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Hussein Mesmar, Frédéric Robert

► **To cite this version:**

Hussein Mesmar, Frédéric Robert. Concentration analysis for elliptic critical equations with no boundary control: ground-state blow-up. *Discrete and Continuous Dynamical Systems - Series S*, 2024, 17 ((4)), pp.1599-1620. 10.3934/dcdss.2023199 . hal-04254969v2

**HAL Id: hal-04254969**

**<https://hal.science/hal-04254969v2>**

Submitted on 24 Apr 2024

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**CONCENTRATION ANALYSIS FOR ELLIPTIC CRITICAL  
EQUATIONS WITH NO BOUNDARY CONTROL:  
GROUND-STATE BLOW-UP**

HUSSEIN MESMAR AND FRÉDÉRIC ROBERT

ABSTRACT. We perform the apriori analysis of solutions to critical nonlinear elliptic equations on manifolds with boundary. The solutions are of minimizing type. The originality is that we impose no condition on the boundary, which leads us to assume  $L^2$ -concentration. We also analyze the effect of a non-homogeneous nonlinearity that results in the fast convergence of the concentration point.

*Dedicated to Yihong Du on the occasion of his 60th birthday*

1. INTRODUCTION

1.1. **Context and main results.** Let  $(M, g)$  be a Riemannian manifold of dimension  $n \geq 3$ , with or without boundary  $\partial M$ . When  $\partial M \neq \emptyset$ ,  $M$  denotes the interior of the manifold and  $\overline{M}$  denotes its closure, so that  $\overline{M} = M \cup \partial M$ : in particular,  $M$  is open in  $\overline{M}$ . We let  $a, f \in C^0(\overline{M})$  be functions and we consider  $u \in C^2(M)$  solution to

$$(1) \quad \Delta_g u + au = fu^{2^*-1}; u > 0 \text{ in } M.$$

where  $\Delta_g := -\operatorname{div}_g(\nabla)$  is the Laplacian with minus sign convention and  $2^* := \frac{2n}{n-2}$  is critical for the Sobolev embeddings  $H_1^2(M) \hookrightarrow L^{2^*}(M)$ . Here, the Sobolev space  $H_1^2(M)$  is the completion of  $\{u \in C^\infty(M) / \|u\|_{H_1^2} < \infty\}$  for the norm  $\|\cdot\|_{H_1^2} := \|\nabla \cdot\|_2 + \|\cdot\|_2$ . In the case of a Euclidean smooth domain  $\Omega \subset \mathbb{R}^n$ , then  $a, f \in C^0(\overline{\Omega})$  and we consider  $u \in C^2(\Omega)$  solution to

$$(2) \quad \Delta u + au = fu^{2^*-1}; u > 0 \text{ in } \Omega.$$

where  $\Delta := -\operatorname{div}(\nabla)$  is the Euclidean Laplacian. Due to the critical exponent  $2^*$ , there might be families of solutions to (1) that are not relatively compact in  $C_{loc}^2(M)$ . For instance, given  $x_0 \in \mathbb{R}^n$  and  $\mu > 0$ , define the *Bubble* as

$$x \mapsto U_{\mu, x_0}(x) := \left( \frac{\mu}{\mu^2 + \frac{c|x-x_0|^2}{n(n-2)}} \right)^{\frac{n-2}{2}}.$$

Then for any domain  $\Omega \subset \mathbb{R}^n$ ,  $U_{\mu, x_0}$  is a solution to (2) when  $a \equiv 0$  and  $f \equiv c$ . Moreover, if  $x_0 \in \overline{\Omega}$ , then  $\sup_{\Omega} U_{\mu, x_0} \rightarrow +\infty$  as  $\mu \rightarrow +\infty$ . In the Riemannian

context, for  $x \in M$  and  $\mu > 0$ , the *Bubble* writes

$$x \mapsto U_{\mu, x_0}^{(M, g)}(x) := \left( \frac{\mu}{\mu^2 + \frac{cd_g(x, x_0)^2}{n(n-2)}} \right)^{\frac{n-2}{2}}.$$

Concerning terminology, we say that a family  $(u_\epsilon)_\epsilon \in C^0(M)$  blows-up if  $\lim_{\epsilon \rightarrow 0} \|u_\epsilon\|_\infty = +\infty$ .

When  $\partial M = \emptyset$ , the description of blowing-up families of (1) with bounded  $L^{2^*}$ -norm has been performed by Druet-Hebey-Robert [7]. The main result in [7] is that blowing-up families are controled from above by  $(\sum_i U_{\mu_{i,\epsilon}, x_{i,\epsilon}})_\epsilon$ , for given families  $(x_{i,\epsilon})_\epsilon \in M$  and  $(\mu_{i,\epsilon})_\epsilon \rightarrow 0$ . With this control, it is possible to give informations on the localization of the limits  $x_{i,\infty} := \lim_{\epsilon \rightarrow 0} x_{i,\epsilon}$ : see Druet [6]. This analysis extends to manifolds with boundary provided a boundary condition like Dirichlet (see Ghoussoub-Mazumdar-Robert [12]) or Neumann (see Druet-Robert-Wei [9]). See Premoselli [17] for a more recent point of view.

The first objective of the present work is to perform an analysis similar to [7] and [6] without condition on  $\partial M \neq \emptyset$ . Tackling such generality requires additional assumption: the relevant notion here is  $L^2$ -concentration that already appeared in Djadli-Druet [5] (see (7) below).

Our second objective is to analyse the effect of a nonconstant function  $f$  in (1). In the case of a single peak, concentration occurs at a critical point. We prove that when this critical point is nondegenerate, then the family of concentration points converges very fast to its limit (see (6) below): this does not generally happen for a constant function  $f$ . A similar control appears in Malchiodi-Mayer [16].

As was shown by Aubin [1], below a threshold, blow-up cannot occur. In this manuscript, we are considering solutions  $(u_\epsilon)$  that carry the minimal energy for blow-up, namely ground-state type solution. The minimal energy is given by the best constant in Sobolev embeddings:

$$(3) \quad \frac{1}{K_0(n)} = \inf_{\varphi \in D_1^2(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla \varphi|^2 dX}{\left( \int_{\mathbb{R}^n} |\varphi|^{2^*} dX \right)^{\frac{2}{2^*}}},$$

where  $D_1^2(\mathbb{R}^n)$  is the completion of  $C_c^\infty(\mathbb{R}^n)$  for the norm  $\varphi \mapsto \|\nabla \varphi\|_2$ . Aubin [1] and Talenti [21] have computed this best constant and have showed that the extremals are exactly  $C \cdot U_{\mu, x_0}$  for  $C \neq 0$ ,  $\mu > 0$  and  $x_0 \in \mathbb{R}^n$ . Our main theorem for ground-state solutions is the following:

**Theorem 1.** *Let  $(M, g)$  be a smooth compact Riemannian manifold of dimension  $n \geq 4$  with nonempty boundary  $\partial M \neq \emptyset$ . We fix  $f \in C^2(\overline{M})$  such that  $f > 0$ . We consider a family  $(h_\epsilon)_\epsilon \in C^1(\overline{M})$  and  $f \in C^2(\overline{M})$ ,  $f > 0$ , such that there exists  $h \in C^1(\overline{M})$  and such that  $\Delta_g + h$  is coercive and*

$$(4) \quad \lim_{\epsilon \rightarrow 0} h_\epsilon = h \text{ in } C^1(\overline{M}).$$

We let  $(u_\epsilon)_\epsilon \in C^2(\overline{M})$  be a family of solutions to

$$(5) \quad \Delta_g u_\epsilon + h_\epsilon u_\epsilon = f u_\epsilon^{2^*-1} \text{ in } M.$$

Let  $x_\epsilon \in \overline{M}$  and  $\mu_\epsilon > 0$  be such that

$$u_\epsilon(x_\epsilon) = \sup_{\overline{M}} u_\epsilon = \mu_\epsilon^{1-\frac{n}{2}}$$

We assume that

- $u_\epsilon \rightarrow 0$  in  $L^2(M)$ ,
- $\lim_{\epsilon \rightarrow 0} x_\epsilon = x_0 \in M$  is an interior point of  $M$ ,
- The solution has minimal-type energy, that is

$$\lim_{\epsilon \rightarrow 0} \int_M f u_\epsilon^{2^*} dv_g = \frac{1}{K_0(n)^{\frac{n}{2}} f(x_0)^{\frac{n-2}{2}}}$$

- The Hessian  $\nabla^2 f(x_0)$  is nondegenerate.

Then  $x_0$  is a critical point of  $f$  and

$$(6) \quad d_g(x_\epsilon, x_0) = o(\mu_\epsilon) \text{ as } \epsilon \rightarrow 0,$$

and for all  $\omega \subset M$  such that  $\bar{\omega} \subset M$  and  $\delta_0 > 0$ , there exists  $C(\omega, \delta_0) > 0$  such that

$$u_\epsilon(x) \leq C(\omega, \delta_0) \left( \frac{\mu_\epsilon}{\mu_\epsilon^2 + d_g(x, x_0)^2} \right)^{\frac{n-2}{2}} + C(\omega, \delta_0) \sup_{\partial B_{\delta_0}(x_0)} u_\epsilon$$

for all  $x \in \omega$ . In addition, assuming that for all  $\delta > 0$ , we have that

$$(7) \quad \lim_{\epsilon \rightarrow 0} \frac{\int_{M \setminus B(x_0, \delta)} u_\epsilon^2 dv_g}{\int_M u_\epsilon^2 dv_g} = 0 \text{ for } n \in \{4, 5, 6\},$$

then

$$(8) \quad h(x_0) = \frac{n-2}{4(n-1)} \left( Scal_g(x_0) - \frac{n-4}{2} \cdot \frac{\Delta_g f(x_0)}{f(x_0)} \right),$$

where  $Scal_g$  is the scalar curvature of  $(M, g)$ .

**Remark:** Theorem 1 applies to the case of a bounded domain of  $\mathbb{R}^n$  endowed with the Euclidean metric  $g := \text{Eucl}$ . In this situation,  $M = \Omega \subset \mathbb{R}^n$  is a domain,  $\Delta_g = -\sum_i \partial_{ii}$ ,  $d_g(x, y) = |x - y|$  is the usual Euclidean norm for  $x, y \in \mathbb{R}^n$  and  $Scal_g = 0$ .

The control (6) is remarkable since it does not hold when  $f$  is degenerate. Indeed, when  $f \equiv 1$  there is an abundance of blowup profiles with various speeds of convergence of the  $(x_\epsilon)$ 's to their limit, see for instance Premoselli [18].

The restriction of dimension  $n \geq 4$  is not surprising: indeed, see Corollary 6.4 in Druet-Hebey [8], (7) does not hold in general for  $n = 3$ . It is known since Aubin and Schoen that for  $n = 3$ , blowup cannot be characterized by local arguments and involves global arguments, like the mass. In the general local context of Theorems 1, no information is known regarding the boundary, which forbids to get any global information.

**1.2. Application to supercritical problems with symmetries.** A natural set application of Theorem 1 is in the context of manifolds invariant under a group of isometries. We consider a compact Riemannian manifold  $(X, g)$  of dimension  $n \geq 3$ , but without boundary ( $\partial X = \emptyset$ ). The critical exponent  $2^*$  can be improved by imposing invariance under the action of an isometry group. Let  $G$  be a compact subgroup of isometries of  $(X, g)$ : we say that a function  $u : X \rightarrow \mathbb{R}$  is  $G$ -invariant if  $u \circ \sigma = u$  for all  $\sigma \in G$ . It follows from Hebey-Vaugon [13] that the critical exponent in this setting is  $2^*(k) := \frac{2(n-k)}{n-k-2}$  where  $k := \min_{x \in X} \dim Gx$  and assuming that  $1 \leq k < n - 2$ . We refer to Hebey-Vaugon [13], Saintier [20] and Faget [10] for extensive considerations on problems invariant under isometries. In general, the

quotient  $X/G$  is not a manifold of dimension  $n - k$ . Following Saintier [20], we make the following assumption on  $G$ :

**Assumption (H):** For any  $x_0 \in X$  such that the orbit  $Gx_0$  is of dimension  $k = \min_{x \in X} \dim Gx \geq 1$  and of volume  $V_m = \min_{x \in X} \{ \text{Vol}_g(Gx) / \dim Gx = k \}$ , there exists  $\delta > 0$ , and  $G'$  a closed subgroup of  $\text{Isom}_g(X)$  such that:

- (1)  $G'x_0 = Gx_0$ ;
- (2) For all  $x \in B_\delta(Gx_0) := \{y \in X / d_g(y; Gx_0) < \delta\}$ , then  $G'x$  is principal and  $G'x \subset Gx$ .

In particular  $\dim G'x = \dim Gx_0 = k$ ,  $\forall x \in B_\delta(Gx_0)$ .

