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IDENTIFIABILITY OF TOTAL EFFECTS FROM ABSTRACTIONS OF TIME SERIES CAUSAL GRAPHS

A PREPRINT

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ABSTRACT

We study the problem of identifiability of the total effect of an intervention from observational time series only given an abstraction of the causal graph of the system. Specifically, we consider two types of abstractions: the extended summary causal graph which conflates all lagged causal relations but distinguishes between lagged and instantaneous relations; and the summary causal graph which does not give any indication about the lag between causal relations. We show that the total effect is always identifiable in extended summary causal graphs and we provide necessary and sufficient graphical conditions for identifiability in summary causal graphs. Furthermore, we provide adjustment sets allowing to estimate the total effect whenever it is identifiable.

1 Introduction

Over the last century and across numerous disciplines, experimentation has emerged as a potent methodology for estimating without bias the total effect of an intervention on a specific component of a given system (Neyman et al., 1990). However, experimentation can be costly, unethical or even unfeasible. Both researchers and experts are thus interested in estimating the effect of an intervention directly from observational data. This can be done under some assumptions when relying on a causal graph (Pearl et al., 2000). Typically, it is tackled through two sequential steps: identifiability and estimation (Pearl, 2019). The identifiability step involves graphically distinguishing cases where a solution is possible and, when it exists, providing an estimand - an expression enabling the estimation of intervention effects from observational data. The subsequent step involves the actual estimation of this estimand from the available data.

The identifiability step received much attention for non-temporal causal graphs (Pearl, 1993, 1995; Spirtes et al., 2000; Pearl et al., 2000; Shpitser and Pearl, 2008; Maathuis and Colombo, 2013; Perkovic, 2020; Anand et al., 2023). Recently, it was also addressed for temporal causal graph. First, Blondel et al. (2016) develop the do-calculus for the full-time causal graphs: each vertex represents a temporal instance of a time series. However, in dynamic systems, experts have difficulties in building time series causal graphs, while they can usually build an abstraction of those graphs where temporal information is omitted. The identifiability problem has only been solved for abstractions of causal graphs either under the assumption that there is no instantaneous relations (Eichler and Didelez, 2007) or under an acyclicity assumption (Assaad et al., 2023).

In this work, we focus on establishing identifiability within abstractions of time series causal graph which may encompass instantaneous causal relationships. Total effects, in time series, take the following form: $P(Y_t = y_t | do(X_{t-\gamma} = x_{t-\gamma}))$, latter denoted $P(y_t | do(x_{t-\gamma}))$ by a slight abuse of notations, where $do(X_{t-\gamma} = x_{t-\gamma})$ represents an intervention (as properly defined in Eichler and Didelez (2007, Assumption 2.3)) on the variable X at time $t - \gamma$, with $\gamma \geq 0$. Here, Y_t corresponds to the response and $do()$ denotes the standard intervention operator (Pearl et al., 2000). Our main contribution lies in demonstrating when, with no hidden confounding (also referred to as causal sufficiency,

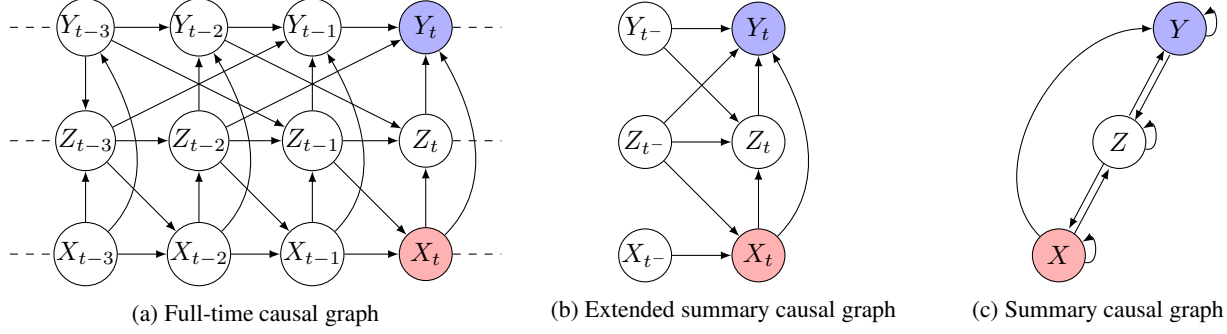


Figure 1: Example of a full-time causal graph (a), its corresponding extended summary causal graph (b) and (c) its corresponding summary causal graph. In (b), t^- represents the past time slice and t the present time slice. In (c), the self loops indicates that a time series causes itself. We illustrate the running example’s total effect $P(y_t | do(x_t))$, with $\gamma = 0$, where the blue vertex represents the response and the red vertex the intervention.

Spirtes et al. (2000)), the total effect can be identified when working with two specific types of abstraction known as the extended summary causal graph and the summary causal graph.

The remainder of the paper is structured as follows: Section 2 presents the problem setup. In Section 3, we first establish that when using extended summary causal graphs, the total effect is always identifiable, and we provide adjustment sets for its estimation. Section 4 delves into the derivation of both necessary and sufficient conditions for identifiability when employing summary causal graphs, and adjustment sets are outlined for cases where identifiability is achieved. Finally, Section 5 concludes the paper.

2 Problem setup

Consider \mathcal{V} a set of p observational time series and $\mathcal{V}^f = \{\mathcal{V}_{t-\ell} | \ell \in \mathbb{Z}\}$ the set of temporal instances of \mathcal{V} where $\mathcal{V}_{t-\ell}$ correspond to the variables of the time series at time $t - \ell$. We suppose that the time series are generated from a dynamic structural causal model (DSCM, Rubenstein et al. (2018)), an extension of structural causal models (SCM, Pearl et al. (2000)) to time series. The graph that is used to qualitatively represent causal relations described in a DSCM is known as the full-time causal graph (FTCG).

Definition 1 (Full-time causal graph (FTCG), Figure 1a). *Let \mathcal{V} be a set of p observation time series. The full-time causal graph (FTCG) $\mathcal{G}^f = (\mathcal{V}^f, \mathcal{E}^f)$ is defined by set of time series for vertices \mathcal{V}^f and the set of edges \mathcal{E}^f defined as follows: $X_{t-\gamma} \rightarrow Y_t$ if and only if X causes Y at time t with a time lag of $\gamma > 0$ if $X = Y$ and with a time lag of $\gamma \geq 0$ for $X \neq Y$.*

The FTCTG is assumed to be acyclic and to satisfy the Markov condition. However, in practice, it is usually impossible to work with FTCTGs and people have resorted to simpler causal graphs, exploiting the fact that causal relations between time series hold throughout time, as formalized in the following assumption.

