Cycle classes on abelian varieties and the geometry of the Abel-Jacobi map

Claire Voisin*

Pour Enrico, avec amitié

Abstract

We discuss two properties of an abelian variety, namely, being a direct summand in a product of Jacobians and the weaker property of being "split". We relate the first property to the integral Hodge conjecture for curve classes on abelian varieties. We also relate both properties to the existence problem for universal zero-cycles on Brauer-Severi varieties over abelian varieties. A similar relation is established for the existence problem of a universal codimension 2 cycle on a cubic threefold.

1 Introduction

The purpose of this paper is to explore some geometric questions related to the integral Hodge conjecture for abelian varieties. This subject has been revisited recently by Beckmann and de Gaay Fortman in [1] who proved the following result (already known by [10] in dimension 3).

Theorem 1.1. *Let A be a principally polarized abelian variety whose minimal class $\gamma_{\min} \in H_2(A, \mathbb{Z})$ is algebraic. Then degree 2 integral Hodge homology classes (or "curve classes") of A are algebraic.*

Here the minimal class is the integral degree 2 Hodge homology class on A defined as follows. Let $g = \dim A$ and $\theta \in H^2(A, \mathbb{Z})$ be the class of the principal polarization. Then $\gamma_{\min} := \frac{\theta^{g-1}}{(g-1)!} \in H^{2g-2}(A, \mathbb{Z}) \cong H_2(A, \mathbb{Z})$.

Remark 1.2. The result proved by Beckman and de Gaay Fortman is in fact more general. In particular, the polarization can be replaced with any line bundle, with no positivity assumption, whose class $c_1(L) \in H^2(A, \mathbb{Z})$ is unimodular.

Remark 1.3. As is well-known, the minimal class of the Jacobian of a curve (equipped with its natural polarization) is algebraic, since it is the class of the curve embedded in its Jacobian. Thus Theorem 1.1 applies to Jacobians of curves, and also products of Jacobians.

We first establish in this paper the following complement to Theorem 1.1.

Theorem 1.4. *(Cf. Proposition 2.1 and Corollary 2.5.) An abelian variety A is a direct summand, as an abelian variety, in a product of Jacobians, if and only if $A \times \widehat{A}$ satisfies the integral Hodge conjecture for curve classes.*

It follows that the integral Hodge conjecture for curve classes on abelian varieties is equivalent to the statement that any abelian variety is a direct summand, as an abelian variety, in a product of Jacobians. The "only if" implication in this theorem is an immediate consequence of [1] (see Remarks 2.4, 2.6). Note that Proposition 2.1 proves a slightly more

*The author is supported by the ERC Synergy Grant HyperK (Grant agreement No. 854361).

precise statement, concerning a principally polarized abelian variety, or more generally an abelian variety which admits a unimodular line bundle.

We will discuss in this paper a notion which is weaker than being a direct summand in a product of Jacobians, namely that of a "split" abelian variety (see Definition 1.9). We first explain our geometric motivation which comes from the study of the stable rationality problem for rationally connected threefolds X. It is classically known since the seminal work of Clemens and Griffiths [3] that the rationality problem for rationally connected threefolds X can be solved by studying the intermediate Jacobian $J := J^3(X)$ of X, which is a principally polarized abelian variety. As discovered in [19], [17], the algebraicity of certain integral Hodge classes on J, and integral Hodge classes on the product $J \times X$, is related to the *stable* rationality of X, via the geometry of the Abel-Jacobi map for cycles algebraically equivalent to 0 on X, and we explore further these phenomena in this paper. As is well-known, for any smooth projective variety X, the group $\mathrm{Pic}^0(X) = \mathrm{CH}^1(X)_{\mathrm{hom}}$ is isomorphic via the Abel-Jacobi map Φ_X^1 to the intermediate Jacobian $J^1(X)$, and furthermore, there exists a universal divisor

$$\mathcal{P} \in \mathrm{Pic}(J^1(X) \times X) = \mathrm{CH}^1(J^1(X) \times X)$$

realizing geometrically this isomorphism as $t \mapsto \Phi_X^1(\mathcal{P}_t - \mathcal{P}_0) \in J^1(X)$. If we now consider a smooth projective variety X with $\mathrm{CH}_0(X) = \mathbb{Z}$ and denote $J := J^3(X)$ its intermediate Jacobian, the Abel-Jacobi map for codimension 2 cycles of X is an isomorphism $\Phi_X : \mathrm{CH}^2(X)_{\mathrm{alg}} \to J^3(X)$ by [2], and a universal codimension 2 cycle is defined in [19] to be a cycle $\Gamma \in \mathrm{CH}^2(J \times X)$, such that the associated Abel-Jacobi map

$$\Phi_\Gamma : \mathrm{Alb}(J) = J \to J$$

is the identity of J. Here Φ_Γ (that we will also denote by Γ_* in the sequel) is induced by the morphism

$$t \mapsto \Phi_X(\Gamma_t - \Gamma_0) \in J$$

of algebraic varieties. An equivalent condition is the fact that

$$[\Gamma]_* : H_1(J, \mathbb{Z}) \to H^3(X, \mathbb{Z})/\mathrm{torsion} =: H^3(X, \mathbb{Z})_{\mathrm{tf}}$$

is the natural isomorphism (recall that, as a complex torus

$$J^3(X) = H^3(X, \mathbb{C})/(F^2H^3(X) \oplus H^3(X, \mathbb{Z})_{\mathrm{tf}}),$$

which defines this natural isomorphism). As by definition, $[\Gamma]_*$ is an isomorphism of Hodge structures, it provides a Hodge class in $H^4(J \times X, \mathbb{Z})$ (see [21, Section 2.2.2]). The existence of a universal codimension 2 cycle on $J \times X$ is thus a particular instance of the integral Hodge conjecture for degree 4 Hodge classes on $J \times X$. As discovered in [18], there are examples of rationally connected threefolds X which do not have a universal codimension 2 cycle. As observed in [19], the existence of a universal codimension 2 cycle for a rationally connected smooth projective variety X is a necessary condition for the existence of a (cohomological) decomposition of the diagonal of X, hence for its stable rationality.

It was however proved in [20] that there always exists, for such an X of dimension 3, a smooth projective variety M of dimension d (that one can in fact take to be a surface) with a codimension 2 cycle $Z \in \mathrm{CH}^2(M \times X)$ inducing an isomorphism

$$\Phi_Z : \mathrm{Alb}(M) \cong J^3(X).$$

It thus follows that the non-existence of a universal codimension 2 cycle for X implies the non-existence of a universal 0-cycle for M, that is, there does not exist a codimension d cycle $\Gamma \in \mathrm{CH}^d(A \times M)$, $A = \mathrm{Alb}(M)$, $d = \dim M$, inducing the identity

$$\Gamma_* : \mathrm{Alb}(A) = A \to \mathrm{Alb}(M) = A.$$

For the same reason as above, these examples provide counterexamples to the integral Hodge conjecture on $\mathrm{Alb}(M) \times M$. Note that Colliot-Thélène [5] constructed related examples for varieties X defined over a non-algebraically closed field.

A natural problem is to try to understand which smooth projective varieties admit a universal 0-cycle in the above sense. Trivially, any abelian variety admits a universal 0-cycle, given by the diagonal. As mentioned above, a curve admits a universal 0-cycle, namely its universal divisor.

Question 1.5. *(i) (Colliot-Thélène [5]) Let $\psi : P \to A$ be a Brauer-Severi variety over an abelian variety. Does P admit a universal 0-cycle?*

(ii) More generally, let $\psi : P \to A$ be a fibration into rationally connected varieties. Does P admit a universal 0-cycle?

Remark 1.6. In both cases, one has $\mathrm{Alb}(P) \stackrel{\psi_*}{=} \mathrm{Alb}(A) \cong A$. In case (i), one might believe that the existence of a universal 0-cycle $\Gamma \in \mathrm{CH}^d(A \times P)$ inducing the identity $\Gamma_* : \mathrm{Alb}(A) = A \to \mathrm{Alb}(P) = A$ forces the Brauer class to be 0, but this is not the case, since the condition on Γ concerns only its action on homology of degree 1 of A and P, namely, $\psi_* \circ \Gamma_*$ must be the identity of $H_1(A, \mathbb{Z})$. It does not say that the Brauer class of P vanishes because the action of $\psi_* \circ \Gamma_*$ on the higher degree homology groups of A (especially the group $H_{2g}(A, \mathbb{Z})$ which controls the index of the fibration ψ, hence the Brauer class) is not specified.

Question 1.5 can be considered as a particular case of the integral Hodge conjecture for Brauer-Severi varieties over abelian varieties. Indeed, as mentioned above, our problem can be formulated as follows:

Question 1.7. *Let $\psi : P \to A$ be a Brauer-Severi variety over an abelian variety. Does there exist a cycle $\Gamma \in \mathrm{CH}^d(A \times P)$, $d = \dim P$, such that $[\Gamma]_* : H_1(A, \mathbb{Z}) \to H_1(P, \mathbb{Z})$ is the inverse of ψ_*?*

The cycle Γ is a cycle on the Brauer-Severi variety $A \times P$ over $A \times A$ and the last condition is a restriction on the Künneth component of type $(1, 2d - 1)$ of $[\Gamma]$. Note that the integral Hodge conjecture for Brauer-Severi varieties $P \to A$ over abelian varieties satisfying the Hodge conjecture has a negative answer in general by [11]. We are going to relate here the problem above to the integral Hodge conjecture on $A \times A$. To start with, an easy general result on Question 1.5(ii) is the following

Proposition 1.8. *Let A be an abelian variety and let $P \to A$ be a fibration with rationally connected general fiber. Assume that A is a direct summand in a product of Jacobians. Then P admits a universal 0-cycle.*

This proposition applied to the case where P is a Brauer-Severi variety provides many examples where there is a universal 0-cycle while the Brauer class is nontrivial (see Remark 1.6). Furthermore, by Theorem 1.4, we conclude that the integral Hodge conjecture for curve classes on abelian varieties implies a positive answer to Question 1.5(i). Our next result is a weak converse to Proposition 1.8 for which we introduce the following

Definition 1.9. *An abelian variety of dimension g is said to be split if there exists a codimension g cycle $\Gamma \in \mathrm{CH}^g(A \times A)$ such that the class $[\Gamma] \in H^{2g}(A \times A, \mathbb{Z})$ acts on $H_*(A, \mathbb{Z})$ as the Künneth projector δ_1 onto $H_1(A, \mathbb{Z})$.*

Remark 1.10. The Künneth projectors δ_i being integral Hodge classes on $A \times A$, they are algebraic if $A \times A$ satisfies the integral Hodge conjecture.