This assumption ensures that  $B_\delta(Gx_0)/G'$  is a Riemannian manifold of dimension  $m := n - k$  with a nontrivial boundary. In the sequel, for any  $p \in \mathbb{N}$ , we define  $C_G^p(X)$  as the space of  $G$ -invariant functions of  $C^p(X)$ . We prove the following in the spirit of Faget [11].

**Theorem 2.** Let  $(X, g)$  be a compact Riemannian manifold of dimension  $n$  without boundary, and let  $G$  be a compact subgroup of isometries of  $X$  which satisfies Assumption (H) and such that  $1 \leq k < n - 2$ . Let  $(h_\epsilon)_\epsilon \in C_G^1(X)$  and  $h \in C_G^1(X)$  be such that  $\Delta_g + h$  is coercive and

$$(9) \quad \lim_{\epsilon \rightarrow 0} h_\epsilon = h > 0 \text{ in } C_G^1(X).$$

Let  $(u_\epsilon)_\epsilon \in C_G^2(X)$  be a family of solutions to

$$(10) \quad \Delta_g u_\epsilon + h_\epsilon u_\epsilon = \lambda_\epsilon u_\epsilon^{2^*(k)-1}; \quad u_\epsilon > 0 \text{ in } X, \quad \int_X u_\epsilon^{2^*(k)} dv_g = 1$$

We assume that

- $u_\epsilon \rightarrow 0$  strongly in  $L^2(X)$ ,
- The energy is of minimal type, that is

$$(11) \quad \lim_{\epsilon \rightarrow 0} \lambda_\epsilon = \frac{V_m^{1 - \frac{2}{2^*(k)}}}{K_0(n-k)}, \quad \text{where } V_m := \min_{x \in X} \{ \text{Vol}_g(Gx) / \dim Gx = k \}.$$

- For all point  $z_0 \in X$  such that  $\dim Gz_0 = k$  and  $\text{Vol}_g(Gz_0) = V_m$ , then the function

$$\left\{ \begin{array}{ll} \bar{v} : B_\delta(Gz_0)/G' & \rightarrow \mathbb{R} \\ & G'x \rightarrow \text{Vol}_g(G'x) \end{array} \right\} \text{ is nondegenerate at } Gz_0.$$

This latest assumption makes sense due to Assumption (H). Let  $(x_\epsilon)_\epsilon \in X$  be such that  $u_\epsilon(x_\epsilon) = \max_X u_\epsilon$  and define  $\mu_\epsilon^{-\frac{n-k-2}{2}} = u_\epsilon(x_\epsilon)$ . Then there exists  $x_0 \in X$  such that  $\dim Gx_0 = k$  and  $\text{Vol}_g(Gx_0) = V_m$  such that  $\lim_{\epsilon \rightarrow 0} x_\epsilon = x_0$  and

$$(12) \quad d_g(x_\epsilon, Gx_0) = o(\mu_\epsilon).$$

Moreover, there exists  $C > 0$  such that

$$(13) \quad u_\epsilon(x) \leq C \left( \frac{\mu_\epsilon}{\mu_\epsilon^2 + d_g(x, Gx_0)^2} \right)^{\frac{n-2}{2}} + \begin{cases} o(\mu_\epsilon) & \text{if } n - k \geq 5 \\ o\left(\mu_\epsilon \sqrt{\ln \frac{1}{\mu_\epsilon}}\right) & \text{if } n - k = 4 \end{cases}$$

and

$$(14) \quad h(x_0) = \frac{n-k-2}{4(n-k-1)} \left( \text{Scal}_{\bar{g}}(\bar{x}_0) + 3 \frac{\Delta_{\bar{g}} \bar{v}(\bar{x}_0)}{\bar{v}(\bar{x}_0)} \right) \text{ when } n - k \geq 4,$$

where  $\bar{g}$  is the metric on  $B_\delta(Gx_0)/G'$  such that the canonical projection  $(B_\delta(Gx_0), g) \rightarrow (B_\delta(Gx_0)/G', \bar{g})$  is a Riemannian submersion.

## 2. POINTWISE CONTROL

We consider  $(u_\epsilon) \in C^2(\bar{M})$ ,  $(h_\epsilon) \in C^1(\bar{M})$ ,  $h \in C^1(\bar{M})$ ,  $f \in C^2(\bar{M})$ ,  $(x_\epsilon)_\epsilon \in M$  and  $(\mu_\epsilon)_\epsilon \in (0, +\infty)$  as in the statement of Theorem 1. In the sequel, we let  $i_g(M, x) > 0$  be the injectivity radius of  $(M, g)$  at an interior point  $x \in M$ .

**Claim 1.** *Set  $\delta \in (0, i_g(M, x_0))$  and define*

$$w_\epsilon(X) := \mu_\epsilon^{\frac{n-2}{2}} u_\epsilon(\exp_{x_\epsilon}(\mu_\epsilon X)) \text{ for any } X \in B_{\frac{\delta}{\mu_\epsilon}}(0) \subset \mathbb{R}^n$$

Then

$$(15) \quad \lim_{\epsilon \rightarrow 0} w_\epsilon(X) = w(X) = \left( \frac{1}{1 + \frac{f(x_0)|X|^2}{n(n-2)}} \right)^{\frac{n-2}{2}} \text{ for all } X \in \mathbb{R}^n.$$

Moreover, the convergence holds in  $C_{loc}^2(\mathbb{R}^n)$ . In addition,

$$\lim_{R \rightarrow +\infty} \lim_{\epsilon \rightarrow 0} \int_{B_{x_\epsilon}(R\mu_\epsilon)} f u_\epsilon^{2^*} dv_g = \frac{1}{K_0(n)^{\frac{n}{2}} f(x_0)^{\frac{n-2}{2}}}.$$

In particular

$$(16) \quad \lim_{R \rightarrow +\infty} \lim_{\epsilon \rightarrow 0} \int_{M \setminus B_{x_\epsilon}(R\mu_\epsilon)} f u_\epsilon^{2^*} dv_g = 0.$$

**Proof of Claim 1:** We define the metric  $g_\epsilon := \exp_{x_\epsilon}^* g(\mu_\epsilon \cdot)$  in  $B_{\frac{\delta}{\mu_\epsilon}}(0) \subset \mathbb{R}^n$ . Since,  $\mu_\epsilon \rightarrow 0$  when  $\epsilon \rightarrow 0$ , then  $g_\epsilon \rightarrow \xi$  in  $C_{loc}^2(\mathbb{R}^n)$  as  $\epsilon \rightarrow 0$  where  $\xi$  is the Euclidean metric. The function  $w_\epsilon$  satisfies the equation

$$(17) \quad \Delta_{g_\epsilon} w_\epsilon + \mu_\epsilon^2 \tilde{h}_\epsilon w_\epsilon = \tilde{f}_\epsilon w_\epsilon^{2^*-1} \text{ in } B_{\frac{\delta}{\mu_\epsilon}}(0)$$

where  $\tilde{h}_\epsilon(X) = h_\epsilon(\exp_{x_\epsilon}(\mu_\epsilon X))$  and  $\tilde{f}_\epsilon(X) = f(\exp_{x_\epsilon}(\mu_\epsilon X))$  for all  $X \in B_{\frac{\delta}{\mu_\epsilon}}(0)$ . Since  $0 < w_\epsilon \leq w_\epsilon(0) = 1$ , there exists  $w \in C^2(\mathbb{R}^n)$  such that the sequence  $w_\epsilon \rightarrow w$  in  $C_{loc}^2(\mathbb{R}^n)$  as  $\epsilon \rightarrow 0$  up to extraction. Passing to the limit in (17), we get that

$$(18) \quad \Delta_\xi w = f(x_0) w^{2^*-1} \text{ in } \mathbb{R}^n, \quad 0 \leq w(0) = 1.$$

It follows from Caffarelli-Gidas-Spruck [3] that  $w(X) = \left(1 + \frac{f(x_0)|X|^2}{n(n-2)}\right)^{-\frac{n-2}{2}}$  for all  $X \in \mathbb{R}^n$ . The change of variable  $x = \exp_{x_\epsilon}(\mu_\epsilon X)$  yields

$$\int_{B_{x_\epsilon}(R\mu_\epsilon)} f u_\epsilon^{2^*} dv_g = \int_{B_R(0)} f(\exp_{x_\epsilon}(\mu_\epsilon X)) w_\epsilon^{2^*} dv_{g_\epsilon}.$$

Therefore,

$$\begin{aligned} \lim_{R \rightarrow +\infty} \lim_{\epsilon \rightarrow 0} \int_{B_{x_\epsilon}(R\mu_\epsilon)} f u_\epsilon^{2^*} dv_g &= \lim_{R \rightarrow +\infty} \lim_{\epsilon \rightarrow 0} \int_{B_R(0)} f(\exp_{x_\epsilon}(\mu_\epsilon x)) w_\epsilon^{2^*} dv_{g_\epsilon} \\ &= f(x_0) \int_{\mathbb{R}^n} w^{2^*} dx = \frac{1}{K_0(n)^{\frac{n}{2}} f(x_0)^{\frac{n-2}{2}}}, \end{aligned}$$

where we have used that  $w$  is a solution to (18) and is an extremal for the Sobolev inequality (3). This proves Claim 1.  $\square$

**Claim 2.**  $u_\epsilon \rightarrow 0$  in  $C_{loc}^0(M \setminus \{x_0\})$ .

*Proof of the claim:* It follows from (16) and  $f > 0$  that for all  $\delta > 0$ , we have that

$$\lim_{\epsilon \rightarrow 0} \int_{M \setminus B_{x_0}(\delta)} u_\epsilon^{2^*} dv_g = 0.$$

Let us fix  $\omega \subset M$  such that  $\bar{\omega} \subset M \setminus \{x_0\}$ . We let  $\omega'$  open such that  $\bar{\omega} \subset \omega'$  and  $\bar{\omega}' \subset M - \{x_0\}$ . Let  $\eta \in C_c^\infty(\omega')$  such that  $\eta(x) = 1$  for all  $x \in \omega$ . Let us take  $l > 1$  to be fixed later. Integrating by parts as in Druet-Hebey ([8], Theorem 6.1), we get that

$$\begin{aligned} \int_M \eta^2 u_\epsilon^l \Delta_g u_\epsilon dv_g &= \int_M \nabla(\eta^2 u_\epsilon^l) \nabla u_\epsilon dv_g = \int_M l \eta^2 u_\epsilon^{l-1} |\nabla u_\epsilon|^2 dv_g + \int_M \nabla \eta^2 \nabla \frac{u_\epsilon^{l+1}}{l+1} dv_g \\ &= \frac{4l}{(l+1)^2} \int_M \eta^2 |\nabla u_\epsilon^{\frac{l+1}{2}}|_g^2 dv_g + \int_M \frac{\Delta \eta^2}{l+1} u_\epsilon^{l+1} dv_g \end{aligned}$$

Independently, for any  $v \in C^1(M)$ , integrating also by parts, we get that

$$\int_M (|\nabla(\eta v)|_g^2 - \eta^2 |\nabla v|_g^2) dv_g = - \int_M \eta v^2 \Delta_g \eta dv_g.$$

Plugging these integrals together yields

$$\int_M |\nabla(\eta u_\epsilon^{\frac{l+1}{2}})|_g^2 dv_g = \frac{(l+1)^2}{4l} \int_M \eta^2 u_\epsilon^l \Delta_g u_\epsilon dv_g + \frac{l+1}{2l} \int_M \left( |\nabla \eta|_g^2 + \frac{l-1}{l+1} \eta \Delta_g \eta \right) u_\epsilon^{l+1} dv_g$$

We then get that

$$\begin{aligned} &\int_M |\nabla(\eta u_\epsilon^{\frac{l+1}{2}})|_g^2 dv_g \\ &= \frac{(l+1)^2}{4l} \int_M \eta^2 u_\epsilon^l \Delta_g u_\epsilon dv_g + \frac{l+1}{2l} \int_M \left( |\nabla \eta|_g^2 + \frac{l-1}{l+1} \eta \Delta_g \eta \right) u_\epsilon^{l+1} dv_g \\ &= \frac{(l+1)^2}{4l} \lambda_\epsilon \int_M \eta^2 u_\epsilon^{l+2^*-1} dv_g - \frac{(l+1)^2}{4l} \int_M \eta^2 h_\epsilon u_\epsilon^{l+1} dv_g \\ &\quad + \frac{l+1}{2l} \int_M \left( |\nabla \eta|_g^2 + \frac{l-1}{l+1} \eta \Delta_g \eta \right) u_\epsilon^{l+1} dv_g \\ &\leq \frac{(l+1)^2}{4l} \lambda_\epsilon \left( \int_M (\eta u_\epsilon^{\frac{l+1}{2}})^{2^*} dv_g \right)^{\frac{2}{2^*}} \left( \int_{B_x(2\delta)} \eta u_\epsilon^{2^*} dv_g \right)^{1-\frac{2}{2^*}} + C \int_\Omega u_\epsilon^{l+1} dv_g \end{aligned}$$

