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HEAT EQUATION FROM A DETERMINISTIC DYNAMICS

GIOVANNI CANESTRARI, CARLANGEOLO LIVERANI, AND STEFANO OLLA

ABSTRACT. We derive the heat equation for the thermal energy under diffusive space-time scaling for a purely deterministic microscopic dynamics satisfying Newton equations perturbed by an external chaotic force acting like a magnetic field.

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1. INTRODUCTION

An outstanding conceptual issue, going back, at least, to the ancient philosophical disputes between Zeno and Democritus 2400 years ago, stems from the fact that the world around us behaves like a continuum (we describe it using partial differential equations), yet it consists of atoms, hence it is discrete in nature (as conclusively proven by Einstein [30] who showed how Brownian motion may emerge from the microscopic dynamics).

The derivation of macroscopic PDE evolution laws from microscopic ones has finally become a well-posed problem with the foundation of statistical mechanics by Boltzmann, Gibbs, and Maxwell. Thanks to the axiomatization of probability by Kolmogorov, this problem can now be seen as a purely mathematical one. The relevance of this line of research in mathematics has been acknowledged by Hilbert in his famous sixth problem [39].

The mathematical problem is outlined as follows: for dynamics with a large number of degrees of freedom, some conserved quantities (e.g. energy, momentum, density of particles) evolve slowly and, properly rescaling space and time, the evolution of their distributions is expected to converge to the solution of macroscopic hydrodynamic equations (such as Euler or heat equations). On the contrary, all other observables are expected to exhibit fast fluctuations for generic systems. The observed macroscopic equation depends on the space-time scaling chosen. In dynamics where momentum is conserved, the first natural space-time scale is the hyperbolic one, and in this limit arises the Euler system of equations for a compressible gas. The first *formal* derivation of the Euler equation from the microscopic Hamiltonian dynamics of interacting particles was done by Morrey [49].

At the moment, there is no rigorous derivation of the Euler equation from a completely deterministic Hamiltonian dynamics. The main problem is the lack of good ergodic properties of the large microscopic dynamics that guarantee that energy, momentum and density of particles are the only conserved quantities that evolve on the macroscopic scale. Progress has been made only by adding to the dynamics conservative noise terms that destroy all other integrals of motion, like local random exchanges of velocities among particles [50, 34, 47, 35].

The mathematical problem is even more difficult in diffusive scaling. This arises in dynamics that do not conserve momentum or have initial conditions with null momentum. In this scaling, if density is fixed and energy is the only

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conserved quantity, the expected macroscopic equation is the heat equation

$$(1.1) \quad \partial_t e = \nabla D(e) \nabla e,$$

where $D(e)$ is the thermal diffusivity that is given by the Green-Kubo formula associated to the space-time correlations of the microscopic energy currents. For completely deterministic models even the proof of the existence of the thermal diffusivity D is missing. In fact, (1.1) is not always valid and in one dimensional systems with momentum conservation it is expected a superdiffusion of the energy driven by a fractional laplacian, as confirmed by numerical simulation and similar superdiffusion proven with conservative noise (cf. [45, 5] and references within).

Important progress has been achieved in the so-called kinetic theory of gases through rarefaction density limits (the so called Grad limit). We can mention the derivation of the Boltzmann equation for a gas in the low-density regime [44], the derivation of the hydrodynamics limit from the Boltzmann equation [18, 3, 54] and important recent progress concerning the direct derivation of the heat equation and fluctuations in the kinetic limit (cf [11, 13, 12] and references within). However, these results are relevant only for extremely rarefied gases.

For systems at fixed particle densities, some important results in diffusive limits have been obtained assuming the microscopic dynamics to be stochastic. This has led to a well-developed theory of hydrodynamics limit for stochastic microscopic dynamics, starting with the work of Fritz [33], Presutti and collaborators [17] and Varadhan [37, 58]. See [56] for a more complete description of the large-scale limits and the main results and [42] for an exposition of some of the main relevant mathematical techniques. Still, when interactions are non-linear, the non-equilibrium diffusive behavior of energy remains an open problem even for stochastic dynamics. One of the main difficulties is that known methods often rely on entropy estimates that fail to control quadratic non-linearities (in particular energy). When stochastic perturbations are present, some results on the existence of thermal diffusivity $D(e)$ exists [8], and equilibrium fluctuations of the energy following linearized heat equations [51] are established.

Diffusive scaling limits have also been established for the densities of *independent* particles with very chaotic dynamics, starting with the seminal work of Bunimovich and Sinai for the evolution of the density of a periodic Lorentz gas [27]. See also [4, 15].

Even more special are (deterministic) *extended completely integrable systems* [57]. In particular, for one dimensional system of hard rods hydrodynamic limits have been proved in hyperbolic and diffusive time scale ([24, 25, 26, 32]). Since in these integrable extended systems there are infinitely many conserved quantities, in the diffusing scaling, after recentering on the hyperbolic evolution, a diffusive system of infinitely many equations arise for the evolution of the energies associated with each conserved quantity.

Another example of an extended, completely integrable system is given by a chain of harmonic oscillators. When this dynamics is perturbed by an energy conserving noise (like the random flip of the sign of the velocities), completely integrability is destroyed and it is possible to prove the diffusive behavior of the energy (1.1) with a thermal diffusivity D independent of ϵ and explicitly computable [7, 6, 43]. Such noise is multiplicative, but the linearity of the dynamics allows to perform the proof by computing the covariances. Unfortunately, this yields a result only on the asymptotic behavior of the average density of the energy, and not as a law of large numbers, as no known technique allows the control of higher moments.

To date, no direct hydrodynamics result for the heat equation (energy conservation) at high-density is available for purely deterministic (Hamiltonian) interacting systems. In this paper, we introduce a mechanical model for which we are able to perform a diffusive scaling and show that, on average, the energy density evolves according to the heat equation (1.1).

To our knowledge, this is the first proof that a mechanical model yields the heat equation in some appropriate scaling limit (without rarefaction), whereby showing that the heat equation can indeed be a consequence of deterministic microscopic classical mechanics.

The model is a harmonic chain in which each particle is subjected to a fast evolving external force acting like a magnetic field. Such a field has a deterministic but chaotic evolution. In particular, each particle has two degrees of freedom and the external force acts rotating the velocity vector. In this way it does not change the energy of the particles. The basic idea is to consider a system with three time scales: the macroscopic scale in which we establish the heat equation, the microscopic one where it takes place the evolution of the positions and velocities of the particles, and a faster time scale describing the evolution of the external field (the *environment*). Hence, the microscopic dynamics can be seen as a fast-slow system. This particular mechanical model was suggested to us by Cedric Bernardin.

Our key idea is that the fast chaotic external forcing acts on the harmonic chain similarly to a stochastic noise that randomly rotates the velocities (similar to the noise in [7, 6, 43]). This phenomenon is intensively studied in fast-slow dynamical systems (under the name of averaging or homogenization for fast-slow systems), and some powerful techniques have been developed [28, 21, 40, 16]. In fact, we prove that, under proper scaling on the fast external field, in the diffusive limit the covariance matrix behaves close to the one of the corresponding stochastic dynamics. A main obstacle being the need to control the difference between the two time evolutions for a time much longer than what is achieved in any existing result.

To obtain our result, we combine ideas inspired by the work of Dolgopyat [28] (see [21] for a simple presentation) and techniques coming from the theory of hydrodynamics limits of stochastic dynamics [43]. In particular, we succeeded in controlling the speed of convergence in a precise quantitative

manner and for times that increase with the size of the system (since we take a diffusive scaling). In a different context, a related point of view has been explored in [22, 20, 23, 46, 14]. Note however that, contrary to the present situation, in all the previous work the fast-slow systems were always systems with few degrees of freedom, with the notable exception of [29], for which, however, no estimates on the speed of the convergence are available.

Although we could model the exterior force evolution by any sufficiently chaotic system, to simplify the presentation, we chose the simplest possible model (a smooth expanding map of the circle). This has the disadvantage that the dynamics is not reversible, but it has the big advantage of considerably reducing the technicalities, hence making the argument accessible also to people with no prior dynamical systems knowledge. Indeed, this allowed us to make the paper essentially self-contained by adding only a couple of appendices; hence catering to a much larger audience. The alternative to having the evolution of the magnetic-like field modeled by an Anosov map or an Anosov flow would have required the introduction of many other technical ideas that would have clouded the main argument without changing the conceptual relevance of the result.

The effect of stationary magnetic fields acting on harmonic chains have also been studied in [55, 9, 10]. Our situation is very different as the forcing is time dependent but does not exchange energy with the system. Hence, contrary to the constant case, we do not consider the energy of this exterior field as part of the system. Finally, in [36] is proposed and numerically studied a discrete time dynamics where a time dependent external field exchanges chaotically the momentum between nearest neighbors particles conserving the energy, but without any other interaction in the dynamics.

1.1. Notations.

Here we collect some notation used in the text that may not be standard. Let $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ and $\mathbb{R}_+ = \{a \in \mathbb{R} : a \geq 0\}$. For $n \in \mathbb{N}$,

- (a) We denote by $\mathcal{M}(n, \mathbb{R})$ the set of $n \times n$ real matrices. For $\mathbb{M} \in \mathcal{M}(n, \mathbb{R})$, we set $\|\mathbb{M}\| = \sup_{|v|_{\mathbb{R}^n}=1} |\mathbb{M}v|$, $|\cdot|$ being the euclidean norm.
- (b) For $A \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, we denote by $DA : \mathbb{R}^n \rightarrow \mathbb{R}^n$ the gradient of A and by $D^2A : \mathbb{R}^n \rightarrow \mathcal{M}(n, \mathbb{R})$ the Hessian of A .
- (c) We use $C_{\#}$ to designate any computable constant depending only on ω_0 , b and f defining the dynamics (2.2) and C_0 in (2.7).
- (d) We use $\mathcal{O}(a)$ to mean that $|\mathcal{O}(a)| \leq C_{\#}a$ for all a in its domain.
- (e) For $x \in \mathbb{Z}_N$, we set $q_x = (q_{x,1}, q_{x,2}) \in \mathbb{R}^2$, $p_x = (p_{x,1}, p_{x,2}) \in \mathbb{R}^2$, representing the position and momentum of the x^{th} -particle. Furthermore, for $A \in \mathcal{C}^1((\mathbb{R}^2 \times \mathbb{R}^2)^N, \mathbb{R})$, we set $\frac{\partial A}{\partial q_x} = \left(\frac{\partial A}{\partial q_{x,1}}, \frac{\partial A}{\partial q_{x,2}} \right)$ and $\frac{\partial A}{\partial p_x} = \left(\frac{\partial A}{\partial p_{x,1}}, \frac{\partial A}{\partial p_{x,2}} \right)$.
- (f) We use the notation $[a]$ for the integer part of a number $a \in \mathbb{R}$.

2. THE DETERMINISTIC MODEL AND THE RESULT

Let¹ $x \in \mathbb{Z}_N$, $\theta = (\theta_x) \in \mathbb{T}^N$, $\varepsilon > 0$, and

$$(2.1) \quad (\theta_\varepsilon(t))_x = f^{\lfloor t\varepsilon^{-1} \rfloor}(\theta_x),$$

where $f \in \mathcal{C}^\infty(\mathbb{T}, \mathbb{T})$, $f' \geq \lambda > 1$. For each $N \in \mathbb{N}$, we denote by $q_x^{N,\varepsilon}(t)$, $p_x^{N,\varepsilon}(t) \in \mathbb{R}^2$ the solution of the following ODE on $\mathbb{R}^{2N} \times \mathbb{R}^{2N}$,

$$(2.2) \quad \begin{aligned} \dot{q}_x^{N,\varepsilon} &= p_x^{N,\varepsilon} \\ \dot{p}_x^{N,\varepsilon} &= (\Delta - \omega_0^2)q_x^{N,\varepsilon} + \varepsilon^{-\frac{1}{2}}b(f^{\lfloor t\varepsilon^{-1} \rfloor}\theta_x)Jp_x^{N,\varepsilon}, \end{aligned}$$

where

$$b \in \mathcal{C}^\infty(\mathbb{T}, \mathbb{R}), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Delta q_x^{N,\varepsilon} = q_{x+1}^{N,\varepsilon} + q_{x-1}^{N,\varepsilon} - 2q_x^{N,\varepsilon} \quad \& \quad \omega_0 > 0.$$

Equation 2.2 corresponds to the classical evolution of a periodic harmonic chain in which each particle is subject to a fast evolving external magnetic-like field. Let ρ_\star be the density of the unique absolutely continuous invariant measure for f , see Appendix A for details. For simplicity, we assume that

$$(2.3) \quad \int_{\mathbb{T}} b(\theta)\rho_\star(\theta)d\theta = 0,$$

i.e., the fluctuations of the external field are zero-average. The following covariance will play an essential role

$$(2.4) \quad \gamma = \int_{\mathbb{T}} b(\theta)^2\rho_\star(\theta)d\theta + 2 \sum_{k=1}^{\infty} \int_{\mathbb{T}} b(\theta)b(f^k\theta)\rho_\star(\theta)d\theta.$$

Note that, $b \in \mathcal{C}^\infty$, Lemma A.3 and equation (2.3) imply that $\gamma < \infty$.

In addition, if we assume that there exists a periodic orbit $\{x_n\}_{n=1}^p$, $p \in \mathbb{N}$, $x_{n+1} = f(x_n)$, $x_p = x_1$, such that

$$(2.5) \quad \sum_{n=1}^p b(x_n) \neq 0,$$

then Lemma B.2 (see also the subsequent discussion) implies that $\gamma > 0$.

Let $\Omega_N = \{z = (\mathbf{q}, \mathbf{p}), \mathbf{q}, \mathbf{p} \in \mathbb{R}^{2N}, \mathbf{q} = (q_x)_x, \mathbf{p} = (p_x)_x\} \simeq \mathbb{R}^{4N}$ be the phase space of the system with N particles. We define the single particle energy $\mathfrak{e}_x \in \mathcal{C}^\infty(\Omega_N, \mathbb{R})$, $x \in \mathbb{Z}_N$, and the total energy \mathcal{E}_N ,

$$(2.6) \quad \mathfrak{e}_x(z) = \frac{p_x^2}{2} + \frac{1}{2}(q_x - q_{x-1})^2 + \frac{\omega_0^2 q_x^2}{2}, \quad \mathcal{E}_N = \sum_{x \in \mathbb{Z}_N} \mathfrak{e}_x.$$

The last crucial ingredient to specify is the class of initial conditions. As we are looking for statistical properties, we have to start with initial conditions described by probability measures. This is reasonable since we are considering systems with a large number of particles and it would be unrealistic

¹We assume *periodic boundary conditions*, in the sense that the site $x = N+1$ coincides with the site $x = 1$.

to pretend to know the exact initial conditions of all the system. This is especially true for the fast-moving magnetic field. Moreover, we want measures that behave reasonably in the thermodynamic limit. We refrain from specifying the most general class of measures to which our results apply, as it would be a rather technical definition, and we content ourselves with the following, already rather general, set.

Definition 2.1. Let $\mathcal{M}_{init} = \mathcal{M}_{init}(C_0)$, $C_0 > 0$, be the set of families of probability measures μ_N on $\Omega_N \times \mathbb{T}^N$, $N \in \mathbb{N}$, such that

$$d\mu_N(\mathbf{q}, \mathbf{p}, \theta) = \rho_N(\theta) \mu_{\theta, N}(d\mathbf{q}, d\mathbf{p}) d\theta,$$

where $\rho_N(\theta) = \prod_{x \in \mathbb{Z}_N} \rho_{x, N}(\theta_x)$, with $\rho_{x, N} \in \mathcal{C}^1(\mathbb{T})$ such that

$$(2.7) \quad \sup_{x \in \mathbb{Z}_N} \sup_{\theta_x \in \mathbb{T}} \frac{|\partial_{\theta_x} \rho_{x, N}(\theta_x)|}{\rho_{x, N}(\theta_x)} \leq C_0.$$

Furthermore, calling \mathbb{E}_{μ_N} the expectation with respect to μ_N , we require that

$$(2.8) \quad \mathbb{E}_{\mu_N} \left[\frac{\mathcal{E}_N(0)}{N} \right] \leq \mathcal{E}_*,$$

for some constant $\mathcal{E}_* \geq 0$.

Let ϕ^t is the flow defined by (2.2), (2.1). An important property of \mathcal{M}_{init} is the following fact, proved at the end of Appendix A.

Lemma 2.2. If C_0 is big enough $\phi_*^t \mathcal{M}_{init} \subseteq \mathcal{M}_{init}$, for any $t \in \mathbb{R}_+$.

From now on, we will assume that C_0 is big enough, as in Lemma 2.2. Let $\mathcal{M}(\mathbb{T})$ be the space of positive measure on \mathbb{T} , endowed with the weak topology, and $\mathcal{M}_{\tilde{\mathcal{E}}}(\mathbb{T}) \subset \mathcal{M}(\mathbb{T})$ the space of measures of total mass $\tilde{\mathcal{E}} \in \mathbb{R}_+$. We are interested in initial distributions $\{\mu_N \in \mathcal{M}_{init}\}$ which correspond to a macroscopic distribution of temperatures $T_0(du) \in \mathcal{M}_{\mathcal{E}_0}(\mathbb{T})$.

Definition 2.3. The class of initial conditions $\mathcal{M}_{init}^*(C_0)$ consists of the sequences $\mu_N \in \mathcal{M}_{init}(C_0)$ such that

$$(2.9) \quad \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\frac{1}{N} \sum_{x \in \mathbb{Z}_N} \varphi\left(\frac{x}{N}\right) \mathbf{e}_x(0) \right] = \int_{\mathbb{T}} \varphi(u) T_0(du),$$

for some $T_0 \in \mathcal{M}_{\mathcal{E}_0}(\mathbb{T})$, $\mathcal{E}_0 > 0$, and any $\varphi \in \mathcal{C}^0(\mathbb{T}, \mathbb{R})$.²

We need to introduce a quantitative version of (2.9) to state our result. Given $\mu_N \in \mathcal{M}_{init}^*$, for $k \in \mathbb{Z} \setminus \{0\}$, $N \in \mathbb{N}$, we set

$$(2.10) \quad \begin{aligned} \varpi(k, N, \mu_N) &= \frac{1}{\mathcal{E}_0} \left| \mathbb{E}_{\mu_N} \left[\frac{1}{N} \sum_{x \in \mathbb{Z}_N} e^{2\pi i k \frac{x}{N}} \mathbf{e}_x(0) \right] - \int_{\mathbb{T}} e^{2\pi i k u} T_0(du) \right|, \\ \Psi(N; \mu_N) &= \sum_{k \in \mathbb{Z} \setminus \{0\}} k^{-2} \varpi(N, k, \mu_N). \end{aligned}$$

² $\mathcal{E}_0 = 0$ corresponds to zero temperature and it is not of present interest.

We call $\Psi(N, \mu_N)$ the *convergence rate to the energy profile associated to the sequence of measures* $\mu_N \in \mathcal{M}_{init}^*$. Note that equation (2.9) implies

$$(2.11) \quad \lim_{N \rightarrow \infty} \Psi(N, \mu_N) = 0.$$

Remark 2.4. *We are interested in taking a joint limit for $\varepsilon \rightarrow 0$ and $N \rightarrow \infty$ in (2.2), so the fact that $q_x^{N,\varepsilon}(t)$, $p_x^{N,\varepsilon}(t)$ depend on N and ε is important. However, to keep the notations as light as possible, we will often suppress the subscripts N and ε if this does not create confusion.*

Remark 2.5. *Our results would also apply to a model where f, b depend on x , as long as they all satisfy some obvious uniformity conditions (e.g. (2.3) and (2.4) with $\gamma_x = \gamma$). We refrain from such a generalization to simplify the exposition.*

Remark 2.6. *Note that, in Definition 2.1 we only demand some regularity in the initial distribution of θ , while we can choose $\mu_{\theta,N}(dq, dp)$ as singular as wanted, even deterministic.*

In order to formulate our main result, we introduce the measure-valued heat equation with initial condition $T_0 \in \mathcal{M}(\mathbb{T})$,

$$(2.12) \quad \begin{aligned} \partial_t T &= \frac{D}{2\gamma} \partial_u^2 T, \quad u \in (0, 1), \\ T(0, du) &= T_0(du), \end{aligned}$$

where γ is given by (2.4) and $D = \frac{2}{2 + \omega_0^2 + \omega_0 \sqrt{\omega_0^2 + 4}}$.

Definition 2.7. *We say that a function $T : [0, +\infty) \rightarrow \mathcal{M}(\mathbb{T})$ is a (weak, measure-valued) solution of (2.12) if: it belongs to $\mathcal{C}^0([0, +\infty), \mathcal{M}(\mathbb{T}))$ and for any $\varphi \in \mathcal{C}^2(\mathbb{T}, \mathbb{R})$ such that $\varphi(0) = \varphi(1)$, $\varphi'(0) = \varphi'(1)$, we have*

$$(2.13) \quad \int_{\mathbb{T}} \varphi(u) T(t, du) - \int_{\mathbb{T}} \varphi(u) T_0(du) = \frac{D}{2\gamma} \int_0^t ds \int_{\mathbb{T}} \varphi''(u) T(s, du).$$

Note that if $T_0 \in \mathcal{M}_{\mathcal{E}_0}(\mathbb{T})$, then $T(t) \in \mathcal{M}_{\mathcal{E}_0}(\mathbb{T})$ for all t . It is also well known that for any $t > 0$ the measure $T(t, du)$ has a smooth density, and that the solution of (2.13) is unique.

We can finally state our main result.

Theorem 2.8. *For each sequence $\{\varepsilon_N\}$ such that $\varepsilon_N \leq N^{-\alpha}$, $\alpha > 6$, and for any initial probability measure $\mu_N \in \mathcal{M}_{init}^*$ there exists $C_\alpha \in \mathbb{R}_+$ such that, for each $N \in \mathbb{N}$ and $\varphi \in \mathcal{C}^8(\mathbb{T}, \mathbb{R})$, we have, for each $t \in \mathbb{R}_+$,*

$$\begin{aligned} & \left| \mathbb{E}_{\mu_N} \left[\frac{1}{N} \sum_{x \in \mathbb{Z}_N} \varphi\left(\frac{x}{N}\right) \mathbf{e}_x(z(N^2 t)) \right] - \int_{\mathbb{T}} \varphi(u) T(t, du) \right| \\ & \leq C_\alpha \mathcal{E}_* \|\varphi\|_{\mathcal{C}^8} \max \left\{ \Psi(N; \mu_N), N^{-\beta_*} (\ln(N+1))^2 \right\}, \end{aligned}$$

where $z(t)$ is the solution of (2.2) with initial distribution μ_N , $T(t, du)$ is the unique solution of (2.12), with initial data given by the measure $T_0 \in \mathcal{M}(\mathbb{T})$ in (2.9), $\Psi(N; \mu_N)$ is defined in (2.10) and $\beta_* = \min\{\frac{1}{2}, \frac{\alpha}{6} - 1\}$.

The rest of the paper is devoted to the proof of Theorem 2.8, proof that, after several preliminaries is concluded in section 6.1.

Before starting let us state an important Corollary of Theorem 2.8.

The Corollary states that on time scales shorter than N^2 the energy does not evolve; on the time scale N^2 the energy evolves according to the heat equation, and on longer time scales, the energy is uniformly distributed.

Corollary 2.9. *For any $\mu_N \in \mathcal{M}_{init}^*$, $\varepsilon_N \leq N^{-\alpha}$, $\alpha > 6$, $t \in \mathbb{R}_+$, and $\beta > 0$ we have*

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{N} \sum_{x \in \mathbb{Z}_N} \mathbb{E}_{\mu_N}(\mathbf{e}_x(z(N^{2-\beta}t))) \delta_{\frac{x}{N}} &= T_0 \\ \lim_{t \rightarrow \infty} \frac{1}{N} \sum_{x \in \mathbb{Z}_N} \mathbb{E}_{\mu_N}(\mathbf{e}_x(z(N^2t))) \delta_{\frac{x}{N}} &= T(t) \\ \lim_{t \rightarrow \infty} \frac{1}{N} \sum_{x \in \mathbb{Z}_N} \mathbb{E}_{\mu_N}(\mathbf{e}_x(z(N^{2+\beta}t))) \delta_{\frac{x}{N}} &= \mathcal{E}_0 \text{ Leb}, \end{aligned}$$

where the limits are meant in the weak topology.

Proof. For each $\varphi \in \mathcal{C}^0$ and $\varepsilon > 0$ let $\varphi_\varepsilon \in \mathcal{C}^8$ be such that $\|\varphi - \varphi_\varepsilon\|_\infty \leq \varepsilon$. Then, by Theorem 2.8, equation (2.8) and the conservation of energy, there exists $\beta_* > 0$ such that, for each $\beta \in \mathbb{R}$,

$$\begin{aligned} \left| \mathbb{E}_{\mu_N} \left[\frac{1}{N} \sum_{x \in \mathbb{Z}_N} \varphi\left(\frac{x}{N}\right) \mathbf{e}_x(z(N^{2+\beta}t)) \right] - \int_{\mathbb{T}} \varphi(u) T(N^\beta t, du) \right| &\leq 2\varepsilon \mathcal{E}_* + \\ &+ C_{\#} \mathcal{E}_* \max \left\{ \Psi(N; \mu_N), N^{-\beta_*} (\ln(N+1))^2 \right\} \|\varphi_\varepsilon\|_{\mathcal{C}^8}. \end{aligned}$$

Therefore, recalling (2.11) and by the arbitrariness of ε , we have

$$\lim_{N \rightarrow \infty} \left| \mathbb{E}_{\mu_N} \left[\frac{1}{N} \sum_{x \in \mathbb{Z}_N} \varphi\left(\frac{x}{N}\right) \mathbf{e}_x(z(N^{2+\beta}t)) \right] - \int_{\mathbb{T}} \varphi(u) T(N^\beta t, du) \right| = 0.$$

If $\beta = 0$, the second statement of the Lemma follows. If $\beta < 0$, then

$$\lim_{N \rightarrow \infty} \int_{\mathbb{T}} \varphi(u) T(N^\beta t, du) = \int_{\mathbb{T}} \varphi(u) T_0(du)$$

yielding the first statement. If $\beta > 0$, then the last statement follows since³

$$\lim_{N \rightarrow \infty} \int_{\mathbb{T}} \varphi(u) T(N^\beta t, du) = \mathcal{E}_0 \int_{\mathbb{T}} \varphi(u) du. \quad \square$$

³One can easily check it by studying the behavior of the equation (2.12) when applied to the Fourier modes $e^{2\pi i k x}$.

