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WHAT MEANS “SLIGHTLY SUPERCRITICAL” IN NAVIER-STOKES EQUATIONS?

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ABSTRACT. The goal of this paper is to clarify some differences between the critical Lebesgue space L^3 and the critical weak Lebesgue space $L^{3,\infty}$, when these are considered in the hypothesis of classical statements for the 3D homogeneous incompressible Navier-Stokes equations.

1. INTRODUCTION

Recent interesting technical progress in the analysis of the velocities solving the Navier-Stokes equations appear associated to critical hypothesis. Let us fix some notions. The homogeneous incompressible Navier-Stokes equations

$$(NS) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p \\ \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}(0, \cdot) = \mathbf{u}_0, \end{cases} \quad (1.1)$$

satisfy the following scaling property : If (\mathbf{u}, p) is a solution on $\mathbb{R}^3 \times [0, T)$ with initial velocity \mathbf{u}_0 , then defining

$$\mathbf{u}_\lambda(x, t) := \lambda \mathbf{u}(\lambda x, \lambda^2 t), \quad p_\lambda(x, t) := \lambda^2 p(\lambda x, \lambda^2 t) \quad \text{and} \quad \mathbf{u}_{0,\lambda} = \lambda \mathbf{u}_0(\lambda \cdot),$$

we obtain $(\mathbf{u}_\lambda, p_\lambda)$ is a solution on $\mathbb{R}^3 \times [0, T/\lambda^2)$ with initial data $\mathbf{u}_{0,\lambda}$.

A norm or quasi-norm $\|\cdot\|_{X_T}$, on the time-dependent vector fields with domain $\mathbb{R}^3 \times [0, T)$, is called critical (for the velocity) if for all time-dependent vector field \mathbf{u} on $\mathbb{R}^3 \times [0, T)$, for all $\lambda > 1$, $\|\mathbf{u}_\lambda \mathbf{1}_{\mathbb{R}^3 \times [0, T/\lambda^2)}\|_{X_T} = \|\mathbf{u}\|_{X_T}$. Restricting our attention to the initial data, we say that a norm or semi-norm $\|\cdot\|_{X_0}$ defined on the vector fields with domain \mathbb{R}^3 , is critical if for all vector field \mathbf{u}_0 on \mathbb{R}^3 , for all $\lambda > 0$, $\|\mathbf{u}_{0,\lambda}\|_{X_0} = \|\mathbf{u}_0\|_{X_0}$.

The most classical results concerning strong solutions of (NS) are limited by critical hypothesis (we refer to [10], [20] and [12]). For example, the local existence of mild solutions with initial data in the critical Lebesgue space L^3 is known (see [10], [15] and [13]), but in the case of initial data belonging to the critical weak Lebesgue space $L^{3,\infty}$, smallness conditions are required:

$$\|\mathbf{u}\|_{L^{3,\infty}(\mathbb{R}^3)} := \sup_{\alpha > 0} \alpha \left[\mu \left(\{x \in \mathbb{R}^3 : |\mathbf{u}(x)| > \alpha\} \right) \right]^{1/3} < \epsilon_{\text{univ}}, \quad \text{being } \mu \text{ the Lebesgue measure.}$$

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Regularity of Leray solutions belonging to $L^\infty((0, T)L^3)$ has been proved (see [8], [18], [19], [22]), however under the “slightly” weaker hypothesis $L^\infty((0, T)L^{3,\infty})$, regularity is an open problem when the solution is not axisymmetric, we refer to [2].

