perp(t)$  so that if we take  $\mathcal{S}_1 = \{\mathcal{S}_{1,x} = (\xi_{1,x}, \xi_{1,x}^\perp)\}_{x \in M}$  as a smooth trivialization along unstable manifolds of  $\mathcal{E}$ , then there is a measurable family of smooth functions  $r_x : (-1, 1) \rightarrow \mathbb{R}$  so that:*

$$(3.3) \quad A^{S_1}(x, t) = \begin{pmatrix} \alpha_x(0) & r_x(t) \\ 0 & \beta_x(0) \end{pmatrix}.$$

*Proof.* The proof is similar to the unstable manifold theorem. Let us spell out the computations.

First we construct  $\xi_{1,x}$  from  $\xi_{0,x}$ . We can put coordinates  $(t, s)_x$  on the one-dimensional bundle  $\mathbb{R}\xi_{0,x}$  so that  $(t, s)_x$  represents the vector  $v := s\xi_{0,x}(t) \in \mathcal{E}_{\Phi_x^1(t)}$ . This way, we can write, for  $t \in (-1, 1)$  and  $s \in \mathbb{R}$ :

$$\Psi_x(t, s)_x := (\lambda_{1,x}t, \alpha_x(t)s)_{f(x)}.$$

that corresponds to the action of  $A$  on vectors in the chosen coordinates (from now on we will remove the subindex of the point where the coordinates are chosen in the notation).

Write  $\alpha_x(t) = \pm \exp(b_x(t))$  for some positive smooth function  $b_x : (-1, 1) \rightarrow \mathbb{R}$  (we will assume from now on that  $\alpha_x(t)$  is positive for simplicity).

We need to find functions  $c_x : (-1, 1) \rightarrow \mathbb{R}$  so that  $c_x(0) = 0$  with the property that  $\Psi_x(t, \exp(c_x(t))) = (\lambda_{1,x}t, \alpha_x(0)\exp(c_{f(x)}(\lambda_{1,x}t)))$  which we can write as:

$$(3.4) \quad c_{f(x)}(\lambda_{1,x}t) + b_x(0) = b_x(t) + c_x(t).$$

This holds for almost every  $x \in M$ , so we can solve  $c_x$  as follows: Denote  $b_m = b_{f^{-m}(x)}$ ,  $c_m = c_{f^{-m}(x)}$ ,  $\lambda_{1,m}^- = \|D_{f^{-m}(x)}f^m|_{E^1}\|^{-1} = (\lambda_{f^{-1}(x)}^1 \cdots \lambda_{f^{-m}(x)}^1)^{-1}$ .

<sup>9</sup>Note that actually, the bundle associated to  $\alpha$  is defined on backward regular points, so it would make sense to say it is well defined for all  $t$ , though we will not use this fact in the proof.



Then we get that for every  $k \in \mathbb{Z}_{>0}$  that

$$(3.5) \quad c_x(t) = c_k(\lambda_{1,k}^- t) + \sum_{j=1}^k \left( b_j(\lambda_{1,j}^- t) - b_j(0) \right).$$

Since  $\lambda_{1,j}^-$  tends to 0 exponentially fast and  $c_k$  is smooth satisfying  $c_k(0) = 0$ , we get that  $c_k(\lambda_{1,k}^- t)$  tends to 0 exponentially fast for a  $\mu$ -typical  $x$  by Birkhoff's theorem. Similarly, we have that the value of  $b_j(\lambda_{1,j}^- t) - b_j(0)$  is also exponentially small so that the sum converges uniformly in  $t$ . Thus for a typical  $x$  we have

$$c_x(t) = \sum_{j=1}^{\infty} \left( b_j(\lambda_{1,j}^- t) - b_j(0) \right).$$

It is clear that the above expression gives the unique solution of equation (3.5). Notice also that the derivatives of  $c_x$  can be solved by an analogous computation. Consequently we can show that the functions  $c_x$  are  $C^\infty$  smooth and derivatives of all orders vary measurably on the point  $x$ .

To get the family of sections  $\xi_{1,x}^\perp$  one argues in the same spirit by looking at the bundle  $\mathcal{E}_{\Phi_x^1(t)}$  quotiented by  $\mathbb{R}\xi_{1,x}(t)$ , the same considerations on the smoothness apply.  $\square$

Finally, we will need to add some component of  $\xi_{1,x}$  to  $\xi_{1,x}^\perp$  in order to change the function  $r_x$  in (3.3) into a polynomial. In the following let us abbreviate  $\alpha_x(0)$  and  $\beta_x(0)$  as  $\alpha_x$  and  $\beta_x$  respectively.

**Claim 3.5.** *There exists a measurable family of smooth functions  $\{u_x : (-1, 1) \rightarrow \mathbb{R}\}_{x \in M}$  such that if we take  $\xi_x(t) = \xi_{1,x}$  and  $\xi'_x(t) = u_x(t)\xi_x(t) + \xi_{1,x}^\perp(t)$  and  $\mathcal{S} = \{\mathcal{S}_x = (\xi_x, \xi'_x)\}_{x \in M}$  we get that:*

$$(3.6) \quad A^{\mathcal{S}}(x, t) = \begin{pmatrix} \alpha_x & p_x(t) \\ 0 & \beta_x \end{pmatrix}$$

where  $p_x$  is a polynomial of degree  $\leq d_0$  where<sup>10</sup>  $d_0 = \lfloor \frac{\alpha - \beta}{\chi_1} \rfloor + 1$ .

*Proof.* Let us then compute the map  $A$  in the coordinates  $\mathcal{S}_1$ . We have

$$\xi'_x(t) = u_x(t)\xi_x(t) + \xi_x^\perp(t) \mapsto (r_x(t) + \alpha_x u_x(t))\xi_{f(x)}(\lambda_{1,x}t) + \beta_x \xi_{f(x)}^\perp(\lambda_{1,x}t).$$

We can write  $r_x(t) = p_x(t) + \hat{r}_x(t)$  where  $p_x(t)$  is a polynomial of degree  $\leq d_0$  and  $\hat{r}_x(t) = O(t^{d_0+1})$ . We need to solve:

$$(3.7) \quad u_{f(x)}(\lambda_{1,x}t) = \frac{1}{\beta_x} (\hat{r}_x(t) + \alpha_x u_x(t)).$$

Let us then solve  $u_x$  formally to see that one can only get a solution for sufficiently large  $d_0$ . This is why one can only get to remove  $r_x(t)$  up to a polynomial of that degree. Write  $u_n(t) = u_{f^{-n}(x)}(t)$ ,  $\hat{r}_n(t) = \hat{r}_{f^{-n}(x)}(t)$ ,  $\lambda_{1,n}^- =$

<sup>10</sup>Caution, here  $\alpha$  and  $\beta$  denote the Lyapunov exponents of the cocycle, which are the integral of the functions  $x \mapsto \log \alpha_x$  and  $x \mapsto \log \beta_x$ . Similarly,  $\chi_1$  denotes the first Lyapunov exponent of  $(f, \mu)$  and can be computed as the integral of  $x \mapsto \log \lambda_{1,x}$ .

$\|D_{f^{-m}(x)}f^m|_{E^1}\|^{-1} = (\lambda_{f^{-1}(x)}^1 \cdots \lambda_{f^{-m}(x)}^1)^{-1}$ ,  $\alpha_n = \alpha_{f^{-1}(x)} \cdots \alpha_{f^{-n}(x)}$  and  $\beta_n = \beta_{f^{-1}(x)} \cdots \beta_{f^{-n}(x)}$  so that:

$$(3.8) \quad u_x(t) = \frac{\alpha_k}{\beta_k} u_k(\lambda_{1,k}^- t) + \sum_{j=1}^k \left( \frac{\alpha_{j-1}}{\beta_j} \hat{r}_j(\lambda_{1,j}^- t) \right).$$

Note that  $\frac{1}{j} \log(\lambda_{1,j}^-)^d$  converges to  $-d\chi_1$  while  $\frac{1}{j} \log \frac{\alpha_j}{\beta_j}$  converges to  $\alpha - \beta$  we can then choose  $d_0$  so that for every  $x$  and  $d \geq d_0$  we have that  $(\lambda_{1,j}^-)^d \beta_j^{-1} \alpha_{j-1}$  converges exponentially fast to 0 (uniformly in  $t$ ) as  $j \rightarrow +\infty$ . This happens as long as  $d_0$  verifies that  $\alpha - \beta - d_0\chi_1 < 0$ .

Using that  $\hat{r}_j(\lambda_{1,j}^- t) = (\lambda_{1,j}^-)^{d_0+1} O(t^{d_0+1})$  we deduce that the function

$$(3.9) \quad u_x(t) = \sum_{j=1}^{\infty} \left( \frac{\alpha_{j-1}}{\beta_j} \hat{r}_j(\lambda_{1,j}^- t) \right)$$

is well defined and smooth along unstable manifolds.  $\square$

This completes the proof of the proposition.  $\square$

*Remark 3.6.* Note that if  $f$  is uniformly partially hyperbolic;  $\mathcal{E}$  is Hölder continuous over  $M$  and smooth along unstable manifolds of  $f$ ; and  $A : \mathcal{E} \rightarrow \mathcal{E}$  is Hölder continuous over  $M$ , and smooth along unstable manifolds of  $f$ , then we can choose both sections  $\xi_x, \xi_x^\perp$  to depend Hölder continuously on  $x$  near any predetermined point in  $M$ .

**3.2. Construction of 0-good unstable coordinate charts.** In this subsection we complete the proof of Proposition 2.12. Let us recall the statement.

**Proposition 3.7.** *Every partially hyperbolic measure  $\mu$  admits a family of 0-good unstable charts.*

*Proof.* Fix first a family of Pesin charts  $\{i_x^0\}_x$  (which are smooth charts varying measurably) from  $(-100\lambda_{1,x}, 100\lambda_{1,x})^3 \rightarrow M$  as in [BP]. These are chosen to verify:

- $i_x^0(0, 0, 0) = x$ ,
- $\partial_i i_x^0(0, 0, 0) \in E_x^i$  are unit vectors.

Using the normal form coordinates, we can make a coordinate change (which we still call  $\{i_x^0\}_x$ ) and further assume that the charts verify:

- $i_x^0(t, 0, 0) = \Phi_x^1(t)$ ,
- $i_x^0(0, 0, t) = \Phi_x^3(t)$ .

Recall the construction in Example 3.2. The derivative map  $Df$  on  $TM$  descends to a vector bundle automorphism on  $\mathcal{E} = TM/E^1$  over  $f$ . This vector bundle automorphism is clearly smooth along the unstable manifolds. We now write  $F_x^0 := (i_{f(x)}^0)^{-1} \circ f \circ (i_x^0) = (F_{x,1}^0, F_{x,2}^0, F_{x,3}^0)$ . Then  $\{i_x^0\}_x$  gives us a smooth trivialization of  $\mathcal{E}$  along unstable manifolds under which the bundle map takes form

$$t \mapsto \begin{pmatrix} \partial_2 F_{x,2}^0(t, 0, 0) & \partial_3 F_{x,2}^0(t, 0, 0) \\ \partial_2 F_{x,3}^0(t, 0, 0) & \partial_3 F_{x,3}^0(t, 0, 0) \end{pmatrix}.$$

Now, if we apply Proposition 3.3 to this cocycle we can find a change of coordinates of the form:

$$(x_1, x_2, x_3) \mapsto (x_1, x_2 + a_2(x_1)x_2 + a_3(x_1)x_3, x_3 + b_2(x_1)x_2 + b_3(x_1)x_3)$$

that produces new charts  $\iota_x$  for which the conditions of 0-good charts are verified since it takes the derivative cocycle along unstable manifolds to normal form.  $\square$

*Remark 3.8.* As in Remark 3.6, the 0-good unstable chart in Proposition 3.7 depends Hölder continuously on the base point near any predetermined point in  $M$ .

**3.3. Two-dimensional cocycles for the  $\ell$ -jets of the stable manifolds.** We have the following.

**Proposition 3.9.** *If there is a family of  $(\ell - 1)$ -good unstable charts  $\{\iota_x\}_{x \in M}$  which moreover verify (2.8) for  $\mu$ -almost every  $x \in M$  and some functions  $\mathcal{T}_x^\ell, a_x, b_x$ , then:*

- (i) *the derivatives  $\partial_3^k F_{x,2}(t, 0, 0) = 0$  for  $\hat{\mu}_x^1$ -a.e.  $t \in (-1, 1)$  and  $1 \leq k \leq \ell$ ,*
- (ii) *there is a  $\ell$ -good family of unstable charts (i.e. so that (2.9) also holds).*

The proof of this proposition relies on the study of cocycle normal forms. Related results have appeared in [BEFRH, Appendix A] and [TZ, Section 4].

In the rest of this subsection, we will assume that  $(f, \mu)$  admits a family of  $(\ell - 1)$ -good unstable charts verifying property (2.8) for some  $\ell \geq 2$ . Our goal here is to construct a two-dimensional cocycle in order to apply Proposition 3.3 to obtain Proposition 3.9(ii).

Given  $x \in M$  and set  $y = f(x)$ . We are going to work in the  $(\ell - 1)$ -good charts centered at  $x$  and  $y$ . Let  $F_x = (F_1, F_2, F_3)$  be  $f$  written out in the  $(\ell - 1)$ -good chart coordinates as in equation (2.7), then we have

- $F_x(0, 0, 0) = (0, 0, 0)$ ,
- and

$$(3.10) \quad F_x(x_1, 0, 0) = (\lambda_{1,x}x_1, 0, 0)$$

- and for  $\hat{\mu}_x^1$ -a.e.  $x_1$  we have that

$$\begin{pmatrix} \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \lambda_{2,x} & 0 \\ 0 & \lambda_{3,x} \end{pmatrix}$$

where derivatives in the above expression are evaluated at  $p = (x_1, 0, 0)$ , and  $\lambda_{2,x}$  and  $\lambda_{3,x}$  are constants depending only on  $x$ .

Let  $x_1 \in (-1, 1)$  be a  $\hat{\mu}_x^1$ -typical value. In other words,  $\Phi_x^1(x_1)$  is a  $\mu$ -typical point. In particular, we may assume that  $W_1^3(\Phi_x^1(x_1))$  is defined. By (2.8), we may define  $a(x_1, s), b(x_1, s)$  by

$$\iota_x^{-1}(W_{loc}^3(\Phi_x^1(x_1))) = \{(x_1 + a(x_1, s), b(x_1, s)s^{\ell+1}, s) : s \in (-1, 1)\}.$$

We can deduce Proposition 3.9(i) from the following lemma.

**Lemma 3.10.** *If a family of unstable normal coordinate charts verifies (2.8) then for every multi-index  $\alpha = (\alpha_1, \alpha_3)$  with  $\alpha_1 \geq 0$  and  $0 \leq \alpha_3 \leq \ell$ , if we denote  $\partial X^\alpha = \partial x_1^{\alpha_1} \partial x_3^{\alpha_3}$  we have that  $\frac{\partial^{|\alpha|} F_2}{\partial X^\alpha}(x_1, 0, 0) = 0$  for  $\hat{\mu}_x^1$ -a.e.  $x_1 \in (-1, 1)$ .*

*Proof.* Since  $F_2(x_1, 0, 0) = 0$ , then, for every  $i \geq 0$ , we have that  $\partial_1^i F_2(x_1, 0, 0) = 0$  for all  $i \geq 0$ . Also, by (2.8) and the  $f$ -invariance of  $\mathcal{W}^3$  we have that for  $\hat{\mu}_x^1$ -a.e.  $x_1 \in (-1, 1)$ ,

$$F_2(x_1 + a(x_1, s), b(x_1, s)s^{\ell+1}, s) = O(s^{\ell+1}).$$

Here we allow the implicit constant in  $O(\cdot)$  on the right hand side above to depend on  $F$  and  $x_1$ , but of course independent of  $s$ . We deduce that for every  $i \in \{1, \dots, \ell\}$ , and  $\hat{\mu}_x^1$ -almost every  $x_1 \in (-1, 1)$  we have

$$\begin{aligned} 0 &= \partial_s^i \{F_2(x_1 + a(x_1, s), b(x_1, s)s^{\ell+1}, s)\}_{s=0} \\ &= \partial_3^i F_2(x_1, 0, 0). \end{aligned}$$

By our hypothesis that  $\hat{\mu}_x^1$  is not atomic, there is a subset of  $x_1$  with full  $\hat{\mu}_x^1$ -measure and no isolated points. Then we deduce for every  $i$  as above and every  $m \geq 0$  and  $\hat{\mu}_x^1$ -a.e.  $x_1$  that

$$\frac{\partial^{i+m} F_2}{\partial x_1^m \partial x_3^i}(x_1, 0, 0) = 0.$$

This concludes the proof.  $\square$

For a  $\hat{\mu}_x^1$ -typical  $x_1$ , we consider the collection of (germs of) curves of the form:

$$t \mapsto (x_1 + O(t), bt^{\ell+1} + O(t^{\ell+2}), ct + O(t^2)).$$

We consider two curves to be equivalent if their values of  $b$  and  $c$  coincide. Then for each  $x_1 \in (-1, 1)$ , we may use  $(b, c)$  in  $\mathbb{R}^2$  to parametrize the equivalence classes of smooth curves through  $(x_1, 0, 0)$ . This gives rise to a  $\mathbb{R}^2$  bundle over  $(-1, 1)$  for  $\mu$ -almost every  $x \in M$ .

