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Quasi-Maximum Likelihood Estimation of long-memory linear processes

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Abstract

The purpose of this paper is to study the convergence of the quasi-maximum likelihood (QML) estimator for long memory linear processes. We first establish a correspondence between the long-memory linear process representation and the long-memory $AR(\infty)$ process representation. We then establish the almost sure consistency and asymptotic normality of the QML estimator. Numerical simulations illustrate the theoretical results and confirm the good performance of the estimator.

Keywords: Long memory process; Semiparametric estimation; Linear process

MSC2010: Primary 62F12, 62M10; Secondary 91B84

1 Introduction

Since Hurst's (1953) introduction of long-range dependent processes, much research has focused on estimating the long-range parameter, whether defined on the basis of the asymptotic power-law behavior of the correlogram at infinity or that of the spectral density at zero (see the monographs [5] and [14] for more details).

Two estimation frameworks have been studied extensively. The first focused on the estimation of the long-memory parameter alone, but could be carried out in a semi-parametric framework, i.e. if only the asymptotic behavior of the correlation or spectral density was specified. This led to the first methods proposed historically, such as those based on the R/S statistic, on quadratic variations, on the log-periodogram, or more recent methods such as wavelet or local Whittle (again, see [14] for more details).

Here we are interested in a more parametric framework, and in estimating all the parameters of the process, not just the long memory parameter. The first notable results on the asymptotic behavior of such a parametric estimator were obtained in Fox and Taqqu (1986) (see [16]) in the special case of Gaussian long-memory processes, using the Whittle estimator. These results were extended to linear long-memory processes with a moment of order 4 by Giraitis and Surgailis (1990) (see [17]). In both settings, the asymptotic normality of the estimator was proved, while non-central limit theorems were obtained for functions of Gaussian processes in [18] or for increments of the Rosenblatt process in [3]. The asymptotic normality of the maximum likelihood estimator for Gaussian time

series was also obtained by Dahlhaus (1989, see [12]) using that of the Whittle estimator obtained in Fox and Taqqu (1986).

For weakly dependent time series, especially for conditionally heteroscedastic processes such as GARCH processes, the quasi-maximum likelihood (QML) estimator has become the benchmark for parametric estimation, providing very interesting convergence results where Whittle's estimator would not. This is true for GARCH or ARMA-GARCH processes (see [7] and [15]), but also for many others such as ARCH(∞), AR(∞), APARCH processes, etc. (see [4]). We will also note convergence results for this modified estimator for long-memory squares processes, typically LARCH(∞) processes, see [6], or quadratic autoregressive conditional heteroscedastic processes, see [13]. But for long-memory processes, such as those defined by a non-finite sum of their autocorrelations, to our knowledge only the paper by Boubacar Mainassara *et al.* (2021) (see [11]) has shown the normality of this QML estimator in the special case of a FARIMA(p, d, q) process with weak white noise.

We therefore propose here to study the convergence of the QML estimator in the general framework of long-memory one-sided linear processes. The key point of our approach is to prove that long-memory one-sided linear processes can be written in autoregressive form with respect to their past values, which we can call long-memory linear AR(∞). This is perfectly suited to the use of QMLE, since this estimator is obtained from the conditional expectation and variance of the process. We then show the almost sure convergence of QMLE for these long-memory AR(∞) processes, which generalizes a result obtained in [4] for weakly dependent AR(∞) processes. We also prove the asymptotic normality of this estimator, which provides an alternative to the asymptotic normality of Whittle's estimator obtained in [17].

Finally, we performed simulations of two long-memory time series and examined the performance of the QMLE as a function of the size of the observed trajectories. This showed that the behavior of the QMLE is consistent with theory as the size of the trajectories increases, and provides a very accurate alternative to Whittle's estimator.

This article is organized as follows: the section 2 below presents the AR(∞) notation of an arbitrary long memory one-sided linear process, the section 3 is devoted to the presentation of the QMLE estimator and its asymptotic behavior, numerical applications are treated in the section 4, while all proofs of the various results can be found in the section 5.

2 Long-memory linear causal time series

Assume that $\varepsilon = (\varepsilon_t)_{t \in \mathbb{Z}}$ is a sequence of centered independent random variables such as $\mathbb{E}[\varepsilon_0^2] = 1$ and $(a_i)_{i \in \mathbb{N}}$ is a sequence of real numbers such as:

$$a_i = L_a(i) i^{d-1} \quad \text{for } i \in \mathbb{N}^* \text{ and } a_0 > 0, \quad (1)$$

where $d \in (0, 1/2)$ and with $L_a(\cdot)$ a positive slow varying function satisfying

$$\text{for any } t > 0, \quad \lim_{x \rightarrow \infty} \frac{L_a(xt)}{L_a(x)} = 1.$$

Now, define the causal linear process $(X_t)_{t \in \mathbb{Z}}$ by

$$X_t = \sum_{i=0}^{\infty} a_i \varepsilon_{t-i} \quad \text{for any } t \in \mathbb{Z}. \quad (2)$$

Since $0 < d < 1/2$, it is well known that $(X_t)_{t \in \mathbb{Z}}$ is a second order stationary long-memory process. Indeed, its autocovariance is

$$r_X(k) = \text{Cov}(X_0, X_k) = \sum_{i=0}^{\infty} a_i a_{i+k} \sim C_d L_a^2(k) k^{2d-1} \quad \text{when } k \rightarrow \infty, \quad (3)$$

where $C_d = \int_0^{\infty} (u + u^2)^{d-1} du$ (see for instance Wu *et al.*, 2010).

Then, it is always possible to provide a causal affine representation for $(X_t)_{t \in \mathbb{Z}}$, *i.e.* it is always possible to write $(X_t)_{t \in \mathbb{Z}}$ as an AR(∞) process:

Proposition 2.1. Let $(X_t)_{t \in \mathbb{Z}}$ be a causal linear process defined in (2) where (a_i) satisfies (1). Then, there exists a sequence of real number $(u_i)_{i \in \mathbb{N}^*}$ such as:

$$X_t = a_0 \varepsilon_t + \sum_{i=1}^{\infty} u_i X_{t-i} \quad \text{for any } t \in \mathbb{Z}, \quad (4)$$

where $(u_i)_{i \in \mathbb{N}^*}$ satisfies

$$\sum_{i=1}^{\infty} u_i = 1 \quad \text{and} \quad u_n \underset{n \rightarrow \infty}{\sim} \frac{a_0 d}{\Gamma(d) \Gamma(1-d)} L_a^{-1}(n) n^{-1-d} = L_u(n) n^{-1-d} \quad (5)$$

where L_u is a slow varying function.

Remark 2.1. Using (18), the reciprocal implication of Proposition 2.1 is also true: if (X_t) satisfies the linear affine causal representation (4) where $(u_i)_{i \in \mathbb{N}}$ satisfies (5), then (X_t) is a one-sided long-memory linear process satisfying (2) where (a_i) satisfies (1).

Remark 2.2. It is also known that $\Gamma(d) \Gamma(1-d) = \frac{\pi}{\sin(\pi d)}$ for any $d \in (0, 1)$, and this implies

$$u_n \underset{n \rightarrow \infty}{\sim} \frac{a_0 d \sin(\pi d)}{\pi L_a(n)} n^{-1-d}.$$

As a consequence, every long-memory one-sided linear process is a long-memory AR(∞) process with the special property that the sum of the autoregressive coefficients equals 1. This is the key point for the use of quasi-maximum likelihood estimation in the following section.

Example of the FARIMA process: Let $(X_t)_{t \in \mathbb{Z}}$ be a standard FARIMA(0, d , 0) with $d \in (0, 1/2)$, which means $X = (I_d - B)^{-d} \varepsilon$, where B is the usual backward linear operator on $\mathbb{R}^{\mathbb{Z}}$. Then, using the power series of $(1-x)^{-d}$, it is known that

$$X_t = \sum_{i=0}^{\infty} a_i \varepsilon_{t-i} \quad \text{with} \quad a_i = \frac{\Gamma(i+d)}{\Gamma(i+1)\Gamma(d)} \quad \text{for } t \in \mathbb{Z}.$$

Using the Stirling expansion of the Gamma function, *i.e.* $\Gamma(x) \underset{x \rightarrow \infty}{\sim} \sqrt{2\pi} e^{-x} x^{x-1/2}$, we obtain $a_n \underset{n \rightarrow \infty}{\sim} \frac{1}{\Gamma(d)} n^{d-1}$, which is (1) with $L_a(n) \underset{n \rightarrow \infty}{\sim} \frac{1}{\Gamma(d)}$.

Moreover, the decomposition $X = \varepsilon + (I_d - (I_d - B)^d) X$ implies:

$$X_t = \varepsilon_t + d \sum_{i=1}^{\infty} \frac{\Gamma(i-d)}{\Gamma(1-d)\Gamma(i+1)} X_{t-i} \quad \text{for } t \in \mathbb{Z}.$$

The expansion $\frac{\Gamma(n-d)}{\Gamma(n+1)} \underset{n \rightarrow \infty}{\sim} n^{-1-d}$ provides $X_t = \varepsilon_t + \sum_{n=1}^{\infty} u_n X_{t-n}$ with $u_n \underset{n \rightarrow \infty}{\sim} \frac{d}{\Gamma(1-d)} n^{-1-d}$, which is equivalent to (5) when $a_0 = 1$ and $L_a(n) \underset{n \rightarrow \infty}{\sim} \frac{1}{\Gamma(d)}$.

3 Asymptotic behavior of the Gaussian Quasi-Maximum Likelihood Estimator

3.1 Definition of the estimator

We will assume that $(X_t)_{t \in \mathbb{Z}}$ is a long-memory one-sided linear process written as an AR(∞) process, *i.e.*

$$X_t = \sigma^* \varepsilon_t + \sum_{k=1}^{\infty} u_k(\theta^*) X_{t-k} \quad \text{for any } t \in \mathbb{Z}, \quad (6)$$

where

- $(\varepsilon_t)_{t \in \mathbb{Z}}$ is a white noise, such that ε_0 has an absolutely continuous probability measure with respect to the Lebesgue measure and such that $\mathbb{E}[\varepsilon_0^2] = 1$, $\mathbb{E}[\varepsilon_0^3] = 0$ and $\mu_4 = \mathbb{E}[\varepsilon_0^4] < \infty$;
- for $\theta = {}^t(\gamma, \sigma^2) \in \Theta$ a compact subset of $\mathbb{R}^{p-1} \times (0, \infty)$, $(u_n(\theta))_{n \in \mathbb{N}}$ is a sequence of real numbers satisfying for any $\theta \in \Theta$,

$$u_n(\theta) = L_\theta(n) n^{-d(\theta)-1} \quad \text{for } n \in \mathbb{N}^* \text{ and } \sum_{n=1}^{\infty} u_n(\theta) = 1. \quad (7)$$

with $d(\theta) \in (0, 1/2)$. We also assume that **the sequence $(u_n(\theta))$ does not depend on σ^2** ;

- $\theta^* = {}^t(\gamma^*, \sigma^{*2})$, θ^* is in the interior of Θ , with $\sigma^* > 0$ an unknown real parameter and $\gamma^* \in \mathbb{R}^{p-1}$ an unknown vector of parameters.