In attempting to deal with supercritical hypotheses, the use of quantitative arguments to study concentration of some frequencies in presence of a singularity, permits to precise a blow-up rate slightly supercritical “with respect to the L^3 norm”. The idea is due to Tao in [21], who proved that for a finite energy solution \mathbf{v} with viscosity $\nu = 1$, which first blows up at T^* , we have for some absolute constant $c > 0$,

$$\limsup_{t \uparrow T^*} \frac{\|\mathbf{v}(\cdot, t)\|_{L^3(\mathbb{R}^3)}}{\left(\log \log \log \frac{1}{T^* - t}\right)^c} = +\infty. \quad (1.2)$$

A quantitative strategy has been established in the local setting, considering type I conditions $\|\mathbf{v}\|_{L^\infty((0, T^*), L^{3,\infty})} < +\infty$, by Barker and Prange in [5], and also by Kang, Miura and Tsai in [11]. Barker and Prange demonstrate under type I conditions that

$$\|\mathbf{v}(\cdot, t)\|_{L^3(B_{x_0}(\delta))} \geq C_M \log \left(\frac{1}{T^* - t} \right), \quad \text{where } \delta = O((T^* - t)^{\frac{1}{2}-}),$$

based on localized smoothing estimates in [4] while Kang, Miura and Tsai propose a direct approach in paper [11]. Very recently, in [3] Barker shows how to quantify the main result in [8]. He proves for δ small enough,

$$\limsup_{t \uparrow T^*} \frac{\|\mathbf{v}(\cdot, t)\|_{L^3(B_{x_0}(\delta))}}{\left(\log \log \log \frac{1}{(T^* - t)^{\frac{1}{4}}}\right)^{\frac{1}{1129}}} = +\infty,$$

improving thus (1.2). All these advances have a “supercritical” appearance however a less optimistic and more accurate reading would be : these advances are slightly supercritical **with respect to the L^3 norm**. With respect to the $L^{3,\infty}$ norm, these results have not a slightly supercritical relationship. Although $L^{3,\infty}$ is still a critical space, analogous results to those cited above, using $L^{3,\infty}$ instead of L^3 , have not been demonstrated, except when conditions of smallness or axisymmetry are imposed. Thus, $L^{3,\infty}$ type conditions for the profiles, and $L^\infty((0, T), L^{3,\infty})$ type conditions in space-time, represent a big obstacle for the understanding of the Navier-Stokes equations.

To reinforce our opinion on the local well behavior of Navier-Stokes solutions, considering L^3 norms in the hypothesis, we will revisit some results.

2. REMOVING SMALLNESS OF THE L^3 NORM IN LOCALIZED SMOOTHING FOR THE NAVIER-STOKES EQUATIONS

In which concerns localized smoothing, we revisit Theorem 1 in [4] where a local smallness hypothesis of L^3 nature appear. To remove this smallness hypothesis the price to be paid is roughly the non-quantification of this result. Let us consider the definitions and notations in [4] (almost standard), in particular consider the constant γ_{univ} described in Theorem 1 in [4]. Our main result reads as follow :

Theorem 2.1. *Let \mathbf{u}_0 be a divergence free vector field belonging to $L^2_{\text{uloc}}(\mathbb{R}^3)$. Consider a compact set $K \subset \mathbb{R}^3$. We suppose that for each $x \in K$, there exists $\lambda_x > 0$, such that $\int_{B_{2\lambda_x}(x)} |\mathbf{u}_0|^3 dy < \gamma_{\text{univ}}^3$ and $\|\lambda_x \mathbf{u}_0(\lambda_x \cdot + x)\|_{L^2_{\text{uloc}}} < +\infty$.*

Suppose \mathbf{u} to be a local energy solution of (NS) with initial data \mathbf{u}_0 (Definition 16 in [4]). Then, there exists $T = T_{\mathbf{u}_0, K}$ such that $\mathbf{u} \in L^\infty(K \times (0, T))$.

Proof We will apply a rescaling argument. Let $x \in K$. We take λ_x such that

$$\int_{B_{2\lambda_x}(x)} |\mathbf{u}_0|^3 dy < \gamma_{\text{univ}}^3$$

and $\|\lambda_x \mathbf{u}_0(\lambda_x \cdot + x)\|_{L^2_{\text{uloc}}} < +\infty$. We have

$$\int_{B_{2\lambda_x}(x)} |\mathbf{u}_0(y)|^3 dy = \int_{B_2(0)} |\lambda_x \mathbf{u}_0(\lambda_x y + x)|^3 dy$$

and $\mathbf{u}_{\lambda_x, x}(t, y) := \lambda_x \mathbf{u}(\lambda_x^2 t, \lambda_x y + x)$ is a solution of the Navier Stokes equations with initial data $\mathbf{u}_{0, \lambda_x, x}(y) = \lambda_x \mathbf{u}_0(\lambda_x y + x)$.

