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# COMPARISON OF DIFFERENT DEFINITIONS OF PSEUDOCHARACTERS

KATHLEEN EMERSON AND SOPHIE MOREL

ABSTRACT. We prove that the definitions of a  $d$ -dimensional pseudocharacter (or pseudorepresentation) given by Chenevier and V. Lafforgue agree over any ring. We also compare the scheme of Lafforgue's  $G$ -pseudocharacters of a group with its  $G$ -character variety.

Pseudocharacters (also known as pseudorepresentations) were first introduced by Wiles ([19]) and Taylor ([17]) to study congruences modulo power of prime numbers between representations of Galois groups. They defined congruence between representations as congruence between their characters, and this led them to the introduction of pseudocharacters, which are functions on a group whose properties mimick those of the character of a finite-dimensional representation. More precisely, let  $\Gamma$  be a group,  $A$  be a commutative unital ring and  $d$  be a positive integer. A map  $T : \Gamma \rightarrow A$  is a  $d$ -dimensional pseudocharacter if  $d! \in A^\times$ ,  $T(1) = d$ ,  $T(\gamma_1\gamma_2) = T(\gamma_2\gamma_1)$  for all  $\gamma_1, \gamma_2 \in \Gamma$  and  $T$  satisfies the  $d$ -dimensional pseudo-character identity: for all  $\gamma_1, \dots, \gamma_{d+1} \in \Gamma$ ,

$$\sum_{\sigma \in \mathfrak{S}_{d+1}} \text{sgn}(\sigma) T^\sigma(\gamma_1, \dots, \gamma_{d+1}) = 0,$$

where, if the decomposition into cycles of  $\sigma \in \mathfrak{S}_{d+1}$  (including trivial cycles) is

$$\sigma = (i_{1,1}i_{1,2} \dots i_{1,n_1}) \dots (i_{k,1}i_{k,2} \dots i_{k,n_k}),$$

then

$$T^\sigma(\gamma_1, \dots, \gamma_{d+1}) = T(\gamma_{i_{1,1}}\gamma_{i_{1,2}} \dots \gamma_{i_{1,n_1}}) \dots T(\gamma_{i_{k,1}}\gamma_{i_{k,2}} \dots \gamma_{i_{k,n_k}}).$$

We denote by  $\text{PChar}_{\Gamma,d}(A)$  the set of  $d$ -dimensional pseudo-characters on  $\Gamma$  with values in  $A$ . This is a contravariant functor from the category of commutative  $\mathbb{Z}[1/d!]$ -algebras to that of sets, and it is not hard to see that it is representable, unlike the functor sending  $A$  to the set of equivalence classes of  $d$ -dimensional representations  $\Gamma \rightarrow \mathbf{GL}_d(A)$ . Moreover, if  $A$  is an algebraically closed field (in which  $d!$  is invertible), then Taylor has shown in [17, Theorem 1] that the trace induces a bijection from the set of equivalence classes of  $d$ -dimensional *semi-simple* representations of  $\Gamma$  over  $A$  and  $\text{PChar}_{\Gamma,d}(A)$  and, if  $k$  is a field of characteristic 0, then Chenevier has shown in [5, Proposition 2.3] that  $\text{PChar}_{\Gamma,d,\text{Spec } k}$  is naturally isomorphic to the  $\mathbf{GL}_d$ -character variety of  $\Gamma$  over  $\text{Spec } k$ .

The original definition of a  $d$ -dimensional pseudocharacter does not work well in characteristic  $p$  dividing  $d!$ , because a semi-simple  $d$ -dimensional representation is no longer determined by its

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trace. However, it is determined if we know all the coefficients of its characteristic polynomial, and Chenevier ([4]) came up with a compact way of packaging all that data in his theory of determinants. Determinants make sense over an arbitrary ring  $A$  without the requirement that  $d! \in A^\times$ , and the functor  $\text{Det}_{\Gamma,d}$  that they define is representable. Moreover, if  $\rho : \Gamma \rightarrow \mathbf{GL}_d(A)$  is a morphism, then it defines a determinant  $D_\rho \in \text{Det}_{\Gamma,d}(A)$ . Chenevier shows that, if  $k$  is an algebraically closed field, then  $\rho \mapsto D_\rho$  is a bijection from the set of  $d$ -dimensional semi-simple representations of  $\Gamma$  over  $k$  to  $\text{Det}_{\Gamma,k}(k)$ . However, it is not known whether the determinant functor  $\text{Det}_{\Gamma,d}$  is isomorphic to the  $\mathbf{GL}_d$ -character variety of  $\Gamma$ .

Vincent Lafforgue also gave a very general definition of pseudocharacters, that this time makes sense for representations with values in any affine group scheme  $G$  over a base ring  $C$ . The definition of his pseudocharacters is less compact, as it requires data for every invariant function on any power  $G$ . Again, we obtain a contravariant functor  $\text{LPC}_{\Gamma,G}$  on the category of commutative  $C$ -algebras, which turns out to be representable by an affine scheme. Also, if  $\rho : \Gamma \rightarrow G(A)$  is a morphism, then it defines a Lafforgue pseudocharacter  $\Theta_\rho \in \text{LPC}_{\Gamma,G}(A)$ . Lafforgue proves in [8, Proposition 11.7] (see also [9, Proposition 8.2]) that, if  $A = k$  is an algebraically closed field, then  $\rho \mapsto \Theta_\rho$  is a bijection from the set of  $G(k)$ -conjugacy classes of  $G$ -completely reducible morphisms  $\Gamma \rightarrow G(k)$  to  $\text{LPC}_{\Gamma,G}(k)$ ; see also Theorem 4.5 of the paper [2] of Böckle, Harris, Khare and Thorne for the details of the proof in positive characteristic, where they also give the more general definition of a Lafforgue pseudocharacter (Lafforgue himself was working over a field).

In Section 2, we construct a morphism of schemes from  $\text{LPC}_{\Gamma,G}$  to the  $G$ -character variety of  $\Gamma$  compatible with the bijections of the previous paragraph. We prove that, for  $G$  a geometrically reductive group scheme with connected geometric fibers over a Dedekind domain or a field, this morphism is an *adequate homeomorphism* in the sense of Alper (see [1, Definition 3.3.1]), that is, an integral universal homeomorphism which is a local isomorphism at all points with residue characteristic 0; see Proposition 2.11.

We come back to the case  $G = \mathbf{GL}_d$ . The main result of this note is that, over any ring, Lafforgue pseudocharacters of  $\Gamma$  are in bijection with  $d$ -dimensional determinants. More precisely, we prove the following theorem:

**Theorem .** (See Theorem 4.1.) *There exists an isomorphism  $\alpha : \text{LPC}_{\Gamma,\mathbf{GL}_d} \rightarrow \text{Det}_{\Gamma,d}$  such that, for any commutative ring  $A$  and any morphism  $\rho : \Gamma \rightarrow \mathbf{GL}_d(A)$ ,  $\alpha$  sends  $\Theta_\rho$  to  $D_\rho$ .*

The plan of the paper is as follows. In Section 1, we recall Chenevier's definition of determinants and the properties of determinants that we will need; in particular, we give Vaccarino's formula for the universal determinant ring of a free monoid (see [18] and [5, Theorem 1.15]). In Section 2, we define Lafforgue pseudocharacters and give some of their basic properties, in particular the (easy) fact that they define a representable functor and the relationship with the character variety (Proposition 2.11). We construct the morphism  $\alpha$  in Section 3 and, in Section 4, we prove that this morphism is invertible. The main ingredients are Donkin's ([6]) description of generators of  $\mathcal{O}(\mathbf{GL}_{d,\mathbb{Z}}^n)^{\mathbf{GL}_{d,\mathbb{Z}}}$  and the result of Vaccarino recalled in Section 1.

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## 1. DETERMINANTS OF ALGEBRAS

In this section, we recall the definition of determinants by Chenevier ([5]) and some basic properties of the functor of determinants. Throughout this section,  $A$  denotes a commutative unital ring, and  $A$ -algebras are always supposed to be associative and unital. Let  $\mathcal{C}_A$  be the category of commutative  $A$ -algebras and  $\mathbf{Set}$  be the category of sets. We first recall Roby's definition of a polynomial law ([14], I.2), a homogeneous polynomial law ([14], I.8) and a multiplicative polynomial law ([15], III.4).

**Definition 1.1.** Let  $M$  and  $N$  be  $A$ -modules. A *polynomial law*  $f : M \rightarrow N$  is a morphism of functors from  $\mathcal{C}_A$  to  $\mathbf{Set}$  between the functors  $B \mapsto M \otimes_A B$  and  $B \mapsto N \otimes_A B$ . For every object  $B$  of  $\mathcal{C}_A$ , we denote the resulting map  $M \otimes_A B \rightarrow N \otimes_A B$  by  $f_B$ .

**Definition 1.2.** Let  $M$  and  $N$  be  $A$ -modules, and let  $f : M \rightarrow N$  be a polynomial law. Let  $d \in \mathbb{N}$ .

- (1) We say that  $f$  is *homogeneous of degree  $d$*  if, for all  $B \in \mathcal{C}_A$ ,  $x \in M \otimes_A B$  and  $b \in B$ , we have

$$f_B(bx) = b^d f_B(x).$$

- (2) Suppose that  $M$  and  $N$  are  $A$ -algebras. We say that  $f$  is *multiplicative* if, for every  $B \in \mathcal{C}_A$ , we have  $f_B(1) = 1$  and  $f_B(xy) = f_B(x)f_B(y)$  if  $x, y \in M \otimes_A B$ .

If  $M$  and  $N$  are  $A$ -algebras and  $d \in \mathbb{N}$ , we write  $\mathcal{M}_A^d(M, N)$  for the set of multiplicative polynomial laws from  $M$  to  $N$  that are homogeneous of degree  $d$ .

The following definition is due to Chenevier (see [5], 1.5).