It follows from the Sobolev inequality that there exists  $C(\omega') > 0$  independent of  $\epsilon$  such that

$$\left( \int_{\omega'} (\eta u_\epsilon^{\frac{l+1}{2}})^{2^*} dv_g \right)^{\frac{2}{2^*}} \leq C(\omega') \left( \int_{\omega'} |\nabla(\eta u_\epsilon^{\frac{l+1}{2}})|_g^2 dv_g + B \int_{\omega'} \eta^2 u_\epsilon^{l+1} dv_g \right)$$

Combining these inequalities yields

$$\left( \int_\omega u_\epsilon^{\frac{l+1}{2} 2^*} dv_g \right)^{\frac{2}{2^*}} \leq \left( \int_{\omega'} (\eta u_\epsilon^{\frac{l+1}{2}})^{2^*} dv_g \right)^{\frac{2}{2^*}} \leq C \int_{\omega'} u_\epsilon^{l+1} dv_g$$

for  $\epsilon > 0$  small enough and where  $C$  is independent of  $\epsilon$ . Taking  $1 < l < 2^* - 1$ , we then get that  $u_\epsilon \rightarrow 0$  in  $L^q(\omega)$  for some  $q > 2^*$ . Since  $u_\epsilon$  satisfies (5), it is classical that  $u_\epsilon \rightarrow 0$  in  $C_{loc}^0(\omega)$ . This proves Claim 2.  $\square$

**Claim 3.** For all  $\omega \subset M$  such that  $\bar{\omega} \subset M$ , there exists  $C(\omega)$  such that

$$(19) \quad d_g(x, x_\epsilon)^{\frac{n-2}{2}} u_\epsilon(x) \leq C(\omega) \text{ for all } \epsilon > 0 \text{ and } x \in \omega.$$

Moreover,

$$(20) \quad \lim_{R \rightarrow 0} \lim_{\epsilon \rightarrow 0} \sup_{x \in \omega \setminus B_{x_\epsilon}(R\mu_\epsilon)} d_g(x, x_\epsilon)^{\frac{n-2}{2}} |u_\epsilon(x)| = 0$$

*Proof of the Claim:* We argue by contradiction and we let  $(y_\epsilon)_\epsilon \in \bar{\omega}$  be such that

$$d_g(y_\epsilon, x_\epsilon)^{\frac{n-2}{2}} u_\epsilon(y_\epsilon) = \sup_{x \in \bar{\omega}} d_g(x, x_\epsilon)^{\frac{n-2}{2}} u_\epsilon(x) \rightarrow +\infty \text{ as } \epsilon \rightarrow 0.$$

It follows from Claim 2 that  $\lim_{\epsilon \rightarrow 0} y_\epsilon = x_0$ . Arguing as in Step 2 of Chapter 4 in Druet-Hebey-Robert [7] and using (16), we get (19). The second estimate (20) follows also from [7].  $\square$

We now state and prove the main result of this section:

**Proposition 1.** Let  $\delta > 0$  be such that  $B_{2\delta}(x_0) \subset M$ . Under the assumptions of Theorem 1, we have that

$$(21) \quad u_\epsilon(y_\epsilon) = \left( \frac{\mu_\epsilon}{\mu_\epsilon^2 + \frac{f(x_0)}{n(n-2)} d_g(x_\epsilon, y_\epsilon)^2} \right)^{\frac{n-2}{2}} (1 + o(1)) + O(\theta_\epsilon) \text{ when } \lim_{\epsilon \rightarrow 0} y_\epsilon = x_0.$$

Moreover, there exists  $C(\delta) > 0$  independent of  $\epsilon$  such that

$$(22) \quad u_\epsilon(x) \leq C \frac{\mu_\epsilon^{\frac{n-2}{2}}}{(\mu_\epsilon + d_g(x, x_\epsilon))^{n-2}} + C\theta_\epsilon$$

$$(23) \quad |\nabla u_\epsilon|(x) \leq C \frac{\mu_\epsilon^{\frac{n-2}{2}}}{(\mu_\epsilon + d_g(x, x_\epsilon))^{n-1}} + C\theta_\epsilon$$

for all  $x \in B_\delta(x_0)$  where

$$(24) \quad \theta_\epsilon := \sup_{x \in \partial B_\delta(x_0)} u_\epsilon(x) \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

*Proof of Proposition 1:* We let  $\nu \in (0, 1)$  to be fixed later. We let  $\alpha_0 > 0$  such that  $\Delta_g + \frac{h - \alpha_0}{1 - \nu}$  is coercive on  $B_{2\delta}(x_0)$  where  $h$  is as in (4): up to taking  $\delta > 0$  small, this is always possible. We let  $\tilde{G}_\nu$  be the Green's function of  $\Delta_g + \frac{h - \alpha_0}{1 - \nu}$  on  $B_{2\delta}(x_0)$  with Dirichlet boundary condition. It follows from Robert [19] that there exists  $c_1, c_2 > 0$  such that

$$(25) \quad c_1 d_g(x, y)^{2-n} \leq \tilde{G}_\nu(x, y) \leq c_2 d_g(x, y)^{2-n} \text{ for all } x, y \in B_\delta(x_0), x \neq y.$$

We define the operator

$$u \mapsto L_\epsilon u := \Delta_g u + h_\epsilon u - f u_\epsilon^{2^*-2} u,$$

so that (5) reads  $L_\epsilon u_\epsilon = 0$ . A straightforward computation yields

$$(26) \quad \frac{L_\epsilon \tilde{G}_\nu^{1-\nu}}{\tilde{G}_\nu^{1-\nu}}(x, x_\epsilon) = \alpha_0 + h_\epsilon(x) - h(x) + \nu(1 - \nu) \left| \frac{\nabla \tilde{G}_\nu}{\tilde{G}_\nu} \right|_g^2 - f u_\epsilon^{2^*-2}$$

By standard properties of Green's function [19], there exists  $c_1, \rho > 0$ , such that

$$(27) \quad \frac{|\nabla_{g,x} \tilde{G}_\nu|_g}{\tilde{G}_\nu}(x, x_\epsilon) \geq \frac{c_1}{d_g(x, x_\epsilon)} \text{ for all } x \in B_\rho(x_\epsilon) - \{x_\epsilon\}.$$



Since  $u_\epsilon \rightarrow 0$  in  $C_{loc}^0(M \setminus \{x_0\})$  and  $h_\epsilon \rightarrow h$  in  $C_{loc}^0(M \setminus \{x_0\})$ , (26) yields

$$L_\epsilon \tilde{G}_\nu^{1-\nu} \geq 0 \text{ in } B_{2\delta}(x_0) \setminus B_\rho(x_0)$$

Let  $R > 0$  to be fixed later. It follows from (20) that

$$d_g(x, x_\epsilon)^2 u_\epsilon^{2^*-2}(x) \leq \eta(R) \text{ for all } x \in B(x_\epsilon, \rho) \setminus B(x_\epsilon, R\mu_\epsilon),$$

where  $\lim_{R \rightarrow +\infty} \eta(R) = 0$ . Now, using  $h_\epsilon \rightarrow h$  in  $C^0(M)$ , (26) and (27), for any  $x \in B(x_\epsilon, \rho) \setminus B(x_\epsilon, R\mu_\epsilon)$ , we get that

$$\begin{aligned} \frac{L_\epsilon \tilde{G}_\nu^{1-\nu}}{\tilde{G}_\nu^{1-\nu}}(x, x_\epsilon) &\geq \frac{\alpha_0}{2} + \nu(1-\nu) \frac{c_1^2}{d_g(x, x_\epsilon)^2} - f u_\epsilon^{2^*-2}(x) \\ &\geq \frac{\alpha_0}{2} + \frac{\nu(1-\nu)c_1^2 - f\eta(R)}{d_g(x, x_\epsilon)^2} \geq 0 \end{aligned}$$

for  $R > 0$  large enough. Therefore, we get that

$$(28) \quad L_\epsilon \tilde{G}_\nu^{1-\nu}(x, x_\epsilon) \geq 0 \text{ for all } x \in B_\delta(x_0) \setminus B_{R\mu_\epsilon}(x_\epsilon).$$

We fix  $\nu_1 \in (0, 1)$ . It follows from (25) and  $\|u_\epsilon\|_\infty = \mu_\epsilon^{1-n/2}$ , that there exists  $c_3 > 0$  such that

$$(29) \quad u_\epsilon(x) \leq c_3 \mu_\epsilon^{\frac{n-2}{2} - \nu_1(n-2)} \tilde{G}_{\nu_1}^{1-\nu_1}(x, x_\epsilon) \text{ for all } x \in \partial B_{R\mu_\epsilon}(x_\epsilon).$$

We set  $\theta_\epsilon := \sup_{x \in \partial B_\delta(x_0)} u_\epsilon(x)$ . It follows from Claim 2 that  $\lim_{\epsilon \rightarrow 0} \theta_\epsilon = 0$ . We fix  $\nu_2 \in (0, 1)$  and we consider the Green's function  $\tilde{G}_{\nu_2}$ . It follows from (25) that there exists  $c_4 > 0$  such that

$$(30) \quad u_\epsilon(x) \leq c_4 \theta_\epsilon \tilde{G}_{\nu_2}^{1-\nu_2}(x, x_\epsilon) \text{ for all } x \in \partial B_\delta(x_0).$$

We define

$$H_\epsilon(x) := c_3 \mu_\epsilon^{\frac{n-2}{2} - \nu_1(n-2)} \tilde{G}_{\nu_1}^{1-\nu_1}(x, x_\epsilon) + c_4 \theta_\epsilon \tilde{G}_{\nu_2}^{1-\nu_2}(x, x_\epsilon) \text{ for } x \in B_{2\delta}(x_\epsilon) - \{x_\epsilon\}.$$

It follows from (28), (29) and (30) that

$$\begin{cases} L_\epsilon u_\epsilon = 0 \leq L_\epsilon H_\epsilon & \text{in } B_\delta(x_0) \setminus B_{R\mu_\epsilon}(x_\epsilon) \\ 0 < u_\epsilon \leq H_\epsilon & \text{on } \partial(B_\delta(x_0) \setminus B_{R\mu_\epsilon}(x_\epsilon)) \end{cases}$$

Since  $L_\epsilon u_\epsilon \geq 0$  in  $\overline{B_\delta(x_0) \setminus B_{R\mu_\epsilon}(x_\epsilon)}$ , it follows from [2] that

$$u_\epsilon \leq H_\epsilon \text{ in } B_\delta(x_0) \setminus B_{R\mu_\epsilon}(x_\epsilon).$$

Using the pointwise control (25) and that  $\|u_\epsilon\|_\infty = \mu_\epsilon^{1-n/2}$ , we get that for all  $\nu_1, \nu_2 \in (0, 1)$ , there exists  $C_{\nu_1, \nu_2} > 0$  such that

$$(31) \quad u_\epsilon(x) \leq C_{\nu_1, \nu_2} \left( \frac{\mu_\epsilon^{\frac{n-2}{2} - \nu_1(n-2)}}{(\mu_\epsilon + d_g(x, x_\epsilon))^{(n-2)(1-\nu_1)}} + \theta_\epsilon d_g(x, x_\epsilon)^{(2-n)(1-\nu_2)} \right)$$

for all  $x \in B_\delta(x_0)$ . Our next step is to prove (21). We let  $(y_\epsilon)_\epsilon \in M$  such that  $\lim_{\epsilon \rightarrow 0} y_\epsilon = x_0$ .

We first assume that  $d_g(x_\epsilon, y_\epsilon) = O(\mu_\epsilon)$  as  $\epsilon \rightarrow 0$ . Then, (21) is a direct consequence of (15).