Let us denote $M_{\mathbf{u}_0, x} := \|\mathbf{u}_{0, \lambda_x, x}\|_{L^2_{\text{uloc}}}$. By Theorem 1 in [4], we conclude that there exists $S^*(M_{\mathbf{u}_0, x})$ such that

$$\mathbf{u}_{\lambda_x, x} \in L^\infty(B_{\frac{1}{3}}(0) \times (\beta, S^*(M_{\mathbf{u}_0, x})))$$

for all $\beta \in (0, S^*(M_{\mathbf{u}_0, x}))$. It follows that

$$\mathbf{u} \in L^\infty(B_{\frac{\lambda_x}{3}}(x) \times (\beta, \lambda_x^2 S^*(M_{\mathbf{u}_0, x})))$$

for all $\beta \in (0, \lambda_x^2 S^*(M_{\mathbf{u}_0, x}))$.

As $\{B_{\frac{\lambda_x}{3}}(x); x \in K\}$ cover the compact K , we can take a finite family $\{x_{(1)}, x_{(2)}, \dots, x_{(k)}\} \subset \mathbb{R}^3$ such that

$$\left\{ B_{\frac{\lambda_{x_{(i)}}}{3}}(x_{(i)}); i \in \{1, \dots, k\} \right\}$$

cover K . Then, we define

$$T = T(\mathbf{u}_0, K) = \min_{i \in \{1, \dots, k\}} \lambda_{x_{(i)}}^2 S^*(M_{\mathbf{u}_0, x_{(i)}})$$

so that $\mathbf{u} \in L^\infty(K \times (\beta, T))$, for all $\beta \in (0, T)$.

Remark 2.1. *If K is not compact, to make this proof works, when we take for each $x \in K$ the positive number λ_x such that*

$$\int_{B_{2\lambda_x}(x)} |\mathbf{u}_0|^3 dy < \gamma_{\text{univ}}^3,$$

we need at the same time $\inf_{x \in K} \lambda_x^2 S^(M_{\mathbf{u}_0, x})$ to be a positive number. So, we can impose as hypothesis λ_x to be bounded from below and $M_{\mathbf{u}_0, x}$ to be bounded from above. Then the following Corollary follows immediately from the proof of Theorem 2.1:*

Corollary 2.1. *Let \mathbf{u}_0 be a divergence free vector field belonging to $L^2_{\text{uloc}}(\mathbb{R}^3)$. We assume that*

$$\sup_{x \in \mathbb{R}^3} \sup_{\lambda > 0} \|\lambda \mathbf{u}_0(\lambda \cdot + x)\|_{L^2_{\text{uloc}}} < +\infty \quad (2.1)$$

and there exists $\underline{\lambda} > 0$ such that for each $x \in \mathbb{R}^3$, there exists $\lambda_x > \underline{\lambda}$, such that

$$\int_{B_{2\lambda_x}(x)} |\mathbf{u}|^3 dy < \gamma_{\text{univ}}^3.$$

Suppose \mathbf{u} to be a local energy solution of the Navier-Stokes equations with initial data \mathbf{u}_0 . Then, there exists $T = T_{\mathbf{u}_0}$ such that $\mathbf{u} \in L^\infty(\mathbb{R}^3 \times (0, T))$.

Remark 2.2. We observe that (2.1) is satisfied by the functions in the critical Morrey space $\dot{M}^{2,3}$, which contains $L^{3,\infty}$, where the norm is given by

$$\|f\|_{\dot{M}^{2,3}} = \sup_{x \in \mathbb{R}^3} \sup_{R > 0} \left(\frac{1}{R} \int_{|y-x| < R} |f|^2 dy \right)^{1/2}.$$

This remark is also used for exemple in [13]. Observe that we could use Theorem 1.1 in [11] in order to prove Theorem 3.1.