Remark 2.10. *The ε dependence in Theorem 2.8 is not optimal. For example, simply including more terms in the expansion stated in Lemma 3.4 one could probably establish the same result with ε_N bounded by a lower inverse power of N , at the price of considerably more work. However, to have a result in which ε does not vanish when $N \rightarrow \infty$ seems to require substantially new ideas.*

3. FUNCTIONS CONTROLLED BY THE ENERGY

Recalling that $z = (\mathbf{q}, \mathbf{p})$, $\mathbf{q}, \mathbf{p} \in \mathbb{R}^{2N}$, $\theta \in \mathbb{T}^N$, we can write equation (2.2) as

$$(3.1) \quad \dot{z}(t) = \left(-\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(t, \theta) \right) z(t),$$

where

$$(3.2) \quad \begin{aligned} \mathbb{A} &= \begin{pmatrix} \mathbb{0} & -\mathbb{1} \\ -\Delta + \omega_0^2 \mathbb{1} & \mathbb{0} \end{pmatrix}_{4N \times 4N}, \quad \mathbb{B}_\varepsilon(t, \theta) = \begin{pmatrix} \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbf{B}_\varepsilon(t, \theta) \end{pmatrix}_{4N \times 4N}, \\ \Delta &= \begin{pmatrix} -2\mathbb{1} & \mathbb{1} & 0 & 0 & \cdots & 0 & \mathbb{1} \\ \mathbb{1} & -2\mathbb{1} & \mathbb{1} & 0 & \cdots & 0 & 0 \\ 0 & \mathbb{1} & -2\mathbb{1} & \mathbb{1} & \cdots & 0 & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \mathbb{1} & 0 & 0 & 0 & \cdots & \mathbb{1} & -2\mathbb{1} \end{pmatrix}, \\ \mathbf{B}_\varepsilon(t, \theta) &= \begin{pmatrix} b(f^{\lfloor t\varepsilon^{-1} \rfloor} \theta_1) J & 0 & \cdots & 0 \\ 0 & b(f^{\lfloor t\varepsilon^{-1} \rfloor} \theta_2) J & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b(f^{\lfloor t\varepsilon^{-1} \rfloor} \theta_N) J \end{pmatrix}, \end{aligned}$$

and $\mathbb{0}$ is a $2N \times 2N$ null matrix. Denote by $z(t, z_0, \theta_0) = (\mathbf{q}(t, z_0, \theta_0), \mathbf{p}(t, z_0, \theta_0))$ the solution of (3.1) with initial conditions $z_0 \in \Omega_N$ and $\theta_0 \in \mathbb{T}^N$.

Lemma 3.1. *For all $z_0 \in \Omega_N$, $\theta_0 \in \mathbb{T}^N$ and $t \in \mathbb{R}^+$, $\mathcal{E}_N(z(t, z_0, \theta_0)) = \mathcal{E}_N(z_0)$.*

Proof. We define the current as

$$(3.3) \quad j_{x,x+1} = -p_x(q_{x+1} - q_x).$$

Recalling the definition of the energy (2.6) and the equations (2.2), we have

$$(3.4) \quad \frac{d}{dt} \mathbf{e}_x = j_{x-1,x} - j_{x,x+1}.$$

The above implies that $\frac{d}{dt} \sum_{x \in \mathbb{Z}_N} \mathbf{e}_x = 0$ and the Lemma. \square

It will be instrumental for our purposes to introduce energy shells,

$$\Sigma_N(E) = \{z \in \Omega_N : \mathcal{E}_N(z) = E\}, \quad E \in \mathbb{R}_+,$$

and we also set

$$\|z\| = \sqrt{\langle \mathbf{q}, \mathbf{q} \rangle + \langle \mathbf{p}, \mathbf{p} \rangle},$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product. Note that $C_{\#}^{-1}\sqrt{\mathcal{E}_N(z)} \leq \|z\| \leq C_{\#}\sqrt{\mathcal{E}_N(z)}$ and so if $z_0 \in \Sigma_N(E)$, then Lemma 3.1 implies that for all $t \in \mathbb{R}_+$,

$$(3.5) \quad C_{\#}^{-1}\sqrt{E} \leq \|z(t, z_0)\| \leq C_{\#}\sqrt{E}.$$

Let $A_N \in C^\infty(\Omega_N, \mathbb{R})$, be a sequence of functions and define, for $E \in \mathbb{R}_+$, the following seminorms,⁴

$$(3.6) \quad \begin{aligned} \|A_N\|_{0,E} &= \sup_{z \in \Sigma_N(E)} |A_N(z)| \\ \|DA_N\|_{1,E} &= \sup_{z \in \Sigma_N(E)} \|DA_N(z)\|, \\ \|D^2A_N\|_{2,E} &= \sup_{z \in \Sigma_N(E)} \sup_{\mathbf{v} \in \mathbb{R}^{4N}} \frac{\|D^2A_N(z)\mathbf{v}\|}{\|\mathbf{v}\|}, \end{aligned}$$

where in the second and third line $\|\cdot\|$ stands for the euclidean norm. We say that the sequence $\{A_N\}_N$ is *controlled by the energy* if

$$(3.7) \quad \sup_{N \in \mathbb{N}} \sup_{E \in \mathbb{R}_+} \left\{ \frac{\|A_N\|_{0,E}}{E}, \frac{\|DA_N\|_{1,E}}{\sqrt{E}}, \|D^2A_N\|_{2,E} \right\} < \infty.$$

Definition 3.2. *If the sequence $\{A_N\}_N$ is controlled by the energy, we set*

$$\|A\|_{\mathfrak{E}} = \sup_{N \in \mathbb{N}} \sup_{E \in \mathbb{R}_+} \left\{ \frac{\|A_N\|_{0,E}}{E}, \frac{\|DA_N\|_{1,E}}{\sqrt{E}}, \|D^2A_N\|_{2,E} \right\}.$$

Let \mathfrak{E} be the set of degree two polynomials' sequences controlled by the energy.

Whenever it does not create confusion, we will write $\|A\|_{\mathfrak{E}}$ in place of $\|A\|_{\mathfrak{E}}$. We now introduce the class of functions $\{A_N\} \in \mathfrak{E}$ that will be interesting for us. We denote by S_N the covariance matrix

$$(3.8) \quad S_N = \begin{bmatrix} S_N^{(q)} & S_N^{(q,p)} \\ S_N^{(p,q)} & S_N^{(p)} \end{bmatrix}_{2N \times 2N},$$

where

$$(3.9) \quad \begin{aligned} [S_N^{(q)}]_{x,y} &= \langle q_x, q_y \rangle, & [S_N^{(q,p)}]_{x,y} &= \langle q_x, p_y \rangle, \\ [S_N^{(p)}]_{x,y} &= \langle p_x, p_y \rangle & \text{and} & S_N^{(p,q)} = [S_N^{(q,p)}]^T. \end{aligned}$$

We will be mainly interested in functions of the type

$$(3.10) \quad A_N = \text{Trace}(\beta_N S_N),$$

for some $2N \times 2N$ matrix β_N . Note that

$$\|A_N\|_{0,E} \leq C_{\#}\|\beta_N\|E, \quad \|DA_N\|_{1,E} \leq C_{\#}\|\beta_N\|\sqrt{E}, \quad \|D^2A_N\|_{2,E} \leq C_{\#}\|\beta_N\|.$$

⁴Recall the definition of DA and D^2A in Section 1.1.

Thus, whenever $\sup_{N \in \mathbb{N}} \|\beta_N\| < \infty$, we have that $\{A_N\} \in \mathfrak{E}$ and

$$(3.11) \quad \|A\|_{\mathfrak{E}} \leq C_{\#} \sup_{N \in \mathbb{N}} \|\beta_N\| =: \|\beta\|.$$

We conclude the section with two useful Lemmata.

Lemma 3.3. *For all $N \in \mathbb{N}$, $t \in \mathbb{R}_+$, $\theta \in \mathbb{T}^N$, $\max\{\|A\|, \|\mathbb{B}_\varepsilon(t, \theta)\|\} \leq C_{\#}$.*

Proof. Since $(q_{x+1} + q_{x-1} - (\omega_0^2 + 2)q_x)^2 \leq 3(q_{x+1}^2 + q_{x-1}^2 + (\omega_0^2 + 2)^2 q_x^2)$,

$$\|\mathbb{A}z\|^2 = \sum_{x \in \mathbb{Z}_N} p_x^2 + (q_{x+1} + q_{x-1} - (\omega_0^2 + 2)q_x)^2 \leq 9(\omega_0^2 + 2)^2 \|z\|^2.$$

Also, since $(b(f^{t\varepsilon^{-1}}\theta_x)Jp_x)^2 \leq \|b\|_{\infty}^2 p_x^2$,

$$\|\mathbb{B}_\varepsilon(t, \theta)z\|^2 = \sum_{x \in \mathbb{Z}_N} (b(f^{\lfloor t\varepsilon^{-1} \rfloor} \theta_x)Jp_x)^2 \leq \|b\|_{\infty}^2 \|z\|^2.$$

□

The next Lemma is a boring, but useful, Taylor expansion.

Lemma 3.4. *For any $A \in \mathfrak{E}$, $z_0 \in \Sigma_N(E)$, $\theta_0 \in \mathbb{T}^N$, $h \in (0, \varepsilon^{\frac{1}{2}})$,*

$$\begin{aligned} A(z(h, z_0, \theta_0)) &= A(z_0) - \langle DA(z_0), \mathbb{A}z_0 \rangle h \\ &\quad + \varepsilon^{-\frac{1}{2}} \int_0^h \langle DA(z_0), \mathbb{B}_\varepsilon(s, \theta_0)z_0 \rangle ds \\ &\quad + \varepsilon^{-1} \int_0^h \int_0^s \langle DA(z_0), \mathbb{B}_\varepsilon(s, \theta_0)\mathbb{B}_\varepsilon(u, \theta_0)z_0 \rangle duds \\ &\quad + \varepsilon^{-1} \int_0^h \int_0^s \langle D^2 A \mathbb{B}_\varepsilon(u, \theta_0)z_0, \mathbb{B}_\varepsilon(s, \theta_0)z_0 \rangle duds \\ &\quad + \mathcal{O}\left(\|A\|_{\mathfrak{E}} E \left(\varepsilon^{-\frac{3}{2}} h^3 + \varepsilon^{-\frac{1}{2}} h^2\right)\right). \end{aligned}$$

Proof. Since the initial condition θ_0 will not play any role in this proof, we will omit it. By the fundamental theorem of calculus and the chain rule,

$$(3.12) \quad \begin{aligned} A(z(h, z_0)) &= A(z_0) \\ &\quad + \int_0^h \left\langle DA(z(s, z_0)), \left(-\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(s)\right) z(s, z_0) \right\rangle ds. \end{aligned}$$

Using the previous equation for $A_i(z) = z_i$, $i \in \{1, \dots, 4N\}$, equation (3.5) and Lemma 3.3, we have

$$(3.13) \quad \|z(h, z_0) - z_0\| \leq C_{\#} h \varepsilon^{-\frac{1}{2}} \sqrt{E}.$$

Next, for $A \in \mathfrak{E}$, we Taylor expand $DA(z(s, z_0))$ and use that D^2A is constant, obtaining

$$\begin{aligned}
(3.14) \quad A(z(h, z_0)) &= A(z_0) - \int_0^h \langle DA(z_0), \mathbb{A}z(s, z_0) \rangle ds \\
&+ \varepsilon^{-\frac{1}{2}} \int_0^h \langle DA(z_0), \mathbb{B}_\varepsilon(s)z(s, z_0) \rangle ds \\
&+ \varepsilon^{-1} \int_0^h \int_0^s \langle D^2A \mathbb{B}_\varepsilon(u)z(u, z_0), \mathbb{B}_\varepsilon(s)z(s, z_0) \rangle dud s \\
&+ \int_0^h \int_0^s \langle D^2A \mathbb{A}z(u, z_0), \mathbb{A}z(s, z_0) \rangle dud s \\
&- \varepsilon^{-\frac{1}{2}} \int_0^h \int_0^s \langle D^2A \mathbb{A}z(u, z_0), \mathbb{B}_\varepsilon(s)z(s, z_0) \rangle dud s \\
&- \varepsilon^{-\frac{1}{2}} \int_0^h \int_0^s \langle D^2A \mathbb{B}_\varepsilon(u)z(u, z_0), \mathbb{A}z(s, z_0) \rangle dud s.
\end{aligned}$$

Let us study each term separately. First, note that

$$\int_0^h \langle DA(z_0), \mathbb{A}z(s, z_0) \rangle ds = \langle DA(z_0), \mathbb{A}z_0 \rangle h + \mathcal{O}(h^2 \varepsilon^{-\frac{1}{2}} \|A\|_{\mathfrak{E}} E)$$

where we have used (3.13), (3.7) and Lemma 3.3. The second line of (3.14) is equal to

$$\begin{aligned}
&\varepsilon^{-\frac{1}{2}} \int_0^h \langle DA(z_0), \mathbb{B}_\varepsilon(s)z(s, z_0) \rangle ds = \varepsilon^{-\frac{1}{2}} \int_0^h \langle DA(z_0), \mathbb{B}_\varepsilon(s)z_0 \rangle ds \\
&+ \varepsilon^{-\frac{1}{2}} \int_0^h \int_0^s \left\langle DA(z_0), \mathbb{B}_\varepsilon(s) \left(-\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(u) \right) z(u, z_0) \right\rangle dud s \\
&= \varepsilon^{-\frac{1}{2}} \int_0^h \langle DA(z_0), \mathbb{B}_\varepsilon(s)z_0 \rangle ds + \varepsilon^{-1} \int_0^h \int_0^s \langle DA(z_0), \mathbb{B}_\varepsilon(s) \mathbb{B}_\varepsilon(u)z_0 \rangle dud s \\
&+ \mathcal{O} \left(\|A\|_{\mathfrak{E}} E (h^3 \varepsilon^{-\frac{3}{2}} + h^2 \varepsilon^{-\frac{1}{2}}) \right).
\end{aligned}$$

The third line of (3.14) is equal to, by (3.13),

$$\begin{aligned}
&\varepsilon^{-1} \int_0^h \int_0^s \langle D^2A \mathbb{B}_\varepsilon(u)z(u, z_0), \mathbb{B}_\varepsilon(s)z(s, z_0) \rangle dud s \\
&= \varepsilon^{-1} \int_0^h \int_0^s \langle D^2A \mathbb{B}_\varepsilon(u)z_0, \mathbb{B}_\varepsilon(s)z_0 \rangle dud s + \mathcal{O}(\|A\|_{\mathfrak{E}} E h^4 \varepsilon^{-2}), \\
&= \varepsilon^{-1} \int_0^h \int_0^s \langle D^2A \mathbb{B}_\varepsilon(u)z_0, \mathbb{B}_\varepsilon(s)z_0 \rangle dud s + \mathcal{O}(\|A\|_{\mathfrak{E}} E h^3 \varepsilon^{-\frac{3}{2}}),
\end{aligned}$$

where, in the last line, we have used that $h \leq \varepsilon^{\frac{1}{2}}$. The remaining terms of (3.14) can be estimated similarly and yield smaller errors. \square

We conclude the section with a simple, but useful consequence.

Corollary 3.5. *For any $A \in \mathfrak{E}$, $z_0 \in \Sigma_N(E)$, $t \in \mathbb{R}_+$, $h \in (0, \varepsilon^{\frac{1}{2}})$,*

$$\int_t^{t+h} A(z(u, z_0)) du = A(z(t, z_0))h + \mathcal{O}\left(\|A\|_{\mathfrak{E}} E h^2 \varepsilon^{-\frac{1}{2}}\right).$$

4. TIME EVOLUTION OF AVERAGES

Our goal is to study the evolution of the averages of the energy. To this end, it is useful to have a class of measures that behaves well under the dynamics and allows to do something equivalent to conditioning for stochastic processes.

Such a class of measures has a long history and multiple incarnations, starting at least from the work of Pesin and Sinai [52]. However, it has reached maturity in the formulation of Dolgopyat [28], under the name of *standard pairs*. In appendix C we describe how to adapt the standard pairs ideology to the present work.

We will show that the evolution of the average with respect to a standard pair is similar to a stochastic evolution. Hence, to state the result we need first to describe the generator of such a stochastic process.

4.1. The stochastic generator and the main estimate.

Let us consider the following generator

$$(4.1) \quad \begin{aligned} \mathcal{G} &= \mathcal{A} + \gamma \mathcal{S} \\ \mathcal{A} &= \sum_{x \in \mathbb{Z}_N} (p_x \cdot \partial_{q_x} + (\Delta q_x - \omega_0^2 q_x) \cdot \partial_{p_x}) \\ \mathcal{S} &= \sum_{x \in \mathbb{Z}_N} (p_{x,2} \partial_{p_{x,1}} - p_{x,1} \partial_{p_{x,2}})^2 = \sum_{x \in \mathbb{Z}_N} (J p_x \cdot \partial_{p_x})^2, \end{aligned}$$

where $\gamma > 0$ is given by (2.4). This correspond to the SDEs

$$(4.2) \quad \dot{\tilde{q}}_x = \tilde{p}_x, \quad d\tilde{p}_x = (\Delta - \omega_0^2) \tilde{q}_x dt - \gamma \tilde{p}_x dt + \sqrt{2\gamma} J \tilde{p}_x dw_x(t),$$

where $\Delta \tilde{q}_x = \tilde{q}_{x+1} + \tilde{q}_{x-1} - 2\tilde{q}_x$ with periodic boundary, and $\{w_x(t), x = 1, \dots, N\}$ are independent one-dimensional standard Wiener processes. The SDEs (4.2) are written in the Ito's formulation. In Stratonovitch formulation they read

$$(4.3) \quad \dot{\tilde{q}}_x = \tilde{p}_x, \quad d\tilde{p}_x = (\Delta - \omega_0^2) \tilde{q}_x dt + \sqrt{2\gamma} J \tilde{p}_x \circ dw_x(t).$$

We will not use directly the SDEs (4.2) or (4.3), but only the generator \mathcal{G} . A useful first estimate on the generator is provided by the following Lemma.

Lemma 4.1. *Let $A \in \mathfrak{E}$, then $\mathcal{G}A \in \mathfrak{E}$ and $\|\mathcal{G}A\|_{\mathfrak{E}} \leq C_{\#} \|A\|_{\mathfrak{E}}$.*

Proof. By equations (4.1) and (3.1) we can write

$$(4.4) \quad \begin{aligned} \mathcal{G}A(z) = & \langle DA(z), \mathbb{A}z \rangle - \gamma \sum_{x \in \mathbb{Z}_N} \left\langle \frac{\partial A}{\partial p_x}, p_x \right\rangle \\ & + \gamma \sum_{x \in \mathbb{Z}_N} \left\langle \frac{\partial^2 A}{\partial p_x^2} Jp_x, Jp_x \right\rangle. \end{aligned}$$

and the Lemma follows recalling Definition 3.2 and Lemma 3.3. \square

Notation. To simplify notations, from now, given a function $A \in \mathcal{C}^0(\Omega_N, \mathbb{R})$ and a measure ν on $\Omega_N \times \mathbb{T}^N$, we will write $\nu(A(t))$ for the expectation $\int A(z(t, z_0, \theta_0)) \nu(dz_0, d\theta_0) = \mathbb{E}_\nu(A(t))$.

The main result of this section is the following theorem.

Theorem 4.2. Let $\mu_N \in \mathcal{M}_{init}$ and $A \in \mathfrak{E}$. For all $N \in \mathbb{N}$, $\varepsilon \in (0, N^{-\frac{1}{33}})$, $h \in (\varepsilon^{\frac{99}{100}}, \varepsilon^{\frac{1}{2}})$, and $t \geq h$,⁵

$$\left| \mu_N(A(t+h)) - \mu_N(A(t)) - \int_t^{t+h} \mu_N(\mathcal{G}A(s)) ds \right| \leq C_{\#} \mu_N(\mathfrak{R}_A(h, \varepsilon, \mathcal{E}_N)) h,$$

where

$$(4.5) \quad \mathfrak{R}_A(h, \varepsilon, E) = \|A\|_{\mathfrak{E}} \left\{ E \left(h^2 \varepsilon^{-\frac{3}{2}} + \varepsilon (\ln \varepsilon^{-1})^2 h^{-1} \right) \right\}.$$

Before proving Theorem 4.2 we mention how we will use it in the following.

Corollary 4.3. Let μ_N , A and ε as in Theorem 4.2. Then, for any $t \in \mathbb{R}_+$,

$$\begin{aligned} \int_0^t \mu_N(\mathcal{G}A(N^2s)) ds &= \frac{\mu_N(A(N^2t)) - \mu_N(A(0))}{N^2} \\ &+ \mathcal{O} \left(\|A\|_{\mathfrak{E}} \mathcal{E}_{\star} \left(t \varepsilon^{\frac{1}{6}} (\ln \varepsilon^{-1})^2 N + \varepsilon^{\frac{1}{3}} N^{-1} \right) \right). \end{aligned}$$

Proof. Let $h \in (\varepsilon^{\frac{5}{6}}, 2\varepsilon^{\frac{5}{6}})$. Then

$$(4.6) \quad \begin{aligned} \int_0^t \mu_N(\mathcal{G}A(N^2s)) ds &= \frac{1}{N^2} \left(\int_0^{\min\{h, tN^2\}} \mu_N(\mathcal{G}A(s)) ds \right. \\ &\quad \left. + \int_{\min\{h, tN^2\}}^{tN^2} \mu_N(\mathcal{G}A(s)) ds \right). \end{aligned}$$

Using Lemma 4.1, Corollary 3.5, $h \in (\varepsilon^{\frac{5}{6}}, 2\varepsilon^{\frac{5}{6}})$, and equation (2.8),

$$(4.7) \quad \begin{aligned} \left| \int_0^{\min\{h, tN^2\}} \frac{\mu_N(\mathcal{G}A(s))}{N^2} ds \right| &\leq C_{\#} \varepsilon^{\frac{5}{6}} \frac{\mu_N(\mathcal{E}_N)}{N^2} \|A\|_{\mathfrak{E}} \\ &\leq C_{\#} \frac{\mathcal{E}_{\star}}{N} \varepsilon^{\frac{5}{6}} \|A\|_{\mathfrak{E}}. \end{aligned}$$

⁵Recall the definition of \mathcal{E}_N in equation (2.6).

By (3.12), one gets that

$$(4.8) \quad \left| \frac{\mu_N(A(\min\{h, tN^2\})) - \mu_N(A(0))}{N^2} \right| \leq C_{\#} h \varepsilon^{-\frac{1}{2}} \frac{\mu_N(\mathcal{E}_N)}{N^2} \|A\|_{\mathfrak{E}} \\ \leq C_{\#} \varepsilon^{\frac{1}{3}} \frac{\mathcal{E}_{\star}}{N} \|A\|_{\mathfrak{E}}.$$

If $t \leq 2\varepsilon^{\frac{5}{6}} N^{-2}$, equations (4.7) and (4.8) prove the statement.

If $t > 2\varepsilon^{\frac{5}{6}} N^{-2}$, then we choose $h \in (\varepsilon^{\frac{5}{6}}, 2\varepsilon^{\frac{5}{6}})$ so that $\frac{N^2 t}{h} = K \in \mathbb{N}$. Then, by Theorem 4.2, the second term of (4.6) is equal to

$$(4.9) \quad \frac{1}{N^2} \sum_{k=1}^{K-1} \int_{kh}^{(k+1)h} \mu_N(\mathcal{G}A(s)) ds = \\ = \sum_{k=1}^{K-1} \frac{[\mu_N(A((k+1)h)) - \mu_N(A(kh)) + h\mathcal{O}(\mu_N(\mathfrak{R}_A(h, \varepsilon, \mathcal{E}_N)))]}{N^2} \\ = \frac{\mu_N(A(N^2 t)) - \mu_N(A(h))}{N^2} + \mathcal{O}\left(\frac{h(K-1)}{N^2} \mu_N(\mathfrak{R}_A(h, \varepsilon, \mathcal{E}_N))\right).$$

Recalling equation (2.8), and that $h \in (\varepsilon^{\frac{5}{6}}, 2\varepsilon^{\frac{5}{6}})$, yields

$$(4.10) \quad \frac{h(K-1)}{N^2} |\mu_N(\mathfrak{R}_A(h, \varepsilon, \mathcal{E}_N))| \leq \\ \leq t \|A\|_{\mathfrak{E}} \mathcal{E}_{\star} N \left(h^2 \varepsilon^{-3/2} + \varepsilon (\ln \varepsilon^{-1})^2 h^{-1} \right) \\ \leq t \mathcal{E}_{\star} \|A\|_{\mathfrak{E}} N \varepsilon^{\frac{1}{6}} (\ln \varepsilon^{-1})^2.$$

Therefore, by (4.9) and (4.7),

$$(4.11) \quad \int_0^t \mu_N(\mathcal{G}A(N^2 s)) ds = N^{-2} \int_0^{N^2 t} \mu_N(\mathcal{G}A(s)) ds \\ = \frac{\mu_N(A(N^2 t)) - \mu_N(A(h))}{N^2} \\ + \mathcal{O}\left(\|A\|_{\mathfrak{E}} \mathcal{E}_{\star} \left(t \varepsilon^{\frac{1}{6}} (\ln \varepsilon^{-1})^2 N + N^{-1} \varepsilon^{\frac{5}{6}} \right)\right).$$

Using (4.8), we have

$$\int_0^t \mu_N(\mathcal{G}A(N^2 s)) ds = \frac{\mu_N(A(N^2 t)) - \mu_N(A(0))}{N^2} \\ + \mathcal{O}\left(\|A\|_{\mathfrak{E}} \mathcal{E}_{\star} \left(t \varepsilon^{\frac{1}{6}} (\ln \varepsilon^{-1})^2 N + \varepsilon^{\frac{1}{3}} N^{-1} \right)\right),$$

concluding the proof. \square

4.2. Evolution of standard pairs.

Our strategy is to prove Theorem 4.2 for a special class of initial measures: probability measures μ_{ℓ} determined by *standard pairs* $\ell \in \mathfrak{S}(N, E, \delta, \varepsilon, \overline{C})$; we use loosely the name standard pair both for ℓ and the associated measure

μ_ℓ . See Appendix C for details.