**Definition 1.3.** Let  $R$  be an  $A$ -algebra. An  *$A$ -valued  $d$ -dimensional determinant on  $R$*  is an element of  $\mathcal{M}_A(R, A)$ . When  $R = A[\Gamma]$  for some group  $\Gamma$ , we also talk about  *$A$ -valued  $d$ -dimensional determinants on  $\Gamma$* .

We write  $\text{Det}_{R,d}(A)$  (resp.  $\text{Det}_{\Gamma,d}(A)$ ) for the set of  $A$ -valued  $d$ -dimensional determinants on  $R$  (resp.  $\Gamma$ ); so  $\text{Det}_{\Gamma,d}(A) = \text{Det}_{A[\Gamma],d}(A)$ .

**Example 1.4.** (1) Let  $X$  be a set, and write  $A\{X\}$  for the free unital associative  $A$ -algebra on  $X$ . Any map  $\rho : X \rightarrow M_d(A)$  gives rise to an  $A$ -valued  $d$ -dimensional determinant  $D_\rho$  on  $A\{X\}$  as follows. For any object  $B$  of  $\mathcal{C}_A$ , the map  $\rho$  induces a morphism of  $B$ -algebras  $\rho_B : B\{X\} \rightarrow M_d(B)$ , where  $M_d$  is the scheme of  $d \times d$  matrices; moreover, if  $B \rightarrow B'$  is a morphism of commutative  $A$ -algebras, then we get a commutative diagram:

$$\begin{array}{ccc} B\{X\} & \xrightarrow{\rho_B} & M_d(B) \\ \downarrow & & \downarrow \\ B'\{X\} & \xrightarrow{\rho_{B'}} & M_d(B') \end{array}$$

We define  $D_{\rho,B} : B\{X\} \rightarrow B$  by  $D_{\rho,B}(\omega) = \det(\rho_B(\omega))$ , where  $\det$  is the usual determinant on  $M_d(B)$ .

- (2) Let  $\Gamma$  be a group and  $\rho : \Gamma \rightarrow \mathbf{GL}_d(A)$  be a morphism of groups. Then we get an  $A$ -valued  $d$ -dimensional determinant  $D_\rho$  on  $A[\Gamma]$  in a similar way: for every commutative  $A$ -algebra  $B$  and every  $x \in B[\Gamma]$ , we set  $D_{\rho,B} = \det(\rho_B(x))$ , where  $\rho_B : B[\Gamma] \rightarrow M_d(B)$  sends an element  $\sum_{i=1}^n b_i \gamma_i$  of  $B[\Gamma]$ , with  $b_i \in B$  and  $\gamma_i \in \Gamma$ , to  $\sum_{i=1}^n b_i \rho(\gamma_i)$ .

**Example 1.5.** Let  $R$  be an  $A$ -algebra and  $D \in \mathcal{M}_A^d(R, A)$ . Following Chenevier ([5], 1.10), we can define polynomial laws  $\Lambda_i : R \rightarrow A$ , for  $0 \leq i \leq d$ , as follows. For any commutative  $A$ -algebra  $B$  and any  $x \in R \otimes_A B$ , set

$$D_{B[T]}(T - x) = \sum_{i=0}^d \Lambda_{i,B}(x) T^{d-i}.$$

Note that  $\Lambda_i$  is homogeneous of degree  $i$ . If  $D = D_\rho$  for  $\rho$  as in Example 1.4(1) or (2), then  $\Lambda_{i,B}(x)$  is the coefficient of the degree  $d - i$  term in the characteristic polynomial of the matrix  $\rho_B(x) \in M_d(B)$ .

**Definition 1.6.** Let  $R$  be an  $A$ -algebra and  $d \in \mathbb{N}$ . The *determinant functor*  $\text{Det}_{R,d} : \mathcal{C}_A \rightarrow \mathbf{Set}$  is defined by

$$\text{Det}_{R,d}(B) = \mathcal{M}_B^d(R \otimes_A B, B) = \mathcal{M}_A^d(R, B)$$

(the equality  $\mathcal{M}_B^d(R \otimes_A B) = \mathcal{M}_A^d(R, B)$  is proved in [5, 3.4]).

If  $R = A[\Gamma]$  with  $\Gamma$  a group, we also write  $\text{Det}_{\Gamma,d}$  instead of  $\text{Det}_{R,d}$ .

**Remark 1.7.** If  $D_1 \in \text{Det}_{R,d_1}$  and  $D_2 \in \text{Det}_{R,d_2}$ , we can define  $D_1 \times D_2 \in \text{Det}_{R,d_1+d_2}$  as follows. For every commutative  $A$ -algebra  $B$  and every  $x \in R \otimes_A B$ , take

$$(D_1 \times D_2)_B(x) = D_{1,B}(x) D_{2,B}(x).$$

If  $\Gamma$  is a group,  $\rho_1 : \Gamma \rightarrow \mathbf{GL}_{d_1}(A)$ ,  $\rho_2 : \Gamma \rightarrow \mathbf{GL}_{d_2}(A)$  are representations and  $D_1 = D_{\rho_1}$ ,  $D_2 = D_{\rho_2}$ , then  $D_1 \times D_2 = D_{\rho_1 \oplus \rho_2}$ .

**Example 1.8.** Let  $\Gamma$  be a group, and let  $X$  be the underlying set of  $\Gamma$ . Then the canonical surjective  $A$ -algebra morphism  $A\{X\} \rightarrow A[\Gamma]$  gives rise to an injective morphism of functors  $\text{Det}_{\Gamma,d} \rightarrow \text{Det}_{A\{X\},d}$ .

**Theorem 1.9** (Roby, see [14] III.1 and [5] 1.6). *Let  $R$  be an  $A$ -algebra and  $d \in \mathbb{N}$ . Then the functor  $\text{Det}_{R,d}$  is representable by the  $A$ -algebra  $(\Gamma_A^d(R))^{\text{ab}}$ , where  $\Gamma_A^d(R)$  is the  $A$ -algebra of divided powers of order  $d$  relative to  $A$  and  $(-)^{\text{ab}}$  denotes the abelianization.*

See Roby's papers ([14] III.1 and [15] II) for the definition of  $\Gamma_A^d(M)$  if  $M$  is an  $A$ -module, and the  $A$ -algebra structure on  $\Gamma_A^d(R)$  if  $R$  is an  $A$ -algebra.

We will also write  $\text{Det}_{R,d}$  for the affine scheme representing the functor  $\text{Det}_{R,d}$ .

**Remark 1.10.** The scheme  $\text{Det}_{R,d}$  is a contravariant functor of the  $A$ -algebra  $R$ . Indeed, if  $u : R \rightarrow S$  is a morphism of  $R$ -algebras, then, for every commutative  $A$ -algebra  $B$  and every  $D \in \text{Det}_{S,d}(B) = \mathcal{M}_A^d(S, B)$ , the maps  $u^*(D)_C : R \otimes_A C \rightarrow B \otimes_A C$ ,  $x \mapsto D((u \otimes \text{id}_C)(x))$ , for  $C$  an object of  $\mathcal{C}_A$ , define an element of  $\mathcal{M}_A^d(R, B) = \text{Det}_{R,d}(B)$ .

In particular, the scheme  $\text{Det}_{A\{X\},d}$  (resp.  $\text{Det}_{\Gamma,d}$ ) is a contravariant functor of the set  $X$  (resp. the group  $\Gamma$ ).

**Remark 1.11.** For every commutative  $A$ -algebra  $B$ , we have a natural morphism of schemes  $\text{Det}_{R \otimes_A B, d} \rightarrow \text{Det}_{R,d} \times_{\text{Spec } A} \text{Spec } B$ . This is an isomorphism by Theorem III.3 on page 262 of [14].

Let  $X$  be a set. We recall Vaccarino's construction of the universal ring of  $\text{Det}(\mathbb{Z}\{X\}, d)$ , following Chenevier's presentation in [5, 1.15]. Let  $F_X(d) = \mathbb{Z}[x_{i,j}, x \in X, 1 \leq i, j \leq d]$  be the ring of polynomials on the variables  $x_{i,j}$  for  $x \in X$  and  $i, j \in \{1, 2, \dots, d\}$ . The *generic matrices representation* is the ring morphism

$$\rho^{\text{univ}} : \mathbb{Z}\{X\} \rightarrow M_d(F_X(d))$$

sending each  $x \in X$  to the matrix  $(x_{ij})_{1 \leq i, j \leq d}$ . This defines a degree  $d$  homogeneous multiplicative polynomial law  $D_{\rho^{\text{univ}}} : \mathbb{Z}\{X\} \rightarrow E_X(d)$ , where  $E_X(d)$  is the subring of  $F_X(d)$  generated by the coefficients of the characteristic polynomials of the  $\rho^{\text{univ}}(p)$ ,  $p \in \mathbb{Z}\{X\}$ .

**Theorem 1.12** (Vaccarino, see Theorem 1.15 of [5]). *The ring  $E_X(d)$  and  $D_{\rho^{\text{univ}}} \in \text{Det}_{\mathbb{Z}\{X\}, d}(E_X(d))$  represent the functor  $\text{Det}_{\mathbb{Z}\{X\}, d}$ .*

Using results of Donkin, we deduce the following corollary.