A simple example of such a sequence $(u_n(\theta))$ is $u_n(\theta) = (\zeta(1+d))^{-1} n^{-1-d}$ for $n \in \mathbb{N}^*$, with $\theta = (d, \sigma^2) \in (0, 1/2) \times (0, \infty)$ where $\zeta(\cdot)$ is the Riemann zeta function. Then $\Theta = [\sigma_m^2, \sigma_M^2] \times [d_m, d_M]$, with $0 < \sigma_m^2 < \sigma_M^2$ and $0 < d_m < d_M < 1/2$.

For ease of reading, denote $d^* = d(\theta^*)$ the long-memory parameter of (X_t) . Denote also $d_+^* = d^* + \varepsilon$ where $\varepsilon \in (0, 1/2 - d^*)$ is chosen as small as possible. Since $(u_n(\theta))_{n \in \mathbb{N}}$ satisfies (7), we know from Remark 2.1 that there exists C_a such that for any $t \in \mathbb{Z}$,

$$X_t = \sum_{i=0}^{\infty} a_i(\theta^*) \varepsilon_{t-i} \quad \text{with } |a_i(\theta^*)| \leq \frac{C_a}{i^{1-d_+^*}} \quad \text{for all } i \in \mathbb{N}^*. \quad (8)$$

We also deduce from (3) that there exists $C_c > 0$ satisfying

$$|r_X(k)| = |\text{Cov}(X_0, X_k)| \leq \frac{C_c}{(1+k)^{1-2d_+^*}} \quad \text{for all } k \in \mathbb{N}. \quad (9)$$

In the sequel we will also denote for any $\theta \in \Theta$,

$$m_t(\theta) = \sum_{k=1}^{\infty} u_k(\theta) X_{t-k} \quad \text{for any } t \in \mathbb{Z}. \quad (10)$$

We want to estimate θ^* from an observed trajectory (X_1, \dots, X_n) , where (X_t) is defined by (6). For such an autoregressive causal process, a Gaussian quasi-maximum likelihood estimator is really appropriate, since it is built on the assumption that (ε_t) is Gaussian white noise, and it is well known that an affine function of ε_t is still a Gaussian random variable (see for example Bardet and Wintenberger, 2009). It consists in considering the log-conditional density $I_n(\theta)$ of (X_1, \dots, X_n) when (ε_t) is a standard Gaussian white noise and with $X_t = \sigma \varepsilon_t + m_t(\theta)$, *i.e.*

$$I_n(\theta) = \sum_{t=1}^n q_t(\theta) = -\frac{1}{2} \sum_{t=1}^n \left(\log(\sigma^2) + \frac{(X_t - m_t(\theta))^2}{\sigma^2} \right) \quad \text{for any } \theta \in \Theta. \quad (11)$$

However, such conditional log-likelihood is not a feasible statistic since $m_t(\theta)$ depends on $(X_k)_{k \leq 0}$ which is unobserved. Hence it is usual to replace $m_t(\theta)$ by the following approximation:

$$\hat{m}_t(\theta) = \sum_{i=1}^{t-1} u_i(\theta) X_{t-i} \quad \text{for any } t \in \mathbb{N}^*, \quad (12)$$

with the convention $\sum_{t=1}^0 = 0$. Then, a quasi conditional log-likelihood $\hat{I}_n(\theta)$ can be defined:

$$\hat{I}_n(\theta) = -\frac{1}{2} \sum_{t=1}^n \left(\log(\sigma^2) + \frac{(X_t - \hat{m}_t(\theta))^2}{\sigma^2} \right). \quad (13)$$

If Θ is a subset of \mathbb{R}^p such as for all $\theta \in \Theta$ there exists an almost surely stationary solution of the equation $X_t = \sigma \varepsilon_t + m_t(\theta)$ for any $t \in \mathbb{Z}$, we define the Gaussian quasi maximum likelihood estimator (QMLE) of θ by

$$\hat{\theta}_n = \underset{\theta \in \Theta}{\text{Argmax}} \hat{I}_n(\theta). \quad (14)$$

Note that a direct implication of the assumption that $(u_n(\theta))$ does not depend on σ^2 is that if we denote $\hat{\theta}_n = {}^t(\hat{\gamma}_n, \hat{\sigma}_n^2)$ the QMLE, then:

$$\hat{\gamma}_n = \underset{(\gamma, \sigma^2) \in \Theta}{\text{Argmin}} \sum_{t=1}^n \left(X_t - \sum_{k=1}^{t-1} u_k(\gamma) X_{t-k} \right)^2 \quad \text{and} \quad \hat{\sigma}_n^2 = \frac{1}{n} \sum_{t=1}^n \left(X_t - \sum_{k=1}^{t-1} u_k(\hat{\gamma}_n) X_{t-k} \right)^2,$$

where by writing convention $u_n(\theta) = u_n(\gamma)$. Hence, in this case of long-memory $\text{AR}(\infty)$, $\hat{\gamma}_n$ is also a non-linear least square estimator of the parameter γ .

3.2 Consistency and asymptotic normality of the estimator

The consistency of the QMLE is established under additional assumptions.

Theorem 3.1. Let $(X_t)_{t \in \mathbb{Z}}$ be a process defined by (6) and its assumption. Assume also:

- for any $n \in \mathbb{N}^*$, $\theta \in \Theta \mapsto u_n(\theta)$ is a continuous function on Θ ;
- If $u_n(\theta) = u_n(\theta')$ for all $n \in \mathbb{N}^*$ with $\theta = (\gamma, \sigma^2)$ and $\theta' = (\gamma', \sigma'^2)$, then $\theta = \theta'$.

Let $\widehat{\theta}_n$ be the QMLE defined in (14). Then

$$\widehat{\theta}_n \xrightarrow[n \rightarrow \infty]{a.s.} \theta^*.$$

This result extends the $\widehat{\theta}_n$ consistency obtained in Bardet and Wintenberger (2009) to short-memory time series models, including ARMA, GARCH, and APARCH, among others, including AR(∞) processes. It also applies to long memory AR(∞) processes.

Remark 3.1. Regarding the long-memory linear process example, θ^* could also be estimated using Whittle's estimator, which is constructed from the spectral density and second-order moments of the process. The consistency and asymptotic normality of this estimator were shown by Giraitis and Surgailis (1990).

Having shown the consistency, we would like to show the asymptotic normality of the QML estimator in the case of the long-memory one-sided linear processes considered above. This amounts to proving it for linear processes whose linear filter depends on a vector of parameters. This will be the case, for example, for FARIMA(p, d, q) processes, for which Boubacar *et al.* (2021) have already shown asymptotic normality in the more general case where (ε_t) is weak white noise, *i.e.* in the case of weak FARIMA(p, d, q) processes.

As it is typical to establish the asymptotic normality of an M-estimator, we make assumptions about the differentiability of the sequence of functions $(u_n(\theta))_{n \in \mathbb{N}^*}$ with respect to θ :

(A) Differentiability of $(u_n(\theta))_{n \in \mathbb{N}^*}$: for any $n \in \mathbb{N}^*$, the function $u_n(\theta)$ is a $\mathcal{C}^2(\Theta)$ function and for any $\delta > 0$, there exists $C_\delta > 0$ such that:

$$\sup_{n \in \mathbb{N}} \sup_{\theta \in \Theta} \left\{ n^{1+d(\theta)-\delta} \left(|u_n(\theta)| + \|\partial_\theta u_n(\theta)\| + \|\partial_{\theta^2}^2 u_n(\theta)\| \right) \right\} \leq C_\delta. \quad (15)$$

Moreover we assume that:

$$\text{for } v \in \mathbb{R}^{p-1}, \text{ if for all } k \in \mathbb{N}^*, \quad {}^t v \partial_\gamma u_k(\theta^*) = 0 \implies v = 0. \quad (16)$$

Example (called **LM** in the numerical applications): For the simple example where $(u_n(\theta))$ is such as $u_n(\theta) = \zeta(1+d)^{-1} n^{-1-d}$ for $n \in \mathbb{N}^*$ with $\theta = (d, \sigma^2) \in (0, 1/2) \times (0, \infty)$, we have:

$$\begin{aligned} \partial_d u_n(\theta) &= -\frac{n^{-1-d}}{\zeta^2(1+d)} \left(\zeta(1+d) \log(n) + \zeta'(1+d) \right) \\ \partial_{d^2}^2 u_n(\theta) &= \frac{n^{-1-d}}{\zeta^3(1+d)} \left(\zeta^2(1+d) \log^2(n) + 2\zeta'(1+d)\zeta(1+d) \log(n) + 2(\zeta'(1+d))^2 - \zeta''(1+d)\zeta(1+d) \right). \end{aligned}$$

Therefore (15) of **(A)** is satisfied with $d(\theta) = d$ (note also that $\delta = 0$ is not possible). Moreover (16) is also clearly satisfied.

Theorem 3.2. Consider the assumptions of Theorem 3.1 and also that $\mathbb{E}[\varepsilon_0^3] = 0$ and $\mu_4 = \|\varepsilon_0\|_4^4 < \infty$. Then with $\widehat{\theta}_n$ defined in (14), and if **(A)** holds, ,

$$\sqrt{n}(\widehat{\theta}_n - \theta^*) = \sqrt{n} \left(\begin{pmatrix} \widehat{\gamma}_n \\ \widehat{\sigma}_n^2 \end{pmatrix} - \begin{pmatrix} \gamma^* \\ \sigma^{*2} \end{pmatrix} \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N} \left(0, \begin{pmatrix} (M^*)^{-1} & 0 \\ 0 & \sigma^{*4}(\mu_4^* - 1) \end{pmatrix} \right), \quad (17)$$

where $M^* = \frac{1}{\sigma^{*2}} \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \partial_{\gamma} u_k((\gamma^*, 0))^t (\partial_{\gamma} u_{\ell}((\gamma^*, 0))) r_X(\ell - k)$.

It is clear that $\widehat{\theta}_n$ satisfies (17) in the case of the FARIMA processes or in the case of the **LM** processes example. It is also worth noting that the central limit theorem is written in exactly the same way as the one obtained in [4], although the latter dealt only with weakly dependent AR(∞) processes.