We will work with standard pairs for which $\delta = \delta_\varepsilon = \varepsilon^{100}$.⁶

Remark 4.4. *Intuitively, standard pairs are the measures closer to a δ function for which the deterministic dynamics still exhibit statistical properties. Indeed, if one uses a δ function as an initial condition, then, the dynamics being deterministic, no temporal decay of correlation can ever take place. The basic idea is that the standard pairs have a marginal on the fast variables θ which is absolutely continuous w.r.t. Lebesgue with a density of bounded variation. On the contrary, conditioning on the fast variable yields a δ function (or close to it) in the \mathbf{q}, \mathbf{p} variables. Finally, the marginal on the slow variables \mathbf{q}, \mathbf{p} is supported on an energy surface $\Sigma_N(E)$.*

We will also consider finite convex combinations of standard pairs (these are called *standard families* and we will designate them by \mathfrak{F}) defined as

$$\mu_{\mathfrak{F}} = \sum_{\ell} p_{\ell} \mu_{\ell}$$

where p is a finite vector of positive entries with $\sum_{\ell} p_{\ell} = 1$. The next Lemma shows that these measures are general enough for our purposes.

Lemma 4.5. *For any $\varepsilon, \delta \geq 0$, the measures $\mu_N \in \mathcal{M}_{init}$ (see 2.1 for the definition) are the weak limit of standard families in $\mathfrak{F}_{N, \delta, \varepsilon}$.⁷*

Proof. By Definition 2.1, for each $A \in \mathcal{C}^0$,

$$\mu_N(A) = \int_{\mathbb{T}^N} \int_{\Omega_N} A(\mathbf{q}, \mathbf{p}, \theta) \mu_{\theta, N}(d\mathbf{q}d\mathbf{p}) \prod_{x \in \mathbb{Z}_N} \rho_{x, N}(\theta_x) d\theta_x.$$

We can then divide \mathbb{T} in intervals I_j of size between δ and 2δ and, setting $\bar{j} = (j_1, \dots, j_N)$, $I_{\bar{j}} = I_{j_1} \times \dots \times I_{j_N}$, write

$$\mu_N(A) = \sum_{\bar{j}} \mu_N(I_{\bar{j}} \times \Omega_N) \int_{I_{\bar{j}}} \int_{\Omega_N} A(\mathbf{q}, \mathbf{p}, \theta) \mu_{\theta, N}(d\mathbf{q}d\mathbf{p}) \prod_{x \in \mathbb{Z}_N} \frac{\rho_{x, N}(\theta_x) d\theta_x}{\mu_N(I_{\bar{j}} \times \Omega_N)}.$$

Next, for each $n \in \mathbb{N}$, let $\{\Delta_{n, k}\}_{k \in \mathbb{N}}$ be a partition of Ω_N in cubes of size $\frac{1}{n}$ and let $\{(\mathbf{q}_k, \mathbf{p}_k)\}_{k \in \mathbb{N}}$ be the center of the cube Δ_k . Then,

$$\mu_N(A) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu_N(\Delta_{n, k}) \sum_{\bar{j}} \mu_N(I_{\bar{j}} \times \Omega_N) \int_{I_{\bar{j}}} \prod_{x \in \mathbb{Z}_N} \frac{\rho_{x, N}(\theta_x)}{\mu_N(I_{\bar{j}} \times \Omega_N)} A(\mathbf{q}_k, \mathbf{p}_k, \theta) d\theta_x,$$

which provides an approximation via a convex combination of standard pairs

$\ell_{\bar{j}, k} = \left(G_{\bar{j}, k} \mathbb{1}_{I_{\bar{j}}}(\theta) \prod_{x \in \mathbb{Z}_N} \frac{\rho_{x, N}(\theta_x)}{\mu_N(I_{\bar{j}} \times \Omega_N)} \right)$, where $G_{\bar{j}, k}(\theta) = (\mathbf{q}_k, \mathbf{p}_k)$ is a constant function. Note that (2.7) implies that $\mathbb{1}_{I_{\bar{j}}}(\theta) \prod_{x \in \mathbb{Z}_N} \frac{\rho_{x, N}(\theta_x)}{\mu_N(I_{\bar{j}} \times \Omega_N)} \in D_{C_0}$.

Finally, $\|DG_{\bar{j}, k}\|_{\infty} = 0$, so that $G_{\bar{j}, k} \in \mathfrak{G}(E, N, \delta, \varepsilon, \bar{C})$ for $E = \mathcal{E}_N(\mathbf{q}_k, \mathbf{p}_k)$ and every $\varepsilon \geq 0$. \square

⁶Of course, 100 is just a number big enough, any sufficiently high power will do.

⁷See Appendix C for the exact definition of standard families.

4.3. Proof of Theorem 4.2.

It suffices to prove Theorem 4.2 for a generic standard pair $\ell \in \mathfrak{S}(N, E, \delta, \varepsilon, \overline{C})$. Indeed, by Lemma 4.5, for any $\varepsilon \geq 0$, there exists a sequence $\mathfrak{f}_n \in \mathfrak{F}_{N, \delta, \varepsilon, \varepsilon}$ of standard families that converges to μ_N . Hence, assuming that the statement is true for all $\ell \in \mathfrak{f}_n$,

$$\begin{aligned} \mu_N(A(t+h)) &= \lim_{n \rightarrow \infty} \sum_{\ell \in \mathfrak{f}_n} p_\ell \mu_\ell(A(t+h)) \\ &= \lim_{n \rightarrow \infty} \sum_{\ell \in \mathfrak{f}_n} p_\ell \left\{ \mu_\ell(A(t)) + \int_t^{t+h} \mu_\ell(\mathcal{G}A(s)) ds + \mathcal{O}(\mu_\ell(\mathfrak{R}_A(h, \varepsilon, \mathcal{E}_N)))h \right\} \\ &= \mu_N(A(t)) + \int_t^{t+h} \mu_N(\mathcal{G}A(s)) ds + \mathcal{O}(\mu_N(\mathfrak{R}_A(h, \varepsilon, \mathcal{E}_N)))h. \end{aligned}$$

To prove the statement for a standard pair note that, by Theorem C.1, the pushforward $\phi_*^t \mu_\ell$, $t \in \mathbb{R}_+$, is described by a standard family $\mathfrak{f}_t \in \mathfrak{F}_{N, E, \delta, \varepsilon}$.

Note that it suffices to prove the Theorem for the case $t = h$. Indeed,

$$\begin{aligned} \mu_\ell(A(z(t+h))) &= \sum_{\ell' \in \mathfrak{F}_{t-h}} p_{\ell'} \mu_{\ell'}(A(z(2h))) \\ &= \sum_{\ell' \in \mathfrak{F}_{t-h}} p_{\ell'} \left(\int_h^{2h} \mu_{\ell'}(\mathcal{G}A(z(s))) ds + \mu_{\ell'}(\mathfrak{R}_A(h, \varepsilon, E)) \right) \\ &= \int_t^{t+h} \mu_\ell(\mathcal{G}A(z(s))) ds + \mu_\ell(\mathfrak{R}_A(h, \varepsilon, \mathcal{E}_N)). \end{aligned}$$

We are then left with the study of

$$\mu_\ell(A \circ \phi^{2h}) = \sum_{\ell' \in \mathfrak{f}_h} p_{\ell'} \mu_{\ell'}(A \circ \phi^h) = \sum_{\ell' \in \mathfrak{f}_h} p_{\ell'} \int_{\mathbb{I}_{\ell'}} A(z(h, G_{\ell'}(\theta))) \rho_{\ell'}(\theta) d\theta.$$

By Lemma 3.4, with $z(0) = G_{\ell'}(\theta)$ and $h > \varepsilon^{\frac{99}{100}}$, we can write

$$\begin{aligned} &\sum_{\ell' \in \mathfrak{f}_h} p_{\ell'} \int_{\mathbb{I}_{\ell'}} \rho_{\ell'}(\theta) \left(A(G_{\ell'}(\theta)) - \langle DA(G_{\ell'}(\theta)), \mathbb{A}G_{\ell'}(\theta) \rangle h \right. \\ &\quad + \varepsilon^{-\frac{1}{2}} \int_0^h \langle DA(G_{\ell'}(\theta)), \mathbb{B}_\varepsilon(s, \theta) G_{\ell'}(\theta) \rangle ds \\ &\quad + \varepsilon^{-1} \int_0^h \int_0^s \langle DA(G_{\ell'}(\theta)), \mathbb{B}_\varepsilon(s, \theta) \mathbb{B}_\varepsilon(u, \theta) G_{\ell'}(\theta) \rangle duds \\ &\quad \left. + \varepsilon^{-1} \int_0^h \int_0^s \langle D^2A \mathbb{B}_\varepsilon(u, \theta) G_{\ell'}(\theta), \mathbb{B}_\varepsilon(s, \theta) G_{\ell'}(\theta) \rangle duds \right) d\theta \\ &\quad + \mathcal{O}\left(\varepsilon^{-\frac{3}{2}} h^3 \|A\|_{\mathfrak{E}} E\right). \end{aligned} \tag{4.12}$$

We want to recognize in (4.12) the action of the generator, see (4.4). To this end, we analyze the terms one by one. The second term in the first line is

part of the generator. Using that $\sum_{\ell' \in \mathfrak{I}_h} p_{\ell'} \mu_{\ell'} = \phi_\star^h \mu_\ell$, and Lemma 4.6,

$$\begin{aligned}
& \varepsilon^{-\frac{1}{2}} \sum_{\ell' \in \mathfrak{I}_h} p_{\ell'} \int_{\mathbb{I}_{\ell'}} \int_0^h \langle DA(G_{\ell'}(\theta)), \mathbb{B}_\varepsilon(s, \theta) G_{\ell'}(\theta) \rangle \rho_{\ell'}(\theta) ds d\theta \\
(4.13) \quad & = \varepsilon^{-\frac{1}{2}} \mu_\ell \left(\int_0^h \langle DA(z(h)), \mathbb{B}_\varepsilon(s+h, \theta) z(h) \rangle ds \right) \\
& = \mathcal{O}(\|A\|_{\mathfrak{E}} E(\varepsilon (\ln \delta_\varepsilon^{-1})^2 + h^3 \varepsilon^{-\frac{3}{2}})).
\end{aligned}$$

For convenience let us set, for each $x, y \in \mathbb{Z}_N$, $\mathbf{m}_x(\mathbf{q}, \mathbf{p}) = \left\langle \frac{\partial A}{\partial p_x}, p_x \right\rangle$ and $\mathfrak{l}_{x,y}(\mathbf{q}, \mathbf{p}) = \left\langle \frac{\partial^2 A}{\partial p_x \partial p_y} J p_y, J p_x \right\rangle$. Observe that $\mathbf{m}_x, \mathfrak{l}_{x,y} \in \mathfrak{E}$, and

$$\begin{aligned}
(4.14) \quad & \sum_{x \in \mathbb{Z}_N} \sum_{y \in \mathbb{Z}_N} |\mathfrak{l}_{x,y}| + \sum_{x \in \mathbb{Z}_N} |\mathbf{m}_x| \leq C_\# \|A\|_{\mathfrak{E}} \cdot \mathcal{E}_N, \\
& \|\mathbf{m}_x\|_{\mathfrak{E}} + \|\mathfrak{l}_{x,y}\|_{\mathfrak{E}} \leq C_\# \|A\|_{\mathfrak{E}}.
\end{aligned}$$

By (4.1) and (4.4),

$$(4.15) \quad \gamma \mathcal{S}A = \gamma \sum_{x \in \mathbb{Z}_N} (\mathfrak{l}_{x,x} - \mathbf{m}_x).$$

Using (3.2), one has

$$\begin{aligned}
& \langle DA(G_{\ell'}(\theta)), \mathbb{B}_\varepsilon(s, \theta) \mathbb{B}_\varepsilon(u, \theta) G_{\ell'}(\theta) \rangle \\
& = - \sum_{x \in \mathbb{Z}_N} b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) \mathbf{m}_x(G_{\ell'}(\theta)),
\end{aligned}$$

so that the expression in the third line of equation (4.12) is equal to

$$\begin{aligned}
(4.16) \quad & -\varepsilon^{-1} \sum_{\ell' \in \mathfrak{I}_h} p_{\ell'} \sum_{x \in \mathbb{Z}_N} \int_{\mathbb{I}_{\ell'}} \int_0^h \int_0^s \mathbf{m}_x(G_{\ell'}(\theta)) \\
& \quad \cdot b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) \rho_{\ell'}(\theta) d\theta du ds.
\end{aligned}$$

Since $|I_{\ell'}^x| \leq \delta_\varepsilon$, using the intermediate value theorem, for any $x \in \mathbb{Z}_N$,

$$\begin{aligned}
& \sup_{\theta \in \mathbb{I}_{\ell'}^x} \left| \int_{I_{\ell'}^x} \mathbf{m}_x(G_{\ell'}(\theta)) b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) \rho_{\ell'}^x(\theta_x) d\theta_x - \right. \\
& \quad \left. \int_{I_{\ell'}^x} b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) \rho_{\ell',x}(\theta_x) d\theta_x \int_{I_{\ell'}^x} \mathbf{m}_x(G_{\ell'}(\theta)) \rho_{\ell',x}(\theta_x) d\theta_x \right| \\
& \leq \delta_\varepsilon \sup_{\theta \in \mathbb{I}_{\ell'}^x} \left| \frac{\partial}{\partial \theta_x} \mathbf{m}_x(G_{\ell'}(\theta)) \right|.
\end{aligned}$$

By Lemma C.4 and the second of (4.14), the previous quantity is less than $C_{\#}\varepsilon^{\frac{1}{2}}\delta_{\varepsilon}E\|A\|_{\mathfrak{E}}$. Thus, for any $x \in \mathbb{Z}_N$, since $\rho_{\ell'}$ is a product,

$$\begin{aligned} & \int_{\mathbb{I}_{\ell'}} \int_0^h \int_0^s \mathbf{m}_x(G_{\ell'}(\theta)) \left(b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) \right) \rho_{\ell'}(\theta) d\theta duds \\ &= \int_{I_{\ell'}^x} \int_0^h \int_0^s \left(b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) \right) \rho_{\ell'}^x(\theta_x) d\theta_x duds \int_{\mathbb{I}_{\ell'}} \mathbf{m}_x(G_{\ell'}(\theta)) \rho_{\ell'}(\theta) d\theta \\ & \quad + \mathcal{O}\left(\varepsilon^{\frac{1}{2}} h^2 \delta_{\varepsilon} E \|A\|_{\mathfrak{E}}\right). \end{aligned}$$

Accordingly, (4.16) equals

$$\begin{aligned} & -\varepsilon^{-1} \sum_{\ell' \in \mathfrak{J}_h} p_{\ell'} \sum_{x \in \mathbb{Z}_N} \int_{I_{\ell'}^x} \int_0^h \int_0^s \left(b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) \right) \rho_{\ell',x}(\theta) d\theta_x duds \\ & \cdot \int_{\mathbb{I}_{\ell'}} \mathbf{m}_x(G_{\ell'}(\theta)) \rho_{\ell'}(\theta) d\theta + \mathcal{O}\left(N\varepsilon^{-\frac{1}{2}} h^2 \delta_{\varepsilon} E \|A\|_{\mathfrak{E}}\right). \end{aligned}$$

By Lemma D.1 and the first one of (4.14), and recalling that $\varepsilon \in (0, N^{-\frac{1}{33}})$,

$$\begin{aligned} (4.17) \quad & \varepsilon^{-1} \int_0^h \int_0^s \langle DA(G_{\ell'}(\theta)), \mathbb{B}_{\varepsilon}(s, \theta) \mathbb{B}_{\varepsilon}(u, \theta) G_{\ell'}(\theta) \rangle duds \\ & = -h\gamma \int_{\mathbb{I}_{\ell'}} \sum_{x \in \mathbb{Z}_N} \mathbf{m}_x(G_{\ell'}(\theta)) \rho_{\ell'}(\theta) d\theta + \mathcal{O}\left(\varepsilon \|A\|_{\mathfrak{E}} E (\ln \delta_{\varepsilon}^{-1})^2\right). \end{aligned}$$

Finally, the last term of (4.12) reads

$$\begin{aligned} (4.18) \quad & \sum_{x \in \mathbb{Z}_N} \int_0^h \int_0^s \int_{\mathbb{I}_{\ell'}} \frac{b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_x)}{\varepsilon} \mathfrak{I}_{x,x}(G_{\ell'}(\theta)) \rho_{\ell'}(\theta) d\theta duds \\ & + \sum_{\substack{x,y \in \mathbb{Z}_N \\ x \neq y}} \int_0^h \int_0^s \int_{\mathbb{I}_{\ell'}} \frac{b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_y)}{\varepsilon} \mathfrak{I}_{x,y}(G_{\ell'}(\theta)) \rho_{\ell'}(\theta) d\theta duds. \end{aligned}$$

Since $\rho_{\ell'}$ is a product measure, for $x \neq y$,

$$\begin{aligned} (4.19) \quad & \int_{\mathbb{I}_{\ell'}} b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_y) \rho_{\ell'}(\theta) d\theta = \\ & = \left(\int_{I_{\ell'}^x} b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) \rho_{\ell',x}(\theta_x) d\theta_x \right) \left(\int_{I_{\ell'}^y} b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_y) \rho_{\ell',y}(\theta_y) d\theta_y \right). \end{aligned}$$

Also, setting $z_{\ell'} = \int_{\mathbb{I}_{\ell'}} G_{\ell'} \rho_{\ell'}$, by Lemma C.4 and the second of (4.14),

$$\begin{aligned} & \sup_{\substack{\theta \in \mathbb{I}_{\ell'} \\ x,y \in \mathbb{Z}_N \\ x \neq y}} |\mathfrak{I}_{x,y}(G_{\ell'}(\theta)) - \mathfrak{I}_{x,y}(z_{\ell'})| \leq \delta_{\varepsilon} N \sup_{x' \in \mathbb{Z}_N} \sup_{\theta \in \mathbb{I}_{\ell'}} \sum_{\substack{x,y \in \mathbb{Z}_N \\ x \neq y}} |\partial_{\theta_{x'}} \mathfrak{I}_{x,y}(G_{\ell'}(\theta))| \\ & \leq \delta_{\varepsilon} N^3 \overline{C} \varepsilon^{\frac{1}{2}} E \|A\|_{\mathfrak{E}} = C_{\#} \varepsilon^{\frac{1}{2}} \delta_{\varepsilon} N^3 E \|A\|_{\mathfrak{E}}. \end{aligned}$$

Accordingly, the second line of (4.18) reads

$$\begin{aligned}
& \varepsilon^{-1} \sum_{\substack{x,y \in \mathbb{Z}_N \\ x \neq y}} \int_0^h \int_0^s \int_{\mathbb{I}_{\ell'}} b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_y) \mathfrak{I}_{x,y}(G_{\ell'}(\theta)) \rho_{\ell'}(\theta) d\theta duds \\
(4.20) \quad &= \varepsilon^{-1} \sum_{\substack{x,y \in \mathbb{Z}_N \\ x \neq y}} \mathfrak{I}_{x,y}(z_{\ell'}) \int_0^h \int_0^s \int_{I_{\ell'}^x} b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) \rho_{\ell',x}(\theta_x) d\theta_x \\
& \quad \cdot \int_{I_{\ell'}^y} b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_y) \rho_{\ell',y}(\theta_y) d\theta_y duds + \mathcal{O}(h^2 \delta_\varepsilon \varepsilon^{-\frac{1}{2}} EN^3 \|A\|_{\mathfrak{E}}).
\end{aligned}$$

Using Lemma A.4 and the first one of (4.14),

$$\begin{aligned}
(4.21) \quad & \sum_{\substack{x,y \in \mathbb{Z}_N \\ x \neq y}} \int_0^h \int_0^s \int_{\mathbb{I}_{\ell'}} \frac{b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_y)}{\varepsilon} \mathfrak{I}_{x,y}(G_{\ell'}(\theta)) \rho_{\ell'}(\theta) d\theta duds \\
&= \mathcal{O}\left(\varepsilon (\ln(\delta_\varepsilon^{-1}))^2 E \|A\|_{\mathfrak{E}}\right) + \mathcal{O}\left(h^2 \varepsilon^{-\frac{1}{2}} \delta_\varepsilon EN^3 \|A\|_{\mathfrak{E}}\right).
\end{aligned}$$

It remains to consider the first term in (4.18). Using again Lemma C.4,

$$\begin{aligned}
(4.22) \quad & \sum_{x \in \mathbb{Z}_N} \int_0^h \int_0^s \int_{\mathbb{I}_{\ell'}} \frac{b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_x)}{\varepsilon} \mathfrak{I}_{x,x}(G_{\ell'}(\theta)) \rho_{\ell'}(\theta) d\theta duds \\
&= \sum_{x \in \mathbb{Z}_N} \int_0^h \int_0^s \int_{I_{\ell'}^x} \frac{b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_x)}{\varepsilon} \rho_{\ell',x}(\theta_x) d\theta_x duds \\
& \quad \cdot \int_{\mathbb{I}_{\ell'}} \mathfrak{I}_{x,x}(G_{\ell'}(\theta)) \rho_{\ell'}(\theta) d\theta + \mathcal{O}(\delta_\varepsilon N^2 \varepsilon^{-\frac{1}{2}} E h^2 \|A\|_{\mathfrak{E}}).
\end{aligned}$$

Finally, by Lemma D.1 and the first of (4.14), equation (4.22) is equal to

$$\begin{aligned}
(4.23) \quad & h\gamma \int_{\mathbb{I}_{\ell'}} \sum_{x \in \mathbb{Z}_N} \mathfrak{I}_{x,x}(G_{\ell'}(\theta)) \rho_{\ell'}(\theta) d\theta + \mathcal{O}(\varepsilon (\ln(\delta_\varepsilon^{-1}))^2 \|A\|_{\mathfrak{E}} E) \\
& \quad + \mathcal{O}(\delta_\varepsilon N^2 \varepsilon^{-\frac{1}{2}} E h^2 \|A\|_{\mathfrak{E}}).
\end{aligned}$$

Collecting (4.13), (4.17), (4.23), and recalling (4.15), yields

$$\mu_\ell(A(z(2h))) = \mu_\ell(\mathcal{G}A(z(h))) h + \mu_\ell(\mathfrak{R}_A(h, \varepsilon, E)),$$

where we have also used that $h \in (\varepsilon^{\frac{99}{100}}, \varepsilon^{\frac{1}{2}})$ and $\varepsilon \in (0, N^{-\frac{1}{33}})$. Hence, by Lemma 4.1 and Corollary 3.5,

$$(4.24) \quad \mu_\ell(A(z(2h))) = \int_h^{2h} \mu_\ell(\mathcal{G}A(z(s))) ds + \mu_\ell(\mathfrak{R}_A(h, \varepsilon, E)),$$

concluding the proof.

4.4. Average of a first order term.

Here, we show that the contribution of the second line of (4.12) is negligible.

