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► **To cite this version:**

François Lemaire, Louis Roussel. Integration of Integro-differential Equations and Deep Learning. Differential Algebra and Related Topics XI, Jun 2023, London, United Kingdom. <hal-04254563>

**HAL Id: hal-04254563**

**<https://hal.science/hal-04254563v1>**

Submitted on 23 Oct 2023

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# Integration of Integro-differential Equations and Deep Learning

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## Integral equation modelling

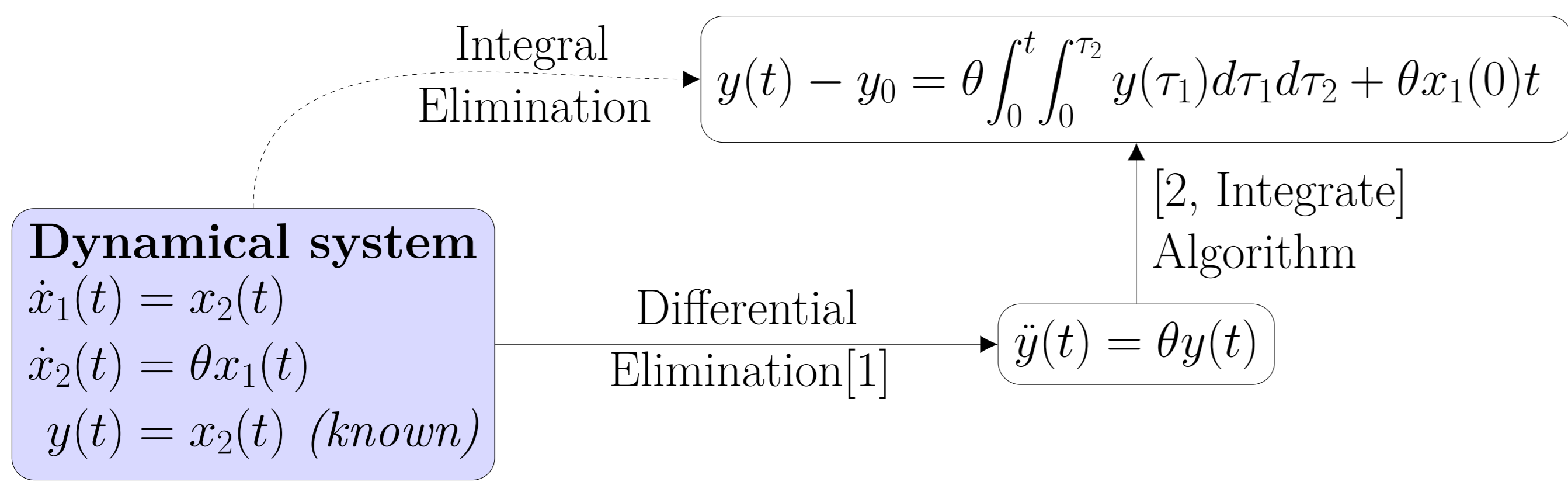


Figure 1: Modelling, elimination and [2, Integrate] algorithm.

Many models (such as the SIR epidemiology model) consist of non-linear differential equations, involving unknown numerical parameters. Recent work investigates the treatment of integro-differential equations i.e. equations involving both derivatives and integration operators. The introduction of integral equations helps to increase the expressiveness of the models, improve the estimation of parameters and reduce the size of the intermediate equations.