Assumption 1 (Consistency throughout time). *An FTCTG \mathcal{G}^f is said to be consistent throughout time if all the causal relationships remain constant in direction throughout time.*

This assumption allows to focus on a finite number of past slices, which is given by the maximum lag. We fix it to γ_{\max} in the following.

Experts are used to working with abstractions of causal graphs, that summarize the information into a graph that is interpretable, often with the omission of precise temporal information details. An extended summary causal graph (Assaad et al., 2022a) serves as a representation for causal relationships among time series. It distinguishes between past time slices, denoted as $\mathcal{V}_{t^-}^e$, and present time slices, denoted as \mathcal{V}_t^e , thus enabling the differentiation between lagged and instantaneous causal relations.

Definition 2 (Extended summary causal graph (ECG), Figure 1b). *Let $\mathcal{G}^f = (\mathcal{V}^f, \mathcal{E}^f)$ be an FTCTG satisfying Assumption 1 with maximal temporal lag γ_{\max} . The underlying extended summary causal graph (ECG) $\mathcal{G}^e = (\mathcal{V}^e, \mathcal{E}^e)$ is given by $\mathcal{V}^e = (\mathcal{V}_{t^-}^e, \mathcal{V}_t^e)$ and \mathcal{E}^e defined as follows:*

- for all $X_t, Y_t \in \mathcal{V}_t^e$, $X_t \rightarrow Y_t \in \mathcal{E}^e$ if and only if $X_t \rightarrow Y_t \in \mathcal{E}^f$;

- for all $X, Y \in \mathcal{V}_{t-}^e$, $X_{t-} \rightarrow Y_t \in \mathcal{E}^e$ if and only if there exists at least one temporal lag $0 < \gamma \leq \gamma_{\max}$ such that $X_{t-\gamma} \rightarrow Y_t \in \mathcal{E}^f$.

In that case, we say that \mathcal{G}^f is compatible with \mathcal{G}^e .

At a higher level of abstraction, a summary causal graph (Peters et al., 2013; Meng et al., 2020; Assaad et al., 2022b,c) represents causal relationships among time series, regardless of the time delay between the cause and its effect.

Definition 3 (Summary causal graph (SCG), Figure 1c). Let $\mathcal{G}^f = (\mathcal{V}^f, \mathcal{E}^f)$ be an FTCTG satisfying Assumption 1 with maximal temporal lag γ_{\max} . The underlying summary causal graph (SCG) $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ is such that

- \mathcal{V}^s corresponds to the set of time series,
- $X \rightarrow Y \in \mathcal{E}^s$ if and only if there exists at least one temporal lag $0 \leq \gamma \leq \gamma_{\max}$ such that $X_{t-\gamma} \rightarrow Y_t \in \mathcal{E}^f$.

In that case, we say that \mathcal{G}^f is compatible with \mathcal{G}^s .

The abstractions of SCGs and ECGs entail that if there is a single ECG or SCG corresponding to a given FTCTG, there are often multiple FTCTGs compatible with a given ECG or SCG. Additionally, it is worth noting that an SCG is considered an abstraction of an ECG. However, in cases where there is no instantaneous relation, ECGs and SCGs convey the same information.

Since the FTCTG is assumed to be acyclic, the ECG is inherently acyclic. In contrast, SCGs may include directed cycles and even self-loops. We use the notation $X \rightleftarrows Y$ to indicate situations where there are time lags where X causes Y and other lags where Y causes X . We denote $Cycles(X, \mathcal{G}^s)$ as the set of all directed cycles containing X in \mathcal{G}^s , and $Cycles^>(X, \mathcal{G}^s)$ with at least 2 vertices (i.e., excluding self-loops). By a slight abuse of notation, we denote $\mathcal{G}^s \setminus \{Y\}$ as the subgraph of \mathcal{G}^s when removing the vertex Y and corresponding edges.

We focus in this paper on the *total effect* (Pearl et al., 2000) of $X_{t-\gamma}$ on Y_t given by $P(y_t | do(x_{t-\gamma}))$, the distribution of Y_t while $X_{t-\gamma}$ is held constant at $x_{t-\gamma}$ and all other variables are permitted to run their natural course according to the DSCM. The goal of this paper is to discuss the identifiability of the total effect given an ECG or an SCG, without the knowledge of the true FTCTG. In general, a total effect $P(y_t | do(x_{t-\gamma}))$ is said to be *identifiable* from a graph if it can be uniquely computed from the observed distribution, without any further assumption on the distribution (Pearl, 1995; Perkovic, 2020). More specifically, in our context, $P(y_t | do(x_{t-\gamma}))$ is *identifiable* if it can be rewritten with a do-free formula that is valid for all FTCTGs compatible with the given ECG or SCG. This consists on finding for all FTCTGs a set of variables known as *adjustment set* (Shpitser et al., 2010) which permits to block all non-causal paths without blocking any causal path between $X_{t-\gamma}$ and Y_t . This leads to the following graphical definition of identifiability.

Definition 4 (Identifiability of total effects in ECGs and SCGs). *With no hidden confounding, given an ECG or an SCG, the total effect $P(y_t | do(x_{t-\gamma}))$ is identifiable if and only if there exists a set that:*

- (i) blocks all back-door paths between $X_{t-\gamma}$ and Y_t in all compatible FTCTGs,
- (ii) does not block any directed path between $X_{t-\gamma}$ and Y_t in all compatible FTCTGs,
- (iii) does not contain any descendant of Y_t in all compatible FTCTGs.

This set is called an adjustment set for $P(y_t | do(x_{t-\gamma}))$ in the ECG or the SCG.

Note that, without hidden confounding, the total effect $P(y_t | do(x_{t-\gamma}))$ in any FTCTG \mathcal{G}^f is identifiable (Blondel et al., 2016; Shpitser and Pearl, 2008). However, since an ECG or an SCG usually represents different possible FTCTGs, it is not a priori certain that a given total effect $P(y_t | do(x_{t-\gamma}))$ will yield equivalent do-free formulas across the various FTCTGs.

3 Identifiability in ECGs

With no hidden confounding, the total effect is always identifiable in ECGs, as established by the following theorem.