Lemma 4.6. *For $\varepsilon \in (0, N^{-\frac{1}{33}})$, $h \in (\varepsilon^{\frac{99}{100}}, \varepsilon^{\frac{1}{2}})$, $\ell \in \mathfrak{S}(N, E, \delta_\varepsilon, \varepsilon, \overline{C})$, and $A \in \mathfrak{E}$, we have*

$$\begin{aligned} & \varepsilon^{-\frac{1}{2}} \mu_\ell \left(\int_0^h \langle DA(z(h)), \mathbb{B}_\varepsilon(h+s)z(h) \rangle ds \right) \\ &= \mathcal{O} \left(\|A\|_{\mathfrak{E}} E \left[\varepsilon (\ln \delta_\varepsilon^{-1})^2 + h^3 \varepsilon^{-\frac{3}{2}} \right] \right). \end{aligned}$$

Proof. By expanding $DA(z(h, G_\ell(\theta)))$ and $z(h, G_\ell(\theta))$ as in (3.12) and since $h \in (\varepsilon^{\frac{99}{100}}, \varepsilon^{\frac{1}{2}})$, we can write $\varepsilon^{-\frac{1}{2}} \mu_\ell \left(\int_0^h \langle DA(z(h)), \mathbb{B}_\varepsilon(h+s)z(h) \rangle ds \right)$ as

$$\begin{aligned} & \varepsilon^{-\frac{1}{2}} \mu_\ell \left(\int_0^h \langle DA(z), \mathbb{B}_\varepsilon(h+s)z \rangle ds \right) \\ (4.25) \quad & + \varepsilon^{-\frac{1}{2}} \mu_\ell \left(\int_0^h \int_0^h \langle DA(z), \mathbb{B}_\varepsilon(h+s) \left(-\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(u) \right) z(u) \rangle dud s \right) \\ & + \varepsilon^{-\frac{1}{2}} \mu_\ell \left(\int_0^h \int_0^h \langle D^2 A \left(-\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(s) \right) z(s), \mathbb{B}_\varepsilon(h+u)z \rangle dud s \right) \\ & + \mathcal{O} \left(\|A\|_{\mathfrak{E}} E h^3 \varepsilon^{-\frac{3}{2}} \right). \end{aligned}$$

We analyze the above terms one by one. Let us set $\mathfrak{g}_x(\mathbf{q}, \mathbf{p}) = \langle \partial_{p_x} A(\mathbf{q}, \mathbf{p}), Jp_x \rangle$, $x \in \mathbb{Z}_N$. Note that $\mathfrak{g}_x \in \mathfrak{E}$, and

$$(4.26) \quad \sum_{x \in \mathbb{Z}_N} |\mathfrak{g}_x| \leq \|A\|_{\mathfrak{E}} \cdot \mathcal{E}_N \quad \& \quad \|\mathfrak{g}_x\|_{\mathfrak{E}} \leq \|A\|_{\mathfrak{E}}.$$

Furthermore,

$$\langle DA(G_\ell(\theta)), \mathbb{B}_\varepsilon(h+s)G_\ell(\theta) \rangle = \sum_{x \in \mathbb{Z}_N} b(f^{\lfloor (h+s)\varepsilon^{-1} \rfloor} \theta_x) \mathfrak{g}_x(G_\ell(\theta)).$$

Using the second of (4.26) and Lemma C.4, the first term of (4.25) equals

$$\begin{aligned} & \varepsilon^{-\frac{1}{2}} \sum_{x \in \mathbb{Z}_N} \int_0^h \int_{\mathbb{T}^N} b(f^{\lfloor (h+s)\varepsilon^{-1} \rfloor} \theta_x) \mathfrak{g}_x(G_\ell(\theta)) \mathbb{1}_{\mathbb{I}_\ell}(\theta) \rho_\ell(\theta) d\theta = \\ (4.27) \quad & = \varepsilon^{-\frac{1}{2}} \sum_{x \in \mathbb{Z}_N} \int_0^h \int_{\mathbb{T}} b(f^{\lfloor (h+s)\varepsilon^{-1} \rfloor} \theta_x) \mathbb{1}_{I_\ell^x}(\theta) \rho_{\ell, x}(\theta) d\theta_x \\ & \cdot \int_{\mathbb{I}_\ell} \mathfrak{g}_x(G_\ell(\theta)) \rho_\ell(\theta) d\theta + \mathcal{O}(h \delta_\varepsilon E N \|A\|_{\mathfrak{E}}). \end{aligned}$$

By Lemma C.3, equation (A.3) and since $h > \varepsilon^{\frac{99}{100}}$, for $\nu < 1$,

$$\left\| \mathcal{L}^{\lfloor h\varepsilon^{-1} \rfloor} \left(\mathbb{1}_{I_\ell^x} \rho_{\ell', x} - \rho_\star \right) \right\|_{BV} \leq C_\# \delta_\varepsilon^{-1} \nu^{\lfloor h\varepsilon^{-1} \rfloor} \leq C_\# \nu^{\varepsilon^{-\frac{1}{200}}},$$

and, since b_x is zero average with respect to ρ_\star ,

$$\begin{aligned}
& \int_0^h \int_{\mathbb{T}} b(f^{\lfloor (h+s)\varepsilon^{-1} \rfloor} \theta_x) \mathbb{1}_{I_\ell^x}(\theta_x) \rho_{\ell,x}(\theta_x) d\theta_x ds = \\
(4.28) \quad & = \int_0^h \int_{\mathbb{T}} b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_x) \left(\mathcal{L}^{\lfloor h\varepsilon^{-1} \rfloor} \left(\mathbb{1}_{I_\ell^x}(\theta_x) \rho_{\ell,x}(\theta_x) \right) - \rho_\star(\theta_x) \right) d\theta_x ds \\
& = \mathcal{O}(\nu^\varepsilon^{-\frac{1}{200}}).
\end{aligned}$$

By (4.28) and the first of (4.26), it follows

$$(4.29) \quad \varepsilon^{-\frac{1}{2}} \left| \mu_\ell \left(\int_0^h \langle DA(z), (\mathbb{B}_\varepsilon(h+s)z) \rangle ds \right) \right| \leq C_\# h \delta_\varepsilon EN \|A\|_{\mathfrak{E}}.$$

Using (3.13), the second term of (4.25) reads

$$\begin{aligned}
& \varepsilon^{-\frac{1}{2}} \mu_\ell \left(\int_0^h \int_0^h \langle DA(z), \mathbb{B}_\varepsilon(h+s) \left(-\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(u) \right) z \rangle dudz \right) \\
& + \mathcal{O} \left(\|A\|_{\mathfrak{E}} E \varepsilon^{-\frac{3}{2}} h^3 \right) \\
& = \varepsilon^{-1} \mu_\ell \left(\int_0^h \int_0^h \langle DA(z), \mathbb{B}_\varepsilon(h+s) \mathbb{B}_\varepsilon(u) z \rangle dudz \right) + \mathcal{O} \left(\varepsilon^{-\frac{3}{2}} h^3 E \|A\|_{\mathfrak{E}} \right),
\end{aligned}$$

where, in the last equality, we have used $h > \varepsilon$. Let $\mathbf{n}_x(\mathbf{q}, \mathbf{p}) = -\langle p_x, \partial_{p_x} A \rangle$, $x \in \mathbb{Z}_N$. Note that $\mathbf{n}_x \in \mathfrak{E}$, and,

$$(4.30) \quad \sum_{x \in \mathbb{Z}_N} |\mathbf{n}_x| \leq \|A\|_{\mathfrak{E}} \cdot \mathcal{E}_N \quad \& \quad \|\mathbf{n}_x\|_{\mathfrak{E}} \leq \|A\|_{\mathfrak{E}}.$$

We have

$$\begin{aligned}
& \varepsilon^{-1} \int_0^h \int_0^h \mu_\ell \left(\langle DA(z), \mathbb{B}_\varepsilon(h+s) \mathbb{B}_\varepsilon(u) z \rangle \right) dudz \\
& = \varepsilon^{-1} \sum_{x \in \mathbb{Z}_N} \int_0^h \int_0^h \int_{I_\ell^x} b(f^{\lfloor (h+s)\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) \mathbf{n}_x(G_\ell(\theta)) \rho_\ell(\theta) d\theta dudz.
\end{aligned}$$

Using Lemma C.4 and the second of (4.30), this is equal to

$$\begin{aligned}
(4.31) \quad & \varepsilon^{-1} \sum_{x \in \mathbb{Z}_N} \int_0^h \int_0^h \int_{I_\ell^x} b(f^{\lfloor (h+s)\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta_x) \rho_{\ell,x}(\theta_x) d\theta_x dudz \\
& \cdot \int_{I_\ell} \mathbf{n}_x(G_\ell(\theta)) \rho_\ell(\theta) d\theta + \mathcal{O}(\varepsilon^{-\frac{1}{2}} h^2 \delta_\varepsilon EN \|A\|_{\mathfrak{E}}).
\end{aligned}$$

By Lemma D.2 and the first of (4.30), the above is $\mathcal{O}(\|A\|_{\mathfrak{E}} E \varepsilon (\ln \delta_\varepsilon^{-1})^2)$. Hence,

$$\begin{aligned}
(4.32) \quad & \varepsilon^{-\frac{1}{2}} \left| \mu_\ell \left(\int_0^h \int_0^h \langle DA(z), \mathbb{B}_\varepsilon(h+s) \left(\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(u) \right) z(u) \rangle dudz \right) \right| \\
& \leq C_\# E \|A\|_{\mathfrak{E}} \left(\varepsilon (\ln \delta_\varepsilon^{-1})^2 + \varepsilon^{-\frac{3}{2}} h^3 + \varepsilon^{-\frac{1}{2}} h^2 \delta_\varepsilon N \right).
\end{aligned}$$

Next, we analyze the third term in (4.25). By (3.13), we can write it as

$$(4.33) \quad \varepsilon^{-1} \mu_\ell \left(\int_0^h \int_0^h \langle D^2 A \mathbb{B}_\varepsilon(s) z, \mathbb{B}_\varepsilon(h+u) z \rangle dud s \right) \\ + \mathcal{O}(\|A\|_{\mathfrak{E}} \varepsilon^{-\frac{3}{2}} h^3 E).$$

For $x, y \in \mathbb{Z}_N$, let $\mathfrak{o}_{x,y}(\mathbf{q}, \mathbf{p}) = \left\langle \frac{\partial^2 A}{\partial p_x \partial p_y} J p_y, J p_x \right\rangle$. Note that $\mathfrak{o}_{x,y} \in \mathfrak{E}$, and,

$$(4.34) \quad \sum_{x \in \mathbb{Z}_N} \sum_{y \in \mathbb{Z}_N} |\mathfrak{o}_{x,y}| \leq \|A\|_{\mathfrak{E}} \cdot \mathcal{E}_N \quad \& \quad \|\mathfrak{o}_{x,y}\|_{\mathfrak{E}} \leq \|A\|_{\mathfrak{E}}.$$

With the above notation we can write the first line of (4.33) as

$$\varepsilon^{-1} \sum_{x \in \mathbb{Z}_N} \sum_{y \in \mathbb{Z}_N} \int_0^h \int_0^h \int_{\mathbb{I}_\ell} \mathfrak{o}_{x,y}(G_\ell(\theta)) b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_y) b(f^{\lfloor (u+h)\varepsilon^{-1} \rfloor} \theta_x) \rho_\ell(\theta) d\theta dud s.$$

By Lemma C.4 and the second of (4.34), this is equal to

$$(4.35) \quad \varepsilon^{-1} \sum_{x \in \mathbb{Z}_N} \sum_{y \in \mathbb{Z}_N} \int_0^h \int_0^h \int_{\mathbb{I}_\ell} b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_y) b(f^{\lfloor (u+h)\varepsilon^{-1} \rfloor} \theta_x) \rho_\ell(\theta) d\theta dud s \\ \cdot \int_{\mathbb{I}_\ell} \mathfrak{o}_{x,y}(G_\ell(\theta)) \rho_\ell(\theta) d\theta + \mathcal{O}\left(N^3 \delta_\varepsilon E \varepsilon^{-\frac{1}{2}} \|A\|_{\mathfrak{E}} h^2\right).$$

For $x \neq y$, since ρ_ℓ is a product measure,

$$\int_{\mathbb{I}_\ell} b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_y) b(f^{\lfloor (u+h)\varepsilon^{-1} \rfloor} \theta_x) \rho_\ell(\theta) d\theta \\ = \int_{I_\ell^y} b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_y) \rho_{\ell,x}(\theta_y) d\theta_y \int_{I_\ell^x} b(f^{\lfloor (u+h)\varepsilon^{-1} \rfloor} \theta_x) \rho_{\ell,x}(\theta_x) d\theta_x.$$

However, by Lemma A.3 and C.3, for any $u \geq 0$,

$$\int_{I_\ell^x} b(f^{\lfloor (u+h)\varepsilon^{-1} \rfloor} \theta_x) \rho_{\ell,x}(\theta_x) d\theta_x = \int_{\mathbb{T}} b(\theta_x) \mathcal{L}^{\lfloor (h+u)\varepsilon^{-1} \rfloor} (\rho_{\ell,x}(\theta_x) \mathbb{1}_{I_\ell^x}(\theta_x)) d\theta_x \\ \leq C_\# \|\rho_{\ell,x} \mathbb{1}_{I_\ell^x}\|_{BV} \nu^{\lfloor (h+u)\varepsilon^{-1} \rfloor} \leq C_\# \delta_\varepsilon^{-1} \nu^{\lfloor h\varepsilon^{-1} \rfloor} \leq C_\# \nu^{\varepsilon^{-\frac{1}{200}}}.$$

For $x = y$,

$$\varepsilon^{-1} \int_0^h \int_0^h \int_{\mathbb{I}_\ell} b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_y) b(f^{\lfloor (u+h)\varepsilon^{-1} \rfloor} \theta_x) \rho_\ell(\theta) d\theta dud s = \\ = \varepsilon^{-1} \int_0^h \int_0^h \int_{I_\ell^x} b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta_x) b(f^{\lfloor (u+h)\varepsilon^{-1} \rfloor} \theta_x) \rho_{\ell,x}(\theta_x) d\theta_x dud s,$$

and by Lemma D.2 this is of order $\mathcal{O}(\varepsilon(\ln \delta_\varepsilon^{-1})^2)$. Therefore, recalling equation (4.35) and using the first of (4.34), we have

$$(4.36) \quad \varepsilon^{-\frac{1}{2}} \mu_\ell \left(\int_0^h \int_0^h \left\langle D^2 A \left(-\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(s) \right) z(s), \mathbb{B}_\varepsilon(h+u) z \right\rangle dud s \right) \\ \leq C_\# \|A\|_{\mathfrak{E}E} \left(\varepsilon(\ln \delta_\varepsilon^{-1})^2 + \varepsilon^{-\frac{3}{2}} h^3 + N^3 \delta_\varepsilon \varepsilon^{-\frac{1}{2}} h^2 \right).$$

Collecting the estimates (4.29), (4.32) e (4.36) and using $\varepsilon \in (0, N^{-\frac{1}{33}})$ and $h \in (\varepsilon^{\frac{99}{100}}, \varepsilon^{\frac{1}{2}})$ the Lemma follows. \square

5. HYDRODYNAMIC LIMIT

For any test function $\varphi \in \mathcal{C}^0(\mathbb{T}^1, \mathbb{R})$, define

$$(5.1) \quad \xi_N(t, \varphi) = \frac{1}{N} \sum_{x \in \mathbb{Z}_N} \varphi \left(\frac{x}{N} \right) \mathbb{E}_{\mu_N}(\mathfrak{e}_x(N^2 t)).$$

That is, $\xi_N(t)$ is a measure that describes the, rescaled, averaged energy distribution at the macroscopic time t . We will show that ξ_N satisfies an integral equation with some error, as stated in the following Proposition.

Proposition 5.1. *For each $\varepsilon \in (0, N^{-\frac{1}{33}})$, $N \in \mathbb{N}$, and initial probability measure $\mu_N \in \mathcal{M}_{init}$ we have, for any $s, t \in \mathbb{R}_+$, $s < t$, and $\varphi \in \mathcal{C}^6(\mathbb{T}, \mathbb{R})$,*

$$\xi_N(t, \varphi) - \xi_N(s, \varphi) = \frac{D}{2\gamma} \int_s^t \xi_N(u, \varphi'') du \\ + \mathcal{E}_* \|\varphi\|_{\mathcal{C}^6} \mathcal{O} \left(N^{-1} + (t-s) \left(N^{-1} + N \varepsilon^{\frac{1}{6}} (\ln \varepsilon^{-1})^2 \right) \right).$$

The first step in the proof of Proposition 5.1 is to find a more convenient expression for $\xi_N(t, \varphi) - \xi_N(s, \varphi)$. By equations (3.3), (3.4) we have, setting $\nabla_{\mathbb{Z}_N} g(x) = g(x+1) - g(x)$,

$$(5.2) \quad \xi_N(t, \varphi) - \xi_N(0, \varphi) = N \int_0^t \sum_{x \in \mathbb{Z}_N} \varphi \left(\frac{x}{N} \right) \mu_N \left(j_{x-1, x}(N^2 s) - j_{x, x+1}(N^2 s) \right) ds \\ = N \int_0^t \sum_{x \in \mathbb{Z}_N} \nabla_{\mathbb{Z}_N} \varphi \left(\frac{x}{N} \right) \mu_N(j_{x, x+1}(N^2 s)) ds.$$

It is convenient to write the above equation in terms of the covariance matrix.

Recalling (3.8) and (3.9), we define

$$(5.3) \quad \mathbf{S}^{(\alpha)}(t) = \int_0^t \mu_N \left(S_N^{(\alpha)}(N^2 s) \right) ds, \quad (\alpha) \in \{(q), (p), (q, p), (p, q)\}.$$

Then, recalling (3.3), we can write

$$(5.4) \quad \xi_N(t, \varphi) - \xi_N(0, \varphi) = N \sum_{x \in \mathbb{Z}_N} \nabla_{\mathbb{Z}_N} \varphi \left(\frac{x}{N} \right) \left[\mathbf{S}_{x, x}^{(q, p)}(t) - \mathbf{S}_{x+1, x}^{(q, p)}(t) \right].$$

Next, we need an interlude dedicated to the evolution of the covariances.

5.1. The dynamics of the covariance matrix.

To study equation (5.4) we need to understand the evolution of the covariance matrices \mathbf{S} . This is the goal of this section, whose main result is equation (5.18).

We use the discrete Fourier transform U (see Appendix E for the definition) to define, for each $(\alpha) \in \{(q), (p), (q, p), (p, q)\}$, $j, j' \in \mathbb{Z}_N$,

$$(5.5) \quad \widehat{\mathbf{S}}_{j,j'}^\alpha = \sum_{x,x' \in \mathbb{Z}_N} \mathbf{S}_{x,x'}^\alpha \psi_j^*(x) \psi_{j'}^*(x') = (U \mathbf{S}^\alpha U)_{j,j'}$$

with inverse

$$(5.6) \quad \mathbf{S}_{x,x'}^\alpha = \sum_{j,j' \in \mathbb{Z}_N} \widehat{\mathbf{S}}_{j,j'}^\alpha \psi_j(x) \psi_{j'}(x') = (U^{-1} \widehat{\mathbf{S}}^\alpha U^{-1})_{x,x'}.$$

In analogy with (3.2) we define the $2N \times 2N$ matrices

$$(5.7) \quad \mathbf{A} = \begin{pmatrix} \mathbb{0} & -\mathbb{1} \\ -\Delta + \omega_0^2 \mathbb{1} & \gamma \mathbb{1} \end{pmatrix},$$

where Δ are the operators such that, for all $g : \mathbb{Z}_N \rightarrow \mathbb{R}^d$, $d \in \mathbb{N}$, acts as $(\Delta g)_x = g(x+1) - 2g(x) + g(x-1)$. Also, we define

$$(5.8) \quad \Sigma(S) = \begin{bmatrix} \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \text{Diag}(S) \end{bmatrix},$$

where $\mathbb{0}$ is $N \times N$ null matrix and for each $N \times N$ matrix M

$$(5.9) \quad \text{Diag}(M) = \begin{bmatrix} M_{1,1} & 0 & 0 & \dots & 0 \\ 0 & M_{2,2} & 0 & \dots & 0 \\ 0 & 0 & M_{3,3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & M_{N,N} \end{bmatrix}.$$

By a straightforward computation:

$$(5.10) \quad \mathcal{G}S_N(t) = -\mathbf{A}S_N(t) - S_N(t)\mathbf{A}^T + 2\gamma\Sigma(S_N^{(p)}(t)).$$

Let

$$\boldsymbol{\beta}_N = \begin{pmatrix} \beta_N^{(q)} & \beta_N^{(q,p)} \\ \beta_N^{(p,q)} & \beta_N^{(p)} \end{pmatrix}$$

where $\beta_N^{(\alpha)}$ are $N \times N$ matrices such that $\sup_N \|\boldsymbol{\beta}_N\| < \infty$. By (3.11),

$$(5.11) \quad \|\text{Tr}(S_N \boldsymbol{\beta}_N)\|_\epsilon \leq C_\# \|\boldsymbol{\beta}\|.$$

Integrating in time and averaging with respect to the initial conditions equation (5.10), and then using Corollary 4.3 with $A_N = \text{Tr}(S_N(t)\boldsymbol{\beta}_N)$, and

equation (5.11), we obtain,

$$(5.12) \quad \begin{aligned} & \text{Tr} \left(\beta_N \left[\mathbf{A} \mathbf{S}_N(t) + \mathbf{S}_N(t) \mathbf{A}^T - 2\gamma \Sigma \left(\mathbf{S}_N^{(p)}(t) \right) \right] \right) \\ &= \frac{\text{Tr} \left(\beta_N \left[\mu_N \left(S_N(0) - S_N(N^2 t) \right) \right] \right)}{N^2} \\ & \quad + \mathcal{O} \left(\|\beta\| \mathcal{E}_* \left(N \varepsilon^{\frac{1}{6}} (\ln \varepsilon^{-1})^2 t + N^{-1} \varepsilon^{\frac{1}{3}} \right) \right). \end{aligned}$$

Since we fix t , in the following we may write \mathbf{S}_N instead of $\mathbf{S}_N(t)$. Let

$$(5.13) \quad \mathcal{R} := \mathbf{A} \mathbf{S}_N(t) + \mathbf{S}_N(t) \mathbf{A}^T - 2\gamma \Sigma \left(\mathbf{S}_N^{(p)}(t) \right),$$

which can be written in components as

$$(5.14) \quad \begin{aligned} \mathcal{R}^{(q)} &= \mathbf{S}_N^{(p,q)} + \mathbf{S}_N^{(q,p)} \\ \mathcal{R}^{(q,p)} &= \mathbf{S}_N^{(q)} (\omega_0^2 I - \Delta) + \gamma \mathbf{S}_N^{(q,p)} - \mathbf{S}_N^{(p)} \\ \mathcal{R}^{(p,q)} &= (\omega_0^2 I - \Delta) \mathbf{S}_N^{(q)} + \gamma \mathbf{S}_N^{(p,q)} - \mathbf{S}_N^{(p)} \\ \mathcal{R}^{(p)} &= (\omega_0^2 I - \Delta) \mathbf{S}_N^{(q,p)} + \mathbf{S}_N^{(p,q)} (\omega_0^2 I - \Delta) + 2\gamma \mathbf{S}_N^{(p)} - 2\gamma \Sigma \left(\mathbf{S}_N^{(p)} \right). \end{aligned}$$

Lemma 5.2. *We have the estimates*

$$\sup_{l \in \mathbb{Z}_N} \sum_{j \in \mathbb{Z}_N} |\widehat{\mathcal{R}}_{j, -j-l}^{(\alpha)}| \leq C_{\#} \mathcal{E}_* \left(N^{-1} + t N \varepsilon^{\frac{1}{6}} (\ln \varepsilon^{-1})^2 \right).$$

Proof. To obtain the above estimates we must make appropriate choices of β . For any $(\alpha) \in \{(q), (p), (q, p), (p, q)\}$ and $l \in \mathbb{Z}_N$, let

$$\beta_{N,x,y}^{l,(\alpha)} = \sum_{j \in \mathbb{Z}_N} \psi_j^*(x) \psi_{-j-l}^*(y) \text{sign}(\widehat{\mathcal{R}}_{j, -j-l}^{(\alpha)}), \quad x, y \in \mathbb{Z}_N,$$

for the (α) block and $\beta_N^{l,(\alpha')} = 0$ for $(\alpha') \neq (\alpha)$. Note that, for all $w, v \in \mathbb{R}^N$,

$$\begin{aligned} |\langle w, \beta_N^{l,(\alpha)} v \rangle| &\leq \left| \sum_{x,y \in \mathbb{Z}_N} \sum_{j \in \mathbb{Z}_N} \psi_j^*(x) \psi_{-j-l}^*(y) w_x v_y \text{sign}(\widehat{\mathcal{R}}_{j, -j-l}^{(\alpha)}) \right| \\ &\leq \sum_{j \in \mathbb{Z}_N} |\widehat{w}_j| |\widehat{v}_{-j-l}| \leq \|\widehat{w}\| \|\widehat{v}\| = \|w\| \|v\|, \end{aligned}$$

which implies that $\|\beta\| \leq 1$. For any $\mathcal{R}^{(\alpha)}$ in (5.14),

$$(5.15) \quad \sup_{l \in \mathbb{Z}_N} \sum_{j \in \mathbb{Z}_N} |\widehat{\mathcal{R}}_{j, -j-l}^{(\alpha)}| = \sup_{l \in \mathbb{Z}_N} \text{Tr}(\mathcal{R} \beta_N^{l,(\alpha)}).$$

However, using (5.12), (5.13) and $\|\beta\| \leq 1$, the right hand side is less than

$$\begin{aligned} & \sup_{l \in \mathbb{Z}_N} \frac{\text{Tr} \left(\beta_N^{l,(\alpha)} \left[\mu_N \left(S_N(0) - S_N(N^2 t) \right) \right] \right)}{N^2} \\ & \quad + \mathcal{O} \left(\mathcal{E}_* \left(N \varepsilon^{\frac{1}{6}} (\ln \varepsilon^{-1})^2 t + N^{-1} \varepsilon^{\frac{1}{3}} \right) \right). \end{aligned}$$

To conclude, note that, by (5.11), for any $t \in \mathbb{R}_+$,

$$\begin{aligned} \sup_{l \in \mathbb{Z}_N} \left| \text{Tr} \left(\beta_N^{l,(\alpha)} \mu_N(S_N(t)) \right) \right| &= \sup_{l \in \mathbb{Z}_N} \left| \mu_N \left(\text{Tr} \left(\beta_N^{l,(\alpha)} S_N(t) \right) \right) \right| \\ &\leq C_{\#} \|\beta\| \mu_N(\mathcal{E}_N(t)) \leq C_{\#} \mathcal{E}_{\star} N, \end{aligned}$$

which concludes the proof of the Lemma. \square

Notation. *The actual values of the matrices $\mathcal{R}^{(\alpha)}$ are irrelevant; so, from now on, we will use the notation \mathcal{R} for any matrix satisfying Lemma 5.2.*

Using the above notation, we can rewrite the equations (5.12) as

$$\begin{aligned} \widehat{\mathcal{R}}_{j,j'} &= \widehat{\mathcal{S}}_{j,j'}^{(q,p)} + \widehat{\mathcal{S}}_{j,j'}^{(p,q)} \\ \widehat{\mathcal{R}}_{j,j'} &= (\mu_j + \mu_{j'}) \widehat{\mathcal{S}}_{j,j'}^{(q)} - 2\widehat{\mathcal{S}}_{j,j'}^{(p)} \\ \widehat{\mathcal{R}}_{j,j'} &= 2\gamma \widehat{\mathcal{S}}_{j,j'}^{(q,p)} - \widehat{\mathcal{S}}_{j,j'}^{(q)} (\mu_j - \mu_{j'}) \\ \widehat{\mathcal{R}}_{j,j'} &= (\mu_j - \mu_{j'}) \widehat{\mathcal{S}}_{j,j'}^{(q,p)} - 2\gamma \widehat{F}_{j,j'} + 2\gamma \widehat{\mathcal{S}}_{j,j'}^{(p)}, \\ \widehat{F}_{j,j'} &:= \sum_{y \in \mathbb{Z}_N} \psi_j^*(y) \psi_{j'}^*(y) \mathbf{S}_{y,y}^{(p)} = N^{-\frac{1}{2}} \sum_{y \in \mathbb{Z}_N} \psi_{j+j'}^*(y) \mathbf{S}_{y,y}^{(p)}, \end{aligned} \tag{5.16}$$

where we have used, see (E.2),

$$(5.17) \quad (U(\omega_0^2 - \Delta)\mathcal{R}U)_{j,j'} = \mu_j \widehat{\mathcal{R}}_{j,j'} \quad \& \quad (U\mathcal{R}(\omega_0^2 - \Delta)U)_{j,j'} = \mu_{j'} \widehat{\mathcal{R}}_{j,j'}.$$

Indeed, the second equation follows by summing the second and third line of (5.14) and using the first line of (5.16). The third equation is obtained similarly by taking the difference of the second and third line of (5.14), and the fourth equation follows from the fourth of (5.14) and the first of (5.16). Equations (5.16) imply

$$\begin{aligned} (\mu_j^2 - \mu_{j'}^2) \widehat{\mathcal{S}}_{j,j'}^{(q)} - 2(\mu_j - \mu_{j'}) \widehat{\mathcal{S}}_{j,j'}^{(p)} &= \widehat{\mathcal{R}}_{j,j'} \\ 2\gamma(\mu_j + \mu_{j'}) \widehat{\mathcal{S}}_{j,j'}^{(q,p)} - (\mu_j^2 - \mu_{j'}^2) \widehat{\mathcal{S}}_{j,j'}^{(q)} &= \widehat{\mathcal{R}}_{j,j'}. \end{aligned}$$

Summing the above two equations yields

$$\gamma(\mu_j + \mu_{j'}) \widehat{\mathcal{S}}_{j,j'}^{(q,p)} - (\mu_j - \mu_{j'}) \widehat{\mathcal{S}}_{j,j'}^{(p)} = \widehat{\mathcal{R}}_{j,j'}.$$