We now show that the above construction gives rise to a  $\mathbb{R}^2$  bundle defined  $\mu$ -almost everywhere.

**Proposition 3.11.** *Let  $\{\tilde{\iota}_x\}_{x \in M}$  be another  $(\ell-1)$ -good unstable chart which verifies (2.8). For  $\mu$ -a.e.  $x$ , we define  $H_x = \tilde{\iota}_x^{-1} \circ \iota_x$ . Given a  $\mu$ -typical  $x$  and  $\hat{\mu}_x^1$ -typical  $x_1$ ,  $H_x$  maps an equivalence class of curves through  $p = (x_1, 0, 0)$  in the chart  $\iota_x$  to an equivalence class of curves through  $p$  in the chart  $\tilde{\iota}_x$ .*

*Proof.* First, we notice that the following analogous statement as in Lemma 3.10 holds.

**Lemma 3.12.** *For every multi-index  $\alpha = (\alpha_1, \alpha_3)$  with  $\alpha_1 \geq 0$  and  $0 \leq \alpha_3 \leq \ell$ , if we denote  $\partial X^\alpha = \partial x_1^{\alpha_1} \partial x_3^{\alpha_3}$  we have that  $\frac{\partial^{|\alpha|} H_x}{\partial X^\alpha}(x_1, 0, 0) = 0$  for  $\hat{\mu}_x^1$ -a.e.  $x_1 \in (-1, 1)$ .*

We omit the proof of Lemma 3.12, which is almost identical to that of Lemma 3.10.

We write  $H_x = (H_1, H_2, H_3)$ . By definition, we have for every  $s \in (-1, 1)$  that

$$(3.11) \quad H_1(s, 0, 0) = s, \quad H_2(s, 0, 0) = H_3(s, 0, 0) = 0.$$

Take a curve  $\gamma(t) = (x_1(t), x_2(t), x_3(t))$  of form

$$\gamma(t) = (x_1 + O(t), bt^{\ell+1} + O(t^{\ell+2}), ct + O(t^2)).$$

Then by Lemma 3.12 and (3.11) we can write  $H_x \circ \gamma(t)$  as

$$(x_1 + O(t), [\frac{\partial H_2}{\partial x_2}(p)b + \frac{1}{(\ell+1)!} \frac{\partial^{\ell+1} H_2}{\partial x_3^{\ell+1}}(p)c^{\ell+1}]t^{\ell+1} + O(t^{\ell+2}), \frac{\partial H_3}{\partial x_3}(p)ct + O(t^2)).$$

From the above expression it is straightforward to conclude the proof.  $\square$

We can thus construct a measurable  $\mathbb{R}^2$  bundle  $\tilde{\mathcal{E}}_\ell$  over a  $\mu$ -full measure subset of  $M$  that is smooth along unstable manifolds, and the action of  $f$  corresponds to a *self-map*  $\tilde{F}$  of  $\tilde{\mathcal{E}}_\ell$  over  $f$ . Notice that  $\tilde{F}$  is *not* a linear bundle automorphism.

The following proposition says that  $(\tilde{\mathcal{E}}_\ell, \tilde{F})$  has  $(\mathcal{E}_\ell, F)$  a linear bundle automorphism as a factor, after a polynomial change of coordinates  $(b, c) \mapsto (b, c^{\ell+1})$ .

**Proposition 3.13.** *Let  $F_x$  denote the action of  $f$  in  $(\ell-1)$ -good unstable charts  $\{v_x\}_{x \in M}$  as above and assume that equation (2.8) is verified, then, for every  $p = (x_1, 0, 0)$  we have*

$$(3.12) \quad \begin{pmatrix} \partial^{(\ell+1)} y_2 \\ (\partial y_3)^{\ell+1} \end{pmatrix} = \begin{pmatrix} \lambda_{2,x} & \frac{\partial^{\ell+1} F_2}{\partial x_3^{\ell+1}}(p) \\ 0 & \lambda_{3,x}^{\ell+1} \end{pmatrix} \begin{pmatrix} \partial^{(\ell+1)} x_2 \\ (\partial x_3)^{\ell+1} \end{pmatrix}$$

where  $t \mapsto (x_1(t), x_2(t), x_3(t))$  represents a curve through  $p$  in the chart at  $x$ , and  $t \mapsto (y_1(t), y_2(t), y_3(t)) = F_x(x_1(t), x_2(t), x_3(t))$  represents a curve through  $F_x(p)$  in the chart at  $f(x)$ .

*Proof.* Consider  $p = (x_1, 0, 0)$  with a  $\hat{\mu}_x^1$ -typical  $x_1$ . We look at the image by  $F_x$  of a curve  $\gamma : (-\varepsilon, \varepsilon) \rightarrow (-1, 1)^3$  through  $p$  of the form (for some values of  $b, c \in \mathbb{R}$ ):

$$t \mapsto (x_1 + O(t), bt^{\ell+1} + O(t^{\ell+2}), ct + O(t^2)).$$

Using Lemma 3.10 and Taylor's expansion we get that the map  $F_x \circ \gamma$  is of the form:

$$t \mapsto (\lambda_{1,x}(x_1 + O(t)), \lambda_{2,x}bt^{\ell+1} + \frac{1}{(\ell+1)!} \frac{\partial^{\ell+1} F_2}{\partial x_3^{\ell+1}}(p)c^{\ell+1}t^{\ell+1} + O(t^{\ell+2}), \lambda_{3,x}ct + O(t^2)).$$

By a substitution  $(b, \hat{c}) = (b, c^{\ell+1})$ , we have

$$(b, \hat{c}) \mapsto (\lambda_{2,x}b + \frac{1}{(\ell+1)!} \frac{\partial^{\ell+1} F_2}{\partial x_3^{\ell+1}}(p)\hat{c}, \lambda_{3,x}^{\ell+1}\hat{c}).$$

Since we have  $(b, c) = (\frac{1}{(\ell+1)!} \partial^{(\ell+1)} x_2(t)|_{t=0}, \partial x_3(t)|_{t=0})$ , this completes the proof.  $\square$

This will be used in the next subsection to complete the proof of Proposition 3.9.

*Remark 3.14.* The bundles defined in this section correspond to some components of the  $(\ell+1)$ -jet bundle of curves through typical points in unstable manifolds of generic points.

**3.4. Construction of  $\ell$ -good charts.** Proposition 3.9 is a consequence of the following proposition.

**Proposition 3.15.** *Assume that  $f$  admits  $(\ell-1)$ -good unstable charts  $\{\iota_x\}_x$  and that equation (2.8) is verified, then there is a smooth change of coordinates which produces  $\ell$ -good unstable charts for  $f$ .*

Besides proving Proposition 3.9, the proof of this proposition allows us to obtain a formula for the change of coordinates.

*Proof.* We are in the situation of Proposition 3.13 and thus we can write the action on  $(\ell+1)$ -jets as a cocycle as in formula (3.12). Applying Proposition 3.3 one can obtain a smooth change of coordinates of the form

$$(3.13) \quad (x_1, x_2, x_3) \mapsto (x_1, x_2 + u_{x,\ell}(x_1)x_3^{\ell+1}, x_3)$$

giving that the action on  $(\ell+1)$ -jets is polynomial and thus providing  $\ell$ -good unstable charts as desired.  $\square$

*Remark 3.16.* Assume that  $f$  is uniformly partially hyperbolic, then we can inductively show that  $\ell$ -good unstable charts, if they exist, can be made to depend Hölder continuously in a neighborhood of any predetermined  $x \in M$ .

When  $\ell = 0$ , this is the content of Remark 3.8. Now we consider the general case. Given an arbitrary  $x \in M$ , and a family of  $(\ell-1)$ -good unstable charts  $\{\iota_x\}_{x \in M}$  depending Hölder continuously on the base point near  $x$ , both the bundle  $\tilde{\mathcal{E}}_\ell$  and  $\tilde{F}$  constructed above depend Hölder continuously on the base point near  $x$ . Since  $x$  is arbitrary, the bundle  $\tilde{\mathcal{E}}_\ell$  and  $\tilde{F}$  are Hölder, and smooth along the unstable manifolds. Then by Remark 3.6, the chart we obtained by applying Proposition 3.15 satisfies the inductive hypothesis: they can be made Hölder, possibly after a coordinate change, in a neighborhood of any predetermined  $x$ .

**3.5. Improvement of charts.** Here we prove the following proposition that is the starting point of the proof of Theorem 2.24.

**Proposition 3.17.** *Let  $\mu$  be a partially hyperbolic measure and  $\{\iota_x\}$  a family of  $\ell$ -good unstable charts. If there exists an integer  $d_0 > 0$  such that the stable templates of  $(\ell+1)$ -jets  $\mathcal{T}_x^\ell$  (given by (2.8)) are polynomials of degree at most  $d_0$  for almost every  $x \in M$ , then  $\mu$  admits  $(\ell+1)$ -good unstable charts. The symmetric statement holds for  $\ell$ -good stable charts.*

*Proof.* By equation (2.8) we have:

$$\iota_x^{-1}(\mathcal{W}_{loc}^3(\Phi_x^1(t))) = \{(t + O(s), \mathcal{T}_x^\ell(t)s^{\ell+1} + O(s^{\ell+2}), s) : s \in (-1, 1)\}$$

for some  $\rho' > 0$  depending only on  $f$  and  $\ell$ .

Since we know by assumption that  $\mathcal{T}_x^\ell$  is a polynomial, we can consider the new smooth charts  $\iota'_x = \iota_x \circ \psi_x$  where:

$$(3.14) \quad \psi_x(t, u, s) = (t, u + \mathcal{T}_x^\ell(t)s^{\ell+1}, s).$$

We have  $\psi_x^{-1}(t, u, s) = (t, u - \mathcal{T}_x^\ell(t)s^{\ell+1}, s)$ . Then

$$(\iota'_x)^{-1}(\mathcal{W}_{\rho'}^3(\Phi_x^1(t))) = \psi_x^{-1} \circ \iota_x^{-1}(\mathcal{W}_{\rho'}^3(\Phi_x^1(t))) = \{(t + O(s), O(s^{\ell+2}), s) : s \in (-1, 1)\}.$$

Thus the new charts verify condition (2.8). Using Proposition 3.9 we complete the proof of the proposition.  $\square$

**3.6. The uniform case.** The results in this section extend to the uniform setting with minor modifications. Let us state the results we will use and discuss briefly the adaptations needed to obtain such statements.

We first need the notion of  $\ell$ -good uniform unstable charts, where the measurability is changed to continuity. This makes sense in view of the uniformity and the fact that normal form coordinates vary continuously with the point. (This is one of the main issues in extending this study to higher dimensions.)

**Definition 3.18** (0-good uniform unstable charts). Let  $\Lambda$  be a partially hyperbolic set of a smooth diffeomorphism  $f$  on a closed 3-manifold. A continuous collection of smooth diffeomorphisms  $\{\iota_x : (-2\|Df\|, 2\|Df\|)^3 \rightarrow M\}_{x \in \Lambda}$  is a family of *uniform unstable coordinate charts* if it verifies that for every  $x \in \Lambda$  we have that  $\iota_x(t_1, 0, 0) = \Phi_x^1(t_1)$ ,  $\iota_x(0, 0, t_3) = \Phi_x^3(t_3)$  for  $t_1, t_3 \in (-1, 1)$ ,  $\partial_2 \iota_x(0, 0, 0)$  is a unit vector in  $E^2(x)$  and if we write  $F_x := \iota_{f(x)}^{-1} \circ f \circ \iota_x = (F_{x,1}, F_{x,2}, F_{x,3})$  then we have that  $F_x : (-1, 1)^3 \rightarrow \mathbb{R}^3$  verifies:

- (i)  $\partial_2 F_{x,2}(t, 0, 0) = \lambda_{2,x}$  for all  $t \in (-1, 1)$ ,
- (ii)  $\partial_3 F_{x,3}(t, 0, 0) = \lambda_{3,x}$  for all  $t \in (-1, 1)$ .
- (iii)  $\partial_2 F_{x,3}(t, 0, 0) = 0$  for all  $t \in (-1, 1)$ .

A family of unstable uniform coordinate charts is called *0-good* if moreover, there is some  $d \geq 1$  (independent of  $x \in \Lambda$ ) such that

$$(3.15) \quad \partial_3 F_{x,2}(t, 0, 0) \text{ is a polynomial of degree } \leq d \text{ in } t \in (-1, 1).$$

*Remark 3.19.* Technically, since  $\Lambda$  may have some topology, it is possible that the tangent space  $T_\Lambda M$  which splits in 3-bundles  $E^1 \oplus E^2 \oplus E^3$  cannot be coherently oriented. This imposes an obstruction for the existence of uniform coordinate charts. There are several solutions for this issue. One is to take a finite cover of (a neighborhood of)  $\Lambda$  and work there. Note that our results are independent of this finite cover and thus this will not result in a loss of generality. Taking charts defined on a fixed square is convenient to avoid charging the notation. We will thus implicitly assume throughout that the bundles are orientable and therefore this obstruction is not existing. The reader not comfortable with this assumption can consider either local families of smooth diffeomorphisms or directly parametrize the charts in cubes defined in the tangent space of each point.

We can use the cocycle normal forms which admit continuous versions to improve the charts and get:

**Definition 3.20** ( $\ell$ -good uniform unstable charts). Let  $\{\iota_x\}_{x \in \Lambda}$  be a family of 0-good unstable charts a partially hyperbolic set  $\Lambda$ . We say the family is  *$\ell$ -good* if for every  $x \in \Lambda$  there are (unique) continuous functions  $\mathcal{J}_x^\ell : (-1, 1) \rightarrow \mathbb{R}$ ,  $a_x : (-1, 1)^2 \rightarrow \mathbb{R}$  and  $b_x : (-1, 1)^2 \rightarrow \mathbb{R}$  such that for every  $t \in (-1, 1)$  so that  $\Phi_x^1(t) \in \Lambda$  we have that:

$$(3.16) \quad \iota_x^{-1}(W_1^3(\Phi_x^1(t))) = \{(t + a_x(t, s)s, \mathcal{J}_x^\ell(t)s^{\ell+1} + b_x(t, s)s^{\ell+2}, s) : s \in (-1, 1)\}.$$

and for some uniform constant  $d := d(\ell, f)$  (independent of  $x$ ) we have that

$$(3.17) \quad \partial_3^\ell F_{x,2}(t, 0, 0) \text{ is a polynomial of degree } \leq d \text{ in } t \in (-1, 1).$$

We will need the following result whose proof in the uniform setting is identical and left to the reader (note in particular that the same steps can be used as explained in Remark 3.16).

**Proposition 3.21.** *Every partially hyperbolic set  $\Lambda$  admits a family of 1-good uniform unstable charts. Moreover, if  $\Lambda$  admits a family of  $\ell$ -good uniform unstable charts and the template  $\mathcal{T}_x^\ell$  given by equation (3.16) is a polynomial of degree  $d(\ell)$  for every  $x$ , then  $\Lambda$  admits  $(\ell + 1)$ -good uniform unstable charts.*

#### 4. PROOF OF THE DICHOTOMY: PROPOSITION 2.19

We let  $f : M \rightarrow M$  be a smooth diffeomorphism and let  $\mu$  be a non-degenerate partially hyperbolic ergodic measure (cf. Definition 2.1) with  $\ell$ -good unstable charts (cf. Definition 2.13).

