HAL open science

Partition sum of thermal, under-constrained systems

Cheng-Tai Lee, Matthias Merkel

▶ To cite this version:

Cheng-Tai Lee, Matthias Merkel. Partition sum of thermal, under-constrained systems. 2023. hal-04251711

HAL Id: hal-04251711

https://hal.science/hal-04251711v1

Preprint submitted on 20 Oct 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution 4.0 International License

Partition sum of thermal, under-constrained systems

Cheng-Tai Lee and Matthias Merkel*
Aix Marseille Univ, Université de Toulon, CNRS, CPT (UMR 7332),
Turing Center for Living Systems, Marseille, France
(Dated: April 17, 2023)

Athermal (i.e. zero-temperature) under-constrained systems are typically floppy, but they can be rigidified by the application of external strain. Following our recently-developed analytical theory for the athermal limit, here and in the companion paper, we extend this theory to under-constrained systems at finite temperatures close to the athermal transition point. We derive from first principles the partition sum for a broad class of under-constrained systems, from which we obtain analytic expressions for elastic material properties such as isotropic tension t and shear modulus G in terms of isotropic strain ε, shear strain γ, and temperature T. Our work unifies the physics of systems as diverse as polymer fibers & networks, membranes, and vertex models for biological tissues.

arXiv:2304.07264v1 [cond-mat.soft] 14 Apr 2023

I. INTRODUCTION

Under-constrained systems possess more degrees of freedom (dofs), $N_{\rm dof}$, than constraints (or springs), N_s. In the athermal limit, i.e. at zero temperature, under-constrained systems are generally floppy. Conversely, over-constrained systems, which have $N_{\rm dof} < N_s$, are generally rigid [1, 2]. We recently developed a generic analytic framework to predict the elastic properties of athermal, under-constrained systems [3, 4]. However, such a framework does not yet exist for *thermal*, i.e. finite-temperature, under-constrained systems.

In the athermal limit, under-constrained systems are typically floppy due to the existence of zero modes. For instance, there are *infinitesimal* zero modes, i.e. collective displacements of the dofs that do not affect any of the constraints *to linear order*. The number of linearly independent infinitesimal zero modes, N_0, is given by [1, 2]:

$$N_0 = N_{\rm dof} - N_s + N_{SSS}, \tag{1}$$

where N_{SSS} is the number of linearly independent states of self-stress (SSS). A SSS is a combination of virtual tensions put on the springs of a network that does not result in any net force on the dofs. According to Eq. (1), under-constrained systems always have a positive number of infinitesimal zero modes.

There are three possibilities regarding the nature of infinitesimal zero modes. First, they could be *trivial* zero modes, i.e. global translations or rotations of the system, which leave the "shape" of the system unchanged. Second, they could correspond to *mechanisms*, i.e. *finite* displacements that change the "shape" of the system, but do not affect any of the constraints [5, 6]. Third, while inducing no changes to the constraints to linear order, they could induce changes to higher order, corresponding to higher-order rigidity [7, 8]. Here, we refer to this third kind as *lower-order* zero mode. At zero temperature, under-constrained are typically floppy because many of the infinitesimal zero modes often correspond to mechanisms.

Still, under-constrained systems can be rigidified by global strain, such as isotropic expansion or shear [2, 3, 7–11]. This athermal rigidity transition is due to the shrinkage of the hyper-volume of dof configurations that are geometrically compatible with the constraints. At this rigidity transition, a SSS is created [5], and a positive number of mechanisms turns into lower-order zero modes. Beyond this transition point, the constraints cannot be fulfilled any more, i.e. the external strain is geometrically incompatible with the constraints, and in the case of a spring network, any further strain creates prestresses in the springs. These prestresses, together with the SSS created at the transition, rigidify the system [2, 3, 5, 9, 12, 13]. In previous work, we used these insights to analytically predict the elastic material properties close to the rigidity transition in the athermal limit [3, 4]. These ideas apply not only to spring networks, but also to other systems such as polymer networks [14–17] and vertex models for biological tissues [3, 13, 18–20].

Despite these advances for the athermal limit, there is still relatively little known about *thermal*, i.e. finite-temperature, under-constrained systems [21–23]. Recent work used effective-medium theory (EMT) to obtain analytical expressions for the shear modulus of spring networks [24, 25], and Mannattil et al. discuss the effect of singularities in the configuration space on the mechanics of the system [6]. However, we still lack a generic theory for the elastic properties of thermal, under-constrained systems.

Here, we analytically derive from first principles the partition sum and elastic properties of thermal, under-constrained systems close to the athermal transition for varying isotropic strain ε, shear strain γ, and small temperatures T. After defining the class of under-constrained systems that we discuss here (section II), we first revisit the athermal limit (section III). This will provide the foundations for discussing at finite temperature both the limit of infinitely stiff springs (section IV) and the general case of finite spring stiffness (section V). In the companion paper [26], we numerically verify the analytical results obtained here.

* matthias.merkel@cnrs.fr

II. MODEL

A. Under-constrained system

We consider a Hookean network with fixed connectivity of N_s generalized springs with length L_i, labeled by $i = 1, \ldots, N_s$. The system energy is:

$$E = \frac{1}{2} \sum_{i=1}^{N_s} K_i \big(L_i - L_{0i}\big)^2, \tag{2}$$

where the sum is over all springs i with spring constants K_i and rest lengths L_{0i}.

Each generalized spring length L_i has dimension d_i. For example, angular springs have $d_i = 0$, ordinary linear springs or perimeter springs in the vertex model have $d_i = 1$, and area and volume springs in the vertex model have $d_i = 2$ and $d_i = 3$, respectively. Thus, also models that, for example, combine 2D or 3D vertex models with spring networks are accounted for [27]. The notion of the dimension d_i of a generalized spring length L_i is further formalized in Eq. (5) below.

The spring network is embedded in D spatial dimensions with periodic boundary conditions with total system volume V. The periodic box can be sheared with shear strain variable γ. Our framework applies both to simple and pure shear [3].

The generalized spring lengths depend on N_{node} node positions $\boldsymbol{R}_a$ with $a = 1, \ldots, N_{\text{node}}$, which correspond to the $N_{\text{dof}} = DN_{\text{node}}$ degrees of freedom. We denote the collection of all components of all node positions by the N_{dof}-dimensional vector $\vec{R}$. Here we consider an under-constrained spring network, where $N_{\text{dof}} > N_s$. To be precise, we discuss here a slightly stricter class of under-constrained models that will be specified later.

Throughout this article, we treat all K_i and L_{0i} as constant and study the behavior of the L_i depending on system size V, shear γ, and temperature T.

B. Athermal transition

We consider a network that in the athermal limit transitions from floppy (zero shear modulus) to rigid (finite shear modulus) at some critical volume. For spring networks or vertex models, this is typically the case when the network percolates the periodic box (in the sense of connectivity percolation). For zero shear strain, $\gamma = 0$, we denote this critical volume by V^*, where, consistent with earlier findings [3, 4], we assume here that the system is floppy for $V < V^*$ and rigid for $V > V^*$.

For simplicity, we assume here and in the following that there are no states of self-stress (SSS) in the floppy regime. Also, we assume that only a single SSS forms at the transition, which is generally true for disordered networks.

C. Isotropic strain

We define isotropic strain ε with respect to the critical volume V^*:

$$\varepsilon := \frac{1}{D} \log\left(\frac{V}{V^*}\right). \tag{3}$$

Thus, for $\gamma = 0$, the transition occurs at $\varepsilon = 0$.

We introduce the dimensionless degrees of freedom (ddofs) $\vec{r}$ as:

$$\vec{r} := V^{-1/D} \vec{R} = e^{-\varepsilon} (V^*)^{-1/D} \vec{R}. \tag{4}$$

This allows us to formalize the notion of spring dimension. We say that a spring i has dimension d_i if the spring length $L_i(\vec{R}, \varepsilon, \gamma)$ is homogeneous of degree d_i with respect to an isotropic rescaling. In other words, L_i can be written as:

$$L_i(\vec{R}, \varepsilon, \gamma) = L_{0i} e^{d_i \varepsilon} \ell_i(\vec{r}, \gamma) \tag{5}$$

with a function $\ell_i(\vec{r}, \gamma)$ that depends on the ddofs $\vec{r}$ and shear strain γ, *but not on isotropic strain* ε. We call ℓ_i the dimensionless spring length. At the transition, where $L_i = L_{0i}$, we have $\ell_i = 1$.

Finally, we define the dimensionless weights

$$w_i := \sqrt{\frac{K_i L_{0i}^2}{E_0}}, \tag{6}$$

where we have introduced the energy scale $E_0 := (\sum_i K_i L_{0i}^2)/N_s$ such that $\sum_i w_i^2 = N_s$. This allows us to rewrite the system energy as

$$E = \frac{E_0}{2} \sum_{i=1}^{N_s} w_i^2 \big(e^{d_i \varepsilon} \ell_i - 1\big)^2. \tag{7}$$

To lowest order in ε, and with $\Delta\ell_i = \ell_i - 1$, we thus have:

$$E = \frac{E_0}{2} \sum_{i=1}^{N_s} w_i^2 \big(\Delta\ell_i + d_i \varepsilon\big)^2. \tag{8}$$

To obtain Eq. (8) from Eq. (7), we have factored out the prefactor $e^{d_i \varepsilon}$ in front of the parentheses, which scale to lowest order as $\sim \varepsilon$. The prefactor of $e^{d_i \varepsilon}$ only adds higher-order terms and is thus neglected.

III. ATHERMAL LIMIT ($T \to 0$)

While in previous work we derived the elastic system properties in the athermal limit based on a minimal-length function [3, 4], we present here an alternative derivation, which will prepare our discussions in the subsequent sections.

A. Alternative derivation

We denote the values of the ddofs $\vec{r}$ at the transition by $\vec{r}^*$. For simplicity, we assume here that $\vec{r}^*$ is uniquely defined. Global translations do not change any of the arguments below, and we focus on networks where mechanisms do not have any influence on the SSS that forms at the transition.

We now expand the dimensionless spring lengths ℓ_i around the transition point at $\vec{r}^*$ and for small γ. This is possible because each dimensionless spring length ℓ_i is an analytical function of $\vec{r}$ and γ [28]. Denoting the components of $\vec{r}$ by r_n, we can write to second order in $\Delta r_n := r_n - r_n^*$ and γ:

$$w_{\dot{i}}\Delta\ell_{\dot{i}} = C_{in}\Delta r_n + \frac{1}{2}M_{imn}\Delta r_m \Delta r_n + B_i^{(1)}\gamma + \frac{1}{2}B_i^{(2)}\gamma^2. \tag{9}$$

Here and in the following, we imply summation over equal indices (except for indices with underdots, and in the absence of explicit sums). On the right-hand side, C_{in} is a generalized compatibility matrix, and the matrix M_{imn} is symmetric in the last two indices. We introduce the shear strain dependence through the coefficients $B_i^{(1/2)}$. In general, matrices C_{in} and M_{imn} may depend on γ. However, including this dependency turns out not to be important to lowest order in our final results, both here and in the following sections (appendix B). There are no ε-dependent terms in Eq. (9), because the ℓ_i functions are independent of ε.