Using the above in the fourth of (5.16) we have

$$\left[2\gamma^2(\mu_j + \mu_{j'}) + (\mu_j - \mu_{j'})^2 \right] \widehat{\mathcal{S}}_{j,j'}^{(q,p)} - 2\gamma(\mu_j - \mu_{j'}) \widehat{F}_{j,j'} = \widehat{\mathcal{R}}_{j,j'}$$

which we find convenient to write as

$$\begin{aligned} \mathbf{S}_{x,y}^{(q,p)} &= \frac{1}{2\gamma} B_{x,y} + \mathcal{R}_{x,y} \\ B_{x,y} &:= \sum_{j,j' \in \mathbb{Z}_N} \widetilde{\Phi}(\mu_j, \mu_{j'}) \widehat{F}_{j,j'} \psi_j(x) \psi_{j'}(y) \\ \widetilde{\Phi}(\mu_j, \mu_{j'}) &= \frac{4\gamma^2(\mu_j - \mu_{j'})}{2\gamma^2(\mu_j + \mu_{j'}) + (\mu_j - \mu_{j'})^2}. \end{aligned} \tag{5.18}$$

Indeed a term $\left[(\mu_j - \mu_{j'})^2 + 2\gamma^2(\mu_j + \mu_{j'}) \right]^{-1} \widehat{\mathcal{R}}_{j,j'}$ satisfies Lemma 5.2, since $(\mu_j - \mu_{j'})^2 + 2\gamma^2(\mu_j + \mu_{j'}) \geq 4\gamma\omega_0^2$, see (E.2). So it can be seen as a generic matrix \mathcal{R} . For further use, note that

$$\begin{aligned} B_{x+1,x} - B_{x,x} &= \sum_{j,j' \in \mathbb{Z}_N} \widetilde{\Phi}(\mu_j, \mu_{j'}) \widehat{F}_{j,j'} \left(e^{2\pi ij/N} - 1 \right) \psi_j(x) \psi_{j'}(x) \\ (5.19) \quad &= \frac{1}{\sqrt{N}} \sum_{j,k \in \mathbb{Z}_N} \widetilde{\Phi}(\mu_j, \mu_{k-j}) \widehat{F}_{j,k-j} \left(e^{2\pi ij/N} - 1 \right) \psi_k(x). \end{aligned}$$

5.2. The heat equation.

We continue our study of equation (5.4). First, by (2.8), for any (α) ,

$$(5.20) \quad \sum_{x \in \mathbb{Z}_N} |\mathbf{S}_{x,x}^{(\alpha)}(t)| \leq C_{\#} \mathcal{E}_{\star} N t.$$

From now on we consider $\varphi \in \mathcal{C}^6(\mathbb{T})$, as in the statement of Proposition 5.1. Taylor expanding φ in (5.4), and using (5.18), and (5.20), we can write

$$\begin{aligned} \xi_N(t, \varphi) - \xi_N(0, \varphi) &= - \sum_{x \in \mathbb{Z}_N} \varphi' \left(\frac{x}{N} \right) \left(\mathbf{S}_{x+1,x}^{q,p} - \mathbf{S}_{x,x}^{q,p} \right) \\ &\quad - \frac{1}{2N} \sum_{x \in \mathbb{Z}_N} \varphi'' \left(\frac{x}{N} \right) \left(\mathbf{S}_{x+1,x}^{q,p} - \mathbf{S}_{x,x}^{q,p} \right) + \mathcal{O}(N^{-1} \mathcal{E}_{\star} \|\varphi\|_{\mathcal{C}^3} t) \\ (5.21) \quad &= - \sum_{x \in \mathbb{Z}_N} \varphi' \left(\frac{x}{N} \right) \frac{(B_{x+1,x} - B_{x,x})}{2\gamma} + \mathcal{O}(N^{-1} \mathcal{E}_{\star} \|\varphi\|_{\mathcal{C}^3} t) \\ &\quad - \frac{1}{2N} \sum_{x \in \mathbb{Z}_N} \varphi'' \left(\frac{x}{N} \right) \frac{(B_{x+1,x} - B_{x,x})}{2\gamma} + Z_N(\varphi') + \frac{1}{2N} Z_N(\varphi'') \\ Z_N(\phi) &= \sum_{x \in \mathbb{Z}_N} \phi \left(\frac{x}{N} \right) [\mathcal{R}_{x,x} - \mathcal{R}_{x,x+1}] \end{aligned}$$

We will prove in section 5.3 the following Lemma.

Lemma 5.3. *For $\phi \in \mathcal{C}^2(\mathbb{T}, \mathbb{R})$, we have*

$$|Z_N(\phi)| \leq C_{\#} \mathcal{E}_{\star} \|\phi\|_{\mathcal{C}^2} \left(N^{-1} + t N \varepsilon^{\frac{1}{6}} (\ln \varepsilon^{-1})^2 \right).$$

Accordingly, to estimate (5.21) it suffices to compute, for each $\phi \in \mathcal{C}^4(\mathbb{T}, \mathbb{R})$,

$$(5.22) \quad \sum_{x \in \mathbb{Z}_N} \phi \left(\frac{x}{N} \right) \left[\frac{B_{x+1,x} - B_{x,x}}{2\gamma} \right].$$

Equation (5.22) reads, using (5.19) and (5.16),

$$\begin{aligned}
& \frac{1}{2\gamma\sqrt{N}} \sum_{j,k \in \mathbb{Z}_N} \widehat{\phi}_{-k} \widetilde{\Phi}(\mu_j, \mu_{k-j}) \widehat{F}_{j,k-j} \left(e^{2\pi ij/N} - 1 \right) = \\
(5.23) \quad & = \sum_{y \in \mathbb{Z}_N} \frac{\mathbf{S}_{y,y}^{(p)}}{2\gamma N} \sum_{k \in \mathbb{Z}_N} \widehat{\phi}_{-k} \psi_k^*(y) \left[\sum_{j \in \mathbb{Z}_N} \widetilde{\Phi}(\mu_j, \mu_{k-j}) \left(e^{2\pi ij/N} - 1 \right) \right].
\end{aligned}$$

To continue it is convenient to choose an explicit representation of \mathbb{Z}_N . We choose $\mathbb{J}_N = \{-\frac{N-1}{2}, \frac{N-1}{2}\}$ for N odd and $\mathbb{J}_N = \{-\frac{N}{2} + 1, \frac{N}{2}\}$ if N is even. With this notation, we can state the following Lemma, which proof we postpone to section 5.4.

Lemma 5.4. *For each $k \in \mathbb{J}_N$ we have*

$$\sum_{j \in \mathbb{Z}_N} \left(e^{2\pi ij/N} - 1 \right) \widetilde{\Phi}(\mu_j, \mu_{k-j}) = 2\pi ikD + \mathcal{O}(k^2 N^{-1})$$

where the diffusion coefficient D is defined in appendix F.

Using the above Lemma, (5.23) and renaming k as $-k$ in the sum,

$$\begin{aligned}
& \sum_{x \in \mathbb{Z}_N} \phi \left(\frac{x}{N} \right) \frac{\left(B_{x+1,x}^{q,p} - B_{x,x}^{q,p} \right)}{2\gamma} \\
(5.24) \quad & = -\frac{1}{N} \sum_{y \in \mathbb{Z}_N} \mathbf{S}_{y,y}^{(p)} \sum_{-k \in \mathbb{J}_N} \widehat{\phi}_k \psi_k(y) \left[2\pi ik \frac{D}{2\gamma} + \mathcal{O}(k^2 N^{-1}) \right] \\
& = -\frac{1}{N} \sum_{y \in \mathbb{Z}_N} \mathbf{S}_{y,y}^{(p)} \sum_{k \in \mathbb{Z}_N} \widehat{\phi}_k \psi_k(y) \frac{e^{2\pi ik/N} - 1}{N} \frac{D}{2\gamma} \\
& \quad + \frac{1}{N} \sum_{y \in \mathbb{Z}_N} \mathbf{S}_{y,y}^{(p)} \sum_{-k \in \mathbb{J}_N} \mathcal{O}(k^2 |\widehat{\phi}_k| N^{-\frac{3}{2}})
\end{aligned}$$

Using Lemma E.1, the assumption $\phi \in \mathcal{C}^4$ and (5.20) yields

$$\begin{aligned}
& \sum_{x \in \mathbb{Z}_N} \phi \left(\frac{x}{N} \right) \frac{\left(B_{x+1,x}^{q,p} - B_{x,x}^{q,p} \right)}{2\gamma} = - \sum_{y \in \mathbb{Z}_N} \mathbf{S}_{y,y}^{(p)} \sum_{k \in \mathbb{Z}_N} \widehat{\phi}_k [\psi_k(y+1) - \psi_k(y)] \frac{D}{2\gamma} \\
& \quad + \mathcal{E}_\star \mathcal{O}(N^{-1}t) \|\phi\|_{\mathcal{C}^4} \\
& = -\frac{D}{2\gamma} \sum_{y \in \mathbb{Z}_N} \mathbf{S}_{y,y}^{(p)} \nabla_{\mathbb{Z}_N} \phi \left(\frac{y}{N} \right) + \mathcal{E}_\star \mathcal{O}(N^{-1}t) \|\phi\|_{\mathcal{C}^4} \\
& = -\frac{D}{2\gamma} \frac{1}{N} \sum_{y \in \mathbb{Z}_N} \mathbf{S}_{y,y}^{(p)} \phi' \left(\frac{y}{N} \right) + \mathcal{E}_\star \mathcal{O}(N^{-1}t) \|\phi\|_{\mathcal{C}^4}.
\end{aligned}$$

Substituting the above in (5.21) (with $\phi = \varphi'$ and $\phi = \varphi''$, respectively) and using Lemma 5.3, yields

$$\begin{aligned} \xi_N(t, \varphi) - \xi_N(0, \varphi) &= \frac{D}{2\gamma} \frac{1}{N} \sum_{y \in \mathbb{Z}_N} \mathbf{S}_{y,y}^{(p)} \varphi'' \left(\frac{y}{N} \right) \\ &\quad + \mathcal{E}_* \|\varphi\|_{\mathcal{C}^6} \mathcal{O} \left(t \left(N^{-1} + N \varepsilon^{\frac{1}{6}} (\ln \varepsilon^{-1})^2 \right) + N^{-1} \right). \end{aligned}$$

Using the equipartition proved in Proposition 5.5, we have

$$(5.25) \quad \begin{aligned} \xi_N(t, \varphi) - \xi_N(0, \varphi) &= \frac{D}{2\gamma} \int_0^t \xi_N(s, \varphi'') ds \\ &\quad + \mathcal{E}_* \|\varphi\|_{\mathcal{C}^6} \mathcal{O} \left(t \left(N^{-1} + N \varepsilon^{\frac{1}{6}} (\ln \varepsilon^{-1})^2 \right) + N^{-1} \right). \end{aligned}$$

which concludes the proof of Proposition 5.1 for $s = 0$.

To prove the Proposition for $s \in (0, t)$ it suffices to note that $\phi_{\star}^s \mu_N \in \mathcal{M}_{\text{init}}$, by Lemma 2.2.

5.3. On the error Z_N .

Here we prove Lemma 5.3. Note that it is enough to estimate one term, the other being similar. Recalling (5.18) and using (E.5), we have for Z_N defined in (5.21)

$$\begin{aligned} \left| \sum_{x \in \mathbb{Z}_N} \phi \left(\frac{x}{N} \right) \mathcal{R}_{x,x} \right| &= \left| \sum_{x \in \mathbb{Z}_N} \sum_{j,k,l \in \mathbb{Z}_N} \widehat{\phi}_l \psi_l(x) \widehat{\mathcal{R}}_{j,k} \psi_j(x) \psi_k(x) \right| \\ &= \left| N^{-1} \sum_{x \in \mathbb{Z}_N} \sum_{j,k,l \in \mathbb{Z}_N} \widehat{\phi}_l \psi_{l+j+k}(x) \widehat{\mathcal{R}}_{j,k} \right| \leq N^{-\frac{1}{2}} \sum_{l \in \mathbb{Z}_N} |\widehat{\phi}_l| \sum_{j \in \mathbb{Z}_N} |\widehat{\mathcal{R}}_{j,-j-l}|. \end{aligned}$$

By Lemma 5.2, this is less than

$$\begin{aligned} C_{\#} \mathcal{E}_* \left(N^{-\frac{3}{2}} + t N^{\frac{1}{2}} \varepsilon^{\frac{1}{6}} (\ln \varepsilon^{-1})^2 \right) \sum_{l \in \mathbb{Z}_N} |\widehat{\phi}_l| \\ \leq C_{\#} \|\phi\|_{\mathcal{C}^2} \mathcal{E}_* \left(N^{-1} + t N \varepsilon^{\frac{1}{6}} (\ln \varepsilon^{-1})^2 \right), \end{aligned}$$

where in the last inequality, we used Lemma E.1 and $\phi \in \mathcal{C}^2$, so that $\sum_{l \in \mathbb{Z}_N} |\widehat{\phi}_l|$ is uniformly bounded by \sqrt{N} .

5.4. Effective derivative.

Here, we prove Lemma 5.4 whereby showing how the needed extra derivative appears. We start with an explicit computation to rewrite $\Phi(\mu_j, \mu_{j'})$ in (5.18) more explicitly. To this end, we use (E.2) and the prosthaphaeresis

identities to compute

$$\begin{aligned}
(5.26) \quad \mu_j + \mu_{j'} &= 2\omega_0^2 + 4 \sin^2\left(\frac{\pi j}{N}\right) + 4 \sin^2\left(\frac{\pi j'}{N}\right) \\
&= 2\omega_0^2 + 4 - 2 \cos\left(\frac{2\pi j}{N}\right) - 2 \cos\left(\frac{2\pi j'}{N}\right) \\
&= 2\omega_0^2 + 4 - 4 \cos\left(\frac{\pi(j+j')}{N}\right) \cos\left(\frac{\pi(j-j')}{N}\right),
\end{aligned}$$

and

$$\begin{aligned}
(5.27) \quad \mu_j - \mu_{j'} &= 4 \sin^2\left(\frac{\pi j}{N}\right) - 4 \sin^2\left(\frac{\pi j'}{N}\right) \\
&= -2 \cos\left(\frac{2\pi j}{N}\right) + 2 \cos\left(\frac{2\pi j'}{N}\right) \\
&= 4 \sin\left(\frac{\pi(j+j')}{N}\right) \sin\left(\frac{\pi(j-j')}{N}\right).
\end{aligned}$$

This allows to write

$$\tilde{\Phi}(\mu_j, \mu_{j'}) = \frac{4 \sin\left(\frac{\pi(j+j')}{N}\right) \sin\left(\frac{\pi(j-j')}{N}\right)}{\omega_0^2 + 2 - 2 \cos\left(\frac{\pi(j+j')}{N}\right) \cos\left(\frac{\pi(j-j')}{N}\right) + \frac{4}{\gamma^2} \sin^2\left(\frac{\pi(j+j')}{N}\right) \sin^2\left(\frac{\pi(j-j')}{N}\right)}.$$

Hence, for each $k \in \mathbb{J}_N$, we have

$$\begin{aligned}
&\sum_{j \in \mathbb{Z}_N} \left(e^{2\pi i j/N} - 1\right) \tilde{\Phi}(\mu_j, \mu_{k-j}) \\
&= \sum_{j \in \mathbb{Z}_N} \frac{(e^{2\pi i j/N} - 1) 4 \sin\left(\frac{\pi k}{N}\right) \sin\left(\frac{\pi(2j-k)}{N}\right)}{\omega_0^2 + 2 - 2 \cos\left(\frac{\pi k}{N}\right) \cos\left(\frac{\pi(2j-k)}{N}\right) + \frac{4}{\gamma^2} \sin^2\left(\frac{\pi k}{N}\right) \sin^2\left(\frac{\pi(2j-k)}{N}\right)} \\
&= 2k \sum_{j \in \mathbb{Z}_N} \frac{(e^{2\pi i j/N} - 1) \sin\left(\frac{2\pi j}{N}\right) 2\pi}{\omega_0^2 + 2 - 2 \cos\left(\frac{2\pi j}{N}\right) N} + \mathcal{O}(k^2 N^{-1}) \\
&= 2k \int_0^{2\pi} \frac{(e^{i\vartheta} - 1) \sin \vartheta}{\omega_0^2 + 2 - 2 \cos \vartheta} d\vartheta + \mathcal{O}(k^2 N^{-1}) \\
&= i2k \int_0^{2\pi} \frac{\sin^2(\vartheta)}{\omega_0^2 + 2 - 2 \cos \vartheta} d\vartheta + \mathcal{O}(k^2 N^{-1})
\end{aligned}$$

where we have used that $\sin\left(\frac{\pi(2j-k)}{N}\right) = \sin\left(\frac{2\pi j}{N}\right) + \mathcal{O}\left(\frac{k}{N}\right)$ in the second equality and the fact that the integral of an odd function is zero in the last equality. To compute the integral, we notice that we can see it as an integral on the unit circle on the complex plan and write

$$\begin{aligned}
\int_0^{2\pi} \frac{\sin^2(\vartheta)}{\omega_0^2 + 2 - 2 \cos \vartheta} d\vartheta &= \frac{i}{4} \int_{|z|=1} \frac{(z - z^{-1})^2 z^{-1}}{\omega_0^2 + 2 - (z + z^{-1})} dz \\
&= -\frac{i}{4} \int_{|z|=1} \frac{(z^2 - 1)^2}{z^2 [z^2 + 1 - (\omega_0^2 + 2)z]} dz.
\end{aligned}$$

Note that the poles inside the unit circle of the integrand are at zero and at $z = 1 + \frac{\omega_0^2}{2} - \sqrt{\omega_0^2 + \frac{\omega_0^4}{4}}$. In particular, setting $a_{\pm} = 1 + \frac{\omega_0^2}{2} \pm \sqrt{\omega_0^2 + \frac{\omega_0^4}{4}}$ we have

$$z^2 + 1 - (\omega_0^2 + 2)z = (z - a_-)(z - a_+),$$

hence, noting that $a_- a_+ = 1$ and recalling (F.5),

$$\begin{aligned} \int_0^{2\pi} \frac{\sin^2(\vartheta)}{\omega_0^2 + 2 - 2\cos\vartheta} d\vartheta &= \frac{\pi}{2} \left[2 + \omega_0^2 + \frac{(a_-^2 - 1)^2}{a_-^2(a_- - a_+)} \right] \\ &= \frac{\pi}{2} \left[2 + \omega_0^2 + \frac{a_-^2(a_- - a_+)^2}{a_-^2(a_- - a_+)} \right] \\ &= \frac{\pi}{2} [2 + \omega_0^2 - (a_+ - a_-)] = \pi a_- = \frac{\pi}{a_+} = \pi D, \end{aligned}$$

which, substituting backward, yields the Lemma.

5.5. Equipartition.

The last item we need to prove, to conclude the argument, is the energy equipartition (essentially, this is a quantitative version of Virial Theorem).

Proposition 5.5. *Suppose that $\varphi \in C^1(\mathbb{T})$. Then,*

$$\begin{aligned} \left| \frac{1}{N} \sum_{x \in \mathbb{Z}_N} \varphi\left(\frac{x}{N}\right) \int_0^t \mu_N \left[p_x^2(N^2 s) - \mathbf{e}_x(N^2 s) \right] ds \right| &\leq C_{\#} \|\varphi\|_{C^1} \mathcal{E}_{\star} \left(N^{-1} \right. \\ &\quad \left. + t \left(N^{-1} + N \varepsilon^{\frac{1}{6}} (\ln \varepsilon^{-1})^2 \right) \right). \end{aligned}$$

Proof. A direct computation yields, setting $Q_x = q_{x-1} \cdot (q_x - q_{x-1})$,

$$(5.28) \quad 2p_x^2 - 2\mathbf{e}_x = -(\nabla_{\mathbb{Z}_N} Q)_x + \mathcal{G} \left(q_x \cdot p_x + \frac{\gamma}{2} q_x^2 \right).$$

Since

$$\left\| \frac{1}{N} \sum_{x \in \mathbb{Z}_N} \varphi\left(\frac{x}{N}\right) \left[q_x \cdot p_x + \frac{\gamma}{2} q_x^2 \right] \right\|_{\mathbf{e}} \leq C_{\#} \|\varphi\|_{\infty},$$

we can apply again Corollary 4.3 and write

$$\begin{aligned}
& \frac{2}{N} \sum_{x \in \mathbb{Z}_N} \varphi\left(\frac{x}{N}\right) \int_0^t \mu_N \left[p_x^2(N^2s) - \mathfrak{e}_x(N^2s) \right] ds \\
&= \frac{1}{N} \sum_{x \in \mathbb{Z}_N} \nabla_{\mathbb{Z}_N} \varphi\left(\frac{x}{N}\right) \int_0^t \mu_N [Q_x(N^2s)] ds \\
(5.29) \quad &+ \frac{1}{N^3} \sum_{x \in \mathbb{Z}_N} \varphi\left(\frac{x}{N}\right) \mu_N \left[q_x(N^2t) \cdot p_x(N^2t) + \frac{\gamma}{2} q_x^2(N^2t) \right] \\
&- \frac{1}{N^3} \sum_{x=0}^n \varphi\left(\frac{x}{N}\right) \mu_N \left[q_x(0) \cdot p_x(0) + \frac{\gamma}{2} q_x^2(0) \right] \\
&+ \mathcal{E}_* \|\varphi\|_\infty \mathcal{O} \left(t N \varepsilon^{\frac{1}{6}} (\ln \varepsilon^{-1})^2 + N^{-1} \varepsilon^{\frac{1}{3}} \right).
\end{aligned}$$

Using the energy bound (2.8) we have, for each $s \in \mathbb{R}_+$,

$$(5.30) \quad \left| \frac{1}{N^3} \sum_{x \in \mathbb{Z}_N} \varphi\left(\frac{x}{N}\right) \mu_N \left[q_x(N^2s) \cdot p_x(N^2s) + \frac{\gamma}{2} q_x^2(N^2s) \right] \right| \leq C_\# \frac{\|\varphi\|_\infty \mathcal{E}_*}{N^2}.$$

While for the term in the second line of (5.29) we have

$$(5.31) \quad \left| \int_0^t \frac{1}{N} \sum_{x \in \mathbb{Z}_N} \nabla_{\mathbb{Z}_N} \varphi\left(\frac{x}{N}\right) \mu_N [Q_x(N^2s)] ds \right| \leq C_\# \frac{\|\varphi'\|_\infty \mathcal{E}_* t}{N}.$$

□

6. PROOF OF THEOREM 2.8

Given a sequence $\{\varepsilon_N\}$ such that $\varepsilon_N \leq N^{-\alpha}$, $\alpha > 6$, Proposition 5.1 implies that, for all $\varphi \in \mathcal{C}^6$, $t \in \mathbb{R}_+$,

$$(6.1) \quad \lim_{N \rightarrow \infty} \left[\xi_N(t, \varphi) - \xi_N(0, \varphi) - \frac{D}{2\gamma} \int_0^t \xi_N(s, \varphi'') ds \right] = 0.$$

For each $t_* > 0$, let $\mathcal{C}_{\mathcal{M}, t_*}^0 = \mathcal{C}^0([0, t_*], \mathcal{M}_{\mathcal{E}_0}(\mathbb{T}))$, where $\mathcal{M}_{\mathcal{E}_0}(\mathbb{T})$ is endowed with metric (G.1) which metrizes the weak topology. Note that $\xi_N \in \mathcal{C}_{\mathcal{M}, t_*}^0$. Let $\mu_N \in \mathcal{M}_{\text{init}}^*$, T_0 be given by (2.9) and $T(t) \in \mathcal{C}_{\mathcal{M}, t_*}^0$ be solution of the heat equation with initial condition T_0 . Since $\mathcal{M}_{\mathcal{E}_0}$ is compact, relative compactness of the sequence $\{\xi_N\}$ follows from equicontinuity in time of ξ_N proved in appendix G and by Ascoli's Theorem [41, pg. 223, Theorem 17]. Accordingly, there exists accumulation points $\xi_*(\cdot) \in \mathcal{C}_{\mathcal{M}, t_*}^0$ of the sequence $\{\xi_N(\cdot)\}$. By (6.1) such accumulation points satisfy the weak heat equation (2.12). Furthermore, by (2.9), $\lim_{N \rightarrow \infty} \xi_N(0) = T_0$. Therefore, since the

heat equation has a unique solution,⁸ one has

$$(6.2) \quad \lim_{N \rightarrow \infty} \xi_N = T$$

uniformly in $[0, t_*]$ in the weak topology.

To extend the result in time, note that $(\tilde{q}(t), \tilde{p}(t), \theta(t)) = (q(t + N^2 t_*), p(t + N^2 t_*), \theta(t + N^2 t_*))$ satisfies our equations (2.2), moreover the distribution μ_N^* of $(\tilde{q}(0), \tilde{p}(0), \theta(0))$ belongs to $\mathcal{M}_{\text{init}}$ by Lemma 2.2. In addition, μ_N^* satisfies (2.9), that is, its energy profile converges to $T(t_*)$. It follows that we can apply our argument to such a process and extend it for another time t_* . Iterating the argument yields

$$(6.3) \quad \lim_{N \rightarrow \infty} \xi_N(t) = T(t) \quad \forall t \in \mathbb{R}_+.$$

6.1. Computing the error.

To conclude the proof of our main result, we must compute the difference between ξ_N and T .

End of Proof of Theorem 2.8.