We will establish a few general facts in Section 2.1. In particular we will prove Proposition 2.8 which says that splitness of abelian varieties is implied by the integral Hodge conjecture for curve classes on abelian varieties.

To state our next result, let us say that an abelian variety is "Mumford-Tate general" if $\rho(A) = 1$ and the Mumford-Tate group of the Hodge structure on $H^1(A, \mathbb{Q})$ is the symplectic group of the skew-pairing given by the polarization. This assumption is satisfied by the polarized abelian variety parameterized by a very general point in the moduli space of polarized abelian varieties with polarization of a given type, but it is in fact a more precise statement, which is satisfied for example by a very general Jacobian of curve or intermediate Jacobian of a cubic threefold. When $\rho(A) = 1$, the group of integral Hodge classes of degree $2g-2$ is also cyclic generated by a class $\gamma_{\rm min}$. We will prove in Section 3.2 the following

Theorem 1.11. *Let A be a Mumford-Tate general abelian variety with $\rho(A) = 1$. Assume that the intersection number $c_1(L)\cdot\gamma_{\rm min}$ is even. Then, if any Brauer-Severi variety $P \to A$ admits a universal 0-cycle, A is split.*

Remark 1.12. If A is principally polarized, the condition that $c_1(L)\cdot\gamma_{\rm min}$ is even is equivalent to the dimension of A being even. If the polarization is of type $(1, \ldots, 1, d)$, this condition says that $d\dim A$ is even.

To summarize our results, for a Mumford-Tate general even dimensional abelian variety A with Picard number 1, we have implications as follows: A being a direct summand in a product of Jacobians implies a positive answer to Question 1.5(i) and (ii), and in the other direction, a positive answer to Question 1.5(i) for any P implies that A is split. The question whether "split" implies that A is a direct summand in a product of Jacobians (so that the three statements are equivalent) remains open.

Question 1.5(ii) is motivated by the stable rationality problem for the cubic threefold, and more specifically, by the following

Question 1.13. *Let X be a smooth cubic threefold. Does X admit a universal codimension 2 cycle?*

Questions 1.5 and 1.13 are directly linked by the Iliev-Markushevich-Tikhomirov construction [14], [12] which describes a Zariski open set of the Hilbert scheme $H_{5,1}$ of elliptic curves of degree 5 in a cubic threefold X as a fibration into $\mathbb{P}^5$ over a Zariski open set of the intermediate Jacobian $J = J^3(X)$ of X. More precisely, we will prove

Proposition 1.14. *Let X be a smooth cubic threefold and let $\widetilde{H}_{5,1}$ be a smooth projective model of $H_{5,1}$. Then X admits a universal codimension 2 cycle if and only if $\widetilde{H}_{5,1}$ admits a universal 0-cycle.*

In the case of the fibration $\widetilde{H}_{5,1} \to J$, the argument leading to the proof of Proposition 1.8 had been used in [19] to prove

Theorem 1.15. *A smooth cubic threefold admits a universal codimension 2 cycle if the minimal class of its intermediate Jacobian is algebraic.*

Our last result is a partial converse to Theorem 1.15, which will be proved in Section 3.3. We will say that a cubic threefold is Mumford-Tate general if its intermediate Jacobian J is.

Theorem 1.16. *Let X be a Mumford-Tate general cubic threefold. Then if X admits a universal codimension 2 cycle, J is split.*

Thanks. *I thank Jean-Louis Colliot-Thélène and James Hotchkiss for inspiring discussions and correspondence. I also thank Olivier de Gaay Fortman for interesting exchanges and the referee for his/her careful reading.*

2 Curve classes on abelian varieties

We establish in this section Theorem 1.4. In the case of abelian varieties equipped with a unimodular line bundle, the following stronger statement holds.

Proposition 2.1. *Let A be a g-dimensional abelian variety equipped with a line bundle L such that $L^g = \pm g!$ (that is, L is unimodular). Then if A satisfies the integral Hodge conjecture for degree 2 integral Hodge homology classes (or "curve classes"), it is a direct summand in a product of Jacobians.*

Remark 2.2. We do not ask that L is ample in Proposition 2.1, so A is not necessarily a principally polarized abelian variety.

Remark 2.3. Even if L is ample, that is, a principal polarization, we just ask that A is a direct summand as an abelian variety, and not that the natural polarization on the product of Jacobians restricts to L. Indeed, the last condition is much too strong by the usual Clemens-Griffiths argument: if A is simple, this would imply that A is isomorphic to the Jacobian of a curve.

Remark 2.4. It is proved in [1] (see also Theorem 1.1), that a product of Jacobians satisfies the integral Hodge conjecture for curve classes. If $j : A \hookrightarrow J$ is the inclusion of a direct summand in such a product J, then for any Hodge class α on A, $j_*\alpha$ is an integral Hodge class on J, and α is algebraic on A if and only if it is algebraic on J, using a left inverse $\pi : J \to A$ of j. Hence the implication in Proposition 2.1 is in fact an equivalence.

Corollary 2.5. *(Cf. Theorem 1.4.) Let A be any abelian variety. Then the integral Hodge conjecture for curve classes on $A \times \widehat{A}$ implies that A is a direct summand in a product of Jacobians.*

The integral Hodge conjecture for curve classes on abelian varieties thus implies that any abelian variety is a direct summand in a product of Jacobians.

Proof. If A is an abelian variety, $A \times \widehat{A}$ admits a line bundle L as in Proposition 2.1, namely the Poincaré divisor (this is the starting point in the Zarhin trick). As A is a direct summand in $A \times \widehat{A}$, it is a direct summand in a product of Jacobians if so is $A \times \widehat{A}$. Proposition 2.1 applies to $A \times \widehat{A}$ and thus the integral Hodge conjecture for curve classes on $A \times \widehat{A}$ implies that $A \times \widehat{A}$ is a direct summand in a product of Jacobians. □

Remark 2.6. Again, the implication of the corollary is in fact an equivalence by [1], since if A is a direct summand in a product of Jacobians, then $A \times \widehat{A}$ is also a direct summand in a product of Jacobians, hence satisfies the integral Hodge conjecture for curve classes as explained in Remark 2.4.

Proof of Proposition 2.1. We consider the minimal class $\gamma_{\min} := \frac{c_1(L)^{g-1}}{(g-1)!}$. By assumption, it is algebraic on A, hence there exist smooth projective curves C_i, and morphisms $j_i : C_i \to A$, such that

$$\gamma_{\min} = \sum_i \epsilon_i j_{i*}[C_i]_{\text{fund}} \text{ in } H_2(A, \mathbb{Z}), \tag{1}$$

where $\epsilon_i = \pm 1$, and where we used the complex orientation of A to define the Poincaré duality isomorphism $H^{2g-2}(A, \mathbb{Z}) \cong H_2(A, \mathbb{Z})$. Using L, the dual abelian variety $\widehat{A} := \mathrm{Pic}^0(A)$ is isomorphic to A. We have a natural morphism

$$\hat{j} : A \cong \widehat{A} \stackrel{(\hat{j}_i)}{\to} \prod_i \mathrm{Pic}^0(C_i) = \prod_i J(C_i) \tag{2}$$

which is induced by the pull-back maps $\hat{j}_i := j_i^* : \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(C_i)$. The abelian variety $J := \prod_i J(C_i)$ admits the divisor $\Theta_{\epsilon.}$ defined as

$$\Theta_{\epsilon.} := \sum_i \epsilon_i \mathrm{pr}_i^* \Theta_i, \tag{3}$$

where $\mathrm{pr}_i : J \to J(C_i)$ is the natural projection and Θ_i is the natural Theta-divisor on $J(C_i)$. The classes $[\Theta_{\epsilon.}] \in H^2(J, \mathbb{Z})$, resp. $c_1(L) \in H^2(A, \mathbb{Z})$ provide equivalently skew-symmetric intersection pairings $\langle\, , \rangle_J$ on $H_1(J, \mathbb{Z})$, resp. $\langle\, , \rangle_L$ on $H_1(A, \mathbb{Z})$. We now have

Lemma 2.7. *The restriction $\hat{j}^*\Theta_{\epsilon.}$ is cohomologous to $\epsilon c_1(L)$, where $\epsilon = \pm 1$ is the sign of $c_1(L)^g$. Equivalently, the restriction of $\langle\, , \,\rangle_J$ to $\hat{j}_* H_1(A, \mathbb{Z})$ is equal to $\epsilon\langle\, , \,\rangle_L$.*

We postpone the proof of the lemma and conclude the proof of the proposition. As $\frac{L^g}{g!} = \pm 1$, the intersection pairing $\langle\, , \,\rangle_A$ is unimodular. By Lemma 2.7, the restriction of $\langle\, , \,\rangle_J$ to $\hat{j}_* H_1(A, \mathbb{Z})$ is unimodular. It follows that there is a direct sum decomposition

$$H_1(J, \mathbb{Z}) = H_1(A, \mathbb{Z}) \oplus H_1(A, \mathbb{Z})^{\perp}, \tag{4}$$

which is orthogonal with respect to $\langle\, , \,\rangle_J$. As the pairing is induced by a $(1,1)$-class, it satisfies the Hodge-Riemann relations and thus the orthogonal decomposition (4) is compatible with the Hodge decomposition, hence induces a direct sum decomposition

$$J = A \oplus A^{\perp}.$$

□

Proof of Lemma 2.7. The class $c_1(L)$, or equivalently the pairing $\langle\, , \,\rangle_A$, induces an isomorphism $\iota_L : H^1(A, \mathbb{Z}) \cong H_1(A, \mathbb{Z})$. We claim that for any α, $\beta \in H^1(A, \mathbb{Z})$,

$$\int_A \gamma_{\min} \cup \alpha \cup \beta = \langle \iota_L(\alpha), \iota_L(\beta)\rangle_L, \tag{5}$$

where the orientation of A is chosen in such a way that $\int_A c_1(L)^g = g!$. This formula is standard (see [7]) in the context of principally polarized abelian varieties. It is proved using a basis $e_1, \ldots, e_{2g}$ of $H^1(A, \mathbb{Z})$ for which $c_1(L) = \sum_{i=1}^{g} e_i \wedge e_{i+g}$ in $\wedge^2 H^1(A, \mathbb{Z})$. Then the isomorphism ι_L maps e_i to e^*_{i+g} and e_{i+g} to $-e_i$ for $i \leq g$. Furthermore for i, $j \leq g$

$$\langle e_i^*, e_{g+j}^*\rangle_L = \delta_{ij} \tag{6}$$

while $\langle e_i^*, e_j^*\rangle_L = 0$ for $i \leq g$, $j \leq g$. By definition,

$$\gamma_{\min} = \sum_i e_1 \wedge e_{1+g} \wedge \ldots e_i \widehat{\wedge e_{i+g}} \ldots \wedge e_g \wedge e_{2g} \text{ in } \bigwedge^{2g-2} H^1(A, \mathbb{Z}) \cong H^{2g-2}(A, \mathbb{Z})$$

so that one gets for any i, j

$$\int_A \gamma_{\min} \cup e_i \cup e_j = 0 \text{ for } i < j,\ j \neq i+g \tag{7}$$

$$\int_A \gamma_{\min} \cup e_i \cup e_{i+g} = 1.$$

Comparing (6) and (7) gives the result.