Remark 3.2. Of course, in this specific context of linear long-memory processes, we would like to make a comparison between the asymptotic results for the convergence of the QMLE estimator and those obtained with Whittle's estimator in [17]. In this paper, more precisely in Theorem 4, the asymptotic covariance matrix of $\widehat{\gamma}_n$ is given by the spectral density f_{γ} and is written as $(4\pi)^{-1} \int_{-\pi}^{\pi} (\partial_{\gamma} \log(f_{\gamma}(\lambda)))^t (\partial_{\gamma} \log(f_{\gamma}(\lambda))) d\lambda$. However, Dahlhaus in [12] has shown that this asymptotic covariance matrix is also that of the maximum likelihood estimator in the case of a Gaussian process, the latter also being $(M^*)^{-1}$ if (ε_t) is Gaussian white noise. This means that asymptotically, the QML and Whittle estimators behave identically. However, we will see a slight numerical advantage due to the convergence of the QMLE in the case of observed trajectories whose size is not too large.

4 Numerical applications

In this section, we report the results of Monte Carlo experiments conducted with different long-memory causal linear processes. More specifically, we considered:

- Two different processes generated from Gaussian standard white noises:
 1. A FARIMA(0, d , 0) process, denoted **FARIMA**, with parameters $\sigma^2 = 4$ and $d = 0.1, 0.2, 0.3$ and 0.4 ;
 2. A long-memory causal affine process, denoted **LM**, defined by:

$$X_t = a_0 \varepsilon_t + \zeta(1 + d)^{-1} \sum_{k=1}^{\infty} k^{-1-d} X_{t-k} \quad \text{for any } t \in \mathbb{Z},$$

with parameters $\sigma^2 = 4$ and $d = 0.1, 0.2, 0.3$ and 0.4 .

- Several trajectory lengths: $n = 300, 1000, 3000$ and 10000 .
- In the case of the **FARIMA** process, we compared the accuracy of the QMLE with the one of the Whittle estimator which also satisfies a central limit theorem (see [17]). We denote $\widehat{\theta}_W = (\widehat{\sigma}_W^2, \widehat{d}_W)$ this estimator.

n		$\sigma^2 = 4, d = 0.1$	$\sigma^2 = 4, d = 0.2$	$\sigma^2 = 4, d = 0.3$	$\sigma^2 = 4, d = 0.4$				
300	$\hat{\theta}_n = (\hat{\sigma}_n^2, \hat{d}_n)$	0.327	0.045	0.317	0.045	0.318	0.046	0.327	0.050
	$\hat{\theta}_W = (\hat{\sigma}_W^2, \hat{d}_W)$	0.327	0.050	0.318	0.050	0.319	0.051	0.332	0.053
1000	$\hat{\theta}_n = (\hat{\sigma}_n^2, \hat{d}_n)$	0.179	0.024	0.179	0.024	0.183	0.025	0.184	0.025
	$\hat{\theta}_W = (\hat{\sigma}_W^2, \hat{d}_W)$	0.179	0.026	0.179	0.026	0.183	0.026	0.185	0.026
3000	$\hat{\theta}_n = (\hat{\sigma}_n^2, \hat{d}_n)$	0.103	0.014	0.105	0.014	0.103	0.014	0.100	0.015
	$\hat{\theta}_W = (\hat{\sigma}_W^2, \hat{d}_W)$	0.103	0.014	0.106	0.015	0.103	0.014	0.100	0.015
10000	$\hat{\theta}_n = (\hat{\sigma}_n^2, \hat{d}_n)$	0.056	0.007	0.056	0.007	0.056	0.008	0.052	0.008
	$\hat{\theta}_W = (\hat{\sigma}_W^2, \hat{d}_W)$	0.056	0.007	0.056	0.007	0.057	0.008	0.052	0.008

Table 1: Square roots of the MSE computed for the QMLE $\hat{\theta}_n$ and the Whittle estimator in the case of a **FARIMA** process computed from 1000 independent replications.

n		$\sigma^2 = 4, d = 0.1$	$\sigma^2 = 4, d = 0.2$	$\sigma^2 = 4, d = 0.3$	$\sigma^2 = 4, d = 0.4$				
300	$\hat{\theta}_n = (\hat{\sigma}_n^2, \hat{d}_n)$	0.082	0.048	0.083	0.054	0.080	0.059	0.080	0.065
1000	$\hat{\theta}_n = (\hat{\sigma}_n^2, \hat{d}_n)$	0.045	0.025	0.047	0.032	0.045	0.032	0.046	0.038
3000	$\hat{\theta}_n = (\hat{\sigma}_n^2, \hat{d}_n)$	0.025	0.014	0.027	0.017	0.024	0.018	0.026	0.020
10000	$\hat{\theta}_n = (\hat{\sigma}_n^2, \hat{d}_n)$	0.013	0.008	0.013	0.010	0.015	0.011	0.014	0.012

Table 2: Square roots of the MSE computed for the QMLE $\hat{\theta}_n$ in the case of the **LRD** process computed from 1000 independent replications.

The results are presented in tables 1 and 2.

Conclusions of the simulations:

1. The results of the simulations show that the consistency of the QML estimator $\hat{\theta}_n$ is satisfied and also that its $1/\sqrt{n}$ convergence rate of the estimators almost occurs.
2. In the case of the FARIMA process, the comparison between the QML and Whittle estimators leads to very similar results for large n , but for $n = 300$ the QML estimator provides slightly more accurate estimates.

5 Proofs

5.1 Proofs of the main results

Proof of Proposition 2.1. Using B the lag or backshift linear operator on $\mathbb{R}^{\mathbb{Z}}$, we can denote $X = S(B)\varepsilon$, where $X = (X_t)_{t \in \mathbb{Z}}$ and $\varepsilon = (\varepsilon_t)_{t \in \mathbb{Z}}$ and $S(B) = \sum_{i=0}^{\infty} a_i B^i$. We know that there exists a linear operator denoted S^{-1} such as $\varepsilon = S^{-1}(B)X$. As a consequence, $X = a_0 \varepsilon + (S(B) - a_0 I_d)\varepsilon = a_0 \varepsilon + (S(B) - a_0 I_d)S^{-1}(B)X = a_0 \varepsilon + (I_d - a_0 S^{-1}(B))X$ which is the affine causal representation of X .

Let $X_t = a_0 \varepsilon_t + \sum_{i=1}^{\infty} u_i X_{t-i}$. Then, for any $t \in \mathbb{Z}$,

$$\begin{aligned} X_t &= a_0 \varepsilon_t + \sum_{i=1}^{\infty} u_i X_{t-i} \\ &= a_0 \varepsilon_t + \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} u_i a_j \varepsilon_{t-i-j} \\ &= a_0 \varepsilon_t + \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k-1} u_{k-j} a_j \right) \varepsilon_{t-k}. \end{aligned}$$

As a consequence, denoting $u_0 = -1$, for any $k \in \mathbb{N}^*$,

$$\sum_{j=0}^{k-1} u_{k-j} a_j = a_k \implies \left(\sum_{i=0}^k u_i \right) \left(\sum_{j=0}^k a_j \right) = 0. \quad (18)$$

Finally, since the convergence radius of the power series $\sum_{\ell=0}^{\infty} a_{\ell} z^{\ell}$ is 1 from asymptotic expansion (1), we deduce that for any $z \in \mathbb{C}$, $|z| < 1$,

$$\left(\sum_{k=0}^{\infty} u_k z^k \right) \left(\sum_{\ell=0}^{\infty} a_{\ell} z^{\ell} \right) = -a_0. \quad (19)$$

Now, we are going to use a Karamata Tauberian theorem as it is stated in Corollary 1.7.3 of Bingham *et al.* (1987):

Fix $\rho > 0$ and let L a slow varying function. Then if $(\alpha_n)_{n \in \mathbb{N}}$ is a sequence of nonnegative real numbers and the power series $A(s) = \sum_{n=0}^{\infty} \alpha_n s^n$ converges for any $s \in [0, 1)$, then

$$\sum_{k=0}^n \alpha_k \underset{n \rightarrow \infty}{\sim} L(n) n^{\rho} \iff A(s) \sim \frac{\Gamma(1 + \rho)}{(1-s)^{\rho}} L((1-s)^{-1}) \text{ as } s \rightarrow 1^{-}. \quad (20)$$

Note that this result is also established if there exists $N_0 \in \mathbb{N}$ such as $(\alpha_n)_{n \geq N_0}$ is a sequence of nonnegative real numbers. We first apply (20) to $(\alpha_n) = (a_n)$. Indeed, from (1) and with $\rho = d$, there exists $N_0 \in \mathbb{N}$ such as $(a_n)_{n \geq N_0}$ is a sequence of nonnegative real numbers and $\sum_{k=0}^n a_k \underset{n \rightarrow \infty}{\sim} L(n) n^{\rho}$ with $L(\cdot) = \frac{L_a(\cdot)}{d}$. Therefore, we deduce that

$$\sum_{n=0}^{\infty} a_n s^n \sim \frac{\Gamma(1+d)}{d(1-s)^d} L_a((1-s)^{-1}) \text{ as } s \rightarrow 1^{-}. \quad (21)$$

Therefore, from (19), the following expansion can be deduced:

$$\sum_{n=0}^{\infty} u_n s^n \sim -\frac{a_0 (1-s)^d}{\Gamma(d)} L_a^{-1}((1-s)^{-1}) \text{ as } s \rightarrow 1^{-}. \quad (22)$$

On the other hand, if we consider (19) when $s \rightarrow 1^{-}$, $\sum_{\ell=0}^{\infty} a_{\ell} s^{\ell} \rightarrow \infty$ since (a_n) satisfies (1). As a consequence,

$$\sum_{n=0}^{\infty} u_n s^n \rightarrow 0 = \sum_{n=0}^{\infty} u_n \text{ when } s \rightarrow 1^{-}.$$

We deduce that $u_n \xrightarrow{n \rightarrow \infty} 0$ and the sequence $(U_n)_{n \in \mathbb{N}}$ can be defined where we denote $U_n = \sum_{k=n+1}^{\infty} u_k$. But since $\sum_{n=0}^{\infty} u_n = 0$, for any $s \in [0, 1]$,

$$\sum_{k=0}^{\infty} u_k s^k = (s-1) \sum_{k=0}^{\infty} U_k s^k.$$

Using (22), we deduce

$$\sum_{k=0}^{\infty} U_k s^k \sim \frac{a_0 (1-s)^{d-1}}{\Gamma(d)} L_a^{-1}((1-s)^{-1}) \text{ as } s \rightarrow 1^-.$$