Since $\varepsilon \in (0, N^{-\alpha})$, $\alpha > 6$, and recalling that $\beta_* = \min\{\frac{1}{2}, \frac{\alpha}{6} - 1\}$, one has

$$(6.4) \quad \left(N^{-1} + N\varepsilon^{\frac{1}{6}}(\ln \varepsilon^{-1})^2\right) \leq C_{\#} N^{-\beta_*} (\ln(N+1))^2.$$

For $t \in \mathbb{R}_+$, $k \in \mathbb{Z} \setminus \{0\}$, we set, recalling (2.10) for the definition of ϖ ,

$$\begin{aligned} \eta_N(t, \varphi) &= \xi_N(t, \varphi) - T(t, \varphi) \quad \& \quad \psi_k(\theta) = e^{2i\pi k\theta}, \\ \mathcal{Q}_N^k &= \max \left\{ \varpi(N, k, \mu_N), N^{-\beta_*} (\ln(N+1))^2 \right\}. \end{aligned}$$

Since $\varpi(N, k, \mu_N) \leq 1 + \frac{\varepsilon_*}{\varepsilon_0}$, for any $\alpha > 6$, there exists $\tilde{C}_\alpha \in \mathbb{R}_+$ such that

$$(6.5) \quad \mathcal{Q}_N^k \leq \tilde{C}_\alpha,$$

for all $k \in \mathbb{Z} \setminus \{0\}$, $N \in \mathbb{N}$. Furthermore, for all $t \in \mathbb{R}_+$ we have

$$(6.6) \quad |\eta_N(t, \varphi)| \leq 2\mathcal{E}_* \|\varphi\|_{C^0} \quad \& \quad \eta_N(t, 1) = 0.$$

By Proposition 5.1 and inequality (6.4), we have

$$(6.7) \quad \begin{aligned} \eta_N(t, \varphi) &= \eta_N(s, \varphi) + \frac{D}{2\gamma} \int_s^t \eta_N(u, \varphi'') du \\ &+ \mathcal{O} \left(\|\varphi\|_{C^6} \mathcal{E}_* \left(N^{-\beta_*} (\ln(N+1))^2 (t-s) + N^{-1} \right) \right). \end{aligned}$$

Evaluating this on $\psi_k(\theta) = e^{2i\pi k\theta}$ we obtain

$$(6.8) \quad \begin{aligned} \eta_N(t, \psi_k) &= \eta_N(s, \psi_k) - 4\pi^2 k^2 \frac{D}{2\gamma} \int_s^t \eta_N(u, \psi_k) du \\ &+ \mathcal{O} \left(k^6 \mathcal{E}_* \left(N^{-\beta_*} (\ln(N+1))^2 (t-s) + N^{-1} \right) \right). \end{aligned}$$

⁸Just compute its Fourier coefficients $T_k(t) := \int_{\mathbb{T}} e^{2\pi i k x} T(t, dx)$.

Note that $|\eta_N(0, \psi_k)| = \mathcal{E}_0 \varpi(N, k, \mu_N)$ and set

$$t_{max} = \inf \left\{ s \in \mathbb{R}_+ : |\eta_N(s, \psi_k)| \geq C \mathcal{E}_* k^6 \mathcal{Q}_N^k \right\},$$

where $C \in (1, \infty)$ is a constant to be fixed later. If $t_{max} = \infty$, then $|\eta_N(t, \psi_k)| \leq C \mathcal{E}_* k^6 \mathcal{Q}_N^k$ for all $t \in \mathbb{R}_+$. Recalling the second of (6.6), it follows that if $\varphi \in \mathcal{C}^8$, then,⁹

$$(6.9) \quad \begin{aligned} \sup_{t \in \mathbb{R}_+} |\xi_N(t, \varphi) - T(t, \varphi)| &\leq \sup_{t \in \mathbb{R}_+} \sum_{k \in \mathbb{Z} \setminus \{0\}} \widehat{\varphi}(k) |\eta_N(t, \psi_k)| \\ &\leq C_{\#} C \mathcal{E}_* \|\varphi\|_{\mathcal{C}^8} \max \left\{ \sum_{k \in \mathbb{Z} \setminus \{0\}} k^{-2} \varpi(N, k, \mu_N), N^{-\beta_*} (\ln(N+1))^2 \right\}, \end{aligned}$$

which proves the Theorem. If instead $t_{max} < \infty$, then, by continuity of η_N ,

$$|\eta_N(t_{max}, \psi_k)| = C \mathcal{E}_* k^6 \mathcal{Q}_N^k.$$

Moreover, by (6.8) and the first of (6.6), for any α , there exists $C_1 \in (0, \infty)$ such that

$$(6.10) \quad \begin{aligned} |\eta_N(s, \psi_k)| &\geq |\eta_N(t_{max}, \psi_k)| - C_1 \mathcal{E}_* k^6 (|s - t_{max}| + N^{-1}), \\ |\eta_N(s, \psi_k)| &\leq |\eta_N(t_{max}, \psi_k)| + C_1 \mathcal{E}_* k^6 (|s - t_{max}| + N^{-1}). \end{aligned}$$

Provided that $|s - t_{max}| \leq \tau \mathcal{Q}_N^k$, $\tau := \frac{C}{3C_1}$, $C \geq 20C_1$ and noticing that $3\mathcal{Q}_N^k \geq N^{-1}$, equation (6.10) implies

$$(6.11) \quad \begin{aligned} \frac{C \mathcal{E}_* k^6 \mathcal{Q}_N^k}{2} = \frac{|\eta_N(t_{max}, \psi_k)|}{2} &\leq |\eta_N(s, \psi_k)| \\ &\leq \frac{3|\eta_N(t_{max}, \psi_k)|}{2} = \frac{3C \mathcal{E}_* k^6 \mathcal{Q}_N^k}{2}, \end{aligned}$$

and, in particular, for $|s - t_{max}| \leq \tau \mathcal{Q}_N^k$,

$$(6.12) \quad \text{sign}(\eta_N(s, \psi_k)) = \text{sign}(\eta_N(t_{max}, \psi_k)).$$

Set

$$\delta t_* = \tau \rho_k \mathcal{Q}_N^k, \quad \rho_k := \min \left\{ 1, \frac{2}{3\widetilde{C}_\alpha \tau} \left(4\pi^2 k^2 \frac{D}{2\gamma} \right)^{-1} \right\}.$$

Note that the choice of ρ_k , (6.11) and (6.5) imply that

$$(6.13) \quad |\eta_N(t_{max}, \psi_k)| - 4\pi^2 k^2 \frac{D}{2\gamma} \int_{t_{max}}^{t_{max} + \delta t_*} |\eta_N(s, \psi_k)| ds \geq 0.$$

⁹Setting $\widehat{\varphi}(k) = \int_{\mathbb{T}} e^{-2\pi i k \theta} \varphi(\theta) d\theta$ for the Fourier coefficients.

By equations (6.8), (6.11), (6.12) and (6.13), for some $C_2 \in (0, \infty)$,

$$\begin{aligned}
(6.14) \quad & |\eta_N(t_{max} + \delta t_*, \psi_k)| \leq |\eta_N(t_{max}, \psi_k)| - 4\pi^2 k^2 \frac{D}{2\gamma} \int_{t_{max}}^{t_{max} + \delta t_*} |\eta_N(s, \psi_k)| ds \\
& + k^6 \mathcal{E}_* C_2 \left(N^{-1} + (\ln(N+1))^2 N^{-\beta_*} \mathcal{Q}_N^k \right) \\
& \leq |\eta_N(t_{max}, \psi_k)| - \left(4\pi^2 k^2 \frac{D}{2\gamma} \right) \left(\frac{C \mathcal{E}_*}{2} k^6 \mathcal{Q}_N^k \right) \left(\tau \rho_k \mathcal{Q}_N^k \right) \\
& + C_2 \mathcal{E}_* k^6 \left(N^{-1} + (\ln(N+1))^2 N^{-\beta_*} \mathcal{Q}_N^k \right).
\end{aligned}$$

Note that $N^{-1} \leq 50 (\ln(N+1))^2 N^{-\beta_*} \mathcal{Q}_N^k$, for all $N \in \mathbb{N}$, and the previous quantity is less than

$$\begin{aligned}
(6.15) \quad & |\eta_N(t_{max}, \psi_k)| - \frac{C \mathcal{E}_* \pi^2 k^8 D}{\gamma} \tau \rho_k \left(\mathcal{Q}_N^k \right)^2 \\
& + 100 C_2 \mathcal{E}_* k^6 (\ln(N+1))^2 N^{-\beta_*} \mathcal{Q}_N^k.
\end{aligned}$$

Note also that for any $\alpha > 6$ there is C big enough such that for any $k \in \mathbb{Z} \setminus \{0\}$,

$$\tau \rho_k = \frac{2}{3 \tilde{C}_\alpha} \left(4\pi^2 k^2 \frac{D}{2\gamma} \right)^{-1}.$$

Therefore, we can choose C big enough, such that the last two terms in (6.15) sum up to a negative contribution, i.e.,

$$\begin{aligned}
& - \frac{C \mathcal{E}_* \pi^2 k^8 D}{\gamma} \tau \rho_k \mathcal{Q}_N^k + 100 C_2 \mathcal{E}_* k^6 N^{-\beta_*} (\ln(N+1))^2 \\
& = - \frac{C \mathcal{E}_* k^6}{3 \tilde{C}_\alpha} \mathcal{Q}_N^k + 100 C_2 \mathcal{E}_* k^6 N^{-\beta_*} (\ln(N+1))^2 \leq 0,
\end{aligned}$$

since $\mathcal{Q}_N^k \geq N^{-\beta_*} (\ln(N+1))^2$. This implies that $|\eta_N(t_{max} + \delta t_*, \psi_k)| \leq |\eta_N(t_{max}, \psi_k)|$. Furthermore, by (6.11), and $\delta t_* = \tau \rho_k \mathcal{Q}_N^k \leq \tau \mathcal{Q}_N^k$,

$$\sup_{t \in [t_{max}, t_{max} + \delta t_*]} |\eta_N(t, \psi_k)| \leq \frac{3C \mathcal{E}_*}{2} k^6 \mathcal{Q}_N^k.$$

Then $|\eta_N(t, \psi_k)| \leq \frac{3C \mathcal{E}_*}{2} k^6 \mathcal{Q}_N^k$, for all $t \geq 0$, follows by iteration. Finally, the theorem follows by equation (6.9). \square

APPENDIX A. DECAY OF CORRELATIONS FOR EXPANDING MAP

In this section, we quickly recap the information used in this paper pertaining to the statistical properties of smooth expanding maps. For simplicity, we confine ourselves to one dimensional maps, but the same theory holds in higher dimensions and for hyperbolic maps, see [19] for a quick exposition or [1, 2] for exhaustive details.

Let $f \in \mathcal{C}^2(\mathbb{T}, \mathbb{T})$ such that $f' \geq \lambda > 1$. A trivial consequence of expansivity is the following.

Lemma A.1. *For any $\bar{x} \in \mathbb{T}$, $\{f^{-n}\bar{x}\}_{n \in \mathbb{N}}$ is dense.*

Proof. Let $(a, b) \subset \mathbb{T}$. Note that f^n , $n \in \mathbb{N}$, belongs to $\mathcal{C}^2(\mathbb{T}, \mathbb{T})$, and $\inf_{[0,1]}(f^n)' \geq \lambda^n$. Therefore, for any $n \in \mathbb{N}$, either $f^n((a, b)) = \mathbb{T}$ or

$$1 > |f^n((a, b))| = \int_a^b (f^n)'(\theta) d\theta \geq |b - a| \lambda^n,$$

which can happen only for finitely many n . Hence, there exists $\bar{n} \in \mathbb{N}$ such that $f^{\bar{n}}((a, b)) = \mathbb{T}$, so $f^{-\bar{n}}(\bar{x}) \cap (a, b) \neq \emptyset$, thus proving the Lemma. \square

We are interested in studying the decay of correlation for observables of bounded variation. To this end, it is useful to introduce the concept of transfer operator defined as, for each $g \in \mathcal{C}^0$, $\rho \in L^1$,

$$(A.1) \quad \int_{\mathbb{T}} \rho(\theta) \cdot g \circ f(\theta) d\theta = \int_{\mathbb{T}} g(\theta) \cdot \mathcal{L}\rho(\theta) d\theta.$$

Note that, by definition $\int_{\mathbb{T}} \mathcal{L}\rho(\theta) d\theta = \int_{\mathbb{T}} \rho(\theta) d\theta$ for each $\rho \in L^1$. A simple change of variables shows that

$$\mathcal{L}\rho(\theta) = \sum_{\theta' \in f^{-1}(\theta)} \frac{\rho(\theta')}{f'(\theta')}.$$

Note also that $|\mathcal{L}\rho(\theta)| \leq \mathcal{L}|\rho|(\theta)$, that implies $\|\mathcal{L}\rho\|_{L^1} \leq \|\rho\|_{L^1}$. All the properties used in this paper are summarized as follows.

Theorem A.2. *Let BV be the space of functions of bounded variation from \mathbb{T} to \mathbb{C} . Then $\mathcal{L} \in L(BV, BV)$, that is \mathcal{L} is a bounded operator when acting on such a space. Moreover, there exists $\nu \in (0, 1)$ such that the spectrum*

$$\sigma_{BV}(\mathcal{L}) \subset \{1\} \cup \{z \in \mathbb{C} : |z| < \nu\},$$

and 1 is a simple eigenvalue of \mathcal{L} with eigenvector $\rho_\star \in \mathcal{C}^1$, $\rho_\star > 0$.

The above theorem, together with equation (A.1), implies that there exists a unique $\rho_\star \in BV$ such that $\mathcal{L}\rho_\star = \rho_\star$, $\int \rho_\star = 1$, and that \mathcal{L}^n , $n \in \mathbb{N}$, has the spectral decomposition

$$(A.2) \quad \mathcal{L}^n \rho = \rho_\star \int_{\mathbb{T}} \rho(\theta) d\theta + Q^n \rho$$

where $Q \in L(BV, BV)$ has spectral radius strictly smaller than ν , hence

$$(A.3) \quad \|Q^n\|_{BV} \leq C_{\#} \nu^n.$$

The above decomposition allows us to establish the following basic result on the decay of correlations.

Lemma A.3. *For each $\rho \in BV$, $g \in L^1$ and $n \in \mathbb{N}$, we have*

$$\left| \int_{\mathbb{T}} \rho(\theta) \cdot g \circ f^n(\theta) d\theta - \int_{\mathbb{T}} g(\theta) \rho_{\star}(\theta) d\theta \int_{\mathbb{T}} \rho(\theta') d\theta' \right| \leq C_{\#} \nu^n \|g\|_{L^1} \|\rho\|_{BV}.$$

Moreover, if $\rho, g \in L^2$, then

$$\lim_{n \rightarrow \infty} \left| \int_{\mathbb{T}} \rho(\theta) \cdot g \circ f^n(\theta) d\theta - \int_{\mathbb{T}} g(\theta) \rho_{\star}(\theta) d\theta \int_{\mathbb{T}} \rho(\theta') d\theta' \right| = 0.$$

Proof. Let $\rho \in BV$ and $g \in L^1$. We just compute, recalling (A.2),

$$\begin{aligned} & \left| \int_{\mathbb{T}} \rho(\theta) \cdot g \circ f^n(\theta) d\theta - \int_{\mathbb{T}} g(\theta) \rho_{\star}(\theta) d\theta \int_{\mathbb{T}} \rho(\theta) d\theta \right| \\ &= \left| \int_{\mathbb{T}} g(\theta) \mathcal{L}^n \rho(\theta) d\theta - \int_{\mathbb{T}} g(\theta) \rho_{\star}(\theta) d\theta \int_{\mathbb{T}} \rho(\theta) d\theta \right| \\ &= \left| \int_{\mathbb{T}} g(\theta) Q^n \rho(\theta) d\theta \right| \leq C_{\#} \nu^n \|g\|_{L^1} \|\rho\|_{BV}. \end{aligned}$$

To prove the second statement, first note that, for any $n \in \mathbb{N}$, $\rho \in L^2$,

$$\|\mathcal{L}^n \rho\|_{L^2}^2 = \int_{\mathbb{T}} \rho(\theta) (\mathcal{L}^n \rho) \circ f^n(\theta) d\theta \leq \|(\mathcal{L}^n \rho) \circ f^n\|_{L^2} \|\rho\|_{L^2},$$

and, since (A.2) and (A.3) imply

$$\|\mathcal{L}^n 1\|_{\infty} \leq \|\mathcal{L}^n 1 - \rho_{\star}\|_{BV} + \|\rho_{\star}\|_{BV} = \|Q^n 1\|_{BV} + \|\rho_{\star}\|_{BV} \leq C_{\#},$$

we have

$$\|(\mathcal{L}^n \rho) \circ f^n\|_{L^2}^2 = \int_{\mathbb{T}} |\mathcal{L}^n \rho|^2 \mathcal{L}^n 1 \leq \|\mathcal{L}^n 1\|_{\infty} \|\mathcal{L}^n \rho\|_{L^2}^2 \leq C_{\#} \|\mathcal{L}^n \rho\|_{L^2}^2.$$

These above inequalities imply that \mathcal{L} is power-bounded on L^2 , i.e.,

$$(A.4) \quad \|\mathcal{L}^n \rho\|_{L^2} \leq C_{\#} \|\rho\|_{L^2}, \quad \text{for any } n \in \mathbb{N}.$$

For each $g, \rho \in L^2$, $\varepsilon > 0$, let $\rho_{\varepsilon} \in BV$ such that $\|\rho - \rho_{\varepsilon}\|_{L^2} \leq \varepsilon$. Using (A.4), and the first part of the Lemma,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \int_{\mathbb{T}} g \mathcal{L}^n \rho - \int_{\mathbb{T}} \rho \int_{\mathbb{T}} g \rho_{\star} \right| &\leq \lim_{n \rightarrow \infty} \left| \int_{\mathbb{T}} g \mathcal{L}^n \rho_{\varepsilon} - \int_{\mathbb{T}} \rho_{\varepsilon} \int_{\mathbb{T}} g \rho_{\star} \right| + C_{\#} \|g\|_{L^2} \varepsilon \\ &= C_{\#} \|g\|_{L^2} \varepsilon, \end{aligned}$$

which by the arbitrariness of ε concludes the proof. \square

Similar results, that will be used in the main text, can be obtained for multiple times correlations.

Theorem A.2 is well-known (e.g. see [19, Chapter 1]). However, we provide a brief proof for the reader's convenience.

Proof of Theorem A.2. For all $\rho \in \mathcal{C}^1$ we have, by a direct computation,

$$(A.5) \quad (\mathcal{L}\rho)' = \mathcal{L}\left(\frac{\rho'}{f'}\right) - \mathcal{L}\left(\frac{f''}{(f')^2}\rho\right).$$

This implies

$$\|(\mathcal{L}\rho)'\|_{L^1} \leq \left\| \mathcal{L}\left(\frac{\rho'}{f'}\right) \right\|_{L^1} + \left\| \mathcal{L}\left(\frac{f''}{(f')^2}\rho\right) \right\|_{L^1} \leq \lambda^{-1}\|\rho'\|_{L^1} + B\|\rho\|_{L^1}$$

where $B = \left\| \frac{f''}{(f')^2} \right\|_{\infty}$.

More generally, equation (A.5) implies the following basic property of \mathcal{L} : for each $\rho \in \mathcal{C}^1$, we have¹⁰

$$(A.6) \quad \begin{aligned} \|\mathcal{L}\rho\|_{L^1} &\leq \|\rho\|_{L^1} \\ \|\mathcal{L}\rho\|_{W^{1,1}} &= \|(\mathcal{L}\rho)'\|_{L^1} + \|(\mathcal{L}\rho)\|_{L^1} \leq \lambda^{-1}\|\rho\|_{W^{1,1}} + (B+1)\|\rho\|_{L^1}, \end{aligned}$$

these are the so-called Lasota-Yorke, or Doeblin-Fortet, inequalities. Hence, $\mathcal{L} \in L(L^1, L^1) \cap L(W^{1,1}, W^{1,1})$ and, iterating, equations (A.6) imply that

$$(A.7) \quad \begin{aligned} \|\mathcal{L}^n \rho\|_{W^{1,1}} &\leq \lambda^{-n}\|\rho\|_{W^{1,1}} + (B+1)(1-\lambda^{-1})^{-1}\|\rho\|_{L^1} \\ &\leq (\lambda^{-n} + (B+1)(1-\lambda^{-1})^{-1})\|\rho\|_{W^{1,1}}. \end{aligned}$$

Thus $\sigma_{W^{1,1}}(\mathcal{L}) \subset \{z \in \mathbb{C} : |z| \leq 1\}$, and by (A.7) the sequence

$$(A.8) \quad \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}^k 1$$

is compact in L^1 and hence has accumulation points, which belong to $W^{1,1}$. Since $\mathcal{L}^k 1 \geq 0$, if ρ_* is one such accumulation point, then $\rho_* \geq 0$ and $\mathcal{L}\rho_* = \rho_*$, that is $1 \in \sigma_{W^{1,1}}(\mathcal{L})$. To prove that ρ_* is strictly positive, note that if there exists \bar{x} such that $\rho_*(\bar{x}) = 0$, then, for any $n \in \mathbb{N}$,

$$(A.9) \quad 0 = \rho_*(\bar{x}) = \mathcal{L}^n \rho_*(\bar{x}) = \sum_{y \in f^{-n}(\bar{x})} \frac{\rho_*(y)}{(f^n)'(y)}.$$

The right hand side is a sum of non negative terms, hence $\rho_*(y) = 0$ for any $y \in f^{-n}(\bar{x})$, $n \in \mathbb{N}$, which, by Lemma A.1, forms a dense set. Since $\rho_* \in W^{1,1} \subset \mathcal{C}^0$, this means that $\rho_* \equiv 0$, which is a contradiction.

The above also implies that 1 must be a simple eigenvalue. Indeed, suppose that $\mathcal{L}\rho = \rho$, $\rho \in W^{1,1} \setminus \{\alpha\rho_* : \alpha \in \mathbb{R}\}$.¹¹ Then there exists $b > 0$ such that $\rho_* - b\rho \geq 0$ but there exists $\bar{x} \in \mathbb{T}$ such that $\rho_*(\bar{x}) - b\rho(\bar{x}) = 0$. Then the same argument as before yields the contradiction $\rho_* = b\rho$.

¹⁰Recall that $\|\rho\|_{W^{1,1}} = \|\rho\|_{L^1} + \|\rho'\|_{L^1}$.

¹¹Note that if $\mathcal{L}(\alpha + i\beta) = \alpha + i\beta$, then it must be $\mathcal{L}\alpha = \alpha$ and $\mathcal{L}\beta = \beta$, so we can restrict to real functions.

To study the rest of the spectrum, note that if $\mathbb{V} = \{\rho \in W^{1,1} : \int_{\mathbb{T}} \rho = 0\}$, then, by (A.1), $\mathcal{L}\mathbb{V} \subset \mathbb{V}$. Moreover, for all $\rho \in \mathbb{V}$, $\|\rho\|_{L^1} \leq \|\rho'\|_{L^1}$. This implies that, for each $\rho \in \mathbb{V}$,

$$(A.10) \quad \|(\mathcal{L}\rho)'\|_{L^1} \leq (\lambda^{-1} + B) \|\rho'\|_{L^1}.$$

The above implies that the spectrum $\sigma_{\mathbb{V}}(\mathcal{L}) \subset \{z \in \mathbb{C} : |z| \leq \lambda^{-1} + B\}$. If $\lambda^{-1} + B =: \nu < 1$, which is the case if the map is close to linear, then equation (A.10) implies

$$(A.11) \quad \sigma_{W^{1,1}}(\mathcal{L}) \subset \{1\} \cup \{z \in \mathbb{C} : |z| \leq \nu\}.$$

This proves the Theorem apart from the fact that \mathcal{L} is acting on $W^{1,1}$ rather than BV , an issue that we will discuss shortly.

It is necessary to make a more sophisticated argument to prove the theorem in the general case, in which B may be large.

From (A.6) and Hennion's Theorem ([38], but see [19, Appendix B] for a self contained exposition) it follows that \mathcal{L} , when acting on $W^{1,1}$, has essential spectral radius bounded by λ^{-1} .

If $\mathcal{L}\rho = e^{i\theta_\star} \rho$, $\int_{\mathbb{T}} |\rho| = 1$, $\rho \neq \rho_\star$. Then,

$$|\rho| \leq \mathcal{L}|\rho|$$

implying $|\rho| = \mathcal{L}|\rho|$, since $\int_{\mathbb{T}} (\mathcal{L}|\rho| - |\rho|) = 0$ and the integrand is non negative. Hence, $\rho = \rho_\star e^{i\psi}$ for some function $\psi \in C^0([0, 1], \mathbb{R})$, $\psi(0) - \psi(1) = 2k\pi$, and $k \in \mathbb{N}$. Moreover $\mathcal{L}(\rho_\star e^{i\psi}) = e^{i\theta_\star} e^{i\psi} \rho_\star$ i.e., $\mathcal{L}(e^{i\psi - i\psi \circ f - i\theta_\star} \rho_\star) - \rho_\star = 0$. Integrating and taking the real part one has

$$\int_{\mathbb{T}} \rho_\star (1 - \cos(\psi - \psi \circ f - \theta_\star)) = 0,$$

which is possible only if $\psi - \psi \circ f - \theta_\star = 2k\pi$. Since f has at least one fixed point, we have $\theta_\star = 2k\pi$ and 1 is the only eigenvalue on the unit circle.

Thus there exists $\nu \in (\lambda^{-1}, 1)$ such that $\sigma_{W^{1,1}} \mathcal{L} \setminus \{1\} \subset \{z \in \mathbb{C} : |z| \leq \nu\}$ (see [19, Proposition 1.3] for a more detailed discussion). Accordingly, we have the spectral decomposition (A.2) (for more details, see, e.g., [19, Section 1.2]). The fact that $\rho_\star \in \mathcal{C}^1$ can be found in [19, Problem 1.7].

To extend \mathcal{L} to BV note that for each $\rho \in BV$ there exists $\{\rho_n\} \subset W^{1,1}$ such that $\|\rho_n\|_{W^{1,1}} \leq \|\rho\|_{BV}$, ρ_n converges to ρ in L^1 , and $\liminf \|\rho_n\|_{W^{1,1}} \geq \|\rho\|_{BV}$ (e.g. see [31, Section 5.2.2 Theorem 2]). This and (A.6) immediately implies that, for each $\rho \in BV$,

$$(A.12) \quad \begin{aligned} \|\mathcal{L}\rho\|_{L^1} &\leq \|\rho\|_{L^1} \\ \|\mathcal{L}\rho\|_{BV} &\leq \lambda^{-1} \|\rho\|_{BV} + (B+1) \|\rho\|_{L^1}, \end{aligned}$$

and by Hennion's Theorem the spectral decomposition (A.2) holds also for $\mathcal{L} \in L(BV, BV)$. \square

Note that, iterating (A.12), we have, for all $n \in \mathbb{N}$,

$$(A.13) \quad \begin{aligned} \|\mathcal{L}^n \rho\|_{L^1} &\leq \|\rho\|_{L^1} \\ \|\mathcal{L}^n \rho\|_{BV} &\leq \lambda^{-n} \|\rho\|_{BV} + \frac{B+1}{1-\lambda^{-1}} \|\rho\|_{L^1}. \end{aligned}$$

Another very useful fact is that densities with bounded logarithmic derivatives constitute an invariant set: let

$$(A.14) \quad C_a = \left\{ \rho \in \mathcal{C}^1 : \rho(\theta) > 0, \quad \frac{|\rho'(\theta)|}{\rho(\theta)} \leq a, \quad \forall \theta \in \mathbb{T} \right\}.$$

Then, since $|\rho'| \leq a\rho$ if $\rho \in C_a$, by (A.5), we have

$$(A.15) \quad \left| \frac{d}{d\theta} \mathcal{L}\rho(\theta) \right| \leq \lambda^{-1} \mathcal{L}|\rho'|(\theta) + B\mathcal{L}\rho(\theta) \leq (\lambda^{-1}a + B)\mathcal{L}\rho(\theta)$$

so $\mathcal{L}C_a \subset C_a$, provided $a > \frac{B}{1-\lambda^{-1}}$.