3. REMOVING BACKWARD DISCRETELY SELF-SIMILAR SINGULARITIES

In this section we want to show how scaling arguments permit to remove easily L^3 profiles in backward discretely selfsimilar singularities. We will remove even $L^{3,q}(\mathbb{R}^3)$ profiles with $1 \leq q < +\infty$, where we consider the Lorentz spaces $L^{p,q}$, for $1 \leq p, q < +\infty$, on measurable sets $\Omega \subset \mathbb{R}^3$, defined by the quantity

$$\|f\|_{L^{p,q}(\Omega)} = \left(p \int_0^{+\infty} \alpha^q d_{f,\Omega}(\alpha)^{q/p} \frac{d\alpha}{\alpha} \right)^{1/q}$$

where,

$$d_{f,\Omega}(\alpha) = \mu(\{x \in \Omega : |f(x)| > \alpha\}),$$

denoting μ the Lebesgue measure. That permits to precise some statements of recent results as Theorem 2 in [6] or Corollary 1.1 in [5].

We start by remember the definition of the λ -discretely self-similarity (see [7, 9]):

Definition 3.1. Let $\lambda > 1$.

- A vector field $\mathbf{u}_0 \in L_{\text{loc}}^2(\mathbb{R}^3)$ is λ -discretely self-similar (λ -DSS) if $\lambda \mathbf{u}_0(\lambda x) = \mathbf{u}_0(x)$.
- A time dependent vector field $\mathbf{u} \in L_{\text{loc}}^2((-\infty, 0) \times \mathbb{R}^3)$ is λ -DSS if $\lambda \mathbf{u}(\lambda^2 t, \lambda x) = \mathbf{u}(t, x)$.
- A pressure $p \in L_{\text{loc}}^1((-\infty, 0) \times \mathbb{R}^3)$ is λ -discretely self-similar if $\lambda^2 p(\lambda^2 t, \lambda x) = p(t, x)$.

In the next statement, the notion of suitable Leray solution admits the standard definitions, including infinite energy solutions, we can consider for example the definition in [5], [6], [9], [14] or [15].

Theorem 3.1. Let $\lambda \in (1, +\infty)$ and let \mathbf{u}_0 be a non trivial divergence-free vector field which is λ -DSS. Suppose, there exists a backward λ -DSS suitable Leray solution \mathbf{u} of (NS) on $(-\infty, 0] \times \mathbb{R}^3$. Then, for all $t_0 \in (-\infty, 0)$ we have,

$$\liminf_{n \rightarrow \infty} \int_{B_1 \setminus B_{\lambda^{-1}}} |\lambda^n \mathbf{u}(\lambda^n x, t_0)| > 0. \quad (3.1)$$

Remark 3.1. Observe that condition (3.1) implies the profile in negative times can't belong to $L^{3,q}$, with $1 \leq q < \infty$. In fact, by change of variables and by Hölder inequality (for Lorentz spaces),

$$\int_{B_1 \setminus B_{\lambda^{-1}}} |\lambda^n \mathbf{u}(\lambda^n x, t_0)| dx = \frac{1}{\lambda^{2n}} \int_{B_{\lambda^n} \setminus B_{\lambda^{n-1}}} |\mathbf{u}(y, t_0)| dy \quad (3.2)$$

$$\leq \frac{1}{\lambda^{2n}} \|\mathbf{1}_{B_{\lambda^n} \setminus B_{\lambda^{n-1}}}\|_{L^{\frac{3}{2}, \frac{q}{q-1}}} \|\mathbf{u}(\cdot, t_0)\|_{L^{3,q}(B_{\lambda^n} \setminus B_{\lambda^{n-1}})}, \quad (3.3)$$

and since $\|\mathbf{1}_{B_{\lambda^n} \setminus B_{\lambda^{n-1}}}\|_{L^{\frac{3}{2}, \frac{q}{q-1}}} \leq C\lambda^{2n}$, we find

$$\liminf_n \int_{B_1 \setminus B_{\lambda^{-1}}} |\lambda^n \mathbf{u}(\lambda^n x, t_0)| \leq C \liminf_n \|\mathbf{u}(\cdot, t_0)\|_{L^{3,q}(B_{\lambda^n} \setminus B_{\lambda^{n-1}})}. \quad (3.4)$$