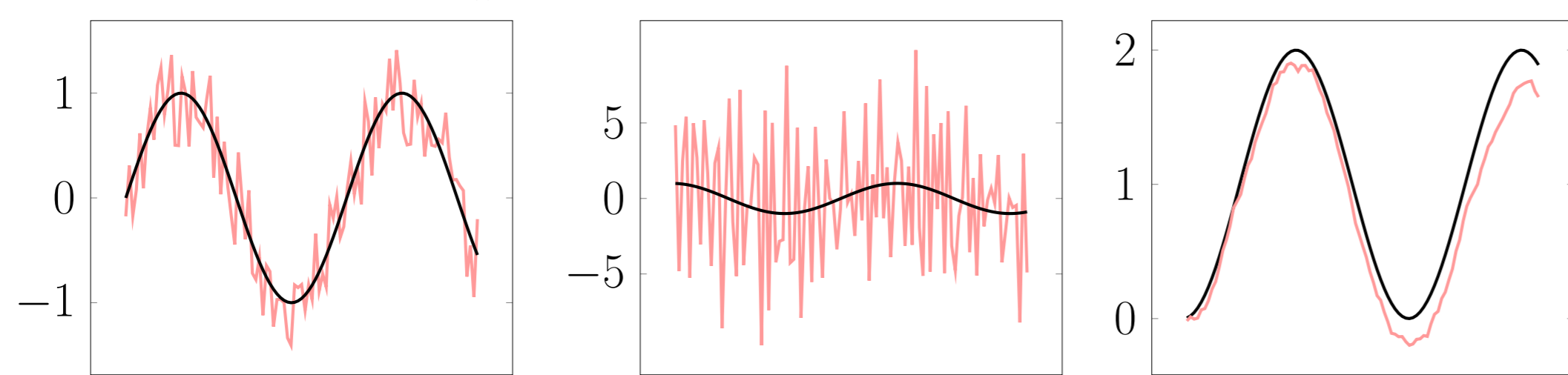


Figure 2: A signal, its derivative and its integral (with noise in red).

## Integration of equations with deep learning [3]

Through learning techniques used for the translation of texts, Lample and Charton [3] succeed in calculating primitives of mathematical expressions, and solutions of differential equations.

The model used is called Transformer and is widely used in natural language processing [4]:

- It takes a list of words as input and returns the list of translated words as output,
- Lample and Charton have created a data generator to generate the train and test datasets, composed of pairs  $(f', f)$  in Polish notation.