**Corollary 1.13.** *The morphism  $M_d^X \rightarrow \text{Det}_{\mathbb{Z}\{X\}, d}$  sending  $\rho$  to  $D_\rho$  (see Example 1.4(1)) induces an isomorphism  $\text{Spec}(\mathcal{O}(M_d^X)^{\text{GL}_d}) \xrightarrow{\sim} \text{Det}_{\mathbb{Z}\{X\}, d}$ .*

*Proof.* Note that  $M_d^X = \text{Spec}(F_X(d))$ , and that the morphism  $M_d^X \rightarrow \text{Det}_{\mathbb{Z}\{X\}, d}$  of the statement corresponds to the multiplicative polynomial law  $\mathbb{Z}\{X\} \xrightarrow{D_{\rho^{\text{univ}}}} E_X(d) \subset F_X(d)$ . On the other hand, by the results of Donkin on the generators of  $\mathcal{O}(M_d^X)^{\text{GL}_d}$  (see [6, 3.1]), we have  $E_X(d) = \mathcal{O}(M_d^X)^{\text{GL}_d}$ . The corollary follows. □

## 2. LAFFORGUE'S DEFINITION OF PSEUDOCHARACTERS

Let  $A$  be a commutative unital ring and  $G$  be an affine group scheme over  $A$ .

As in Section 1, we denote by  $\mathcal{C}_A$  the category of commutative unital  $A$ -algebras. If  $B$  is a commutative  $A$ -algebra and  $X$  is a set, we write  $\mathcal{C}(X, B)$  for the  $B$ -algebra of functions  $X \rightarrow B$ .

For every set  $X$  (resp. integer  $n \in \mathbb{N}$ ), we make  $G$  act on the scheme  $G^X$  (resp.  $G^n$ ) by diagonal conjugation, and we denote by  $\mathcal{O}(G^X)^G$  (resp.  $\mathcal{O}(G^n)^G$ ) the  $A$ -algebra of  $G$ -invariant regular functions on  $G^X$  (resp.  $G^n$ ).

**Definition 2.1** (See [8] Proposition 11.7 and [2] Definition 4.1). Let  $X$  be a set and  $B$  be an object of  $\mathcal{C}_A$ . A  $B$ -valued  $G$ -pseudocharacter for the set  $X$  is a family of  $A$ -algebra morphisms  $\mathcal{O}(G^n)^G \rightarrow \mathcal{C}(X, B)$ , for  $n \geq 1$ , satisfying the following condition:

(LPC1) For all  $n, m \geq 1$ , every map  $\zeta : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$  and every  $f \in \mathcal{O}(G^m)^G$ , if we define  $f^\zeta \in \mathcal{O}(G^n)^G$  by  $f^\zeta(g_1, \dots, g_n) = f(g_{\zeta(1)}, \dots, g_{\zeta(m)})$ , then we have

$$\Theta_m(f)(x_{\zeta(1)}, \dots, x_{\zeta(m)}) = \Theta_n(f^\zeta)(x_1, \dots, x_n)$$

for all  $x_1, \dots, x_n \in X$ .

Suppose that  $X$  is the underlying set of a group  $\Gamma$ . Then a  $B$ -valued  $G$ -pseudocharacter for the group  $\Gamma$  is a  $B$ -valued  $G$ -pseudocharacter for the set  $X$  satisfying the following additional condition:

(LPC2) For all  $n \geq 1$  and  $f \in \mathcal{O}(G^n)^G$ , if we define  $\widehat{f} \in \mathcal{O}(G^{n+1})^G$  by  $\widehat{f}(g_1, \dots, g_{n+1}) = f(g_1, \dots, g_{n-1}, g_n g_{n+1})$ , then we have

$$\Theta_{n+1}(\widehat{f})(\gamma_1, \dots, \gamma_{n+1}) = \Theta_n(f)(\gamma_1, \dots, \gamma_{n-1}, \gamma_n \gamma_{n+1})$$

for all  $\gamma_1, \dots, \gamma_{n+1} \in \Gamma$ .

We denote by  $\text{LPC}_{X,G}^1$  (resp.  $\text{LPC}_{\Gamma,G}$ ) the functor  $\mathcal{C}_A \rightarrow \mathbf{Set}$  sending  $B$  to the set of  $B$ -valued  $G$ -pseudocharacters for the set  $X$  (resp. for the group  $\Gamma$ ). If  $X$  is the underlying set of a group  $\Gamma$ , then  $\text{LPC}_{\Gamma,G}$  is a subfunctor of  $\text{LPC}_{X,G}^1$ .

**Example 2.2.** (1) If  $X$  is a set and  $\rho : X \rightarrow G(A)$  is a map, then we define an  $A$ -valued  $G$ -pseudocharacter  $\Theta_\rho^1$  for the set  $X$  by

$$\Theta_{\rho,n}^1(f)(x_1, \dots, x_n) = f(\rho(x_1), \dots, \rho(x_n)),$$

for every  $n \geq 1$ , every  $f \in \mathcal{O}(G^n)^G$  and all  $x_1, \dots, x_n \in X$ . Note that  $\Theta_\rho^1$  only depends on the conjugacy class of  $\rho$ .

(2) Similarly, if  $\Gamma$  is a group and  $\rho : \Gamma \rightarrow G(A)$  is a morphism of groups, then we define an  $A$ -valued  $G$ -pseudocharacter  $\Theta_\rho$  for the group  $\Gamma$  by

$$\Theta_{\rho,n}(f)(\gamma_1, \dots, \gamma_n) = f(\rho(\gamma_1), \dots, \rho(\gamma_n)),$$

for every  $n \geq 1$ , every  $f \in \mathcal{O}(G^n)^G$  and all  $\gamma_1, \dots, \gamma_n \in \Gamma$ . Again,  $\Theta_\rho$  only depends on the conjugacy class of  $\rho$ .

**Remark 2.3.** The functor  $\text{LPC}_{X,G}^1$  depends contravariantly on the set  $X$ . Indeed, if  $u : X \rightarrow Y$  is a map, then, for every commutative  $A$ -algebra  $B$  and every  $\Theta = (\Theta_n)_{n \geq 1} \in \text{LPC}_{Y,G}^1(B)$ , the family  $u^*(\Theta) = (u^*(\Theta)_n)_{n \geq 1}$  defined by

$$u^*(\Theta)_n(f)(x_1, \dots, x_n) = \Theta_n(f)(u(x_1), \dots, u(x_n))$$

for  $n \geq 1$ ,  $f \in \mathcal{O}(G^n)^G$  and  $x_1, \dots, x_n \in X$  is a  $B$ -valued  $G$ -pseudocharacter for the set  $X$ .

Similarly, the functor  $\text{LPC}_{\Gamma,G}$  depends contravariantly on the group  $\Gamma$ .

If  $X$  is a set,  $x_1, \dots, x_n \in X$  and  $f \in \mathcal{O}(G^n)$ , we define a regular function  $f_{x_1, \dots, x_n} \in \mathcal{O}(G^X)$  by

$$f_{x_1, \dots, x_n}((g_x)_{x \in X}) = f(g_{x_1}, \dots, g_{x_n}).$$

Note that  $f \in \mathcal{O}(G^n)^G$  if and only if  $f_{x_1, \dots, x_n} \in \mathcal{O}(G^X)^G$ .

**Proposition 2.4.** Let  $X$  be a set and let  $R_{X,G}^1 = \mathcal{O}(G^X)^G$ . Consider the element  $\Theta^{\text{univ}}$  of  $\text{LPC}_{X,G}^1(R_{X,G}^1)$  defined by

$$\Theta_n^{\text{univ}}(f)(x_1, \dots, x_n) = f_{x_1, \dots, x_n} \in R_{X,G}^1,$$

for every  $n \geq 1$ , every  $f \in \mathcal{O}(G^n)^G$  and all  $x_1, \dots, x_n \in X$ . Then  $R_{X,G}^1$  represents the functor  $\text{LPC}_{X,G}^1$ , and  $\Theta^{\text{univ}} \in \text{LPC}_{X,G}^1(R_{X,G}^1)$  is the universal element.

*Proof.* The morphism of functors  $\text{Spec}(R_{X,G}^1) \rightarrow \text{LPC}_{X,G}^1$  corresponding to  $\Theta^{\text{univ}}$  sends a morphism of rings  $u : R_{X,G}^1 \rightarrow B$  to the  $G$ -pseudocharacter  $\Theta_u$  defined by

$$\Theta_{u,n}(f)(x_1, \dots, x_n) = u(\Theta_n^{\text{univ}}(f)(x_1, \dots, x_n)),$$

for  $n \geq 1$ ,  $f \in \mathcal{O}(G^n)^G$  and  $x_1, \dots, x_n \in X$ . We check that it is an isomorphism. The central point is that, by definition of the product  $G^X$ , we have  $\mathcal{O}(G^X) = \varinjlim_{Y \subset X \text{ finite}} \mathcal{O}(G^Y)$ ; in particular, every element of  $\mathcal{O}(G^X)^G$  is of the form  $f_{x_1, \dots, x_n}$ , for some  $n \geq 1$ ,  $x_1, \dots, x_n \in X$  and  $f \in \mathcal{O}(G^n)^G$ .