From (18), we also have for any $n \in \mathbb{N}$

$$\left(\sum_{k=0}^n u_k \right) \left(\sum_{\ell=0}^n a_\ell \right) = -a_0 \quad (23)$$

Since (a_n) satisfies (1), we know that there exists N_0 such as $a_n > 0$ and $\sum_{\ell=0}^n a_\ell > 0$ for any $n \geq N_0$. Therefore we know from (23) that for any $n \geq N_0$, $\sum_{k=0}^n u_k < 0$ and thus $U_n > 0$ since $\sum_{k=0}^{\infty} u_k = 0$. Thus we can apply (20) to $(\alpha_n) = (U_n)$ with $\rho = 1-d$ and this induces

$$\sum_{k=0}^n U_k \underset{n \rightarrow \infty}{\sim} \frac{a_0}{\Gamma(d)\Gamma(2-d)} L_a^{-1}(n) n^{1-d}.$$

Since for $n \geq N_0$, $u_n > 0$, we deduce that (U_n) is a positive decreasing sequence for $n \geq N_0$. Using again Bingham *et al.* (1987), we deduce that

$$U_n \underset{n \rightarrow \infty}{\sim} \frac{a_0 (1-d)}{\Gamma(d)\Gamma(2-d)} L_a^{-1}(n) n^{-d} = \frac{a_0}{\Gamma(d)\Gamma(1-d)} L_a^{-1}(n) n^{-d}.$$

To finish with, since (U_n) is a positive decreasing sequence for $n \geq N_0$, we deduce:

$$u_n = \frac{a_0 d}{\Gamma(d)\Gamma(1-d)} L_a^{-1}(n) n^{-1-d},$$

and this achieves the proof. \square

Proof of Theorem 3.1. In the sequel, we will denote for any $t \in \mathbb{N}^*$ and $\theta \in \Theta$,

$$\tilde{m}_t(\theta) = m_t(\theta) - \hat{m}_t(\theta) = \sum_{k=t}^{\infty} u_k(\theta) X_{t-k}. \quad (24)$$

For a random variable Z and $r \geq 1$, denote $\|Z\|_r = (\mathbb{E}[|Z|^r])^{1/r}$.

1. Firstly we prove some useful inequalities.

From the Cauchy-Schwarz Inequality, for any $\theta \in \Theta$ and $t \in \mathbb{Z}$,

$$\begin{aligned} (m_t(\theta))^2 &\leq \left(\sum_{k=1}^{\infty} |u_k(\theta)| \right) \left(\sum_{k=1}^{\infty} |u_k(\theta)| X_{t-k}^2 \right) \\ &\leq \sup_{\theta \in \Theta} \left\{ \sum_{k=1}^{\infty} |u_k(\theta)| \right\} \sup_{\theta \in \Theta} \left\{ \sum_{k=1}^{\infty} |u_k(\theta)| X_{t-k}^2 \right\} \\ \implies \left\| \sup_{\theta \in \Theta} m_t(\theta) \right\|_2^2 &\leq \left(\sup_{\theta \in \Theta} \left\{ \sum_{k=1}^{\infty} |u_k(\theta)| \right\} \right)^2 \|X_0\|_2^2 < \infty, \end{aligned}$$

since (u_k) follows (7), Θ is a compact subset, $\theta \in \Theta \mapsto u_k(\theta)$ is a continuous function for any $k \geq 1$ and $d(\theta) \in (0, 1/2)$.

Using the same inequalities we also obtain that there exists $C_2 > 0$ such that for any $t \geq 1$,

$$\left\| \sup_{\theta \in \Theta} |\widehat{m}_t(\theta)| \right\|_2^2 < \infty \quad \text{and} \quad \left\| \sup_{\theta \in \Theta} |\widetilde{m}_t(\theta)| \right\|_2^2 \leq \left(\sup_{\theta \in \Theta} \left\{ \sum_{k=t+1}^{\infty} |u_k(\theta)| \right\} \right)^2 \|X_0\|_2^2 \leq C_2 t^{-2\underline{d}}, \quad (25)$$

with $0 < \underline{d} < \inf_{\theta \in \Theta} d(\theta)$ from the condition (7) on $(u_n(\theta))$.

Finally with

$$\begin{cases} q_t(\theta) &= -\frac{1}{2} \left(\log(\sigma^2) + \frac{(X_t - m_t(\theta))^2}{\sigma^2} \right) \\ \widehat{q}_t(\theta) &= -\frac{1}{2} \left(\log(\widehat{\sigma}^2) + \frac{(X_t - \widehat{m}_t(\theta))^2}{\widehat{\sigma}^2} \right) \end{cases}, \quad (26)$$

we obtain from the previous bounds and $\sigma^2 \in [\sigma_m^2, \sigma_M^2]$ where $0 < \sigma_m^2 < \sigma_M^2$,

$$\begin{aligned} \sup_{\theta \in \Theta} |q_t(\theta)| &\leq \sup_{\theta \in \Theta} \left\{ \frac{1}{\sigma_m^2} (X_t^2 + m_t^2(\theta)) + \frac{1}{2} |\log(\sigma_M^2)| \right\} \\ \implies \left\| \sup_{\theta \in \Theta} |q_t(\theta)| \right\|_1 &\leq \frac{1}{\sigma_m^2} (\|X_t\|_2^2 + \left\| \sup_{\theta \in \Theta} |m_t(\theta)| \right\|_2^2) + \frac{1}{2} |\log(\sigma_M^2)| \\ &< \infty. \end{aligned} \quad (27)$$

And to conclude with these preliminary bounds, using Cauchy-Schwarz and the triangular inequality, there exists $C > 0$ such as for $t \geq 1$,

$$\begin{aligned} \left\| \sup_{\theta \in \Theta} |q_t(\theta) - \widehat{q}_t(\theta)| \right\|_1 &\leq \frac{1}{2} \left\| \sup_{\theta \in \Theta} |2X_t + m_t(\theta) + \widehat{m}_t(\theta)| \right\|_2 \left\| \sup_{\theta \in \Theta} |\widetilde{m}_t(\theta)| \right\|_2 \\ &\leq \frac{1}{2} \left(2 \|X_0\|_2^2 + \left\| \sup_{\theta \in \Theta} |m_t(\theta)| \right\|_2^2 + \left\| \sup_{\theta \in \Theta} |\widehat{m}_t(\theta)| \right\|_2^2 \right) (C_2 t^{-2\underline{d}})^{1/2} \\ &\leq C t^{-\underline{d}}. \end{aligned} \quad (28)$$

2. From its AR(∞) representation (1), and since $\|X_0\|_2 < \infty$, then $(X_t)_{t \in \mathbb{Z}}$ is a second order ergodic stationary sequence (see Theorem 36.4 in Billingsley, 1995). But for any $\theta \in \Theta$, there exists $H_\theta^q : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$ such that

$$q_t(\theta) = H_\theta^q((\varepsilon_{t-j})_{j \geq 0}),$$

with also $\mathbb{E}[|q_t(\theta)|] < \infty$ from (27). Then using Theorem 36.4 in Billingsley (1995), $(q_t(\theta))_{t \in \mathbb{Z}}$ is an ergodic stationary sequence for any $\theta \in \Theta$ and therefore

$$I_n(\theta) \xrightarrow[n \rightarrow \infty]{a.s.} \mathbb{E}[q_0(\theta)] \quad \text{for any } \theta \in \Theta,$$

with $I_n(\theta)$ defined in (11). Moreover, since Θ is a compact set and since we have $\mathbb{E}[\sup_{\theta \in \Theta} |q_t(\theta)|] < \infty$ from (27), using Theorem 2.2.1. in Straumann (2005), we deduce that $(q_t(\theta))_{t \in \mathbb{Z}}$ also follows a uniform ergodic theorem and we obtain

$$\sup_{\theta \in \Theta} |I_n(\theta) - \mathbb{E}[q_0(\theta)]| \xrightarrow[n \rightarrow \infty]{a.s.} 0. \quad (29)$$

Now, using $\widehat{I}_n(\theta)$ defined in (13), we can write

$$\sup_{\theta \in \Theta} |I_n(\theta) - \widehat{I}_n(\theta)| \leq \frac{1}{n} \sum_{t=1}^n \sup_{\theta \in \Theta} |q_t(\theta) - \widehat{q}_t(\theta)|. \quad (30)$$

In Corollary 1 of Kounias and Weng (1969), it is established that for a \mathbb{L}^1 sequence of r.v. $(Z_t)_t$ and a sequence of positive real numbers $(b_n)_{n \in \mathbb{N}^*}$ such as $b_n \xrightarrow{n \rightarrow \infty} \infty$, then $\sum_{t=1}^{\infty} \frac{\mathbb{E}[|Z_t|]}{b_t} < \infty$ implies

$$\frac{1}{b_n} \sum_{t=1}^n Z_t \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

Therefore, with $b_t = t$ and $Z_t = \sup_{\theta \in \Theta} |q_t(\theta) - \widehat{q}_t(\theta)|$ for $t \in \mathbb{N}^*$, using the inequality (28),

$$\sum_{t=1}^{\infty} \frac{1}{t} \mathbb{E}[\sup_{\theta \in \Theta} |q_t(\theta) - \widehat{q}_t(\theta)|] \leq C \sum_{t=1}^{\infty} t^{-d-1} < \infty \implies \frac{1}{n} \sum_{t=1}^n \sup_{\theta \in \Theta} |q_t(\theta) - \widehat{q}_t(\theta)| \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

Then, using (30) and (29), we deduce:

$$\sup_{\theta \in \Theta} |\widehat{I}_n(\theta) - \mathbb{E}[q_0(\theta)]| \xrightarrow[n \rightarrow \infty]{a.s.} 0. \quad (31)$$

3. Finally, the same argument already detailed in the proof of Theorem 1 of Bardet and Wintenberger (2009) is used: $\theta \in \Theta \mapsto \mathbb{E}[q_0(\theta)]$ has a unique maximum reached in $\theta = \theta^* \in \Theta$ because it is assumed that if $u_n(\theta) = u_n(\theta')$ for all $n \in \mathbb{N}^*$ with $\theta = (\gamma, \sigma^2)$ and $\theta' = (\gamma', \sigma^2)$, then $\theta = \theta'$. This property and the uniform almost sure consistency (31) lead to $\widehat{\theta}_n \xrightarrow[n \rightarrow \infty]{a.s.} \theta^*$. \square

Proof of Theorem 3.2. As a preamble to this proof, since $\widehat{\theta}_n \xrightarrow[n \rightarrow \infty]{a.s.} \theta^*$ by Theorem 3.1, we will be able to reduce the Θ domain. Let $\widetilde{\Theta} \subset \Theta$ be a compact set of \mathbb{R}^p such that:

$$\widetilde{\Theta} = \left\{ \theta \in \Theta, 2d(\theta^*) - 1/2 < \inf_{\theta \in \widetilde{\Theta}} d(\theta) < d(\theta^*) \right\}.$$

Note that $2d(\theta^*) - 1/2 < d(\theta^*)$, so it's still possible to determine $\widetilde{\Theta}$.