Theorem 1 (Identifiability in ECG). *Consider an ECG \mathcal{G}^e . The total effect $P(y_t | do(x_{t-\gamma}))$, with $\gamma \geq 0$ is identifiable in \mathcal{G}^e .*

Proof. If $X \notin Anc(Y, \mathcal{G}^e)$, then in every FTCTG \mathcal{G}^f compatible with \mathcal{G}^e , $X_{t-\gamma} \notin Anc(Y_t, \mathcal{G}^f)$. Thus, $P(y_t | do(x_{t-\gamma}))$ is always identifiable in \mathcal{G}^e , and $P(y_t | do(x_{t-\gamma})) = P(y_t)$.

Assume now that $X \notin \text{Anc}(Y, \mathcal{G}^e)$. Let γ_{\max} be the maximal lag, and \mathcal{G}^f be an FTCTG compatible with \mathcal{G}^e . We prove that

$$\mathcal{B}_\gamma = \{(Z_{t-\gamma-\ell})_{1 \leq \ell \leq \gamma_{\max}} | Z_{t-} \in \text{Par}(X_t, \mathcal{G}^e)\} \\ \cup \{Z_{t-\gamma} | Z_t \in \text{Par}(X_t, \mathcal{G}^e)\}$$

is an adjustment set for $P(y_t | do(x_{t-\gamma}))$ in \mathcal{G}^f .

First, we have to prove that $\text{Par}(X_{t-\gamma}, \mathcal{G}^f) \subseteq \mathcal{B}_\gamma$. Let $Z_{t-\gamma-\ell} \in \text{Par}(X_{t-\gamma}, \mathcal{G}^f)$. If $\ell = 0$, then Z_t causes X_t in \mathcal{G}^e by consistency throughout time, which means that $Z_{t-\gamma} \in \mathcal{B}_\gamma$. If $\ell > 0$, then Z_{t-} causes X_t in \mathcal{G}^e , that is $Z_{t-\ell-\gamma} \in \mathcal{B}_\gamma$. This shows that the set \mathcal{B}_γ blocks all back-door paths relatively to $P(y_t | do(x_{t-\gamma}))$.

Then, we have to prove \mathcal{B}_γ does not contain any descendant of $X_{t-\gamma}$ in \mathcal{G}^f . If this is true, there exists $W_{t-\gamma} \in \mathcal{B}_\gamma \cap \text{Desc}(X_{t-\gamma}, \mathcal{G}^f)$, at time slice $t-\gamma$ because it is a parent and a descendant of $X_{t-\gamma}$. By consistency throughout time, $W_t \in \text{Desc}(X_t, \mathcal{G}^f)$. However, by definition of \mathcal{B}_γ , $X_t \in \text{Desc}(W_t, \mathcal{G}^f)$, which contradicts the acyclicity assumption of \mathcal{G}^e . It means that neither it blocks any directed path between $X_{t-\gamma}$ and Y_t , nor it contains any descendant of Y_t . \square

As every total effect is identifiable in ECG, it is relevant to determine an adjustment set. The set \mathcal{B}_γ , as defined in the proof of Theorem 1, can serve as an adjustment set for $P(y_t | do(x_{t-\gamma}))$, though this set is very large. The following proposition gives a range of adjustment sets, which are potentially smaller. This proposition is based on the back-door criterion of the densest compatible FTCTG, defined as follows: for an ECG \mathcal{G}^e and a maximal lag γ_{\max} , the *densest compatible FTCTG* is the FTCTG compatible with \mathcal{G}^e with all potential edges.

Proposition 1. *Consider an ECG \mathcal{G}^e and a maximal lag γ_{\max} and let $\gamma \geq 0$. Any adjustment set \mathcal{B}'_γ for the total effect $P(y_t | do(x_{t-\gamma}))$ given by the back-door criterion on the densest compatible FTCTG is a valid adjustment set for the total effect in \mathcal{G}^e and*

$$P(y_t | do(x_{t-\gamma})) = \sum_{\mathbf{b} \in \mathcal{B}'_\gamma} P(\mathbf{b}) P(y_t | x_{t-\gamma}, \mathbf{b}),$$

where the sum covers all values \mathbf{b} in the support of \mathcal{B}'_γ .

Proof. Let \mathcal{G}_d^f be the densest FTCTG compatible with \mathcal{G}^e , and \mathcal{B}'_γ be an adjustment set in \mathcal{G}_d^f . Let \mathcal{G}^f be another FTCTG compatible with \mathcal{G}^e . By definition of \mathcal{G}_d^f , any back-door path in \mathcal{G}^f is also in \mathcal{G}_d^f (the last graph contains all possible edges). Then, \mathcal{B}'_γ blocks all back-door paths in \mathcal{G}^f . Moreover, since no vertex in \mathcal{B}'_γ is a descendant of $X_{t-\gamma}$ in \mathcal{G}_d^f , the same holds for \mathcal{G}^f . Thus, \mathcal{B}'_γ is also an adjustment set in \mathcal{G}^f . \square

Note that \mathcal{B}_γ is in this set and that this adjustment set \mathcal{B}'_γ is distinct from the back-door criterion for the total effect $P(y_t | do(x_{t-\gamma}))$ as applied to the true FTCTG, which we do not know.

4 Identifiability in SCGs

SCGs are the most abstract version of causal graphs considered in this paper. Due to the potential cycles, we use the notion of σ -blocked path introduced in Forré and Mooij (2017) but we restrict it to the case of no hidden confounding.

Definition 5 (σ -blocked path with no hidden confounding). *Consider an SCG $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$. A path $\pi = \langle X, \dots, Y \rangle$ in \mathcal{G}^s is σ -blocked by a set $\mathcal{Z} \subseteq \mathcal{V}^s$ if:*

- at least one of the endnodes X, Y is in \mathcal{Z} ; or
- there exists a vertex $Z \in \pi$, such that $Z \notin \text{Anc}(Z, \mathcal{G}^s)$ with two adjacent edges in π that form a collider at Z , (i.e., $\rightarrow Z \leftarrow$); or
- there exists a vertex $Z \in \pi \cap \mathcal{Z}$ with two adjacent edges in π that forms a non-collider at Z , and such that there exists a directed edge pointing from Z to a vertex not in the strongly connected component $\text{Sc}(Z, \mathcal{G}^s) = \text{Anc}(Z, \mathcal{G}^s) \cap \text{Desc}(Z, \mathcal{G}^s)$.

If none of the above holds then the path $\langle X, \dots, Y \rangle$ is σ -active by \mathcal{Z} .