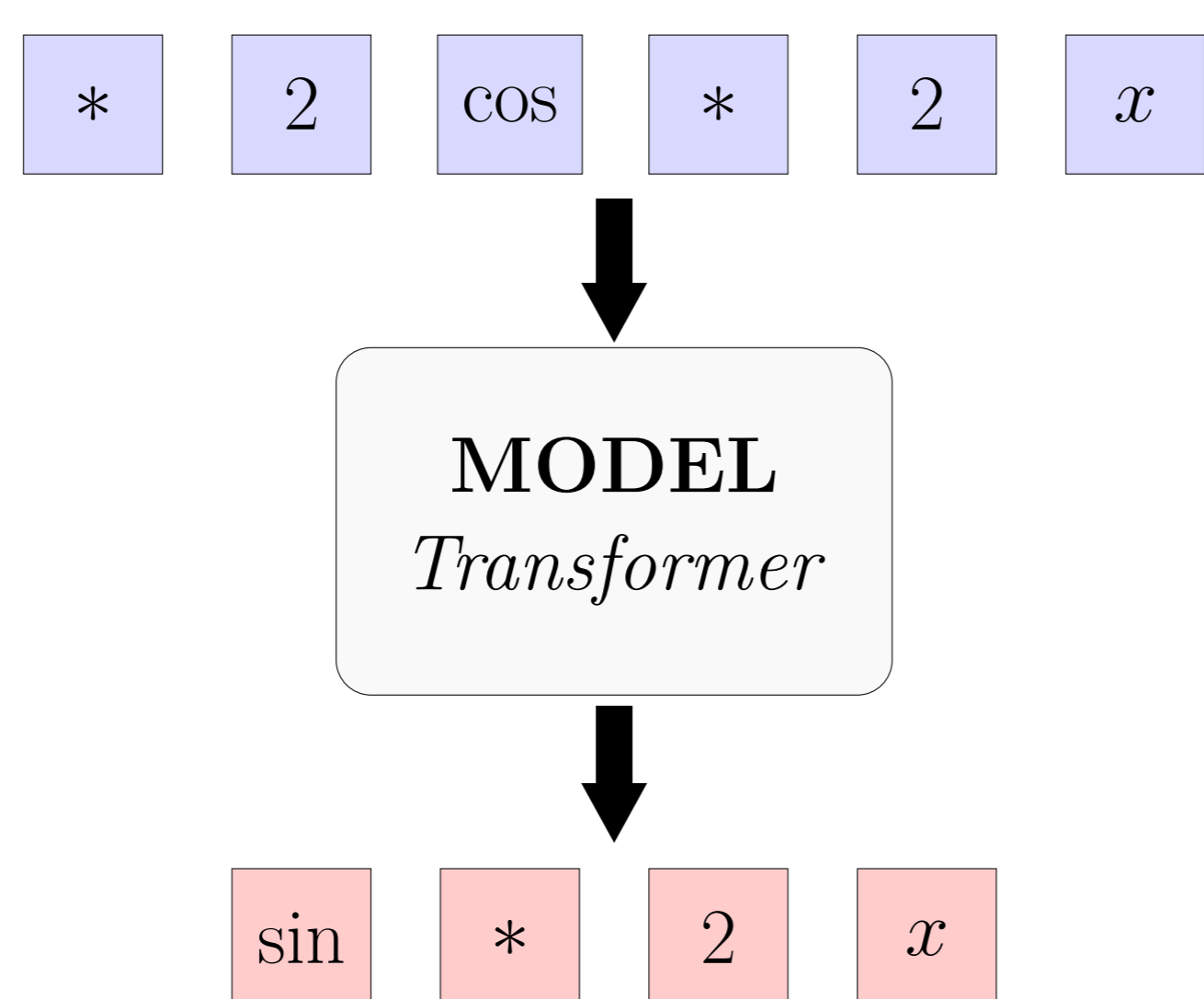


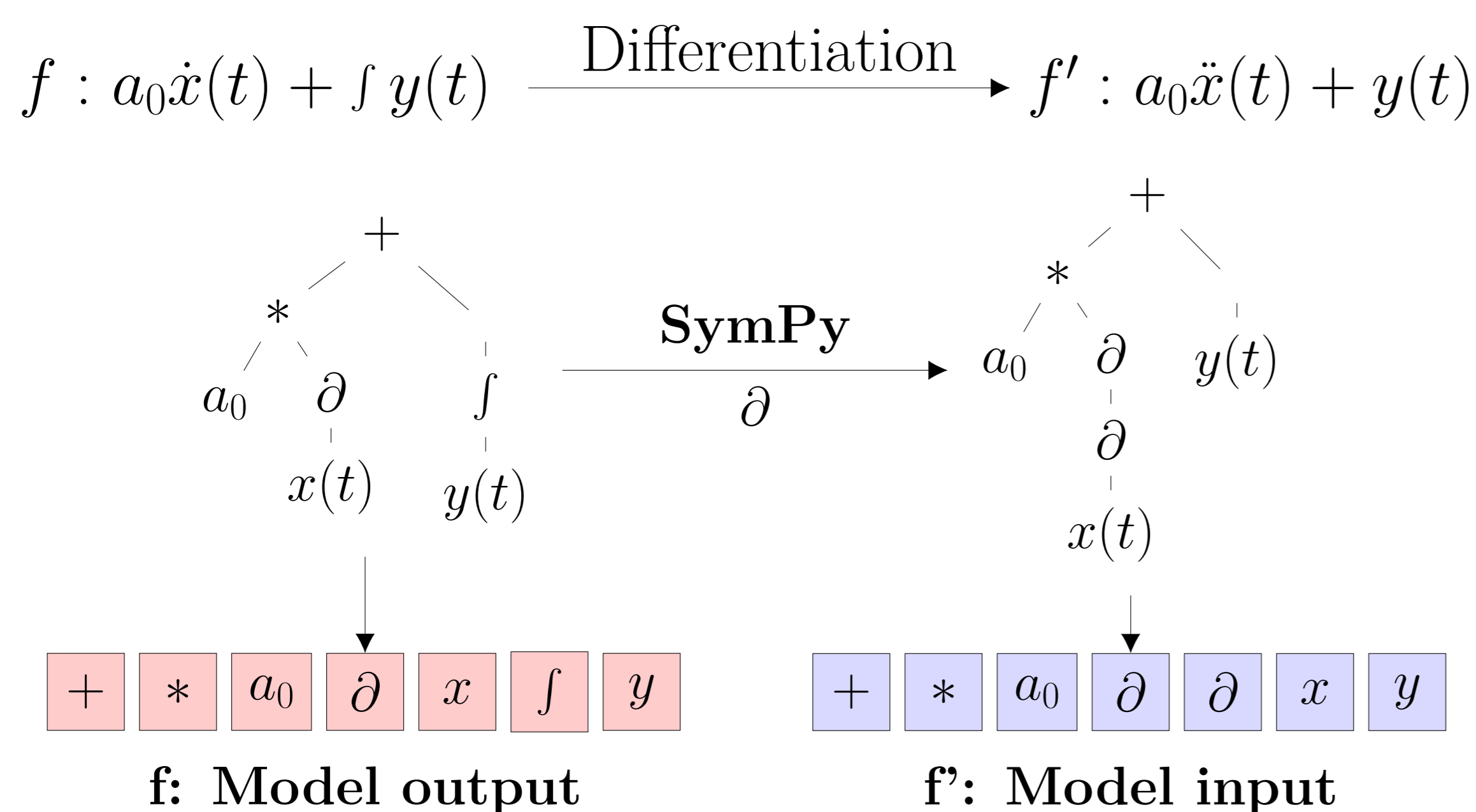
Figure 3: Integration of  $2 \cos(2x)$  with Deep Learning.