For a compact set  $K \subset M$  and  $x \in K$  we denote  $\hat{K}_x = (\Phi_x^1|_{(-1,1)})^{-1}(K)$ . Under our non-degeneracy assumption we have:

**Lemma 4.1.** *For every compact subset  $K \subset M$  with  $\mu(K) > 0$  we have that for  $\mu$ -almost every  $x \in K$  the set  $W_1^1(x) \cap K$  is infinite.*

*Proof.* Let  $\mathcal{A} = \{x \in M : \hat{\mu}_x^1 \text{ has at least one atom}\}$ . Since  $\mathcal{A}$  is  $f$ -invariant, by ergodicity it either has zero or full  $\mu$ -measure. Since  $\mu$  is non-degenerate,  $\mathcal{A}$  has zero measure. For almost every  $x \in K$ , we have  $x \notin \mathcal{A}$  and  $\mu_x^1(W_1^1(x) \cap K) > 0$  since  $\mu(K) > 0$ . For any such  $x$ ,  $W_1^1(x) \cap K$  is infinite.  $\square$

Proposition 2.19 is a consequence of the following:

**Proposition 4.2.** *Let  $\mu$  be a partially hyperbolic measure admitting  $\ell$ -good unstable charts. Then there exists an integer  $d := d(\ell, f, \mu) > 0$  such that:*

- (i) *either for  $\mu$ -a.e.  $x \in M$  we have that  $\mathcal{T}_x^\ell$  is a polynomial of degree  $d$  when restricted to a full measure set with respect to  $\hat{\mu}_x^1$ ,*
- (ii) *or for  $\mu$ -a.e.  $x \in M$  if  $S_x \subset (-1, 1)$  is a subset with positive  $\hat{\mu}_x^1$  measure, then  $\mathcal{T}_x^\ell|_{S_x}$  is not smooth in the sense of Whitney.*

*Proof.* We write  $\lambda_{i,x}^{(n)} := \lambda_{i,f^{n-1}(x)} \cdots \lambda_{i,x} \in \{\pm \|D_x f^n|_{E^i(x)}\| \}$  for  $i \in \{1, 3\}$ , for each integer  $n \geq 0$  and for  $\mu$ -a.e.  $x$ . We will use the notation  $J_x^{(n)} := (-\lambda_{1,x}^{(n)-1}, \lambda_{1,x}^{(n)-1})$ .

Since we have  $\ell$ -good unstable charts, by definition there is an integer  $d$  such that (2.9) holds. Iterating (2.11) we get the following formula for  $t \in J_x^{(n)}$ :

$$(4.1) \quad \mathcal{T}_{f^n(x)}^\ell(\lambda_{1,x}^{(n)} t) = \frac{\lambda_{2,x}^{(n)}}{(\lambda_{3,x}^{(n)})^{\ell+1}} \mathcal{T}_x^\ell(t) + P_x^{(n)}(t)$$

where  $P_x^{(n)}$  is a polynomial of degree  $\leq d$ . After a change of variables in (4.1) we get:

$$(4.2) \quad \mathcal{T}_x^\ell(t) = \alpha_x^{(n)} \mathcal{T}_{f^{-n}(x)}^\ell(\beta_x^{(n)} t) + Q_x^{(n)}(t)$$

where

$$\alpha_x^{(n)} = \frac{\lambda_{2,f^{-n}(x)}^{(n)}}{(\lambda_{3,f^{-n}(x)}^{(n)})^{\ell+1}}, \quad \beta_x^{(n)} = (\lambda_{1,f^{-n}(x)}^{(n)})^{-1}$$



and  $Q_x^{(n)}(t) = P_{f^{-n}(x)}^{(n)}(\beta_x^{(n)}t)$  is also a polynomial of  $t$  of degree  $\leq d$ . By enlarging  $d$  if necessary, we may assume without loss of generality that

$$(4.3) \quad \lim_n \frac{1}{n} \log(\alpha_x^{(n)}(\beta_x^{(n)})^m) < 0 \quad \text{for every } m \geq d \text{ and } \mu\text{-a.e. } x \in M.$$

Note that it suffices to take

$$d > \frac{\chi_2 - (\ell + 1)\chi_3 + (\ell + 2)\epsilon}{\chi_1 - \epsilon}.$$

We denote by  $\mathcal{A} \subset M$  the set of  $x \in M$  with the following property: there is a compact set  $S_x$  of positive  $\hat{\mu}_x^1$ -measure such that  $\mathcal{T}_x^\ell$  is smooth in the sense of Whitney on  $S_x$ . We assume that  $\mu(\mathcal{A}) > 0$ , for otherwise we already have (ii). Then by ergodicity and by (2.11), we have  $\mu(\mathcal{A}) = 1$ . By definition, it is clear that for every  $x \in \mathcal{A}$  and for almost every  $y$  (with respect to  $\mu_x^1$ ) in a neighborhood of  $x$ , we have  $y \in \mathcal{A}$ .

We may upgrade the set  $\mathcal{A}$  in the following the way. We denote by  $\mathcal{B} \subset M$  the subset of  $x \in M$  such that for every  $x \in \mathcal{B}$ , there is a compact set  $S_x$  of positive  $\hat{\mu}_x^1$ -measure such that  $\mathcal{T}_x^\ell$  is smooth in the sense of Whitney on  $S_x$  and moreover  $x$  is a density point of  $S_x$  with respect to  $\hat{\mu}_x^1$ . By definition, we see that for every  $x \in \mathcal{A}$ , the set  $\mathcal{B} \cap W_{loc}^1(x)$  has positive  $\hat{\mu}_x^1$ -measure. Since we have seen that  $\mathcal{A}$  is a full measure set. This means that  $\mu(\mathcal{B}) > 0$ . Then by ergodicity, we have  $\mu(\mathcal{B}) = 1$ .

We fix some small constant  $\varepsilon > 0$ . By Lusin's lemma, there is a compact subset  $\mathcal{Q} \subset \mathcal{B}$  such that  $\mu(\mathcal{Q}) > 1 - \frac{\varepsilon}{100}$ , and the conditional measure  $\mu_x^1$  depend continuously on  $x \in \mathcal{Q}$ . Moreover, by slightly reducing the size of  $\mathcal{Q}$  if necessary, we may assume in addition to the above that  $\mathcal{T}_x^\ell$ , as a function defined  $\hat{\mu}_x^1$ -almost everywhere, depends continuously on  $x \in \mathcal{Q}$ , in the following sense. For every Cauchy sequence  $\{x_n\}_{n \geq 0}$  in  $\mathcal{Q}$  converging to some  $x \in \mathcal{Q}$ , there exists a compact subset  $E_n \subset (-1, 1)$  for each  $n \geq 0$  such that, as  $n$  tends to infinity,  $\hat{\mu}_{x_n}^1(E_n)$  converges to 1, and  $E_n$  converges in Hausdorff's distance to a compact subset  $E$  of  $\hat{\mu}_x^1$ -measure 1, such that for every sequence  $\{t_n \in E_n\}_{n \geq 0}$  converging to  $t \in E$ , we have that  $\mathcal{T}_{x_n}^\ell(t_n)$  converges to  $\mathcal{T}_x^\ell(t)$ .

Summarizing the above, we deduce that there is a point  $x \in \mathcal{Q}$  with the following properties:

- one has that  $\mu(\overline{\{f^n(x)\}_{n \geq 0}} \cap \mathcal{Q}) = \mu(\mathcal{Q})$ ,
- there is a compact set  $\hat{S}_x \subset \text{supp}(\hat{\mu}_x^1)$  so that  $\mathcal{T}_x^\ell|_{\hat{S}_x}$  is smooth in the sense of Whitney, and
- $\frac{\hat{\mu}_x^1(\hat{S}_x \cap J_x^{(n)})}{\hat{\mu}_x^1(J_x^{(n)})}$  tends to 1 as  $n$  tends to infinity.

We can write (cf. (2.12)) for some  $c > 1$  that:

$$(4.4) \quad \mathcal{T}_x^\ell(t) = a_{x,1}t + \dots + a_{x,d}t^d + \hat{\mathcal{T}}_x^\ell(t) \text{ where } |\hat{\mathcal{T}}_x^\ell(t)| < c|t|^{d+1} \text{ if } t \in \left(-\frac{1}{c}, \frac{1}{c}\right) \cap \hat{S}_x.$$

Pick  $y \in \mathcal{Q}$  so that  $\mu(B_\varepsilon(y) \cap \mathcal{Q}) > 0$  for all  $\varepsilon > 0$ , and a sequence  $n_i \rightarrow \infty$  so that  $f^{n_i}(x) \in \mathcal{Q}$ , and converges to  $y$ . Notice that we can deduce from (4.2) and (4.4):

$$(4.5) \quad \mathcal{T}_{f^{n_i}(x)}^\ell(t) = \hat{Q}_{f^{n_i}(x)}^{(n_i)}(t) + \alpha_{f^{n_i}(x)}^{(n_i)} \hat{\mathcal{T}}_x^\ell(\beta_{f^{n_i}(x)}^{(n_i)}t)$$

where  $\hat{Q}_x^{(n_i)}(t)$  is a polynomial of degree  $\leq d$ . If  $\beta_{f^{n_i}(x)}^{(n_i)}t \in \hat{S}_x \cap J_x^{(n_i)}$  we have that

$$(4.6) \quad \alpha_{f^{n_i}(x)}^{(n_i)} \hat{\mathcal{T}}_x^\ell(\beta_{f^{n_i}(x)}^{(n_i)}t) \leq \alpha_{f^{n_i}(x)}^{(n_i)} c(\beta_{f^{n_i}(x)}^{(n_i)})^{d+1} |t|^{d+1}.$$

Notice that the  $\hat{\mu}_{f^{n_i}(x)}^1$ -measure of the set of  $t$  satisfying  $\beta_{f^{n_i}(x)}^{(n_i)}t \in \hat{S}_x \cap J_x^{(n_i)}$  is at least

$$\hat{\mu}_x^1(\hat{S}_x \cap J_x^{(n_i)}) / \hat{\mu}_x^1(J_x^{(n_i)})$$

which tends to 1 as  $i$  tends to infinity.

Up to passing to a subsequence of  $(n_i)_{i \geq 0}$ , we have that for every  $i \geq 0$  there exist a polynomial  $R_i$  of degree  $\leq d$ , and a subset  $E_i \subset (-1, 1)$  such that: as  $i$  tends to infinity,  $\hat{\mu}_{f^{n_i}(x)}^1(E_i)$  converges to 1;  $E_i$  converges in the Hausdorff's distance to a subset  $E \subset (-1, 1)$  of full  $\hat{\mu}_y^1$ -measure; and for every sequence  $\{t_i \in E_i\}_{i \geq 0}$ , we have  $R_i(t_i)$  converges to  $\mathcal{T}_y^\ell(t)$ .

Since a polynomial with degree  $\leq d$  is determined by its values at  $d + 1$  points, we deduce that  $\mathcal{T}_y^\ell$  is a polynomial of degree  $d$  on a full  $\hat{\mu}_y^1$ -measure set for every  $y \in \Omega$ . By letting  $\varepsilon$  tend to 0 we deduce that  $\mathcal{T}_y^\ell$  is a polynomial of degree  $d$  when restricted to the support of  $\hat{\mu}_x^1$  for  $\mu$ -a.e.  $x \in M$ .  $\square$

## 5. POLYNOMIALS AND RATIONAL FUNCTIONS

We consider the collections of functions  $\text{Poly}^d = \{p : [-1, 1] \rightarrow \mathbb{R} : p \text{ is a polynomial of degree } \leq d\}$  and  $\text{Rat}^d = \{\frac{q}{p} : [-1, 1] \rightarrow \mathbb{R} : p, q \in \text{Poly}^d; p(t) \neq 0 \forall t \in [-1, 1]\}$ . Clearly we have that  $\text{Poly}^d \subset \text{Rat}^d$ . We note that  $\text{Poly}^d$  is a linear subspace of  $C^0([-1, 1])$ , but  $\text{Rat}^d$  is not.

We will need a compactness result which is standard for polynomials. We first give a definition. Given constants  $k \in \mathbb{Z}_{>0}$ ,  $\sigma, \eta > 0$ , we say that a subset  $E \subset [-1, 1]$  is  $(k, \sigma, \eta)$ -spread if for any intervals  $I_0, \dots, I_k$  such that  $\sum_i |I_i| < \eta$  we have that  $E \setminus \bigcup I_i$  has at least  $k + 1$  points with pairwise distances strictly larger than  $\sigma$ .

**Proposition 5.1.** *For every  $d \in \mathbb{Z}_{>0}$ ,  $\sigma, \eta > 0$  there is  $C := C(d, \sigma, \eta) > 0$  such that for any  $(d, \sigma, \eta)$ -spread subset  $E \subset [-1, 1]$ , and any  $R \in \text{Rat}^d$  satisfying  $\sup_{t \in E} |R(t)| = 1$ , the following is true:*

- (i) *there are intervals  $I_0, \dots, I_d$  such that  $\sum_i |I_i| < \eta$  and  $|R'(t)| < C$  for every  $t \in [-1, 1] \setminus \bigcup I_i$ ,*
- (ii) *there are intervals  $J_0, \dots, J_{2d}$  such that  $\sum_i |J_i| < \eta$  and  $|R(t)| > C^{-1}$  for every  $t \in [-1, 1] \setminus \bigcup J_i$ .*

*Remark 5.2.* We will use this result in intervals of varying length (not always  $[-1, 1]$ ) and for rational functions with possibly different normalizations (not always  $\sup_{t \in E} |R(t)| = 1$ ). If the rational function  $R$  is defined in  $[a, b]$  and  $\sup_{t \in E} |R(t)| = A$  with  $E \subset [a, b]$  is such that  $\xi(E)$  is  $(d, \sigma, \eta)$ -spread where  $\xi : [a, b] \rightarrow [-1, 1]$  is the unique affine bijection, then we can apply the result to  $\hat{R}(t) = \frac{1}{A} R(\xi^{-1}(t))$  which is a rational function of the same degree defined on  $[-1, 1]$ . We obtain that the derivative of  $\hat{R}$  is less than  $C = C(d, \sigma, \eta)$  except in a finite family of intervals which cover a small proportion (less

than  $\eta$ ) and thus the derivative of  $R$  is less than  $\frac{C}{A(b-a)}$  by the chain rule. In the same way, the lower bound for  $|R(t)|$  in (ii) becomes  $\frac{A}{C}$ .

To prove this proposition we will need the following elementary result that will also serve other purposes:

**Lemma 5.3.** *Let  $C_0, \sigma > 0$  and let  $(R_n \in \text{Rat}^d)_{n \geq 1}$  be a sequence of rational functions such that for every  $n \geq 1$  there exist points  $t_{0,n}, \dots, t_{d,n} \in [-1, 1]$  with pairwise distances strictly larger than  $\sigma$  verifying that  $\sup_i |R_n(t_{i,n})| \leq C_0$ . Then, there exist a subsequence  $n_j \rightarrow \infty$ , points  $s_1, \dots, s_d \in \mathbb{D}_2$ , and a rational function  $R_\infty \in \text{Rat}^d$  (whose poles are contained in  $\{s_1, \dots, s_d\}$ ) such that  $R_{n_j}$  converges to  $R_\infty$  uniformly on compact subsets of  $\mathbb{D}_2 \setminus \{s_1, \dots, s_d\}$  (where  $\mathbb{D}_2 = \{z \in \mathbb{C} : |z| \leq 2\}$ ).*

To see the need to take out some points from the interval, we may consider the sequence  $\{R_n(z) = \frac{1}{nz^2+1}\}_{n \geq 1}$ .

*Proof.* We can write

$$(5.1) \quad R_n(z) = c_n \frac{\prod_{i=1}^{k_n} (z - a_{i,n})}{\prod_{j=1}^{m_n} (z - b_{j,n})}$$

where  $c_n, a_{i,n}, b_{j,n} \in \mathbb{C}$  and  $0 \leq k_n, m_n \leq d$ <sup>11</sup>. Up to considering a subsequence, we can assume that  $k_n = k$  and  $m_n = m$  are constant for all  $n$  and that  $c_n \rightarrow c_\infty$ ,  $a_{i,n} \rightarrow a_{i,\infty}$ ,  $b_{j,n} \rightarrow b_{j,\infty}$  all converge in  $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ .

We order  $a_{i,n}$  and  $b_{j,n}$  so that they decrease in modulus. We let  $\hat{k} \in \{1, \dots, k+1\}$  and  $\hat{m} \in \{1, \dots, m+1\}$  the smallest integers so that  $a_{\hat{k},\infty}, b_{\hat{m},\infty} \in \mathbb{C}$  (so that  $a_{i,\infty} = \infty$  if  $i < \hat{k}$  and  $b_{j,\infty} = \infty$  if  $j < \hat{m}$ ; if  $\hat{k} = k+1$  or  $\hat{m} = m+1$  means that all coefficients diverge).