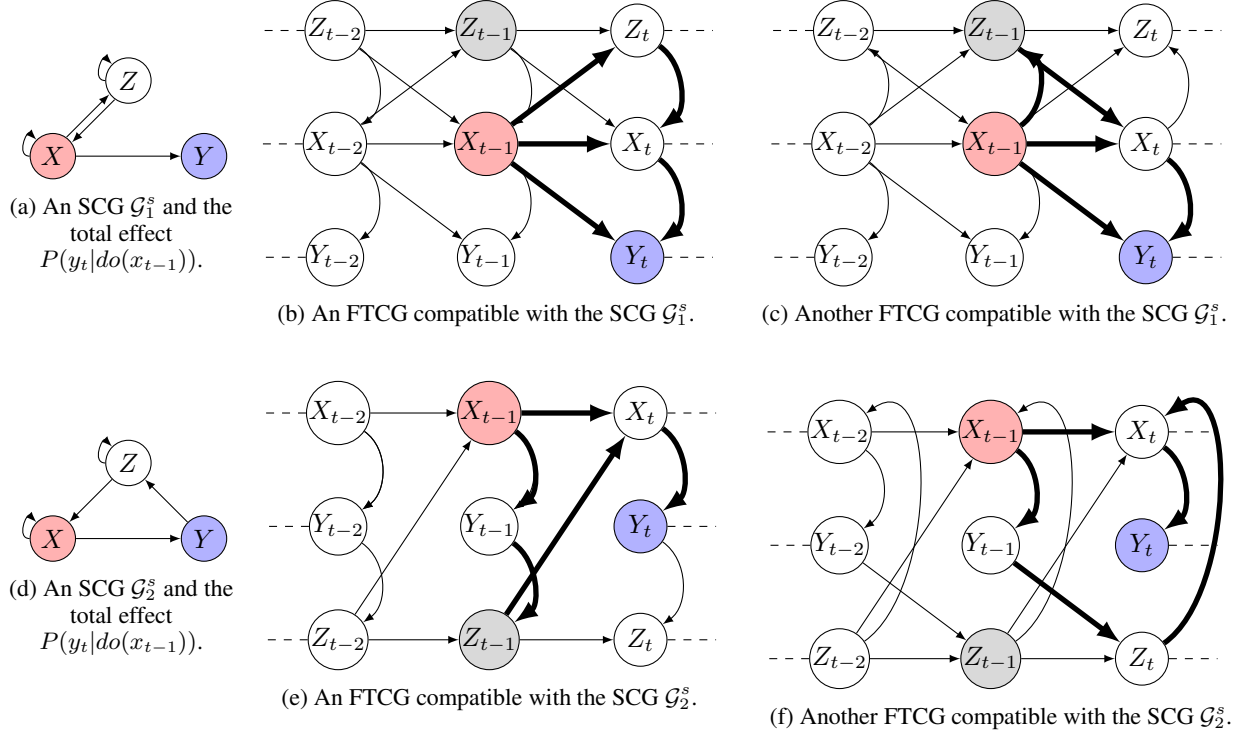


Figure 2: An example of an SCG \mathcal{G}_1^s in (a) (resp. \mathcal{G}_2^s in (d)) with two of its compatible FTCCGs in (b) and (c) (resp. in (e) and (f)). Each pair of red and blue vertices in the FTCCGs represents one total effect we are interested in. Gray vertices are ambiguous, and bold edges correspond to direct path from $X_{t-\gamma}$ to Y_t . In both examples, the total effect is not identifiable given the SCG as stated in Lemma 1 (resp. Lemma 2).

Note that a back-door path can start with the edge \rightleftharpoons , and particularly $X \rightleftharpoons Y$ is a σ -active back-door path in \mathcal{G}^s .

Given the above definition, we can now present the main result of this section, which states necessary and sufficient conditions for the identifiability in SCG.

Theorem 2 (Identifiability in SCG). *Consider an SCG $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$. The total effect $P(y_t | do(x_{t-\gamma}))$, with $\gamma \geq 0$, is not identifiable if and only if $X \in Anc(Y, \mathcal{G}^s)$ and one of the following holds:*

- $\gamma \neq 0$ and $Cycles^>(X, \mathcal{G}^s \setminus \{Y\}) \neq \emptyset$, or
- there exists a σ -active back-door path

$$\pi^s = \langle V^1 = X, \dots, V^n = Y \rangle$$

from X to Y in \mathcal{G}^s such that $\langle V^2, \dots, V^{n-1} \rangle \subseteq Desc(X, \mathcal{G}^s)$ and one of the following holds:

- $n > 2$, ie $\langle V^2, \dots, V^{n-1} \rangle \neq \emptyset$, or
- $\gamma \neq 1$, or
- $\gamma = 1$, $n = 2$ and $Cycles(Y, \mathcal{G}^s \setminus \{X\}) \neq \emptyset$.

Proof. To prove this theorem, we present necessary conditions in Lemmas 1-4 and sufficient conditions for the identifiability in Lemmas 5-7. \square

We first introduce the notion of ambiguous vertices that will be useful for the proofs of most of the lemmas.

Definition 6 (Ambiguous vertices). *Consider an SCG \mathcal{G}^s and the total effect $P(y_t | do(x_{t-\gamma}))$, for $\gamma \geq 0$. A vertex $V_{t'}$ is ambiguous for $P(y_t | do(x_{t-\gamma}))$ if there exist two FTCCGs \mathcal{G}_1^f and \mathcal{G}_2^f compatible with \mathcal{G}^s such that:*

- $V_{t'}$ belongs to a back-door path of $(X_{t-\gamma}, Y_t)$ in \mathcal{G}_1^f ; and
- $V_{t'}$ belongs to a directed path from $X_{t-\gamma}$ to Y_t in \mathcal{G}_2^f or $V_{t'} \in Desc(Y_t, \mathcal{G}_2^f)$.