From now on, we assume that

$$\lim_{\epsilon \rightarrow 0} d_g(x_\epsilon, y_\epsilon) = 0 \text{ and } \lim_{\epsilon \rightarrow 0} \frac{d_g(x_\epsilon, y_\epsilon)}{\mu_\epsilon} = +\infty.$$

We let  $G_\epsilon$  be the Green's function for  $\Delta_g + h_\epsilon$  in  $B_\delta(x_0)$  with Dirichlet boundary condition. We let  $(y_\epsilon)_\epsilon \in B_{\delta/2}(x_0)$ . Green's representation formula yields

$$(32) \quad u_\epsilon(y_\epsilon) = \int_{B_\delta(x_0)} G_\epsilon(y_\epsilon, x) (\Delta_g u_\epsilon + h_\epsilon u_\epsilon)(x) dv_g(x) - \int_{\partial B_\delta(x_0)} \partial_{\bar{n}} G_\epsilon(y_\epsilon, z) u_\epsilon(z) d\sigma_g(z).$$

It follows from Robert [19] that there exists  $c_5, c_6 > 0$  such that

$$(33) \quad d_g(x, y)^{n-2} |G_\epsilon(x, y)| + d_g(x, y)^{n-1} |\nabla_x G_\epsilon(x, y)| \leq c_5 \text{ for all } x, y \in B_\delta(x_0), x \neq y$$

for all  $\epsilon > 0$ . Combining these estimates with equation (5) and (24), we get that

$$(34) \quad u_\epsilon(y_\epsilon) = \int_{B_{R\mu_\epsilon}(x_\epsilon)} G_\epsilon(y_\epsilon, x) f(x) u_\epsilon^{2^*-1}(x) dv_g(x) + A_\epsilon(R) + B_\epsilon$$

where

$$|A_\epsilon(R)| \leq C \int_{B_\delta(x_0) \setminus B_{R\mu_\epsilon}(x_\epsilon)} d_g(x, y_\epsilon)^{2-n} u_\epsilon^{2^*-1}(x) dv_g(x)$$

$$\text{and } |B_\epsilon| \leq C \int_{\partial B_\delta(x_0)} d_g(z, y_\epsilon)^{1-n} u_\epsilon(z) d\sigma_g(z) \leq C\theta_\epsilon.$$

We deal with the first term of (34). With a change of variable and (15), we get that

$$\begin{aligned} & \int_{B_{R\mu_\epsilon}(x_\epsilon)} G_\epsilon(y_\epsilon, x) f(x) u_\epsilon^{2^*-1}(x) dv_g(x) \\ &= \mu_\epsilon^{n-2} \int_{B_R(0)} G_\epsilon(y_\epsilon, \exp_{x_\epsilon}(\mu_\epsilon X)) f(\exp_{x_\epsilon}(\mu_\epsilon X)) w_\epsilon^{2^*-1}(X) dv_{g_\epsilon}(X) \end{aligned}$$

It follow from [19] that for any  $(z_\epsilon)_\epsilon \in M$  such that  $\lim_{\epsilon \rightarrow 0} d_g(z_\epsilon, x_\epsilon) = 0$  we have that

$$\lim_{\epsilon \rightarrow 0} d_g(x_\epsilon, z_\epsilon)^{n-2} G_\epsilon(x_\epsilon, z_\epsilon) = \frac{1}{(n-2)\omega_{n-1}}.$$

Since  $\mu_\epsilon = o(d_g(x_\epsilon, y_\epsilon))$ , we then get that

$$(35) \quad \begin{aligned} & \int_{B_{R\mu_\epsilon}(x_\epsilon)} G_\epsilon(y_\epsilon, x) f(x) u_\epsilon^{2^*-1}(x) dv_g(x) \\ &= \frac{f(x_0) \mu_\epsilon^{n-2}}{(n-2)\omega_{n-1} d_g(x_\epsilon, y_\epsilon)^{n-2}} \left( \int_{B_R(0)} w^{2^*-1}(X) dX + o(1) \right) \\ &= \frac{f(x_0) \mu_\epsilon^{n-2}}{(n-2)\omega_{n-1} d_g(x_\epsilon, y_\epsilon)^{n-2}} \left( \int_{\mathbb{R}^n} w^{2^*-1}(X) dX + o(1) + \eta(R) \right) \end{aligned}$$

where  $\lim_{R \rightarrow +\infty} \eta(R) = 0$ . With (18) and (15), we get that

$$(36) \quad \begin{aligned} f(x_0) \int_{\mathbb{R}^n} w^{2^*-1}(X) dX &= \lim_{R \rightarrow +\infty} \int_{B(0, R)} \Delta w dX = \lim_{R \rightarrow +\infty} \int_{\partial B(0, R)} (-\partial_\nu w) d\sigma \\ &= \left( \frac{n(n-2)}{f(x_0)} \right)^{\frac{n-2}{2}} (n-2)\omega_{n-1}. \end{aligned}$$

We now deal with  $A_\epsilon(R)$ . Using the pointwise control (31), we get that

$$(37) \quad |A_\epsilon(R)| \leq C_{\nu_1, \nu_2} \int_{B_\delta(x_0) \setminus B_{R\mu_\epsilon}(x_\epsilon)} d_g(x, y_\epsilon)^{2-n} \frac{\mu_\epsilon^{\frac{n+2}{2} - \nu_1(n+2)}}{(\mu_\epsilon + d_g(x, x_\epsilon))^{(n+2)(1-\nu_1)}} dv_g(x) \\ + C_{\nu_1, \nu_2} \theta_\epsilon^{2^* - 1} \int_{B_\delta(x_0)} d_g(x, y_\epsilon)^{2-n} d_g(x, x_\epsilon)^{-(n+2)(1-\nu_2)} dv_g(x) + C\theta_\epsilon.$$

It follows from Giraud's lemma (see Appendix A of [7] for instance) that for  $\alpha, \beta \in (0, n-2)$  such that  $\alpha + \beta > n$ , there exists  $C > 0$  such that

$$\int_{B_\delta(x_0)} d_g(y, x)^{\alpha-n} d_g(x, y)^{\beta-n} dv_g(x) \leq C \text{ for all } y, z \in B_\delta(x_0).$$

Taking  $1 - \nu_2 > 0$  close to 0, we then get that

$$(38) \quad \theta_\epsilon^{2^* - 1} \int_{B_\delta(x_0)} d_g(x, y_\epsilon)^{2-n} d_g(x, x_\epsilon)^{-(n+2)(1-\nu_2)} dv_g(x) \leq C\theta_\epsilon^{2^* - 1} \leq C\theta_\epsilon.$$

We now deal with the remaining term of (37). We split the domain  $B_\delta(x_0) = D_\epsilon^1 \cup D_\epsilon^2$  where

$$D_\epsilon^1 := \{x \in B_\delta(x_0) \text{ s.t. } d_g(x, y_\epsilon) \geq d_g(x_\epsilon, y_\epsilon)/2\}$$

$$\text{and } D_\epsilon^2 := \{x \in B_\delta(x_0) \text{ s.t. } d_g(x, y_\epsilon) < d_g(x_\epsilon, y_\epsilon)/2\}.$$

We fix  $R > 0$ . With the change of variable  $x := \exp_g(\mu_\epsilon X)$ , we get that

$$(39) \quad \int_{D_\epsilon^1 \setminus B_{R\mu_\epsilon}(x_\epsilon)} d_g(x, y_\epsilon)^{2-n} \frac{\mu_\epsilon^{\frac{n+2}{2} - \nu_1(n+2)}}{(\mu_\epsilon + d_g(x, x_\epsilon))^{(n+2)(1-\nu_1)}} dv_g(x) \\ \leq C d_g(x_\epsilon, y_\epsilon)^{2-n} \int_{B_{2\delta}(x_\epsilon) \setminus B_{R\mu_\epsilon}(x_\epsilon)} \frac{\mu_\epsilon^{\frac{n+2}{2} - \nu_1(n+2)}}{(\mu_\epsilon + d_g(x, x_\epsilon))^{(n+2)(1-\nu_1)}} dv_g(x) \\ \leq C d_g(x_\epsilon, y_\epsilon)^{2-n} \mu_\epsilon^{\frac{n-2}{2}} \int_{\mathbb{R}^n \setminus B_R(0)} \frac{1}{(1 + |X|)^{(n+2)(1-\nu_1)}} dX \\ \leq \eta(R) d_g(x_\epsilon, y_\epsilon)^{2-n} \mu_\epsilon^{\frac{n-2}{2}} \text{ where } \lim_{R \rightarrow +\infty} \eta(R) = 0$$

when  $\nu_1 < 2/(n+2)$ . Concerning the other integral, note that for all  $x \in D_\epsilon^2$ , we have that  $d_g(x, x_\epsilon) \geq d_g(x_\epsilon, y_\epsilon)/2$ . Therefore

$$(40) \quad \int_{D_\epsilon^2} d_g(x, y_\epsilon)^{2-n} \frac{\mu_\epsilon^{\frac{n+2}{2} - \nu_1(n+2)}}{(\mu_\epsilon + d_g(x, x_\epsilon))^{(n+2)(1-\nu_1)}} dv_g(x) \\ \leq C \frac{\mu_\epsilon^{\frac{n+2}{2} - \nu_1(n+2)}}{d_g(y_\epsilon, x_\epsilon)^{(n+2)(1-\nu_1)}} \int_{d_g(x, y_\epsilon) < d_g(x_\epsilon, y_\epsilon)/2} d_g(x, y_\epsilon)^{2-n} dv_g(x) \\ \leq C \frac{\mu_\epsilon^{\frac{n+2}{2} - \nu_1(n+2)}}{d_g(y_\epsilon, x_\epsilon)^{(n+2)(1-\nu_1)}} d_g(x_\epsilon, y_\epsilon)^2 \\ \leq C \frac{\mu_\epsilon^{\frac{n-2}{2}}}{d_g(x_\epsilon, y_\epsilon)^{n-2}} \left( \frac{\mu_\epsilon}{d_g(y_\epsilon, x_\epsilon)} \right)^{2-\nu_1(n+2)} = o\left( \frac{\mu_\epsilon^{\frac{n-2}{2}}}{d_g(x_\epsilon, y_\epsilon)^{n-2}} \right)$$

when  $\nu_1 < 2/(n+2)$ . Putting (35), (36), (37), (38), (39) and (40) together yields

$$u_\epsilon(y_\epsilon) = \left( \frac{n(n-2)}{f(x_0)} \right)^{\frac{n-2}{2}} \frac{\mu_\epsilon^{\frac{n-2}{2}}}{d_g(x_\epsilon, y_\epsilon)^{n-2}} (1 + o(1)) + O(\theta_\epsilon)$$

This yields (21) since  $d_g(x_\epsilon, y_\epsilon)/\mu_\epsilon \rightarrow +\infty$  as  $\epsilon \rightarrow 0$ .

When  $d_g(x_\epsilon, y_\epsilon) = o(1)$ , (22) is a direct consequence of (21). Since  $u_\epsilon \rightarrow 0$  in  $C_{loc}^0(M - \{x_0\})$  and  $u_\epsilon > 0$ , it follows from Harnack's inequality that there exists  $c(\tau) > 0$  such that  $u_\epsilon(x) \leq c(\tau)u_\epsilon(y)$  for all  $x, y \in B_{2\delta}(x_0) \setminus B_\tau(x_0)$ . Therefore, if  $(y_\epsilon)_\epsilon \in B_{2\delta}(x_0)$  is such that  $y_\epsilon \not\rightarrow x_0$ , we have that  $u_\epsilon(y_\epsilon) = O(\theta_\epsilon)$ . This proves (22) when  $x$  is far from  $x_0$ . This proves (21) holds in all cases, which yields (22).

Concerning the gradient estimate, differentiate Green's representation formula (32) to obtain

$$\nabla u_\epsilon(y_\epsilon) = \int_{B_\delta(x_0)} \nabla_y G_\epsilon(y_\epsilon, x) (\Delta_g u_\epsilon + h_\epsilon u_\epsilon)(x) dv_g(x) - \int_{\partial B_\delta(x_0)} \partial_{\bar{n}} \nabla_y G_\epsilon(y_\epsilon, z) u_\epsilon(z) d\sigma_g(z).$$

Using the pointwise control (33), we then get that

$$|\nabla u_\epsilon(y_\epsilon)| \leq C \int_{B_\delta(x_0)} d_g(y_\epsilon, x)^{1-n} u_\epsilon^{2^*-1}(x) dv_g(x) + C \int_{\partial B_\delta(x_0)} d_g(y_\epsilon, z)^{-n} u_\epsilon(z) d\sigma_g(z).$$

We get (23) arguing as in the proof of (22). This proves Proposition 1.  $\square$

### 3. SPEED OF CONVERGENCE OF $(x_\epsilon)_\epsilon$

Let  $\Omega \subset \mathbb{R}^n$  be a smooth bounded domain. Let  $u \in C^2(\bar{\Omega})$ ,  $u > 0$ , and  $f \in C^1(\bar{\Omega})$  be functions and  $c \in \mathbb{R}$ . Then for all  $z \in \mathbb{R}^n$ , the Pohozaev identity writes

$$\begin{aligned} (41) \quad & \int_{\Omega} \left( (x-z)^i \partial_i u + \frac{n-2}{2} u \right) \left( \Delta_\xi u - c f u^{2^*-1} \right) dx \\ & = \int_{\partial\Omega} \left[ (x-z, \nu) \left( \frac{|\nabla u|_\xi^2}{2} - \frac{c f u^{2^*}}{2^*} \right) - \left( (x-z)^i \partial_i u + \frac{n-2}{2} u \right) \partial_\nu u \right] d\sigma \\ & \quad + \frac{1}{2^*} \int_{\Omega} c \langle \nabla f(x), x-z \rangle_\xi u^{2^*} dx \end{aligned}$$

Differentiating with respect to  $z$ , we get that for any  $j \in \{1, \dots, n\}$

$$\begin{aligned} (42) \quad & - \int_{\Omega} \partial_j u \left( \Delta_\xi u - c f u^{2^*-1} \right) dx \\ & = \int_{\partial\Omega} \left[ -\nu_j \left( \frac{|\nabla u|_\xi^2}{2} - \frac{c f u^{2^*}}{2^*} \right) + \partial_j u \partial_\nu u \right] d\sigma - \frac{c}{2^*} \int_{\Omega} \partial_j f(x) u^{2^*} dx \end{aligned}$$

We refer to Ghoussoub-Robert [12] for a proof. We fix  $\delta \in (0, i_g(M, x_0))$ . We define

$$\hat{u}_\epsilon(X) := u_\epsilon(\exp_{x_\epsilon}(X)) \text{ for all } X \in B_\delta(0) \subset \mathbb{R}^n.$$

Therefore, equation (5) rewrites

$$\Delta_{\hat{g}_\epsilon} \hat{u}_\epsilon + \hat{u}_\epsilon u_\epsilon = \hat{f}_\epsilon \hat{u}_\epsilon^{2^*-1} \text{ in } B_\delta(0).$$

where  $\hat{h}_\epsilon(X) := h_\epsilon(\exp_{x_\epsilon}(X))$  and  $\hat{f}_\epsilon(X) := f(\exp_{x_\epsilon}(X))$  for all  $X \in B_\delta(0) \subset \mathbb{R}^n$  and  $\hat{g}_\epsilon := \exp_{x_\epsilon}^* g$  is the pull-back of  $g$  via the exponential map.