In the spirit of (14), let's define

$$\widetilde{\theta}_n = \underset{\theta \in \widetilde{\Theta}}{\text{Argmax}} \widehat{I}_n(\theta).$$

Using Theorem 3.1, it is clear that $\widetilde{\theta}_n \xrightarrow[n \rightarrow \infty]{a.s.} \theta^*$. Moreover, for all $x = (x_1, \dots, x_p) \in \mathbb{R}^p$,

$$\begin{aligned} \mathbb{P}\left(\sqrt{n}(\widehat{\theta}_n - \theta^*) \overset{p}{\times} (-\infty, x_j]\right) &= \mathbb{P}\left(\sqrt{n}(\widehat{\theta}_n - \theta^*) \in \overset{p}{\times}_{j=1} (-\infty, x_j] \mid \widehat{\theta}_n \in \widetilde{\Theta}\right) \mathbb{P}(\widehat{\theta}_n \in \widetilde{\Theta}) \\ &\quad + \mathbb{P}\left(\sqrt{n}(\widehat{\theta}_n - \theta^*) \in \overset{p}{\times}_{j=1} (-\infty, x_j] \mid \widehat{\theta}_n \notin \widetilde{\Theta}\right) \mathbb{P}(\widehat{\theta}_n \notin \widetilde{\Theta}) \\ &= \mathbb{P}\left(\sqrt{n}(\widetilde{\theta}_n - \theta^*) \in \overset{p}{\times}_{j=1} (-\infty, x_j]\right) \mathbb{P}(\widehat{\theta}_n \in \widetilde{\Theta}) \\ &\quad + \mathbb{P}\left(\sqrt{n}(\widetilde{\theta}_n - \theta^*) \in \overset{p}{\times}_{j=1} (-\infty, x_j]\right) \mathbb{P}(\widehat{\theta}_n \notin \widetilde{\Theta}) \end{aligned}$$

Since $\widehat{\theta}_n \xrightarrow[n \rightarrow \infty]{a.s.} \theta^*$ by Theorem 3.1 and therefore $\mathbb{P}(\widehat{\theta}_n \notin \widetilde{\Theta}) \xrightarrow[n \rightarrow \infty]{} 0$ because $\theta^* \in \widetilde{\Theta}$, it is clear that the asymptotic distribution of $\sqrt{n}(\widehat{\theta}_n - \theta^*)$ is the same as the one of $\sqrt{n}(\widetilde{\theta}_n - \theta^*)$. Consequently,

throughout the rest of the proof, Θ will be replaced by $\tilde{\Theta}$ and $\hat{\theta}_n$ by $\tilde{\theta}_n$.

In the sequel, for $\theta \in \tilde{\Theta}$, we will denote $d = d(\theta) - \varepsilon$ and $d_+^* = d^* + \varepsilon$ where $d^* = d(\theta^*)$ is the unknown long-memory parameter, and we chose $\varepsilon > 0$ such as $\varepsilon \leq \frac{1}{6} (1 - 4d(\theta) + 2d(\theta^*))$. Hence, from the definition of $\tilde{\Theta}$, $1 - 4d(\theta) + 2d(\theta^*) > 0$ and

$$4d_+^* - 2d - 1 < 0. \quad (32)$$

From Assumption **(A)**, for any $\theta \in \tilde{\Theta}$ and $t \in \mathbb{Z}$, $\partial_\theta m_t(\theta)$ and $\partial_{\theta^2}^2 m_t(\theta)$ a.s. exist with

$$\partial_\theta m_t(\theta) = \sum_{k=1}^{\infty} \partial_\theta u_k(\theta) X_{t-k} \quad \text{and} \quad \partial_{\theta^2}^2 m_t(\theta) = \sum_{k=1}^{\infty} \partial_{\theta^2}^2 u_k(\theta) X_{t-k}.$$

And the same for $\partial_\theta \hat{m}_t(\theta)$, $\partial_\theta \tilde{m}_t(\theta)$, $\partial_{\theta^2}^2 \hat{m}_t(\theta)$ and $\partial_{\theta^2}^2 \tilde{m}_t(\theta)$. However, note that for any $\theta \in \tilde{\Theta}$, $(m_t(\theta))_t$, $(\partial_\theta m_t(\theta))_t$ and $(\partial_{\theta^2}^2 m_t(\theta))_t$ are stationary processes while $(\hat{m}_t(\theta))_t$, $(\tilde{m}_t(\theta))_t$ and their derivatives are not.

Due to these results, for any $\theta \in \tilde{\Theta}$:

$$\partial_\theta q_t(\theta) = \begin{pmatrix} \partial_\gamma q_t(\theta) \\ \partial_{\sigma^2} q_t(\theta) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sigma^2} \partial_\gamma m_t(\theta) (X_t - m_t(\theta)) \\ \frac{1}{2\sigma^4} ((X_t - m_t(\theta))^2 - \sigma^2) \end{pmatrix}, \quad (33)$$

and the same for $\partial_\theta \hat{q}_t(\theta)$ by replacing $m_t(\theta)$ by $\hat{m}_t(\theta)$. Once again for any $\theta \in \tilde{\Theta}$, $(\partial_\theta q_t(\theta))_t$ is a stationary process, while $(\partial_\theta \hat{q}_t(\theta))_t$ is not. Finally, for all $\theta \in \tilde{\Theta}$, define

$$\partial_\theta L_n(\theta) = \frac{1}{n} \sum_{t=1}^n \partial_\theta q_t(\theta) \quad \text{and} \quad \partial_\theta \hat{L}_n(\theta) = \frac{1}{n} \sum_{t=1}^n \partial_\theta \hat{q}_t(\theta).$$

Following the same reasoning it can be shown that for any $t \in \mathbb{Z}$, $\theta \in \tilde{\Theta} \mapsto q_t(\theta)$ and $\theta \in \tilde{\Theta} \mapsto \hat{q}_t(\theta)$ are a.s. $\mathcal{C}^2(\tilde{\Theta})$ functions and therefore the random matrices $\partial_{\theta^2}^2 L_n(\theta)$ and $\partial_{\theta^2}^2 \hat{L}_n(\theta)$ a.s. exist.

The proof of Theorem 3.2 will be decomposed in 3 parts:

1. First, as it was already established in Bardet and Wintenberger (2009), $(\partial_\theta q_t(\theta^*))_t$ is a stationary ergodic martingale difference since with the σ -algebra $\mathcal{F}_t = \sigma\{(X_{t-k})_{k \geq 1}\}$,

$$\mathbb{E}[\partial_\theta q_t(\theta^*) \mid \mathcal{F}_t] = 0,$$

because (X_t) is a causal process and ε_t is independent of \mathcal{F}_t and $\mathbb{E}[\varepsilon_0^2] = 1$.

Now since $\mathbb{E}[\|\partial_\theta q_0(\theta^*)\|^2] < \infty$ from the same arguments as in the proof of the consistency of the estimator. Then the central limit for stationary ergodic martingale difference, Theorem 18.3 of Billingsley (1968) can be applied and

$$\sqrt{n} \partial_\theta L_n(\theta^*) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, G^*), \quad (34)$$

since $\mathbb{E}[\partial_\theta q_0(\theta^*)] = 0$ and where $G^* := \mathbb{E}[\partial_\theta q_0(\theta^*) \times {}^t(\partial_\theta q_0(\theta^*))]$.

2. We are going to prove that:

$$n \mathbb{E} \left[\sup_{\theta \in \tilde{\Theta}} \|\partial_{\theta} \widehat{L}_n(\theta) - \partial_{\theta} L_n(\theta)\|^2 \right] \xrightarrow{n \rightarrow \infty} 0. \quad (35)$$

Using a line of reasoning already used in Beran and Schützner (2009, Lemma 1 and 2) and Bardet (2023, Lemma 5.1 3.), and derived from Parzen (1995, Theorem 3.B), there exists $C > 0$ such that:

$$\mathbb{E} \left[\sup_{\theta \in \tilde{\Theta}} \|\partial_{\theta} \widehat{L}_n(\theta) - \partial_{\theta} L_n(\theta)\|^2 \right] \leq C \sup_{\theta \in \tilde{\Theta}} \mathbb{E} \left[\|\partial_{\theta} \widehat{L}_n(\theta) - \partial_{\theta} L_n(\theta)\|^2 \right],$$

because we assumed that $\theta \rightarrow u_n(\theta)$ is a $\mathcal{C}^{p+1}(\tilde{\Theta})$ function and therefore $\partial_{\theta} \widehat{L}_n(\theta) - \partial_{\theta} L_n(\theta)$ is a $\mathcal{C}^p(\tilde{\Theta})$ function.

Then, for $\theta \in \tilde{\Theta}$,

$$\partial_{\gamma} q_t(\theta) - \partial_{\gamma} \widehat{q}_t(\theta) = \frac{1}{\sigma^2} \left(\partial_{\gamma} \widetilde{m}_t(\theta) (X_t - m_t(\theta)) + \partial_{\gamma} \widehat{m}_t(\theta) \widetilde{m}_t(\theta) \right).$$

As a consequence, for $\theta \in \tilde{\Theta}$,

$$\begin{aligned} & n \mathbb{E} \left[\|\partial_{\theta} \widehat{L}_n(\theta) - \partial_{\theta} L_n(\theta)\|^2 \right] \\ &= \frac{1}{n \sigma^4} \left(2 \sum_{1 \leq s < t \leq n} \mathbb{E} \left[t \left(\partial_{\gamma} \widetilde{m}_t(\theta) (X_t - m_t(\theta)) + \partial_{\gamma} \widehat{m}_t(\theta) \widetilde{m}_t(\theta) \right) \left(\partial_{\gamma} \widetilde{m}_s(\theta) (X_s - m_s(\theta)) + \partial_{\gamma} \widehat{m}_s(\theta) \widetilde{m}_s(\theta) \right) \right] \right. \\ & \quad \left. + \sum_{t=1}^n \mathbb{E} \left[t \left(\partial_{\gamma} \widetilde{m}_t(\theta) (X_t - m_t(\theta)) + \partial_{\gamma} \widehat{m}_t(\theta) \widetilde{m}_t(\theta) \right) \left(\partial_{\gamma} \widetilde{m}_t(\theta) (X_t - m_t(\theta)) + \partial_{\gamma} \widehat{m}_t(\theta) \widetilde{m}_t(\theta) \right) \right] \right) \\ &= \frac{1}{n \sigma^4} (I_1 + I_2). \quad (36) \end{aligned}$$