Next result is used multiple time in the text.

Lemma A.4. *For all $K \in \mathbb{N}$, $\{q_j\}_{j=0}^{K-1} \in BV$, $\max_j \|q_j\|_{BV} \geq 2$,*

$$\begin{aligned} &\sum_{j=0}^{K-1} \left\| \mathcal{L}^j \left(q_j - \rho_\star \int_{\mathbb{T}} q_j(\theta) d\theta \right) \right\|_{L^1} \\ &\leq C_\# \left(\max_j \|q_j\|_{L^1} \right) \ln \left(\max_j \|q_j\|_{BV} \right) + C_\#. \end{aligned}$$

Proof. Denote $\tilde{q}_j = q_j - \rho_\star \int_{\mathbb{T}} q_j(\theta) d\theta$. Let $M = \lfloor C_1 \ln(\max_j \|q_j\|_{BV}) \rfloor \geq \lfloor C_1 \ln 2 \rfloor$, where $C_1 \in \mathbb{R}_+$ will be chosen later. Then,

$$\sum_{j=0}^{K-1} \|\mathcal{L}^j \tilde{q}_j\|_{L^1} = \sum_{j=0}^{\min\{M, K-1\}} \|\mathcal{L}^j \tilde{q}_j\|_{L^1} + \sum_{j=\min\{M+1, K\}}^{K-1} \|\mathcal{L}^j \tilde{q}_j\|_{BV}.$$

Using the first of (A.6) we have

$$\sum_{j=0}^{\min\{M, K-1\}} \|\mathcal{L}^j \tilde{q}_j\|_{L^1} \leq 2M \max_j \|q_j\|_{L^1}.$$

On the other hand, by equation (A.3),

$$\begin{aligned} \sum_{j=\min\{M+1, K\}}^{K-1} \|\mathcal{L}^j \tilde{q}_j\|_{BV} &= \sum_{j=\min\{M+1, K\}}^{K-1} \|Q^j \tilde{q}_j\|_{BV} \leq C_\# \sum_{j=M+1}^{K-1} \nu^j \|q_j\|_{BV} \\ &\leq C_\# \nu^{M+1} \max_j \|q_j\|_{BV} \leq C_\# \left(\max_j \|q_j\|_{BV} \right)^{1-C_1 \ln \nu^{-1}}, \end{aligned}$$

and choosing $C_1 \geq (\ln \nu^{-1})^{-1}$ yields the result. \square

Note that, for $g \in L^\infty$ and q_j as in Lemma A.4, we have, for all N ,

$$(A.16) \quad \sum_{j=0}^{N-1} \left| \int_{\mathbb{T}} g(f^j \theta) q_j(\theta) d\theta - \int_{\mathbb{T}} g(\theta) \rho_\star(\theta) d\theta \sum_{j=0}^{N-1} \int_{\mathbb{T}} q_j(\theta') d\theta' \right| \\ \leq C_\# \|g\|_\infty \left(\max_j \|q_j\|_{L^1} \right) \ln \left(\max_j \|q_j\|_{BV} \right) + C_\# \|g\|_\infty.$$

We conclude the Appendix with a simple, but important, application of the above theory.

Proof of Lemma 2.2. By the conservation of energy implies, for any $t \in \mathbb{R}^+$,

$$(A.17) \quad \mathbb{E}_{\phi_\star^t \mu_N} [\mathcal{E}_N(0)] = \mathbb{E}_{\mu_N} [\mathcal{E}_N(t)] = \mathbb{E}_{\mu_N} [\mathcal{E}_N(0)] \leq \mathcal{E}_\star N.$$

Furthermore, note that $d\phi_\star^t \mu_N(q, p, \theta) = \tilde{\rho}_N(\theta) \tilde{\mu}_{\theta, N}(dq, dp) d\theta$, where

$$\tilde{\rho}_N(\theta) = \prod_{x \in \mathbb{Z}_N} \mathcal{L}^{\lfloor t\varepsilon^{-1} \rfloor} \rho_{x, N}(\theta_x).$$

The fact that $f \in \mathcal{C}^\infty(\mathbb{T})$ implies that $\mathcal{L}^{\lfloor t\varepsilon^{-1} \rfloor} \rho_{x, N} \in \mathcal{C}^1(\mathbb{T})$ and equation (A.15) implies (2.7) if C_0 is big enough. \square

APPENDIX B. COBOUNDARIES

We say that a function $\mathbf{c} \in \mathcal{C}^0(\mathbb{T}, \mathbb{R})$ is a continuous coboundary if there exists $\zeta \in \mathcal{C}^0(\mathbb{T}, \mathbb{R})$ such that $\mathbf{c} = \zeta - \zeta \circ f$. We call $L^2(\rho_\star)$ the L^2 functions on \mathbb{T} with respect to the measure $\rho_\star(\theta) d\theta$.

Lemma B.1. For all $n \in \mathbb{N}$,

$$\left| \left\| \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} b \circ f^k \right\|_{L^2(\rho_\star)}^2 - \gamma \right| \leq \frac{C_\#}{n}.$$

Proof. By a direct computation, and recalling (2.4),

$$\begin{aligned} \left\| \sum_{k=0}^{n-1} b \circ f^k \right\|_{L^2(\rho_\star)}^2 &= \sum_{k, j=0}^{n-1} \int_{\mathbb{T}} (b \circ f^k)(\theta) \cdot (b \circ f^j)(\theta) \cdot \rho_\star(\theta) d\theta \\ &= \sum_{k=0}^{n-1} \int_{\mathbb{T}} b^2(\theta) \cdot \rho_\star(\theta) d\theta + 2 \sum_{j=0}^{n-2} \sum_{k>j}^{n-1} \int_{\mathbb{T}} b \circ f^{k-j}(\theta) b(\theta) \rho_\star(\theta) d\theta \\ &= n \int_{\mathbb{T}} b^2(\theta) \cdot \rho_\star(\theta) d\theta + 2 \sum_{\ell=1}^{n-1} (n-\ell) \int_{\mathbb{T}} b \circ f^\ell(\theta) \cdot b(\theta) \rho_\star(\theta) d\theta \\ &= n\gamma - 2n \sum_{\ell=n}^{\infty} \int_{\mathbb{T}} b \circ f^\ell(\theta) \cdot b(\theta) \rho_\star(\theta) d\theta - 2 \sum_{\ell=1}^{n-1} \ell \int_{\mathbb{T}} b \circ f^\ell(\theta) \cdot b(\theta) \rho_\star(\theta) d\theta. \end{aligned}$$

Finally, by Lemma A.3, $\left| \int_{\mathbb{T}} b \circ f^\ell \cdot b \cdot \rho_\star \right| \leq C_\# \nu^\ell$, for some $\nu < 1$. \square

Lemma B.2. $\gamma = 0$ if and only if b is a continuous coboundary.

Proof. If b is a continuous coboundary, then Lemma B.1 implies

$$\gamma = \lim_{n \rightarrow \infty} \left\| \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} b \circ f^k \right\|_{L^2(\rho_*)}^2 = \lim_{n \rightarrow \infty} \left\| \frac{1}{\sqrt{n}} (\zeta \circ f^n - \zeta) \right\|_{L^2(\rho_*)}^2.$$

Conversely, if $\gamma = 0$, then Lemma B.1 implies $\left\| \sum_{k=0}^{n-1} b \circ f^k \right\|_{L^2(\rho_*)} \leq C\#$, for all $n \in \mathbb{N}$. Thus, the sequence $\left\{ \sum_{k=0}^{n-1} b \circ f^k \right\}_n$ admits weakly convergent subsequences in $L^2(\rho_*)$. Let $\zeta \in L^2$ be an accumulation point. Then, recalling Lemma A.3 and that b is zero average, for each $\psi \in W^{1,1} \subset \mathcal{C}^0$, and some $\{n_j\}_j$, we have

$$\begin{aligned} \int_{\mathbb{T}} \zeta \circ f(\theta) \psi(\theta) \rho_*(\theta) d\theta &= \int_{\mathbb{T}} \zeta(\theta) \mathcal{L}(\psi \rho_*)(\theta) d\theta \\ &= \lim_{j \rightarrow \infty} \sum_{k=0}^{n_j-1} \int_{\mathbb{T}} b \circ f^{k+1}(\theta) \psi(\theta) \rho_*(\theta) d\theta \\ &= - \int_{\mathbb{T}} b(\theta) \psi(\theta) \rho_*(\theta) d\theta + \lim_{j \rightarrow \infty} \sum_{k=0}^{n_j} \int_{\mathbb{T}} b \circ f^k(\theta) \psi(\theta) \rho_*(\theta) d\theta \\ &= - \int_{\mathbb{T}} b(\theta) \psi(\theta) \rho_*(\theta) d\theta + \int_{\mathbb{T}} \zeta(\theta) \psi(\theta) \rho_*(\theta) d\theta. \end{aligned}$$

Since $W^{1,1}$ is dense in L^2 and $\rho_* > 0$ (see Theorem A.2) it follows that

$$(B.1) \quad b = \zeta - \zeta \circ f,$$

as elements of L^2 , and we assume without loss of generality that $\int \zeta \rho_* = 0$. It remains to show that $\zeta \in \mathcal{C}^0$. Multiplying by ρ_* and applying \mathcal{L} to (B.1) yields

$$\mathcal{L}(b\rho_*) = -(\mathbb{1} - \mathcal{L})(\zeta\rho_*).$$

Hence, for all $\psi \in L^2$, using the decay of correlation in Lemma A.3, it follows

$$\sum_{k=0}^{\infty} \int \psi \mathcal{L}^{k+1}(b\rho_*) = \lim_{n \rightarrow \infty} \int \psi [\mathcal{L}^n(\zeta\rho_*) - (\zeta\rho_*)] = - \int \psi \zeta \rho_*.$$

Since $\int_{\mathbb{T}} b(\theta) \rho_*(\theta) d\theta = 0$, $\sum_{k=0}^{\infty} \mathcal{L}^{k+1}(b\rho_*) = (\mathbb{1} - \mathcal{L})^{-1} \mathcal{L}(b\rho_*)$. Hence,

$$(\mathbb{1} - \mathcal{L})^{-1} \mathcal{L}(b\rho_*) = -\zeta\rho_*$$

as elements of L^2 . Finally, since $\rho_* > 0$, we have

$$\zeta = -\frac{1}{\rho_*} (\mathbb{1} - \mathcal{L})^{-1} \mathcal{L}(b\rho_*) \in W^{1,1} \subset \mathcal{C}^0.$$

□

The importance of Lemma B.2 is that being a continuous coboundary is relatively easy to check. Indeed, if \mathfrak{c} is a continuous coboundary, for any f -invariant measure ν , one has that $\int_{\mathbb{T}} \mathfrak{c} d\nu = 0$. In particular, if there exists just one periodic orbit $\{p, f(p), \dots, f^M(p) = p\}$, $M \in \mathbb{N}$, such that $\sum_{k=0}^{M-1} b(f^k p) \neq 0$, then b cannot be a coboundary and $\gamma > 0$.

APPENDIX C. STANDARD PAIRS

In this Appendix, we provide the definition and the properties of the standard pairs used in this paper.

Let $\delta > 0$ be small enough so that f is invertible on each interval of size δ . For each numbers $a_x, b_x \in \mathbb{T}$, $b_x - a_x \in [\delta/2, \delta]$, $x \in \mathbb{Z}_N$, we define $\mathbb{I}(a, b) = \times_{x \in \mathbb{Z}_N} [a_x, b_x]$. For $G \in \mathcal{C}^2(\mathbb{I}(a, b), \Omega_N)$, we denote its components as

$$G(\theta) = \{G_x(\theta) = (G_{q_x^1}(\theta), G_{q_x^2}(\theta), G_{p_x^1}(\theta), G_{p_x^2}(\theta))\}_{x \in \mathbb{Z}_N}.$$

and we set,

$$\|DG\|_{\infty} = \sup_{x \in \mathbb{Z}_N} \sup_{\theta_x \in [a_x, b_x]} \left(\sum_{y=1}^N \left(\frac{\partial G_y(\theta)}{\partial \theta_x} \right)^2 \right)^{1/2}.$$

For $\bar{C} \in \mathbb{R}_+$ and $E, \delta \in \mathbb{R}_+$ and $N \in \mathbb{N}$, consider the set of functions

$$\mathfrak{S}(E, N, \delta, \varepsilon, \bar{C}) = \left\{ G \in \mathcal{C}^2(\mathbb{I}(a, b), \Sigma_N(E)) \mid \|DG\|_{\infty} \leq \bar{C} \sqrt{E\varepsilon} \right\}.$$

We call the graph of $G \in \mathfrak{S}$,¹² a standard surface. Our goal is to construct a class of measures supported on such surfaces, see Figure 1 for a picture. The condition $\|DG\|_{\infty} \leq \bar{C} \sqrt{E\varepsilon}$ implies that the surfaces are almost flat. That is, there is little correlation between the state of the magnetic field and the macroscopic positions of the particles $z \in \Omega_N$.

For $C_0 > \frac{B}{1-\lambda-\Gamma}$, let us define the standard probability densities as

$$D_{C_0}(a, b) = \left\{ \rho \in \mathcal{C}^1(\mathbb{I}(a, b), \mathbb{R}_+) \mid \rho(\theta) = \prod_{x \in \mathbb{Z}_N} \rho_x(\theta_x), \int_{a_x}^{b_x} \rho_x(\theta_x) d\theta_x = 1, \left\| \frac{\rho'_x}{\rho_x} \right\|_{\mathcal{C}^0} \leq 2C_0 \right\}.$$

A standard pair ℓ is given by $\ell = (G_{\ell}, \rho_{\ell})$, where $G_{\ell} \in \mathfrak{S}$ and $\rho_{\ell} \in D_{C_0}(a, b)$ is a standard probability density, and both are defined on the same domain

$$\mathbb{I}(a, b) =: \mathbb{I}_{\ell} = \times_x I_{\ell}^x \subset \mathbb{T}^N \quad I_{\ell}^x = [a_x, b_x] \subset \mathbb{T}.$$

With a slight abuse of notations, we write $\ell \in \mathfrak{S}$.

A standard pair ℓ induces the probability measure μ_{ℓ} on $\mathbb{T}^N \times \Sigma_N(E)$

$$\mu_{\ell}(g) = \int_{\mathbb{I}_{\ell}} g(\theta, G_{\ell}(\theta)) \rho_{\ell}(\theta) d\theta,$$

¹²We may write \mathfrak{S} rather than $\mathfrak{S}(E, N, \delta, \varepsilon, \bar{C})$ if the parameters are not relevant.

for any continuous function $g \in \mathcal{C}^0(\mathbb{T}^N \times \Omega_N, \mathbb{R})$ and where $d\theta$ is the normalized Lebesgue measure on \mathbb{T}^N .

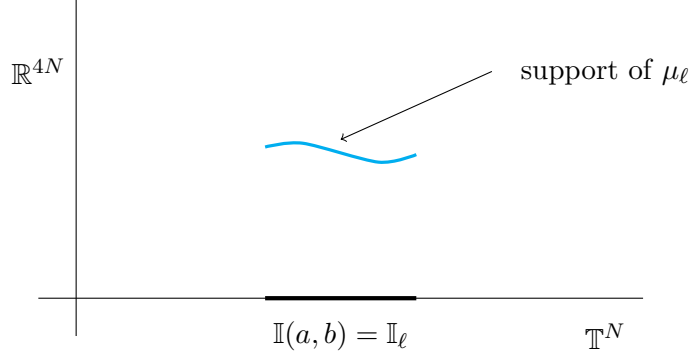


Figure 1: The graph of G_ℓ (in cyan) supporting the measure μ_ℓ .

We define a standard family $(\{\ell_{j=1}^K\}, \nu) \in \mathfrak{F}_{N, \delta, \varepsilon, \overline{C}}$ (resp., $\mathfrak{F}_{E, N, \delta, \varepsilon, \overline{C}}$) as a finite collection of standard pairs $\{\ell_j\}$, $\ell_j \in \bigcup_{E \in \mathbb{R}_+} \mathfrak{S}(E, N, \delta, \varepsilon, \overline{C})$, (resp., $\mathfrak{S}(E, N, \delta, \varepsilon, \overline{C})$) endowed with a finite factor measure $\nu = (p_1, \dots, p_K)$. Any standard family $\{\ell_j, p_j\}$ naturally induces the measure μ on $\mathbb{T}^N \times \Omega_N$

$$\mu(g) = \sum_{j=1}^K p_j \mu_{\ell_j}(g), \quad g \in \mathcal{C}^0(\mathbb{T}^N \times \Omega_N, \mathbb{R}).$$

Note that the measures defined by standard families form, by construction, a convex set whose extremal points are standard pairs.

Let ϕ^t be the flow defined by (2.2), (2.1).¹³ We are interested in the pushforward of a measure defined as $\phi_*^t \mu(A) := \mu((\phi^t)^{-1}(A))$ for each measurable set $A \subset \Omega_N \times \mathbb{T}^N$.

Theorem C.1. *There exists $\overline{C} > 0$ big enough and $\varepsilon_0, \delta_0 > 0$ small enough, depending only on ω_0, b , and f , such that if $\mu \in \mathfrak{F}_{E, N, \delta, \varepsilon, \overline{C}}$, then $\phi_*^\varepsilon \mu \in \mathfrak{F}_{E, N, \delta, \varepsilon, \overline{C}}$, for all $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \delta_0)$ and all $E \in \mathbb{R}_+$, $N \in \mathbb{N}$.*

Proof. Without loss of generality, we can assume that $\mu \in \mathfrak{F}_{E, N, \delta, \varepsilon, \overline{C}}$ consists of a single standard pair $\ell \in \mathfrak{S}(E, N, \delta, \varepsilon, \overline{C})$.

Note that $z(\varepsilon, G_\ell(\theta), \theta) = e^{(-\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(0, \theta))\varepsilon} G_\ell(\theta)$ is the solution of (3.1) at

¹³While the flow is discontinuous in the θ variable, it is differentiable, with respect to time, in the z variable.

time ε starting at $G_\ell(\theta)$. Thus, for any $g \in \mathcal{C}^0(\mathbb{T}^N \times \Omega_N, \mathbb{R})$:

$$\begin{aligned} \phi_\star^\varepsilon \mu_\ell(g) &= \mu_\ell(g \circ \phi^\varepsilon) \\ &= \int_{\mathbb{I}(a,b)} g(f(\theta), z(\varepsilon, G_\ell(\theta))) \rho_\ell(\theta) d\theta \\ &= \int_{\mathbb{I}(a,b)} g\left(f(\theta), e^{(-\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(0, \theta))^\varepsilon} G_\ell(\theta)\right) \prod_{x \in \mathbb{Z}_N} \rho_x^\ell(\theta_x) d\theta. \end{aligned}$$

For all δ small enough, f is invertible on each $[a_x, b_x]$ and we call φ_x the inverse of $f|_{[a_x, b_x]}$. Setting $\tilde{\theta} = f(\theta)$,

$$(C.1) \quad \int_{\mathbb{T}^N} g(\tilde{\theta}, e^{(-\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(0, \varphi(\tilde{\theta})))^\varepsilon} G_\ell(\varphi(\tilde{\theta}))) \prod_{x \in \mathbb{Z}_N} (\mathcal{L}(\rho_x^\ell \mathbb{1}_{[a_x, b_x]}))(\tilde{\theta}_x) d\tilde{\theta},$$

where \mathcal{L} is the SRB transfer operator associated to the map f , see Appendix A for details. Note that, $\mathcal{L}(\rho_x^\ell \mathbb{1}_{[a_x, b_x]}) = \mathcal{L}(\rho_x^\ell) \mathbb{1}_{f([a_x, b_x])}$. By equation (A.15), and our choice of C_0 , one has that $\mathcal{L}\rho$ belong to $D_{C_0}(f(a), f(b))$. However, it can happen that $|f([a_x, b_x])| > \delta$, since f is expanding. In this case we can break the image in finitely many pieces indexed by $q \in \Omega$, such that $|b_x^{(q)} - a_x^{(q)}| \in [\frac{\delta}{2}, \delta]$ and define

$$Z_q = \int_{\mathbb{I}(a^{(q)}, b^{(q)})} \prod_{x \in \mathbb{Z}_N} \mathcal{L}\rho_x^\ell(\theta_x) d\theta_x \quad \& \quad \rho^{(q)}(\theta) = Z_q^{-1} (\mathcal{L}\rho_\ell(\theta)) \mathbb{1}_{\mathbb{I}(a^{(q)}, b^{(q)})}(\theta).$$

This allows to re-write (C.1) as

$$\sum_{q \in \Omega} Z_q \int_{\mathbb{I}(a^{(q)}, b^{(q)})} \rho^{(q)}(\tilde{\theta}) g(\tilde{\theta}, e^{(-\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(0, \varphi(\tilde{\theta})))^\varepsilon} G_\ell(\varphi(\tilde{\theta}))) d\tilde{\theta},$$

where $\rho_x^{(q)} \in D_{C_0}(a^{(q)}, b^{(q)})$, $\sum_{q \in \Omega} Z_q = 1$. Let us set

$$\tilde{G}^{(q)}(\tilde{\theta}) = e^{(-\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(0, \varphi(\tilde{\theta})))^\varepsilon} G_\ell(\varphi(\tilde{\theta})).$$

First, notice that, by the conservation of energy, $\tilde{G}^{(q)} \in \mathcal{C}^2(\mathbb{I}(a^{(q)}, b^{(q)}), \Sigma_N(E))$. To conclude the proof, we need to bound, the derivative of \tilde{G} . For all θ ,

$$\begin{aligned} \left\| \frac{\partial e^{(-\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(0, \theta))^\varepsilon}}{\partial \theta_x} \right\| &\leq \sum_{k=1}^{\infty} \left\| \frac{\partial (-\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(0, \theta))^k}{\partial \theta_x} \right\| \frac{\varepsilon^k}{k!} \\ &\leq \sum_{k=1}^{\infty} k C_\#^k \frac{\varepsilon^{\frac{k}{2}}}{k!} = C_\# \sqrt{\varepsilon} e^{C_\# \sqrt{\varepsilon}}. \end{aligned}$$

Using the above inequality, since $\left\| \frac{\partial \varphi_x^{(q)}}{\partial \theta_x} \right\|_\infty \leq \lambda^{-1}$, yields

$$\begin{aligned} \left\| \frac{\partial \tilde{G}^{(q)}}{\partial \tilde{\theta}_x} \right\| &\leq \left\| \frac{\partial e^{(-\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(0, \varphi(\tilde{\theta})))_\varepsilon}}{\partial \tilde{\theta}_x} G_\ell^{(q)}(\varphi(\tilde{\theta})) \right. \\ &\quad \left. + e^{(-\mathbb{A} + \varepsilon^{-\frac{1}{2}} \mathbb{B}_\varepsilon(0, \varphi(\tilde{\theta})))_\varepsilon} \frac{\partial G_\ell^{(q)}(\varphi(\tilde{\theta}))}{\partial \tilde{\theta}_x} \right\| \\ &\leq \lambda^{-1} C_\# \sqrt{\varepsilon} e^{C_\# \sqrt{\varepsilon}} \left\| G_\ell(\varphi(\tilde{\theta})) \right\| + \lambda^{-1} e^{C_\# \sqrt{\varepsilon}} \left\| \frac{\partial G_\ell(\theta)}{\partial \theta_x} \right\| \\ &\leq \lambda^{-1} C_\# e^{C_\# \sqrt{\varepsilon}} \sqrt{E\varepsilon} + \lambda^{-1} e^{C_\# \sqrt{\varepsilon}} \bar{C} \sqrt{E\varepsilon}. \end{aligned}$$

If ε is small enough, we obtain the claim by choosing \bar{C} sufficiently large. \square

We end this section with the following useful lemmata.

Lemma C.2. *Let ρ a probability density on \mathbb{T} , $\text{supp}(\rho) = I$, where $I = [a, b]$, $|b - a| = \delta$ such that $\rho \in \mathcal{C}^1(\dot{I}) \cap \mathcal{C}^0(I)$, and $\int_I |\rho'(u)| du \leq C$, for some $C > 0$. Then*

$$(C.2) \quad \|\rho\|_{BV} \leq 2\delta^{-1} + 3C + 1.$$

Proof. Since ρ can only have jumps at the boundaries of $[a, b]$, its total variation is bounded by C plus $\rho(a) + \rho(b)$. On the other hand by the mean value theorem, there exists $x_* \in I$ such that $\rho(x_*) = \delta^{-1}$, hence for $x \in \{a, b\}$

$$|\rho(x)| = \delta^{-1} + \left| \int_{x_*}^x \rho'(y) dy \right| \leq \delta^{-1} + C.$$

$$(C.3) \quad \|\rho\|_{BV} \leq \int \rho + \rho(a) + \rho(b) + \int |\rho'| \leq 1 + 2\delta^{-1} + 3C.$$

\square

Lemma C.3. *Let $\ell \in \mathfrak{S}(E, N, \delta, \varepsilon, \bar{C})$ and $\rho_{\ell, x} \in D_{C_0}$ be the associated density. Then,*

$$\|\rho_{\ell, x} \mathbb{1}_{I_\ell^x}\|_{BV} \leq (6C_0 + 3)\delta^{-1}.$$

Proof. We have that $|\rho'_{\ell, x}| \leq 2C_0 \rho_{\ell, x}$, so that $\int_I |\rho'_{\ell, x}(\theta)| d\theta \leq 2C_0$, and $\delta < 1$, so the statement follows directly from Lemma C.2. \square

Lemma C.4. *Let $\ell \in \mathfrak{S}(E, N, \delta, \varepsilon)$ and let $\mathfrak{h} \in \mathfrak{E}$. Then,*

$$\sup_{x \in \mathbb{Z}_N} \sup_{\theta \in \mathbb{I}_\ell} \left| \frac{\partial}{\partial \theta_x} \mathfrak{h}(G(\theta)) \right| \leq \bar{C} \varepsilon^{\frac{1}{2}} E \|\mathfrak{h}\|_{\mathfrak{E}}.$$

Proof. Using Holder inequality and the definition of standard pair,

$$\begin{aligned} \left| \frac{\partial}{\partial \theta_x} \mathfrak{h}(G_\ell(\theta)) \right| &= \left| \sum_{k=1}^{4N} \frac{\partial}{\partial z_k} \mathfrak{h}(G_\ell(\theta)) \cdot \frac{\partial G_\ell(\theta)_k}{\partial \theta_x} \right| \\ &\leq \left(\sum_{k=1}^{4N} \left(\frac{\partial}{\partial z_k} \mathfrak{h}(G_\ell(\theta)) \right)^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^{4N} \left(\frac{\partial G_\ell(\theta)_k}{\partial \theta_x} \right)^2 \right)^{\frac{1}{2}} \\ &\leq \|DG_\ell\|_\infty \left(\|\mathfrak{h}\|_{\mathfrak{E}} \sqrt{E} \right) \leq \bar{C} \sqrt{\varepsilon} \|\mathfrak{h}\|_{\mathfrak{E}} E. \end{aligned}$$

□

APPENDIX D. CORRELATION ESTIMATES

Not to disrupt the main text, we prove here two Lemmata used in Section 4.