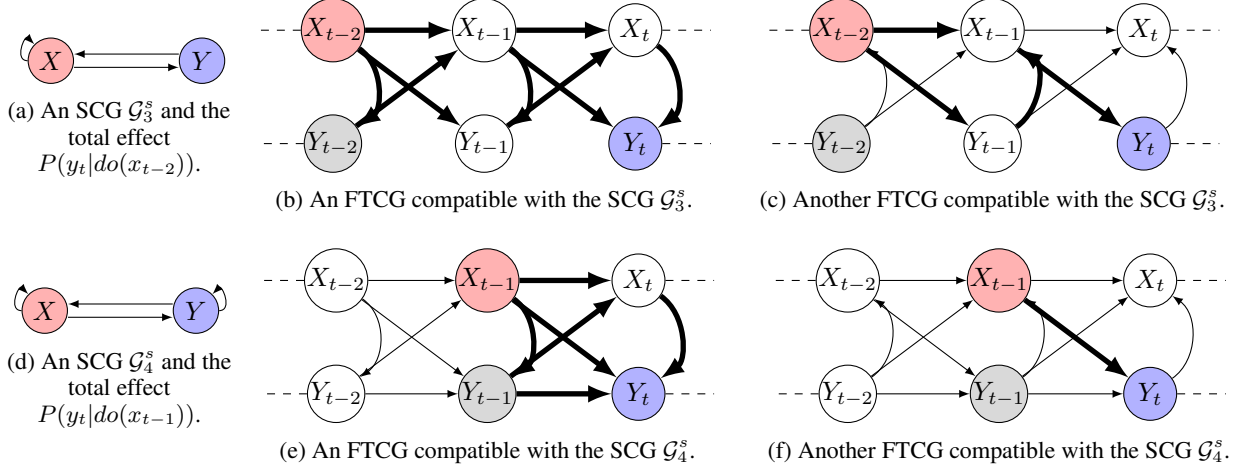


Figure 3: An example of an SCG \mathcal{G}_3^s in (a) (resp. \mathcal{G}_4^s in (d)) with two of its compatible FTCGs in (b) and (c) (resp. in (e) and (f)). Each pair of red and blue vertices in the FTCGs represents one total effect we are interested in. Gray vertices are ambiguous, and bold edges correspond to direct path from $X_{t-\gamma}$ to Y_t . In both examples, the total effect is not identifiable given the SCG as stated in Lemma 3 (resp. Lemma 4).

Ambiguous vertices are the one that prevent identifiability, as shown in the next property.

Property 1. Consider an SCG \mathcal{G}^s and the total effect $P(y_t | do(x_{t-\gamma}))$ for $\gamma \geq 0$. If there exists a back-door path π^f in an FTCG compatible with \mathcal{G}^s between $X_{t-\gamma}$ and Y_t in which all vertices are ambiguous then $P(y_t | do(x_{t-\gamma}))$ is not identifiable.

Proof. Let's assume that the total effect is identifiable. Then, there exists a set of vertices \mathcal{W} which blocks all back-door paths between $X_{t-\gamma}$ and Y_t of all FTCGs compatible with \mathcal{G}^s and which does not block any directed paths between $X_{t-\gamma}$ and Y_t nor contains any descendant of Y_t . Let's consider an FTCG compatible with \mathcal{G}^s and a back-door path π^f between $X_{t-\gamma}$ and Y_t in this FTCG. All vertices which are in \mathcal{W} and π^f block π^f . As they are in \mathcal{W} , they cannot belong, in any FTCG compatible with \mathcal{G}^s , to a directed path from $X_{t-\gamma}$ and Y_t , as otherwise they would block this path, and cannot be a descendant of Y_t . They are thus unambiguous according to Definition 6. \square

This enables us to prove that the conditions stated in Theorem 2 are necessary. We detail in Lemmas 3 and 4 all the cases, and depict them in Figures 2-3. We present an SCG and two FTCGs compatible with the SCG that do not share any adjustment set. Ambiguous vertices are highlighted in gray. The proofs of the lemmas follow a similar approach: we provide two FTCGs that are compatible with the SCG, along with associated paths in which all vertices are ambiguous. Thus, by Property 1, the total effect is not identifiable.

Lemma 1. Consider an SCG \mathcal{G}^s and the total effect $P(y_t | do(x_{t-\gamma}))$ for $\gamma > 0$. If $X \in Anc(Y, \mathcal{G}^s)$ and $Cycles^>(X, \mathcal{G}^s \setminus \{Y\}) \neq \emptyset$ then $P(y_t | do(x_{t-\gamma}))$ is not identifiable.

This lemma is illustrated in Figure 2 (top row) with the SCG \mathcal{G}_1^s and the total effect $P(y_t | do(x_{t-1}))$.

Proof. Let $\pi^s = X \rightarrow W^2 \rightarrow \dots \rightarrow W^{m-1} \rightarrow Y$ be a directed path in \mathcal{G}^s and $C = \langle X, V^2, \dots, V^{n-1}, X \rangle \in Cycles^>(X, \mathcal{G}^s \setminus \{Y\})$.

There exists an FTCG \mathcal{G}_1^f compatible with \mathcal{G}^s such that $\pi_1^f = X_{t-\gamma} \rightarrow \dots \rightarrow V_{t-\gamma}^{n-1} \rightarrow X_{t-\gamma+1} \rightarrow W_{t-\gamma+1}^2 \dots \rightarrow Y_t$. There also exists an FTCG \mathcal{G}_2^f compatible with \mathcal{G}^s with the path $\pi_2^f = X_{t-\gamma} \leftarrow V_{t-\gamma}^{n-1} \leftarrow \dots \leftarrow X_{t-\gamma+1} \rightarrow \dots \rightarrow Y_t$. Thus, by Property 1, $P(y_t | do(x_{t-\gamma}))$ is not identifiable. \square

Lemma 2. Consider an SCG \mathcal{G}^s , and the total effect $P(y_t | do(x_{t-\gamma}))$, for $\gamma \geq 0$. If $X \in Anc(Y, \mathcal{G}^s)$ and there exists a σ -active back-door path

$$\pi^s = \langle V^1 = X, \dots, V^n = Y \rangle$$

from X to Y in \mathcal{G}^s such that $\langle V^2, \dots, V^{n-1} \rangle \subseteq Desc(X, \mathcal{G}^s)$ and $n > 2$ then $P(y_t | do(x_{t-\gamma}))$ is not identifiable.

This lemma is illustrated in Figure 2 (bottom row) with the SCG \mathcal{G}_2^s and the total effect $P(y_t | do(x_{t-1}))$.

Proof. Let $\pi^s = \langle V^1 = X, \dots, V^n = Y \rangle$ be such a σ -active backdoor path.