Concerning I_1 , since $X_t = \sigma^* \varepsilon_t + m_t(\theta^*)$ and since ε_t is independent to all the other terms because $s < t$, we deduce that $(X_t - m_t(\theta))$ can be replaced by $n_t(\theta, \theta^*) = (m_t(\theta^*) - m_t(\theta))$. As a consequence, after its expansion, I_1 can be written as a sum of 6 expectations of products of 4 linear combinations of (ε_t) . Moreover, if for $j = 1, \dots, 4$, $Y_{t_j}^{(j)} = \sum_{k=0}^{\infty} \beta_k^{(j)} \xi_{t_j-k}$, where $t_1 \leq t_2 \leq t_3 \leq t_4$, $(\beta_n^{(j)})_{n \in \mathbb{N}}$ are 4 real sequences and $(\xi_t)_{t \in \mathbb{Z}}$ is a white noise such as $\mathbb{E}[\xi_0^2] = 1$ and $\mathbb{E}[\xi_0^4] = \mu_4 < \infty$, then:

$$\begin{aligned} \mathbb{E} \left[\prod_{j=1}^4 Y_{t_j}^{(j)} \right] &= (\mu_4 - 3) \sum_{k=0}^{\infty} \beta_k^{(1)} \beta_{t_2-t_1+k}^{(2)} \beta_{t_3-t_1+k}^{(3)} \beta_{t_4-t_1+k}^{(4)} + \mathbb{E}[Y_{t_1}^{(1)} Y_{t_2}^{(2)}] \mathbb{E}[Y_{t_3}^{(3)} Y_{t_4}^{(4)}] \\ & \quad + \mathbb{E}[Y_{t_1}^{(1)} Y_{t_3}^{(3)}] \mathbb{E}[Y_{t_2}^{(2)} Y_{t_4}^{(4)}] + \mathbb{E}[Y_{t_1}^{(1)} Y_{t_4}^{(4)}] \mathbb{E}[Y_{t_2}^{(2)} Y_{t_3}^{(3)}]. \end{aligned}$$

Now, consider for example $Y_{t_1}^{(1)} = \partial_{\gamma} \widetilde{m}_s(\theta)$, $Y_{t_2}^{(2)} = (X_s - m_s(\theta))$, $Y_{t_3}^{(3)} = \partial_{\gamma} \widehat{m}_t(\theta)$ and $Y_{t_4}^{(4)} = \widetilde{m}_t(\theta)$. From Lemma 5.1 and for any used sequence $(\beta_k^{(j)})_{k \in \mathbb{N}}$, there exists $C > 0$ such as for any $k \in \mathbb{N}$:

$$|\beta_k^{(1)}| \leq \frac{C}{s^d (k+1)^{1-d^*}}, \quad |\beta_k^{(4)}| \leq \frac{C}{t^d (k+1)^{1-d^*}} \quad \text{and} \quad \max(|\beta_k^{(2)}|, |\beta_k^{(3)}|) \leq \frac{C}{(k+1)^{1-d^*}}.$$

As a consequence, with $s < t$,

$$\begin{aligned} \left| (\mu_4 - 3) \sum_{k=0}^{\infty} \beta_k^{(1)} \beta_k^{(2)} \beta_{t-s+k}^{(3)} \beta_{t-s+k}^{(4)} \right| &\leq \frac{C}{s^d t^d} \sum_{k=1}^{\infty} \frac{1}{k^{2-2d_+^*}} \frac{1}{(k+t-s)^{2-2d_+^*}} \\ &\leq \frac{C}{s^d t^d (t-s)^{2-2d_+^*}}. \end{aligned} \quad (37)$$

And we obtain the same bound for any quadruple products appearing in I_1 .

Consider now the other terms of I_1 . Using Lemmas 5.3 and 5.4, we obtain for any $\theta \in \tilde{\Theta}$ and $s < t$:

$$\begin{aligned} \bullet \quad &\left| \mathbb{E}[Y_{t_1}^{(1)} Y_{t_2}^{(2)}] \mathbb{E}[Y_{t_3}^{(3)} Y_{t_4}^{(4)}] \right| = \left| \mathbb{E}[\partial_\gamma \tilde{m}_s(\theta) (X_s - m_s(\theta))] \mathbb{E}[\partial_\gamma \hat{m}_t(\theta) \tilde{m}_t(\theta)] \right| \\ &= \left| \mathbb{E}[\partial_\gamma \tilde{m}_s(\theta) n_s(\theta, \theta^*)] \right| \left| \mathbb{E}[\partial_\gamma \hat{m}_t(\theta) \tilde{m}_t(\theta)] \right| \\ &\leq C \frac{1}{s^{1+d-2d_+^*}} \frac{1}{t^{1+d-2d_+^*}}; \\ \bullet \quad &\left| \mathbb{E}[Y_{t_1}^{(1)} Y_{t_3}^{(3)}] \mathbb{E}[Y_{t_2}^{(2)} Y_{t_4}^{(4)}] \right| = \left| \mathbb{E}[\partial_\gamma \tilde{m}_s(\theta) \partial_\gamma \hat{m}_t(\theta)] \mathbb{E}[(X_s - m_s(\theta)) \tilde{m}_t(\theta)] \right| \\ &= \left| \mathbb{E}[\partial_\gamma \tilde{m}_s(\theta) \partial_\gamma \hat{m}_t(\theta)] \right| \left| \mathbb{E}[n_s(\theta, \theta^*) \tilde{m}_t(\theta)] \right| \\ &\leq C \left(\frac{1}{s^d t^{1-2d_+^*}} + \frac{1}{s^{1+2d-2d_+^*}} \right) \left(\frac{1}{t^{1+d_s-2d_+^*}} + \frac{1}{t^{1+2d-2d_+^*}} \right) \\ \bullet \quad &\left| \mathbb{E}[Y_{t_1}^{(1)} Y_{t_4}^{(4)}] \mathbb{E}[Y_{t_2}^{(2)} Y_{t_3}^{(3)}] \right| = \left| \mathbb{E}[\partial_\gamma \tilde{m}_s(\theta) \tilde{m}_t(\theta)] \mathbb{E}[(X_s - m_s(\theta)) \partial_\gamma \hat{m}_t(\theta)] \right| \\ &= \left| \mathbb{E}[\partial_\gamma \tilde{m}_s(\theta) \tilde{m}_t(\theta)] \right| \left| \mathbb{E}[n_s(\theta, \theta^*) \partial_\gamma \hat{m}_t(\theta)] \right| \\ &\leq C \frac{1}{s^d t^{1-2d_+^*+d}} \frac{1}{(t-s)^{1-2d_+^*}} \end{aligned}$$

Using these inequalities as well as (37), we deduce from classical comparisons between sums and integrals:

$$\begin{aligned} &\sum_{1 \leq s < t \leq n} \mathbb{E} \left[{}^t \left(\partial_\gamma \tilde{m}_t(\theta) (X_t - m_t(\theta)) \partial_\gamma \hat{m}_s(\theta) \tilde{m}_s(\theta) \right) \right] \\ &\leq C \sum_{1 \leq s < t \leq n} \frac{\mu_4 - 3}{s^d t^d (t-s)^{2-2d_+^*}} + \left(\frac{1}{s^d t^{1-2d_+^*}} + \frac{1}{s^{1+2d-2d_+^*}} \right) \left(\frac{1}{t^{1+d_s-2d_+^*}} + \frac{1}{t^{1+2d-2d_+^*}} \right) \\ &\quad + \frac{1}{s^{1+d-2d_+^*}} \frac{1}{t^{1+d-2d_+^*}} + \frac{1}{s^d t^{1-2d_+^*+d}} \frac{1}{(t-s)^{1-2d_+^*}} \\ &\leq C \left(\int_1^n x^{2d_+^*-1-2d} dx + \int_1^n \frac{dx}{x^{2+d-2d_+^*}} \int_1^x \frac{dy}{y^{d-2d_+^*}} + \int_1^n \frac{dx}{x^{2+2d-4d_+^*}} \int_1^x \frac{dy}{y^d} + \int_1^n \frac{dx}{x^{1+d}} \int_1^x \frac{dy}{y^{1+2d-4d_+^*}} \right. \\ &\quad \left. + \int_1^n \frac{dx}{x^{1+2d-2d_+^*}} \int_1^x \frac{dy}{y^{1+2d-2d_+^*}} + \int_1^n \frac{dx}{x^{1+d-2d_+^*}} \int_1^x \frac{dy}{y^{1+d-2d_+^*}} + \int_1^n \frac{dx}{x^{1+d-2d_+^*}} \int_1^x \frac{dy}{y^d (x-y)^{1-2d_+^*}} \right) \\ &\leq C (n^{2d_+^*-2d} + n^{4d_+^*-2d} + n^{4d_+^*-3d} + n^{4d_+^*-3d} + n^{4d_+^*-4d} + n^{4d_+^*-2d} + n^{4d_+^*-2d}) \\ &\leq C n^{4d_+^*-2d}. \end{aligned}$$

We obtain exactly the same bounds if we consider the 3 others expectations, *i.e.*

$$\mathbb{E} \left[{}^t \left(\partial_\gamma \tilde{m}_t(\theta) (X_t - m_t(\theta)) \right) \partial_\gamma \tilde{m}_s(\theta) (X_s - m_s(\theta)) \right], \mathbb{E} \left[{}^t \left(\partial_\gamma \hat{m}_t(\theta) \tilde{m}_t(\theta) \right) \partial_\gamma \tilde{m}_s(\theta) (X_s - m_s(\theta)) \right]$$

or $\mathbb{E} \left[t \left(\partial_\gamma \widehat{m}_t(\theta) \widetilde{m}_t(\theta) \right) \partial_\gamma \widehat{m}_s(\theta) \widetilde{m}_s(\theta) \right]$. As a consequence, we finally obtain:

$$\frac{1}{\sigma^4 n} I_1 \leq C n^{4d^* - 2d - 1} \quad \text{for any } n \in \mathbb{N}^*. \quad (38)$$

Now consider the term I_2 in (36) and therefore the case $s = t$. For $Y_{t_1}^{(1)} = \partial_\gamma \widetilde{m}_t(\theta)$, $Y_{t_2}^{(2)} = (X_t - m_t(\theta))$, $Y_{t_3}^{(3)} = \partial_\gamma \widehat{m}_t(\theta)$ and $Y_{t_4}^{(4)} = \widetilde{m}_t(\theta)$, and the coefficient $(\beta_k^{(j)})$ defined previously, we obtain:

$$\left| (\mu_4 - 3) \sum_{k=0}^{\infty} \beta_k^{(1)} \beta_k^{(2)} \beta_k^{(3)} \beta_k^{(4)} \right| \leq C \sum_{k=1}^{\infty} \frac{1}{t^{2d}} \frac{1}{k^{4-4d^*}} \leq C \frac{1}{t^{2d}}. \quad (39)$$