We further simplify the problem of minimizing the system energy by performing a singular-value decomposition (SVD) on the compatibility matrix:

$$C_{in} = \sum_{p=1}^{N_s} U_{ip} s_p V_{np}, \tag{10}$$

where each of U_{ip} and V_{np} is an orthogonal square matrix, and the singular values s_p are sorted in decreasing order: $s_1 \geq \cdots \geq s_{N_s}$. Because at the transition a single SSS is created, we have $s_{N_s-1} > 0$ and $s_{N_s} = 0$, where U_{iN_s} is the corresponding SSS. Furthermore, all V_{np} with $p = N_s, \ldots, N_{\text{dof}}$ are infinitesimal zero modes.

Based on the SVD of the compatibility matrix, Eq. (10), we define changes of spring lengths and of degrees of freedom in the "eigen bases" of C_{in}:

$$\Delta\tilde{\ell}_p := \sum_i U_{ip} w_i \Delta\ell_i \tag{11}$$

$$\Delta\tilde{r}_p := V_{np}\Delta r_n. \tag{12}$$

Thus, the expansion in Eq. (9) transforms into:

$$\Delta\tilde{\ell}_p = s_{\dot{p}}\Delta\tilde{r}_{\dot{p}} + \frac{1}{2}\tilde{M}_{pmn}\Delta\tilde{r}_m\Delta\tilde{r}_n + \tilde{B}_p^{(1)}\gamma + \frac{1}{2}\tilde{B}_p^{(2)}\gamma^2, \tag{13}$$

where we have also transformed M and $B^{(1/2)}$ into the eigenbases of C: $\tilde{M}_{pmn} := U_{ip}M_{im'n'}V_{m'm}V_{n'n}$ and $\tilde{B}_p^{(1/2)} := U_{ip}B_i^{(1/2)}$.

We now insert the Taylor expansion of $w_{\dot{i}}\Delta\ell_{\dot{i}}$ into the system energy, Eq. (8). First, using Parseval's theorem, we have:

$$\begin{aligned} E &= \frac{E_0}{2}\sum_{i=1}^{N_s}\left(w_{\dot{i}}\Delta\ell_{\dot{i}} + d_{\dot{i}}w_{\dot{i}}\varepsilon\right)^2 \\ &= \frac{E_0}{2}\sum_{p=1}^{N_s}\left(\Delta\tilde{\ell}_p + \tilde{w}_p\varepsilon\right)^2. \end{aligned} \tag{14}$$

Here, we have introduced $\tilde{w}_p := \sum_i U_{ip}d_i w_i$. Note that Eq. (10) is invariant with respect to a sign flip in the SSS, whose components are given by U_{iN_s} for each spring i. For convenience, we choose this sign such that $\tilde{w}_{N_s} = \sum_i U_{iN_s} d_i w_i \geq 0$.

Inserting Eq. (13) into the energy, Eq. (14), we obtain:

$$E = \frac{E_0}{2}\sum_{p=1}^{N_s} G_p^2. \tag{15}$$

with

$$\begin{aligned} G_p &:= \Delta\tilde{\ell}_p + \tilde{w}_p\varepsilon \\ &= s_{\dot{p}}\Delta\tilde{r}_{\dot{p}} + \frac{1}{2}\tilde{M}_{pmn}\Delta\tilde{r}_m\Delta\tilde{r}_n \\ &\qquad + \tilde{w}_p\varepsilon + \tilde{B}_p^{(1)}\gamma + \frac{1}{2}\tilde{B}_p^{(2)}\gamma^2. \end{aligned} \tag{16}$$

To discuss the energy minimum we focus here for clarity on the case where $\tilde{B}_p^{(1)} = 0$ for all p. The general case of arbitrary $\tilde{B}_p^{(1)}$ is discussed in appendix B. As we discuss in appendix A, at the energy minimum, the $\Delta\tilde{r}_n$ do not scale in the same way with the strain variables ε and γ for different n. For $p < N_s$ we find that $\Delta\tilde{r}_p$ is of order of at least $\mathcal{O}(\varepsilon) + \mathcal{O}(\gamma^2)$, and for $n \geq N_s$ we find that $\Delta\tilde{r}_n$ is of order of at least $\mathcal{O}(|\varepsilon|^{1/2}) + \mathcal{O}(\gamma)$. Consequentially, all of the $\tilde{M}$ terms involving any of the $\Delta\tilde{r}_p$ with $p < N_s$ can be neglected in all the G_q with $q \leq N_s$. Thus, each of the $\Delta\tilde{r}_p$ with $p < N_s$ only appears in total once, namely as linear term in the respective G_p. Thus, to the lowest relevant order, when minimizing the energy in Eq. (15), each of the first $N_s - 1$ terms G_p for $p < N_s$ can become zero by adjusting $\Delta\tilde{r}_p$ accordingly:

$$\begin{aligned} \Delta\tilde{r}_p = -\frac{1}{s_{\dot{p}}}\Bigg(&\frac{1}{2}\sum_{m,n=N_s}^{N_{\text{dof}}}\tilde{M}_{\dot{p}mn}\Delta\tilde{r}_m\Delta\tilde{r}_n \\ &+ \tilde{w}_p\varepsilon + \frac{1}{2}\tilde{B}_{\dot{p}}^{(2)}\gamma^2\Bigg). \end{aligned} \tag{17}$$

The remaining term is:

$$G_{N_s} = \frac{1}{2}\sum_{m,n=N_s}^{N_{\text{dof}}}\tilde{M}_{N_s mn}\Delta\tilde{r}_m\Delta\tilde{r}_n + \tilde{w}_{N_s}\varepsilon + \frac{1}{2}\tilde{B}_{N_s}^{(2)}\gamma^2. \tag{18}$$

We know that for $\gamma = 0$ and $\varepsilon > 0$, where the system is in the rigid regime, the energy minimum is strictly positive, $E = E_0 G_{N_s}^2/2 > 0$. This is only possible if the matrix $\tilde{M}_{N_s mn}$ is positive semi-definite; otherwise G_{N_s} could become zero. As a consequence, for the athermal rigid regime, $\varepsilon + b_\varepsilon \gamma^2 > 0$, the energy is minimal if $\Delta \tilde{r}_n = 0$ for all $N_s \leq n \leq N_{\mathrm{dof}}$, and it has the value

$$E = \frac{E_0 \tilde{w}_{N_s}^2}{2} \left[\varepsilon + b_\varepsilon \gamma^2\right]^2, \tag{19}$$

where we introduced

$$b_\varepsilon := \frac{B_{N_s}^{(2)}}{2\tilde{w}_{N_s}} \qquad \text{if } \tilde{B}_p^{(1)} = 0 \text{ for all } p. \tag{20}$$

In the more general case where the $\tilde{B}_p^{(1)}$ do not vanish, the definition of b_ε includes more terms, while Eq. (19) remains the same up to a shift in γ (appendix B).

Note that the matrix $\tilde{M}_{N_s mn}$ is closely linked to the Hessian of the system and the emergence of rigidity in the athermal limit (appendix C).

B. Comparison to earlier results

The published expression for the energy of a network with only d-dimensional springs in the rigid regime corresponds to [3, 4]:

$$E = \frac{E_0 N_s}{2(L_0^*)^2(1 + a_\ell^2)} \left[- \Delta L_0 + b\gamma^2 \right]^2, \tag{21}$$

where $\Delta L_0 = -dL_0^* \varepsilon$, and L_0^*, a_ℓ, and b are constants that depend on the network structure. We included here the prefactor of 1/2 in our energy definition as compared to [3, 4]. Comparing Eqs. (19) and (21), we find:

$$\tilde{w}_{N_s} = d \left[\frac{N_s}{1 + a_\ell^2} \right]^{1/2} \tag{22}$$

$$b_\varepsilon = \frac{b}{dL_0^*}. \tag{23}$$

In addition to the earlier work [3, 4], our approach here also clarifies why the result is analytic in γ; it is inherited from the analytic nature of the ℓ_i functions (Eq. (9)).

C. Elastic properties

In the rigid regime of the athermal limit, i.e. for $\varepsilon + b_\varepsilon \gamma^2 > 0$, tension $t_E = \partial E/\partial V = (\partial E/\partial \varepsilon)/DV$ and shear modulus $G_E = (\partial^2 E/\partial \gamma^2)/V$ are to lowest order in ε and γ:

$$t_E = \kappa_E \left[\varepsilon + b_\varepsilon \gamma^2\right] \tag{24}$$

$$G_E = 2Db_\varepsilon \kappa_E \left[\varepsilon + 3b_\varepsilon \gamma^2\right] \tag{25}$$

$$= 2Db_\varepsilon t \left(1 + \frac{2b_\varepsilon \gamma^2}{|\varepsilon + b_\varepsilon \gamma^2|}\right), \tag{26}$$

where we defined

$$\kappa_E := \frac{E_0 \tilde{w}_{N_s}^2}{DV^*}. \tag{27}$$

The index E indicates purely energetic rigidity.

IV. STIFF-SPRING LIMIT ($K_i \to \infty$)

In the limit of infinitely stiff springs but at finite temperature T, elasticity is created purely by entropic effects. For $\varepsilon + b_\varepsilon \gamma^2 > 0$ (rigid regime in the athermal limit), there are no configurations where all springs can attain their respective rest lengths. Thus, this regime is inaccessible in the stiff-spring limit. Conversely, for $\varepsilon + b_\varepsilon \gamma^2 \leq 0$ (floppy regime in the athermal limit) there *are* configurations where the springs can attain their rest lengths. In this section, we derive an expression for the accessible configurational phase space volume Ω, from which we can directly derive the elastic network properties. We focus again on the limit of small ε and γ, in which the network is *almost rigid* [29].

A. Configurational phase space volume

Up to a constant factor, the configurational phase space volume is given by (appendix D)

$$\Omega \sim \int \left(\prod_{a=1}^{N_{\mathrm{node}}} \mathrm{d}^D \boldsymbol{R}_a \right) \prod_{i=1}^{N_{\mathrm{s}}} \delta(L_i - L_{0i}), \tag{28}$$

which, using the rescaling from Eqs. (4) and (5), transforms into

$$\Omega \sim \int \left(\prod_{n=1}^{N_{\mathrm{dof}}} \mathrm{d}r_n \right) \prod_{i=1}^{N_{\mathrm{s}}} \delta(e^{d_i \varepsilon} \ell_i - 1), \tag{29}$$

where we ignore any prefactors in Ω that are powers of e^ε, because these factors only create higher-order terms in the final result.

We aim to obtain an analytical expression for Ω in the vicinity of the transition point, i.e. for $|\varepsilon| \ll 1$ and $|\gamma| \ll 1$. We can thus rewrite Ω as:

$$\Omega \sim \int \left(\prod_{n=1}^{N_{\mathrm{dof}}} \mathrm{d}\Delta r_n \right) \prod_{i=1}^{N_{\mathrm{s}}} \delta\Big(w_i \Delta \ell_i + d_i w_i \varepsilon\Big). \tag{30}$$

Here, we have included a constant factor of w_i in each of the Dirac deltas, and we again ignored prefactors of e^ε in Ω.