**Lemma 1.** *Let  $(\phi_\epsilon)_\epsilon \in C^0(B_\delta(0))$  such that*

$$\begin{cases} \lim_{\epsilon \rightarrow 0} \phi_\epsilon(0) = s \in \mathbb{R}, \\ |\phi_\epsilon(X) - \phi_\epsilon(0)| \leq C|X| \text{ for all } X \in B_\delta(0) \text{ and } \epsilon > 0. \end{cases}$$

*We fix  $p \geq 0$  and  $q \geq 1$ . Then*

$$\begin{aligned} & \int_{B_\delta(0)} \phi_\epsilon |X|^p \hat{u}_\epsilon^q dX \\ &= \begin{cases} \mu_\epsilon^{n+p-\frac{q(n-2)}{2}} \left( s \int_{\mathbb{R}^n} |X|^p w^q dX + o(1) \right) + O(\theta_\epsilon^q) & \text{if } (n-2)q > p+n, \\ \left( s \left( \frac{n(n-2)}{f(x_0)} \right)^{q\frac{n-2}{2}} \omega_{n-1} + o(1) \right) \mu_\epsilon^{\frac{q(n-2)}{2}} \ln \left( \frac{1}{\mu_\epsilon} \right) + O(\theta_\epsilon^q) & \text{if } (n-2)q = p+n, \\ O(\mu_\epsilon^{q\frac{n-2}{2}}) + O(\theta_\epsilon^q) & \text{if } (n-2)q < p+n \end{cases} \end{aligned}$$

*Moreover, for any family  $(\delta_\epsilon)_\epsilon \in (0, 1)$  such that  $\lim_{\epsilon \rightarrow 0} \delta_\epsilon = \lim_{\epsilon \rightarrow 0} \frac{\mu_\epsilon}{\delta_\epsilon} = 0$ , we have that*

$$(43) \quad \int_{B_{\delta_\epsilon}(0)} \phi_\epsilon |X|^p \hat{u}_\epsilon^q dX = \mu_\epsilon^{n+p-\frac{q(n-2)}{2}} \left( s \int_{\mathbb{R}^n} |X|^p w^q dX + o(1) \right) + O(\theta_\epsilon^q) \text{ if } q > \frac{p+n}{n-2}.$$

*Proof:* We fix  $\nu > 0$ . It follows from (21) and (22) that there exists  $\alpha \in (0, \delta)$  such that

$$\left| \hat{u}_\epsilon^q(X) - \left( \frac{\mu_\epsilon}{\mu_\epsilon^2 + \frac{f(x_0)}{n(n-2)}|X|^2} \right)^{q\frac{n-2}{2}} \right| \leq \nu \left( \frac{\mu_\epsilon}{\mu_\epsilon^2 + \frac{f(x_0)}{n(n-2)}|X|^2} \right)^{q\frac{n-2}{2}} + C\theta_\epsilon^q$$

for all  $X \in B_\rho(0) \setminus B_\alpha(0)$ . Note that for all  $\alpha \in (0, \delta)$ , it follows from the Harnack inequality that

$$\int_{B_\delta(0) \setminus B_\alpha(0)} \phi_\epsilon |X|^p \hat{u}_\epsilon^q dX = O(\theta_\epsilon^q).$$

We then get that

$$\begin{aligned} & \left| \int_{B_\delta(0)} \phi_\epsilon |X|^p \hat{u}_\epsilon^q dX - \int_{B_\alpha(0)} \phi_\epsilon |X|^p \left( \frac{\mu_\epsilon}{\mu_\epsilon^2 + \frac{f(x_0)}{n(n-2)}|X|^2} \right)^{q\frac{n-2}{2}} dX \right| \\ & \leq C\nu \int_{B_\delta(0)} |X|^p \left( \frac{\mu_\epsilon}{\mu_\epsilon^2 + |X|^2} \right)^{q\frac{n-2}{2}} dX + C\theta_\epsilon^q \end{aligned}$$

We then get Lemma 1 when  $q(n-2) < n+p$ . With the change of variable  $X = \mu_\epsilon Y$ , we get that

$$\begin{aligned} & \int_{B_\alpha(0)} \phi_\epsilon |X|^p \left( \frac{\mu_\epsilon}{\mu_\epsilon^2 + \frac{f(x_0)}{n(n-2)}|X|^2} \right)^{q\frac{n-2}{2}} dX \\ &= \mu_\epsilon^{n+p-q\frac{n-2}{2}} \int_{B_{\alpha/\mu_\epsilon}(0)} \phi_\epsilon(\mu_\epsilon Y) |Y|^p \left( \frac{1}{1 + \frac{f(x_0)}{n(n-2)}|Y|^2} \right)^{q\frac{n-2}{2}} dY \\ &= \mu_\epsilon^{n+p-q\frac{n-2}{2}} \begin{cases} s \int_{\mathbb{R}^n} |Y|^p w^q(Y) dY + o(1) & \text{if } q(n-2) > p+n \\ s \left( \frac{n(n-2)}{f(x_0)} \right)^{q\frac{n-2}{2}} \omega_{n-1} \ln \left( \frac{1}{\mu_\epsilon} \right) + o(\ln \mu_\epsilon) & \text{if } q(n-2) = p+n \end{cases} \end{aligned}$$

The proof of (43) is similar by taking  $\alpha := \delta_\epsilon$ . Putting these estimates together yields Lemma 1.  $\square$

We now prove (6). We fix  $l \in \{1, \dots, n\}$ . We define

$$\delta_\epsilon := \mu_\epsilon^{\frac{1}{n-1}}.$$

Pohozaev's identity (42) applied to  $\hat{u}_\epsilon$  reads

$$(44) \quad A_\epsilon = -B_\epsilon + C_\epsilon - D_\epsilon$$

With

$$\begin{aligned} A_\epsilon &:= -\frac{1}{2^*} \int_{B_{\delta_\epsilon}(0)} \partial_l \hat{f}_\epsilon \hat{u}_\epsilon^{2^*} dX \\ B_\epsilon &:= \int_{\partial B_{\delta_\epsilon}(0)} - \left[ \nu_l \left( \frac{|\nabla \hat{u}_\epsilon|^2}{2} - \frac{\hat{f}_\epsilon \hat{u}_\epsilon^{2^*}}{2^*} \right) + \partial_l \hat{u}_\epsilon \partial_\nu \hat{u}_\epsilon \right] d\nu \\ C_\epsilon &:= \int_{B_{\delta_\epsilon}(0)} \partial_l \hat{u}_\epsilon \hat{h}_\epsilon \hat{u}_\epsilon dX \text{ and } D_\epsilon := \int_{B_{\delta_\epsilon}(0)} \partial_l \hat{u}_\epsilon (\Delta_\xi \hat{u}_\epsilon - \Delta_{\hat{g}_\epsilon} \hat{u}_\epsilon) dX \end{aligned}$$

We estimate these terms separately. It follows from (22) and (23) that

$$B_\epsilon = O \left( \mu_\epsilon \left( \left( \frac{\mu_\epsilon}{\delta_\epsilon} \right)^{n-3} + \frac{\mu_\epsilon^{n-1}}{\delta_\epsilon^{n+1}} + \frac{\delta_\epsilon^{n-1}}{\mu_\epsilon} \theta_\epsilon^2 \right) \right) = o(\mu_\epsilon) \text{ as } \epsilon \rightarrow 0.$$

Concerning  $C_\epsilon$ , integrating by parts, we have that

$$C_\epsilon = \int_{B_{\delta_\epsilon}(0)} \partial_l \hat{u}_\epsilon \hat{h}_\epsilon \hat{u}_\epsilon dX = - \int_{B_{\delta_\epsilon}(0)} \partial_l \frac{\hat{h}_\epsilon}{2} \hat{u}_\epsilon^2 dX + \int_{\partial B_{\delta_\epsilon}(0)} \hat{h}_\epsilon \frac{\hat{u}_\epsilon^2}{2} \bar{\nu}_l d\sigma$$

With (22) and Lemma 1, we then get that

$$C_\epsilon = O \left( \mu_\epsilon \left( o(1) + \left( \frac{\mu_\epsilon}{\delta_\epsilon} \right)^{n-3} + \frac{\delta_\epsilon^{n-1}}{\mu_\epsilon} \theta_\epsilon^2 \right) \right) = o(\mu_\epsilon) \text{ as } \epsilon \rightarrow 0.$$

We now estimate  $D_\epsilon$ . We write

$$-(\Delta_{\hat{g}_\epsilon} - \Delta_\xi) = (\hat{g}_\epsilon^{ij} - \delta^{ij}) \partial_{ij} - \hat{g}_\epsilon^{ij} \hat{\Gamma}_{ij}^k(\hat{g}_\epsilon) \partial_k$$

where the  $\hat{\Gamma}_{ij}^k$ 's are the Christoffel symbols of the metric  $\hat{g}_\epsilon$ . The following lemma is reminiscent in such problems:

**Lemma 2.** *Let  $\Omega$  be a smooth domain of  $\mathbb{R}^n$ . For any  $i, j, k \in \{1, \dots, n\}$ , let us consider  $a^{ijk} \in C^1(\mathbb{R}^n)$ . We assume that  $a^{ijk} = a^{jik}$  for all  $i, j, k \in \{1, \dots, n\}$ . Then for all  $u \in C^2(\mathbb{R}^n)$ , we have that*

$$\begin{aligned} \int_{\Omega} a^{ijk} \partial_{ij} u \partial_k u dx &= \int_{\Omega} \left( -\partial_l a^{lij} + \frac{1}{2} \partial_l a^{ijl} \right) \partial_i u \partial_j u dx \\ &\quad + \int_{\partial\Omega} \left( -\frac{1}{2} a^{ijl} \bar{\nu}_l + a^{lji} \bar{\nu}_l \right) \partial_i u \partial_j u d\sigma \end{aligned}$$

where  $\bar{\nu}$  is the outer normal vector at  $\partial\Omega$  and Einstein's summation convention has been used.