Moreover, using the same inequalities as in the case $s < t$, we obtain:

$$\begin{aligned} \bullet \quad & \left| \mathbb{E}[Y_{t_1}^{(1)} Y_{t_2}^{(2)}] \mathbb{E}[Y_{t_3}^{(3)} Y_{t_4}^{(4)}] \right| \leq C \frac{1}{t^{2+2d-4d^*}}; \\ \bullet \quad & \left| \mathbb{E}[Y_{t_1}^{(1)} Y_{t_3}^{(3)}] \mathbb{E}[Y_{t_2}^{(2)} Y_{t_4}^{(4)}] \right| \leq C \frac{1}{t^{2+2d-4d^*}} \\ \bullet \quad & \left| \mathbb{E}[Y_{t_1}^{(1)} Y_{t_4}^{(4)}] \mathbb{E}[Y_{t_2}^{(2)} Y_{t_3}^{(3)}] \right| \leq C \frac{1}{t^{1-2d^*+2d}}. \end{aligned}$$

Therefore,

$$\sum_{t=1}^n \mathbb{E} \left[t \left(\partial_\gamma \widetilde{m}_t(\theta) (X_t - m_t(\theta)) \partial_\gamma \widehat{m}_t(\theta) \widetilde{m}_t(\theta) \right) \right] \leq C \sum_{t=1}^n \frac{\mu_4 - 3}{t^{2d}} + \frac{1}{t^{1-2d^*+2d}} \leq C n^{1-2d}.$$

As a consequence, we finally obtain that there exists $C > 0$ such that:

$$\frac{1}{\sigma^4 n} I_2 \leq C n^{-2d} \quad \text{for any } n \in \mathbb{N}^*. \quad (40)$$

Therefore, from (38) and (40), we deduce that there exists $C > 0$ such that for any $n \in \mathbb{N}^*$:

$$n \mathbb{E} \left[\left\| \partial_\theta \widehat{L}_n(\theta) - \partial_\theta L_n(\theta) \right\|^2 \right] \leq C (n^{-2d} + n^{4d^* - 2d - 1}) \xrightarrow[n \rightarrow \infty]{} 0, \quad (41)$$

from (32).

3. For $\theta \in \widetilde{\Theta}$ and $n \in \mathbb{N}^*$, since $\partial_{\theta^2}^2 \widehat{L}_n(\theta)$ is a.s. a $\mathcal{C}^2(\widetilde{\Theta})$ function, the Taylor-Lagrange expansion implies:

$$\sqrt{n} \partial_\theta \widehat{L}_n(\theta^*) = \sqrt{n} \partial_\theta \widehat{L}_n(\widetilde{\theta}_n) + \partial_{\theta^2}^2 \widehat{L}_{\theta^2}(\widetilde{\theta}_n) \times \sqrt{n} (\theta^* - \widetilde{\theta}_n)$$

where $\widetilde{\theta}_n = c \widetilde{\theta}_n + (1 - c) \theta^*$ and $0 < c < 1$. But $\partial_\theta \widehat{L}_n(\widetilde{\theta}_n) = 0$ because $\widetilde{\theta}_n$ is the unique local extremum of $\theta \rightarrow \widehat{L}_n(\theta)$. Therefore,

$$\sqrt{n} \partial_\theta \widehat{L}_n(\theta^*) = \partial_{\theta^2}^2 \widehat{L}_n(\widetilde{\theta}_n) \times \sqrt{n} (\theta^* - \widetilde{\theta}_n). \quad (42)$$

Now, $\mathbb{E} \left[\left\| \partial_{\theta^2}^2 q_0(\theta) \right\| \right] < \infty$ from the same arguments as in the proof of the consistency of the estimator, and using Theorem 36.4 in Billingsley (1995), $(\partial_{\theta^2}^2 q_t(\theta))_{t \in \mathbb{Z}}$ is an ergodic stationary sequence for any $\theta \in \widetilde{\Theta}$. Moreover $\widetilde{\theta}_n \xrightarrow[n \rightarrow \infty]{a.s.} \theta^*$ since $\widetilde{\theta}_n \xrightarrow[n \rightarrow \infty]{a.s.} \theta^*$. Hence:

$$\partial_{\theta^2}^2 L_n(\widetilde{\theta}_n) \xrightarrow[n \rightarrow \infty]{a.s.} \mathbb{E} \left[\partial_{\theta^2}^2 q_0(\theta) \right] = F(\theta^*).$$

Moreover, using the same arguments as in Lemma 4 of [4], we have:

$$\sup_{\theta \in \tilde{\Theta}} \left\| \partial_{\theta^2}^2 L_n(\theta) - \partial_{\theta^2}^2 \widehat{L}_n(\theta) \right\| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 \implies \partial_{\theta^2}^2 \widehat{L}_n(\bar{\theta}_n) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} F(\theta^*). \quad (43)$$

Usual calculations show that:

$$F(\theta^*) = - \begin{pmatrix} M^* & 0 \\ 0 & \frac{1}{2\sigma^{*4}} \end{pmatrix} \quad \text{and} \quad G(\theta^*) = \begin{pmatrix} M^* & 0 \\ 0 & \frac{\mu_4^* - 1}{4\sigma^{*4}} \end{pmatrix},$$

$$\text{with} \quad M^* = \frac{1}{\sigma^{*2}} \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \partial_{\gamma} u_k((\gamma^*, 0))^t (\partial_{\gamma} u_{\ell}((\gamma^*, 0))) r_X(\ell - k)$$

where $G(\theta^*) = \mathbb{E} [\partial_{\theta} q_0(\theta^*)^t \partial_{\theta} q_0(\theta^*)]$ has already been defined in (34).

Thanks to the formula for M^* , we can deduce that F^* is invertible. Indeed, M^* is invertible if and only if $\mathbb{E} [\partial_{\theta} q_0(\theta^*)^t \partial_{\theta} q_0(\theta^*)]$ is invertible and therefore if and only if for all $v \in \mathbb{R}^{p-1}$, ${}^t v \mathbb{E} [\partial_{\gamma} q_0(\theta^*)^t \partial_{\theta} q_0(\theta^*)] v = \mathbb{E} [({}^t v \partial_{\gamma} q_0(\theta^*))^2] = 0$ or ${}^t v \partial_{\gamma} q_0(\theta^*) = 0$ a.s. implies $v = 0$. Or, pour $v \in \mathbb{R}^{p-1}$,

$$\begin{aligned} {}^t v \partial_{\gamma} q_0(\theta^*) = 0 \quad \text{a.s.} &\implies \frac{1}{\sigma^{*2}} \varepsilon_0 \sum_{k=1}^{\infty} {}^t v \partial_{\gamma} u_k(\theta^*) X_{-k} = 0 \quad \text{a.s.} \\ &\implies \sum_{k=1}^{\infty} {}^t v \partial_{\gamma} u_k(\theta^*) X_{-k} = 0 \quad \text{a.s.} \quad (\varepsilon_0 \text{ is independent to } \mathcal{F}_0) \\ &\implies {}^t v \partial_{\gamma} u_k(\theta^*) = 0 \quad \text{for all } k \in \mathbb{N}^* \\ &\implies v = 0 \quad \text{from (16).} \end{aligned}$$

Now, from (34) and (41), we deduce that:

$$\sqrt{n} \partial_{\theta} \widehat{L}_n(\theta^*) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, G(\theta^*)),$$

and since $F(\theta^*)$ is a definite negative matrix, from (42) we deduce that

$$\sqrt{n} (\tilde{\theta}_n - \theta^*) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, F(\theta^*)^{-1} G(\theta^*) F(\theta^*)^{-1}). \quad (44)$$

Finally, from the previous computations of $G(\theta^*)$ and $F(\theta^*)$, we deduce (17). □

5.2 Proofs of additional lemmas

Lemma 5.1. Under the assumptions of Theorem 3.1, for any $\theta \in \Theta$ and $t \in \mathbb{Z}$ or $t \in \mathbb{N}^*$, with $m_t(\theta)$, $\widehat{m}_t(\theta)$ and $\widetilde{m}_t(\theta)$ respectively defined in (10), (12) and (24), we have:

$$m_t(\theta) = \sum_{k=1}^{\infty} \alpha_k(\theta, \theta^*) \varepsilon_{t-k}, \quad \widehat{m}_t(\theta) = \sum_{k=1}^{\infty} \widehat{\alpha}_{k,t}(\theta, \theta^*) \varepsilon_{t-k} \quad \text{and} \quad \widetilde{m}_t(\theta) = \sum_{k=0}^{\infty} \widetilde{\alpha}_{k,t}(\theta, \theta^*) \varepsilon_{-k},$$

where there exists $C > 0$ such as for any $k \geq 1$ and $t \in \mathbb{N}^*$,

$$\max(|\alpha_k(\theta, \theta^*)|, |\widehat{\alpha}_{k,t}(\theta, \theta^*)|) \leq \frac{C}{k^{1-d_+^*}} \quad \text{and} \quad |\widetilde{\alpha}_{k,t}(\theta, \theta^*)| \leq \frac{C}{t^d k^{1-d_+^*}}.$$

Moreover, under the assumptions of Theorem 3.2, the same properties also hold for $\partial_{\theta} m_t(\theta)$, $\partial_{\theta} \widehat{m}_t(\theta)$ and $\partial_{\theta} \widetilde{m}_t(\theta)$.