To insert the Taylor expansion, Eq. (13), into Eq. (30), we first apply a rotation by V_{np} to the integration variables Δr_n, and a rotation by U_{ip} on the argument of the

multi-dimensional Dirac delta, neither of which yields an additional factor in Ω:

$$\begin{aligned}\Omega &\sim \int \left(\prod_{n=1}^{N_{\rm dof}} \mathrm{d}\Delta\tilde{r}_n\right) \prod_{p=1}^{N_s} \delta\left(\Delta\tilde{\ell}_p + \tilde{w}_p\varepsilon\right) \\ &= \int \left(\prod_{n=1}^{N_{\rm dof}} \mathrm{d}\Delta\tilde{r}_n\right) \prod_{p=1}^{N_s} \delta(G_p).\end{aligned} \tag{31}$$

The constraints by the Dirac deltas in this equations imply $G_p = 0$ for all $p \leq N_s$.

In the case where all $B_p^{(1)} = 0$, we can again use the same arguments as in appendix A to show that all terms involving the $\Delta\tilde{r}_p$ with $p < N_s$ can be neglected in all G_q with $q \leq N_s$. Thus, like in the athermal limit, the value of each of the $\Delta\tilde{r}_p$ with $p < N_s$ is adjusted by the respective equation $G_p = 0$, and the remaining G_{N_s} simplifies to Eq. (18). In other words, we can split the integrations in Eq. (31) as follows:

$$\Omega \sim \int \left(\prod_{n=N_s}^{N_{\rm dof}} \mathrm{d}\Delta\tilde{r}_n\right) \delta(G_{N_s}) \prod_{p=1}^{N_s-1} \int \mathrm{d}\Delta\tilde{r}_p\, \delta(G_p). \tag{32}$$

Here, each of the $N-1$ inner integrals evaluates to a constant factor of $1/s_p$. We are thus left with:

$$\Omega \sim \Omega_{N_s}(\varepsilon, \gamma) \tag{33}$$

with

$$\begin{aligned}\Omega_{N_s}(\varepsilon,\gamma) &:= \int \left(\prod_{n=N_s}^{N_{\rm dof}} \mathrm{d}\Delta\tilde{r}_n\right) \delta\left(G_{N_s}\right) \\ &= \int \left(\prod_{n=N_s}^{N_{\rm dof}} \mathrm{d}\Delta\tilde{r}_n\right) \\ &\quad \times \delta\left(\frac{1}{2} \sum_{m,n=N_s}^{N_{\rm dof}} \tilde{M}_{N_s mn} \Delta\tilde{r}_m \Delta\tilde{r}_n \right. \\ &\qquad \left. + \tilde{w}_{N_s}\left[\varepsilon + b_\varepsilon \gamma^2\right]\right).\end{aligned} \tag{34}$$

To evaluate $\Omega_{N_s}(\varepsilon,\gamma)$, we first diagonalize the matrix $\tilde{M}_{N_s mn}$:

$$\tilde{M}_{N_s mn} = \sum_{k=1}^{N_{\rm dof}-N_s+1} \mu_k v_m^k v_n^k, \tag{35}$$

with $m, n = N_s, \ldots, N_{\rm dof}$, and where μ_k are the eigenvalues and v_m^k are the associated orthonormal eigenvectors. We sort the eigenvalues in decreasing order, $\mu_1 \geq \cdots \geq \mu_{N_{\rm dof}-N_s+1}$, where $\mu_k \geq 0$, since the matrix $\tilde{M}_{N_s mn}$ is positive semi-definite.

With $\Delta q_k := v_m^k \Delta\tilde{r}_m$, the equation $G_{N_s} = 0$ transforms into:

$$-\tilde{w}_{N_s}\left[\varepsilon + b_\varepsilon\gamma^2\right] = \frac{1}{2} \sum_{k=1}^{N_{\rm dof}-N_s+1} \mu_k \Delta q_k^2. \tag{36}$$

Because the v_m^k represent a rotation, the differential transforms without prefactor:

$$\begin{aligned}\Omega_{N_s} &= \int \left(\prod_{k=1}^{N_{\rm dof}-N_s+1} \mathrm{d}\Delta q_k\right) \\ &\quad \times \delta\left(\frac{1}{2} \sum_{k=1}^{N_{\rm dof}-N_s+1} \mu_k \Delta q_k^2 + \tilde{w}_{N_s}\left[\varepsilon + b_\varepsilon\gamma^2\right]\right).\end{aligned} \tag{37}$$

To evaluate this integral, we first discuss the zero modes of $\tilde{M}_{N_s mn}$, i.e. modes k with $\mu_k = 0$, for which there are two possibilities: First, such a mode k could be one of the D global translations, or it could be related to a mechanism. In the first case, the integral over a global translation leads to a factor of e^ε, which can again be neglected in the end result. Similarly, the integral over the manifold corresponding to the mechanisms that exists at the transition yields a constant (describing the hypervolume of that manifold) plus, possibly, a higher-order term in ε and/or γ (describing the change of the manifold volume upon strain), which can be neglected. Another possibility is however that any change of the zero mode Δq_k *does* affect $\Delta\ell_{N_s}$, but only to higher than quadratic order. We believe this is an exceptional case and do not consider it further in the main text, but we discuss it in appendix E.

We denote the number of non-zero eigen values of $\tilde{M}_{N_s mn}$ by N_e. Integrating out the zero modes in Eq. (37), we have:

$$\begin{aligned}\Omega_{N_s} &\sim \int \left(\prod_{k=1}^{N_e} \mathrm{d}\Delta q_k\right) \\ &\quad \times \delta\left(\frac{1}{2} \sum_{k=1}^{N_e} \mu_k \Delta q_k^2 + \tilde{w}_{N_s}\left[\varepsilon + b_\varepsilon\gamma^2\right]\right).\end{aligned} \tag{38}$$

The Dirac delta ensures that the integral is over the surface $\partial\mathcal{E}$ of an ellispoid $\mathcal{E}$ in N_e-dimensional space, defined by the equation:

$$\sum_{k=1}^{N_e} \frac{\Delta q_k^2}{\rho_k^2} = 1 \tag{39}$$

with half axes $\rho_k := (-2\tilde{w}_{N_s}[\varepsilon + b_\varepsilon\gamma^2]/\mu_k)^{1/2} \sim [-(\varepsilon + b_\varepsilon\gamma^2)]^{1/2}$. Using standard transformation rules of the Dirac delta, we get:

$$\Omega_{N_s} \sim \int_{\partial\mathcal{E}} \mathrm{d}S \left|\frac{\partial G_{N_s}}{\partial \boldsymbol{\Delta q}}\right|^{-1}, \tag{40}$$

where

$$\begin{aligned}\left|\frac{\partial G_{N_s}}{\partial \boldsymbol{\Delta q}}\right|^2 &= \sum_{k=1}^{N_e} \left(\frac{\partial G_{N_s}}{\partial \Delta q_k}\right)^2 \\ &= \sum_{k=1}^{N_e} \mu_k^2 \Delta q_k^2.\end{aligned} \tag{41}$$

With $\mu_{\max} := \mu_1$ and $\mu_{\min} := \mu_{N_e} > 0$, we thus have on the ellipsoid surface:

$$-\mu_{\min}\left[\varepsilon + b_\varepsilon \gamma^2\right] \leq \frac{1}{2\tilde{w}_{N_s}} \left|\frac{\partial G_{N_s}}{\partial \boldsymbol{\Delta q}}\right|^2 \leq -\mu_{\max}\left[\varepsilon + b_\varepsilon \gamma^2\right]. \tag{42}$$

Thus, $|\partial G_{N_s}/\partial \boldsymbol{\Delta q}| \sim [-(\varepsilon + b_\varepsilon \gamma^2)]^{1/2}$. As a consequence, we obtain that the integral in Eq. (40) scales as the hypersurface area of the ellipsoid divided by $[-(\varepsilon + b_\varepsilon \gamma^2)]^{1/2}$:

$$\Omega_{N_s}(\varepsilon, \gamma) \sim \Big[-(\varepsilon + b_\varepsilon \gamma^2)\Big]^{(N_e-2)/2}. \tag{43}$$

Hence, $\Omega \sim \Omega_{N_s}$ scales as a power law with the distance to the transition point as long as $N_e > 2$.

In appendix B, we discuss the more general case where the $\tilde{B}_p^{(1)}$ do not vanish. The definition of b_ε changes like in the athermal case, while the result Eq. (43) remains the same up to an additional term $B_{N_s}^{(1)}\gamma$, which can be removed through redefining γ.

B. Free energy, tension, and shear modulus

The free energy of the system in the stiff-spring limit, F_S, is up to a constant:

$$\begin{aligned} F_S &= -\frac{1}{\beta}\log\Omega \\ &= -\frac{N_e - 2}{2\beta}\log\Big[-(\varepsilon + b_\varepsilon\gamma^2)\Big], \end{aligned} \tag{44}$$

where $\beta := k_B T$ with k_B being the Boltzmann constant. The index S stresses that F_S results entirely from entropic effects.

We then obtain for the tension $t_S = (\partial F_S/\partial\varepsilon)/DV$ and the shear modulus $G_S = (\partial^2 F_S/\partial\gamma^2)/V$, to lowest order:

$$t_S = -\frac{\kappa_S T}{\varepsilon + b_\varepsilon\gamma^2} \tag{45}$$

$$G_S = -2Db_\varepsilon\kappa_S T\frac{\varepsilon - b_\varepsilon\gamma^2}{(\varepsilon + b_\varepsilon\gamma^2)^2} \tag{46}$$

$$= 2Db_\varepsilon t\left(1 + \frac{2b_\varepsilon\gamma^2}{|\varepsilon + b_\varepsilon\gamma^2|}\right) \tag{47}$$

with

$$\kappa_S := \frac{k_B(N_e - 2)}{2DV^*}. \tag{48}$$

Note that only the regime with $\varepsilon + b_\varepsilon\gamma^2 < 0$ is accessible here, which corresponds to the athermal floppy regime.

V. GENERAL CASE

A. Partition sum

To study how entropic and energetic elasticity interact, we evaluate the partition sum of the system

$$Z = \int\left(\prod_{a=1}^{N_{\text{node}}} \mathrm{d}^D\boldsymbol{R}_a\right)e^{-\beta E}. \tag{49}$$

In terms of rotated dimensionless node positions relative to the transition point, $\Delta\tilde{r}_n$, we have:

$$Z \sim \int\left(\prod_{k=1}^{N_{\text{dof}}} \mathrm{d}\Delta\tilde{r}_n\right)e^{-\beta E}, \tag{50}$$

where we again ignore prefactors proportional to e^{ε}.

We evaluate the partition sum for small ε, γ, and T. Because $\varepsilon \ll 1$, we can use the energy expression from Eq. (15):

$$E = \frac{E_0}{2}\sum_{p=1}^{N_s} G_p^2. \tag{51}$$

Here we discuss the case where all $\tilde{B}_p^{(1)} = 0$, while the more general case of arbitrary $\tilde{B}_p^{(1)}$ is discussed in appendix B.