Lemma D.1. *Let I be an interval of size δ , $\rho \in \mathcal{C}^1(I, \mathbb{R})$ be a probability density such that $\|\rho'\|_\infty \leq 2C_0$. Then for $h \in (\varepsilon, 1)$, and ε small enough,*

$$\left| 2\varepsilon^{-1} \int_0^h \int_0^s \int_I b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta) b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta) \rho(\theta) d\theta ds du - h\gamma \right| \leq C_{\#} \varepsilon (\ln \delta^{-1})^2.$$

Proof. We first divide the integral in the two terms

$$(D.1) \quad \int_0^h \int_0^{\varepsilon \lfloor \varepsilon^{-1} s \rfloor} \int_I b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta) b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta) \rho(\theta) d\theta ds du,$$

and

$$(D.2) \quad \int_0^h \int_{\varepsilon \lfloor \varepsilon^{-1} s \rfloor}^s \int_I b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta) b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta) \rho(\theta) d\theta ds du.$$

Recalling that ρ_\star is the invariant measure of f , we can write (D.2) as,

$$\begin{aligned} &\int_0^{\varepsilon \lfloor h\varepsilon^{-1} \rfloor} (s - \varepsilon \lfloor \varepsilon^{-1} s \rfloor) \int_{\mathbb{T}} b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta)^2 \rho(\theta) \mathbb{1}_I(\theta) d\theta ds + \mathcal{O}(\varepsilon^2) \\ (D.3) \quad &= \frac{\varepsilon^2}{2} \sum_{j=0}^{\lfloor h\varepsilon^{-1} \rfloor} \int_{\mathbb{T}} b(f^j \theta)^2 (\rho(\theta) \mathbb{1}_I(\theta) - \rho_\star(\theta)) d\theta ds + \mathcal{O}(\varepsilon^2) \\ &\quad + \frac{\varepsilon^2}{2} \sum_{j=0}^{\lfloor h\varepsilon^{-1} \rfloor} \int_{\mathbb{T}} b(f^j \theta)^2 \rho_\star(\theta) d\theta ds. \end{aligned}$$

Note that $\|\rho \mathbb{1}_I\|_{L^1} = 1$ and, by Lemma C.3, $\|\rho \mathbb{1}_I\|_{BV} \leq C_{\#} \delta^{-1}$. Hence, by (A.16) with $q_j = \rho \mathbb{1}_I - \rho_\star$, for all j (hence $\int_{\mathbb{T}} q_j = 0$), and $g = b^2$, we have that the first term of (D.3) is $\mathcal{O}(\varepsilon^2 \ln \delta^{-1})$. Hence, by the invariance of ρ_\star , the quantity (D.3) is equal to

$$(D.4) \quad \frac{\varepsilon h}{2} \int_{\mathbb{T}} b(\theta)^2 \rho_\star(\theta) d\theta + \mathcal{O}(\varepsilon^2 \ln \delta^{-1}).$$

The term (D.1) is equal to

$$\begin{aligned}
& \sum_{j=1}^{\lfloor h\varepsilon^{-1} \rfloor - 1} \int_{j\varepsilon}^{(j+1)\varepsilon} ds \sum_{k=0}^{j-1} \int_{k\varepsilon}^{(k+1)\varepsilon} du \int_I b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta) b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta) \rho(\theta) d\theta \\
& + \int_{\varepsilon \lfloor h\varepsilon^{-1} \rfloor}^h ds \sum_{k=0}^{\lfloor h\varepsilon^{-1} \rfloor - 1} \int_{k\varepsilon}^{(k+1)\varepsilon} du \int_I b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta) b(f^{\lfloor s\varepsilon^{-1} \rfloor} \theta) \rho(\theta) d\theta \\
\text{(D.5)} \quad & = \varepsilon^2 \sum_{j=1}^{\lfloor h\varepsilon^{-1} \rfloor - 1} \sum_{k=0}^{j-1} \int_I b(f^k \theta) b(f^j \theta) \rho(\theta) d\theta \\
& + (h - \varepsilon \lfloor h\varepsilon^{-1} \rfloor) \varepsilon \sum_{k=0}^{\lfloor h\varepsilon^{-1} \rfloor - 1} \int_I b(f^{\lfloor h\varepsilon^{-1} \rfloor} \theta) b(f^k \theta) \rho(\theta) d\theta.
\end{aligned}$$

The last line of (D.5) can be written as

$$\begin{aligned}
& (h - \varepsilon \lfloor h\varepsilon^{-1} \rfloor) \varepsilon \sum_{k=0}^{\lfloor h\varepsilon^{-1} \rfloor - 1} \int_{\mathbb{T}} b(f^{\lfloor h\varepsilon^{-1} \rfloor - k} \theta) b(\theta) \left[\mathcal{L}^k (\rho \mathbb{1}_I) \right](\theta) d\theta \\
\text{(D.6)} \quad & = (h - \varepsilon \lfloor h\varepsilon^{-1} \rfloor) \varepsilon \sum_{j=1}^{\lfloor h\varepsilon^{-1} \rfloor} \int_{\mathbb{T}} b(f^j \theta) b(\theta) \left[\mathcal{L}^{\lfloor h\varepsilon^{-1} \rfloor - j} (\rho \mathbb{1}_I) \right](\theta) d\theta.
\end{aligned}$$

Recalling (A.13), and Lemma C.3,

$$\begin{aligned}
& \|b \mathcal{L}^{\lfloor h\varepsilon^{-1} \rfloor - j} (\rho \mathbb{1}_I)\|_{L^1} \leq \|b\|_{\infty} \\
& \|b \mathcal{L}^{\lfloor h\varepsilon^{-1} \rfloor - j} (\rho \mathbb{1}_I)\|_{BV} \leq \|b\|_{BV} \|\mathcal{L}^{\lfloor h\varepsilon^{-1} \rfloor - j} (\rho \mathbb{1}_I)\|_{BV} \leq C_{\#} \delta_{\varepsilon}^{-1}.
\end{aligned}$$

Using equation (A.16), with $q_j = b \mathcal{L}^{\lfloor h\varepsilon^{-1} \rfloor - j} (\rho \mathbb{1}_I)$ and $g = b$, and since $(h - \varepsilon \lfloor h\varepsilon^{-1} \rfloor) \leq \varepsilon$, allows to write the expression (D.6) as

$$\begin{aligned}
\text{(D.7)} \quad & (h - \varepsilon \lfloor h\varepsilon^{-1} \rfloor) \varepsilon \sum_{j=1}^{\lfloor h\varepsilon^{-1} \rfloor} \int_{\mathbb{T}} b(f^j \theta) \rho_{\star}(\theta) d\theta \int_{\mathbb{T}} b(\theta') \left[\mathcal{L}^{\lfloor h\varepsilon^{-1} \rfloor - j} (\rho \mathbb{1}_I) \right](\theta'_x) d\theta' \\
& + \mathcal{O}(\varepsilon^2 \ln \delta^{-1}) = \mathcal{O}(\varepsilon^2 \ln \delta^{-1})
\end{aligned}$$

where we have used the invariance of ρ_{\star} and the fact that b is zero average.

The line next to the last in (D.5) can be written as

$$\begin{aligned}
& \varepsilon^2 \sum_{j=1}^{\lfloor h\varepsilon^{-1} \rfloor - 1} \sum_{k=0}^{j-1} \int_{\mathbb{T}} b(f^k \theta) b(f^j \theta) \rho_{\star}(\theta) d\theta \\
& - \varepsilon^2 \sum_{j=1}^{\lfloor h\varepsilon^{-1} \rfloor - 1} \sum_{k=0}^{j-1} \int_{\mathbb{T}} b(f^{j-k} \theta) b(\theta) \left[\mathcal{L}^k (\rho_{\star} - \rho_x \mathbb{1}_I) \right](\theta) d\theta = \\
\text{(D.8)} \quad & = \varepsilon^2 \sum_{j=1}^{\lfloor h\varepsilon^{-1} \rfloor - 1} (\lfloor h\varepsilon^{-1} \rfloor - 1 - j) \int_{\mathbb{T}} b(f^j \theta) b(\theta) \rho_{\star}(\theta) d\theta \\
& - \varepsilon^2 \sum_{j=1}^{\lfloor h\varepsilon^{-1} \rfloor - 1} \int_{\mathbb{T}} b(f^j \theta) b(\theta) \sum_{k=0}^{\lfloor h\varepsilon^{-1} \rfloor - 1 - j} \left[\mathcal{L}^k (\rho_{\star} - \rho_x \mathbb{1}_I) \right](\theta) d\theta.
\end{aligned}$$

For any $p \in \mathbb{N}$, using Lemma A.4 with $q_j = \rho_{\star} - \rho_x \mathbb{1}_I$,

$$\left\| b \sum_{k=0}^{\lfloor h\varepsilon^{-1} \rfloor - 1 - j} \mathcal{L}^k (\rho_{\star} - \rho_x \mathbb{1}_I) \right\|_{L^1} \leq C_{\#} \ln \delta^{-1},$$

where we have used Lemma C.3 and $\|\rho_{\star} - \rho_x \mathbb{1}_I\|_{L^1} \leq 2$. Furthermore, using the spectral decomposition (A.2),

$$\begin{aligned}
& \left\| b \sum_{k=0}^{\lfloor h\varepsilon^{-1} \rfloor - 1 - j} \mathcal{L}^k (\rho_{\star} - \rho_x \mathbb{1}_I) \right\|_{BV} \\
& \leq \|b\|_{BV} C_{\#} \delta^{-1} \sum_{k=0}^{\lfloor h\varepsilon^{-1} \rfloor - 1 - j} \nu^k \leq C_{\#} \delta^{-1}.
\end{aligned}$$

Therefore, using again (A.16) with $q_j = b \sum_{k=0}^{\lfloor h\varepsilon^{-1} \rfloor - 1 - j} \mathcal{L}^k (\rho_{\star} - \rho_x \mathbb{1}_I)$ and $g = b$, we can rewrite (D.8) as

$$\text{(D.9)} \quad h\varepsilon \sum_{j=1}^{\infty} \int_{\mathbb{T}} b(f^j \theta) b(\theta) \rho_{\star}(\theta) d\theta + \mathcal{O}(\varepsilon^2 (\ln \delta^{-1})^2).$$

The Lemma follows by summing (D.4), (D.7) and (D.9) and recalling the definition of γ in (2.4). \square

Lemma D.2. *Let I be an interval of size δ , $\rho \in \mathcal{C}^1(I, \mathbb{R})$ be a probability density such that $\|\rho'\|_{\infty} \leq 2C_0$. Then, for all ε small enough and $h \in (\varepsilon, 1)$*

$$\left| \varepsilon^{-1} \int_0^h ds \int_0^h du \int_I b(f^{\lfloor (h+s)\varepsilon^{-1} \rfloor} \theta) b(f^{\lfloor u\varepsilon^{-1} \rfloor} \theta) \rho(\theta) d\theta \right| \leq C_{\#} \varepsilon (\ln \delta^{-1})^2.$$

Proof. We assume that $h\varepsilon^{-1} = K \in \mathbb{N}$ as the general case can be treated similarly to Lemma D.1. By adding and subtracting ρ_{\star} , the integral in the

statement can be written as

$$(D.10) \quad \begin{aligned} & \varepsilon \sum_{k=0}^{K-1} \sum_{j=0}^{K-1} \int_{\mathbb{T}} b(f^{\lfloor h\varepsilon^{-1} \rfloor + k} \theta) b(f^j \theta) (\rho(\theta) \mathbb{1}_I(\theta) - \rho_*(\theta)) d\theta + \\ & + \varepsilon \sum_{k=0}^{K-1} \sum_{j=0}^{K-1} \int_{\mathbb{T}} b(f^{\lfloor h\varepsilon^{-1} \rfloor + k} \theta) b(f^j \theta) \rho_*(\theta) d\theta. \end{aligned}$$

Let us consider the first term of (D.10). Setting $p = k - j + K$,

$$\varepsilon \sum_{p=1}^{2K-1} \int_{\mathbb{T}} b(f^p \theta) b(\theta) \sum_{j=\max\{K-p, 0\}}^{\min\{K-1, 2K-1-p\}} \mathcal{L}^j (\rho(\theta) \mathbb{1}_I(\theta) - \rho_*(\theta)) d\theta.$$

By Lemma A.4, we have, for any $p \in \{0, 1, \dots, 2K-1\}$,

$$\left\| b \sum_{j=\max\{K-p, 0\}}^{\min\{K-1, 2K-1-p\}} \mathcal{L}^j (\rho \mathbb{1}_I - \rho_*) \right\|_{L^1} \leq C_{\#} \ln \delta^{-1},$$

and, using the spectral decomposition (A.2),

$$\left\| b \sum_{j=\max\{K-p, 0\}}^{\min\{K-1, 2K-1-p\}} \mathcal{L}^j (\rho \mathbb{1}_I - \rho_*) \right\|_{BV} \leq C_{\#} \|b\|_{BV} \delta^{-1} \sum_{j=0}^{2K-1} \nu^j \leq C_{\#} \delta^{-1}.$$

Therefore, applying (A.16), the first term of (D.10) is of order $\mathcal{O}(\varepsilon (\ln \delta^{-1})^2)$.

Setting $p = k - j + K$ and noting that

$$\min\{K-1, 2K-1-p\} - \max\{K-p, 0\} \leq p-1,$$

the second term of (D.10) can be bounded by

$$(D.11) \quad \varepsilon \sum_{p=1}^{2K-1} (p-1) \left| \int_{\mathbb{T}} b(f^p \theta) b(\theta) \rho_*(\theta) d\theta \right|.$$

Note that by Lemma A.3, $\left| \int_{\mathbb{T}} b(f^p \theta) b(\theta) \rho_*(\theta) d\theta \right| \leq C_{\#} \nu^p$, from which the Lemma follows. \square

APPENDIX E. DISCRETE GRADIENT, LAPLACIAN AND FOURIER

For any $g : \mathbb{Z}_N \rightarrow \mathbb{R}$, $N \in \mathbb{N} \cup \{\infty\}$, the lattice gradient, its adjoint and the laplacian are defined as

$$(E.1) \quad \nabla_{\mathbb{Z}_N} g_x = f_{x+1} - f_x, \quad \nabla_{\mathbb{Z}_N}^* g_x = f_{x-1} - f_x$$

and $\Delta_{\mathbb{Z}_N} g_x = -\nabla_{\mathbb{Z}_N}^* \nabla_{\mathbb{Z}_N} g_x = g_{x+1} + g_{x-1} - 2g_x$, $x \in \mathbb{Z}_N$, respectively.

The orthonormal base of eigenvectors of $-\Delta$ consists of

$$(E.2) \quad \begin{aligned} \psi_j(x) &= \frac{e^{i2\pi jx/N}}{\sqrt{N}}, & \lambda_j &= 4 \sin^2 \left(\frac{\pi j}{N} \right), & j &\in \mathbb{Z}_N \\ (\omega_0^2 - \Delta) \psi_j &= \mu_j \psi_j = (\omega_0^2 + \lambda_j) \psi_j. \end{aligned}$$

In other words, the laplacian is diagonalized by the discrete Fourier transform U defined, for each $\varphi : \mathbb{Z}_N \rightarrow \mathbb{C}$, by

$$(E.3) \quad \begin{aligned} \widehat{\varphi}_j &:= (U\varphi)_j = \sum_{x \in \mathbb{Z}_N} \varphi_x \psi_j^*(x) \\ (U^{-1}\widehat{\varphi})_x &= \sum_{j \in \mathbb{Z}_N} \widehat{\varphi}_j \psi_j(x), \end{aligned}$$

where ψ_j^* is the complex conjugate of ψ_j . For any matrix $\mathcal{A} : (\mathbb{Z}_N)^2 \rightarrow \mathbb{C}$, let

$$(E.4) \quad \begin{aligned} \widehat{\mathcal{A}}_{j,j'} &:= (U\mathcal{A}U)_{j,j'} = \sum_{x,x' \in \mathbb{Z}_N} \mathcal{A}_{x,x'} \psi_j^*(x) \psi_{j'}^*(x') \\ (U^{-1}\widehat{\mathcal{A}}U^{-1})_{x,x'} &= \sum_{j,j' \in \mathbb{Z}_N} \widehat{\mathcal{A}}_{j,j'} \psi_j(x) \psi_{j'}(x'). \end{aligned}$$

Note that if $\varphi \in \mathcal{C}^k(\mathbb{T}, \mathbb{C})$, then, for each $N \in \mathbb{N}$, it induces naturally functions from $\mathbb{Z}_N \rightarrow \mathbb{C}$, which we call with the same name, i.e. $\varphi_x = \varphi\left(\frac{x}{N}\right)$. Remark that, in the limit $N \rightarrow \infty$, we obtain

$$\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \widehat{\varphi}_j = \int_{\mathbb{T}} \varphi(u) e^{-i2\pi ju} du,$$

the usual Fourier transform of φ . Note also that, for any $q \in \mathbb{Z}_N$,

$$(E.5) \quad \sum_{x \in \mathbb{Z}_N} \psi_q(x) = N^{\frac{1}{2}} \delta_{q,0}.$$

Lemma E.1. *Let $\varphi \in \mathcal{C}^k(\mathbb{T})$ and $\widehat{\varphi}_j$ be defined by (E.3) with $\varphi_x = \varphi\left(\frac{x}{N}\right)$. Then, for each $N \in \mathbb{N}$ and $|j| \leq N/2$,*

$$(E.6) \quad |\widehat{\varphi}_j| \leq \frac{C_{\#} \|\varphi\|_{\mathcal{C}^k} N^{\frac{1}{2}}}{|j|^k}.$$

Proof. Note that,

$$\begin{aligned} |N^{-\frac{1}{2}} \widehat{\varphi}_j| &= \left| \frac{1}{N(e^{-i2\pi j/N} - 1)} \sum_{x \in \mathbb{Z}_N} e^{-i2\pi jx/N} \nabla_{\mathbb{Z}_N} \varphi\left(\frac{x}{N}\right) \right| \\ &\leq \frac{1}{N|e^{-i2\pi j/N} - 1|} \sum_{x \in \mathbb{Z}_N} \frac{\|\varphi'\|_{\infty}}{N}. \end{aligned}$$

Since for $\theta \in [-\pi/2, \pi/2]$ we have $|1 - e^{i\theta}|^2 = 2|1 - \cos\theta| \geq \frac{\theta^2}{4}$, we have

$$N^{-\frac{1}{2}} |\widehat{\varphi}_j| \leq \frac{1}{\pi j} \|\varphi'\|_{\infty},$$

that proves the statement for $k = 1$. Then iterate for general k . \square

The Green's function of $-\Delta_{\mathbb{Z}} + \omega_0^2$, where $\Delta_{\mathbb{Z}}$ is the laplacian on the integer lattice \mathbb{Z} , is given by, see e.g. [53, (27)],

$$(E.7) \quad \begin{aligned} G_{\omega_0}(x) &= (-\Delta_{\mathbb{Z}} + \omega_0^2)^{-1}(x) = \int_0^1 \frac{\cos(2\pi ux)}{4 \sin^2(\pi u) + \omega_0^2} du \\ &= \frac{1}{\omega_0 \sqrt{\omega_0^2 + 4}} \left\{ 1 + \frac{\omega_0^2}{2} + \omega_0 \sqrt{1 + \frac{\omega_0^2}{4}} \right\}^{-|x|}, \quad x \in \mathbb{Z}. \end{aligned}$$

APPENDIX F. ON THE DIFFUSION COEFFICIENT D

The thermal diffusivity D can be written differently. There is a kinetic expression that relates it to the dispersion relation of the harmonic lattice $\omega(k) = \sqrt{\omega_0^2 + 4 \sin^2(\pi k)}$. In the kinetic limit the velocity of a *phonon* of wavenumber k is $\frac{\omega'(k)}{2\pi}$, and the diffusion coefficient is related to the kinetic energy of the phonons by (see Section 6 in [5] for more details)

$$(F.1) \quad D = \frac{1}{2\pi^2} \int [\omega'(k)]^2 dk,$$

that is valid for a generic harmonic lattice. For nearest-neighbor interaction

$$(F.2) \quad \begin{aligned} D &= \frac{1}{2\pi^2} \int [\omega'(k)]^2 dk = \int \frac{8 \sin^2(\pi k) \cos^2(\pi k)}{\omega^2(k)} dk \\ &= \int \frac{1 - \cos(4\pi k)}{\omega^2(k)} dk. \end{aligned}$$

Another expression comes from the fluctuation-dissipation relation [43]:

$$(F.3) \quad \begin{aligned} D &= 1 + 2G_{\omega_0}(1) - (1 + \omega_0^2)G_{\omega_0}(0) - G_{\omega_0}(2) \\ &= 1 + (\omega_0^2 - \Delta) G_{\omega_0}(1) - \omega_0^2 (G_{\omega_0}(0) + G_{\omega_0}(1)) \\ &= 1 - \omega_0^2 (G_{\omega_0}(0) + G_{\omega_0}(1)), \end{aligned}$$

where G_{ω_0} is defined in (E.7). From the first line of (F.3)

$$(F.4) \quad \begin{aligned} D &= \int \left(1 + \frac{2 \cos(2\pi k) - 1 - \omega_0^2 - \cos(4\pi k)}{\omega^2(k)} \right) dk \\ &= \int \left(\frac{4 \sin^2(\pi k) + 2 \cos(2\pi k) - 1 - \cos(4\pi k)}{\omega^2(k)} \right) dk \\ &= \int \frac{1 - \cos(4\pi k)}{\omega^2(k)} dk. \end{aligned}$$

From the last line of (F.3) and (E.7) we also have

$$(F.5) \quad \begin{aligned} D &= 1 - \frac{\omega_0^2}{\omega_0 \sqrt{\omega_0^2 + 4}} \left(1 + \left\{ 1 + \frac{1}{2} \omega_0^2 + \frac{1}{2} \omega_0 \sqrt{4 + \omega_0^2} \right\}^{-1} \right) \\ &= \frac{2}{2 + \omega_0^2 + \omega_0 \sqrt{4 + \omega_0^2}}. \end{aligned}$$

APPENDIX G. COMPACTNESS

Let $\mathcal{M}_{\mathcal{E}_0}(\mathbb{T})$ the set of the positive measures on \mathbb{T} with total mass bounded by \mathcal{E}_0 , endowed by the weak topology. Under this topology $\mathcal{M}_{\mathcal{E}_0}(\mathbb{T})$ is complete, separable, metrizable and compact. We choose the metric

$$(G.1) \quad d(\mu, \mu') = \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} \min \left\{ 1, \left| \int \psi_k d\mu - \int \psi_k d\mu' \right| \right\},$$

where $\psi_k(\theta) = e^{2\pi i \theta k}$, $k \in \mathbb{Z}$, which are a countable dense set in $\mathcal{C}^0(\mathbb{T})$.

We have that $\xi_N \in \mathcal{C}([0, t_*], \mathcal{M}_{\mathcal{E}_0}(\mathbb{T}))$, that we endow with the uniform topology induced by the metric (G.1). Since $\mathcal{M}_{\mathcal{E}_0}(\mathbb{T})$ is compact, relative compactness of the sequence $\{\xi_N\}$ follows from the following equicontinuity in time of ξ_N , by Ascoli's Theorem [41, pg. 223, Theorem 17].

Proposition G.1. *Provided $\varepsilon \leq N^{-\alpha}$, $\alpha > 6$, we have, for all $t_* > 0$,*

$$(G.2) \quad \lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{0 \leq s, t \leq t_*, |t-s| \leq \delta} d(\xi_N(t), \xi_N(s)) = 0$$

Proof. Let $\varphi \in \mathcal{C}^6(\mathbb{T})$. Recalling the definition of ξ_N in (5.1), by Proposition (5.1) and (2.8), we have, for all $0 \leq s < t \leq t_*$,

$$\begin{aligned} \|\xi_N(t, \varphi)\| &\leq \|\varphi\|_{\infty} \mathcal{E}_* \\ \xi_N(t, \varphi) - \xi_N(s, \varphi) &= \frac{D}{2\gamma} \int_s^t \xi_N(u, \varphi'') du + \mathcal{E}_* \|\varphi\|_{\mathcal{C}^6} \mathcal{O}((t-s) + N^{-1}). \end{aligned}$$

It follows

$$(G.3) \quad |\xi_N(t, \varphi) - \xi_N(s, \varphi)| \leq C_{\#} \mathcal{E}_* (|t-s| + N^{-1}) \|\varphi\|_{\mathcal{C}^6}.$$

Then using the metric (G.1) we have

$$(G.4) \quad d(\xi_N(t), \xi_N(s)) \leq \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} \min \{1, C_{\#} \mathcal{E}_* k^6 (|t-s| + N^{-1})\},$$

and, for any $\delta \leq 1$,

$$(G.5) \quad \begin{aligned} \sup_{|t-s| \leq \delta} d(\xi_N(t), \xi_N(s)) &\leq C_{\#} \mathcal{E}_* \left(\sum_{k=0}^{\infty} \frac{1}{2^k} \min \{1, \delta k^6\} + N^{-1} \right) \\ &\leq C_{\#} \mathcal{E}_* \left(\sum_{k=0}^{\delta^{-\frac{1}{12}}} \frac{1}{2^k} \sqrt{\delta} + \sum_{k=\delta^{-\frac{1}{12}}}^{\infty} \frac{1}{2^k} + N^{-1} \right) \leq C_{\#} \mathcal{E}_* (\sqrt{\delta} + N^{-1}), \end{aligned}$$

from which the Proposition follows. \square

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