Let  $B$  be a commutative  $A$ -algebra. Let  $u, v : R_{X,G}^1 \rightarrow B$  be two morphisms of  $A$ -algebras such that  $\Theta_u = \Theta_v$ . Let  $h \in R_{X,G}^1$ , and choose an integer  $n \geq 1$ ,  $x_1, \dots, x_n \in X$  and  $f \in \mathcal{O}(G^n)^G$  such that  $h = f_{x_1, \dots, x_n}$ . Then

$$u(h) = \Theta_{u,n}(f)(x_1, \dots, x_n) = \Theta_{v,n}(f)(x_1, \dots, x_n) = v(h).$$

This proves that  $u = v$ . Now let  $\Theta \in \text{LPC}_{X,G}^1(B)$ ; we want to find a morphism of  $A$ -algebras  $w : R_{X,G}^1 \rightarrow B$  such that  $\Theta = \Theta_w$ . Let  $Y$  be a finite subset of  $X$ . If  $\zeta : Y \xrightarrow{\sim} \{1, 2, \dots, n\}$  is a bijection, then we get an isomorphism of  $A$ -algebras  $\mathcal{O}(G^Y)^G \xrightarrow{\sim} \mathcal{O}(G^n)^G$  sending  $h \in \mathcal{O}(G^Y)^G$  to the regular function  $h^\zeta : (g_1, \dots, g_n) \mapsto h((g_{\zeta(y)})_{y \in Y})$ , and we define  $w_Y : \mathcal{O}(G^Y)^G \rightarrow B$  by

$$w_Y(h) = \Theta_n(h^\zeta)(\zeta^{-1}(1), \dots, \zeta^{-1}(n)).$$

By condition (LPC1), this does not depend on the choice of  $\zeta$  and, if  $Y \subset Z$  are finite subsets of  $X$ , then  $w_{Z|\mathcal{O}(G^Y)^G} = w_Y$ . So the family  $(w_Y)_{Y \subset X \text{ finite}}$  defines a morphism of  $A$ -algebras  $w : R_{X,G}^1 \rightarrow B$ , and it follows immediately from the definition of  $\Theta^{\text{univ}}$  that we have  $\Theta = \Theta_w$ .  $\square$

Let  $\Gamma$  be a group. For all  $\gamma, \delta \in \Gamma$ , we define a  $G$ -equivariant morphism of  $A$ -modules  $\varphi_{\gamma, \delta} : \mathcal{O}(G \times G^\Gamma) \rightarrow \mathcal{O}(G^\Gamma)$  by

$$\varphi_{\gamma, \delta}(f)((g_\alpha)_{\alpha \in \Gamma}) = f(g_\gamma \delta, (g_\alpha)_{\alpha \in \Gamma}) - f(g_\gamma g_\delta, (g_\alpha)_{\alpha \in \Gamma}).$$

We have a morphism of  $A$ -algebras  $\mathcal{O}(G^\Gamma) \rightarrow \mathcal{O}(G \times G^\Gamma)$  induced by the projection of  $G \times G^\Gamma$  on its second factor, and this morphism is equivariant for the action of  $G$  by diagonal conjugation. The map  $\varphi_{\gamma, \delta}$  becomes  $\mathcal{O}(G^\Gamma)$ -linear for this action of  $\mathcal{O}(G^\Gamma)$  on  $\mathcal{O}(G \times G^\Gamma)$ . In particular,  $\varphi_{\gamma, \delta}(\mathcal{O}(G \times G^\Gamma))$  (resp.  $\varphi_{\gamma, \delta}(\mathcal{O}(G \times G^\Gamma)^G)$ ) is an ideal of  $\mathcal{O}(G^\Gamma)$  (resp.  $\mathcal{O}(G^\Gamma)^G$ ).

**Proposition 2.5.** (i) Let  $J_{\Gamma, G} \subset \mathcal{O}(G^\Gamma)$  be the sum of the images of the  $\varphi_{\gamma, \delta}$ , for all  $\gamma, \delta \in \Gamma$ . Then  $J_{\Gamma, G}$  is an ideal of  $\mathcal{O}(G^\Gamma)$ , and  $\text{Spec}(\mathcal{O}(G^\Gamma)/J_{\Gamma, G}) \subset \text{Spec}(\mathcal{O}(G^\Gamma)) = G^\Gamma$  sends any commutative  $B$ -algebra  $A$  to the set of maps  $\rho : \Gamma \rightarrow G(B)$  that are morphisms of groups.  
(ii) Let  $I_{\Gamma, G} \subset J_{\Gamma, G}^G$  be the sum of the  $\varphi_{\gamma, \delta}(\mathcal{O}(G \times G^\Gamma)^G)$ , for all  $\gamma, \delta \in \Gamma$ . Then  $I_{\Gamma, G}$  is an ideal of  $R_{\Gamma, G}^1 = \mathcal{O}(G^\Gamma)^G$ , and, if we set  $R_{\Gamma, G} = R_{\Gamma, G}^1/I_{\Gamma, G}$ , then the isomorphism  $\text{Spec}(R_{\Gamma, G}^1) \xrightarrow{\sim} \text{LPC}_{\Gamma, G}^1$  of Proposition 2.4 induces an isomorphism between the closed subscheme  $\text{Spec}(R_{\Gamma, G})$  of  $\text{Spec}(R_{\Gamma, G}^1)$  and  $\text{LPC}_{\Gamma, G} \subset \text{LPC}_{\Gamma, G}^1$ .

*Proof.* We already know that  $J_{\Gamma, G}$  and  $I_{\Gamma, G}$  are ideals, because they are sums of ideals.

Let  $B$  be a commutative  $A$ -algebra and  $\rho : \Gamma \rightarrow G(B)$  be a map; this corresponds to a morphism of  $A$ -algebras  $u : \mathcal{O}(G^\Gamma) \rightarrow B$ , and we have

$$u(f) = f((\rho(\gamma))_{\gamma \in \Gamma})$$



for every  $f \in \mathcal{O}(G^\Gamma)$ . We want to prove that  $\rho$  is a morphism of groups if and only if  $u(J_{\Gamma,G}) = 0$ . Suppose that  $\rho$  is a morphism of groups. Let  $\gamma, \delta \in \Gamma$  and  $f \in \mathcal{O}(G \times G^\Gamma)$ . Then

$$u(\varphi_{\gamma,\delta}(f)) = f(\rho(\gamma\delta), (\rho(\alpha)_{\alpha \in \Gamma})) - f(\rho(\gamma)\rho(\delta), (\rho(\alpha)_{\alpha \in \Gamma})) = 0$$

as  $\rho$  is a morphism. This shows that  $J_{\Gamma,G} \subset \text{Ker } u$ . Conversely, suppose that  $J_{\Gamma,G} \subset \text{Ker } u$ . Let  $\gamma, \delta \in \Gamma$ . Then, for every  $f \in \mathcal{O}(G)$ , we have

$$0 = u(\varphi_{1,(\gamma,\delta)}(f)) = f(\rho(\gamma\delta)) - f(\rho(\gamma)\rho(\delta)),$$

hence  $\rho(\gamma\delta) = \rho(\gamma)\rho(\delta)$ . This finishes the proof of (i).

We now prove the second statement of (ii). Let  $B$  be a commutative  $A$ -algebra, let  $\Theta \in \text{LPC}_{\Gamma,G}^1(B)$ , and let  $u : R_{\Gamma,G}^1 \rightarrow B$  be the morphism of  $A$ -algebras corresponding to  $\Theta$ . Suppose that  $\Theta$  satisfies condition (LPC2). Let  $\gamma, \delta \in \Gamma$  and  $f \in \mathcal{O}(G \times G^\Gamma)^G$ . Choose a finite subset  $\{\gamma_1, \dots, \gamma_n\}$  of  $\Gamma$  and  $h \in \mathcal{O}(G^{n+1})^G$  such that

$$f(g, (g_\alpha)_{\alpha \in \Gamma}) = h(g_{\gamma_1}, \dots, g_{\gamma_n}, g).$$

Then we have

$$f(g_\gamma g_\delta, (g_\alpha)_{\alpha \in \Gamma}) = \widehat{h}((g_\alpha)_{\alpha \in \Gamma}, g_\gamma, g_\delta),$$

so

$$u(\varphi_{\gamma,\delta}(f)) = \Theta_{n+1}(h)(\gamma_1, \dots, \gamma_n, \gamma\delta) - \Theta_{n+2}(\widehat{h})(\gamma_1, \dots, \gamma_n, \gamma, \delta) = 0.$$

This proves that  $I_{\Gamma,G} \subset \text{Ker } u$ .

Conversely, suppose that  $I_{\Gamma,G} \subset \text{Ker } u$ . Let  $n \geq 1$ ,  $h \in \mathcal{O}(G^n)^G$  and  $\gamma_1, \dots, \gamma_{n+1} \in \Gamma$ . Define  $f \in \mathcal{O}(G \times G^\Gamma)^G$  by

$$f(g, (g_\alpha)_{\alpha \in \Gamma}) = h(g_{\gamma_1}, \dots, g_{\gamma_{n-1}}, g).$$

Then

$$0 = u(\varphi_{\gamma_n, \gamma_{n+1}}(f)) = \Theta_n(h)(\gamma_1, \dots, \gamma_{n-1}, \gamma_n \gamma_{n+1}) - \Theta_n(\widehat{h})(\gamma_1, \dots, \gamma_{n-1}, \gamma_n, \gamma_{n+1}).$$

This implies that  $\Theta$  satisfies condition (LPC2). □

Next we discuss the behavior of the functors  $\text{LPC}_{X,G}^1$  and  $\text{LPC}_{\Gamma,G}$  under change of the base ring  $A$ .

We will use the notion of *adequate homeomorphism* defined by Alper (see [1, Definition 3.3.1]): a morphism of schemes is an adequate homeomorphism if it is integral, a universal homeomorphism and a local isomorphism at all points whose residue field is of characteristic 0. We will also use Alper's notion of *geometrically reductive group schemes*, see Definition 9.1.1 of [1], which generalizes that of reductive group schemes. In particular, if  $G$  is an affine smooth algebraic group over a field  $k$ , then it is geometrically reductive if and only if it is reductive (Lemma 9.2.8 of [1]), and, if  $G \rightarrow S$  is a smooth group scheme with connected fibers, then it is a geometrically reductive group scheme if and only if it is reductive (Theorem 9.7.5 of [1]).