The proof is by integrations by parts and goes back to Hebey-Vaugon [14] and can also be found in Cheikh-Ali [4]. It follows from Lemma 2 that

$$\begin{aligned} \int_{B_{\delta_\epsilon}(0)} (\hat{g}_\epsilon^{ij} - \delta^{ij}) \partial_{ij} \hat{u}_\epsilon \partial_l \hat{u}_\epsilon dx &= \int_{B_{\delta_\epsilon}(0)} \left( -\partial_m \hat{g}_\epsilon^{mj} \delta_{j,l} + \frac{1}{2} \partial_l \hat{g}_\epsilon^{ij} \delta_{m,l} \right) \partial_i \hat{u}_\epsilon \partial_j \hat{u}_\epsilon dx \\ &\quad + \int_{\partial B_{\delta_\epsilon}(0)} \left( -\frac{1}{2} (\hat{g}_\epsilon^{ij} - \delta^{ij}) \delta_{m,l} \vec{\nu}_m + (\hat{g}_\epsilon^{mj} - \delta^{mj}) \delta_{i,l} \vec{\nu}_l \right) \partial_i \hat{u}_\epsilon \partial_j \hat{u}_\epsilon d\sigma \end{aligned}$$

Using (22) to control the boundary terms, we get that

$$\begin{aligned} D_\epsilon &= \int_{B_{\delta_\epsilon}(0)} \left( -\partial_m \hat{g}_\epsilon^{mj} \delta_{j,l} + \frac{1}{2} \partial_m \hat{g}_\epsilon^{ij} \delta_{m,l} \right) \partial_i \hat{u}_\epsilon \partial_j \hat{u}_\epsilon dx \\ &\quad - \int_{B_{\delta_\epsilon}(0)} \hat{g}_\epsilon^{ij} \Gamma_{ij}^k(\hat{g}_\epsilon) \partial_k \hat{u}_\epsilon \partial_l \hat{u}_\epsilon dX + O \left( \underbrace{\mu_\epsilon \left( \left( \frac{\mu_\epsilon}{\delta_\epsilon} \right)^{n-3} + \frac{\delta_\epsilon^{n-1}}{\mu_\epsilon} \theta_\epsilon^2 \right)}_{=o(\mu_\epsilon)} \right) \end{aligned}$$

as  $\epsilon \rightarrow 0$ . Since

$$\Gamma_{ij}^k(\hat{g}_\epsilon) = \frac{1}{2} \hat{g}_\epsilon^{km} (\partial_i(\hat{g}_\epsilon)_{jm} + \partial_j(\hat{g}_\epsilon)_{im} - \partial_m(\hat{g}_\epsilon)_{ij})$$

and  $\hat{g}_\epsilon$  is normal at 0 (that is  $\partial_m(\hat{g}_\epsilon)_{ij}(0) = 0$  for all  $i, j, m \in \{1, \dots, n\}$ ), we then get that there exists  $a_{ij\alpha} \in \mathbb{R}$ ,  $i, j, \alpha \in \{1, \dots, n\}$  such that

$$D_\epsilon = \int_{B_{\delta_\epsilon}(0)} a_{ij\alpha} X^\alpha \partial_i \hat{u}_\epsilon \partial_j \hat{u}_\epsilon dx + O \left( \int_{B_{\delta_\epsilon}(0)} |X|^2 |\nabla \hat{u}_\epsilon|^2 dX \right) + o(\mu_\epsilon)$$

With (23), we get that

$$\begin{aligned} \int_{B_{\delta_\epsilon}(0)} |X|^2 |\nabla \hat{u}_\epsilon|^2 dX &\leq C \int_{B_{\delta_\epsilon}(0)} |X|^2 \frac{\mu_\epsilon^{n-2}}{(\mu_\epsilon + |X|)^{2(n-1)}} dX + C \theta_\epsilon^2 \delta_\epsilon^n \\ &\leq C \mu_\epsilon^2 \int_{B_{\delta_\epsilon/\mu_\epsilon}(0)} \frac{|X|^2 dX}{(1 + |X|)^{2(n-1)}} + C \theta_\epsilon^2 \delta_\epsilon^n = o(\mu_\epsilon) \end{aligned}$$

With (23), given  $R > 0$ , using (46) and  $n > 3$ , we have that

$$\begin{aligned} \left| \int_{B_{\delta_\epsilon}(0) \setminus B_{R\mu_\epsilon}(0)} X^\alpha \partial_i \hat{u}_\epsilon \partial_j \hat{u}_\epsilon dx \right| &\leq C \int_{B_{\delta_\epsilon}(0) \setminus B_{R\mu_\epsilon}(0)} \frac{\mu_\epsilon^{n-2} |X| dX}{(\mu_\epsilon + |X|)^{2(n-1)}} + C \theta_\epsilon^2 \delta_\epsilon^n \\ &\leq C \mu_\epsilon \int_{\mathbb{R}^n \setminus B_R(0)} \frac{|Y| dX}{(1 + |Y|)^{2(n-1)}} + C \theta_\epsilon^2 \delta_\epsilon^n \leq \eta(R) \mu_\epsilon + o(\mu_\epsilon) \end{aligned}$$

where  $\lim_{R \rightarrow +\infty} \eta(R) = 0$ . Using the change of variable  $X = \mu_\epsilon Y$ , the convergence (15) and the radial symmetry of  $w$ , we have that

$$\begin{aligned} \int_{B_{R\mu_\epsilon}(0)} X^\alpha \partial_i \hat{u}_\epsilon \partial_j \hat{u}_\epsilon dx &= \mu_\epsilon \int_{B_R(0)} Y^\alpha \partial_i w_\epsilon \partial_j w_\epsilon dY \\ &= \mu_\epsilon \left( \int_{B_R(0)} X^\alpha \partial_i w \partial_j w dY + o(1) \right) = o(\mu_\epsilon) \text{ since } n > 3. \end{aligned}$$

Therefore, we get that  $\int_{B_{\delta_\epsilon}(0)} X^\alpha \partial_i \hat{u}_\epsilon \partial_j \hat{u}_\epsilon dx = o(\mu_\epsilon)$ , and then

$$D_\epsilon = o(\mu_\epsilon) \text{ for } n \geq 4.$$

We now deal with  $A_\epsilon$ . With a Taylor expansion of  $f$ , we get

$$\begin{aligned} A_\epsilon &= -\frac{1}{2^\star} \int_{B_{\delta_\epsilon}(0)} \partial_l \hat{f}_\epsilon \hat{u}_\epsilon^{2^\star} dX \\ &= -\frac{1}{2^\star} \partial_l \hat{f}_\epsilon(0) \int_{B_{\delta_\epsilon}(0)} \hat{u}_\epsilon^{2^\star} dX - \frac{1}{2^\star} \partial_{lj} \hat{f}_\epsilon(0) \int_{B_{\delta_\epsilon}(0)} X^j \hat{u}_\epsilon^{2^\star} dX \\ &\quad + O\left(\int_{B_{\delta_\epsilon}(0)} |X|^2 \hat{u}_\epsilon^{2^\star} dX\right) \end{aligned}$$

Arguing as above, we get that

$$\int_{B_{\delta_\epsilon}(0)} |X|^2 \hat{u}_\epsilon^{2^\star} dX = o(\mu_\epsilon) \text{ and } \int_{B_{\delta_\epsilon}(0)} X^j \hat{u}_\epsilon^{2^\star} dX = o(\mu_\epsilon) \text{ as } \epsilon \rightarrow 0.$$

With (21), we get that there exists  $C_0 > 0$  such that

$$\int_{B_{\delta_\epsilon}(0)} \hat{u}_\epsilon^{2^\star} dX = C_0 + o(1) \text{ as } \epsilon \rightarrow 0.$$

Therefore, we get that

$$A_\epsilon = \left(-\frac{C_0}{2^\star} + o(1)\right) \partial_l \hat{f}_\epsilon(0) + o(\mu_\epsilon) \text{ as } \epsilon \rightarrow 0.$$

Putting the estimates of  $A_\epsilon$ ,  $B_\epsilon$ ,  $C_\epsilon$  and  $D_\epsilon$  into (44) yields

$$(45) \quad \partial_l \hat{f}_\epsilon(0) = o(\mu_\epsilon) \text{ as } \epsilon \rightarrow 0 \text{ for all } l \in \{1, \dots, n\}.$$

Passing to the limit, we get that  $\nabla f(x_0) = 0$ . We now express  $\partial_l \hat{f}_\epsilon(0)$  more precisely. We write

$$\hat{f}_\epsilon(X) = f \circ \exp_{x_\epsilon}(X) = \tilde{f} \circ \varphi(X_\epsilon, X) \text{ for } X \in \mathbb{R}^n$$

where  $\tilde{f} := f \circ \exp_{x_0}$  and  $\varphi(Z, X) := \exp_{x_0}^{-1} \circ \exp_{\exp_{x_0}(Z)}(X)$  for  $X, Z \in \mathbb{R}^n$ . We set  $X_\epsilon := \exp_{x_0}^{-1}(x_\epsilon)$ . Since  $\nabla \tilde{f}(0) = 0$ , we get that

$$\partial_l (f \circ \exp_{x_\epsilon})(0) = \frac{\partial^2 (f \circ \exp_{x_0})}{\partial x_l \partial x_j}(0) X_\epsilon^j + o(|X_\epsilon|)$$

therefore, with (45), we get that

$$\frac{\partial^2 (f \circ \exp_{x_0})}{\partial x_l \partial x_j}(0) X_\epsilon^j = o(|X_\epsilon|) + o(\mu_\epsilon) \text{ as } \epsilon \rightarrow 0 \text{ for all } l = 1, \dots, n.$$

Since  $\nabla^2 f(x_0)$  is nondegenerate, we then get that  $|X_\epsilon| = o(\mu_\epsilon)$ , in other words  $d_g(x_\epsilon, x_0) = o(\mu_\epsilon)$  as  $\epsilon \rightarrow 0$ . This proves (6).

**Lemma 3.** *Under the assumptions of Theorem 1, we have that*

$$(46) \quad \theta_\epsilon = \begin{cases} o(1) & \text{if } n \geq 7, \\ o(\mu_\epsilon) & \text{if } n \in \{5, 6\}, \\ o\left(\mu_\epsilon \sqrt{\ln\left(\frac{1}{\mu_\epsilon}\right)}\right) & \text{if } n = 4. \end{cases}$$



*Proof:* The case  $n \geq 7$  is simply (24). It follows from (22) that

$$\begin{aligned} \int_{B_\delta(x_0)} u_\epsilon^2 dv_g &\leq \int_{B_{2\delta}(x_\epsilon)} \frac{\mu_\epsilon^{n-2}}{(\mu_\epsilon + d_g(x, x_\epsilon))^{2(n-2)}} dv_g + C\theta_\epsilon^2 \\ &\leq \mu_\epsilon^2 \int_{B_{2\delta/\mu_\epsilon}(0)} \frac{1}{(1 + |X|)^{2(n-2)}} dv_g + C\theta_\epsilon^2 \\ &\leq C \begin{cases} \mu_\epsilon^2 & \text{if } n \geq 5 \\ \mu_\epsilon^2 \ln\left(\frac{1}{\mu_\epsilon}\right) & \text{if } n = 4 \end{cases} \end{aligned}$$

Equation (5) rewrites  $\Delta_g u_\epsilon + (h_\epsilon - f u_\epsilon^{2^*-2})u_\epsilon = 0$  in  $M$ . Since  $u_\epsilon \rightarrow 0$  in  $C_{loc}^0(M - \{x_0\})$  and  $u_\epsilon > 0$ , it follows from Harnack's inequality that there exists  $c > 0$  such that  $u_\epsilon(x) \leq c u_\epsilon(y)$  for all  $x, y \in B_{2\delta}(x_0) \setminus B_{\delta/3}(x_0)$ . Therefore, with the definition (24) of  $\theta_\epsilon$ , we get that

$$\int_{B_{2\delta}(x_0) \setminus B_{\delta/3}(x_0)} u_\epsilon^2 dv_g \geq c^{-2} \theta_\epsilon^2.$$

When  $4 \leq n \leq 6$ , it follows from  $L^2$ -concentration assumption (7) that

$$\int_{B_{2\delta}(x_0) \setminus B_{\delta/3}(x_0)} u_\epsilon^2 dv_g \leq \int_{M \setminus B_{\delta/3}(x_0)} u_\epsilon^2 dv_g = o\left(\int_{B_\delta(x_0)} u_\epsilon^2 dv_g\right) \text{ as } \epsilon \rightarrow 0.$$

Putting these inequalities together, we get (46). This proves Lemma 3.  $\square$

#### 4. INTERACTION WITH THE SCALAR CURVATURE: PROOF OF (8)

This part is strongly inspired by Cheikh-Ali [4]. We define

$$\delta_\epsilon := \begin{cases} \mu_\epsilon^{\frac{2}{n-2}} & \text{if } n \geq 7 \\ \delta & \text{if } n \in \{4, 5, 6\}. \end{cases}$$

Writing the Pohozaev identity (41) for  $\hat{u}_\epsilon$  that satisfies (5), we get that

$$(47) \quad A_\epsilon + B_\epsilon = C_\epsilon + D_\epsilon$$

where

$$\begin{aligned} B_\epsilon &:= \int_{\partial B_{\delta_\epsilon}(0)} \left[ (X, \nu) \left( \frac{|\nabla \hat{u}_\epsilon|^2}{2} - \frac{\hat{f}_\epsilon \hat{u}_\epsilon^{2^*}}{2^*} \right) - \left( X^l \partial_l \hat{u}_\epsilon + \frac{n-2}{2} \hat{u}_\epsilon \right) \partial_\nu \hat{u}_\epsilon \right] d\nu \\ C_\epsilon &:= - \int_{B_{\delta_\epsilon}(0)} \left( X^l \partial_l \hat{u}_\epsilon + \frac{n-2}{2} \hat{u}_\epsilon \right) \hat{h}_\epsilon \hat{u}_\epsilon dX \\ D_\epsilon &:= - \int_{B_{\delta_\epsilon}(0)} \left( X^l \partial_l \hat{u}_\epsilon + \frac{n-2}{2} \hat{u}_\epsilon \right) (\Delta_{\hat{g}_\epsilon} \hat{u}_\epsilon - \Delta_\xi \hat{u}_\epsilon) dX \\ A_\epsilon &:= \frac{1}{2^*} \int_{B_{\delta_\epsilon}(0)} (\nabla \hat{f}_\epsilon, X) \hat{u}_\epsilon^{2^*} dX \end{aligned}$$