We will use the fact that the functions are bounded to show the following:

**Claim 5.4.** *Up to taking further a subsequence we have that the sequence of functions*

$$\hat{c}_n(z) := c_n \frac{\prod_{i=1}^{\hat{k}-1} (z - a_{i,n})}{\prod_{j=1}^{\hat{m}-1} (z - b_{j,n})}$$

*converges uniformly in  $\mathbb{D}_2$  to a constant function  $\hat{c}_\infty \in \mathbb{C}$ .*

*Proof.* Up to taking a subsequence we can assume that the tuple of points  $(t_{i,n})_{i=0}^d$  converge to the tuple of points  $(t_{i,\infty})_{i=0}^d \in [-1, 1]^d$  which are pairwise at distance  $\geq \sigma$ . By  $k - \hat{k} \leq d$ , we can assume without loss of generality that  $t_{0,n}$  is uniformly far from  $a_{i,n}$  for all  $\hat{k} \leq i \leq k$  (and therefore for all  $1 \leq i \leq k$  as  $a_{i,n} \rightarrow \infty$  if  $i < \hat{k}$ ).

It is enough to show that the functions  $\hat{c}_n(z)$  are bounded uniformly in some point of  $\mathbb{D}_2$  since one can compute the logarithmic derivative as:

$$\frac{\hat{c}'_n(z)}{\hat{c}_n(z)} = \sum_{i=1}^{\hat{k}-1} \frac{1}{z - a_{i,n}} - \sum_{j=1}^{\hat{m}-1} \frac{1}{z - b_{j,n}},$$

which converges uniformly to 0 in  $\mathbb{D}_2$  because the coefficients  $a_{i,n}$  and  $b_{j,n}$  diverge.

<sup>11</sup>We use the convention that  $\prod_{i=1}^0 (z - \gamma_i) = 1$ .

To get the uniform boundedness, we compute the value of  $\hat{c}_n$  in the point  $t_{0,n} \in [-1, 1] \subset \mathbb{D}_2$ . Notice that

$$R_n(t_{0,n}) = \hat{c}_n(t_{0,n}) \frac{\prod_{i=\hat{k}}^k (t_{0,n} - a_{i,n})}{\prod_{i=\hat{m}}^m (t_{0,n} - b_{i,n})}$$

is uniformly bounded. Since the product  $\prod_{i=\hat{k}}^k (t_{0,n} - a_{i,n})$  is uniformly bounded from below and  $R_n(t_{0,n})$  is uniformly bounded from above, we get the desired result.  $\square$

Now it is easy to show that outside any given neighborhood of  $\{b_{\hat{m},\infty}, \dots, b_{m,\infty}\}$  in  $\mathbb{D}_2$  the sequence  $(R_n)_{n \geq 1}$  converges uniformly to

$$(5.2) \quad R_\infty(z) = \hat{c}_\infty \frac{\prod_{i=\hat{k}}^k (z - a_{i,\infty})}{\prod_{i=\hat{m}}^m (z - b_{i,\infty})}.$$

The rational function  $R_\infty$  verifies the desired properties.  $\square$

*Proof of Proposition 5.1.* We proceed by contradiction and find a sequence  $R_n = \frac{Q_n}{P_n} \in \text{Rat}^d$  so that there are  $(d, \delta, \eta)$ -spread sets  $E_n \subset [-1, 1]$  so that  $|R_n(t)| \leq 1$  for all  $t \in E_n$  and for every family of intervals  $I_1, \dots, I_d$  whose sum of lengths do not exceed  $\delta$  there is some  $t \in [-1, 1] \setminus \bigcup I_i$  so that  $|R'_n(t)| > n$ .

Using Lemma 5.3 we can find a rational function  $R_\infty \in \text{Rat}^d$ , a subsequence  $n_j \rightarrow \infty$  and points  $s_1, \dots, s_d \in \mathbb{C}$  containing the poles of  $R_\infty$  such that  $R_{n_j} \rightarrow R_\infty$  on every compact subset of  $\mathbb{D}_2 \setminus \{s_1, \dots, s_d\}$ . In particular, on every compact subset of  $\mathbb{D}_2 \setminus \{s_1, \dots, s_d\}$  we have that  $R'_{n_j} \rightarrow R'_\infty$  uniformly. By covering the points  $s_1, \dots, s_d \in \mathbb{R}$  with small intervals whose lengths add up to less than  $\sigma$  we find a contradiction since  $R'_\infty$  is bounded away of those intervals. This proves (i).

To prove (ii), one can use that  $R_\infty$  has the form given by equation (5.2) and use logarithmic derivatives (i.e. consider the derivative of  $\log(R_\infty)$ ) to see that

$$\frac{R'_\infty(z)}{R_\infty(z)} = \sum_i \frac{1}{z - a_i} - \sum_j \frac{1}{z - b_j}, \quad a_i, b_j \in \mathbb{C}.$$

It follows that by a contradiction argument we can bound  $R_\infty(z)$  from below away from the  $2d$  points  $a_i, b_j$  (counted with multiplicity).

This concludes the proof of the proposition.  $\square$

## 6. DISTANCE TO RATIONAL FUNCTIONS

This section is devoted to showing the following statement.

**Proposition 6.1.** *Let  $\mu$  be a partially hyperbolic measure with  $\ell$ -good unstable charts  $\{v_x\}$  such that the stable templates  $\mathcal{T}_x^\ell$  (cf. (2.8)) are not in  $\text{Poly}^d$  for some  $d = d(f, \ell) = O(\ell)$ . Then  $\mu$  has QNI.*

As a consequence, using Proposition 3.17 we deduce the following.

**Corollary 6.2.** *If  $\mu$  does not have QNI then it admits  $\ell$ -good stable and unstable charts for every integer  $\ell \geq 1$ .*

*Remark 6.3.* The proof of Proposition 6.1 can be simplified if one knows that the center unstable direction is more regular, for instance, if  $f$  where an Anosov diffeomorphism with expanding center direction, then it gets simpler as the full unstable foliation is of class  $C^{1+}$ . This allows us to consider only polynomials instead of general rational functions in the proof below. However, to apply the result for  $f^{-1}$  one would need to deal with the lack of integrability and regularity of the center stable subspaces (note that it is very rare for both the center stable and the center unstable subspaces be more regular than Hölder).

**6.1. Some uniform distance.** We now deduce some consequences from the hypothesis that the functions  $\mathcal{T}_x^\ell$  are not polynomials.

Let  $d_1$  be a sufficiently large positive integer depending only on  $f, \ell$ , to be determined later. The underlying assumption of this section is that  $\mu$  is a non-degenerate partially hyperbolic measure with  $\ell$ -good charts and the functions  $\mathcal{T}_x^\ell$  are not polynomials of degree  $\leq d_1$  restricted to the support of  $\hat{\mu}_x^1$  for almost every  $x$ . By ergodicity, we see that item (ii) of Proposition 4.2 holds.

**Proposition 6.4.** *For every  $\varepsilon > 0$ , for every integer  $d > 0$ , there is a compact set  $K \subset M$  with  $\mu(K) > 1 - \varepsilon$  such that for every  $\nu > 0$  there is  $c := c(d, \nu, \varepsilon) > 0$  such that for any  $x \in K$  and any polynomials  $P, Q$  of degree  $\leq d$ , the set*

$$(6.1) \quad I_{x,c}^{P,Q} = \left\{ t \in (-1, 1) : \left| \mathcal{T}_x^\ell(t) - \frac{Q(t)}{P(t)} \right| \leq c \right\}$$

satisfies that  $\hat{\mu}_x^1(I_{x,c}^{P,Q}) \leq \nu$ .

Note that since  $\mathcal{T}_x^\ell$  is defined on a  $\hat{\mu}_x^1$ -full measure set, the set  $I_{x,c}^{P,Q}$  is also only defined up to a  $\hat{\mu}_x^1$ -null measure set (also recall that  $\hat{\mu}_x^1$  is normalized so that it is a probability measure in  $(-1, 1)$ ).

*Proof.* We proceed by contradiction. We notice that if  $c < c'$  then  $I_{x,c}^{P,Q} \subset I_{x,c'}^{P,Q}$ . If the result does not hold then there exist an integer  $d > 0$ , a constant  $\nu > 0$ , and a compact set  $K_0 \subset M$  with  $\mu(K_0) > 0$  such that for every  $x \in K_0$ , for every integer  $n > 0$  there exist polynomials  $P_n, Q_n : (-1, 1) \rightarrow \mathbb{R}$  of degree  $\leq d$  such that  $\hat{\mu}_x^1(I_{x,1/n}^{P_n, Q_n}) > \nu$ . By reducing the size of  $K_0$  if necessary, we may assume in addition that all objects we will consider vary continuously on  $K_0$  (cf. Proposition B.1).

We now show that there exists  $\delta > 0$  such that each set  $I_{x,1/n}^{P_n, Q_n}$  contains  $d + 1$  points with pairwise distances larger than  $\delta$ .

Since by assumption  $\mu$  is non-degenerate, we may assume that  $\hat{\mu}_x^1$  are non-atomic probabilities varying continuously on  $x$  restricted to the compact set  $K_0$ , for any  $\nu > 0$  there exists  $\delta > 0$  (which depends on  $\mu, \nu, d$  and  $K_0$ ) so that for every  $x \in K_0$ , every subset of  $(-1, 1)$  with  $\hat{\mu}_x^1$ -measure larger than  $\nu$  must contain  $d + 1$ -points with pairwise distances larger than  $\delta$ .

Let us fix an arbitrary  $x \in K_0$ . Up to taking some subsequence, we can apply Lemma 5.3 to obtain a rational function  $R_\infty \in \text{Rat}^d$  such that  $\frac{Q_n}{P_n} \rightarrow R_\infty$  uniformly away from finitely many points in  $[-1, 1]$ . We deduce that  $\mathcal{T}_x^\ell$  coincides with a rational function in a set of positive  $\hat{\mu}_x^1$  measure

(this is because we can remove intervals of uniform size around the points where the convergence is not uniform, and this will cover no more than half the measure of  $\hat{\mu}_x^1$ , so there is a positive measure set where the template coincides with a rational function, in particular smooth). Since  $\mu(K_0) > 0$  and  $x$  is arbitrary, we can apply Proposition 4.2 to get a contradiction.  $\square$

*Remark 6.5.* Note that we cannot ensure with the limiting process that the template will coincide with a smooth function in some open set of the support a priori. This is why we need to deal with density points and apply Proposition 4.2.

Before stating the next proposition, we recall the notation (2.3):  $W_1^{1,k}(x) = f^{-k}(W_1^1(f^k(x)))$  and  $W_1^{3,k}(x) = f^k(W_1^3(f^{-k}(x)))$ .

**Proposition 6.6.** *There exists  $\delta := \delta(f, \ell, \mu) > 0$  such that for every integer  $d_1 > 0$ , every  $\varepsilon > 0$  there is  $\Omega \subset M$  with  $\mu(\Omega) > 1 - \frac{\varepsilon}{10}$  such that for every  $\nu > 0$ , there is  $c := c(f, \mu, d_1, \varepsilon, \nu) > 0$  such that for every  $x \in \Omega$ , every  $k > 0$  such that  $f^k(x) \in \Omega$  and every pair of polynomials  $Q, P$  of degree  $\leq d_1$  there is a set  $U_{Q,P,x,k} \subset W_1^{1,k}(x)$  such that  $\mu_x^1(U_{Q,P,x,k} \cap W_1^{1,k}(x)) > (1 - \nu)\mu_x^1(W_1^{1,k}(x))$  and*

$$(6.2) \quad \left| \mathcal{J}_x^\ell(t) - \frac{Q(t)}{P(t)} \right| > ce^{-\delta k}, \quad \forall t \in (\Phi_x^1)^{-1}(U_{Q,P,x,k}).$$

*Proof.* Let  $\Omega_0$  be a compact set such that  $\mu(\Omega_0) > 1 - \frac{\varepsilon}{100}$  and every object we will consider varies continuously as in Proposition B.1.

We apply Proposition 6.4 to  $\varepsilon/100$  and get a set  $\Omega_1$  verifying Proposition 6.4 in place of  $K$  (in particular,  $\mu(\Omega_1) > 1 - \frac{\varepsilon}{100}$ ). Then the set  $\Omega = \Omega_0 \cap \Omega_1$  satisfies  $\mu(\Omega) > 1 - \frac{\varepsilon}{10}$ . Proposition 6.4 gives a constant  $c_0 > 0$  so that for every pair of polynomials  $P_0, Q_0$  of degree  $\leq d_1$  and a point  $x \in M$  with  $f^k(x) \in \Omega$ , we have that  $\hat{\mu}_{f^k(x)}^1(I_{f^k(x),c_0}^{P_0,Q_0}) < \nu \mu_{f^k(x)}^1(W_1^1(f^k(x))) = \nu$  (recall that  $\mu_{f^k(x)}^1$  is of unit mass restricted to  $W_1^1(f^k(x))$ ).

Let  $d_1$  be sufficiently large so that  $\partial_3^{\ell+1} F_{x,2}(\cdot, 0, 0)$  is a polynomial of degree  $\leq d_1$  for  $\mu$ -a.e.  $x$ . Now fix some point  $x \in \Omega$  such that  $f^k(x) \in \Omega$  and polynomials  $P, Q$  of degree  $\leq d_1$ . By formula (2.11) and the fact that  $f$  has  $\ell$ -good charts, we see that there is a polynomial of degree  $\leq d_1$ , denoted by  $R$ , such that for every  $t \in (-\lambda_{1,x}^{(k)})^{-1}, (\lambda_{1,x}^{(k)})^{-1}$  we have that

$$(6.3) \quad \mathcal{J}_x^\ell(t) = \frac{(\lambda_{3,x}^{(k)})^{\ell+1}}{\lambda_{2,x}^{(k)}} \mathcal{J}_{f^k(x)}^\ell(\lambda_{1,x}^{(k)} t) + R(\lambda_{1,x}^{(k)} t).$$

Therefore, to estimate  $\left| \mathcal{J}_x^\ell(t) - \frac{Q(t)}{P(t)} \right|$  for  $t \in (-\lambda_{1,x}^{(k)})^{-1}, (\lambda_{1,x}^{(k)})^{-1}$  it is enough to estimate:

$$(6.4) \quad \left| \frac{(\lambda_{3,x}^{(k)})^{\ell+1}}{\lambda_{2,x}^{(k)}} (\mathcal{J}_{f^k(x)}^\ell(\lambda_{1,x}^{(k)} t) - \frac{Q_0(t)}{P_0(t)}) \right|$$

for some polynomials  $P_0, Q_0$  of degree at most  $2d_1$ .

We let  $U_{P,Q,x,k}$  be the set of points in  $W_1^{1,k}(x)$  such that their images under  $f^k$  do not belong to  $I_{f^k(x),c_0}^{P_0,Q_0}$ . Since the measure  $\mu$  is invariant, we have that

$$\frac{\mu_x^1(U_{P,Q,x,k})}{\mu_x^1(W_1^{1,k}(x))} = 1 - \frac{\mu_{f^k(x)}^1(\Phi_{f^k(x)}^1(I_{f^k(x),c_0}^{P_0,Q_0}))}{\mu_{f^k(x)}^1(W_1^1(f^k(x)))} \geq 1 - \nu.$$

Since  $x$  and  $f^k(x)$  both belongs to  $\mathcal{Q}$ , there exist  $\delta = \delta(f, \ell, \mu) > 0$  and  $c_1 = c_1(f, \mathcal{Q}) > 0$ , so that  $\frac{\lambda_{2,x}^{(k)}}{(\lambda_{3,x}^{(k)})^{\ell+1}} \geq c_1 e^{-\delta k}$ . By (6.4) and (6.3), we can choose  $c = c_0 c_1$  so that (6.2) holds for points in  $U_{P,Q,x,k}$ .  $\square$

**6.2. Proof of Proposition 6.1.** To show that Definition 2.8 is verified we will use the equivalent characterization of QNI in Lemma A.1.