We focus on small temperatures, $k_B T \ll E_0[\varepsilon + b_\varepsilon\gamma^2]^2$, such that states with smaller energies E receive an exponentially higher Boltzmann weight. Using our discussion in appendix A, we find again that in all G_q with $q \leq N_s$ all the $\Delta\tilde{r}_p$ with $p < N_s$ can be neglected in the quadratic terms. Thus, to lowest order each of the $\Delta\tilde{r}_p$ with $p < N_s$ appears only once, namely in the linear term of in the respective G_p. As a consequence, the integrals in the partition sum in Eq. (50) can be rearranged as follows:

$$Z \sim \int\left(\prod_{n=N_s}^{N_{\text{dof}}} \mathrm{d}\Delta\tilde{r}_n\right) e^{-\frac{\beta E_0 G_{N_s}^2}{2}} \prod_{p=1}^{N_s-1}\int \mathrm{d}\Delta\tilde{r}_p\, e^{-\frac{\beta E_0 G_p^2}{2}}. \tag{52}$$

The innermost first $N_s - 1$ integrals are Gaussian integrals, each of which evaluates to the constant $Z_p = \sqrt{2\pi/\beta E_0 s_p^2}$. When combined with the thermal wavelength of the nodes, each of the factors Z_p corresponds to the partition sum of a harmonic oscillator with eigen frequency $\omega_p = s_p\sqrt{E_0/m}$ with m being the mass of the nodes. This means that each of the nonzero eigen modes of the Hessian at the transition point just creates a harmonic oscillator, like for isostatic and over-constrained systems. However, for under-constrained systems, we additionally have the last integral, which stems from the SSS that forms at the transition:

$$Z_{N_s} := \int\left(\prod_{n=N_s}^{N_{\text{dof}}} \mathrm{d}\Delta\tilde{r}_n\right) e^{-\frac{\beta E_0 G_{N_s}^2}{2}}. \tag{53}$$

Neglecting the factors Z_p with $p < N_s$, which solely depend on temperature, but not on strain, we have $Z \sim Z_{N_s}$.

B. Saddle point approximation

To simplify the remaining integral Z_{N_s} in Eq. (53), we introduce in its integrand an additional integral over some variable ε_S that trivially evaluates to one:

$$Z_{N_s} = \int \left(\prod_{n=N_s}^{N_{\rm dof}} \mathrm{d}\Delta\tilde{r}_n \right) e^{-\frac{\beta E_0 G_{N_s}^2}{2}} \times \tilde{w}_{N_s} \int_{-\infty}^{\infty} \mathrm{d}\varepsilon_S \, \delta\Big(G_{N_s} + \tilde{w}_{N_s}[\varepsilon_S - \varepsilon] \Big). \tag{54}$$

After exchanging the order of integration and substituting Ω_{N_s}, defined in Eq. (34), we have:

$$Z_{N_s} = \tilde{w}_{N_s} \int_{-\infty}^{\infty} \mathrm{d}\varepsilon_S \, \Omega_{N_s}(\varepsilon_S, \gamma) e^{-\beta E_{N_s}(\varepsilon - \varepsilon_S)}, \tag{55}$$

with:

$$E_{N_s}(\varepsilon - \varepsilon_S) := \frac{E_0 \tilde{w}_{N_s}^2}{2} \big(\varepsilon - \varepsilon_S\big)^2 = \frac{DV^* \kappa_E}{2} \big(\varepsilon - \varepsilon_S\big)^2. \tag{56}$$

Here, we used the definition for κ_E in Eq. (27). From Eq. (43) in the previous section, we have for $\varepsilon_S + b_\varepsilon \gamma^2 \leq 0$:

$$\Omega_{N_s}(\varepsilon_S, \gamma) \sim \Big[-(\varepsilon_S + b_\varepsilon \gamma^2) \Big]^{(N_e-2)/2}, \tag{57}$$

while for $\varepsilon_S + b_\varepsilon \gamma^2 > 0$, no configurations are possible, $\Omega_{N_s}(\varepsilon_S, \gamma) = 0$. These transformations identify ε_S as the entropic strain discussed in the companion paper [26].

We use a saddle point approximation to simplify the integral in Eq. (55):

$$Z_{N_s} = \tilde{w}_{N_s} \int_{-\infty}^{\infty} \mathrm{d}\varepsilon_S \, e^{-\beta \bar{F}}, \tag{58}$$

where $\bar{F}$ is the free energy of the system for imposed ε_S:

$$\bar{F}(\varepsilon_S; \varepsilon, \gamma) := E_{N_s}(\varepsilon - \varepsilon_S) - \frac{1}{\beta} \log \Omega_{N_s}(\varepsilon_S, \gamma) = DV^* \bigg(\frac{\kappa_E}{2} \big(\varepsilon - \varepsilon_S\big)^2 - \kappa_S T \log \Big[-(\varepsilon_S + b_\varepsilon \gamma^2) \Big] \bigg) \tag{59}$$

On the second line, we used the definition of κ_S, Eq. (48), and ignored a constant offset.

For small temperatures, we can apply the saddle point approximation, where we Taylor expand $\bar{F}(\varepsilon_S)$ to second order around its minimum at $\varepsilon_S = \hat{\varepsilon}_S$, which results in:

$$Z_{N_s} \sim \int_{-\infty}^{\infty} \mathrm{d}\varepsilon_S \, e^{-\beta \bar{F}} = e^{-\beta \bar{F}(\hat{\varepsilon}_S)} \sqrt{\frac{2\pi}{\beta \bar{F}''(\hat{\varepsilon}_S)}}, \tag{60}$$

where $\bar{F}'' := \partial^2 F / \partial \varepsilon_S^2$:

$$\bar{F}''(\hat{\varepsilon}_S) = DV^* \bigg(\kappa_E + \frac{\kappa_S T}{[\hat{\varepsilon}_S + b_\varepsilon \gamma^2]^2} \bigg). \tag{61}$$

To find the minimum $\hat{\varepsilon}_S$, we transform $\partial \bar{F} / \partial \varepsilon_S = 0$, and obtain:

$$\hat{\varepsilon}_S + b_\varepsilon \gamma^2 = \frac{\varepsilon + b_\varepsilon \gamma^2}{2} - \sqrt{1+\theta} \left| \frac{\varepsilon + b_\varepsilon \gamma^2}{2} \right|, \tag{62}$$

with

$$\theta := \frac{4 \kappa_S T}{\kappa_E (\varepsilon + b_\varepsilon \gamma^2)^2} = \frac{2(N_e - 2) k_B T}{E_0 \tilde{w}_{N_s}^2 (\varepsilon + b_\varepsilon \gamma^2)^2}. \tag{63}$$

Note that for $\gamma = 0$, the equation $\partial \bar{F}(\hat{\varepsilon}_S) / \partial \varepsilon_S = 0$, defining the minimum $\hat{\varepsilon}_S$ of $\bar{F}(\varepsilon_S) = E(\varepsilon - \varepsilon_S) + F_S(\varepsilon_S)$, corresponds to equating energetic and entropic tension, like in our intuitive explanation in the companion paper [26].

C. Free energy, tension, and shear modulus

Up to terms that depend only on temperature and higher-order terms in the strain variables, the free energy of the system, $F = -(\log Z)/\beta$, is:

$$F = \bar{F}(\hat{\varepsilon}_S) + \frac{1}{2\beta} \log \bar{F}''(\hat{\varepsilon}_S). \tag{64}$$

After some transformations, we find for the isotropic tension $t = (\partial F / \partial \varepsilon)/DV$ and the shear modulus $G = (\partial^2 F / \partial \gamma^2)/V$, to lowest order in ε and γ (appendix F):

$$t = \kappa_E \big(\varepsilon - \hat{\varepsilon}_S\big) \left[1 + \frac{1}{2(N_e - 2)} \frac{\theta}{1+\theta} \right] \tag{65}$$

and

$$G = 2Db_\varepsilon \kappa_E \big(\varepsilon - \hat{\varepsilon}_S\big) \Bigg[1 + \frac{2b_\varepsilon \gamma^2}{|\varepsilon + b_\varepsilon \gamma^2| \sqrt{1+\theta}} + \frac{1}{2(N_e-2)} \frac{\theta}{1+\theta} \bigg(1 + \frac{2b_\varepsilon \gamma^2}{|\varepsilon + b_\varepsilon \gamma^2| \sqrt{1+\theta}} - \frac{4 b_\varepsilon \gamma^2}{(\varepsilon + b_\varepsilon \gamma^2)(1+\theta)} \bigg) \Bigg]. \tag{66}$$

Note that the last terms in both t and G scale as $\sim (N_e - 2)^{-1}$. These terms stem from the term $\sim \log \bar{F}''$

in the free energy. For $N_e \gg 1$, these contributions can be neglected, and we have:

$$t = \kappa_E(\varepsilon - \hat{\varepsilon}_S) \tag{67}$$

$$G = 2Db_\varepsilon\kappa_E(\varepsilon - \hat{\varepsilon}_S)\left[1 + \frac{2b_\varepsilon\gamma^2}{|\varepsilon + b_\varepsilon\gamma^2|\sqrt{1+\theta}}\right] \tag{68}$$

$$= 2Db_\varepsilon t\left[1 + \frac{2b_\varepsilon\gamma^2}{|\varepsilon + b_\varepsilon\gamma^2|\sqrt{1+\theta}}\right]. \tag{69}$$

D. Limits

1. Small temperature, $\theta \ll 1$

For $\theta \ll 1$, Eq. (62) becomes:

$$\hat{\varepsilon}_S + b_\varepsilon\gamma^2 = \frac{\varepsilon + b_\varepsilon\gamma^2}{2} - \left(1 + \frac{\theta}{2}\right)\left|\frac{\varepsilon + b_\varepsilon\gamma^2}{2}\right| = (\varepsilon + b_\varepsilon\gamma^2) \times \begin{cases} 1 + \theta/4 & \text{for } \varepsilon + b_\varepsilon\gamma^2 < 0, \\ -\theta/4 & \text{for } \varepsilon + b_\varepsilon\gamma^2 > 0. \end{cases} \tag{70}$$

Thus, to lowest order in θ:

$$\varepsilon - \hat{\varepsilon}_S = (\varepsilon + b_\varepsilon\gamma^2) \times \begin{cases} -\theta/4 & \text{for } \varepsilon + b_\varepsilon\gamma^2 < 0, \\ 1 & \text{for } \varepsilon + b_\varepsilon\gamma^2 > 0, \end{cases} \tag{71}$$

and we recover the purely entropic and energetic tensions and shear moduli, respectively (compare Eqs. (24), (26), (45), and (47)):

$$t = \begin{cases} t_S & \text{for } \varepsilon + b_\varepsilon\gamma^2 < 0, \\ t_E & \text{for } \varepsilon + b_\varepsilon\gamma^2 > 0, \end{cases} \tag{72}$$

$$G = \begin{cases} G_S & \text{for } \varepsilon + b_\varepsilon\gamma^2 < 0, \\ G_E & \text{for } \varepsilon + b_\varepsilon\gamma^2 > 0. \end{cases} \tag{73}$$

2. Small strain, $\theta \gg 1$

For $\theta \gg 1$, Eq. (62) becomes:

$$\hat{\varepsilon}_S + b_\varepsilon\gamma^2 = -\sqrt{\frac{\kappa_S T}{\kappa_E}} + \frac{\varepsilon + b_\varepsilon\gamma^2}{2} \tag{74}$$

And thus:

$$\varepsilon - \hat{\varepsilon}_S = \sqrt{\frac{\kappa_S T}{\kappa_E}} + \frac{\varepsilon + b_\varepsilon\gamma^2}{2}. \tag{75}$$