If we see as in (1) $\gamma_{\min}$ as a degree 2 homology class (rather than a degree $2g-2$ cohomology class) on A, (5) rewrites as

$$\int_{\gamma_{\min}} \alpha \cup \beta = \epsilon \langle \iota_L(\alpha), \iota_L(\beta)\rangle_L, \tag{8}$$

since the complex orientation of A and the orientation used in (5) differ by the sign ϵ. Using (1), we rewrite (8) as

$$\langle \iota_L(\alpha), \iota_L(\beta)\rangle_L = \epsilon \sum_i \epsilon_i \int_{C_i} j_i^*\alpha \cup j_i^*\beta. \tag{9}$$

We apply in turn (8) to each Jacobian $J(C_i)$ equipped with its principal polarization Θ_i and minimal class $[C_i]$ and get

$$\langle \iota_L(\alpha), \iota_L(\beta)\rangle_L = \epsilon \sum_i \epsilon_i \langle \iota_{\Theta_i}(j_i^*\alpha), \iota_{\Theta_i}(j_i^*\beta)\rangle_{\Theta_i}. \tag{10}$$

This concludes the proof of Lemma 2.7 by general duality, recalling that we are looking at the dual embedding

$$A \cong \widehat{A} \stackrel{\hat{j}=(\hat{j}_i)}{\rightarrow} \prod J(C_i),$$

where $\hat{j}_i = j_i^* : \widehat{A} \to J(C_i)$. One observes that, by definition of $\hat{j}_i : A \to J(C_i)$, for any $\alpha \in H^1(A, \mathbb{Z})$,

$$\iota_{\Theta_i}(j_i^* \alpha) = \hat{j}_{i*}(\iota_L(\alpha)) \text{ in } H_1(J(C_i), \mathbb{Z}). \tag{11}$$

This allows to rewrite the right hand side of (10) as follows

$$\begin{aligned} \epsilon \sum_i \epsilon_i \langle \iota_{\Theta_i}(j_i^* \alpha), \iota_{\Theta_i}(j_i^* \beta) \rangle_{\Theta_i} &= \epsilon \sum_i \epsilon_i \langle \hat{j}_{i*}(\iota_L(\alpha)), \hat{j}_{i*}(\iota_L(\beta))) \rangle_{\Theta_i} \\ &= \epsilon \langle \hat{j}_*(\iota_L(\alpha)), \hat{j}_*(\iota_L(\beta)) \rangle_J, \end{aligned} \tag{12}$$

which concludes the proof by (10). □

2.1 Split abelian varieties

The algebraicity with $\mathbb{Q}$-coefficients of the Künneth projectors of any g-dimensional abelian variety A is well-known (see [13]) and follows from the fact that $A \times A$ contains codimension g subvarieties Γ_i defined as the graph of multiplication by i for any integer i. It also contains their transpose ${}^t\Gamma_i$. As $[\Gamma_i]_*$ acts by multiplication by i^k on $H_k(A \times A, \mathbb{Z})$, one gets

$$\begin{aligned} [\Gamma_i] &= \sum_{k=0}^{2g} i^k \delta_k, \\ [{}^t\Gamma_i] &= \sum_{k=0}^{2g} i^{2g-k} \delta_k. \end{aligned} \tag{13}$$

These equations imply that the δ_i are algebraic with $\mathbb{Q}$-coefficients, as shows the nonvanishing of a Vandermonde determinant (for adequate choices of integers $i_0, \ldots, i_{2g}$), which will appear in the denominator. If we want to analyze the situation with $\mathbb{Z}$-coefficients, (forgetting about the polarization which can be of very large degree and not bring any further information), we argue as follows. Equations (13) show that, in the sublattice L of $H^{2g}(A \times A, \mathbb{Z})$ generated by the δ_k, the group of algebraic classes contains $\sum_{k=0}^{2g} i^k \delta_k$ and $\sum_{k=0}^{2g} i^{2g-k} \delta_k$ for any i. It seems possible that (if the polarization has sufficiently divisible degree) no other combination of the δ_k is algebraic on $A \times A$. The subgroup $L' \subset L$ generated by $\sum_{k=0}^{2g} i^k \delta_k$ and $\sum_{k=0}^{2g} i^{2g-k} \delta_k$ for any i is not the whole group generated by the δ_k. Indeed, consider the dual lattice L^*. An element of L^* is a combination

$$P = \sum_{k=0}^{2g} \alpha_k \delta_k^*$$

and we associate to P the polynomial $P(x) = \sum_{k=0}^{2g} \alpha_k x^k$. We now consider the group $M \cong (L')^*$ of elements $P \in L^* \otimes \mathbb{Q}$ that restrict to elements of $(L')^* \subset (L')^* \otimes \mathbb{Q}$. If a Künneth projector δ_k satisfies $\mu \delta_k \in L'$, one has

$$\mu \alpha_k \in \mathbb{Z}$$

for any $P = \sum_{k=0}^{2g} \alpha_k \delta_k^* \in M$, so we need to know what are the denominators of elements $P \in M$. Such an element P satisfies by (13) the conditions

$$\sum_{k=0}^{2g} \alpha_k i^k \in \mathbb{Z}\ \forall i \in \mathbb{N}, \tag{14}$$

$$\sum_{k=0}^{2g} \alpha_k i^{2g-k}\ \forall i \in \mathbb{N}.$$

The corresponding polynomial $P(x)$ thus has the property that

$$P(i) \in \mathbb{Z}\ \forall i \in \mathbb{N},\ {}^tP(i) \in \mathbb{Z}\ \forall i \in \mathbb{N}, \tag{15}$$

where ${}^tP$ is the reciprocal polynomial of P. Polynomials $P(x) = \sum_{k=0}^{2g} \alpha_k x^k$ with rational coefficients taking integral values on integers are well-known to be combinations with integral coefficients of binomial polynomials. We also have to take into account the reciprocal condition to compute the denominators of the elements in M. In dimension 2, we get that $2\delta_i$ is algebraic for any i.

Recall from Definition 1.9 that an abelian variety A is split if its first Künneth projector δ_1 on $H_1(A, \mathbb{Z})$ is algebraic. This property is related to Theorem 1.4 by the following

Proposition 2.8. *Let A be an abelian variety.*

(i) Assume A is principally polarized (or has a unimodular line bundle) and the minimal class $\gamma_{\min}$ is algebraic. Then A is split.

(ii) Assume A is a direct summand in a product of Jacobians, then A is split.

(iii) Assume the integral Hodge conjecture holds for curve classes on abelian varieties. Then any abelian variety is split.

We will use the following

Lemma 2.9. *Let $A = J(C)$ be the Jacobian of a curve. Then A is split.*

Proof. Indeed, let $j : C \hookrightarrow A$ be the canonical morphism determined by a 0-cycle of degree 1 on C. Recalling that A is isomorphic to its dual $\widehat{A} = \mathrm{Pic}^0(A)$, denote by $\mathcal{P}$ the Poincaré divisor on $A \times A$. We consider the restriction of $\mathcal{P}$ to $A \times C$, that we denote by $\mathcal{P}_C \in \mathrm{CH}^1(A \times C)$. As $[\mathcal{P}]_*$ acts as the Poincaré duality isomorphism $H_1(A, \mathbb{Z}) \cong H^1(A, \mathbb{Z})$ and trivially on other cohomology groups, $[\mathcal{P}_C]_*$ induces an isomorphism $H_1(A, \mathbb{Z}) \cong H^1(C, \mathbb{Z})$, since the restriction map $j^* : H^1(A, \mathbb{Z}) \to H^1(C, \mathbb{Z})$ is an isomorphism. Furthermore $[\mathcal{P}_C]_*$ acts trivially on the other cohomology groups. As $j_* : H_1(C, \mathbb{Z}) \to H_1(A, \mathbb{Z})$ is an isomorphism, the class of the g-cycle $(Id_A, j)_* \mathcal{P}_C \in \mathrm{CH}_g(A \times A)$ acts as the identity of $H_1(A, \mathbb{Z})$ and trivially on the other cohomology groups. □

Proof of Proposition 2.8. In both cases (i) and (iii), it follows from Proposition 2.1 and Corollary 2.5 that A is a direct summand in a product J of Jacobians, so we only have to prove (ii). Let $g := \dim A$ and $g' := \dim J$. By Lemma 2.9, J is split, hence there is a cycle $\Gamma \in \mathrm{CH}^{g'}(J \times J)$ such that $[\Gamma]$ acts as the projector onto $H_1(J, \mathbb{Z})$. Let $j : A \to J$ be the inclusion and $\pi : J \to A$ be the projection. Then if

$$\Gamma' := (Id_A, \pi)_*(j, Id_J)^* \Gamma \in \mathrm{CH}^g(A \times A),$$

we have

$$[\Gamma']_* = \pi_* \circ j_* : H_*(A, \mathbb{Z}) \to H_*(A, \mathbb{Z}),$$

hence $[\Gamma']$ acts as the projector onto $H_1(A, \mathbb{Z})$ and A is split. □

Remark 2.10. If A is a direct summand in a Jacobian, one can easily prove by the same arguments as above as above that all Künneth projectors $\delta_i : H_*(A, \mathbb{Z}) \to H_i(A, \mathbb{Z}) \hookrightarrow H_*(A, \mathbb{Z})$ are algebraic. It is not so clear however that this last property holds if we only assume that A is split.