Fix a commutative ring  $A$  and a flat affine group scheme  $G$  over  $A$ . Let  $B$  be a commutative  $A$ -algebra. For every  $A$ -module  $V$  with an action of  $G$ , we have  $V^G \otimes_A B \subset (V \otimes_A B)^{G_B}$ . As

$\mathcal{O}(G^X) \otimes_A B = \mathcal{O}(G_B^X)$  for every set  $X$ , we deduce that  $J_{\Gamma, G_B} = J_{\Gamma, G} \otimes_A B$  and  $I_{\Gamma, G_B} \supset I_{\Gamma, G} \otimes_A B$ . So we get morphisms of  $B$ -algebras, for  $X$  a set and  $\Gamma$  a group,

$$R_{X, G}^1 \otimes_A B = \mathcal{O}(G^X)^G \otimes_A B \rightarrow \mathcal{O}(G_B^X)^{G_B} = R_{X, G_B}^1$$

and

$$R_{\Gamma, G} \otimes_A B = (\mathcal{O}(G^\Gamma)^G \otimes_A B) / (I_{\Gamma, G} \otimes_A B) \rightarrow R_{\Gamma, G_B}$$

and corresponding morphisms of schemes

$$\beta_X : \text{LPC}_{X, G_B}^1 \rightarrow \text{LPC}_{X, G}^1 \otimes_A B$$

and

$$\beta_\Gamma : \text{LPC}_{\Gamma, G_B} \rightarrow \text{LPC}_{\Gamma, G} \otimes_A B.$$

**Proposition 2.6.** (i) *The morphisms  $\beta_X$  and  $\beta_\Gamma$  are isomorphisms in the following cases:*

(a)  *$B$  is a flat  $A$ -algebra;*

(b)  *$A$  is a Dedekind domain and  $G$  is geometrically reductive over  $A$  and has connected geometric fibers.*

(ii) *If  $G$  is geometrically reductive over  $A$  and has connected geometric fibers, then  $\beta_X$  and  $\beta_\Gamma$  are always adequate homeomorphisms.*

*Proof.* Point (ii) follows from Proposition 5.2.9(3) of [1]. We prove (i). Suppose first that  $B$  is flat over  $A$ . Then, by Lemma 2 of Seshadri's paper [16], for every  $A$ -module  $V$  with an action of  $G$ , the canonical morphism  $V^G \otimes_A B \rightarrow (V \otimes_A B)^{G_B}$  is an isomorphism. Applying this lemma to  $\mathcal{O}(G^X)$ , we see that  $\beta_X$  is an isomorphism. As the functor  $(-)\otimes_A B$  is right exact, Seshadri's lemma also implies that  $I_{\Gamma, G} \otimes_A B = I_{\Gamma, G_B}$ , hence that  $\beta_\Gamma$  is an isomorphism. We finally assume that  $A$  is a principal ideal domain and that  $G$  is geometrically reductive over  $A$ . By Lemma 2.7, the morphism  $\mathcal{O}(G^Y)^G \otimes_A B \rightarrow \mathcal{O}(G_B^Y)^{G_B}$  is an isomorphism for every set  $Y$ ; as in the proof of (i)(a), we conclude that  $\beta_X$  and  $\beta_\Gamma$  are isomorphisms. □

**Lemma 2.7.** *Let  $A$  be a Dedekind domain and  $G$  be a geometrically reductive group scheme with connected geometric fibers over  $A$ . Then, for every set  $Y$ , the injective morphism  $\mathcal{O}(G^Y)^G \otimes_A B \rightarrow \mathcal{O}(G_B^Y)^{G_B}$  is an isomorphism.*

**Remark 2.8.** Lemma 2.7 and its application to the functor LPC are probably well-known to many people in some form. We found version of it in a note by Chen (see [3]), as well as in a preprint by Quast (see Section 1.3 of [12]).

*Proof of Lemma 2.7.* We will use a number of results about the cohomology of algebraic groups that are gathered in Jantzen's book [7]; the particular results that we need are due to Donkin and Mathieu, see [7] for the original references.

First, as the tensor product and the functor of invariants commute with direct limits and as  $\mathcal{O}(G^Y)$  is the direct limit of the  $\mathcal{O}(G^Z)$  for  $Z \subset Y$  finite, we may assume that  $Y$  is finite. By the universal coefficients theorem (Proposition 4.18 in Chapter I of [7]), we have an exact sequence

$$0 \rightarrow \mathcal{O}(G^Y)^G \otimes_A B \rightarrow \mathcal{O}(G_B^Y)^{G_B} \rightarrow \text{Tor}_1^A(\text{H}^1(G, \mathcal{O}(G^Y)), B) \rightarrow 0.$$

So it suffices to show that  $\text{H}^1(G, \mathcal{O}(G^Y)) = 0$ . For this, it suffices to prove that the localization of  $\text{H}^1(G, \mathcal{O}(G^Y))$  at every ideal of  $A$  is zero; as localizations are flat, by the universal coefficient

theorem again, we may replace  $A$  by one of its localizations, hence assume that  $A$  is a field or a discrete valuation ring. Now, by Lemma B.9 of [7], it suffices to prove that, for every maximal ideal  $\mathfrak{m}$  of  $A$ , if  $k = A/\mathfrak{m}$ , then  $\mathcal{O}(G_k^Y)$  has a good filtration (in the sense of II.4.16 of [7]) as a  $G_k$ -module. This follows immediately from Propositions 4.20 and 4.21 of Chapter II of [7].

□

We finally investigate the relationship between pseudo-characters and the character variety.

**Definition 2.9.** Let  $A$  be a commutative ring,  $G$  be a flat affine group scheme over  $A$  and  $\Gamma$  be a group. The  $G$ -character variety of  $\Gamma$  is the affine scheme  $\text{Char}_{\Gamma,G} = \text{Spec}((\mathcal{O}(G^\Gamma)/J_{\Gamma,G})^G)$ .

If  $B$  is an  $A$ -algebra, then we have  $I_{\Gamma,G_B} \supset I_{\Gamma,G} \otimes_A B$ , so we get a morphism of  $B$ -algebras

$$(\mathcal{O}(G^\Gamma)/J_{\Gamma,G})^G \otimes_A B \rightarrow ((\mathcal{O}(G^\Gamma)/J_{\Gamma,G}) \otimes_A B)^{G_B} = (\mathcal{O}(G_B^\Gamma)/I_{\Gamma,G_B})^{G_B},$$

and a corresponding morphism of schemes

$$\beta'_\Gamma : \text{Char}_{\Gamma,G_B} \rightarrow \text{Char}_{\Gamma,G} \otimes_A B.$$

**Lemma 2.10.** *Suppose that  $G$  is geometrically reductive over  $A$  and has connected geometric fibers.*

- (i) *If  $B$  is a flat  $A$ -algebra, then  $\beta'_\Gamma$  is an isomorphism.*
- (ii) *In general,  $\beta'_\Gamma$  is an adequate homeomorphism.*

*Proof.* Point (i) follows from Proposition 5.2.9(1) of [1], and point (ii) from Proposition 5.2.9(3) of the same paper.

□

**Proposition 2.11.** *Suppose that  $G$  is a geometrically reductive group scheme with connected geometric fibers over  $A$  and that  $A$  is a Dedekind domain or a field. Let  $\iota$  be the morphism  $\text{Char}_{\Gamma,G} \rightarrow \text{LPC}_{\Gamma,G}$  induced by  $\mathcal{O}(G^\Gamma)^G/I_{\Gamma,G} \rightarrow \mathcal{O}(G^\Gamma)^G/J_{\Gamma,G}^G \hookrightarrow (\mathcal{O}(G^\Gamma)/J_{\Gamma,G})^G$ .*

- (i) *If  $A$  is a field of characteristic 0, then  $\iota$  is an isomorphism.*
- (ii) *In general,  $\iota$  is an adequate homeomorphism.*

The statement of the proposition is very close to that of Proposition 11.7 of [8] and Theorem 4.5 of [2], and its proof uses the same kind of ideas. We still include it here because it is not very hard.

*Proof.* We prove (i). As  $A$  is a field, the group  $G$  is reductive over  $A$ . So, for every algebraic representation  $V$  of  $G$  over  $A$  (not necessarily finite-dimensional), we have the Reynolds operator  $E = E_V : V \rightarrow V$  (see [10] Definition 1.5), which is a  $G$ -equivariant projection with image  $V^G$ , compatible with any morphism of representations  $V \rightarrow W$ ; also, if  $V$  is a  $A$ -algebra and the action of  $G$  preserves its multiplication, then  $E_V$  is  $V^G$ -linear. We claim that  $I_{\Gamma,G} = J_{\Gamma,G}^G$ . We already know that  $I_{\Gamma,G} \subset J_{\Gamma,G}^G$ . Conversely, let  $h \in J_{\Gamma,G}^G$ . Write  $h = \sum_{i=1}^n \varphi_{\gamma_i, \delta_i}(f_i)$ , with  $\gamma_i, \delta_i \in \Gamma$  and  $f_i \in \mathcal{O}(G \times G^\Gamma)$ . As the  $\varphi_{\gamma_i, \delta_i}$  are  $G$ -equivariant morphisms, they are compatible with the Reynolds operators, so

$$h = E(h) = \sum_{i=1}^n \varphi_{\gamma_i, \delta_i}(E(f_i)) \in I_{\Gamma,G}.$$

It remains to prove that the injective morphism  $\mathcal{O}(G^\Gamma)^G/J_{\Gamma,G}^G \rightarrow (\mathcal{O}(G^\Gamma)/J_{\Gamma,G})^G$  is also surjective. Let  $f \in \mathcal{O}(G^\Gamma)$ , and suppose that the class of  $f$  modulo  $J_{\Gamma,G}$  is  $G$ -invariant. As the canonical surjection  $\mathcal{O}(G^\Gamma) \rightarrow \mathcal{O}(G^\Gamma)/J_{\Gamma,G}$  is  $G$ -equivariant, it is compatible with the Reynolds operators, so we deduce that  $f - E(f) \in J_{\Gamma,G}$ , which means that  $f + J_{\Gamma,G}$  is in the image of  $\mathcal{O}(G^\Gamma)^G/J_{\Gamma,G}^G$ .