We then find:

$$t = \frac{2N_e - 3}{2N_e - 4}\left(\sqrt{\kappa_E\kappa_S T} + \kappa_E\frac{\varepsilon + b_\varepsilon\gamma^2}{2}\right), \tag{76}$$

and for the shear modulus:

$$G = Db_\varepsilon\left[2\sqrt{\kappa_E\kappa_S T} + \kappa_E(\varepsilon + b_\varepsilon\gamma^2)\right] \times \left[\frac{2N_e - 3}{2N_e - 4}\left(1 + b_\varepsilon\gamma^2\sqrt{\frac{\kappa_E}{\kappa_S T}}\right) - \frac{b_\varepsilon\gamma^2}{2N_e - 4}\,\frac{\kappa_E(\varepsilon + b_\varepsilon\gamma^2)}{\kappa_S T}\Big)\right]. \tag{77}$$

For $N_e \gg 1$ both expressions simplify to:

$$t = \sqrt{\kappa_E\kappa_S T} + \kappa_E\frac{\varepsilon + b_\varepsilon\gamma^2}{2} \tag{78}$$

$$G = 2Db_\varepsilon t\left(1 + b_\varepsilon\gamma^2\sqrt{\frac{\kappa_E}{\kappa_S T}}\right). \tag{79}$$

Note that we assumed in our derivation for the general case that $k_B T \ll E_0[\varepsilon + b_\varepsilon\gamma^2]^2$. Using the definitions of θ in Eq. (63), of κ_E in Eq. (27), and of κ_S in Eq. (48), this condition corresponds to $\theta \ll 2(N_e - 2)/\tilde{w}_{N_s}^2$. Thus, this limit corresponds to $1 \ll \theta \ll 2(N_e - 2)/\tilde{w}_{N_s}^2$, which can be fulfilled for large systems. Note that in our numerical results for randomly-cut triangular networks, we also found that our prediction was accurate for finite temperatures even for vanishing strain, $\varepsilon + b_\varepsilon\gamma^2 \to 0$ [26].

VI. DISCUSSION

In this article, we developed a theory for thermal, under-constrained systems with fixed connectivity. We provide analytical expressions for the elastic material properties such as tension t and shear modulus G close to the athermal rigidity transition depending on isotropic strain ε, shear strain γ, and temperature T. The only three parameters are κ_E, which characterizes energetic rigidity, κ_S, which characterizes entropic rigidity, and b_ε, which characterizes the interaction between isotropic and shear strain. These parameters are phenomenological coefficients that depend on the microscopic structure of the system.

We first discussed the limits of zero temperature (athermal limit), and infinitely stiff springs, before discussing the general case of finite temperature and finite spring stiffness. In the athermal limit, the elastic properties were analytically derived before [3, 4], where the system is floppy for $\varepsilon + b_\varepsilon\gamma^2 \leq 0$ and rigid for $\varepsilon + b_\varepsilon\gamma^2 > 0$. We showed that in the stiff-spring limit, the phase space volume is given by the surface of a hyper-ellipsoid, which leads to a free energy that scales logarithmically with the combined strain $\varepsilon + b_\varepsilon\gamma^2$. In both cases, the mechanical properties are inherently linked to the properties of the SSS that is created at the transition. Finally, we derive the partition sum for the general case, where energetic and entropic rigidity are coupled only through the SSS.

We introduced under-constrained system as systems that have $N_{\rm dof} > N_s$, and our results apply to a subset of such systems. In particular, we consider systems that fulfill the following three criteria: (i) there is no SSS in the floppy regime, (ii) only a single SSS forms at the transition, and (iii) $N_e > 2$. Here, N_e is the number of non-zero eigen values of the $\tilde{M}_{N_s mn}$ matrix. In other words, in the athermal limit N_e is the number of eigen values of the Hessian that are zero in the floppy regime and become finite in the rigid regime (appendix C). Lifting the restriction on the number of SSS in the floppy regime probably does not change our results. Similarly, we do not expect the number of SSS that form at the transition to substantially affect our results. Finally, it will be interesting to study the marginal cases of $N_e = 1$ and $N_e = 2$ in future work.

While not explicitly studied here, our results have immediate consequences with respect to the scaling with network connectivity z. For $\gamma = 0$, earlier work by Zhang and Mao using EMT indicated three scaling regimes for the shear modulus G of randomly-cut triangular networks [24]. Remarkably, these scaling regimes are very similar to what we report in section V D, where up to a prefactor, isotropic strain ε is replaced by the relative connectivity $\Delta z = z - z_c$, with z_c being the connectivity at isostaticity [24]. To understand how this could be, we discuss the Δz scaling that results from our findings by separately discussing the Δz scaling for κ_S and κ_E. Eq. (48) predicts that κ_S essentially only depends on N_e, i.e. the number of non-zero eigen values of the matrix $\tilde{M}_{N_s mn}$ with $m, n \geq N_s$. Thus, if for large systems N_e scales to dominating order linearly with $\Delta z = -4(N_{\rm dof} - N_s)/N_{\rm dof}$, then this implies $\kappa_S \sim |\Delta z|^1$. Meanwhile, how energetic elasticity κ_E scales with connectivity Δz depends on the class of the network [4]. For instance, for $|\Delta z| \ll 1$ our results for the a_ℓ scaling in Refs. [3, 4] imply that $\kappa_E \sim |\Delta z|^2$ for Delaunay networks, $\kappa_E \sim |\Delta z|^1$ for randomly-cut, packing-derived networks, and $\kappa_E \sim |\Delta z|^0$ for phantom triangular networks. Because of Eq. (67), these scaling relations of κ_S and κ_E with Δz determine the scaling of tension t. However, for the scaling of the shear modulus G, Eq. (69), the scalings of $b_\varepsilon \kappa_S$ and $b_\varepsilon \kappa_E$ are the relevant ones, and thus the scaling of $b_\varepsilon \sim b$ needs to be taken into account. Based on earlier results for the scaling of b [3, 4], we have $b_\varepsilon \kappa_S \sim |\Delta z|^{-1}$ and $b_\varepsilon \kappa_E \sim |\Delta z|^0$ for Delaunay networks, $b_\varepsilon \kappa_S \sim |\Delta z|^0$ and $b_\varepsilon \kappa_E \sim |\Delta z|^0$ for randomly-cut, packing-derived networks, and $b_\varepsilon \kappa_S \sim |\Delta z|^{0.5}$ and $b_\varepsilon \kappa_E \sim |\Delta z|^{-0.5}$ for phantom triangular networks. Taken together, this suggests that the scaling of the elastic properties with Δz is expected to depend on the class of network studied. For instance, the results by Zhang and Mao [24] can be explained from the perspective of our results if the randomly-cut triangular networks have $b_\varepsilon \kappa_S \sim |\Delta z|^0$ and $b_\varepsilon \kappa_E \sim |\Delta z|^0$ like randomly-cut, packing-derived networks. In this case, the scaling is entirely determined by the scaling of the strain of the network state after network initialization, $\varepsilon_{\rm ini}$, with connectivity Δz, which is generally linear to lowest order: $\varepsilon_{\rm ini} \sim \Delta z$ (remember that we define isotropic strain here with respect to the athermal transition point) [3, 4]. This would be a simple explanation for why the shear modulus of Zhang and Mao has the same form as our results when replacing ε by Δz. We leave a more thorough numerical test of all these predicitons for the scaling with Δz, and a study of their consequences, for future work.

In general, there can be a degeneracy of the possible network configurations $\vec{r}^*$ at the transition, and here we have assumed that this does not affect our result. To this end, we needed to explicitly restrict ourselves to systems where the choice of the transition configuration $\vec{r}^*$ does not affect the properties of the SSS. In particular, while properties like the other singular values, s_p with $p < N_s$, of the compatibility matrix C_{in} may depend on $\vec{r}^*$ [6, 30], our results should be unaffected as long as the SSS (more precisely $\tilde{w}_{N_s}$) and its behavior with respect to shear (more precisely b_ε) are independent of the choice of $\vec{r}^*$. While this seems to be true intuitively, preliminary results suggest that stronger assumptions about the system than those listed in section II A are required to rigorously show it. The tools required for such a proof are outside the scope of the ideas presented here, and their development is left for future work.

Previous work discussing the athermal limit was based on a minimal-length function [3, 4], where a Taylor expansion with a quadratic dependence on γ relied on the assumption that the minimal-length function was analytic in γ. With the approach here we now that the dependence on γ is analytic, because we know that the functions $\ell_i(\vec{r}, \gamma)$ are analytic. However, it is in principle still possible that higher-order terms could create isolated discontinuities in the emergent behavior of the system.

Finally, we note that all results obtained here are based on Taylor expansions for systems of a given finite size. As discussed in Ref. [4], the strain ranges over which these predictions hold may depend on system size. In particular, depending on the type of network studied, the extent of these ranges may decrease with increasing system size, and beyond this strain range, different scaling regimes may appear [31].

ACKNOWLEDGMENTS

We thank Chris Santangelo, Jen Schwarz, and Manu Mannattil for fruitful discussions. We thank the Centre Interdisciplinaire de Nanoscience de Marseille (CINaM) for providing office space. The project leading to this publication has received funding from France 2030, the French Government program managed by the French National Research Agency (ANR-16-CONV-0001), and from the Excellence Initiative of Aix-Marseille University - A*MIDEX. 1

Appendix A: Orders of magnitude of the $\Delta\tilde{r}_n$

We discuss the scaling of the $\Delta\tilde{r}_n$ with ε and γ in the athermal limit at the energy minimum, for which we will also have to discuss the scaling of the G_p. We focus here for clarity on the case where $\tilde{B}_p^{(1)} = 0$ for all p. The general case of arbitrary $\tilde{B}_p^{(1)}$ is discussed in appendix B.

Using Eq. (15), the energy minimum must fulfill:

$$0 = \frac{1}{E_0}\frac{\partial E}{\partial \Delta\tilde{r}_p} = s_p G_p + G_q \tilde{M}_{qpn}\Delta\tilde{r}_n \quad \text{for } p < N_s, \tag{A1}$$

$$0 = \frac{1}{E_0}\frac{\partial E}{\partial \Delta\tilde{r}_m} = G_q \tilde{M}_{qmn}\Delta\tilde{r}_n \qquad \text{for } m \geq N_s. \tag{A2}$$

At the transition all $\Delta\tilde{r}_n = 0$, and so they must be of higher than absolute order in ε or γ. Moreover, let us choose $\hat{p} < N_s$ such that none of the of G_p with $p < N_s$ is of lower order in ε than $G_{\hat{p}}$. Because of Eq. (A1), the order of $G_{\hat{p}}$ in ε is higher than that of the minimum order of all G_q with $q \leq N_s$. Because the order of $G_{\hat{p}}$ is minimal among the G_p with $p < N_s$, this implies that the order of $G_{\hat{p}}$ is higher than that of G_{N_s}. According to Eq. (18), G_{N_s} is of order $\mathcal{O}(\varepsilon)$ or higher [32]. Thus, because the order of $G_{\hat{p}}$ is minimal, all G_p with $p < N_s$ are of higher order than $\mathcal{O}(\varepsilon)$. Analogously, one can argue that all G_p with $p < N_s$ are of higher order than $\mathcal{O}(\gamma^2)$. Thus, up to order $[\mathcal{O}(\varepsilon) + \mathcal{O}(\gamma^2)]^2$, the energy in Eq. (15) becomes $E = E_0 G_{N_s}^2/2$.