Then there exists an FTCTG \mathcal{G}_1^f compatible with \mathcal{G}^s such that $\pi_1^f = X_{t-\gamma} \leftarrow \dots \leftarrow V_{t-\gamma}^i \rightarrow \dots \rightarrow Y_t$ is a backdoor path. As $X \in \text{Anc}(Y, \mathcal{G}^s)$ and all $V^j \subseteq \text{Desc}(X, \mathcal{G}^s)$, $2 \leq j \leq n-1$, all those variables also belong to a direct path from $X_{t-\gamma}$ to Y_t . Thus, by Property 1, $P(y_t | do(x_{t-\gamma}))$ is not identifiable. \square

Lemma 3. Consider an SCG \mathcal{G}^s and the total effect $P(y_t | do(x_{t-\gamma}))$, with $\gamma \neq 1$. If $X \rightleftharpoons Y \in \mathcal{G}^s$ then $P(y_t | do(x_{t-\gamma}))$ is not identifiable.

This lemma is illustrated in Figure 3 (top row) with the SCG \mathcal{G}_3^s and the total effect $P(y_t | do(x_{t-2}))$.

Proof. If $\gamma = 0$, there exists an FTCTG \mathcal{G}_1^f compatible with \mathcal{G}^s such that $X_t \in \text{Par}(Y_t, \mathcal{G}_1^f)$ and another FTCTG \mathcal{G}_2^f compatible with \mathcal{G}^s such that $Y_t \in \text{Par}(X_t, \mathcal{G}_2^f)$, thus the total effect is not identifiable.

Let $\gamma > 1$. There exists an FTCTG \mathcal{G}_1^f compatible with \mathcal{G}^s such that $X_{t-\gamma} \leftarrow Y_{t-\gamma} \rightarrow X_{t-\gamma+1} \rightarrow \dots \rightarrow Y_t$ and $X_{t-\gamma} \rightarrow Y_t$. There also exists an FTCTG \mathcal{G}_2^f compatible with \mathcal{G}^s such that $X_{t-\gamma} \rightarrow Y_{t-\gamma} \rightarrow X_{t-\gamma+1} \rightarrow \dots \rightarrow Y_t$. Then, by Property 1, $P(y_t | do(x_{t-\gamma}))$ is not identifiable. \square

Lemma 4. Consider an SCG \mathcal{G}^s and the total effect $P(y_t | do(x_{t-1}))$ ($\gamma = 1$). If $X \rightleftharpoons Y \in \mathcal{G}^s$ and $\text{Cycles}(Y, \mathcal{G}^s \setminus \{X\}) \neq \emptyset$ then $P(y_t | do(x_{t-1}))$ is not identifiable.

This lemma is illustrated in Figure 3 (bottom row) with the SCG \mathcal{G}_4^s and the total effect $P(y_t | do(x_{t-1}))$.

Proof. Let $\langle Y, V^2, \dots, V^{n-1}, Y \rangle \in \text{Cycles}(Y, \mathcal{G}^s \setminus \{X\})$. There exists an FTCTG \mathcal{G}_1^f compatible with \mathcal{G}^s with a back-door path $X_{t-1} \leftarrow Y_{t-1} \rightarrow \dots \rightarrow V_{t-1}^{n-1} \rightarrow Y_t$ and a directed path $X_{t-1} \rightarrow Y_t$. There also exists an FTCTG \mathcal{G}_2^f compatible with \mathcal{G}^s with a directed $X_{t-1} \rightarrow Y_{t-1} \rightarrow \dots \rightarrow V_{t-1}^{n-1} \rightarrow Y_t$. Then, by Property 1, $P(y_t | do(x_{t-1}))$ is not identifiable. \square

Now, we turn to Lemmas 5-7, that show that conditions in Theorem 2 are sufficient. To do so, we consider the following set:

$$\begin{aligned} \mathcal{A}_\gamma = & \{(Z_{t-\gamma-\ell})_{1 \leq \ell \leq \gamma_{\max}} | Z \in \text{Desc}(X; \mathcal{G}^s)\} \\ & \cup \{(Z_{t-\gamma-\ell})_{0 \leq \ell \leq \gamma_{\max}} | Z \in \mathcal{V}^s \setminus \text{Desc}(X, \mathcal{G}^s)\} \end{aligned}$$

and we prove that it is an adjustment set in each case. As one can note, it contains all possible parents of $X_{t-\gamma}$ in all FTCTGs \mathcal{G}^f compatible with \mathcal{G}^s . Thus, \mathcal{A}_γ blocks any back-door path π between $X_{t-\gamma}$ and Y_t in any FTCTG \mathcal{G}^f compatible with \mathcal{G}^s through the parent of $X_{t-\gamma}$ on that path (which is the first node on the path from $X_{t-\gamma}$ to Y_t), which cannot be a collider.

In addition to ambiguous vertices, one can also define ambiguous paths, as follows.

Definition 7 (Ambiguous paths). Consider an SCG \mathcal{G}^s and an FTCTG \mathcal{G}^f compatible with \mathcal{G}^s . A path $\pi^f \in \mathcal{G}^f$ between $X_{t-\gamma}$ and Y_t , for $\gamma \geq 0$, is an ambiguous path if it does not contain any vertex at time $t - \gamma - \ell$ for $\ell \geq 1$. We note Π_γ^f the set of all ambiguous paths in \mathcal{G}^f .

When π is not in Π_γ^f , then the parent of $X_{t-\gamma}$ is in the past of $X_{t-\gamma}$ and thus cannot be ambiguous. One thus has the following property:

Property 2. Consider an SCG \mathcal{G}^s and the total effect $P(y_t | do(x_{t-\gamma}))$, for $\gamma \geq 0$. Suppose π^f is a back-door path between $X_{t-\gamma}$ and Y_t in a given FTCTG \mathcal{G}^f compatible with \mathcal{G}^s . If $\pi^f \notin \Pi_\gamma^f$, then π^f is blocked by a subset of \mathcal{A}_γ containing at least one non-ambiguous vertex.

We now introduce the notion of compatible path.

Definition 8 (Compatible path). Consider an SCG \mathcal{G}^s , an FTCTG \mathcal{G}^f compatible with \mathcal{G}^s , and the total effect $P(y_t | do(x_{t-\gamma}))$, for $\gamma \geq 0$. We say that a path $\pi^f = \langle X_{t-\gamma}, W_{t_2}^2, \dots, W_{t_{m-1}}^{m-1}, Y_t \rangle$ in \mathcal{G}^f is compatible with a path $\pi^s = \langle X, V^2, \dots, V^{n-1}, Y \rangle$ in \mathcal{G}^s if for all $(W_{t_j}^j)_{2 \leq j \leq m-1}$: either $W^j \in \langle V^2, \dots, V^{n-1} \rangle$ or $\exists V \in \langle V^2, \dots, V^{n-1} \rangle$ such that $W^j \in \text{Cycles}(V, \mathcal{G}^s) \setminus \text{Cycles}(X, \mathcal{G}^s)$.