## Adaptation of [3] to integro-differential equations

In order to integrate integro-differential equations, we rewrote from scratch the implementation of [3] (based on PyTorch) to fit our needs.

### Generation

Our generation process creates unary-binary trees, where nodes are  $+$ ,  $-$ ,  $*$ ,  $\div$  plus the  $\partial$  and  $\int$  operators, and leaves are  $x(t)$ ,  $y(t)$ ,  $a_0$  and  $t$ .



## Retrieve $f$ from $f'$

For this experiment, we generated 601 000 equations that include the following operations/operators:  $+$ ,  $-$ ,  $*$ ,  $\div$ ,  $\partial$ ,  $\int$ . We used two functions:  $x(t)$ ,  $y(t)$  and a parameter,  $a_0$ . The generated equations are then differentiated in order to obtain 601 000 pairs  $(f', f)$ . Training is done using 600 000 pairs and the remaining 1000 pairs are used to evaluate the accuracy (i.e. number of correct integrals/number of equations of the test dataset).

$$f' : a_0 \dot{x} + y \xrightarrow{\text{Transformer}} f : a_0 x + \int y$$

	Accuracy	Comments
Maple	75.8%	We use $\text{int}(f, t)$ . Could be enhanced (+5% ?) with integration by parts, but it is difficult to automate.
Transformer	96%	Our adapted version of [3].

Table 1: Accuracy of the model versus the accuracy of Maple.

Equation $f'$	Solution $f$
$\frac{1}{f x} - \frac{t x}{(f x)^2}$	$\frac{t}{f x}$
$(-a_0 x - 1) f x y + \ddot{y}(t + a_0 f x) + a_0 x y f x + x y t - a_0 \dot{y} x - \dot{y}$ $(t + a_0 f x)^2$	$\frac{\dot{y} + f x y}{t + a_0 f x}$

Table 2: Equations that the model can integrate, but not Maple.

## Retrieve $\frac{A}{B}$ from $A'B - AB'$ (and not $\frac{A'B-AB'}{B^2}$ )

The aim of this experiment is to integrate equations that require the introduction of an integrating factor. Datasets are composed of pairs  $(A'B - AB', A/B)$  where  $A$  and  $B$  are random irreducible polynomials. Here, we used 420 000 pairs for the training and 10 000 for the evaluation of the accuracy.

$$A'B - BA' : \dot{x}y + \dot{x} - x\dot{y} \xrightarrow{\text{Transformer}} \frac{A}{B} : \frac{x}{y+1}$$

	Accuracy	Comments
Maple	68.82 %	We use $\text{dsolve}/\text{intfactor}$ . The algorithm does not apply for 3.2% of equations (i.e equations without derivatives)
Transformer	93.79%	73.37% greedy, 93.79% beam search 10

Table 3: Accuracy of the model versus the accuracy of Maple.

Equation $f'$	Solution $f$
$1 - \dot{y} \int \frac{1}{y} + \dot{y} \int 2x - 2xy$	$\frac{y}{\int \frac{2xy-1}{y}}$
$2y^2 + 2\dot{x} + y^3\dot{y} - \dot{y}^2x - x^2\ddot{y} + 2\dot{x} \int y + \dot{y}^2x \int y +$ $y^2\dot{y}x + \dot{x}y\dot{y} - y^2\ddot{y} \int y - \dot{y}xy - \dot{x}y\dot{y} \int y - \dot{y}xy \int y$	$\frac{x + y \int y}{2 + \dot{y}x + y\dot{y}}$

Table 4: Equations that the model can integrate, but not Maple.

## Conclusion

We achieve a quite good accuracy on the integration of a specific set of integro-differential equations. Our next goal will be focused on the integral elimination of differential systems with deep learning.

## Bibliography

- [1] François Boulier. *The DifferentialAlgebra project*. 2023. URL: <https://codeberg.org/francois.boulier/DifferentialAlgebra>.
- [2] François Boulier et al. "Additive Normal Forms and Integration of Differential Fractions". In: *Journal of Symbolic Computation* (2016). URL: <https://hal.archives-ouvertes.fr/hal-01245378>.
- [3] Guillaume Lample and François Charton. "Deep Learning for Symbolic Mathematics". In: *The Ninth International Conference on Learning Representations*. Virtual, May 2019, p. 24. URL: <http://arxiv.org/abs/1912.01412>.
- [4] A. Vaswani et al. "Attention Is All You Need". In: *CoRR* abs/1706.03762 (2017). arXiv: 1706.03762. URL: <http://arxiv.org/abs/1706.03762>.