3 Existence of universal 0-cycles

3.1 Universal 0-cycle on rationally connected fibrations over abelian varieties

We give in this section the proof of Propositions 1.8 and 1.14.

Proof of Proposition 1.8. Let $\psi : P \to A$ be a fibration with rationally connected general fiber. Assume that A is a direct summand in a direct sum $J = \prod_{i=1}^{k} J(C_i)$ of Jacobians and denote respectively by $j : A \to J$ and $\pi : J \to A$ the inclusion and the projection. We consider the fibered product $\psi_J : P_J := J \times_A P \to J$ and observe that the Graber-Harris-Starr theorem [9] applies to the restriction of ψ_J over any curve passing through the general point of J. Thus for the general translate of any curve $C \subset J$, there is a lift $\sigma : C \to P_J$ of C in P_J, with graph $\Gamma_\sigma \in \mathrm{CH}^n(C \times P_J)$, where $n = \dim P_J$. By assumption, $J = \prod_i J(C_i)$ where C_i is a smooth curve of genus g_i. We thus know that there is an embedding $C_i \subset J$ for each i and a cycle $\Gamma_{\sigma_i} \in \mathrm{CH}^n(C \times P_J)$ as above. We can thus construct a cycle

$$Z \in \mathrm{CH}^n(C_1^{(g_1)} \times \ldots \times C_k^{(g_k)} \times P_J)$$

defined as

$$Z = \sum_i \mathrm{pr}_i^* Z_i,$$

where $Z_i \in \mathrm{CH}^n(C_i^{(g_i)} \times P_J)$ is the cycle whose pull-back to $C^{g_i} \times P_J$ is the symmetric cycle $\sum_{j=1}^{g_i} p_j^* \Gamma_{\sigma_i}$ (the p_j being the projections from $C_i^{g_i}$ to C_i). Furthermore, we have a birational map $\tau : \prod_i C_i^{(g_i)} \to J$ such that $\mathrm{alb}_{C_i} \circ \mathrm{pr}_i \circ \tau^{-1}$ is the projection from J to $J(C_i)$ and we now set

$$Z' := (\tau, Id_P)_* Z \in \mathrm{CH}^n(J \times P).$$

For each i, the cycle Γ_{σ_i} has the property that

$$\Gamma_{\sigma_i *} : \mathrm{Alb}(C_i) \to \mathrm{Alb}(P_J) = J$$

is the inclusion of $J(C_i)$ in J, hence the cycle Z has the property that

$$Z_* : \mathrm{Alb}(\prod_i C_i^{(g_i)}) = J \to \mathrm{Alb}(P_J) = J$$

is the identity. It follows that the cycle Z' satisfies as well the property that

$$Z'_* : \mathrm{Alb}(J) = J \to \mathrm{Alb}(P_J) = J$$

is the identity so Z' is a universal 0-cycle for P_J. Finally, let $\pi_P : P_J \to P$ be the natural projection, and let

$$Z'' := \pi_P \circ Z' \circ j \in \mathrm{CH}^m(A \times P),\ m = \dim P.$$

It is clear that Z'' is a universal 0-cycle for P. □

Using Proposition 2.1, we deduce

Corollary 3.1. *(Cf. [19]) Let $\psi : P \to A$ be a fibration with rationally connected general fiber. If A has a unimodular line bundle such that the minimal class $\gamma_{\min}$ is algebraic, P admits a universal zero-cycle.*

We now turn to the case of the Iliev-Markushevich-Tikhomirov fibration [12], [14]. As in the introduction, X is a smooth cubic threefold, and $\widetilde{H}^{5,1}$ is a smooth projective model of the Hilbert scheme $H^{5,1}$ of degree 5, genus 1, curves in X.

Proof of Proposition 1.14. We have $\mathrm{Alb}(\widetilde{H}_{5,1}) = J^3(X) =: J$ since $\widetilde{H}_{5,1}$ is fibered over $J^3(X)$ (via the Abel-Jacobi map) into rationally connected 5-folds. Assume that $\widetilde{H}_{5,1}$ admits a universal 0-cycle $\Gamma \in \mathrm{CH}^{10}(J \times \widetilde{H}_{5,1})$. The Zariski closure $\mathcal{E} \subset \widetilde{H}_{5,1} \times X$ of the universal elliptic curve of degree 5 gives a codimension 2 cycle in $\widetilde{H}_{5,1} \times X$, which induces the Iliev-Markushevich-Tikhomirov Abel-Jacobi isomorphism (see [12], [14])

$$\mathcal{E}_* : \mathrm{Alb}(\widetilde{H}_{5,1}) \cong J^3(X) = J.$$

Consider the composition

$$\Gamma_X := \mathcal{E} \circ \Gamma \in \mathrm{CH}^2(J \times X).$$

Then

$$\Gamma_{X*} = \mathcal{E}_* \circ \Gamma_* : \mathrm{Alb}(J) = J \to J^3(X) = J$$

is the identity of J, hence X has a universal codimension 2 cycle. In the other direction, suppose that X has a universal codimension 2 cycle $\Gamma_X \in \mathrm{CH}^2(J \times X)$. We claim that there exists a cycle $\Gamma \in \mathrm{CH}^{10}(J \times \widetilde{H}_{5,1})$ such that

$$\Gamma_{X*} = \mathcal{E}_* \circ \Gamma_* : J \to J.$$

The claim immediately implies that Γ_* is a universal 0-cycle for $\widetilde{H}_{5,1}$ since

$$\mathcal{E}_* : \mathrm{Alb}(\widetilde{H}_{5,1}) \to J^3(X) = J \tag{16}$$

is an isomorphism. To prove the claim, we use the Shen universal generation theorem [16], which says the following. Denoting by Σ the surface of lines in X, and by $P_\Sigma \subset \Sigma \times X$ the universal family of lines, there exists a cycle $\Gamma_\Sigma \in \mathrm{CH}^2(J \times \Sigma)$, such that

$$P_{\Sigma*} \circ \Gamma_{\Sigma*} = \Gamma_{X*} : J \to J. \tag{17}$$

Note that by [3],

$$P_{\Sigma*} : \mathrm{Alb}(\Sigma) \to J^3(X) = J \tag{18}$$

is also an isomorphism. In order to construct Γ, we apply the following Lemma 3.2 proved below

Lemma 3.2. *There exists a correspondence* $\Gamma_{1,5} \in \mathrm{CH}^{10}(\Sigma \times \widetilde{H}_{5,1})$ *inducing an isomorphism* $\Gamma_{1,5*} : \mathrm{Alb}(\Sigma) \to \mathrm{Alb}(\widetilde{H}_{5,1})$ *compatible with the Abel-Jacobi isomorphisms (16) and (18).*

Indeed, we set $\Gamma := \Gamma_{1,5} \circ \Gamma_\Sigma \in \mathrm{CH}^{10}(J \times \widetilde{H}_{5,1})$ and it follows from (17) that Γ is a universal 0-cycle for $\widetilde{H}_{5,1}$. □

Proof of Lemma 3.2. We choose a point $x \in X$ and a smooth plane cubic curve E passing through x. Let $\Delta \subset X$ be a general line. Let Q be the plane $\langle \Delta, x \rangle$. Then $Q \cap X$ is the union of Δ and a conic C passing through x. The union $C \cup E$ is a (reducible) elliptic curve of degree 5, which is parameterized by a smooth point of $H^{5,1}$, hence by a point of $\widetilde{H}_{5,1}$. This construction gives a rational map $\phi : \Sigma \dashrightarrow \widetilde{H}_{5,1}$. Let $\Gamma_{1,5} \in \mathrm{CH}^{10}(\Sigma \times \widetilde{H}_{5,1})$ be minus the graph of ϕ. For any Δ as above, the classes in X of the curves Δ, C and E satisfy the relations

$$\Delta + C = h^2 \text{ in } \mathrm{CH}^2(X),\ E = h^2 \text{ in } \mathrm{CH}^2(X),$$

hence the curve $C \cup E$ is rationally equivalent to $2h^2 - \Delta$ in X and we have

$$\Phi_X \circ \Gamma_{1,5*} = -\Phi_X \circ \phi_* = P_* : \mathrm{Alb}(\Sigma) \to J^3(X).$$

□

3.2 Hodge classes and cycles classes on Brauer-Severi varieties

Our goal is to establish Theorem 1.11 concerning the existence of a universal 0-cycle for the total space $P \to A$ of a Brauer-Severi variety over an abelian variety A. We first start with the following easy result concerning the Hodge classes on the total space of a Brauer-Severi variety $p : P \to B$ of relative dimension d, where we assume that B is smooth projective and $H^3(B, \mathbb{Z})$ has no torsion. In this case, the Brauer class α_P belongs to the $(d+1)$-torsion of the group

$$H^2(B, \mathcal{O}_B)/H^2(B, \mathbb{Z}) \hookrightarrow H^2(B, \mathcal{O}_B^*),$$

where the sheaf $\mathcal{O}_B^*$ is the sheaf of invertible holomorphic functions on B equipped with the Euclidean topology, and the inclusion above is induced by the exponential exact sequence. The class α_P can be constructed as follows. The Brauer class measures the obstruction to the existence of an algebraic (or equivalently holomorphic since B is projective) line bundle L on P, whose restriction to the fibers $P_x \cong \mathbb{P}^d$ is the generator $\mathcal{O}(1)$. As there is no torsion in $H^3(B, \mathbb{Z})$, the Leray spectral sequence of p shows that a topological such line bundle H exists on P, and $c_1(H) \in H^2(P, \mathbb{Z})$ is well defined modulo $p^*H^2(B, \mathbb{Z})$. We can choose H to be holomorphic if we can arrange that $c_1(H)$ vanishes in $H^2(P, \mathcal{O}_P)$. The class α_P is defined as the image of $c_1(H)$ in

$$H^2(P, \mathcal{O}_P)/p^*H^2(B, \mathbb{Z}) \cong H^2(B, \mathcal{O}_B)/H^2(B, \mathbb{Z}).$$

This class is of $(d+1)$-torsion because P carries a holomorphic line bundle whose restriction to the fibers $P_x \cong \mathbb{P}^d$ is the line bundle $\mathcal{O}(d+1)$, namely the relative anticanonical bundle.