Following Cheikh-Ali [4] and using the pointwise controls (21), (22), (23) and the control (46) on  $(\theta_\epsilon)_\epsilon$  when  $n \in \{4, 5, 6\}$ , we get that

$$B_\epsilon = \begin{cases} o(\mu_\epsilon^2) & \text{if } n \geq 5 \\ o\left(\mu_\epsilon^2 \ln \frac{1}{\mu_\epsilon}\right) & \text{if } n = 4. \end{cases} \text{ as } \epsilon \rightarrow 0,$$

$$C_\epsilon = \begin{cases} h(x_0)\mu_\epsilon^2 \ln\left(\frac{1}{\mu_\epsilon}\right) \left(\frac{8}{f(x_0)}\right)^2 \omega_3 + o(\mu_\epsilon^2 \ln \frac{1}{\mu_\epsilon}) & \text{if } n = 4 \\ h(x_0)\mu_\epsilon^2 \int_{\mathbb{R}^n} w^2 dX + o(\mu_\epsilon^2) & \text{if } n \geq 5 \end{cases}$$

$$D_\epsilon = \begin{cases} -\mu_\epsilon^2 \ln\left(\frac{1}{\mu_\epsilon}\right) \frac{1}{6} \text{Scal}_g(x_\epsilon) \left(\frac{8}{f(x_0)}\right)^2 \omega_3 + o(\mu_\epsilon^2 \ln \frac{1}{\mu_\epsilon}) & \text{if } n = 4 \\ -\mu_\epsilon^2 \frac{n-2}{4(n-1)} \text{Scal}_g(x_\epsilon) \int_{\mathbb{R}^n} w^2 dX + o(\mu_\epsilon^2) & \text{if } n \geq 5 \end{cases}$$

We are then left with estimating  $A_\epsilon$ . With a Taylor expansion of  $\hat{f}_\epsilon$ , we get that

$$A_\epsilon = \frac{1}{2^*} \int_{B_{\delta_\epsilon}(0)} \partial_i \hat{f}_\epsilon(0) X^i \hat{u}_\epsilon^{2^*} dX + \frac{1}{2^*} \partial_{ij} \hat{f}_\epsilon(0) \int_{B_{\delta_\epsilon}(0)} X^i X^j \hat{u}_\epsilon^{2^*} dX \\ + O\left(\int_{B_{\delta_\epsilon}(0)} |X|^3 \hat{u}_\epsilon^{2^*} dX\right)$$

Since  $\hat{f}_\epsilon := f \circ \exp_{x_\epsilon}$  and  $\nabla f(x_0) = 0$ , we get that  $\nabla \hat{f}_\epsilon(0) = O(d_g(x_\epsilon, x_0))$ . With (6), we then get that  $\nabla \hat{f}_\epsilon(0) = o(\mu_\epsilon)$ . It follows from Lemma 1 that  $\int_{B_{\delta_\epsilon}(0)} |X|^3 \hat{u}_\epsilon^{2^*} dX = o(\mu_\epsilon^2)$  and  $\int_{B_{\delta_\epsilon}(0)} |X| \hat{u}_\epsilon^{2^*} dX = O(\mu_\epsilon)$ . Therefore, we get that

$$A_\epsilon = \frac{1}{2^*} \partial_{ij} \hat{f}_\epsilon(0) \int_{B_{\delta_\epsilon}(0)} X^i X^j \hat{u}_\epsilon^{2^*} dX + o(\mu_\epsilon^2)$$

Arguing as in the proof of Lemma 1, we get

$$\int_{B_{\frac{\delta_\epsilon}{\mu_\epsilon}}(0)} X^i X^j \hat{u}_\epsilon^{2^*} dX = \mu_\epsilon^2 \int_{\mathbb{R}^n} X^i X^j w^{2^*} dX + o(\mu_\epsilon^2) \text{ when } n \geq 4.$$

Since  $w$  is radially symmetric, we get that  $\int_{\mathbb{R}^n} X^i X^j w^{2^*} dX = \frac{\delta_{ij}}{n} \int_{\mathbb{R}^n} |X|^2 w^{2^*} dX$ . Since  $\hat{g}_\epsilon$  is normal at 0, we have that  $\Delta_g f(x_\epsilon) = -\sum_i \partial_{ii} \hat{f}_\epsilon(0)$ , which yields We claim that

$$A_\epsilon = -\frac{1}{2^* n} \Delta_g f(x_0) \mu_\epsilon^2 \int_{\mathbb{R}^n} |X|^2 w^{2^*} dX + o(\mu_\epsilon^2) \quad \text{if } n \geq 4$$

By Jaber [15] we have that

$$\frac{\int_{\mathbb{R}^n} |X|^2 w^{2^*} dX}{\int_{\mathbb{R}^n} w^2 dX} = \frac{n^2(n-4)}{4(n-1)f(x_0)} \text{ for } n \geq 5.$$

Putting the expressions of  $A_\epsilon$ ,  $B_\epsilon$ ,  $C_\epsilon$  and  $D_\epsilon$  in (47) and letting  $\epsilon \rightarrow 0$  yield (8). This ends the proof of Theorem 1.

## 5. APPLICATION TO A SUPER-CRITICAL PROBLEM: PROOF OF THEOREM 2

We follow the notations and assumptions of Theorem 2. We consider a family  $(u_\epsilon)_{\epsilon>0} \in C_G^2(X)$  of  $G$ -invariant solutions to the problem

$$(48) \quad \Delta_g u_\epsilon + h_\epsilon u_\epsilon = \lambda_\epsilon u_\epsilon^{2^*(k)-1}, \quad \int_X u_\epsilon^{2^*(k)} dv_g = 1, \quad \|u_\epsilon\|_2 \rightarrow 0 \text{ as } \epsilon \rightarrow 0$$

where  $(h_\epsilon)_{\epsilon>0} \in C_G^1(X)$  is such that there exists  $h \in C_G^1(X)$  such that (9) holds and  $(\lambda_\epsilon)_\epsilon$  is such that (11) holds.

**Claim 4.** *There exists  $x_0 \in X$  such that*

$$(49) \quad \lim_{\epsilon \rightarrow 0} \int_{B_\delta(Gx_0)} u_\epsilon^{2^*(k)} dv_g = 1 \text{ for all } \delta > 0.$$

*Proof:* We fix a point  $z_0 \in X$ . We choose  $\eta_0 \in C^\infty(\mathbb{R})$  such that  $\eta_0(t) = 1$  for  $t \leq 1$  and  $\eta_0(t) = 0$  for  $t \geq 2$ . Given  $\delta > 0$ , we define  $\eta(x) := \eta_0(\frac{d_g(Gx_0, x)}{\delta})$  for all  $x \in X$ . For  $\delta > 0$  small enough, we have that  $\eta \in C_G^\infty(X)$ . Multiply (10) by  $\eta^2 u_\epsilon^l$  for some  $l > 1$  and integrate over  $X$ , we get that

$$(50) \quad \int_M \eta^2 u_\epsilon^l \Delta_g u_\epsilon dv_g + \int_M \eta^2 h_\epsilon u_\epsilon^{l+1} dv_g = \lambda_\epsilon \int_M \eta^2 u_\epsilon^{l+2^*(k)-1} dv_g$$

As in the proof of Claim 2, we get that

$$\int_M |\nabla(\eta u_\epsilon^{\frac{l+1}{2}})|_g^2 dv_g = \frac{(l+1)^2}{4l} \int_M \eta^2 u_\epsilon^l \Delta_g u_\epsilon dv_g + \frac{l+1}{2l} \int_M \left( |\nabla \eta|_g^2 + \frac{l-1}{l+1} \eta \Delta_g \eta \right) u_\epsilon^{l+1} dv_g$$

Using (50) and Hölder's inequality, we get

$$(51) \quad \int_X |\nabla(\eta u_\epsilon^{\frac{l+1}{2}})|_g^2 dv_g \leq C \int_X u_\epsilon^{l+1} dv_g + \frac{(l+1)^2}{4l} \lambda_\epsilon \left( \int_X (\eta u_\epsilon^{\frac{l+1}{2}})^{2^*(k)} dv_g \right)^{\frac{2}{2^*(k)}} \left( \int_{B_{2\delta}(Gz_0)} u_\epsilon^{2^*(k)} dv_g \right)^{1 - \frac{2}{2^*(k)}}$$

It follows from Faget [10] that for all  $\alpha > 0$ , there exists  $B > 0$  such that, for all  $\epsilon$ ,

$$\left( \int_X (\eta u_\epsilon^{\frac{l+1}{2}})^{2^*(k)} dv_g \right)^{\frac{2}{2^*(k)}} \leq \frac{K_0(n-k)(1+\alpha)}{V_m^{1 - \frac{2}{2^*(k)}}} \int_X |\nabla(\eta u_\epsilon^{\frac{l+1}{2}})|_g^2 dv_g + B \int_X \eta^2 u_\epsilon^{l+1} dv_g$$

where  $K_0(n-k)$  is as in (3) and  $V_m = \min_{x \in X} \{Vol_g(Gx) / \dim Gx = k\}$ . By combining this inequality with (51), we obtain:

$$(52) \quad \left( \int_X (\eta u_\epsilon^{\frac{l+1}{2}})^{2^*(k)} dv_g \right)^{\frac{2}{2^*(k)}} \chi_\epsilon \leq C \|u_\epsilon\|_{l+1}^{l+1}$$

where

$$\chi_\epsilon := 1 - \frac{(l+1)^2}{4l} \lambda_\epsilon \frac{K_0(n-k)(1+\alpha)}{V_m^{1 - \frac{2}{2^*(k)}}} \left( \int_{B_{2\delta}(Gz_0)} u_\epsilon^{2^*(k)} dv_g \right)^{1 - \frac{2}{2^*(k)}}$$

Assume that, up to extraction,

$$\lim_{\epsilon \rightarrow 0} \int_{B_{2\delta}(Gz_0)} u_\epsilon^{2^*(k)} dv_g < 1.$$

Using (11), there exists  $1 < l < 2^*(k) - 1$  such that  $\chi_\epsilon \geq \beta > 0$  for all  $\epsilon > 0$  up to taking  $\alpha$  small. As  $u_\epsilon \rightarrow 0$  in  $L^{l+1}(X)$  since  $l+1 < 2^*(k)$ , with (52), we then get that  $\lim_{\epsilon \rightarrow 0} \int_{B_\delta(Gz_0)} u_\epsilon^{\frac{l+1}{2} 2^*(k)} dv_g = 0$ . With similar arguments, we get that for all  $\delta' < \delta$ ,  $u_\epsilon \rightarrow 0$  in  $L^q(B_{\delta'}(Gz_0))$  for all  $q \geq 1$ . It then follows from (10) and elliptic theory that  $u_\epsilon \rightarrow 0$  in  $C^0(B_{\delta'}(Gz_0))$ . Since  $\int_X u_\epsilon^{2^*(k)} dv_g = 1$  and  $X$  is compact, the existence of  $x_0 \in X$  such that (49) holds follows. This proves the claim.  $\square$

**Claim 5.** *We have that  $\dim Gx_0 = k$  and  $Vol_g(Gx_0) = V_m$ .*

*Proof:* We follow Faget [10]. Assume that  $\dim Gx_0 > k$ . Therefore, there exists  $\delta > 0$  such that  $\dim Gx \geq k_1 > k$  for all  $x \in B_{2\delta}(Gx_0)$ . It then follows from Hebey-Vaugon [13] that  $H_{1,G}^2(B_\delta(Gx_0)) \hookrightarrow L^p(B_\delta(Gx_0))$  is compact for  $1 \leq p < 2^*(k_1)$ . Since  $2^*(k) < 2^*(k_1)$  and  $u_\epsilon \rightarrow 0$  in  $L^2(X)$ , we get that  $u_\epsilon \rightarrow 0$  strongly in