Let  $V, \alpha > 0$  be two constants, and let  $d_1 > 0$  be an integer, chosen depending only on  $f, \mu$  at the end of proof. We fix an arbitrary constant  $\varepsilon > 0$ . Using Proposition B.1, we choose a compact set  $\mathcal{P}_1 \subset M$  with  $\mu(\mathcal{P}_1) > 1 - \varepsilon/100$  which verify the following properties:

- (i)  $W_1^1(x)$  and  $W_1^3(x)$  vary Hölder continuously with respect to  $x \in \mathcal{P}_1$  in the smooth topology (see [BP, §8]); and the chart  $\iota_x$  has uniformly bounded smooth norm for all  $x \in \mathcal{P}_1$ ;
- (ii) given  $\nu > 0$ , we have that for large enough  $j > 0$  and for every  $x \in \mathcal{P}_1$  one has  $\mu_x^i(W_1^{i,j}(x) \cap \mathcal{P}_1) > (1 - \frac{\nu}{10})\mu_x^i(W_1^{i,j}(x))$  for  $i \in \{1, 3\}$ .

Consider  $\nu_n = 2^{-n}$  and let  $\mathcal{Q}_n$  be the set given by Proposition 6.6 for the values  $\nu_n$  so that  $\mu(\mathcal{Q}_n) > 1 - (\varepsilon/100)2^{-n}$ . Consider  $\mathcal{Q} = \bigcap \mathcal{Q}_n$  and  $\mathcal{P}_0 = \mathcal{P}_1 \cap \mathcal{Q}$  which also verifies the previous properties (and  $\mu(\mathcal{P}_0) > 1 - \varepsilon$ ). Moreover, we know that given  $\nu > 0$  we know that if  $x, f^k(x) \in \mathcal{Q}$  then equation (6.2) is verified for every rational function  $\frac{Q}{P}$  of degree at most  $d_1$  with  $\delta$  depending only on  $f, \ell, \mu$ ; and  $c$  depending only on  $f, \ell, \nu, \varepsilon, \mu$ .

We fix some  $\nu \in (0, 1)$  from now on. In the following, we say that a constant  $C$  is uniform if  $C > 0$  and it depends only on  $f, \mu$  and the sets given above. We will use  $c$  to denote a generic uniform constant which may vary from line to line.

We fix an arbitrary  $x' \in \mathcal{P}_0$ .

The Hölder condition in (i) ensures that there exist uniform constants  $c_1, c_2, \gamma_1, \gamma_2, r_0 > 0$  such that for  $y' = \Phi_{x'}^3(s) \in W_{r_0}^3(x') \cap \mathcal{P}_1$ , we may write

$$(6.5) \quad \iota_{x'}^{-1}(W_{loc}^1(y')) = \{(t', \hat{Q}(t'), \hat{P}(t')) : t' \in (-r_0, r_0)\},$$

where  $\hat{Q}$  and  $\hat{P}$  are smooth functions (with uniformly bounded derivatives of any given order) such that for  $t' \in (-r_0, r_0)$

$$(6.6) \quad c_2 |s|^{\gamma_2} < |\hat{P}(t')| + |\hat{Q}(t')| < c_1 |s|^{\gamma_1}.$$

By making  $r_0$  smaller if necessary, for any  $t \in (-r_0, r_0)$ , we denote  $z' = \Phi_{x'}^1(t)$ , and we have a well-defined  $t'$  as the unique constant depending on  $t$  and  $y$  such that  $(t', \hat{P}(t'))$  belongs to  $\pi_{1,3}(\iota_{x'}^{-1}(W_1^3(z')))$  where  $\pi_{1,3}$  is the projection from  $\mathbb{R}^3$  to its 1st and 3rd coordinates. By the Hölder condition, we deduce that  $|t'| \leq C|t|^{\gamma_3}$  for some  $\gamma_3 > 0$  depending only on  $f$  and  $\mu$ . We may write

$$(6.7) \quad \iota_{x'}^{-1}(W_{loc}^3(z')) = \{(t + a(t)u + e_{z'}(t, u), \mathcal{J}_{x'}^\ell(t)u^{\ell+1} + \hat{e}_{z'}(u), u) : u \in (-r_0, r_0)\}.$$

Now assume that  $t$  is chosen as that  $z' \in \mathcal{P}_1$ . Then there is a uniform constant  $c_3 > 0$  such that  $|a(t)| \leq c_3$ ,  $|e_{z'}(u)| \leq c_3 u^2$  and  $|\hat{e}_{z'}(u)| \leq c_3 u^{\ell+2}$ . Notice that we may deduce from the above bound that

$$(6.8) \quad |t - t'| < c_3 |\hat{P}(t')| + c_3 |\hat{P}(t')|^2 \leq 2c_3 |\hat{P}(t')|.$$

**Lemma 6.7.** *There is a uniform constant  $c_4 > 0$  such that we have*

$$(6.9) \quad d(W_1^3(z'), W_1^1(y')) \geq c_4 |\hat{P}(t')|^{\ell+1} \mathcal{J}_{x'}^\ell(t) - \hat{Q}(t')| - c_4^{-1} |\hat{P}(t')|^{\ell+2}.$$

*Proof.* Since  $\{i_x\}$  is a family of  $\ell$ -good unstable coordinates, we can see that the tangent spaces of the curves  $i_{x'}^{-1}(W_{loc}^3(z'))$  and  $i_{x'}^{-1}(W_{loc}^1(y'))$  are both disjoint from a closed cone  $\{(v_1, v_2, v_3) : |v_1| + |v_3| \leq c|v_2|\}$  for some constant  $c > 0$  independent of all choices of  $x' \in \mathcal{P}_0$ ,  $y', z' \in \mathcal{P}_1$  given above. Thus, by matching the 1st and 3rd coordinates of the expressions in (6.5) and (6.7), we have

$$d(W_1^3(z'), W_1^1(y')) \geq c |\hat{P}(t')|^{\ell+1} \mathcal{J}_{x'}^\ell(t) + \hat{e}_{z'}(\hat{P}(t')) - \hat{Q}(t')|.$$

By (6.7) and the choices of  $t, t'$ , we deduce that

$$c |\hat{P}(t')|^{\ell+1} \mathcal{J}_{x'}^\ell(t) + \hat{e}_{z'}(\hat{P}(t')) - \hat{Q}(t')| \geq c_4 |\hat{P}(t')|^{\ell+1} \mathcal{J}_{x'}^\ell(t) - \hat{Q}(t')| - c_4^{-1} |\hat{P}(t')|^{\ell+2}$$

for some uniform constant  $c_4 > 0$ . This concludes the proof.  $\square$

We first consider the case where  $|\hat{P}(t')| < c_2 |s|^{\gamma_2}/2$ . In this case, by (6.6), we have

$$(6.10) \quad |\hat{Q}(t')| > c_2 |s|^{\gamma_2}/2.$$

Then by Lemma 6.7 and by reducing the size of  $r_0$  if necessary, we have

$$(6.11) \quad \begin{aligned} \text{RHS of (6.9)} &\geq c_4 |\hat{Q}(t')| - c_4^{-1} |\hat{P}(t')|^{\ell+1} \mathcal{J}_{x'}^\ell(t) - c_4^{-1} |\hat{P}(t')|^{\ell+2} \\ &\geq c_4 c_2 |s|^{\gamma_2}/2 - c_4^{-1} (c_2 |s|^{\gamma_2}/2)^{\ell+1} \\ &\geq c_4 c_2 |s|^{\gamma_2}/4. \end{aligned}$$

Now it remains to consider the case where  $|\hat{P}(t')| \geq c_2 |s|^{\gamma_2}/2$ . Then we have

$$(6.12) \quad c_2 |s|^{\gamma_2}/2 \leq |\hat{P}(t')| < c_1 |s|^{\gamma_1}.$$

We let  $d_1$  be large depending only on  $f, \mu$  and  $\ell$ . Fix some  $s \in (-r_0, r_0)$ . Denote  $r_s = |s|^{C_*}$  where

$$(6.13) \quad C_* = \frac{2\gamma_2(2\ell+3)}{\gamma_3(d_1+1)}.$$

Now we fix an arbitrary  $t \in (-r_s, r_s)$  such that  $z' \in \mathcal{P}_1$ . Then we have  $|t'| \leq 2r_s$ , and

$$(6.14) \quad \text{RHS of (6.9)} \geq c_4 |\hat{P}(t')|^{\ell+1} \left| \mathcal{J}_{x'}^\ell(t) - \frac{\hat{Q}(t')}{\hat{P}(t')^{\ell+1}} \right| - c_4^{-1} |\hat{P}(t')|^{\ell+2}.$$

Since  $y' \in \mathcal{P}_1$ , there exists a uniform constant  $c_5 > 0$  so that Taylor's expansion gives  $\hat{Q}(\tau) = Q(\tau) + q(\tau)$  and  $\hat{P}^{\ell+1}(\tau) = P(\tau) + p(\tau)$  such that  $|q(\tau)|, |p(\tau)| < c_5 |\tau|^{d_1+1}$ , and  $P, Q$  are polynomials of degree  $\leq d_1$ . By  $C_* > \frac{2\gamma_2(\ell+1)}{\gamma_3(d_1+1)}$  and by reducing  $r_0$  if necessary, we deduce  $|P(t')| \geq$



$|\hat{P}(t')|^{\ell+1}/2 > 0$  by (6.12) and  $|t'| \leq Cr_s^{\gamma_3}$ . Enlarging  $c_5$  if necessary, the function  $\beta_5(t') := \left| \frac{\hat{Q}(t')}{\hat{P}(t')^{\ell+1}} - \frac{Q(t')}{P(t')} \right|$  satisfies that

$$(6.15) \quad \beta_5(t') \leq 2c_1c_5 \frac{|t'|^{d_1+1}|s|^{\gamma_1}}{|\hat{P}^{2(\ell+1)}(t')|}.$$

Denote  $R(t) = \frac{Q(t)}{P(t)} \in \text{Rat}^{d_1}$ . We have  $R(0) = 0$ . By (6.9) and our choice of  $Q, P$ , we get

$$d(W_1^3(z'), W_1^1(y')) \geq c_4 |\hat{P}(t')|^{\ell+1} |\mathcal{J}_{x'}^\ell(t) - R(t')| - c_4^{-1} |\hat{P}(t')|^{\ell+2} - c_4^{-1} |\hat{P}(t')|^{\ell+1} \beta_5(t').$$

Then by (6.15) and by reducing  $c_4$  if necessary, the distance  $d(W_1^3(z'), W_1^1(y'))$  is bounded from below by

$$(6.16) \quad \begin{aligned} & c_4 |\hat{P}(t')|^{\ell+1} |\mathcal{J}_{x'}^\ell(t) - R(t')| - c_4^{-1} (|\hat{P}(t')|^{-(\ell+1)} |t'|^{d_1+1} |s|^{\gamma_1} \\ & \quad + |\hat{P}(t')|^{\ell+2}) \\ & \geq c_4 |\hat{P}(t')|^{\ell+1} |\mathcal{J}_{x'}^\ell(t) - R(t')| - cc_4^{-1} |\hat{P}(t')|^{\ell+2}. \end{aligned}$$

The last inequality above follows from (6.12), (6.13) and  $|t'| \leq Cr_s^{\gamma_3}$ .

Recall that  $\nu > 0$  is small constant that we have fixed at the beginning of the proof. We have the following.

**Claim 6.8.** *There exist constants  $V_0, m_0 > 0$ ,  $d_1 = O(\ell)$  and  $\alpha_0, C_0 > 0$  such that the following is true. Given any  $m \geq m_0$ , denote by  $I^{(m)} = (-\lambda_{1,x'}^{(m)})^{-1}, (\lambda_{1,x'}^{(m)})^{-1}$ . Then, if  $x' \in \mathcal{Q}$  is such that  $f^m(x') \in \mathcal{Q}$ , and  $s \in (-r_0, r_0)$  is such that*

$$(6.17) \quad \frac{-\log s}{\log \lambda_{1,x'}^{(m)}} \in \left[ \frac{3}{5}V_0, \frac{5}{3}V_0 \right],$$

and  $y' = \Phi_{x'}^3(s) \in \mathcal{P}_1$ , there is a subset  $U_{y'}$  of  $W_1^{1,m}(x') = \Phi_{x'}^1(I^{(m)})$  such that  $\mu_{x'}^1(U_{y'}) > (1-\nu)\mu_{x'}^1(W_1^{1,m}(x'))$ , and for any  $z' = \Phi_{x'}^1(t) \in U_{y'}$  we have

$$(6.18) \quad d(W_1^3(z'), W_1^1(y')) > C_0 e^{-\alpha_0 m}.$$

*Proof.* Given some  $s \in (-r_0, r_0)$  with  $\Phi_{x'}^3(s) \in \mathcal{Q}$  we can define the functions  $\hat{P}$  and  $\hat{Q}$  as in equation (6.5). We will find constants  $V_0, m_0$  and  $d_1$  depending only on  $f, \mu, \ell$  and constants  $\alpha_0, C_0$  so that if  $f^m(x') \in \mathcal{Q}$  for some  $m > m_0$  and  $s$  verifies bounds in equation (6.17) with  $y' = \Phi_{x'}^3(s) \in \mathcal{P}_1$ , then we verify (6.18). Note that the functions  $\hat{P}$  and  $\hat{Q}$  are well defined as long as  $y' \in \mathcal{P}_1$  so the rest of the constructions can be made.

We will fix  $V_0 > \frac{2}{\gamma_1}$  and  $d_1 > 10V_0\gamma_3^{-1}\gamma_2(\ell+1)$ . Note that this will ensure that  $C_* < \frac{3}{5V_0}$  from our choice of  $C_*$ .

We will consider  $C_0$  sufficiently small and  $\alpha_0, m_0$  sufficiently large verifying some conditions that will be explicit in the proof. For a given  $m > m_0$  and  $s$  verifying condition (6.17) and  $\Phi_{x'}^1(s) \in \mathcal{P}_1$ , we consider  $U_s$  to be the set of points on which (6.18) holds.

We will divide the set  $I^{(m)} = I_{>}^{(m)} \cup I_{<}^{(m)}$  where

- $t' \in I_{>}^{(m)}$  if  $|\hat{P}(t')| > c_2|s|^{\gamma_2}/2$  (cf. equation (6.12)) and,
- $t' \in I_{<}^{(m)}$  if  $|\hat{P}(t')| \leq c_2|s|^{\gamma_2}/2$ .

Note that if we have  $|\hat{P}(t')| \leq c_2|s|^{\gamma^2/2}$  (i.e.  $t' \in I_{<}^{(m)}$ ), then by (6.9) and (6.11), we have

$$d(W_1^3(z'), W_1^1(y')) \geq c_2 c_4 |s|^{\gamma^2/4}.$$

We can then deduce (6.18) with appropriate  $C_0$ ,  $\alpha_0$  and  $m_0$  for all  $t' \in I_{<}^{(m)}$ . Thus, if  $C_0$  is sufficiently small and  $\alpha_0$  sufficiently large and  $m$  sufficiently large, we have that  $I_{<}^{(m)}$  can be considered fully contained in  $U_s$ .

We will now deal with  $I_{>}^{(m)}$  and show that for an appropriate choice of  $C_0$ ,  $\alpha_0$ , if  $m$  is large we get that  $U_s \cap I_{>}^{(m)}$  covers  $I_{>}^{(m)}$  except for a subset whose measure is at most  $\nu \hat{\mu}_{x'}^1(I^{(m)})$ .

There exists  $\sigma > 0$  so that for every subset  $T \subset I^{(m)}$  with  $\hat{\mu}_{x'}^1(T) \geq (1 - \nu/2) \hat{\mu}_{x'}^1(I^{(m)})$  it verifies that  $T$  is  $(d_1, \sigma, \nu/2)$ -spread in  $I^{(m)}$ .

By Proposition 5.1 (see Remark 5.2) there exists  $C := C(d_1, \sigma, \nu/2) > 1$  such that if  $\hat{R} \in \text{Rat}^{d_1}$  and  $D = \sup_{t \in I^{(m)}} |\hat{R}(t)|$  then  $|\hat{R}'(\tau)| \leq CD \lambda_{1,x'}^{(m)}$  and  $|\hat{R}'(\tau)| > D/C$  for every  $\tau \in I^{(m)} \setminus \cup I_i$  where  $I_i \subseteq \hat{I}_i$ ,  $0 \leq i \leq 3d$ , are open subintervals of  $I^{(m)}$  such that  $\hat{I}_0, \dots, \hat{I}_{3d}$  are mutually disjoint, whose union is of  $\hat{\mu}_{x'}^1$ -measure at most  $\frac{\nu}{2} \hat{\mu}_{x'}^1(I^{(m)})$ . We can without loss of generality assume that  $t' \in I_{>}^{(m)}$  since we already know that  $I_{<}^{(m)} \subset U_s$  for well chosen values of the constants.