We know that for $\gamma = 0$ and for positive ε, where the system is in the rigid regime, the minimum of the energy is strictly positive, $E > 0$. Using Eq. (18), this is only possible if the matrix $\tilde{M}_{N_s mn}$ is positive semi-definite. Minimization of the energy $E = E_0 G_{N_s}^2/2$ implies that $G_{N_s} = 0$ or else $G_{N_s} = 0$ is minimal. As a consequence, using Eq. (18), none of the $\Delta\tilde{r}_n$ with $1 \leq n \leq N_{\text{dof}}$ can be of lower order than $\mathcal{O}(|\varepsilon|^{1/2}) + \mathcal{O}(\gamma)$. Thus, because up to order $\mathcal{O}(\varepsilon) + \mathcal{O}(\gamma^2)$ we have $G_p = 0$ for $p < N_s$, Eq. (16) implies that none of the $\Delta\tilde{r}_p$ with $p < N_s$ is of lower order than $\mathcal{O}(\varepsilon) + \mathcal{O}(\gamma^2)$. Further, this implies that any terms involving the $\Delta\tilde{r}_p$ with $p < N_s$ can be neglected in the G_p with $p \leq N_s$. Thus, each of the $\Delta\tilde{r}_p$ with $p < N_s$ appears in total only once, namely as linear term in the respective G_p. As a consequence, each of the equations $G_p = 0$ for $p < N_s$ can be fulfilled by adjusting $\Delta\tilde{r}_p$ accordingly.

Appendix B: Non-zero first-order terms, $\tilde{B}_p^{(1)}$, in γ

To discuss the general case of non-vanishing $\tilde{B}_p^{(1)}$, we first simplify by applying shifts to both ε and γ,

$$\bar{\varepsilon} := \varepsilon - \Delta\varepsilon \tag{B1}$$

$$\bar{\gamma} := \gamma - \Delta\gamma \tag{B2}$$

such that in the athermal limit $(\bar{\varepsilon}, \bar{\gamma}) = (0, 0)$ is on the transition line and, after again carrying out the Taylor expansion, Eq. (13), the linear-order $\bar{\gamma}$ term in G_{N_s} vanishes at the energy minimum:

$$G_{N_s} = \frac{1}{2}\tilde{M}_{N_s mn}\Delta\tilde{r}_m\Delta\tilde{r}_n + \tilde{w}_{N_s}\bar{\varepsilon} + \frac{1}{2}B_{N_s}^{(2)}\bar{\gamma}^2. \tag{B3}$$

Note that the G_p with $p < N_s$ do in general still have a linear-order term in $\bar{\gamma}$:

$$G_p = s_p\Delta\tilde{r}_p + \frac{1}{2}\tilde{M}_{pmn}\Delta\tilde{r}_m\Delta\tilde{r}_n + \tilde{w}_p\hat{\varepsilon} + B_p^{(1)}\bar{\gamma} + \frac{1}{2}B_p^{(2)}\bar{\gamma}^2, \tag{B4}$$

and that the $\Delta\tilde{r}_n$ are also redefined due to the shifts in ε and γ. If $\Delta\varepsilon$ and $\Delta\gamma$ are sufficiently small such that to lowest order C, M, and $B^{(2)}$ in Eq. (9) are unaffected, it can be shown that the shifts are given by $\Delta\varepsilon = (\tilde{B}_{N_s}^{(1)})^2/4\tilde{w}_{N_s}^2 b_\varepsilon$ and $\Delta\gamma = -\tilde{B}_{N_s}^{(1)}/2\tilde{w}_{N_s} b_\varepsilon$, where $\tilde{B}_{N_s}^{(1)}$ is the linear-order term before the shift, and b_ε is given by Eq. (B8) below.

We first discuss the athermal limit. Using the arguments from appendix A, one finds again that at the energy minimum, all $G_p = 0$ for $p < N_s$ up to order $\mathcal{O}(\bar{\varepsilon}) + \mathcal{O}(\bar{\gamma}^2)$, and none of the $\Delta\tilde{r}_n$ with $1 \leq n \leq N_{\text{dof}}$ can be of lower order than $\mathcal{O}(|\bar{\varepsilon}|^{1/2}) + \mathcal{O}(\bar{\gamma})$. As a consequence, *up to linear order in* $\bar{\gamma}$, we have from Eq. (B4):

$$\Delta\tilde{r}_p = -\frac{\tilde{B}_p^{(1)}}{s_p}\bar{\gamma} \qquad \text{for } p < N_s. \tag{B5}$$

Inserting this into Eq. (B3) together with the diagonalization of the matrix $\tilde{M}_{N_s mn}$ with $m, n \geq N_s$, Eq. (35), we obtain:

$$G_{N_s} = \frac{1}{2}\sum_{k=1}^{N_{\text{dof}}-N_s+1}\mu_k\Big(\Delta q_k - \Delta q_k^0\Big)^2 + \tilde{w}_{N_s}\big[\bar{\varepsilon} + b_\varepsilon\bar{\gamma}^2\big], \tag{B6}$$

where we introduced

$$\Delta q_k^0 := \frac{\bar{\gamma}}{\mu_k}\sum_{\substack{m\geq N_s\\ p<N_s}} v_m^k \tilde{M}_{N_s mp}\frac{\tilde{B}_p^{(1)}}{s_p} \qquad \text{for } \mu_k > 0, \tag{B7}$$

and $\Delta q_k^0 := 0$ for $\mu_k = 0$, and

$$b_\varepsilon := \frac{1}{2\tilde{w}_{N_s}}\Bigg[\tilde{B}_{N_s}^{(2)} + \sum_{p,q<N_s}\frac{\tilde{B}_p^{(1)}}{s_p}\tilde{M}_{N_s pq}\frac{\tilde{B}_q^{(1)}}{s_q} - \sum_{\substack{p,q<N_s\\ m,n\geq N_s\\ k:\mu_k>0}}\frac{\tilde{B}_p^{(1)}}{s_p}\tilde{M}_{N_s pm}\frac{v_m^k v_n^k}{\mu_k}\tilde{M}_{N_s nq}\frac{\tilde{B}_q^{(1)}}{s_q}\Bigg]. \tag{B8}$$

We assumed here that for all k with $\mu_k = 0$ the relation $\sum_{n\geq N_s, p<N_s} v_n^k \tilde{M}_{N_s np} \tilde{B}_p^{(1)}/s_p = 0$. If the second-order zero mode k stems from a mechanism or a trivial zero mode (i.e. linked to a global translation), this relation is necessarily true; otherwise there would be no energy minimum. If the mode k is neither mechanism nor trivial zero mode, but a higher-than-second-order zero mode, then higher-order terms need to be taken into account. However, we regard this as an exceptional case.

The additional terms in b_ε in Eq. (B8) as compared to Eq. (20) scale as $\sim \tilde{B}_p^{(1)} \tilde{B}_q^{(1)}$. The symmetry of the system with respect to the sign of γ combined with the central limit theorem suggest that, in general, these linear terms scale with system size as $\tilde{B}_p^{(1)} \sim N_s^{-1/2}$. Indeed, we numerically found earlier for phanomized triangular networks that $\tilde{B}_{N_s}^{(1)} \sim \Delta\gamma \sim N_{\text{dof}}^{-1/2} \sim N_s^{-1/2}$ [4]. We thus expect that the additional contributions in b_ε scale as N_s^{-1}.

For the stiff-spring limit and the general case of finite spring stiffness and temperature, these arguments are essentially the same. The only difference in the stiff-spring limit is that we immediately have $G_p = 0$. Moreover, for non-vanishing $\tilde{B}_p^{(1)}$, the ellipsoid that describes the possible system configurations is now centered at (Δq_k^0) (compare Eqs. (36) and (B6)). According to Eq. (B7) this ellipsoid center displaces linearly with shear strain $\bar{\gamma}$. In the general case, we can follow the same arguments as in the athermal limit, using that the terms where the G_p with $p < N_s$ are of order $\mathcal{O}(\bar{\varepsilon}) + \mathcal{O}(\bar{\gamma}^2)$ or lower have an exponentially reduced Boltzmann weight and can thus be neglected in the partition sum.

In our Taylor expansion, Eq. (9), we have left out terms that are linear in both $\Delta\tilde{r}_n$ and γ. We expect such terms to also scale with system size as $\sim N_s^{-1/2}$. They create additional contributions to Δq_k^0 and, in interaction with the $\tilde{B}_p^{(1)}$ terms, contributions to b_ε. However, other than adding a correction to the value of b_ε they do not affect the results presented in the main text.

Appendix C: Relation between the Hessian and the matrix $\tilde{M}_{N_s mn}$

We evaluate the Hessian of the system,

$$H_{mn} := \frac{\partial^2 E}{\partial\Delta\tilde{r}_m \partial\Delta\tilde{r}_n}, \tag{C1}$$

in the rigid athermal regime, at an energy minimum close to the transition point, where we assume for simplicity that $\tilde{B}_p^{(1)} = 0$ for all $p \leq N_s$.

Using Eq. (15):

$$H_{mn} = E_0 \sum_{p=1}^{N_s} \left(\frac{\partial G_p}{\partial\Delta\tilde{r}_m} \frac{\partial G_p}{\partial\Delta\tilde{r}_n} + G_p \frac{\partial^2 G_p}{\partial\Delta\tilde{r}_m \partial\Delta\tilde{r}_n} \right). \tag{C2}$$

Further, according to our discussion in section III A, at an energy minimum, $\Delta\tilde{r}_p$ is at least of order $\mathcal{O}(\varepsilon)+\mathcal{O}(\gamma^2)$ for $p < N_s$, and $\Delta\tilde{r}_n = 0$ for $n \geq N_s$. Also, $G_p = 0$ for $p < N_s$ up to order $\mathcal{O}(\varepsilon) + \mathcal{O}(\gamma^2)$. Thus, we have up to order $\mathcal{O}(\varepsilon) + \mathcal{O}(\gamma^2)$:

$$\begin{aligned} H_{mn} = E_0 &\sum_{p=1}^{N_s-1} s_p^2 \delta_{pm}\delta_{pn} \\ &+ E_0 \sum_{p,q=1}^{N_s-1} s_p \Delta\tilde{r}_q \Big(\delta_{pm}\tilde{M}_{pqn} + \delta_{pn}\tilde{M}_{pqm} \Big) \\ &+ E_0 G_{N_s} \tilde{M}_{N_s mn}. \end{aligned} \tag{C3}$$

In this expression, the first term, which is of absolute order in ε and γ, corresponds to the Hessian at the transition point. The second and third term are of order $\mathcal{O}(\varepsilon) + \mathcal{O}(\gamma^2)$.

Eq. (C3) shows that to lowest order, the N_e nonzero eigen modes of $\tilde{M}_{N_s mn}$ for $m, n \geq N_s$, are those eigen modes of H_{mn} that have zero eigen value at the transition point, and whose eigen values become nonzero in the rigid regime, scaling as $\sim \mathcal{O}(\varepsilon) + \mathcal{O}(\gamma^2)$. This is because for $m, n \geq N_s$ only the third term in Eq. (C3) can be nonzero.

Appendix D: Partition sum for stiff-spring limit

In Eq. (29), to define the configurational phase space volume Ω, we use a Dirac delta for each of the N_s spring lengths in the system. One could instead have the idea to use a single Dirac delta for the system energy E, since $E = 0$ is equivalent to all springs simultaneously attaining their rest length. However, both definitions will lead to different results. This is because the integration of a Dirac delta involves the evaluation of the Jacobian of the argument of the Dirac delta with respect to the integration variable. Since the Jacobian of the energy creates additional prefactors that correspond to the prestresses in the system, such prefactors would additionally appear when using a Dirac delta of the energy.