Proof. We know that $X_t = \sum_{\ell=0}^{\infty} a_{\ell}(\theta^*) \varepsilon_{t-\ell}$ for any $t \in \mathbb{Z}$. Then,

$$\begin{aligned} m_t(\theta) &= \sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} u_k(\theta) a_{\ell}(\theta^*) \varepsilon_{t-k-\ell} = \sum_{j=1}^{\infty} \left(\sum_{k=1}^j u_k(\theta) a_{j-k}(\theta^*) \right) \varepsilon_{t-j} = \sum_{j=1}^{\infty} \alpha_j(\theta, \theta^*) \varepsilon_{t-j} \\ \widehat{m}_t(\theta) &= \sum_{k=1}^{t-1} \sum_{\ell=0}^{\infty} u_k(\theta) a_{\ell}(\theta^*) \varepsilon_{t-k-\ell} = \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\min(j, t-1)} u_k(\theta) a_{j-k}(\theta^*) \right) \varepsilon_{t-j} = \sum_{j=1}^{\infty} \widehat{\alpha}_{j,t}(\theta, \theta^*) \varepsilon_{t-j} \\ \widetilde{m}_t(\theta) &= \sum_{k=t}^{\infty} \sum_{\ell=0}^{\infty} u_k(\theta) a_{\ell}(\theta^*) \varepsilon_{t-k-\ell} = \sum_{j=0}^{\infty} \left(\sum_{k=0}^j u_{t+k}(\theta) a_{j-k}(\theta^*) \right) \varepsilon_{t-j} = \sum_{j=0}^{\infty} \widetilde{\alpha}_{j,t}(\theta, \theta^*) \varepsilon_{t-j} \end{aligned}$$

As a consequence, using $|a_{\ell}(\theta^*)| \leq C \ell^{d_+^* - 1}$ and $|u_{\ell}(\theta)| \leq C \ell^{-d-1}$ for any $\ell \in \mathbb{N}^*$, we obtain:

$$\begin{aligned} |\alpha_j(\theta, \theta^*)| &\leq C \sum_{k=1}^j \frac{1}{k^{1+d}} \frac{1}{(1+j-k)^{1-d_+^*}} \\ &\leq C \left(\frac{1}{(j/2)^{1-d_+^*}} \sum_{k=1}^{j/2} \frac{1}{k^{1+d}} + \frac{1}{(j/2)^{1+d}} \sum_{k=j/2}^j \frac{1}{(1+j-k)^{1-d_+^*}} \right) \\ &\leq \frac{C}{j^{1-d_+^*}}. \end{aligned}$$

Using the same kind of decomposition, we obtain the other bounds. \square

Lemma 5.2. For any $\alpha > 1$, $\beta \in (0, 1)$, there exists $C > 0$ such as for any $1 \leq a$,

$$\begin{aligned} I_{\alpha}(a) &= \sum_{k=1}^{\infty} \frac{1}{k^{\alpha} (k+a)^{\alpha}} \leq \frac{C}{a^{\alpha}} \\ I_{\alpha}(a, b) &= \sum_{k=1}^{\infty} \frac{1}{(k+a)^{\alpha} (k+b)^{\alpha}} \leq \frac{C}{a^{\alpha-1} b^{\alpha}} \quad \text{for any } b > a \geq 1 \\ J_{\alpha, \beta}(0, a) &= \sum_{k=1}^{\infty} \frac{1}{(k+a)^{\alpha} k^{\beta}} \leq \frac{C}{a^{\alpha+\beta-1}} \\ J_{\alpha, \beta}(a, 0) &= \sum_{k=1}^{\infty} \frac{1}{k^{\alpha} (k+a)^{\beta}} \leq \frac{C}{a^{\beta}} \\ J_{\alpha, \beta}(a, b) &= \sum_{k=1}^{\infty} \frac{1}{(k+a)^{\beta} (k+b)^{\alpha}} \leq \frac{C}{a^{\beta} b^{\alpha-1}} \min\left(1, \frac{a}{b}\right) \quad \text{for any } b \geq 1 \end{aligned}$$

Lemma 5.3. Under the assumptions of Theorem 3.1, there exists $C > 0$ such as for any $\theta \in \Theta$ and $1 \leq s \leq t \leq n$,

$$|\mathbb{E}[\widetilde{m}_s(\theta) \widetilde{m}_t(\theta)]| \leq \frac{C}{s^d t^{1-2d_+^*+d}}. \quad (45)$$

Proof. Using the bounds of functions I_{1+d} and $J_{1+d,1-2d}$ defined in Lemma 5.2, we obtain

$$\begin{aligned}
\mathbb{E}[\tilde{m}_s(\theta) \tilde{m}_t(\theta)] &= \sum_{k=s}^{\infty} \sum_{\ell=t}^{\infty} u_k(\theta) u_\ell(\theta) r_X(t-s+k-\ell) \\
&\leq C \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{1}{(s+k)^{1+d}} \frac{1}{(t+\ell)^{1+d}} \frac{1}{(1+|k-\ell|)^{1-2d_+^*}} \\
&\leq C \left(\sum_{j=1}^{\infty} \frac{1}{(1+j)^{1-2d_+^*}} \sum_{\ell=1}^{\infty} \frac{1}{(\ell+s+j)^{1+d}(\ell+t)^{1+d}} + \sum_{k=1}^{\infty} \frac{1}{(k+s)^{1+d}(k+t)^{1+d}} \right) \\
&\quad + \sum_{j=1}^{\infty} \frac{1}{(1+j)^{1-2d_+^*}} \sum_{k=1}^{\infty} \frac{1}{(k+s)^{1+d}(k+t+j)^{1+d}} \\
&\leq C \left(I_{1+d}(s, t) + \sum_{j=1}^{\infty} \frac{1}{(1+j)^{1-2d_+^*}} (I_{1+d}(s+j, t) + I_{1+d}(s, t+j)) \right) \\
&\leq C \left(\frac{1}{s^d t^{1+d}} + \frac{1}{s^d} J_{1+d,1-2d_+^*}(0, t) + \frac{1}{t^{d+1}} \sum_{j=1}^{t-s} \frac{1}{(1+j)^{1-2d_+^*}} \frac{1}{(s+j)^d} \right. \\
&\quad \left. + \frac{1}{t^d} \sum_{j=t-s}^{\infty} \frac{1}{(1+j)^{1-2d_+^*}} \frac{1}{(s+j)^{1+d}} \right) \\
&\leq C \left(\frac{1}{s^d t^{1+d}} + \frac{1}{s^d t^{1-2d_+^*+d}} + \frac{1}{s^d t^{d+1}} (t-s+1)^{2d_+^*} + \frac{1}{t^d} J_{1+d,1-2d_+^*}(t-s, t) \right) \\
&\leq C \left(\frac{1}{s^d t^{1+d}} + \frac{1}{s^d t^{1-2d_+^*+d}} + \frac{1}{s^d t^{1-2d_+^*+d}} + \frac{1}{t} J_{1+d,1-2d_+^*}(t-s, t) \right) \leq \frac{C}{s^d t^{1-2d_+^*+d}}.
\end{aligned}$$

□

Lemma 5.4. Under the assumptions of Theorem 3.1, there exists $C > 0$ such as for any $\theta \in \Theta$ and any $1 \leq s$ and $1 \leq t$,

$$\left| \mathbb{E}[\tilde{m}_s(\theta) m_t(\theta)] \right| \leq \begin{cases} C \left(\frac{1}{s^d t^{1-2d_+^*}} + \frac{(1+t-s)^{2d_+^*}}{s^{1+2d}} \right) & \text{if } s \leq t \\ C \left(\frac{t^{2d_+^*}}{s^{1+d}} + \frac{(1+s-t)^{2d_+^*}}{s^{1+d} t^d} \right) & \text{if } s \geq t \end{cases}. \quad (46)$$

Proof.

$$\begin{aligned}
\mathbb{E}[\tilde{m}_s(\theta) m_t(\theta)] &= \sum_{k=s}^{\infty} \sum_{\ell=1}^{\infty} u_k(\theta) u_{\ell}(\theta) r_X(t-s+k-\ell) \\
&\leq C \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{1}{(s+k)^{1+d}} \frac{1}{\ell^{1+d}} \frac{1}{(1+|t+k-\ell|)^{1-2d^*}} \\
&\leq C \left(\sum_{j=1}^{\infty} \frac{1}{(1+t+j)^{1-2d^*}} \sum_{\ell=1}^{\infty} \frac{1}{(\ell+s+j)^{1+d}\ell^{1+d}} \right. \\
&\quad \left. + \sum_{j=1}^t \frac{1}{(1+t-j)^{1-2d^*}} \sum_{k=1}^{\infty} \frac{1}{(k+s)^{1+d}(k+j)^{1+d}} \right. \\
&\quad \left. + \sum_{j=t}^{\infty} \frac{1}{(1+j-t)^{1-2d^*}} \sum_{k=1}^{\infty} \frac{1}{(k+s)^{1+d}(k+j)^{1+d}} \right) \\
&\leq C \left(\sum_{j=1}^{\infty} \frac{1}{(t+j)^{1-2d^*}} I_{1+d}(s+j, 0) + \sum_{j=1}^t \frac{1}{j^{1-2d^*}} I_{1+d}(s, t-j) + \sum_{j=1}^{\infty} \frac{1}{j^{1-2d^*}} I_{1+d}(s, j+t) \right).
\end{aligned}$$

Then, if $s \leq t$,

$$\begin{aligned}
|\mathbb{E}[\tilde{m}_s(\theta) m_t(\theta)]| &\leq C \left(J_{1+d, 1-2d^*}(t, s) + \frac{1}{s^d} \sum_{j=1}^{t-s} \frac{1}{j^{1-2d^*}} \frac{1}{(t-j)^{1+d}} \right. \\
&\quad \left. + \frac{1}{s^{1+d}} \sum_{j=1}^s \frac{1}{(t-j)^{1-2d^*}} \frac{1}{(s-j)^{1+d}} + \frac{1}{s^d} J_{1+d, 1-2d^*}(0, t) \right) \\
&\leq C \left(\frac{1}{s^d t^{1-2d^*}} + \frac{1}{s^d t^{1-2d^*}} + \frac{1}{s^{1+2d-2d^*}} + \frac{1}{s^d t^{1+d-2d^*}} \right) \\
&\leq C \left(\frac{1}{s^d t^{1-2d^*}} + \frac{1}{s^{1+2d-2d^*}} \right).
\end{aligned}$$

And if $s > t$,

$$\begin{aligned}
|\mathbb{E}[\tilde{m}_s(\theta) m_t(\theta)]| &\leq C \left(J_{1+d, 1-2d^*}(t, s) + \frac{1}{s^{1+d}} \sum_{j=1}^t \frac{1}{j^{1-2d^*}} \frac{1}{(t-j)^{1+d}} \right. \\
&\quad \left. + \frac{1}{s^{1+d}} \sum_{j=1}^{s-t} \frac{1}{j^{1-2d^*}} \frac{1}{(t+j)^d} + \frac{1}{s^d} \sum_{j=s-t}^{\infty} \frac{1}{j^{1-2d^*}} \frac{1}{(t+j)^{1+d}} \right) \\
&\leq C \left(\frac{1}{s^{1+d} t^{1-2d^*}} + \frac{1}{s^{1+d} t^{1-2d^*}} + \frac{1}{s^{1+2d-2d^*}} + \frac{1}{s^{1+2d-2d^*}} \right) \\
&\leq C \left(\frac{1}{s^{1+d} t^{1-2d^*}} + \frac{1}{s^{1+2d-2d^*}} \right).
\end{aligned}$$

□

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