We prove (ii). We know that  $\text{Char}_{\Gamma,G} \rightarrow \text{Spec}(\mathcal{O}(G^\Gamma)^G/J_{\Gamma,G}^G)$  is an adequate homeomorphism by Lemma 5.2.12 of [1] (see Remark 5.2.14 of *loc. cit.*). So it remains to prove that  $\iota' : \text{Spec}(\mathcal{O}(G^\Gamma)^G/J_{\Gamma,G}^G) \rightarrow \text{Spec}(\mathcal{O}(G^\Gamma)^G/I_{\Gamma,G})$  is an adequate homeomorphism. This morphism is a closed embedding, hence it is integral, universally injective and universally closed. If  $\text{Frac}(A)$  is of characteristic 0, then  $\iota'$  becomes an isomorphism after we tensor it by  $\text{Frac}(A)$  by Proposition 2.6(ii); otherwise, its source and target have no point with residue field of characteristic 0. So it remains to show that  $\iota'$  is surjective, which is equivalent to the fact that  $\iota$  is surjective.

Let  $x$  be a point of  $\text{Spec}(\mathcal{O}(G^\Gamma)^G/I_{\Gamma,G})$ , which corresponds to a morphism of  $A$ -algebras  $u : \mathcal{O}(G^\Gamma)^G/I_{\Gamma,G} \rightarrow K$ , with  $K$  a field. We want to find an extension  $L$  of  $K$  and a morphism of  $A$ -algebras  $v : (\mathcal{O}(G^\Gamma)/J_{\Gamma,G})^G \rightarrow L$  such that  $v \circ \iota^* = u$ . We may always enlarge  $K$  and  $L$ , so we may assume that they are algebraically closed. Then, by Proposition 2.6(ii) and Lemma 2.10(ii),  $\text{Char}_{\Gamma,G}$  and  $\text{Char}_{\Gamma,G_K}$  (resp.  $\text{LPC}_{\Gamma,G}$  and  $\text{LPC}_{\Gamma,G_K}$ ) have the same points over any algebraically closed extension of  $K$ , so we may assume that  $A = K$  (and so that  $G$  is reductive over  $K$ ). By Theorem 5.13 of [11] (or Lemma 5.2.1 and Remark 5.2.2 of [1]), there exists an extension  $L$  of  $K$  and a morphism of  $K$ -algebras  $w : \mathcal{O}(G^\Gamma) \rightarrow L$  such that  $u$  is induced by  $w|_{\mathcal{O}(G^\Gamma)^G}$ . The morphism  $w$  corresponds to an element  $(g_\gamma) \in G^\Gamma(L)$ , and we denote by  $H$  the closed subgroup of  $G_L$  generated by the set  $\{g_\gamma, \gamma \in \Gamma\}$ . We choose  $w$  such that the dimension of the maximal tori in  $Z_G(H)$  is maximal. We claim that  $H$  is then strongly reductive in  $G$  in the sense of Definition 16.1 of Richardson's paper [13], that is,  $H$  is not contained in any proper parabolic subgroup of  $Z_G(S)$ , where  $S$  is a maximal torus of  $Z_G(H)$ . Indeed, let  $\lambda$  be a cocharacter of  $Z_G(S)$ , and suppose that  $H$  is contained in  $P(\lambda) := \{g \in Z_G(S) \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\}$ . For every  $\gamma \in \Gamma$ , let  $g'_\gamma = \lim_{t \rightarrow 0} \lambda(t)g_\gamma\lambda(t)^{-1} \in G(L)$ . Then  $(g_\gamma)_{\gamma \in \Gamma}$  and  $(g'_\gamma)_{\gamma \in \Gamma}$  have the same image in  $\text{Spec}(\mathcal{O}(G^\Gamma)^G)(L)$ , so the morphism  $w' : \mathcal{O}(G^\Gamma) \rightarrow L$  corresponding to  $(g'_\gamma)_{\gamma \in \Gamma} \in G^\Gamma(L)$  satisfies  $w'|_{\mathcal{O}(G^\Gamma)^G} = w|_{\mathcal{O}(G^\Gamma)^G}$ . On the other hand, the closed subgroup  $H'$  of  $G$  generated by the  $g'_\gamma$  is contained in the centralizer of  $\lambda$  in  $Z_G(S)$ , so  $\lambda$  has to be central in  $Z_G(S)$ , otherwise  $Z_G(H')$ , which contains the group generated by  $S$  and the image of  $\lambda$ , would have a maximal torus of dimension greater than  $\dim(S)$ . So  $H$  is not contained in any proper parabolic subgroup of  $Z_G(S)$ . Also, as  $H$  is a Noetherian scheme, for every big enough finite subset  $X$  of  $\Gamma$ , the closed subgroup of  $G$  generated by  $\{g_\gamma, \gamma \in X\}$  is equal to  $H$ .

We now prove that, for this choice of  $w$ , the map  $\gamma \mapsto g_\gamma$  is a morphism of groups; this implies that  $w$  extends to  $\mathcal{O}(G^\Gamma)/J_{\Gamma,G}$ , hence defines a point  $y$  of  $\text{Char}_{\Gamma,G}(L)$  such that  $\iota(y) = x$ . Let  $\gamma, \delta \in \Gamma$ . Choose a finite subset  $X$  of  $\Gamma$  such that  $\gamma, \delta, \gamma\delta \in X$  and the closed subgroup of  $G$  generated by  $\{g_\alpha, \alpha \in X\}$  is equal to  $H$ . As  $w$  vanishes on  $I_{\Gamma,G}$ , the images of  $(g_{\gamma\delta}, (g_\alpha)_{\alpha \in \Gamma})$  and  $(g_\gamma g_\delta, (g_\alpha)_{\alpha \in \Gamma})$  by the map  $(G \times G^\Gamma)(L) \rightarrow \text{Spec}(\mathcal{O}(G \times G^\Gamma)^G)(L)$  are equal, so the images of  $(g_{\gamma\delta}, (g_\alpha)_{\alpha \in X})$  and  $(g_\gamma g_\delta, (g_\alpha)_{\alpha \in X})$  by the map  $(G \times G^X)(L) \rightarrow \text{Spec}(\mathcal{O}(G \times G^X)^G)(L)$  are also equal. The closed subgroups of  $G$  generated by the families  $(g_{\gamma\delta}, (g_\alpha)_{\alpha \in X})$  and  $(g_\gamma g_\delta, (g_\alpha)_{\alpha \in X})$  are both equal to  $H$ , hence strongly reductive in  $G$ ; so, by Theorem 16.4 of Richardson's paper [13], the  $G$ -orbits of these families in  $(G \times G^X)(L)$  are closed. As they have the same image in  $\text{Spec}(\mathcal{O}(G \times G^X)^G)(L)$ , and as  $G \times G^X$  is of finite type over  $K$ , this implies that they are in the

same  $G$ -conjugacy class. Let  $h \in G(L)$  such that  $h(g_\gamma g_\delta, (g_\alpha)_{\alpha \in X})h^{-1} = (g_\gamma g_\delta, (g_\alpha)_{\alpha \in X})$ . Then  $h$  centralizes all the  $g_\alpha$  for  $\alpha \in X$ , so  $g_\gamma g_\delta = hg_\gamma g_\delta h^{-1} = g_\gamma g_\delta$ . This finishes the proof.  $\square$

**Remark 2.12.** The morphism  $\text{Char}_{\Gamma, G} \rightarrow \text{LPC}_{\Gamma, G}$  of Proposition 2.11 is an isomorphism if and only if both morphisms  $\mathcal{O}(G^\Gamma)^G/I_{\Gamma, G} \rightarrow \mathcal{O}(G^\Gamma)^G/J_{\Gamma, G}^G$  and  $\mathcal{O}(G^\Gamma)^G/J_{\Gamma, G}^G \hookrightarrow (\mathcal{O}(G^\Gamma)/J_{\Gamma, G})^G$  are isomorphisms. This seems unlikely, but we cannot offer a counterexample.

### 3. A LAFFORGUE PSEUDOCHARACTER GIVES RISE TO A DETERMINANT

Let  $A$  be a commutative unital ring and  $d$  be a positive integer. If  $X$  is a set and  $\Gamma$  is a group, we write  $\text{LPC}_{X, d}^1$  and  $\text{LPC}_{\Gamma, d}$  instead of  $\text{LPC}_{X, \mathbf{GL}_{d, A}}^1$  and  $\text{LPC}_{\Gamma, \mathbf{GL}_{d, A}}$ . We will also use the notation of Example 2.2.

Let  $X$  be a set,  $d$  be a positive integer,  $B$  be a commutative  $A$ -algebra and  $\Theta = (\Theta_n)$  be an element of  $\text{LPC}_{X, d}^1(B)$ . We want to construct an element  $\alpha_X^1(\Theta)$  of  $\text{Det}_{A\{X\}, d}(B)$ , that is, a degree  $d$  homogeneous multiplicative polynomial law from  $A\{X\}$  to  $B$  (seen as  $A$ -algebras).

Let  $Y, Z$  be sets and  $\sigma : Y \rightarrow Z$  be a map. If  $C$  is a commutative  $A$ -algebra, then we define a map  $\det_{\sigma, C} : C\{Y\} = A\{Y\} \otimes_A C \rightarrow \mathcal{O}(\mathbf{GL}_{d, C}^Z)^{\mathbf{GL}_{d, C}}$  in the following way. Let  $p \in C\{Y\}$ . Then  $\det_{\sigma, C}(p)$  is the regular function on  $\mathbf{GL}_{d, C}^Z$  sending  $(g_z)_{z \in Z}$  to  $\det(p((g_{\sigma(y)})_{y \in Y}))$  (where  $\det$  is the usual determinant on  $\mathbf{GL}_d$ ), which is clearly  $\mathbf{GL}_{d, C}$ -invariant. Note that this construction is functorial in  $C$ , and that we have  $\mathcal{O}(\mathbf{GL}_{d, C}^X)^{\mathbf{GL}_{d, C}} = \mathcal{O}(\mathbf{GL}_{d, A}^Z)^{\mathbf{GL}_{d, A}} \otimes_A C$  by Lemma 2.7. Hence the family  $(\det_{\sigma, C})_{C \in \text{Ob}(\mathcal{C}_A)}$  defines a degree  $d$  homogeneous multiplicative polynomial law from  $A\{Y\}$  to  $\mathcal{O}(\mathbf{GL}_{d, A}^Z)^{\mathbf{GL}_{d, A}}$ , which we denote by  $\det_\sigma$ .