The following property characterizes any back-door path in an FTCTG compatible with the SCG through graphical conditions on the SCG.

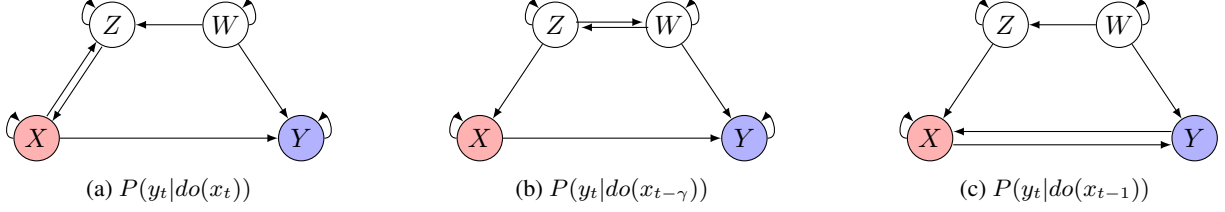


Figure 4: Three SCGs and a total effect which is identifiable. Each pair of red and blue vertices in the FTCGs represents the total effect we are interested in, and we precise the total effect and the lag considered in the caption. This illustrates Lemma 6 (Figure a-b) and Lemma 7 (Figure c).

Property 3. Consider an SCG \mathcal{G}^s and the total effect $P(y_t | do(x_{t-\gamma}))$ for $\gamma \geq 0$. Then (i) \Rightarrow (ii), where:

- (i) $\gamma = 0$ or $Cycles^>(X, \mathcal{G}^s \setminus \{Y\}) = \emptyset$,
- (ii) in an FTCG \mathcal{G}^f compatible with \mathcal{G}^s , there exists no back-door path $\pi^f \in \Pi_\gamma^f$ that is not compatible with any back-door path in \mathcal{G}^s .

Proof. Assume first $Cycles^>(X, \mathcal{G}^s \setminus \{Y\}) = \emptyset$. Suppose $\exists \pi^f = X_{t-\gamma} \leftarrow W_{t-\gamma} \cdots \rightarrow Y_t \in \Pi_\gamma^f$ which is a back-door path between $X_{t-\gamma}$ and Y_t that is not compatible with any back-door path $\pi^s = \langle V^1 = X, V^2, \dots, V^{n-1}, V^n = Y \rangle$ in \mathcal{G}^s .

If $n = 2$, then the path compatible with the cycle $\langle X, X \rangle$ is of the form $X_{t-\gamma} \rightarrow X_{t-\gamma+i} \rightarrow \dots \rightarrow X_{t-\gamma+j} \rightarrow Y_t$: it means that π^f cannot be a back-door path.

If $n > 2$, $W_{t-\gamma}$ is such that $W \notin \{V^2, \dots, V^{n-1}\}$ and $\nexists V \in \{V^2, \dots, V^{n-1}\}$ such that $W \in Cycles(V, \mathcal{G}^s)$. If the path between $W_{t-\gamma}$ and Y_t does not pass by $X_{t-\gamma+\ell}$ with $\ell > 0$, then there exists a back-door path between X and Y passing by W in \mathcal{G}^s as π^f lies in an FTCG compatible with \mathcal{G}^s , which contradicts our assumption. So the path necessarily passes by $X_{t-\gamma+\ell}$. Thus there is a cycle C_x on X such that $size(C_x) > 2$, which leads again to a contradiction. Thus, there does not exist a back-door path $\pi^f \in \Pi_\gamma^f$ between $X_{t-\gamma}$ and Y_t that is not compatible with any back-door path in \mathcal{G}^s .

The case $\gamma = 0$ is treated in the same way, with the fact that the path considered cannot go back to X_t as this would create a cycle in the FTCG. \square

In the following, we prove the lemmas that correspond to the identifiability cases.

Lemma 5. Consider an SCG \mathcal{G}^s , $\gamma \geq 0$ fixed and the total effect $P(y_t | do(x_{t-\gamma}))$. If $X \notin Anc(Y, \mathcal{G}^s)$ then $P(y_t | do(x_{t-\gamma}))$ is identifiable.

Proof. If $X \notin Anc(Y, \mathcal{G}^s)$, then in every FTCG \mathcal{G}^f compatible with \mathcal{G}^s , $X_{t-\gamma} \notin Anc(Y_t, \mathcal{G}^f)$. Thus, $P(y_t | do(x_{t-\gamma}))$ is always identifiable in \mathcal{G}^s , and $P(y_t | do(x_{t-\gamma})) = P(y_t)$. \square

Lemma 6. Consider an SCG \mathcal{G}^s , $\gamma \geq 0$ fixed and the total effect $P(y_t | do(x_{t-\gamma}))$. If $X \in Anc(Y, \mathcal{G}^s)$ and

- (i) either $\gamma = 0$ or $Cycles^>(X, \mathcal{G}^s \setminus \{Y\}) = \emptyset$ and
- (ii) $\nexists \sigma$ -active back-door path $\pi^s = \langle V^1 = X, \dots, V^n = Y \rangle$ from X to Y in \mathcal{G}^s such that $\langle V^2, \dots, V^{n-1} \rangle \subseteq Desc(X, \mathcal{G}^s)$,

then $P(y_t | do(x_{t-\gamma}))$ is identifiable.

This lemma is illustrated in Figure 4a - 4b.

Proof. We will prove that A_γ is an adjustment set for $P(y_t | do(x_{t-\gamma}))$ in any FTCG compatible with \mathcal{G}^s under conditions (i) and (ii). Let \mathcal{G}^f be an FTCG compatible with \mathcal{G}^s , and Π_γ^f the set of ambiguous paths. By Property 2, any back-door path $\pi^f \notin \Pi_\gamma^f$ can be blocked by A_γ . Furthermore, by definition, elements of A_γ cannot block any directed path between $X_{t-\gamma}$ and Y_t and cannot be descendants of Y_t as otherwise they would be descendant of $X_{t-\gamma}$.