By other hand, since $\mathbf{u}(t_0) \in L^{3,q}$, with $1 \leq q < \infty$, using the definition of this Lorentz space and the monotone convergence, we can verify that $\lim_{n \rightarrow \infty} \|\mathbf{u}(\cdot, t_0)\|_{L^{3,q}(\mathbb{R}^3 \setminus B_{\lambda^{n-1}})} = 0$.

In terms of L^2 norms, we have (3.1) also implies

$$\liminf_{R \rightarrow \infty} \frac{1}{R} \int_{B_R} |\mathbf{u}(\cdot, t_0)|^2 > 0.$$

In fact, by the Hölder inequality

$$\liminf_{n \rightarrow \infty} \int_{B_1 \setminus B_{\lambda^{-1}}} |\lambda^n \mathbf{u}(\lambda^n x, t_0)| = \liminf_{n \rightarrow \infty} \frac{1}{\lambda^{2n}} \int_{B_{\lambda^n} \setminus B_{\lambda^{n-1}}} |\mathbf{u}(x, t_0)| \leq \liminf_{n \rightarrow \infty} \frac{c}{\lambda^n} \int_{B_{\lambda^n} \setminus B_{\lambda^{n-1}}} |\mathbf{u}(x, t_0)|^2.$$

Proof of Theorem 3.1 : Suppose there exists $t_0 \in (-\infty, 0)$ such that

$$\liminf_{n \rightarrow \infty} \int_{B_1 \setminus B_{\lambda^{-1}}} |\lambda^n \mathbf{u}(\lambda^n x, t_0)| = 0. \quad (3.5)$$

We will demonstrate $\mathbf{u}(\cdot, 0) = 0$. It is sufficient to show that for almost all $x \in B_1 \setminus B_{\lambda^{-1}}$, $\mathbf{u}(x, 0) = \lim \mathbf{u}(x, t_n) = 0$. Then, by discretely selfsimilarity of the initial profile we obtain $\mathbf{u}(\cdot, 0) = 0$.

By interior estimates of regularity (see [19] and [22]) we know

$$\mathbf{u} \in \mathcal{C}([-1, 0], L^1(B_1)). \quad (3.6)$$

Consider the sequence $t_n = t_0/\lambda^{2n}$. By λ -DSS property we get

$$\lambda^n \mathbf{u}(\lambda^n x, t_0) = \mathbf{u}(x, \frac{1}{\lambda^{2n}} t_0). \quad (3.7)$$

By (3.7) and (3.5) we can extract a subsequence of $\mathbf{u}(\cdot, \frac{1}{\lambda^{2n}} t_0)$ converging to zero pointwise. We so get, up to a subsequence, for almost all $x \in B_1 \setminus B_{\lambda^{-1}}$, $\mathbf{u}(x, 0) = \lim \mathbf{u}(x, t_n) = 0$ by (3.6).

Corollary 3.1. Suppose that \mathbf{u} defined on $(-\infty, 0] \times \mathbb{R}^3$ is a suitable backward discretely selfsimilar solution such that

$$\|\mathbf{u}\|_{L^\infty((-1,0), L^{3,\infty})} < +\infty$$

and there exists $t_0 < 0$ such that $\mathbf{u}(t_0) \in L^{3,q}$, with $3 \leq q < +\infty$, then $\mathbf{u} = 0$.

It is a direct consequence of Theorem 3.1 and Theorem 4.1 in [1] (a Liouville type theorem).

DATA AVAILABILITY STATEMENT : Data sharing isn't applicable to this article as no datasets were generated or analyzed during the current study.

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