To see why the definition Eq. (29) is the correct one in the stiff-spring limit, we start form the partition sum:

$$Z = \int \left(\prod_{a=1}^{N_{\text{node}}} \mathrm{d}^D \boldsymbol{R}_a \right) \exp\left(-\frac{\beta}{2} \sum_{i=1}^{N_s} K_i \big(L_i - L_{0i}\big)^2 \right). \tag{D1}$$

A Gaussian converges to a Dirac Delta in the limit of small standard deviation:

$$\lim_{K_i\to\infty} \sqrt{\frac{\beta K_i}{2\pi}} \exp\left(-\frac{\beta K_i (L_i - L_{0i})^2}{2} \right) = \delta(L_i - L_{0i}). \tag{D2}$$

Thus, whenever all K_i are large enough such that the $(\beta K_i)^{-1/2}$ becomes much smaller than all relevant length scales of the system, the partition sum can be expressed

as:

$$Z = \left(\frac{2\pi}{\beta}\right)^{N_s/2} \left(\prod_{i=1}^{N_s} K_i\right)^{-1/2} \times \int \left(\prod_{a=1}^{N_{\text{node}}} \mathrm{d}^D \boldsymbol{R}_a\right) \prod_{i=1}^{N_s} \delta(L_i - L_{0i}). \tag{D3}$$

This indicates that the correct approach is to use Dirac deltas on the spring lengths rather than the energy E. Note that in Eq. (29), we have ignored the first prefactor that only depends on temperature, and so is not relevant for the elastic system properties. Moreover, while the second prefactor does not depend on strain or temperature, it converges to zero for $K_i \to \infty$. However, to discuss the elastic system properties, we only need derivatives of $\log \Omega$, where this term only creates a constant albeit diverging offset.

Appendix E: Higher-order dependencies of $\Delta\ell_{N_s}$ on the dofs

Here we outline how our arguments change in the presence of eigen vectors with eigen value zero of the matrix $\tilde{M}_{N_s mn}$ with $m, n \geq N_s$, but which affect $\Delta\tilde{\ell}_{N_s}$ to some order $\alpha > 2$. We first note that the exponent α needs to be even, because otherwise, such a mode would allow for states with $G_{N_s} = 0$ for $\varepsilon > 0$ and $\gamma = 0$, even though the system is supposed ot be rigid with $E > 0$ in this regime.

Setting again for simplicity $\tilde{B}_p^{(1)} = 0$, we discuss the following generalization of the expansion of $\Delta\tilde{\ell}_{N_s}$ from Eqs. (13), (36):

$$\Delta\tilde{\ell}_{N_s} = \sum_\alpha \frac{1}{\alpha!} \sum_{k=1}^{N_e^{(\alpha)}} \mu_{\alpha,k} \Delta q_{\alpha,k}^\alpha + \frac{1}{2} \tilde{B}_{N_s}^{(2)} \gamma^2, \tag{E1}$$

where the sum over α is over all positive even integers, $\alpha!$ denotes the faculty of α, the variable $N_e^{(\alpha)}$ denotes the number of modes $\Delta q_{\alpha,k}$ that enter $\Delta\tilde{\ell}_{N_s}$ to lowest order as $\sim \Delta q_{\alpha,k}^\alpha$, and $\mu_{\alpha,k}$ are the associated coefficients.

Thus, the phase space volume is governed by an integral of the following kind:

$$\Omega_{N_s} \sim I(\varepsilon, \gamma) := \int \left(\prod_\alpha \prod_{k=1}^{N_e^{(\alpha)}} \mathrm{d}\Delta q_{\alpha,k}\right) \times \delta\left(\sum_\alpha \frac{1}{\alpha!} \sum_{k=1}^{N_e^{(\alpha)}} \mu_{\alpha,k} \Delta q_{\alpha,k}^\alpha + \tilde{w}_{N_s}\left[\varepsilon + b_\varepsilon \gamma^2\right]\right), \tag{E2}$$

To extract the dependency of I on ε and γ, we first define:

$$I_1 := \int \left(\prod_\alpha \prod_{k=1}^{N_e^{(\alpha)}} \mathrm{d}x_{\alpha,k}\right) \times \delta\left(\sum_\alpha \frac{1}{\alpha!} \sum_{k=1}^{N_e^{(\alpha)}} \mu_{\alpha,k} x_{\alpha,k}^\alpha + \tilde{w}_{N_s}\right), \tag{E3}$$

which depends neither on ε nor on γ. Now we can transform the expression in Eq. (E2) using the following substitutions for all integration variables:

$$\Delta q_{\alpha,k} = \left[\varepsilon + b_\varepsilon \gamma^2\right]^{1/\alpha} x_{\alpha,k}, \tag{E4}$$

which yields:

$$\Omega_{N_s}(\varepsilon, \gamma) \sim I(\varepsilon, \gamma) = \left[\varepsilon + b_\varepsilon \gamma^2\right]^{N_\Omega} I_1 \tag{E5}$$

with:

$$N_\Omega = -1 + \sum_\alpha \frac{N_e^{(\alpha)}}{\alpha}. \tag{E6}$$

The first term in N_Ω arises from removing the factor $[\varepsilon + b_\varepsilon \gamma^2]$ from the Dirac delta, and all the remaining terms arise from the substitution of the integration variables. Note that for the case discussed in the main text, i.e. for $N_e := N_e^{(2)}$ and $N_e^{(\alpha)} = 0$ for $\alpha > 2$, we have indeed $N_\Omega = (N_e - 2)/2$. Hence, also higher-order dependencies of $\Delta\tilde{\ell}_{N_s}$ on the dofs also contribute to the exponent N_Ω, and thus to $\kappa_S = N_\Omega k_B / DV^*$; they are just attenuated according to their order α.

In this argument, we have excluded modes that leave $\Delta\tilde{\ell}_{N_s}$ invariant. They do not contribute to N_Ω. This is because any substitution like in Eq. (E4), while creating a prefactor, also rescales the integral bounds by the same prefactor, both of which exactly cancel each other out. Meanwhile, for any mode that *does* affect $\Delta\tilde{\ell}_{N_s}$, the rescaling of the integral bounds does not matter, since the integration domain of these modes is effectively limited by the Dirac delta, not the integral bounds.

Appendix F: Tension and shear modulus in the general case

To obtain the isotropic tension $t := (\partial F / \partial \varepsilon)/DV$, we first use the chain rule. To lowest order, $V = V^*$, and so we obtain from Eq. (64):

$$t = \frac{1}{DV^*} \left[\frac{\partial \bar{F}}{\partial \hat{\varepsilon}_S} \frac{\partial \hat{\varepsilon}_S}{\partial \varepsilon} + \frac{\partial \bar{F}}{\partial \varepsilon} + \frac{1}{2\beta \bar{F}''} \left(\frac{\partial \bar{F}''}{\partial \hat{\varepsilon}_S} \frac{\partial \hat{\varepsilon}_S}{\partial \varepsilon} + \frac{\partial \bar{F}''}{\partial \varepsilon}\right)\right]. \tag{F1}$$

We compute to lowest order:

$$\frac{\partial \hat{\varepsilon}_S}{\partial \varepsilon} = -\frac{\hat{\varepsilon}_S + b_\varepsilon \gamma^2}{|\varepsilon + b_\varepsilon \gamma^2| \sqrt{1+\theta}} \tag{F2}$$

$$\frac{1}{DV^*}\frac{\partial \bar{F}}{\partial \hat{\varepsilon}_S} = -\kappa_E(\varepsilon - \hat{\varepsilon}_S) - \frac{\kappa_S T}{\hat{\varepsilon}_S + b_\varepsilon \gamma^2} = 0 \tag{F3}$$

$$\frac{1}{DV^*}\frac{\partial \bar{F}}{\partial \varepsilon} = \kappa_E(\varepsilon - \hat{\varepsilon}_S) \tag{F4}$$

$$\frac{1}{DV^*}\frac{\partial \bar{F}''}{\partial \hat{\varepsilon}_S} = -\frac{2\kappa_S T}{(\hat{\varepsilon}_S + b_\varepsilon \gamma^2)^3} \tag{F5}$$

$$\frac{1}{DV^*}\frac{\partial \bar{F}''}{\partial \varepsilon} = 0. \tag{F6}$$

We thus obtain:

$$t = \kappa_E(\varepsilon - \hat{\varepsilon}_S)\left[1 + \frac{1}{2(N_e - 2)}\frac{\theta}{1+\theta}\right], \tag{F7}$$

where we have also used that

$$\bar{F}'' = -DV^*\kappa_E \frac{\left|\varepsilon + b_\varepsilon \gamma^2\right| \sqrt{1+\theta}}{\hat{\varepsilon}_S + b_\varepsilon \gamma^2} \tag{F8}$$

and

$$\kappa_E(\varepsilon - \hat{\varepsilon}_S) = -\frac{\kappa_S T}{\hat{\varepsilon}_S + b_\varepsilon \gamma^2}, \tag{F9}$$

which follows from Eq.(62). The term $\sim (N_e - 2)^{-1}$ in Eq. (F7) corresponds to the terms involving $\bar{F}''$.

For the shear modulus $G := (\partial^2 F/\partial \gamma^2)/V$, we have to lowest order:

$$G = \frac{1}{V^*}\left[\frac{\partial^2 \bar{F}(\hat{\varepsilon}_S(\varepsilon,\gamma);\varepsilon,\gamma)}{\partial \gamma^2} + \frac{1}{2\beta}\frac{\partial^2}{\partial \gamma^2}\log \bar{F}''(\hat{\varepsilon}_S(\varepsilon,\gamma);\varepsilon,\gamma)\right]. \tag{F10}$$

We compute with Eqs. (F3) and (F9):

$$\begin{aligned}\frac{1}{V^*}\frac{\partial \bar{F}(\hat{\varepsilon}_S(\varepsilon,\gamma);\varepsilon,\gamma)}{\partial \gamma} &= \frac{1}{V^*}\frac{\partial \bar{F}(\hat{\varepsilon}_S;\varepsilon,\gamma)}{\partial \gamma} \\ &= -\frac{2D\kappa_S T b_\varepsilon \gamma}{\hat{\varepsilon}_S + b_\varepsilon \gamma^2} \\ &= 2D\kappa_E(\varepsilon - \hat{\varepsilon}_S) b_\varepsilon \gamma,\end{aligned} \tag{F11}$$

and further:

$$\begin{aligned}&\frac{1}{V^*}\frac{\partial^2 \bar{F}(\hat{\varepsilon}_S(\varepsilon,\gamma);\varepsilon,\gamma)}{\partial \gamma^2} \\ &= \frac{1}{V^*}\left[\frac{\partial^2 \bar{F}(\hat{\varepsilon}_S;\varepsilon,\gamma)}{\partial \gamma \partial \hat{\varepsilon}_S}\frac{\partial \hat{\varepsilon}_S}{\partial \gamma} + \frac{\partial^2 \bar{F}(\hat{\varepsilon}_S;\varepsilon,\gamma)}{\partial \gamma^2}\right]\end{aligned} \tag{F12}$$

with

$$\frac{1}{V^*}\frac{\partial^2 \bar{F}(\hat{\varepsilon}_S;\varepsilon,\gamma)}{\partial \gamma \partial \hat{\varepsilon}_S} = -2D\kappa_E b_\varepsilon \gamma \tag{F13}$$

$$\frac{\partial \hat{\varepsilon}_S}{\partial \gamma} = -b_\varepsilon \gamma\left(1 + \frac{\varepsilon + b_\varepsilon \gamma^2}{|\varepsilon + b_\varepsilon \gamma^2|\sqrt{1+\theta}}\right) \tag{F14}$$

$$= -2b_\varepsilon \gamma \frac{\varepsilon - \hat{\varepsilon}_S}{|\varepsilon + b_\varepsilon \gamma^2|\sqrt{1+\theta}} \tag{F15}$$

$$\frac{1}{V^*}\frac{\partial^2 \bar{F}(\hat{\varepsilon}_S;\varepsilon,\gamma)}{\partial \gamma^2} = 2D\kappa_E(\varepsilon - \hat{\varepsilon}_S) b_\varepsilon. \tag{F16}$$