Lemma 3.3. *Let γ be an integral Hodge class of degree $2k$ on B. Then, if there exists an integral Hodge class $\tilde{\gamma} \in \mathrm{Hdg}^{2k+2d}(P, \mathbb{Z})$ such that $p_*\tilde{\gamma} = \gamma$, one has*

$$\gamma \cup \alpha_P = 0 \text{ in } H^{2k+2}(B, \mathbb{C})/(F^{k+1}H^{2k+2}(B, \mathbb{C}) + H^{2k+2}(B, \mathbb{Z})). \tag{19}$$

In (19), $F^{\cdot}$ denotes the Hodge filtration on the Betti cohomology of B with complex coefficients and the cup-product $\gamma \cup \alpha_P$ is defined as follows. An integral Hodge class γ of degree $2k$ on a smooth projective variety Y can be seen as a pair $(\gamma_{\mathbb{Z}}, \gamma_F)$, with $\gamma_{\mathbb{Z}} \in H^{2k}(Y, \mathbb{Z})$, $\gamma_F \in F^kH^{2k}(Y, \mathbb{C})$ such that

$$\gamma_{\mathbb{C}} = \gamma_F \text{ in } H^{2k}(Y, \mathbb{C}). \tag{20}$$

Given such a Hodge class γ on Y and a Brauer class

$$\alpha \in H^2(Y, \mathcal{O}_Y)/H^2(Y, \mathbb{Z}) = H^2(Y, \mathbb{C})/(F^1H^2(Y, \mathbb{C}) + H^2(Y, \mathbb{Z}))$$

with lift $\tilde{\alpha} \in H^2(Y, \mathcal{O}_Y) = H^2(Y, \mathbb{C})/F^1H^2(Y, \mathbb{C})$, we define

$$\gamma \cup \alpha := \gamma_F \cup \tilde{\alpha} \in H^{2k+2}(Y, \mathbb{C})/F^{k+1}H^{2k+2}(Y, \mathbb{C}) \text{ mod } H^{2k+2}(Y, \mathbb{Z}). \tag{21}$$

Proof of Lemma 3.3. We choose as before a lift $\tilde{\alpha}_P$ of α_P in $H^2(Y, \mathcal{O}_Y)$. By construction, the pull-back $p^*\alpha_P$ vanishes in $H^2(P, \mathcal{O}_P)/H^2(P, \mathbb{Z})$. We thus have

$$p^*\tilde{\alpha}_P = \eta \text{ in } H^2(P, \mathbb{C}), \tag{22}$$

where $\eta \in H^2(P, \mathbb{Z})$. If $\tilde{\gamma} \in \mathrm{Hdg}^{2k+2d}(P, \mathbb{Z})$, with de Rham component $\tilde{\gamma}_F \in F^{k+d}H^{2k+2d}(P, \mathbb{C})$ and integral component $\tilde{\gamma}_{\mathbb{Z}} \in H^{2k+2d}(P, \mathbb{Z})$, it follows from (20) and (22) that

$$\tilde{\gamma}_F \cup p^*\tilde{\alpha}_P = \tilde{\gamma}_{\mathbb{Z}} \cup \eta \text{ in } H^{2k+2+2d}(P, \mathbb{C})/F^{k+d+1}H^{2k+2+2d}(P, \mathbb{C}). \tag{23}$$

As the right hand side is an integral cohomology class (modulo torsion) on P, we conclude by push-forward to B that $p_*\tilde{\gamma}_F \cup \tilde{\alpha}_P = p_*(\tilde{\gamma}_F \cup p^*\tilde{\alpha}_P)$ is an integral cohomology class on B, modulo $F^{k+1}H^{2k+2}(B, \mathbb{C})$. Using the description (21) of the cup-product, we conclude that the Hodge class $\gamma = p_*\tilde{\gamma}$ satisfies (19). □

Proof of Theorem 1.11. Let A be an abelian variety of dimension g which is Mumford-Tate general with $\rho(A) = 1$. The Néron-Severi group $\mathrm{NS}(A)$ is 1-dimensional, generated by the class $c_1(L)$ for some ample line bundle L on A. We recall the notation $\gamma_{\min}$ for the generator of the cyclic group $\mathrm{Hdg}^{2g-2}(X, \mathbb{Z})$. Our assumptions are that

(i) the intersection number $\gamma_{\min} \cdot c_1(L)$ is even and

(ii) any Brauer-Severi variety $p : P \to A$ has a universal 0-cycle.

We will choose a Brauer class

$$\beta \in \mathrm{Tors}(H^2(A, \mathcal{O}_A)/H^2(A, \mathbb{Z})) \cong H^2(A, \mathbb{Z})_{\mathrm{tr}} \otimes \mathbb{Q}/\mathbb{Z}$$

with arbitrary divisible order, where $H^2(A, \mathbb{Z})_{\mathrm{tr}} := H^2(A, \mathbb{Z})/\mathrm{NS}(A)$. Let $p : P_\beta \to A$ be a Brauer-Severi variety on A, of Brauer class β. Let d_β be the dimension of P_β. As, by assumption (ii), P_β has a universal zero-cycle for P_β, there is a cycle Z_β in $\mathrm{CH}^{d_\beta}(A \times P_\beta)$, such that the class $[W_\beta] := (Id, p)_*[Z_\beta] \in \mathrm{Hdg}^{2g}(A \times A, \mathbb{Z})$ has (1,1)-Künneth component equal to δ_1, that is, acts as the identity on $H_1(A, \mathbb{Z})$.

We will prove the following

Proposition 3.4. *Assume β comes from a general class in $H^2(A, \mathbb{Q})_{\mathrm{tr}}$ with arbitrarily divisible denominator. Then the Hodge class $[W_\beta]$ on $A \times A$ satisfies*

$$[W_\beta] = \alpha_0\delta_0 + \delta_1 + \alpha_2 c_1(L) \otimes \gamma_{\min} + \gamma_\beta \text{ in } \mathrm{Hdg}^{2g}(A \times A, \mathbb{Z}), \tag{24}$$

where α_0 and α_2 are integral and $\gamma_\beta \in \mathrm{Hdg}^{2g}(A \times A, \mathbb{Z})$ is arbitrarily divisible.

Assuming the proposition, the proof of Theorem 1.11 is concluded as follows. As A is very general, the rational Hodge conjecture is known for $A \times A$ (see the proof of Lemma 3.5). It follows that any integral Hodge class which is sufficiently divisible is algebraic on $A \times A$ and in particular the Hodge classes γ_β of (24) with arbitrarily high divisibility are algebraic on $A \times A$. The class $\alpha_0\delta_0 = \alpha_0[A \times \mathrm{pt}]$ is algebraic, and $[W_\beta] = [(Id, p)_* Z_\beta]$ is also algebraic, so we conclude from (24) that the class

$$\delta_1 + \alpha_2 c_1(L) \otimes \gamma_{\min} \tag{25}$$

is algebraic.

It remains to prove that this implies that δ_1 is algebraic on $A \times A$. If we let act on $A \times A$ the endomorphism $\mu'_- := (Id_A, \mu_-)$, where μ_- is the multiplication by -1 on A, we deduce from the algebraicity of (25) that the class $-\delta_1 + \alpha_2 c_1(L) \otimes \gamma_{\min}$ is also algebraic on $A \times A$, so that $2\alpha_2 c_1(L) \otimes \gamma_{\min}$ is algebraic on $A \times A$. If we now take the square (in the sense of the composition of correspondences) of (25), we find that the class

$$\delta_1 + \alpha_2^2 (c_1(L) \cdot \gamma_{\min}) c_1(L) \otimes \gamma_{\min}$$

is also algebraic. By assumption (i), $c_1(L) \cdot \gamma_{\min}$ is even, so we conclude that the class $\alpha_2^2(c_1(L) \cdot \gamma_{\min}) c_1(L) \otimes \gamma_{\min}$ is algebraic, and finally δ_1 is algebraic, so A is split. $\square$

For the proof of Proposition 3.4, we will use the following Lemma 3.5. Hodge classes of degree $2g$ on $A \times A$ decompose according to the Künneth decomposition

$$\gamma = \sum_i \gamma_i,$$

where $\gamma_i = \gamma \circ \delta_i \in H^i(A, \mathbb{Z}) \otimes H^{2g-i}(A, \mathbb{Z})$. For any class $\tilde{\beta} \in H^2(A, \mathbb{Q})$, the cup-product by $\mathrm{pr}_2^* \tilde{\beta}$ preserves the Künneth decomposition and induces a morphism

$$\overline{\mathrm{pr}_2^* \tilde{\beta} \cup} : \mathrm{Hdg}^{2g}(A \times A, \mathbb{Q}) \to H^{2g+2}(A \times A, \mathbb{Q})/\mathrm{Hdg}^{2g+2}(A \times A, \mathbb{Q}). \tag{26}$$

Lemma 3.5. *For a Mumford-Tate general abelian variety A with Picard number 1 and polarizing class l, and a generic class $\tilde{\beta} \in H^2(A,\mathbb{Q})$, the morphism $\overline{\tilde{\beta}\cup}$ of (26) has for kernel the $\mathbb{Q}$-vector subspace generated by*

$$\delta_0 \in H^0(A,\mathbb{Q}) \otimes H^{2g}(A,\mathbb{Q}),\ \delta_1 \in H^1(A,\mathbb{Q}) \otimes H^{2g-1}(A,\mathbb{Q}),\\ l \otimes \gamma_{\min} \in H^2(A,\mathbb{Q}) \otimes H^{2g-2}(A,\mathbb{Q}). \tag{27}$$