We come back to our  $\Theta \in \text{LPC}_{X, d}^1(B)$ . By Proposition 2.4, it corresponds to a morphism of  $A$ -algebras  $u_\Theta : \mathcal{O}(\mathbf{GL}_{d, A}^X)^G \rightarrow B$ , and we send it to the polynomial law

$$\alpha_X^1(\Theta) = u_\Theta \circ \det_{\text{id}_X} : A\{X\} \rightarrow B.$$

In other words, for every commutative  $A$ -algebra  $C$  and every  $p \in C\{X\}$ , the element  $\alpha_X^1(\Theta)_C(p)$  of  $B \otimes_A C$  is the image by  $u_\Theta \otimes \text{id}_C$  of the element  $(g_x)_{x \in X} \rightarrow \det(p(g_x)_{x \in X})$  of  $\mathcal{O}(\mathbf{GL}_{d, C}^X)^{\mathbf{GL}_{d, C}}$ .

The functoriality of  $\alpha_X^1(\Theta)_C$  in  $C$  follows immediately from its definition, and the fact that it defines a degree  $d$  homogeneous multiplicative polynomial law follows from the properties of the determinant on  $\mathbf{GL}_d$ .

**Proposition 3.1.** (i) Let  $X$  be a set and  $d \geq 1$ . Then the maps  $\text{LPC}_{X, d}^1(B) \rightarrow \text{Det}_{A\{X\}, d}(B) = \mathcal{M}_A(A\{X\}, B)$ ,  $\Theta \mapsto \alpha_X^1(\Theta)$ , form a morphism of functors  $\alpha_X^1 : \text{LPC}_{X, d}^1 \rightarrow \text{Det}_{A\{X\}, d}$  such that, for every commutative  $A$ -algebra  $B$  and every map  $\rho : X \rightarrow \mathbf{GL}_d(B)$ , we have  $\alpha_X^1(\Theta_\rho^1) = D_\rho$ . Moreover, the morphisms  $\alpha_X^1$  are natural in  $X$ .

(ii) Let  $\Gamma$  be a group and  $d \geq 1$ . We denote by  $X$  the underlying set of  $\Gamma$ . Then the morphism  $\alpha_X : \text{LPC}_{X, d}^1 \rightarrow \text{Det}_{A\{X\}, d}$  restricts to a morphism  $\alpha_\Gamma : \text{LPC}_{\Gamma, d} \rightarrow \text{Det}_{A[\Gamma], d}$  such that, for every commutative  $A$ -algebra  $B$  and every morphism of groups  $\rho : \Gamma \rightarrow \mathbf{GL}_d(B)$ , we have  $\alpha_\Gamma(\Theta_\rho) = D_\rho$ . Moreover, the morphisms  $\alpha_\Gamma$  are natural in  $\Gamma$ .

*Proof.* (i) The fact that  $\alpha_X^1$  is a morphism of functors and the naturality in  $X$  follow easily from the definition of  $\alpha_X^1(\Theta)$ . Let  $B$  be a commutative  $A$ -algebra and  $\rho : X \rightarrow \mathbf{GL}_d(B)$  be a map; then the morphism of  $A$ -algebra  $u : \mathcal{O}(\mathbf{GL}_{d,A}^X)^{\mathbf{GL}_{d,A}} \rightarrow B$  corresponding to  $\Theta_\rho$  is given by  $u(f) = f((\rho(x))_{x \in X})$ . So, if  $C$  is a commutative  $A$ -algebra and  $p \in C\{X\}$ , we have

$$\alpha_X^1(\Theta)_C(p) = \det(p((\rho(x))_{x \in X})) = \det(\rho(p)) = D_\rho(p).$$

(ii) It suffices to prove that  $\alpha_X$  sends  $\text{LPC}_{\Gamma,d}$  to  $\text{Det}_{A[\Gamma],d}$ , the other statements then follow immediately from (i).

Let  $B$  be a commutative  $A$ -algebra and  $\Theta \in \text{LPC}_{\Gamma,d}(B)$ . As  $\Theta$  is also in  $\text{LPC}_{X,d}^1(B)$ , we have a degree  $d$  homogeneous polynomial law  $D = \alpha_X^1(\Theta) : A\{X\} \rightarrow B$ . Saying that  $D$  is in the image of  $\mathcal{M}_A(A[\Gamma], B)$  means that, for every commutative  $A$ -algebra  $C$ , the map  $D_C : C\{X\} \rightarrow B \otimes_A C$  factors through the obvious surjection  $\pi_C : C\{X\} \rightarrow C[\Gamma]$ . Fix a commutative  $A$ -algebra  $C$ . We want to prove that, for all  $p, q \in C\{X\}$  such that  $\pi_C(p) = \pi_C(q)$ , we have  $D_C(p) = D_C(q)$ . For  $p \in C\{X\}$ , let  $n(p)$  be the sum over all the nonconstant monomials  $m$  appearing in  $p$  of  $\deg(m) - 1$ ; so  $n(p) = 0$  if and only if  $p$  is of degree  $\leq 1$ . We claim that, for every  $p \in C\{X\}$  such that  $n(p) \geq 1$ , there exists  $p_1 \in C\{X\}$  such that  $n(p_1) = n(p) - 1$ ,  $\pi_C(p_1) = \pi_C(p)$  and  $D_C(p_1) = D_C(p)$ . This claim implies the desired result; indeed, if  $p, q \in C\{X\}$  are such that  $\pi_C(p) = \pi_C(q)$ , then, by the claim, we can find  $p_1, q_1 \in C\{X\}$  such that  $n(p_1) = n(q_1) = 0$ ,  $\pi_C(p_1) = \pi_C(p) = \pi_C(q) = \pi_C(q_1)$ ,  $D_C(p_1) = D_C(p)$  and  $D_C(q_1) = D_C(q)$ ; as  $n(p_1) = n(q_1) = 0$ , the polynomials  $p_1$  and  $q_1$  are of degree  $\leq 1$ , so  $\pi_C(p_1) = \pi_C(q_1)$  implies that  $p_1 = q_1$ , and then we have  $D_C(p) = D_C(p_1) = D_C(q_1) = D_C(q)$ .

So it suffices to prove the claim. Let  $p \in C\{X\}$  such that  $n(p) \geq 1$ , and let  $m$  be a monomial of degree  $\geq 2$  appearing in  $p$ . Write  $m = cx_{\gamma_1} \dots x_{\gamma_k}$  with  $c \in C \setminus \{0\}$ ,  $k \geq 2$  and  $\gamma_1, \dots, \gamma_k \in \Gamma$ , where, for every  $\gamma \in \Gamma$ , we denote by  $x_\gamma$  the corresponding element of  $X$ . Let  $r = p - m$  and set  $p_1 = r + cx_{\gamma_1 \gamma_2} x_{\gamma_3} \dots x_{\gamma_k}$ . Then  $n(p_1) = n(p) - 1$  and  $\pi_C(p_1) = \pi_C(p)$ . It remains to prove that  $D_C(p_1) = D_C(p)$ . Let  $u_\Theta : \mathcal{O}(\mathbf{GL}_{d,A}^X)^{\mathbf{GL}_{d,A}} \rightarrow B$  be the morphism of  $A$ -algebras corresponding to the image of  $\Theta$  in  $\text{LPC}_{\Gamma,d}^1(B)$ . We have

$$D_C(p) = \alpha_X^1(\Theta)_C(p) = u_\Theta(\det_{\text{id}_X}(p)) \quad \text{and} \quad D_C(p_1) = u_\Theta(\det_{\text{id}_X}(p_1)).$$

As  $\Theta$  is in  $\text{LPC}_{\Gamma,d}(B)$ , the morphism  $u_\Theta$  vanishes on  $I_{\Gamma,G}$ . So, to show that  $D_C(p) = D_C(p_1)$ , it suffices to note that  $p_1 - p = \varphi_{\gamma_1, \gamma_2}(F)$ , with  $F \in \mathcal{O}(\mathbf{GL}_{d,A} \times \mathbf{GL}_{d,A}^X)^{\mathbf{GL}_{d,A}}$  the function  $(g, (g_\gamma)_{\gamma \in \Gamma}) \mapsto \det(r((g_\gamma)_{\gamma \in \Gamma}) + cgg_{\gamma_3} \dots g_{\gamma_k})$ .

□

#### 4. FROM DETERMINANTS TO PSEUDOCHARACTERS

The main result of this note is the following theorem.

**Theorem 4.1.** *Let  $A$  be a commutative ring and  $d$  be a positive integer.*

(i) *Let  $X$  be a set. The morphism  $\alpha_X^1 : \text{LPC}_{X,d}^1 \rightarrow \text{Det}(A\{X\}, d)$  of Proposition 3.1(i) corresponds by the isomorphisms  $\text{LPC}_{X,d}^1 \simeq \text{Spec}(\mathcal{O}(\mathbf{GL}_d^X)^{\mathbf{GL}_d})$  and*

$\text{Det}_{A\{X\},d} \simeq \text{Spec}(\mathcal{O}(M_d^X)^{\mathbf{GL}_d})$  of Proposition 2.4 and Corollary 1.13 to the restriction morphism  $\mathcal{O}(M_d^X)^{\mathbf{GL}_d} \rightarrow \mathcal{O}(\mathbf{GL}_d^X)^{\mathbf{GL}_d}$ . Moreover,  $\alpha_X^1$  is injective, and, for every commutative  $A$ -algebra  $B$ , an element  $D$  of  $\text{Det}_{A\{X\},d}(B)$  is in the image of  $\alpha_X^1$  if and only if  $D_B(x) \in B^\times$  for every  $x \in X$ .