We now turn our attention to paths in Π_γ^f . Let $\pi^f \in \Pi_\gamma^f$ be a back-door path between $X_{t-\gamma}$ and Y_t . Since $\gamma = 0$ or $\text{Cycles}^>(X, \mathcal{G}^s \setminus \{Y\}) = \emptyset$ then by Property 3, all back-door paths in Π_γ^f are compatible with back-door paths in \mathcal{G}^s . Let $\pi^s = \langle V^1 = X, \dots, V^n = Y \rangle$ be a σ -active back-door path in \mathcal{G}^s compatible with π^f . By (ii), there exists $m \geq 1$ vertices such that $\{V^{i_1}, \dots, V^{i_m}\} \subseteq \langle V^2, \dots, V^{n-1} \rangle$ and $\{V^{i_1}, \dots, V^{i_m}\} \not\subseteq \text{Desc}(X, \mathcal{G}^s)$. Then, $\forall V_{t-\gamma}$ such that $V \in \{V^{i_1}, \dots, V^{i_m}\}$, $V_{t-\gamma} \notin \text{Desc}(X_{t-\gamma}, \mathcal{G}^f)$ and since $X \in \text{Anc}(Y, \mathcal{G}^s)$ then it must be the case that $V \notin \text{Desc}(Y, \mathcal{G}^s)$ and by consequence $V_{t-\gamma} \notin \text{Desc}(Y_t, \mathcal{G}^f)$. Thus, $V_{t-\gamma}$ cannot be an ambiguous vertex. Its parent in π^f furthermore blocks π^f , is not ambiguous (as otherwise $V_{t-\gamma}$ would be ambiguous) and is a member of \mathcal{A}_γ by definition of \mathcal{A}_γ . Thus \mathcal{A}_γ blocks all back-door paths between $X_{t-\gamma}$ and Y_t in any \mathcal{G}^f compatible with \mathcal{G}^s . Furthermore, no node in \mathcal{A}_γ can block a directed path between $X_{t-\gamma}$ and Y_t or is a descendant of Y_t as nodes in \mathcal{A}_γ are either defined before $t - \gamma$ or are not descendant of $X_{t-\gamma}$, and thus of Y_t . This concludes the proof. \square

Lemma 7. Consider an SCG \mathcal{G}^s and the total effect $P(y_t \mid \text{do}(x_{t-1}))$ ($\gamma = 1$). If the only σ -active back-door path from X to Y in \mathcal{G}^s is $X \rightleftharpoons Y \in \mathcal{G}^s$ and

(i) $\text{Cycles}(Y, \mathcal{G}^s \setminus \{X\}) = \emptyset$ and

(ii) $\text{Cycles}^>(X, \mathcal{G}^s \setminus \{Y\}) = \emptyset$,

then $P(y_t \mid \text{do}(x_{t-1}))$ is identifiable.

This lemma is illustrated in Figure 4c.

Proof. We will prove that \mathcal{A}_1 is an adjustment set for $P(y_t \mid \text{do}(x_{t-1}))$ in any FTCTG compatible with \mathcal{G}^s under conditions (i) and (ii). Let \mathcal{G}^f be an FTCTG compatible with \mathcal{G}^s , and Π_1^f the set of ambiguous paths.

Since (ii) then by Property 3, all back-door paths in Π_1^f are compatible with back-door paths in \mathcal{G}^s . In addition, by Property 2, any path $\pi^f \notin \Pi_1^f$ can be blocked by \mathcal{A}_1 . Therefore, in the following, we focus on paths in Π_1^f compatible with back-door paths in \mathcal{G}^s .

Consider the σ -active back-door path $\pi^s = \langle X, Y \rangle$. As there cannot be a loop on Y by (i), the only path $\pi^f \in \Pi_1^f$ from X_{t-1} to Y_t compatible with π^s that pass by Y_{t-1} is $\pi^f = \langle X_{t-1}, Y_{t-1}, X_t, Y_t \rangle$. Then, under consistency throughout time, acyclicity and temporal priority, the only choices are $X_{t-1} \rightarrow Y_{t-1} \rightarrow X_t \rightarrow Y_t$ and $X_{t-1} \leftarrow Y_{t-1} \rightarrow X_t \leftarrow Y_t$. The first is a directed path, the second a back-door path already blocked due to the collider $Y_{t-1} \rightarrow X_t \leftarrow Y_t$. Thus, all potential back-door paths between X_{t-1} and Y_t in any \mathcal{G}^f compatible with \mathcal{G}^s are blocked, and \mathcal{A}_1 does not activate them. \square

The above lemmas conclude the proof of Theorem 2. When the total effect is identifiable, \mathcal{A}_γ is an adjustment set, but it has a large size, so we provide a smaller adjustment set, defined as follows:

$$\mathcal{A}'_\gamma = \{V_{t'} \in \mathcal{A}_\gamma \mid V \in \text{Anc}(X, \mathcal{G}^s) \cup \text{Anc}(Y, \mathcal{G}^s)\}.$$

Proposition 2. Consider an SCG \mathcal{G}^s and the total effect $P(y_t \mid \text{do}(x_{t-\gamma}))$, with $\gamma \geq 0$. Under assumptions of identifiability provided by Theorem 2, the set \mathcal{A}'_γ is an adjustment set for the total effect.

Proof. Let \mathcal{G}^f be an FTCTG compatible with \mathcal{G}^s . Consider $V_{t'} \in \mathcal{A}_\gamma \setminus \mathcal{A}'_\gamma$: by definition of \mathcal{A}'_γ , it follows that $V_{t'} \notin \text{Anc}(X_{t-\gamma}, \mathcal{G}^f) \cup \text{Anc}(Y_t, \mathcal{G}^f)$. Therefore $V_{t'}$ does not lie on any back-door path between $X_{t-\gamma}$ and Y_t : $V_{t'}$ is not necessary in the adjustment set, confirming that \mathcal{A}'_γ is also an adjustment set. \square

5 Conclusion

In this paper, we showed that the total effect is always identifiable for extended summary causal graphs (ECGs) but not always for summary causal graphs (SCGs). Additionally, we established necessary and sufficient graphical criteria for the identifiability of the total effect in SCGs, despite the potential presence of cycles.

These results have significant implications, such as impact analysis in dynamic systems, particularly in scenarios where experts are unable to provide either a full temporal causal graph (FTCTG) or an ECG. They are also valuable in cases where the assumptions underlying causal discovery methods for inferring causal graphs with time lags are deemed overly restrictive.

We have provided a set of adjustment sets for estimating the total effect in ECGs, and an adjustment set when considering SCGs. However, those sets can potentially be large. While the estimation of the total effect based on such adjustments remains unbiased, there is a possibility of high variance. Determining the optimal adjustment set, that has minimal asymptotic estimation variance, remains an open question in this context.

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