Proof. As the cup-product by $\mathrm{pr}_2^*\tilde{\beta}$ acts as $Id \otimes (\tilde{\beta}\cup)$ on $H^i(A,\mathbb{Q}) \otimes H^{2g-i}(A,\mathbb{Q})$, it is clear for degree or Hodge type reasons that the three classes δ_0, δ_1 and $l \otimes \gamma_{\min}$ belong to the kernel of $\overline{\tilde{\beta}\cup}$. In the other direction, let us describe the rational Hodge classes on $A \times A$. By assumption, the Mumford-Tate group of A is the symplectic group $\mathrm{Sp}(2g)$, hence the Lefschetz decomposition

$$H^i(X,\mathbb{Q}) = \oplus_{i-2j\geq 0} l^j \cup H^{i-2j}(A,\mathbb{Q})_{\mathrm{prim}} \tag{28}$$

for $i \leq g$, is a decomposition into simple Hodge structures and there are no nonzero morphisms of Hodge structures

$$H^{i-2j}(A,\mathbb{Q})_{\mathrm{prim}} \to H^{i-2j'}(A,\mathbb{Q})_{\mathrm{prim}}$$

for $j \neq j'$. It follows that the rational Hodge classes in

$$H^{2g-i}(A,\mathbb{Q}) \otimes H^i(A,\mathbb{Q}) = \mathrm{End}(H^i(A,\mathbb{Q})),$$

namely the morphisms of Hodge structures in $\mathrm{End}(H^i(A,\mathbb{Q}))$, are linear combinations of the projectors $\pi_{i,j}$ of the Lefschetz decompositions (28). It is well-known that these projectors are algebraic with rational coefficients (this is a consequence of the Lefschetz standard conjecture for abelian varieties, see [13]). When $i \geq g$, the discussion is the same, except that we use the Lefschetz isomorphism

$$H^i(X,\mathbb{Q}) \cong H^{2g-i}(X,\mathbb{Q})$$

and the Lefschetz decomposition on $H^{2g-i}(X,\mathbb{Q})$. We now observe that, for any i, and for any Hodge class ϕ in the subspace

$$\mathrm{Hdg}^{2g}(A \times A,\mathbb{Q}) \cap \mathrm{End}(H^i(A,\mathbb{Q})) \subset \mathrm{End}_0(H^*(A,\mathbb{Q})) = H^{2g}(A \times A,\mathbb{Q})$$

generated by all the $\pi_{i,j}$ except the three classes appearing in (27), the image $\mathrm{Im}\,\phi \subset H^i(A,\mathbb{Q})$ is a Hodge structure with a nonzero component $(\mathrm{Im}\,\phi)^{p,q}$ for some $q \leq g-2$. Furthermore, there are only finitely many such Hodge substructures, since they all must be direct sums of Lefschetz components appearing in the Lefschetz decomposition (28). One then easily checks that for a generic $\eta \in H^2(A,\mathcal{O}_A)$, and any ϕ as above, the cup-product map

$$\eta : (\mathrm{Im}\,\phi)^{p,q} \to H^{p,q+2}(A)$$

is nonzero, hence $\eta\cup\phi \neq 0$ in $\mathrm{Hom}(H^{p,q}(A), H^{p,q+2}(A))$. It follows from the above discussion that for a general $\eta \in H^2(A,\mathcal{O}_A)$, the cup-product map

$$\eta\cup : \mathrm{Hdg}^{2g}(A \times A,\mathbb{Q}) \to H^{2g+2}(A \times A,\mathbb{C})/F^{g+1}H^{2g+2}(A \times A,\mathbb{C})$$

has for kernel the space generated by (27). This implies the lemma because the image of $H^2(A,\mathbb{Q})$ in $H^2(A,\mathcal{O}_A)$ is Zariski dense and one has the following commutative diagram for any $\tilde{\beta} \in H^2(A,\mathbb{Q})$ with image $\eta \in H^2(A,\mathcal{O}_A)$

$$\begin{array}{ccc} \mathrm{Hdg}^{2g}(A \times A,\mathbb{Q}) & \stackrel{\overline{\tilde{\beta}\cup}}{\to} & H^{2g+2}(A \times A,\mathbb{Q})/\mathrm{Hdg}^{2g+2}(A \times A,\mathbb{Q}) \\ \| & & \downarrow \\ \mathrm{Hdg}^{2g}(A \times A,\mathbb{Q}) & \stackrel{\eta\cup}{\to} & H^{2g+2}(A,\mathbb{C})/F^{g+1}H^{2g+2}(A,\mathbb{C}). \end{array}$$

□

Proof of Proposition 3.4. Lemma 3.3 applied to the Brauer-Severi variety $A \times P_\beta \to A \times A$ says that

$$[W_\beta] \cup \mathrm{pr}_2^*\beta = 0 \text{ in } \mathrm{Tors}(H^{2g+2}(A \times A, \mathbb{C})/(F^{g+1}H^{2g+2}(A \times A, \mathbb{C}) + H^{2g+2}(A \times A, \mathbb{Z}))) \quad (29)$$
$$= H^{2g+2}(A \times A, \mathbb{Q})/(\mathrm{Hdg}^{2g+2}(A \times A, \mathbb{Q}) + H^{2g+2}(A \times A, \mathbb{Z})). \quad (30)$$

The second equality (30) follows from the fact that

$$\mathrm{Hdg}^{2g+2}(A \times A, \mathbb{Q}) = \mathrm{Ker}\,(H^{2g+2}(A \times A, \mathbb{Q}) \to H^{2g+2}(A \times A, \mathbb{C})/F^{g+1}H^{2g+2}(A \times A, \mathbb{C})).$$

Equation (29) says equivalently that, with the notation of (26),

$$\overline{\mathrm{pr}_2^*\tilde{\beta}\cup}[W_\beta] = 0 \text{ in } (H^{2g+2}(A \times A, \mathbb{Z})/\mathrm{Hdg}^{2g+2}(A \times A, \mathbb{Z})) \otimes \mathbb{Q}/\mathbb{Z}. \quad (31)$$

We now apply the following elementary

Lemma 3.6. *Let H_1, H_2 be two lattices, and $\psi \in \mathrm{Hom}\,(H_1, H_2) \otimes \mathbb{Q}$ be an injective morphism. Then there exists an integer d, such that, for any integer N, and any $h \in H_1$ with $\frac{1}{N}\psi(h) \in H_2 \subset H_2 \otimes \mathbb{Q}$, one has $h \in \frac{N}{d}H_1$. In particular, $h \in H_1$ if d divides N, and $h \in H_1$ is arbitrarily divisible if $\frac{N}{d}$ is.*

We apply this lemma to

$$H_1 = \mathrm{Hdg}^{2g}(A \times A, \mathbb{Z})/\langle \delta_0, \delta_1, l \otimes \gamma_{\min}\rangle, \ H_2 = H^{2g+2}(A \times A, \mathbb{Z})/\mathrm{Hdg}^{2g+2}(A \times A, \mathbb{Z}),$$

taking for ψ the cup-product map $\overline{\mathrm{pr}_2^*\tilde{\beta}\cup}$. It satisfies our assumptions by Lemma 3.5. We thus conclude that (31) implies that, when N is arbitrarily divisible, the class $[W_{\frac{1}{N}\beta}]$ is arbitrarily divisible modulo $\langle \delta_0, \delta_1, l \otimes \gamma_{\min}\rangle$. As we know furthermore that the Künneth component $[W_{\frac{1}{N}\beta}]$ is equal to δ_1, this concludes the proof of Proposition 3.4. □

3.3 On the existence of a universal codimension 2 cycle for a cubic threefold

This section is devoted to the proof of Theorem 1.16. Let X be a cubic threefold and $J = J^3(X)$ its intermediate Jacobian. This is a 5-dimensional principally polarized abelian variety. We assume that X admits a universal codimension 2 cycle $\Gamma \in \mathrm{CH}^2(J \times X)$ and we want to prove, under the assumption that X is Mumford-Tate general, that J is split. Note that, especially in view of Proposition 1.14, the statement has some similarities with Theorem 1.11. There are however two differences. First of all, in the cubic case, the Iliev-Markushevich-Tikhomirov construction does not provide a Brauer-Severi variety but only a Brauer-Severi variety over a Zariski open set of J. Secondly, in the cubic case, we are given only one (generic) Brauer-Severi variety admitting a universal 0-cycle, while in Theorem 1.11, we are given Brauer-Severi varieties admiting a universal 0-cycle, with Brauer class general of arbitrarily high order.

Let $\Sigma = F_1(X)$ be the surface of lines in X. We will first prove some preparatory lemmas. By [3], the universal line

$$P \subset \Sigma \times X,$$

induces an embedding $j = \Phi_X \circ P_* : \Sigma \to J^3(X) = J$ and an isomorphism

$$P_* = j_* : \mathrm{Alb}(\Sigma) \to J. \quad (32)$$

According to [16], given Γ, there exists a correspondence $\Gamma' \in \mathrm{CH}^2(J \times \Sigma)$ such that

$$P_* \circ \Gamma'_* = \Gamma_* : \mathrm{CH}_0(J)_{\mathrm{hom}} \to \mathrm{CH}^2(X)_{\mathrm{hom}}. \quad (33)$$

As $\mathrm{CH}^2(X)_{\mathrm{hom}} \cong J^3(X) = J$ via the Abel-Jacobi map Φ_X (see [3], and [2] for a more general result), (33) is equivalent to the fact that

$$P_* \circ \Gamma'_* = \Gamma_* = Id_J : J \to J. \quad (34)$$

Via the isomorphism (32), we can write (34) as

$$j_* \circ \Gamma'_* = Id_J, \tag{35}$$

and equivalently, looking at the action of these correspondences on homology

$$j_* \circ [\Gamma']_* = Id_{H_1(J,\mathbb{Z})}. \tag{36}$$

Our strategy will be to modify Γ' by composing it with self-correspondences of Σ so as to achieve the condition

$$[j \circ \Gamma'] = \delta_1, \tag{37}$$

where δ_1 is the Künneth projector on $H_1(J,\mathbb{Z})$. We first note the following

Lemma 3.7. *Denote by $\delta_{1,\Sigma} \in H^4(\Sigma \times \Sigma, \mathbb{Z})$ the Künneth projector onto $H_1(\Sigma, \mathbb{Z})$. Then $2\delta_{1,\Sigma}$ is algebraic. Equivalently, twice the Künneth projector $\delta_{3,\Sigma}$ onto $H_3(\Sigma,\mathbb{Z})$ is algebraic.*

Remark 3.8. The integral cohomology of Σ is torsion free (see [4]), so the $\delta_{i,\Sigma}$ are well defined.