We choose the intervals so that there are  $I_i \subseteq \hat{I}_i$  with  $\hat{I}_i$  containing the  $\kappa|I_i|$ -neighborhood of  $I_i$ , where  $\kappa > 0$  depends only on  $\mu$ ,  $f$  and  $\mathcal{Q}$ , but is independent of  $i$ ,  $x$  and  $m$ .

Let the rational function  $R$  be constructed as before so that (6.16) holds. Consider  $D = \sup_{t \in (-1,1)} |R(t)|$  and  $\hat{D} = \sup_{t \in (-1,1)} |\mathcal{J}_{x'}^\ell(t)|$ .

Assume first that  $D \geq C\hat{D} + 1$ . Then, it follows that choosing  $t' \in I^{(m)} \setminus \cup \hat{I}_i$  we have that  $|R(t') - \mathcal{J}_{x'}^\ell(t)| > 1$  and therefore by equation (6.16) we have that:

$$d(W_1^3(z'), W_1^1(y')) \geq \frac{c_4}{2} |\hat{P}(t')|^{\ell+1} \geq C_0 e^{-\alpha_0 m},$$

Where  $C_0 \ll c_4 c_2^{\ell+1}$  and  $\alpha_0$  and  $m_0$  are large enough so that  $|s|^{\gamma^2(\ell+1)} \geq e^{-\alpha_0 m}$  if  $m \geq m_0$ . So in this case it holds that  $U_s$  contains  $I^{(m)} \setminus \cup_i \hat{I}_i$ .

We can therefore assume from now on that  $D \leq C\hat{D} + 1$ .

By Proposition 6.6 and our choice of  $c_\nu$ , there exist  $\delta = \delta(f, \ell, \mu) > 0$  and a subset  $U'_s \subset I^{(m)}$  such that (as long as  $m_0$  is sufficiently large)  $\hat{\mu}_{x'}^1(U'_s) > (1 - \nu/10) \hat{\mu}_{x'}^1(I^{(m)})$  and if  $t \in (\Phi_{x'}^1)^{-1}(U'_s)$ , then

$$|\mathcal{J}_{x'}^\ell(t) - R(t)| > ce^{-\delta m}.$$

We claim that  $U_s$  contains  $U'_s \cap (I^{(m)} \setminus \cup \hat{I}_i)$  for well chosen values of  $C_0$ ,  $\alpha_0$  and sufficiently large  $m$ .

By (6.8), (6.12) and the way we chose the intervals  $I_i \subseteq \hat{I}_i$  by letting  $|s|$  be smaller than some uniform constant, for any  $t \in U_s$  we have that  $t$  and  $t'$  belong to the same component of  $I^{(m)} \setminus \cup_i I_i$ , and consequently we have

$$(6.19) \quad |R(t) - R(t')| \leq CD \lambda_{1,x'}^{(m)} |t - t'| \leq c_6 \lambda_{1,x'}^{(m)} |\hat{P}(t')|$$

for some uniform constant  $c_6 > 0$  (note that here we used that  $D$  is uniformly bounded).

Putting together (6.16) and (6.19), we get that if  $t \notin J$  then there is a constant  $\beta_7(t)$  and a uniform constant  $c_7 > 0$  such that

$$(6.20) \quad d(W_1^3(z'), W_1^1(y')) \geq c_4 |\hat{P}(t')|^{\ell+1} |\mathcal{J}_{x'}^\ell(t) - R(t)| - \beta_7(t)$$

where

$$\begin{aligned} |\beta_7(t)| &\leq c_4^{-1} (|\hat{P}(t')|^{\ell+1} |R(t) - R(t')| + |\hat{P}(t')|^{\ell+2}) \\ &\leq c_7 \lambda_{1,x'}^{(m)} |\hat{P}(t')|^{\ell+2}. \end{aligned}$$

By (6.12), we have

$$|\hat{P}(t')|^{\ell+1} > c_2^{\ell+1} |s|^{\gamma_2(\ell+1)} \quad \text{and} \quad |\hat{P}(t')| \leq c_1 |s|^{\gamma_1}.$$

By (6.20) we deduce that for any  $t' \in U'_s \cap (I^{(m)} \setminus \bigcup \hat{I}_i)$ , we have

$$(6.21) \quad \begin{aligned} d(W_1^3(z'), W_1^1(y')) &\geq c_4 |\hat{P}(t')|^{\ell+1} \left( ce^{-\delta m} - \frac{|\beta_7(t)|}{|\hat{P}(t')|^{\ell+1}} \right) \\ &\geq c_4 |\hat{P}(t')|^{\ell+1} \left( ce^{-\delta m} - c_7 \lambda_{1,x'}^{(m)} |\hat{P}(t')| \right) \\ &\geq c_8 |s|^{\gamma_2(\ell+1)} \left( c_8 e^{-\delta m} - c_8^{-1} \lambda_{1,x'}^{(m)} |s|^{\gamma_1} \right) \end{aligned}$$

for some uniform constant  $c_8 > 0$ .

We fix a large constant  $m_0 > 0$  such that for every  $m > m_0$ , and every  $|s| < (\lambda_{1,x'}^{(m)})^{-3V_0/5}$ , we have

$$\lambda_{1,x'}^{(m)} |s|^{\gamma_1} < \frac{c_8^2}{2} e^{-\delta m}.$$

Then for every  $s$  satisfying (6.17), we have

$$d(W_1^3(z'), W_1^1(y')) \geq \frac{c_8^2}{2} e^{-\delta m} (\lambda_{1,x'}^{(m)})^{-5V_0\gamma_2(\ell+1)/3}.$$

which gives (6.18) for  $C_0 < \frac{c_8^2}{2}$  and  $\alpha_0$  so that  $e^{-\delta m} (\lambda_{1,x'}^{(m)})^{-5V_0\gamma_2(\ell+1)/3} \geq e^{-\alpha_0 m}$ .

Notice that, by the hypothesis that  $|t| < r_s$ , we need to ensure that

$$C_* \log |s| \geq -\frac{5C_*}{3} V_0 \log \lambda_{1,x'}^{(m)} \geq -\log \lambda_{1,x'}^{(m)} \geq \log |t|,$$

which is ensured by our choice of  $V_0$  and  $d_1$ .

This shows that  $U_s$  contains  $U'_s \cap (I^{(m)} \setminus \bigcup \hat{I}_i)$  for well chosen values of  $C_0, \alpha_0, m_0$  and thus completes the proof.  $\square$

Note that Claim 6.8 has put us under the conditions of Lemma A.1 from which we can deduce that QNI is verified for  $f$ . Indeed, let  $V_0, \alpha_0$ , be given by Claim 6.8. Then by slightly reducing the size of  $\mathcal{P}_0$  if necessary, and by letting integer  $k_0 \geq m_0$  be sufficiently large depending only on  $f, \mu$ , we may assume that for any  $x \in \mathcal{P}_0$  and any  $k > k_0$ , we have  $k^{-1} \log \lambda_{3,f-k(x)}^k \in (\frac{99}{100} \chi_3, \frac{100}{99} \chi_3)$  and  $k^{-1} \log \lambda_{1,x}^{(k)} \in (\frac{99}{100} \chi_1, \frac{100}{99} \chi_1)$ . We set  $V = -\chi_1 V_0 / \chi_3$  and  $\alpha = \alpha_0$ . Then for any integers  $k_1, k_2 \geq k_0$  such that  $\frac{k_2}{k_1} \in (\frac{2}{3} V, \frac{3}{2} V)$  we

choose  $S_x = W_1^{3,k_2}(x) \cap \mathcal{P}_1$ . Then for each  $y = \Phi_x^3(s) \in S_x$ , we have (6.17) for  $m = k_1$ . We set  $U_y = \Phi_x^1(U_s)$  where  $U_s$  is given by Claim 6.8. Then we can see that the conditions of Lemma A.1 is satisfied.

## 7. COMPATIBILITY OF GOOD CHARTS OR QNI: PROOF OF THEOREM 2.25

Throughout this section, we let  $\mu$  be a partially hyperbolic measure of  $f$ , which admits  $L$ -good stable charts  $\{\iota_x\}_{x \in M}$  and  $L$ -good unstable charts  $\{\iota'_x\}_{x \in M}$  for some large integer  $L$ , which will be determined later depending on  $f, \mu$  and  $\ell$ .

To facilitate the proof, we introduce the following notation. We denote by  $T_0$  the hyperplane  $\{(t_1, t_2, t_3) : t_2 = 0\}$ . Given a function  $\phi : (-1, 1)^2 \rightarrow \mathbb{R}$ , we denote by  $\tau_\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  the diffeomorphism  $\tau_\phi(x, y, z) = (x, y + \phi(x, z), z)$ . We define  $T_\phi = \tau_\phi(T_0)$ .

Given  $x \in M$ . We define

$$S_{1,x} = \iota_x^{-1} \left( \bigcup_{t \in (-1,1)} W_1^1(\Phi_x^3(t)) \right).$$

In the following, we say that  $S_{1,x}$  and  $T_\phi$  (for some function  $\phi$ ) are *tangent to order  $L$  on a subset  $U \subset W_1^3(x)$*  if there exists  $C > 0$  depending on  $f, \mu, x$  and  $U$  such that for any  $t$  with  $\Phi_x^3(t) \in U$  we have

$$(7.1) \quad \tau_\phi^{-1} \iota_x^{-1} (W_{loc}^1(\Phi_x^3(t))) = \{(s, O(C|s|^L), t + O(Cs)) : s \in (-1, 1)\}.$$

Similarly, we define

$$S'_{3,x} = (\iota'_x)^{-1} \left( \bigcup_{t \in (-1,1)} W_1^3(\Phi_x^1(t)) \right),$$

and say that  $S'_{3,x}$  and  $T_\phi$  are tangent to order  $L$  on a subset  $U' \subset W_1^1(x)$  if there exists  $C' > 0$  depending on  $f, \mu, x$  and  $U'$  such that for any  $t$  with  $\Phi_x^1(t) \in U'$  we have

$$(7.2) \quad \tau_\phi^{-1} (\iota'_x)^{-1} (W_{loc}^3(\Phi_x^1(t))) = \{(t + O(Cs), O(C|s|^L), s) : s \in (-1, 1)\}.$$

Given a  $\mu$ -typical  $x$ , the smooth surface  $(\iota'_x)^{-1} \circ \iota_x(T_0)$  contains a graph of a function  $\psi_x : (-r_x, r_x)^2 \rightarrow \mathbb{R}$  for some  $r_x \in (0, 1)$ . Notice that  $\psi_x$  is only defined for  $x$  in a  $\mu$ -full measure set. We also denote

$$I_x = \{(a, b) \in \mathbb{N}^2 : a + b \leq 2\ell \text{ and } \partial_1^a \partial_3^b \psi_x(0, 0) \neq 0\}.$$

By definition,  $\partial_1^k \psi_x(0, 0) = \partial_3^k \psi_x(0, 0) = 0$  for every integer  $k \geq 0$ , and consequently we have

$$(7.3) \quad (\{(0, i) : 0 \leq i \leq 2\ell\} \cup \{(i, 0) : 0 \leq i \leq 2\ell\}) \cap I_x = \emptyset.$$

We have the following.

**Lemma 7.1.** *Given  $x \in M$  such that  $\psi_x$  is defined. If  $I_x = \emptyset$  then we have  $I_{f(x)} = \emptyset$ , and (2.13) holds at  $x$ .*

*Proof.* Assume that  $x \in M$  satisfies that  $I_x = \emptyset$ . We can deduce (2.13) from Taylor's expansion.

Assume to the contrary that  $I_{f(x)} \neq \emptyset$ . Then by Lemma 2.15, there exists  $C_x > 0$  such that for every  $\varepsilon > 0$  there exist  $t_1, t_3$  with  $|t_1|, |t_3| < \varepsilon$  satisfying

$$d(W_1^3(\Phi_x^1(t_1)), W_1^1(\Phi_x^3(t_3))) \leq C_x \varepsilon^{2\ell+1}$$

and

$$d(f(W_1^3(\Phi_x^1(t_1))), f(W_1^1(\Phi_x^3(t_3)))) \geq \varepsilon^{2\ell}/C_x.$$

We obtain a contradiction by letting  $\varepsilon$  be sufficiently small. Consequently, we deduce that  $I_{f(x)} = \emptyset$ .  $\square$

The main result of this section is the following.

**Proposition 7.2.** *Given an integer  $\ell > 0$ , there exists  $L = L(\mu, f, \ell) > 0$  such that the following is true. Assume that there is a set  $\mathcal{P}_0 \subset M$  with  $\mu(\mathcal{P}_0) > 0$  such that for any  $x \in \mathcal{P}_0$  we have  $I_x \neq \emptyset$ . Then  $\mu$  has QNI.*

*Proof.* By Lemma 7.1, the set of  $x$  such that  $I_x \neq \emptyset$  is  $f$ -invariant. Then by ergodicity we may assume without loss of generality that  $\mu(\mathcal{P}_0) = 1$ .

By Pesin's theory, there is a constant  $\delta > 0$ , depending only on  $f, \mu$ , such that for any  $\varepsilon > 0$ , there is a compact set  $\mathcal{P}_\varepsilon \subset M$  with  $\mu(\mathcal{P}_\varepsilon) \geq 1 - \varepsilon$  such that  $E^1$  and  $E^3$  are uniformly  $\delta$ -Holder continuous on  $\mathcal{P}_\varepsilon$ .

By (7.3), there are numbers  $\frac{-10\chi_1\ell^2}{\chi_3\delta^2} > V > \frac{-10\chi_1}{\chi_3\delta^2}$  and  $K > 1$  such that for any  $V' \in (\frac{-\chi_3}{4\chi_1}V\delta^2, \frac{-4\chi_3}{\chi_1}V\delta^{-2})$ , for any  $x \in \mathcal{P}_0$ , the set  $\{a + bV' : (a, b) \in I_x\}$  admits a unique minimum  $K(V', x) \leq K$ . By the choices of  $V$  and  $K$ , we may assume that there exists a measurable positive function  $x \mapsto c_x$  such that for every  $x \in \mathcal{P}_0$ , for any  $s_1, s_3 \in (-c_x, c_x) \setminus \{0\}$  with  $\frac{\log|s_3|}{\log|s_1|} \in (\frac{-\chi_3}{4\chi_1}V\delta^2, \frac{-4\chi_3}{\chi_1}V\delta^{-2})$ , we have

$$(7.4) \quad |\psi_x(s_1, s_3)| \geq c_x |s_1|^K.$$

We fix a small constant  $\varepsilon > 0$ .

Let  $C_1 > 1$  be a large constant to be determined in due course. By Lusin's theorem and by enlarging  $C_1$  if necessary, we may take a compact subset  $B_0 \subset \mathcal{P}_0$  with

$$(7.5) \quad \mu(B_0) > 1 - \varepsilon/2,$$

satisfying the following properties:

(i) we have

$$(7.6) \quad r_x, c_x > C_1^{-1}, \quad x \in B_0;$$

(ii) the smooth norms of the charts  $\iota_x$  and  $\iota'_x$  are bounded by  $C_1$  whenever  $x \in B_0$ ;

(iii) for any  $n \in \mathbb{Z}$ , for any  $i \in \{1, 3\}$  and any  $x \in B_0$  we have

$$(7.7) \quad C_1^{-1} e^{n(\chi_i - \varepsilon)} < \|D_x f^n|_{E^i(x)}\| < C_1 e^{n(\chi_i + \varepsilon)};$$

(iv)  $E^1$  and  $E^3$  are uniformly  $\delta$ -Holder continuous on  $B_0$ , with  $\delta$ -Holder norms bounded by  $C_1$ .

By Proposition B.1, there is a compact subset  $B \subset B_0$  with

$$(7.8) \quad \mu(B) > 1 - \varepsilon,$$

such that the following holds: for every  $\nu > 0$ , there exist  $m_0 > 0$  such that for every  $x \in B$  and every  $m > m_0$  there exist a subset  $U_x^{1,m} \subset B_0 \cap W_1^{1,m}(x)$  such that

$$(7.9) \quad \mu_x^1(U_x^{1,m}) > (1 - \nu)\mu_x^1(W_1^{1,m}(x)),$$

and  $S'_{3,x}, T_0$  are tangent to order  $L$  on  $U_x^{1,m}$ ; and a subset  $U_x^{3,m} \subset B_0 \cap W_1^{3,m}(x)$  such that

$$(7.10) \quad \mu_x^3(U_x^{3,m}) > (1 - \nu)\mu_x^3(W_1^{3,m}(x)),$$

and  $S_{1,x}, T_0$  are tangent to order  $L$  on  $U_x^{3,m}$ . Moreover, by Lemma 2.15, we may assume that the implicit constants for above tangencies are uniformly bounded.