(ii) For every set  $\Gamma$ , the morphism  $\alpha_\Gamma : \text{LPC}_{\Gamma,d} \rightarrow \text{Det}(A[\Gamma], d)$  of Proposition 3.1(ii) is an isomorphism.

*Proof.* By Remark 1.11 and Proposition 2.6(i)(b), we may assume that  $A = \mathbb{Z}$ .

Let  $X$  be a set. As at the end of Section 1, we denote the universal matrices representation by  $\rho^{\text{univ}} : X \rightarrow M_d(F_X(d))$ , where  $F_X(d) = \mathbb{Z}[x_{i,j}, x \in X, 1 \leq i, j \leq d]$ . The universal element of  $\text{Det}_{\mathbb{Z}\{X\},d}$  is then  $D^{\text{univ}} = D_{\rho^{\text{univ}}} : \mathbb{Z}\{X\} \rightarrow E_X(d)$ , where  $E_X(d)$  is generated by the coefficients of the characteristic polynomials of the  $\rho^{\text{univ}}(p)$ ,  $p \in \mathbb{Z}\{X\}$ .

We prove (i). The first statement follows from the fact that  $\alpha_X^1(D_\rho) = \Theta_\rho$  for every commutative ring  $A$  and every map  $\rho : \Gamma \rightarrow \mathbf{GL}_d(A)$ . For the injectivity of  $\alpha_X^1$ , it suffices to prove that  $\mathcal{O}(\mathbf{GL}_d^X)^{\mathbf{GL}_d}$  is a localization of  $\mathcal{O}(M_d^X)^{\mathbf{GL}_d}$ , but this follows from the fact that  $\mathcal{O}(\mathbf{GL}_d^X)$  is the localization of  $\mathcal{O}(M_d^X)$  by the multiplicative set generated by the functions  $\det_x : (g_y)_{y \in X} \mapsto \det(g_x)$ ,  $x \in X$ , which are in  $\mathcal{O}(M_d^X)^{\mathbf{GL}_d}$ . Finally, let  $A$  be a commutative ring, let  $D \in \text{Det}_{\mathbb{Z}\{X\},d}(A)$ , and let  $u : \mathcal{O}(M_d^X)^{\mathbf{GL}_d} \rightarrow A$  be the morphism of rings corresponding to  $D$ . Then  $D$  is the image of  $\alpha_X^1$  if and only if  $u$  extends to a morphism of rings  $\mathcal{O}(\mathbf{GL}_d^X)^{\mathbf{GL}_d} \rightarrow A$ , which is equivalent to the condition that  $u(\det_x) \in A^\times$  for every  $x \in A$ . As  $u(\det_x) = u(\det(\rho^{\text{univ}}(x))) = D_A(x)$  for every  $x \in X$ , we get the conclusion.

We prove (ii). Let  $X$  be the underlying set of  $\Gamma$ . We have a commutative diagram

$$\begin{array}{ccc} \text{LPC}_{\Gamma,d} & \xrightarrow{\alpha_\Gamma} & \text{Det}_{\mathbb{Z}[\Gamma],d} \\ \downarrow & & \downarrow \\ \text{LPC}_{\Gamma,d}^1 & \xrightarrow{\alpha_X^1} & \text{Det}_{\mathbb{Z}\{X\},d} \end{array}$$

and  $\alpha_X^1$  is injective by (i), so it suffices to check that the image of  $\alpha_\Gamma$  is  $\text{Det}_{\mathbb{Z}[\Gamma],d}$ . For this, we check that the universal element of  $\text{Det}_{\mathbb{Z}[\Gamma],d}$  is in the image of  $\alpha_\Gamma$ . Let  $\varphi : E_X(d) \simeq (\Gamma_{\mathbb{Z}}^d(\mathbb{Z}\{X\}))^{\text{ab}} \rightarrow A := (\Gamma_{\mathbb{Z}}(\mathbb{Z}[\Gamma]))^{\text{ab}}$  be induced by the quotient map  $\pi : \mathbb{Z}\{X\} \rightarrow \mathbb{Z}[\Gamma]$ . The universal element  $D$  of  $\text{Det}_{\mathbb{Z}[\Gamma],d}$  is defined by  $D \circ \pi = \varphi \circ D^{\text{univ}}$ , and its preimage by  $\alpha_X^1$  is the element  $\Theta$  of  $\text{LPC}_{\Gamma,d}^1$  defined by

$$\Theta_n(f)(\gamma_1, \dots, \gamma_n) = \varphi(\Theta_n^{\text{univ}}(f)(\gamma_1, \dots, \gamma_n)),$$

where  $\Theta^{\text{univ}} = (\alpha_X^1)^{-1}(D^{\text{univ}})$ . It suffices to show that  $\Theta$  satisfies condition (LPC2); we will then have  $\Theta \in \text{LPC}_{\Gamma,d}(A)$  and  $\alpha_\Gamma(\Theta) = D$ . Let  $n$  be a positive integer and  $\gamma_1, \dots, \gamma_{n+1} \in \Gamma$ . We want to check that  $\Theta_n(f)(\gamma_1, \dots, \gamma_{n-1}, \gamma_n \gamma_{n+1}) = \Theta_{n+1}(\widehat{f})(\gamma_1, \dots, \gamma_{n+1})$  for every  $f \in \mathcal{O}(\mathbf{GL}_d^n)^{\mathbf{GL}_d}$ . As both sides are  $\mathbb{Z}$ -algebra morphisms in  $f$ , it suffices to check it on generators of  $\mathcal{O}(\mathbf{GL}_d^n)^{\mathbf{GL}_d}$ . So, by results of Donkin (cf. [6, 3.1]), we may assume that

$$f(g_1, \dots, g_n) = \Lambda_k(g_{i_1} \dots g_{i_r}),$$

where  $k \in \{1, \dots, n\}$ ,  $\Lambda_k$  is the  $k$ th coefficient of the characteristic polynomial (i.e.  $\Lambda_k(g) = (-1)^k \text{Tr}(\Lambda^k g)$ ),  $r \in \mathbb{N}$  and  $i_1, \dots, i_r \in \{1, 2, \dots, n\}$ . For every  $\gamma \in \Gamma$ , we denote

by  $x_\gamma$  the element of  $X$  corresponding to  $\gamma$ . For  $i \in \{1, 2, \dots, n-1\}$ , we set  $y_i = z_i = x_{\gamma_i}$ , and we also set  $y_n = x_{\gamma_n \gamma_{n+1}}$ ,  $z_n = x_{\gamma_n} x_{\gamma_{n+1}}$ ; these are all elements of  $\mathbb{Z}\{X\}$ . We then have

$$\begin{aligned}\Theta_n(f)(\gamma_1, \dots, \gamma_{n-1}, \gamma_n \gamma_{n+1}) &= \varphi(\Theta_n^{\text{univ}}(f)(\gamma_1, \dots, \gamma_{n-1}, \gamma_n \gamma_{n+1})) \\ &= \varphi(f(\rho^{\text{univ}}(x_{\gamma_1}), \dots, \rho^{\text{univ}}(x_{\gamma_{n-1}}), \rho^{\text{univ}}(x_{\gamma_n \gamma_{n+1}}))) \\ &= \varphi(\Lambda_k(\rho^{\text{univ}}(y_{i_1} \dots y_{i_r})))\end{aligned}$$

and

$$\begin{aligned}\Theta_{n+1}(\widehat{f})(\gamma_1, \dots, \gamma_{n+1}) &= \varphi(\Theta_{n+1}^{\text{univ}}(\widehat{f})(\gamma_1, \dots, \gamma_{n+1})) \\ &= \varphi(\widehat{f}(\rho^{\text{univ}}(x_{\gamma_1}), \dots, \rho^{\text{univ}}(x_{\gamma_{n+1}}))) \\ &= \varphi(\Lambda_k(\rho^{\text{univ}}(z_{i_1} \dots z_{i_r}))).\end{aligned}$$

As  $\pi \circ \det \circ \rho^{\text{univ}} = \varphi \circ D^{\text{univ}} = D \circ \pi$  as polynomial laws, we get a commutative diagram

$$\begin{array}{ccc}\mathbb{Z}\{X\}[t] & \xrightarrow{\det \circ \rho^{\text{univ}}} & E_X(d)[t] \\ \pi \otimes \text{id}_{\mathbb{Z}[t]} \downarrow & & \downarrow \varphi \otimes \text{id}_{\mathbb{Z}[t]} \\ \mathbb{Z}[\Gamma][t] & \xrightarrow{D} & (\Gamma_{\mathbb{Z}}^d(\mathbb{Z}[\Gamma]))^{\text{ab}}[t]\end{array}$$

As  $(\pi \otimes \text{id}_{\mathbb{Z}[t]})(t - y_{i_1} \dots y_{i_r}) = (\pi \otimes \text{id}_{\mathbb{Z}[t]})(t - z_{i_1} \dots z_{i_r})$ , we get that

$$(**) \quad (\varphi \otimes \text{id}_{\mathbb{Z}[t]})(\det \circ \rho^{\text{univ}}(t - y_{i_1} \dots y_{i_r})) = (\varphi \otimes \text{id}_{\mathbb{Z}[t]})(\det \circ \rho^{\text{univ}}(t - z_{i_1} \dots z_{i_r})).$$

But, for every  $p \in \mathbb{Z}\{X\}$ , we have

$$\det \circ \rho^{\text{univ}}(t - p) = t^d + \sum_{k=1}^d \Lambda_k(\rho^{\text{univ}}(p))t^{d-k} \in E_X(d)[t],$$

so identity (\*\*\*) implies that

$$\varphi(\Lambda_k(\rho^{\text{univ}}(y_{i_1} \dots y_{i_r}))) = \varphi(\Lambda_k(\rho^{\text{univ}}(z_{i_1} \dots z_{i_r}))),$$

as desired. □

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