Corollary 3.9. *Twice the Künneth projector $\delta_{2,\Sigma}$ of Σ is also algebraic.*

Proof. Indeed we have $2\delta_{2,\Sigma} = 2[\Delta_\Sigma] - 2\delta_{1,\Sigma} - 2\delta_{3,\Sigma} - 2\delta_{0,\Sigma} - 2\delta_{4,\Sigma}$, where $2\delta_{1,\Sigma}$ and $2\delta_{3,\Sigma}$ are algebraic by Lemma 3.7, and $\delta_{0,\Sigma} = [\Sigma \times \mathrm{pt}]$, $\delta_{4,\Sigma} = [\mathrm{pt} \times \Sigma]$ are also algebraic, as already mentioned. □

Proof of Lemma 3.7. We know by [3] that the class $l \in H^2(\Sigma,\mathbb{Z})$ of the curve $C_\Delta \subset \Sigma$ of lines meeting a given line $\Delta \subset X$ satisfies

$$2l = j^*\theta, \tag{38}$$

where $\theta \in H^2(J,\mathbb{Z})$ is the class of a Theta-divisor. Let $(j,j)^*\mathcal{P}$ be the pull-back to $\Sigma \times \Sigma$ of a Poincaré divisor $\mathcal{P}$ on $J \times J$, so that

$$[(j,j)^*\mathcal{P}] \in H^1(\Sigma,\mathbb{Z}) \otimes H^1(\Sigma,\mathbb{Z}) \subset H^2(\Sigma,\mathbb{Z})$$

is algebraic. The class

$$\gamma := [(j,j)^*\mathcal{P}] \cup \mathrm{pr}_2{}^* l \in H^1(\Sigma,\mathbb{Z}) \otimes H^3(\Sigma,\mathbb{Z}) \subset H^4(\Sigma,\mathbb{Z}) \tag{39}$$

is thus algebraic. It is clear from (39) that γ acts trivially on $H_i(\Sigma,\mathbb{Z})$ for $i \neq 1$. The action of γ on $H_1(\Sigma,\mathbb{Z})$ is given by

$$\gamma_*(u) = l \cup j^*([\mathcal{P}]_*(j_*u)) \text{ in } H^3(\Sigma,\mathbb{Z}) \cong H_1(\Sigma,\mathbb{Z}). \tag{40}$$

It remains to see that the right hand side is equal to $2u$. Pushing forward to J, we get, using the fact that $l = \frac{1}{2}j^*\theta$,

$$j_*(\gamma_*(u)) = \frac{1}{2}\theta \cup j_*j^*([\mathcal{P}]_*(j_*u)) = \frac{1}{2}\theta \cup [\Sigma] \cup [\mathcal{P}]_*(j_*u) \text{ in } H^9(J,\mathbb{Z}) \cong H_1(J,\mathbb{Z}).$$

The right hand side is equal to $2j_*u$ because

$$[\Sigma] = \frac{\theta^3}{3!},\ \gamma_{\min} = \frac{\theta^4}{4!},$$

where the first equality is proved in [3], and $[\mathcal{P}]_* : H_1(J,\mathbb{Z}) \to H^1(J,\mathbb{Z})$ is the inverse of $\gamma_{\min}\cup : H^1(J,\mathbb{Z}) \to H^9(J,\mathbb{Z}) = H_1(A,\mathbb{Z})$. □

We now study the action of $j_*\circ[\Gamma']_*$ on the other homology groups of J.

Lemma 3.10. *The image of $j_*\circ[\Gamma']_* : H_3(J,\mathbb{Z})\to H_3(J,\mathbb{Z})$ is contained in $2j_*H_3(\Sigma,\mathbb{Z})$.*

Proof. We observe that, with $\mathbb{Q}$-coefficients, the Hodge structure on $H_3(J,\mathbb{Q}) = H^7(J,\mathbb{Q})$ splits by the Lefschetz decomposition as

$$H^7(J,\mathbb{Q}) = \theta^2\cup H^3(J,\mathbb{Q})_{\rm prim}\oplus\theta^3\cup H^1(J,\mathbb{Q}), \tag{41}$$

where the Hodge structures of the two summands are simple, have no nontrivial endomorphisms and admit no non-trivial morphisms from one to the other. These three facts follow from the fact that, by assumption, the Mumford-Tate group of the Hodge structure on $H^1(J,\mathbb{Q})$ is the symplectic group of $(H^1(J,\mathbb{Q}),\langle\,,\,\rangle_\theta)$. Note that, as $[\Sigma]=\frac{\theta^3}{3!}$ and $j^*: H^1(J,\mathbb{Q})\to H^1(\Sigma,\mathbb{Q})$ is surjective, the space $\theta^3H^1(J,\mathbb{Q})$ is equal to $j_*H^1(\Sigma,\mathbb{Q})$. As $j_*\circ[\Gamma']_*$ is a morphism of Hodge structures and the image of $j_*\circ[\Gamma']_*$ is contained in $\operatorname{Im} j_*$, it follows from the decomposition (41) with its stated properties that $j_*\circ[\Gamma']_*$ is a multiple $\lambda\pi$ of the projector π on $j_*H^1(\Sigma,\mathbb{Q})=\theta^3\cup H^1(J,\mathbb{Q})$ associated with the decomposition (41). Our statement is thus that the coefficient λ is an even integer. Using the fact that $[\Sigma]=\frac{\theta^3}{3!}$, one easily shows that the sublattice

$$j_*H^1(\Sigma,\mathbb{Z}) = [\Sigma]\cup H^1(J,\mathbb{Z})\subset H^7(J,\mathbb{Z})$$

is primitive. As $\lambda\pi = j_*\circ[\Gamma']_* : H^7(J,\mathbb{Z})\to H^7(J,\mathbb{Z})$ is equal to λId on $[\Sigma]\cup H^1(J,\mathbb{Z})$, it follows that λ is an integer. To see that λ must be even, we observe that $H^7(J,\mathbb{Z})$ carries a unimodular intersection pairing ω_7 thanks to the principal polarization of J, which provides isomorphisms $H^i(J,\mathbb{Z})\cong H_i(J,\mathbb{Z})$ for all i. Passing to $\mathbb{Z}$-coefficients, the decomposition (41) provides the inclusion of a finite index sublattice

$$= \frac{\theta^3}{3!}\cup H^1(J,\mathbb{Z})\oplus(\frac{\theta^3}{3!}\cup H^1(J,\mathbb{Z}))^{\perp_{\omega_7}}\subset H^7(J,\mathbb{Z}), \tag{42}$$

where the orthogonal decomposition induces (41) after passing to rational coefficients. If there exists an integral endomorphism of the lattice $H^7(J,\mathbb{Z})$ which acts as an odd multiple $\lambda\pi$ of the orthogonal projector from $H^7(J,\mathbb{Z})$ onto the first summand in (42), then the discriminant of the restriction of the pairing ω_7 to $\frac{\theta^3}{3!}\cup H^1(J,\mathbb{Z})$ is odd. But, by Lemma 3.11 proved below, and using as above the isomorphism

$$H^7(J,\mathbb{Z})\cong H_7(J,\mathbb{Z})\cong H^3(J,\mathbb{Z})$$

given by the principal polarization and Poincaré duality, this restriction is equal to four times the unimodular pairing on $H^1(J,\mathbb{Z})$, which is a contradiction. ∎

Lemma 3.11. *Let A be a principally polarized abelian variety of dimension 5 and θ the class of its Theta divisor. Then the unimodular pairing on $H^3(A,\mathbb{Z})$ restricts to 4 times the theta pairing on $H^1(A,\mathbb{Z})\cong\theta\cup H^1(A,\mathbb{Z})\subset H^3(A,\mathbb{Z})$.*

Proof. The unimodular pairing on $H^3(A,\mathbb{Z})$ is given by the composite isomorphism

$$\iota : H^3(A,\mathbb{Z})\cong\bigwedge^3 H^1(A,\mathbb{Z})\cong\bigwedge^3 H_1(A,\mathbb{Z})\cong H_3(A,\mathbb{Z}), \tag{43}$$

where the middle isomorphism is induced by θ, the left isomorphism is given by cup-product and the last isomorphism is given by Pontryagin product. This isomorphism maps $\theta\cup H^1(A,\mathbb{Z})$ to $\gamma_{\rm min}*H_1(A,\mathbb{Z})$. We thus have to compute, for $\alpha\in H^1(A,\mathbb{Z})$, $\beta\in H_1(A,\mathbb{Z})$ the pairing $\langle\theta\cup\alpha,\gamma_{\rm min}*\beta\rangle$. The statement is that it is equal to 4 times the pairing $\langle\alpha,\beta\rangle$. The statement is topological so we can assume that $A=JC$ is the Jacobian of a curve $C\subset A$ whose class $[C]\in H_2(A,\mathbb{Z})$ is the minimal class $\gamma_{\rm min}$. Let $\mu: C\times C\to A$ be the sum map. Let β be represented by the class of an oriented circle $B\subset C$, and let

$$\mu_B : C\times B\to A$$

be the restriction of μ to $C \times B$. We have by definition

$$\gamma_{\min} * \beta = \mu_*([C \times B]_{\mathrm{fund}}) \text{ in } H_3(A, \mathbb{Z}),$$

so that

$$\langle \theta \cup \alpha, \gamma_{\min} * \beta \rangle = \int_{C \times B} \mu_B^*(\theta \cup \alpha). \tag{44}$$

We observe that $\mu_B^* \theta = \mathrm{pr}_1^* \theta_C + \mathcal{P}$, where $\mathcal{P} \in H^1(C, \mathbb{Z}) \otimes H^1(B, \mathbb{Z})$ is now the pull-back to $C \times B$ of the Poincaré divisor of $J \times J$ and pr_1 is the first projection $C \times B \to C$. Furthermore

$$\mu_B^* \alpha = \mathrm{pr}_1^* \alpha_{|C} + \mathrm{pr}_2^* \alpha_{|B}.$$

Furthermore, the degree of θ_C is equal to 5. It follows that

$$\int_{C \times B} \mu_B^*(\theta \cup \alpha) = 5 \int_{\mathrm{pt} \times B} \alpha_{|B} + \int_{C \times B} [\mathcal{P}] \cup \mathrm{pr}_1^* \alpha_{|C}. \tag{45}$$