$L^{2^*(k)}(B_\delta(Gx_0))$ , contradicting (49). Therefore  $\dim Gx_0 = k$ . It follows from Faget (formula (8) in [10]) that for all  $\alpha > 0$ , there exists  $\delta_\alpha > 0$  such that

$$\left( \int_{B_{\delta_\alpha}(Gx_0)} |v|^{2^*(k)} dv_g \right)^{\frac{2}{2^*(k)}} \leq (1 + \alpha) \frac{K_0(n-k)}{\text{Vol}_g(Gx_0)^{1-\frac{2}{2^*(k)}}} \int_{B_{\delta_\alpha}(Gx_0)} |\nabla v|^2 dv_g$$

for all  $v \in C_G^1(B_{\delta_\alpha}(Gx_0))$  with compact support in  $B_{\delta_\alpha}(Gx_0)$ . Let us fix  $\eta_\alpha \in C_G^\infty(X)$  with compact support in  $B_{\delta_\alpha}(Gx_0)$  and such that  $0 \leq \eta_\alpha \leq 1$  and  $\eta_\alpha(x) = 1$  for  $d_g(x, Gx_0) < \delta_\alpha/2$ . We then get that

$$\begin{aligned} \left( \int_{B_{\delta_\alpha/2}(Gx_0)} u_\epsilon^{2^*(k)} dv_g \right)^{\frac{2}{2^*(k)}} &\leq \left( \int_{B_{\delta_\alpha}(Gx_0)} (\eta_\alpha u_\epsilon)^{2^*(k)} dv_g \right)^{\frac{2}{2^*(k)}} \\ (53) \qquad \qquad \qquad &\leq (1 + \alpha) \frac{K_0(n-k)}{\text{Vol}_g(Gx_0)^{1-\frac{2}{2^*(k)}}} \int_X |\nabla(\eta_\alpha u_\epsilon)|^2 dv_g \end{aligned}$$

Integrating by parts and using  $\|u_\epsilon\|_2 \rightarrow 0$ , we get that

$$(54) \quad \int_X |\nabla(\eta_\alpha u_\epsilon)|_g^2 dv_g = \int_X \eta_\alpha^2 |\nabla u_\epsilon|_g^2 dv_g + \int_X \eta(\Delta_g \eta) u_\epsilon^2 dv_g \leq \int_X |\nabla u_\epsilon|_g^2 dv_g + o(1)$$

Multiplying (48) by  $u_\epsilon$ , integrating and using again  $\|u_\epsilon\|_2 \rightarrow 0$ , we get that

$$(55) \quad \lambda_\epsilon = \int_X \lambda_\epsilon u_\epsilon^{2^*(k)} dv_g = \int_X |\nabla u_\epsilon|^2 dv_g + \int_X h_\epsilon u_\epsilon^2 dv_g = \int_X |\nabla u_\epsilon|^2 dv_g + o(1).$$

Putting together (53), (54), (55) and (49), we get that

$$1 \leq (1 + \alpha) \frac{K_0(n-k)}{\text{Vol}_g(Gx_0)^{1-\frac{2}{2^*(k)}}} \lambda_\epsilon + o(1).$$

Using (11), letting  $\epsilon \rightarrow 0$  and then  $\alpha \rightarrow 0$  yields  $\text{Vol}_g(Gx_0) \leq V_m$ . Therefore  $\text{Vol}_g(Gx_0) = V_m$  and the claim is proved.  $\square$

**Claim:** *The following  $L^2$ -concentration holds*

$$(56) \quad \lim_{\epsilon \rightarrow 0} \frac{\int_{X \setminus B_\delta(Gx_0)} u_\epsilon^2 dv_g}{\int_X u_\epsilon^2 dv_g} = 0 \text{ for } n - k \geq 4.$$

We prove the claim by arguing as in Djadli-Druet [5]. We have that

$$\int_{X \setminus B_\delta(Gx_0)} u_\epsilon^2 dv_g \leq \left( \sup_{X \setminus B_\delta(Gx_0)} u_\epsilon \right) \int_{X \setminus B_\delta(Gx_0)} u_\epsilon dv_g$$

Since  $u_\epsilon \rightarrow 0$  in  $C_{loc}^0(X \setminus Gx_0)$ , Harnack's inequality yields  $c > 0$  such that

$$\begin{aligned} \int_{X \setminus B_\delta(Gx_0)} u_\epsilon^2 dv_g &\leq c \inf_{X \setminus B_\delta(Gx_0)} u_\epsilon \int_{X \setminus B_\delta(Gx_0)} u_\epsilon dv_g \\ &\leq c \left( \int_{X \setminus B_\delta(Gx_0)} u_\epsilon^2 dv_g \right)^{\frac{1}{2}} \int_X u_\epsilon dv_g \leq c \|u_\epsilon\|_2 \int_X u_\epsilon dv_g \end{aligned}$$

Integrating (10) yields  $\int_X h_\epsilon u_\epsilon dv_g = \int_X u_\epsilon^{2^*(k)-1} dv_g$ . It follows from (9) that there exists  $\beta > 0$  such that  $h_\epsilon \geq \beta$  for all  $\epsilon > 0$ . Therefore we get that

$$\int_{X \setminus B_\delta(Gx_0)} u_\epsilon^2 dv_g \leq c \beta^{-1} \|u_\epsilon\|_2 \|u_\epsilon\|_{2^*(k)-1}^{2^*(k)-1}.$$

if  $n-k \geq 6$ , we have  $2^*(k)-1 \leq 2$ . Using Hölder inequalities, we have  $\|u_\epsilon\|_{2^*(k)-1}^{2^*(k)-1} \leq c\|u_\epsilon\|_2^{2^*(k)-1}$ . Since  $u_\epsilon \rightarrow 0$  in  $L^2(X)$ , we get that

$$\int_{X \setminus B_\delta(Gx_0)} u_\epsilon^2 dv_g \leq c\|u_\epsilon\|_2^{2^*(k)} = o(\|u_\epsilon\|_2^2)$$

if  $4 \leq n-k \leq 5$ , we have  $2^*(k)-1 = 2\omega + (1-\omega)2^*(k)$  with  $\omega = \frac{n-k-2}{4} > 0$ . Hölder's inequality yields  $\|u_\epsilon\|_{2^*(k)-1}^{2^*(k)-1} \leq \|u_\epsilon\|_2^{2\omega} \|u_\epsilon\|_{2^*(k)}^{(1-\omega)2^*(k)}$ . As a result

$$\int_{X \setminus B_\delta(Gx_0)} u_\epsilon^2 dv_g \leq \|u_\epsilon\|_{2^*(k)-1}^{2^*(k)-1} \|u_\epsilon\|_2 \leq \|u_\epsilon\|_2^{1+2\omega} \|u_\epsilon\|_{2^*(k)}^{(1-\omega)2^*(k)} = o(\|u_\epsilon\|_2^2).$$

This proves the claim.  $\square$

We are now in position to take the quotient. Since  $\dim Gx_0 = k$ , we choose  $\delta > 0$  and  $G' \subset G$  as in Assumption (H). Then  $M := B_\delta(Gx_0)/G'$  is a manifold of dimension  $n-k$  that is endowed with the metric  $\bar{g}$  on  $B_\delta(Gx_0)/G'$  such that the projection  $(B_\delta(Gx_0), g) \rightarrow (B_\delta(Gx_0)/G', \bar{g})$  is a Riemannian submersion. We define  $\bar{u}_\epsilon \in C^2(M)$ ,  $\bar{h}_\epsilon \in C^1(M)$  and  $\bar{v} \in C^2(M)$  be such that

$$\bar{u}_\epsilon(\bar{x}) = u_\epsilon(x), \bar{h}_\epsilon(\bar{x}) = h_\epsilon(x) \text{ and } \bar{v}(\bar{x}) = \text{Vol}_g(G'x) \text{ for all } x \in B_\delta(Gx_0).$$

Let us first rewrite equation (48) as in Saintier [20]. Let  $\bar{\varphi} \in C_c^\infty(M)$  be a function on  $M = B_\delta(Gx_0)/G'$ . Define  $\varphi(x) := \bar{\varphi}(\bar{x})$  for all  $x \in B_\delta(Gx_0)$ : as one checks,  $\varphi \in C_c^\infty(B_\delta(Gx_0))$  and is  $G$ -invariant. It then follows from (48) that

$$\int_X (\nabla u_\epsilon, \nabla \varphi)_g dv_g + \int_X h_\epsilon u_\epsilon \varphi dv_g = \lambda_\epsilon \int_X u_\epsilon^{2^*(k)-1} \varphi dv_g.$$

We define  $\tilde{g} := \bar{v}^{-\frac{2}{n-k-2}} \bar{g}$ . Since  $u_\epsilon, \varphi$  are  $G$ -invariant, we get that

$$\int_X (\nabla u_\epsilon, \nabla \varphi)_g dv_g = \int_{B_\delta(Gx_0)/G'} \bar{v}(\bar{x}) (\nabla \bar{u}_\epsilon, \nabla \bar{\varphi})_{\bar{g}} dv_{\bar{g}} = \int_{B_\delta(Gx_0)/G'} (\nabla \bar{u}_\epsilon, \nabla \bar{\varphi})_{\tilde{g}} dv_{\tilde{g}}.$$

Performing the same computations for the remaining terms, setting  $\tilde{h}_\epsilon := \bar{v}^{-\frac{2}{n-k-2}} \bar{h}_\epsilon$ ,  $\tilde{f} := \bar{v}^{-\frac{2}{n-k-2}}$  and  $\tilde{u}_\epsilon := \lambda_\epsilon^{\frac{1}{2^*(k)-2}} \bar{u}_\epsilon$ , we get that

$$\Delta_{\tilde{g}} \tilde{u}_\epsilon + \tilde{h}_\epsilon \tilde{u}_\epsilon = \tilde{f} \tilde{u}_\epsilon^{2^*(k)-1} \text{ in } M.$$

We deal with the  $L^{2^*(k)}$ -norm. The definitions of  $\tilde{f}$  and  $\tilde{g}$  and (49) yield

$$\lim_{\epsilon \rightarrow 0} \int_M \tilde{f} \tilde{u}_\epsilon^{2^*(k)} dv_{\tilde{g}} = \lim_{\epsilon \rightarrow 0} \lambda_\epsilon^{\frac{2^*(k)}{2^*(k)-2}} = \frac{1}{K_0(n-k)^{\frac{n-k}{2}} \tilde{f}(\bar{x}_0)^{\frac{n-k-2}{2}}}.$$

Concerning the  $L^2$ -concentration, it follows from (56) that for any  $r < \delta$ , for  $n-k \geq 4$ , we have that

$$\int_{B_\delta(Gx_0) \setminus B_r(Gx_0)} u_\epsilon^2 dv_g \leq \int_{X \setminus B_r(Gx_0)} u_\epsilon^2 dv_g = o\left(\int_{B_r(Gx_0)} u_\epsilon^2 dv_g\right).$$

Taking the quotient, we get that

$$\int_{M \setminus B_r(\bar{x}_0)} \tilde{u}_\epsilon^2 dv_{\tilde{g}} = o\left(\int_{B_r(\bar{x}_0)} \tilde{u}_\epsilon^2 dv_{\tilde{g}}\right) \text{ when } n-k \geq 4,$$

which yields the  $L^2$ -concentration on  $M$ . We apply Theorem 1. Taking  $(x_\epsilon)_\epsilon \in X$  such that  $\|u_\epsilon\|_\infty = u_\epsilon(x_\epsilon) = \mu_\epsilon^{-\frac{n-k-2}{2}}$ , we get (12) and (13). Equation (8) rewrites

$$\tilde{h}(\bar{x}_0) = \frac{n-k-2}{4(n-k-1)} \left( \text{Scal}_{\tilde{g}}(\bar{x}_0) - \frac{n-k-4}{2} \cdot \frac{\Delta_{\tilde{g}}\tilde{f}(\bar{x}_0)}{\tilde{f}(\bar{x}_0)} \right)$$

where  $\tilde{h} := \lim_{\epsilon \rightarrow 0} \tilde{h}_\epsilon$ . Using the invariance of the conformal Laplacian, that is

$$\Delta_{\tilde{g}}\varphi + \frac{m-2}{4(m-1)}\text{Scal}_{\tilde{g}}\varphi = \omega^{-\frac{m+2}{m-2}} \left( \Delta_{\tilde{g}}(\omega\varphi) + \frac{m-2}{4(m-1)}\text{Scal}_{\tilde{g}}\omega\varphi \right)$$

for any  $\varphi \in C^2(M)$  and where  $\tilde{g} = \omega^{\frac{4}{m-2}}\bar{g}$ ,  $m = n - k$ , we get (14). This proves Theorem 2.

**Acknowledgement:** The initial version of this article required the  $L^2$ -concentration (7) for all dimensions  $n \geq 4$ . The authors are grateful to the anonymous referee who noticed that this concentration could be bypassed for  $n \geq 7$ .

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HUSSEIN MESMAR, INSTITUT ÉLIE CARTAN, UNIVERSITÉ DE LORRAINE, BP 70239, F-54506  
VANDŒUVRE-LES-NANCY, FRANCE  
*Email address:* `houssein.mesmar91@hotmail.com`

FRÉDÉRIC ROBERT, INSTITUT ÉLIE CARTAN, UNIVERSITÉ DE LORRAINE, BP 70239, F-54506  
VANDŒUVRE-LÈS-NANCY, FRANCE  
*Email address:* `frederic.robert@univ-lorraine.fr`