Let us denote

$$(7.11) \quad S'_{1,x} = (\iota'_x)^{-1} \circ \iota_x(S_{1,x}).$$

Then  $S'_{1,x}$  and  $(\iota'_x)^{-1} \circ \iota_x(T_0) = T_{\psi_x}$  are tangent to order  $L$  on  $U_x^{3,m}$ . By enlarging  $C_1$  if necessary, we may assume that

(v) (7.1) holds for  $C = C_1$  whenever  $x \in B$ ,  $U = U_x^{3,m}$  and  $\phi = \psi_x$ ; and

(7.2) holds for  $C = C_1$  whenever  $x \in B$ ,  $U' = U_x^{1,m}$  and  $\phi = 0$ .

We may assume without loss of generality that  $U_x^{3,m}$ , resp.  $U_x^{1,m}$ , is disjoint from  $W_r^{1,m}(x)$ , resp.  $W_r^{3,m}(x)$ , for some  $r = r(f, \mu, \nu, \varepsilon) > 0$ .

Now take an arbitrary  $x \in B$  and two large integers  $k_1, k_2 > m_0$  such that  $f^{k_1}(x), f^{-k_2}(x) \in B$  and

$$(7.12) \quad \frac{k_2}{k_1} \in \left(\frac{2}{3}V, \frac{3}{2}V\right).$$

Let us now suppose that  $t_1, t_3 \in (-C_1^{-1}, C_1^{-1}) \setminus \{0\}$  satisfy that

$$(7.13) \quad \Phi_x^3(t_3) \in \tilde{U}_x^3 := U_x^{3,k_2} \subset W_1^{3,k_2}(x);$$

and

$$(7.14) \quad \Phi_x^1(t_1) \in \tilde{U}_x^1 := U_x^{1,k_1} \subset W_1^{1,k_1}(x).$$

By (7.9) and (7.10), we have

$$(7.15) \quad \frac{\mu_x^1(\tilde{U}_x^1)}{\mu_x^1(W_1^{1,k_1}(x))}, \frac{\mu_x^3(\tilde{U}_x^3)}{\mu_x^3(W_1^{3,k_2}(x))} > 1 - \nu.$$

Thus, by (7.12) and by enlarging  $k_1, k_2$  if necessary, we may assume that for any  $t_1, t_3$  satisfying (7.13) and (7.14), the following also holds:

$$(7.16) \quad \frac{\log |t_3|}{\log |t_1|} \in \left(\frac{-3\chi_3}{5\chi_1}V, \frac{-5\chi_3}{3\chi_1}V\right).$$

Recall that  $\pi_{1,3} : (-1, 1)^3 \rightarrow (-1, 1)^2$  denotes the projection to the first and the third coordinates. Consider the curves  $\gamma_3 = (\iota'_x)^{-1}(W_1^3(\Phi_x^1(t_1)))$

and  $\gamma_1 = (\iota'_x)^{-1}(W_1^1(\Phi_x^3(t_3)))$ . Their projections  $\pi_{1,3}(\gamma_3)$  and  $\pi_{1,3}(\gamma_1)$  have a unique intersection  $(s_1, s_3)$ . In other words, there exist  $r_1, r_3 \in \mathbb{R}$  such that

$$(7.17) \quad (s_1, r_1, s_3) \in S'_{1,x}, \quad (s_1, r_3, s_3) \in S'_{3,x}.$$

We denote

$$(7.18) \quad r'_1 = \psi_x(s_1, s_3).$$

By definition, we have  $(s_1, r'_1, s_3) \in T_{\psi_x}$ . By the tangency between  $S'_{1,x}$  and  $(\iota'_x)^{-1} \circ \iota_x(T_0) = T_{\psi_x}$  on  $\tilde{U}_x^3$ ; and the tangency between  $S'_{3,x}$  and  $T_0$  on  $\tilde{U}_x^1$ , we have

$$(7.19) \quad |r'_1 - r_1| \leq C_1 |s_1|^L, \quad |r_3| \leq C_1 |s_3|^L.$$

Moreover, by the  $\delta$ -Holder continuity of  $E^1$  and  $E^3$  on  $B_0$ , we have

$$(7.20) \quad C_1^{-1} |t_1|^{1/\delta} < |s_1| + |r_1| < C_1 |t_1|^\delta, \quad |s_3| < C_1 |t_3|^\delta.$$

Let us first assume that  $|r_1| \geq (2C_1)^{-1} |t_1|^{1/\delta}$ . Then by (7.20), the second inequality in (7.19) and a similar argument as in Lemma 6.7, we deduce that

$$\begin{aligned} d(W_1^3(\Phi_x^1(t_1)), W_1^1(\Phi_x^3(t_3))) &\geq C^{-1} |r_1 - r_3| \\ &\geq C^{-1} |r_1| - C^{-1} |r_3| \\ &\geq C^{-1} C_1^{-1} |t_1|^{1/\delta} - 2CC_1 |t_3|^{L\delta}. \end{aligned}$$

By (7.16), and by assuming that

$$L > \frac{-4\chi_1}{\chi_3 \delta^2 V},$$

we have

$$(7.21) \quad d(W_1^3(\Phi_x^1(t_1)), W_1^1(\Phi_x^3(t_3))) > \frac{1}{2} C^{-1} C_1^{-1} |t_1|^{1/\delta}.$$

Now we assume that  $|r_1| < (2C_1)^{-1} |t_1|^{1/\delta}$ . In this case we have that

$$|s_1| > (2C_1)^{-1} |t_1|^{1/\delta}.$$

Then, together with (7.20) and (7.16), we deduce that

$$(7.22) \quad \frac{\log |s_3|}{\log |s_1|} \in \left( \frac{-\chi_3}{4\chi_1} V \delta^2, \frac{-4\chi_3}{\chi_1} V \delta^{-2} \right).$$

In particular, we have  $|s_3| < |s_1|$ . By our choice of  $V$ , by (7.4), (7.18), and by enlarging  $C_1$  if necessary, we have

$$(7.23) \quad |r'_1| > C_1^{-1} |s_1|^K > C^{-1} C_1^{-K-1} |t_1|^{K/\delta}.$$

Thus we have

$$\begin{aligned} d(W_1^3(\Phi_x^1(t_1)), W_1^1(\Phi_x^3(t_3))) &\geq C^{-1} |r_1 - r_3| \\ &\geq C^{-1} |r'_1| - C^{-1} |r'_1 - r_1| - C^{-1} |r_3| \\ &\geq C^{-1} C_1^{-K-1} |t_1|^{K/\delta} - CC_1 (|s_1|^L + |s_3|^L) \\ &\geq C^{-1} C_1^{-K-1} |t_1|^{K/\delta} - 2CC_1 |t_1|^{L\delta}. \end{aligned}$$

By assuming that

$$(7.24) \quad L > 2\delta^{-2} K,$$

we have that for any  $x \in B$ , for any sufficiently large  $k_1, k_2$  satisfying (7.12), for any  $t_1, t_3$  satisfying (7.13), (7.14) and (7.16), we have

$$(7.25) \quad d(W_1^3(\Phi_x^1(t_1)), W_1^1(\Phi_x^3(t_3))) > \frac{1}{2}C^{-1}C_1^{-K-1}|t_1|^{K/\delta}.$$

By Lemma A.1, we see that  $f$  has the QNI property.  $\square$

*Proof of Theorem 2.25.* It suffices to combine Lemma 7.1 and Proposition 7.2.  $\square$

## 8. CONTINUOUS AND UNIFORM VERSIONS FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS

In this section we explain how to adapt the results in the previous sections to the case where the measure is supported in a (uniformly) partially hyperbolic set.

Let  $f : M \rightarrow M$  be a smooth diffeomorphism and  $\Lambda \subset M$  a compact  $f$ -invariant subset. Assume that there is a continuous splitting of  $T_\Lambda M = E^u \oplus E^c \oplus E^s = E^1 \oplus E^2 \oplus E^3$  and consider the functions  $\lambda_{i,x}$  defined in equation (2.15) which are continuous on  $\Lambda$  and verify (for an appropriate metric) that  $|\lambda_x^1| > |\lambda_x^2| > |\lambda_x^3|$  as well as  $|\lambda_x^1| > 1 > |\lambda_x^3|$ .

We wish to show:

**Theorem 8.1.** *Let  $f : M \rightarrow M$  be a smooth diffeomorphism of a closed 3 manifold  $M$  and  $\Lambda \subset M$  a compact  $f$ -invariant partially hyperbolic subset. Then, for every  $\mu$  with full support on  $\Lambda$  there is the following dichotomy:*

- *Either  $\mu$  has QNI (cf. Definition 2.8), or,*
- *for every  $\ell \geq 1$ , the set  $\Lambda$  is jointly integrable up to order  $\ell$  (cf. Definition 1.1).*

Note that the second condition is independent of the measure, and forces every measure with full support on  $\Lambda$  to not verify QNI. Also, while not obvious from the definition of the QNI property, our result implies that having this property for all invariant measures with full support on some partially hyperbolic subset with good continuation properties (e.g. the whole manifold) is an open property in the smooth topology.

**8.1. Proof of Theorem 8.1.** As for the measurable case, the proof has three stages<sup>12</sup>:

- First show that if QNI is not verified, then there are  $\ell$ -good stable and unstable charts for all  $\ell$ . In this case, we will need to check that, since the normal form coordinates vary continuously on the point, these  $\ell$ -good charts will turn out to be continuous. The proof mimics what is done in §6.
- Then, show that if QNI is not verified, then, the approximations of the stable and unstable Hopf brushes are at the same up to order  $\ell$ . This proof mimics the one done in §7 and indeed, in this case, no continuity is needed.

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<sup>12</sup>Note that whenever possible, we will use the results from previous sections, particularly §6 and §7. We note that in those sections, the fact that templates where measurable functions included an extra difficulty that here we could do without if we wanted to show the results here directly. We leave those simplifications to the interested reader.



- Finally, show that this compatibility of charts implies that there is a continuous family of surfaces that approximates well the Hopf brushes up to order  $\ell$ .

Let us give the main arguments and see how to adapt what has already been done:

**Lemma 8.2.** *Let  $\mu$  be a measure of full support on  $\Lambda$  and assume that  $\Lambda$  does not admit  $\ell$ -good uniform unstable charts for some  $\ell \geq 1$ . Then,  $\mu$  has QNI.*

*Proof.* First, let  $k \leq \ell$  be the largest  $k \geq 1$  such that  $\Lambda$  admits  $k$ -good uniform unstable charts. We claim that if the templates  $\mathcal{T}_x^k$  are not polynomial, then  $\mu$  has QNI.

To see this, note first that we can apply Proposition 6.4 since the templates cannot verify option (i) in Proposition 4.2 as continuity of the templates (cf. Remark 3.19) would imply that they are polynomial everywhere and that would allow to construct  $(k+1)$ -good uniform unstable charts using Proposition 3.21.

Now, the rest of the proof of Proposition 6.1 works verbatim.  $\square$

The rest of the proof of Theorem 8.1 follows using results that we previously discussed. Note first that what we have so far allows us to construct continuous families of smooth surfaces  $\mathcal{S}_x^s$  and  $\mathcal{S}_x^u$  that approximate the Hopf brushes  $\mathcal{H}_x^s$  and  $\mathcal{H}_x^u$  to arbitrary good order. These surfaces are constructed by mapping the set  $\{(x, y, z) : y = 0\}$  by the compatible  $\ell$ -good charts as considered in §7. So, if we show that these surfaces are mutually tangent to high order we would be done. Note that the surfaces are constructed using the templates, so, they coincide with the ones constructed in §7, and thus, the proof in §7 shows that if they are not tangent to high order almost everywhere, then, the measure has QNI. Since the measure is full support and the surfaces vary continuously, this concludes (note that this requires the uniform  $\ell$ -good charts to vary continuously, which is the content of Remark 3.19).

#### APPENDIX A. DISCUSSION ON THE NOTION OF QNI

In this appendix we provide some alternative ways to understand the QNI property and prove Proposition 2.9.

*Proof of Proposition 2.9.* Assume  $\mu$  has the QNI property for  $f$  as stated in Definition 2.8. We wish to show that it also verifies the property for  $f^{-1}$ . For this, consider  $\alpha > 0$ ,  $\varepsilon > 0$  and  $\nu > 0$  and we will consider the set  $\mathcal{P}$  given by the fact that  $\mu$  has the QNI property and some value of  $C = C(\nu, \varepsilon)$  (which may differ from the one given for  $\mu$ ) and  $k_0$  as given for  $\mu$ .

To get the result, it is enough to show that there is a function  $\rho(\nu)$  such that  $\rho(\nu) \rightarrow 0$  as  $\nu \rightarrow 0$  so that if  $k > k_0$  and  $x, f^k(x), f^{-k}(x) \in \mathcal{P}$  there is a subset  $\hat{U}_x \subset W_1^{1,k}(x)$  with  $\mu_x^1(\hat{U}_x) > (1 - \rho(\nu))\mu_x^1(W_1^{1,k}(x))$  with the property that given  $z \in \hat{U}_x$  there is  $\hat{S}_z \subset W_1^{3,k}(x)$  with  $\mu_x^3(\hat{S}_z) > (1 - \rho(\nu))\mu_x^3(W_1^{3,k}(x))$  so that if  $y \in \hat{S}_z$  then

$$(A.1) \quad d(W_1^1(y), W_1^3(z)) > \hat{C}e^{-\alpha k}.$$

Consider the set of pairs  $(y, z) \in W_1^{3,k}(x) \times W_1^{1,k}(x)$  which verify equation (A.2). It follows from the fact that  $\mu$  verifies QNI that this set has measure larger than  $(1 - \nu)^2$  with respect to the probability measure  $\frac{\mu_x^3}{\mu_x^3(W_1^{3,k}(x))} \times \frac{\mu_x^1}{\mu_x^1(W_1^{1,k}(x))}$  and thus, by Fubini's theorem it follows that considering  $\rho(\nu) = 2\sqrt{\nu}$  the result follows.  $\square$

The following characterization of QNI is the one we establish to prove our main results.

**Lemma A.1.** *Assume that  $\mu$  is a partially hyperbolic measure for a smooth diffeomorphism  $f$  satisfying the following property.*

- there exist  $V > 2$ ,  $\alpha > 0$  and,
- for every  $\varepsilon > 0$ , there exists a subset  $\mathcal{P}_0 \subset M$  of measure  $\mu(\mathcal{P}_0) > 1 - \varepsilon$   
and,
- for every  $\nu > 0$ , there exist  $k_* = k_*(\nu, \varepsilon)$ , and a constant  $C = C(\nu, \varepsilon)$  so that such that:

if  $k_1, k_2 \geq k_*$  with  $\frac{k_2}{k_1} \in (\frac{2}{3}V, \frac{3}{2}V)$  and  $x, f^{k_1}(x), f^{-k_2}(x) \in \mathcal{P}_0$ , then

- there is a subset  $S_x \subset W_1^{3,k_2}(x)$  with  $\mu_x^3(S_x) > (1 - \nu)\mu_x^3(W_1^{3,k_2}(x))$  with the following property:
- For all  $y \in S_x$  there exists  $U_y \subset W_1^{1,k_1}(x)$  with  $\mu_x^1(U_y) > (1 - \nu)\mu_x^1(W_1^{1,k_1}(x))$  so that if  $z \in U_y$  then

$$(A.2) \quad d(W_1^3(z), W_1^1(y)) > Ce^{-\alpha k_1}.$$

Then  $\mu$  has the QNI property (cf. Definition 2.8).

Note that the condition on  $k_1, k_2$  says that equation (A.2) (up to possibly changing  $\alpha$ ) is the same as asking that  $d(W_1^1(y), W_1^3(z)) > Ce^{-\alpha \min\{k_1, k_2\}}$  or other variations.