The first term is equal to $5\langle \alpha, \beta \rangle$. By definition of the Poincaré divisor, the second term equals $\int_C \alpha_C \cup [\mathcal{P}]^* \beta = -\langle \alpha, \beta \rangle$, hence (44) and (45) give

$$\langle \theta \cup \alpha, \gamma_{\min} * \beta \rangle = 4\langle \alpha, \beta \rangle.$$

□

Proof of Theorem 1.16. We want to prove that the existence of a universal codimension 2-cycle Γ implies that J is split, that is, the Künneth projector δ_1 onto $H_1(J, \mathbb{Z})$ is algebraic. Note the following

Lemma 3.12. *Assuming the existence of a universal codimension* 2*-cycle* Γ*, the algebraicity on* $J \times J$ *of the Künneth projector* δ_1 *onto* $H_1(J, \mathbb{Z})$ *is implied by the algebraicity on* $\Sigma \times \Sigma$ *of the Künneth projector* $\delta_{1,\Sigma}$ *onto* $H_1(\Sigma, \mathbb{Z})$*.*

Proof. Indeed, given Γ, let $\Gamma' \in \mathrm{CH}^2(X \times \Sigma)$ be a Shen cycle, satisfying the equivalent conditions (35), (36). Let $\Delta_{1,\Sigma}$ be a codimension 2 cycle on Σ such that $[\Delta_{1,\Sigma}] = \delta_{1,\Sigma}$. Let

$$\Delta_1 := j_* \circ \Delta_{1,\Sigma} \circ \Gamma' \in \mathrm{CH}^5(J \times J).$$

Then $[\Delta_1]_* = 0$ on $H_i(J, \mathbb{Z})$ for $i \neq 1$, since

$$[\Delta_1]_* = j_* \circ [\Delta_{1,\Sigma}]_* \circ [\Gamma']_* : H_*(J, \mathbb{Z}) \to H_*(J, \mathbb{Z}) \tag{46}$$

and $[\Delta_{1,\Sigma}]_* = 0$ on $H_i(\Sigma, \mathbb{Z})$ for $i \neq 1$. Furthermore, by (46) and (36), $[\Delta_1]_*$ acts as the identity on $H_1(J, \mathbb{Z})$. □

Note also that, by Lemma 3.7, in order to prove that the Künneth projector $\delta_{1,\Sigma}$ is algebraic, it suffices to prove that an odd multiple $\lambda \delta_{1,\Sigma}$ is algebraic, or equivalently an odd multiple $\lambda \delta_{3,\Sigma}$ is algebraic.

We now consider the Künneth components $[\Gamma'']_i$ of the class of the cycle

$$\Gamma'' := \Gamma' \circ j \in \mathrm{CH}^2(\Sigma \times \Sigma).$$

We know by (36) that $[\Gamma'']_1$ acts as the identity on $H_1(\Sigma, \mathbb{Z})$, and, by Lemma 3.10, that $[\Gamma'']_3$ acts by an even multiple $2\lambda_3$ of the identity on $H_3(\Sigma, \mathbb{Z})$. It follows from Lemma 3.7 that there exists an algebraic cycle Γ''_3 on $\Sigma \times \Sigma$ acting as $2\lambda_3 Id$ on $H_3(\Sigma, \mathbb{Z})$ and by 0 on the other homology groups of Σ. Hence

$$\Gamma''' := \Gamma'' - \Gamma''_3$$

acts by 0 on $H_3(\Sigma,\mathbb{Z})$ and by Id on $H_1(\Sigma,\mathbb{Z})$. We can in an obvious way also modify Γ''' so that it acts trivially on $H_0(\Sigma,\mathbb{Z})$ and $H_4(\Sigma,\mathbb{Z})$.

Finally, we have to consider what happens on $H_2(\Sigma,\mathbb{Z})$. In fact, $H_2(\Sigma,\mathbb{Z})$ contains the finite index sublattice

$$\mathbb{Z}l \oplus (\mathbb{Z}l)^{\perp}, \tag{47}$$

where the class l is defined in (38) and satisfies $l^2 = 5$, implying that the index of the sublattice (47) is odd. Furthermore, the decomposition in (47) is a direct sum of Hodge structures, the first one being trivial, and the second one being simple and nontrivial, because X is Mumford-Tate general. It follows that the Künneth component $[\Gamma'']_2$ acts on $H_2(\Sigma,\mathbb{Z})$ preserving the sublattice (47) and its decomposition, hence by multiplication by respective integers λ_1, λ_2 on the summands. The fact that $l^2 = 5$ shows that the cycle $l \times l$ on Σ acts by multiplication by 5 on the first summand, so that the cycle

$$5\Gamma''' - \lambda_1 l \times l$$

has the property that its cohomology class $[5\Gamma''' - \lambda_1 l \times l]$ acts by 0 on l, and by $5\lambda_2$ on $(\mathbb{Z}l)^{\perp}$.

We finally discuss the parity of λ_2.

Case (i). $5\lambda_2 = 2m$ *is even.* We know by Corollary 3.9 that $2\delta_2$ is algebraic, hence there exists a codimension 2 cycle Δ_2 on $\Sigma \times \Sigma$ such that $[\Delta_2]_*$ acts as $2mId$ on $(\mathbb{Z}l)^{\perp}$, and 0 on $\mathbb{Z}l$ and the other homology groups of Σ. But then the cycle

$$5\Gamma''' - \Delta_2 - \lambda_1 l \times l$$

acts by 0 on $H_i(\Sigma,\mathbb{Z})$ for $i \neq 0$ and by an odd multiple of the identity on $H_1(\Sigma,\mathbb{Z})$, which concludes the proof in this case.

Case (ii). $5\lambda_2 = 2m+1$ *is odd.* In this case, $(2m+1)\Delta_\Sigma - 5\Gamma'''$ acts by 0 on $(\mathbb{Z}l)^{\perp}$, by $(2m+1-5)Id$ on $H_1(\Sigma,\mathbb{Z})$, and by $(2m+1)Id$ on $H_3(\Sigma,\mathbb{Z})$. As $2m+1-5 = 2k$ is even, there exists by Lemma 3.7 a codimension 2 cycle Δ_1 on $\Sigma \times \Sigma$ such that $[\Delta_1] = 2k\delta_1$ and then the cycle

$$(2m+1)\Delta_\Sigma - 5\Gamma''' - \Delta_1$$

acts by 0 on $(\mathbb{Z}l)^{\perp}$ and on $H_1(\Sigma,\mathbb{Z})$, and by $(2m+1)Id$ on $H_3(\Sigma,\mathbb{Z})$. Its class is thus an odd multiple of δ_3. An odd multiple of δ_3 is thus algebraic, hence an odd multiple of δ_1 is algebraic. □

References

[1] Th. Beckmann, O. de Gaay Fortman. Integral Fourier transforms and the integral Hodge conjecture for one-cycles on abelian varieties, Compos. Math. 159 (2023), no. 6, 1188-1213.

[2] S. Bloch, V. Srinivas. Remarks on correspondences and algebraic cycles, Amer. J. of Math. 105 (1983) 1235-1253.

[3] H. Clemens, P. Griffiths. The intermediate Jacobian of the cubic threefold, Ann. of Math. 95 (1972), 281-356.

[4] A. Collino. The fundamental group of the Fano surface. I, II. In *Algebraic threefolds (Varenna, 1981)*, pp. 209218, 219-220, Lecture Notes in Math., 947, Springer, Berlin-New York, (1982).

[5] J.-L. Colliot-Thélène. Application d'Albanese et zéro-cycles, à paraître.

[6] J.-L. Colliot-Thélène, A. Skorobogatov. *The Brauer-Grothendieck group.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics 71. Springer, (2021).

[7] O. Debarre. *Tores et variétés abéliennes complexes*, Cours Spécialisés 6, Société Mathématique de France, EDP Sciences, 1999.

[8] S. Druel. Espace des modules des faisceaux de rang 2 semi-stables de classes de Chern $c_1 = 0$, $c_2 = 2$ et $c_3 = 0$ sur la cubique de $\mathbb{P}^4$. Internat. Math. Res. Notices 2000, no. 19, 985-1004.

[9] T. Graber, J. Harris, J. Starr. Families of rationally connected varieties. J. Amer. Math. Soc. 16 (2003), no. 1, 57-67.

[10] C. Grabowski. On the integral Hodge conjecture for 3-folds, PhD Thesis Duke University (2004).

[11] J. Hotchkiss. Hodge theory of twisted derived categories and the period-index problem, in preparation.

[12] A. Iliev,D. Markushevich. The Abel-Jacobi map for a cubic threefold and periods of Fano threefolds of degree 14. Doc. Math. 5 (2000), 23-47.

[13] D. Lieberman. Numerical and homological equivalence of algebraic cycles on Hodge manifolds. Amer. J. Math. 90 (1968), 366-374.

[14] D. Markushevich, A. S. Tikhomirov. The Abel-Jacobi map of a moduli component of vector bundles on the cubic threefold. J. Algebraic Geom. 10 (2001), no. 1, 37-62.

[15] B. Moonen, A. Polishchuk. Divided powers in Chow rings and integral Fourier transforms. Adv. Math. 224 (2010), no. 5, 2216-2236.

[16] M. Shen. Rationality, universal generation and the integral Hodge conjecture. Geom. Topol. 23 (2019), no. 6, 2861-2898.

[17] C. Voisin. On the universal CH_0 group of cubic hypersurfaces JEMS Volume 19, Issue 6 (2017) pp. 1619-1653.

[18] C. Voisin. Unirational threefolds with no universal codimension 2 cycle, Invent math. Vol. 201, Issue 1 (2015), 207-237.

[19] C. Voisin. Abel-Jacobi map, integral Hodge classes and decomposition of the diagonal, J. Algebraic Geom. 22 (2013), 141-174.

[20] C. Voisin. Geometric representability of 1-cycles on rationally connected threefolds, preprint 2022.

[21] C. Voisin. *Chow rings, decomposition of the diagonal and the topology of families*, Annals of Math. Studies 187, Princeton University Press 2014.