Thus:

$$\begin{aligned}&\frac{1}{V^*}\frac{\partial^2 \bar{F}(\hat{\varepsilon}_S(\varepsilon,\gamma);\varepsilon,\gamma)}{\partial \gamma^2} \\ &= 2D\kappa_E(\varepsilon - \hat{\varepsilon}_S) b_\varepsilon\left(1 + \frac{2b_\varepsilon \gamma^2}{|\varepsilon + b_\varepsilon \gamma^2|\sqrt{1+\theta}}\right).\end{aligned} \tag{F17}$$

Furthermore:

$$\begin{aligned}&\frac{\partial \bar{F}''(\hat{\varepsilon}_S(\varepsilon,\gamma);\varepsilon,\gamma)}{\partial \gamma} \\ &= \frac{\partial \bar{F}''(\hat{\varepsilon}_S;\varepsilon,\gamma)}{\partial \hat{\varepsilon}_S}\frac{\partial \hat{\varepsilon}_S}{\partial \gamma} + \frac{\partial \bar{F}''(\hat{\varepsilon}_S;\varepsilon,\gamma)}{\partial \gamma} \\ &= DV^*\kappa_E\left[\left(\frac{1}{\hat{\varepsilon}_S + b_\varepsilon \gamma^2} - \frac{\hat{\varepsilon}_S - \varepsilon}{(\hat{\varepsilon}_S + b_\varepsilon \gamma^2)^2}\right)\right. \\ &\qquad \times\left(-2b_\varepsilon \gamma \frac{\varepsilon - \hat{\varepsilon}_S}{|\varepsilon + b_\varepsilon \gamma^2|\sqrt{1+\theta}}\right) \\ &\qquad \left. - 2b_\varepsilon \gamma \frac{\hat{\varepsilon}_S - \varepsilon}{(\hat{\varepsilon}_S + b_\varepsilon \gamma^2)^2}\right] \\ &= -2DV^*\kappa_E b_\varepsilon \gamma \frac{\varepsilon - \hat{\varepsilon}_S}{(\hat{\varepsilon}_S + b_\varepsilon \gamma^2)^2 |\varepsilon + b_\varepsilon \gamma^2|\sqrt{1+\theta}} \\ &\quad \times\left[\varepsilon + b_\varepsilon \gamma^2 - |\varepsilon + b_\varepsilon \gamma^2|\sqrt{1+\theta}\right] \\ &= -4DV^*\kappa_E b_\varepsilon \gamma \frac{\varepsilon - \hat{\varepsilon}_S}{(\hat{\varepsilon}_S + b_\varepsilon \gamma^2)|\varepsilon + b_\varepsilon \gamma^2|\sqrt{1+\theta}},\end{aligned} \tag{F18}$$

and thus:

$$\frac{\partial}{\partial \gamma}\log \bar{F}''(\hat{\varepsilon}_S(\varepsilon,\gamma);\varepsilon,\gamma) = 4b_\varepsilon \gamma \frac{\varepsilon - \hat{\varepsilon}_S}{(\varepsilon + b_\varepsilon \gamma^2)^2(1+\theta)}. \tag{F19}$$

The second derivative is:

$$\begin{aligned}&\frac{\partial^2}{\partial \gamma^2}\log \bar{F}''(\hat{\varepsilon}_S(\varepsilon,\gamma);\varepsilon,\gamma) \\ &= \frac{4b_\varepsilon(\varepsilon - \hat{\varepsilon}_S)}{(\varepsilon + b_\varepsilon \gamma^2)^2(1+\theta)} \\ &\quad + \frac{4b_\varepsilon \gamma}{(\varepsilon + b_\varepsilon \gamma^2)^2(1+\theta)} \times 2b_\varepsilon \gamma \frac{\varepsilon - \hat{\varepsilon}_S}{|\varepsilon + b_\varepsilon \gamma^2|\sqrt{1+\theta}} \\ &\quad - \frac{16b_\varepsilon^2 \gamma^2(\varepsilon - \hat{\varepsilon}_S)}{(\varepsilon + b_\varepsilon \gamma^2)^3(1+\theta)^2} \\ &= \frac{4b_\varepsilon(\varepsilon - \hat{\varepsilon}_S)}{(\varepsilon + b_\varepsilon \gamma^2)^2(1+\theta)}\left[1 + \frac{2b_\varepsilon \gamma^2}{|\varepsilon + b_\varepsilon \gamma^2|\sqrt{1+\theta}}\right.\end{aligned}$$

$$- \frac{4b_\varepsilon\gamma^2}{(\varepsilon + b_\varepsilon\gamma^2)(1+\theta)}\Bigg]. \tag{F20}$$

Combining Eqs. (F10), (F17), (F20), and (F21), we obtain Eq. (66) in the main text.

From Eqs. (48) and (F9) follows:

$$\frac{1}{2\beta V^*} = \frac{D\kappa_S T}{N_e - 2} = \frac{D\kappa_E}{N_e - 2}\,\theta\,\frac{(\varepsilon + b_\varepsilon\gamma^2)^2}{4} \tag{F21}$$

[1] J. C. Maxwell, Philosophical Magazine Series 4 **27**, 294 (1864).
[2] C. R. Calladine, International Journal of Solids and Structures **14**, 161 (1978).
[3] M. Merkel, K. Baumgarten, B. P. Tighe, and M. L. Manning, Proceedings of the National Academy of Sciences **116**, 6560 (2019).
[4] C.-T. Lee and M. Merkel, Soft Matter **18**, 5410 (2022).
[5] T. C. Lubensky, C. L. Kane, X. Mao, A. Souslov, and K. Sun, Reports on Progress in Physics **78**, 73901 (2015), arxiv:1503.01324.
[6] M. Mannattil, J. M. Schwarz, and C. D. Santangelo, Physical Review Letters **128**, 208005 (2022), arxiv:2112.04279.
[7] O. K. Damavandi, V. F. Hagh, C. D. Santangelo, and M. L. Manning, Physical Review E **105**, 025003 (2022).
[8] O. K. Damavandi, V. F. Hagh, C. D. Santangelo, and M. L. Manning, Physical Review E **105**, 025004 (2022).
[9] S. Alexander, Physics Report **296**, 65 (1998), arxiv:1512.04885.
[10] B. Cui, G. Ruocco, and A. Zaccone, Granular Matter **21**, 69 (2019), arxiv:1901.09582.
[11] S. Zhang, E. Stanifer, V. V. Vasisht, L. Zhang, E. Del Gado, and X. Mao, Physical Review Research **4**, 043181 (2022).
[12] M. Wyart, Annales de Physique **30**, 1 (2005).
[13] M. Merkel and M. L. Manning, New Journal of Physics **20**, 022002 (2018).
[14] P. R. Onck, T. Koeman, T. van Dillen, and E. van der Giessen, Physical Review Letters **95**, 178102 (2005), arxiv:cond-mat/0502397.
[15] C. P. Broedersz and F. C. MacKintosh, Reviews of Modern Physics **86**, 995 (2014), arxiv:1404.4332.
[16] A. J. Licup, S. Münster, A. Sharma, M. Sheinman, L. M. Jawerth, B. Fabry, D. a. Weitz, and F. C. MacKintosh, Proceedings of the National Academy of Sciences **112**, 9573 (2015), arxiv:1503.00924.
[17] A. Sharma, A. J. Licup, K. A. Jansen, R. Rens, M. Sheinman, G. H. Koenderink, and F. C. MacKintosh, Nature Physics **12**, 584 (2016), arxiv:1506.07792.
[18] M. Moshe, M. J. Bowick, and M. C. Marchetti, Physical Review Letters **120**, 268105 (2017), arxiv:1708.07848.
[19] D. M. Sussman and M. Merkel, Soft Matter **14**, 3397 (2018).
[20] X. Wang, M. Merkel, L. B. Sutter, G. Erdemci-Tandogan, M. L. Manning, and K. E. Kasza, Proceedings of the National Academy of Sciences **117**, 13541 (2020), arxiv:2005.07283.
[21] M. Plischke and B. Joós, Physical Review Letters **80**, 4907 (1998).
[22] M. Dennison, M. Sheinman, C. Storm, and F. C. MacKintosh, Physical Review Letters **111**, 095503 (2013).
[23] M. C. Wigbers, F. C. Mackintosh, and M. Dennison, Physical Review E - Statistical, Nonlinear, and Soft Matter Physics **92**, 1 (2015), arxiv:1410.7860.
[24] L. Zhang and X. Mao, Physical Review E **93**, 1 (2016).
[25] S. Chen, T. Markovich, and F. C. MacKintosh, Physical Review Letters **130**, 088101 (2023).
[26] Reference to be inserted later.
[27] A. Parker, M. C. Marchetti, M. L. Manning, and J. M. Schwarz, How does the extracellular matrix affect the rigidity of an embedded spheroid? (2020), arxiv:arXiv:2006.16203.
[28] See, e.g., the appendices of Refs. [13, 33].
[29] M. Holmes-Cerfon, L. Theran, and S. J. Gortler, Communications on Pure and Applied Mathematics **74**, 2185 (2021).
[30] D. Z. Rocklin, V. Vitelli, and X. Mao, Folding mechanisms at finite temperature (2018), arxiv:arXiv:1802.02704.
[31] S. Arzash, J. L. Shivers, and F. C. MacKintosh, Soft Matter **16**, 6784 (2020).
[32] This is because any term of lower order would need to arise from the $\tilde{M}$ terms in Eq. (18). However, related $\tilde{M}$ terms appear also in the G_p for $p < N_s$, implying that also the G_p would be of that same order in ε, which is not possible, since $G_{\hat{p}}$, which has lowest order among the $p < N_s$, should be of higher order than G_{N_s}. There is also the (very marginal) possibility that the $\tilde{M}$ term creating the lower-order term in ε does not appear in *any* of the G_p with $p < N_s$. In this case, energy minimization would ensure that the corresponding $\Delta\tilde{r}_n$ cannot be of lower order than $\mathcal{O}(|\varepsilon|^{1/2})$.
[33] M. Merkel, *From Cells to Tissues: Tissue Remodeling and Polarity Reorientation in Shearing Epithelia*, Ph.D. thesis, Technische Universität Dresden (2014).