*Proof.* We fix some  $\varepsilon \in (0, 1)$ . We set

$$(A.3) \quad a_0 = \frac{1 + \frac{2}{3}V}{\frac{2}{3}V - 1}, \quad b_0 = \frac{1 + \frac{3}{2}V}{\frac{3}{2}V - 1} \in (a_0, 1).$$

We can without loss of generality suppose that  $\mathcal{P}_0$  satisfies that

$$(A.4) \quad \mu(\mathcal{P}_0) > 1 - \min\left(\frac{\varepsilon}{100}, \frac{b_0 - a_0}{4b_0}\right).$$

We define in the following way the set  $\mathcal{P}$  in Definition 2.8. Given a constant  $N > 0$ , we let  $\mathcal{P} = \mathcal{P}(\varepsilon, N)$  be the set of points  $x \in \mathcal{P}_0$  such that for every  $k > N$ ,

$$(A.5) \quad \frac{1}{k} |\{0 \leq j \leq k - 1 : f^j(x) \in \mathcal{P}_0\}| > 1 - \min\left(\frac{\varepsilon}{50}, \frac{b_0 - a_0}{2b_0}\right),$$

$$(A.6) \quad \frac{1}{k} |\{-k \leq j \leq -1 : f^j(x) \in \mathcal{P}_0\}| > 1 - \min\left(\frac{\varepsilon}{50}, \frac{b_0 - a_0}{2b_0}\right).$$

Using Birkhoff's theorem we may assume by letting  $N$  be sufficiently large that

$$(A.7) \quad \mu(\mathcal{P}) > 1 - \varepsilon.$$

We now show that the statement in Definition 2.8 is satisfied for this  $\mathcal{P}$ .

Fix some  $\nu \in (0, 1)$ . We let  $k_* = k_*(\varepsilon, \nu)$  be given by the hypothesis of the lemma.

Let us take some  $x \in \mathcal{P}$  and an integer  $k \geq k_0$  such that  $f^k(x) \in \mathcal{P}$  and  $f^{-k}(x) \in \mathcal{P}$ .

We denote  $k' = (\frac{a_0 + b_0}{2b_0})k$ . By the definition of  $\mathcal{P}$ , we have

$$|\{a_0k < l < b_0k' : f^l(x) \in \mathcal{P}_0\}| > (1 - \frac{b_0 - a_0}{2b_0})b_0k' - a_0k - 1 > 0.$$

Consequently, there exists some  $j \in \{a_0k + 1, \dots, b_0k'\}$  such that  $x' := f^j(x) \in \mathcal{P}_0$ .

Denote  $k_2 = k + j$  and  $k_1 = k - j$ . Then we have

$$f^{k_1}(x') \in \mathcal{P}_0, \quad f^{-k_2}(x') \in \mathcal{P}_0 \quad \text{and} \quad \frac{k_2}{k_1} \in (\frac{2}{3}V, \frac{3}{2}V).$$

By the hypothesis of the lemma, there exists a subset  $S' \in W_1^{3, k_2}(x')$  with  $\mu_{x'}^3(S') > (1 - \nu)\mu_{x'}^3(W_1^{3, k_2}(x'))$  such that for any  $y \in S'$  there exists  $U'_y \subset W_1^{1, k_1}(x')$  with  $\mu_{x'}^1(U'_y) > (1 - \nu)\mu_{x'}^1(W_1^{1, k_1}(x'))$  such that if  $z \in U'_y$ , then

$$(A.8) \quad d(W_1^1(y), W_1^3(z)) > Ce^{-\alpha k}.$$

We define  $S_x = f^{-j}(S')$  and for each  $y \in S_x$ , define  $U_y = f^{-j}(U'_{f^j(y)})$ . Notice that  $\mu_x^3 = (f^j)_*\mu_{x'}^3$  and  $\mu_x^1 = (f^j)_*\mu_{x'}^1$ . Then it is clear that the statement in Definition 2.8 holds for  $x$  by letting  $\alpha$  be larger.  $\square$

We end this appendix by commenting the difference between our definition of QNI and that in [Ka].

The only difference between the definitions is the choice of the notion of local stable/unstable manifolds. We have chosen to work with  $W_1^i(x)$  (with  $i \in \{1, 3\}$ ) to be the unstable/stable manifold of length 1 with respect to the normal form coordinates. Note that the Riemannian length of these manifolds is not continuously varying as the normal form coordinates are just measurable, but they vary continuously in sets of arbitrarily large measure. To choose the scales, we have chosen to use  $W_1^{1, k}(x) = f^{-k}(W_1^1(f^k(x)))$  and  $W_1^{3, k}(x) = f^k(W_1^3(f^{-k}(x)))$ . In [Ka] he first introduces a (sufficiently small) measurable partition  $\mathcal{B}$  of the lamination with a Markov property and defines  $W_{loc}^1(x) = W_1^1(x) \cap \mathcal{B}(x)$ . Then, he takes  $W_{loc}^{1, k}(x)$  to be  $f^{-k}(W_{loc}^1(f^k(x)))$  (a symmetric partition allows to define local stable manifolds). The definition of QNI in [Ka] it is then identical to Definition 2.8 where the sets  $W_1^{1, k}(x)$  and  $W_1^{3, k}(x)$  are replaced by  $W_{loc}^{1, k}(x)$  and  $W_{loc}^{3, k}(x)$ .

As it is usual, to see the equivalence, one considers large measure sets of points where the 'boundary' of the leaves  $W_{loc}^1(x)$  and  $W_{loc}^3(x)$  is 'far' from the center point  $x$ . In those sets, and for iterates which return to those sets there is an easy way to relate the sets  $W_{loc}^{i, k}(x)$  and  $W_1^{i, k}(x)$  and thus one can go from one definition to the other without difficulty.

## APPENDIX B. SOME MEASURABLE CONSIDERATIONS

Here we state an abstract result about measurable dynamics.

**Proposition B.1.** *Let  $\mu$  be a partially hyperbolic measure for a diffeomorphism  $f$  of a 3-dimensional closed manifold  $M$ . Assume that  $c_1, \dots, c_k$  are measurable functions with respect to  $\mu$ . Then, for every  $\varepsilon > 0$  and compact set  $\mathcal{Q} \subset M$  with  $\mu(\mathcal{Q}) > 1 - \varepsilon/2$  there exists compact subsets  $\mathcal{P} \subset \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{Q}$  and  $C, k_0 > 0$  such that  $\mu(\mathcal{P}) > 1 - \varepsilon$  and such that:*

- (i) *all functions  $c_1, \dots, c_k$  are uniformly continuous on  $\mathcal{P}_1$ ,*
- (ii) *for every  $\nu > 0$ , there exists an integer  $m_0 > 1$  such that for every  $m > m_0$ , for every  $x \in \mathcal{P}_0$  and for both  $i \in \{1, 3\}$  we have that  $\mu_x^i(\mathcal{P}_1 \cap W_1^{i,m}(x)) \geq (1 - \nu)\mu_x^i(W_1^{i,m}(x))$ .*

*Proof.* Item (i) is a standard application of Lusin's theorem. Without loss of generality, let us assume that  $\mu(\mathcal{P}_1) > 1 - 2\varepsilon/3$ .

To see item (ii), we define for any integers  $q, m \geq 2$  a subset of  $\mathcal{P}_1$  by the formula

$$\mathcal{Q}_{q,m} = \{x \in \mathcal{P}_1 : \mu_x^i(\mathcal{P}_1 \cap W_1^{i,m'}(x)) \geq (1 - q^{-1})\mu_x^i(W_1^{i,m'}(x)), \forall i \in \{1, 3\}, m' \geq m\}.$$

Fix an arbitrary integer  $q \geq 2$ . We have  $\lim_{m \rightarrow \infty} \mu(\mathcal{P}_1 \setminus \mathcal{Q}_{q,m}) = 0$ . We choose some  $m_q \geq 2$  such that  $\mu(\mathcal{P}_1 \setminus \mathcal{Q}_{q,m_q}) < \varepsilon/(100q^2)$ . Then we take  $\mathcal{P}_0 = \bigcap_{q \geq 2} \mathcal{Q}_{q,m_q}$ . It is clear that  $\mu(\mathcal{P}_0) > 1 - 3\varepsilon/4$ , and satisfies item (ii).  $\square$

### APPENDIX C. SOME STATEMENTS ABOUT COCYCLES

Here we give some proofs of some results which are probably well-known but not available in the literature. The reason is that not many references deal with cocycles which are only smooth along unstable manifolds. We state a particular case since it is the one we will use, but of course it holds in more generality. We use the notation and definitions from §3. Note that this can be seen as just a generalization of the fact that Pesin unstable manifolds are smooth. Note that the following result is implicit in [Rue, Remark 5.2(b)].

**Proposition C.1.** *Let  $f : M \rightarrow M$  be a  $C^\infty$  smooth diffeomorphism preserving an ergodic partially hyperbolic measure  $\mu$ . Let  $\mathcal{E} \rightarrow M$  be a (measurable) two-dimensional vector bundle over  $(M, \mu)$  and let  $A : \mathcal{E} \rightarrow \mathcal{E}$  be a vector bundle automorphism which is smooth along unstable manifolds. Assume that the Lyapunov exponents of  $A$  with respect to  $\mu$  are  $\alpha > \beta$ , corresponding to Oseledets subspaces  $E_\alpha$  and  $E_\beta$  respectively. Then there exists a family of smooth trivializations  $\mathcal{S}_0 = \{\mathcal{S}_{0,x} = (\xi_{0,x}, \xi_{0,x}^\perp)\}_{x \in M}$  such that for  $\mu$ -a.e.  $x$ ,*

$$A^{\mathcal{S}_0}(x, t) = \begin{pmatrix} \alpha_x(t) & r_x^0(t) \\ 0 & \beta_x(t) \end{pmatrix}$$

where  $\alpha_x, \beta_x, r_x : (-1, 1) \rightarrow \mathbb{R}$  are smooth functions. Moreover, for  $\mu$ -a.e.  $x$ , we have that  $\xi_{0,x}(0) \in E_\alpha(x)$ .

Let us recall that the fact that  $A$  is smooth along unstable manifolds implicitly requires the bundle  $\mathcal{E}$  to be defined and be smooth along unstable manifolds (see Remark 3.1). This means that for  $\mu$ -almost every  $x \in M$ , the bundle  $\mathcal{E}$  is defined over  $W_1^1(x)$  and admits a smooth trivialization  $\mathcal{S}$  making  $A^{\mathcal{S}}(x, \cdot)$  smooth as a function from  $(-1, 1)$  to  $\text{GL}(2, \mathbb{R})$ .

*Remark C.2.* Under the assumption that: (1)  $f$  is a partially hyperbolic diffeomorphism on a 3-dimensional closed manifold  $M$ ; (2)  $\mathcal{E} \rightarrow M$  is a Hölder continuous, two-dimensional vector bundle over  $M$  that is smooth along unstable manifolds of  $f$ ; and (3)  $A : \mathcal{E} \rightarrow \mathcal{E}$  is a Hölder continuous, vector bundle automorphism which is smooth along unstable manifolds of  $f$ , we can require  $\mathcal{S}_0$  in Proposition C.1 to consist of sections with uniformly bounded smooth norms. Moreover, all the statements in Proposition C.1 hold for every  $x \in M$ , and  $\alpha_x, \beta_x, r_x$  are smooth functions with uniformly bounded smooth norms. These facts follow immediately from the proof below.

*Proof.* By hypothesis, there is a family of trivializations  $\mathcal{S} = \{\mathcal{S}_x = (\xi_x, \xi_x^\perp)\}_{x \in M}$  such that for  $\mu$  almost every  $x \in M$ ,  $\xi_x, \xi_x^\perp : (-1, 1) \rightarrow \mathcal{E}$  are smooth maps so that  $\xi_x(t), \xi_x^\perp(t) \in \mathcal{E}_{\Phi_x^\perp(t)}$  are linearly independent. Moreover, we may assume without loss of generality that for  $\mu$ -a.e.  $x \in M$ ,  $\xi_x(0) \in E_\alpha(x)$  and  $\xi_x^\perp(0) \in E_\beta(x)$ .

The restriction of the bundle map  $A$  on  $\mathcal{E}|_{W_1^1(x)}$ , seen under the basis  $(\xi_x, \xi_x^\perp)$  and  $(\xi_{f(x)}, \xi_{f(x)}^\perp)$ , is given by the matrix

$$A^{\mathcal{S}}(x, t) = \begin{bmatrix} \alpha_x(t) & r_x(t) \\ q_x(t) & \beta_x(t) \end{bmatrix}.$$

Here functions  $\alpha_x, r_x, \beta_x, q_x$  are smooth. Moreover, we have  $q_x(0) = 0$  by our choices of  $\xi_x(0)$  and  $\xi_x^\perp(0)$ .

Let us define another family of trivializations  $\hat{\mathcal{S}} = \{\hat{\mathcal{S}}_x = (\hat{\xi}_x, \hat{\xi}_x^\perp)\}_{x \in M}$  by setting

$$\hat{\xi}_x = \eta_x \xi_x + p_x \xi_x^\perp$$

where  $p_x$  is a smooth function on  $(-1, 1)$  satisfying  $p_x(0) = 0$ ; and  $\eta_x$  is a non-vanishing smooth function on  $(-1, 1)$  satisfying  $\eta_x(0) = 1$ . Then the restriction of the bundle map  $A$  on  $\mathcal{E}|_{W_1^1(x)}$ , seen under the basis  $(\hat{\xi}_x, \hat{\xi}_x^\perp)$  and  $(\hat{\xi}_{f(x)}, \hat{\xi}_{f(x)}^\perp)$ , is given by the matrix

$$A^{\hat{\mathcal{S}}}(x, t) = \begin{bmatrix} \eta_{f(x)}(\lambda_{1,x}t)^{-1} & 0 \\ -\eta_{f(x)}(\lambda_{1,x}t)^{-1}p_{f(x)}(\lambda_{1,x}t) & 1 \end{bmatrix} A^{\mathcal{S}}(x, t) \begin{bmatrix} \eta_x(t) & 0 \\ p_x(t) & 1 \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_x(t) & \hat{r}_x(t) \\ \hat{q}_x(t) & \hat{\beta}_x(t) \end{bmatrix}$$

where

(C.1)

$$\hat{q}_x(t) = \eta_x(t)q_x(t) + p_x(t)\beta_x(t) - (\eta_{f(x)}^{-1}p_{f(x)})(\lambda_{1,x}t)(\eta_x(t)\alpha_x(t) + p_x(t)r_x(t)).$$

We will choose  $\eta_x$  and  $p_x$  such that for  $\mu$ -a.e.  $x$  we have the equations

$$(C.2) \quad \eta_x(t)q_x(t) + p_x(t)\beta_x(t) = \alpha_x(0)p_{f(x)}(\lambda_{1,x}t),$$

$$(C.3) \quad \eta_x(t)\alpha_x(t) + p_x(t)r_x(t) = \alpha_x(0)\eta_{f(x)}(\lambda_{1,x}t).$$

If we denote  $\lambda_{1,m}^-(x) = (\lambda_{1,f^{-1}(x)} \cdots \lambda_{1,f^{-m}(x)})^{-1}$ , we can solve the equation (C.2) by setting

$$p_x(t) = \sum_{n=1}^{\infty} \left[ \prod_{j=1}^{n-1} \alpha_{f^{-j}(x)}(0) \right]^{-1} \left[ \prod_{j=1}^{n-1} \beta_{f^{-j}(x)}(\lambda_{1,j}^-(x)t) \right] \alpha_{f^{-n}(x)}(0)^{-1} (\eta_{f^{-n}(x)} q_{f^{-n}(x)}) (\lambda_{1,n}^-(x)t).$$

Notice that the above sum converges since  $\alpha > \beta$ , and

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{j=1}^{n-1} \log \alpha_{f^{-j}(x)}(0) = \alpha \text{ and } \lim_{n \rightarrow \infty} n^{-1} \sum_{j=1}^{n-1} \log \beta_{f^{-j}(x)}(0) = \beta.$$

Then we can solve the equation (C.3) by setting

$$\eta_x(t) = \sum_{n=1}^{\infty} \left[ \prod_{j=1}^{n-1} \alpha_{f^{-j}(x)}(0) \right]^{-1} \left[ \prod_{j=1}^{n-1} \alpha_{f^{-j}(x)}(\lambda_{1,j}^-(x)t) \right] \alpha_{f^{-n}(x)}(0)^{-1} (p_{f^{-n}(x)} r_{f^{-n}(x)})(\lambda_{1,n}^-(x)t).$$

Thus the equations (C.2) and (C.3) are simultaneously solvable. We see that  $A^{\hat{s}}(x, \cdot)$  is of form  $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ . This concludes the proof.  $\square$

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