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# DEVIATION RESULTS FOR MANDELBROT'S MULTIPLICATIVE CASCADES WITH (STRETCHED) EXPONENTIAL TAILS

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## Abstract

Let  $W$  be a nonnegative random variable with expectation 1. For all  $r \geq 2$ , we consider the total mass  $Z_r^\infty$  of the associated Mandelbrot multiplicative cascade in the  $r$ -ary tree. For all  $n \geq 1$ , we also consider the total mass  $Z_r^n$  of the measure at height  $n$  in the  $r$ -ary tree. Liu, Rio, Rouault [15, 16, 22] established large deviation results for  $(Z_r^n)_{r \geq 2}$  for all  $n \in \llbracket 1, \infty \rrbracket$  (resp. for  $n = \infty$ ) in the case  $W$  has an everywhere finite cumulant generating function  $\Lambda_W$  (resp.  $W$  is bounded). Here, we extend these results to the case that  $\Lambda_W$  is only assumed finite on a neighborhood of zero, and even to the case that  $W$  has a stretched exponential tail. In addition, we study deviations of all orders. It is noticeable that we obtain recursive definitions of rate functions and that we resort to the moments bound instead of the standard Chernoff bound to establish the upper bounds of deviation in the infinite tree.

**Keywords:** Mandelbrot multiplicative cascades, large deviation principles, moments bound, stretched exponential tails.

**AMS MSC 2010:** 60F10; 60G57; 60J80.

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## 1 Introduction

Multiplicative cascades were introduced by Mandelbrot [18, 17] in order to analyze some problems of turbulence precisely. If  $r \geq 2$  is an integer, a multiplicative cascade is a random measure  $\mu_r^\infty$  on the unit interval, defined as a limit of measures  $\mu_r^n$  on the sub- $\sigma$ -algebra generated by the  $r$ -adic intervals of level  $n \geq 1$ . The study of the  $\mu_r^n$ 's generated a huge literature, notably about the multifractal dimensions of  $\mu_r^\infty$  (see e.g. Holley and Waymire [11] and Barral [2]). It appears that a natural and fundamental object is the total mass  $Z_r^n := \mu_r^n([0, 1])$  of the measure  $\mu_r^n$ , which can be described as follows. Let  $W$  be a nonnegative random variable such that  $\mathbb{E}[W] = 1$ . Now, let  $(W_{i_1, \dots, i_n})_{n \geq 1, 1 \leq i_1, \dots, i_n \leq r}$  be a family of independent and identically distributed random variables distributed as  $W$  indexed by all finite sequences of integers between 1 and  $r$ . Then we define, for all  $n \geq 1$ ,

$$Z_r^n := \frac{1}{r^n} \sum_{1 \leq i_1, \dots, i_n \leq r} W_{i_1} W_{i_1, i_2} \cdots W_{i_1, \dots, i_n} \quad (1)$$

and we set  $Z_r^0 := 1$ . For fixed  $r$ , the properties of  $Z_r^n$  were studied in several works. First, Kahane and Peyrière [12] showed that  $(Z_r^n)_{n \geq 1}$  is a nonnegative martingale with expectation 1 and that the properties of the limit  $Z_r^\infty$  were characterized by the behavior of the moments of  $W$  and the quantity  $\mathbb{E}[W \log(W)]$ . In particular,  $\mathbb{E}[Z_r^\infty] \leq 1$  and it is proved in [12, Théorème 1] that  $\mathbb{E}[Z_r^\infty] = 1$  if and only if  $\mathbb{E}[W \log(W)] < \log(r)$ . Moreover the distribution of  $Z_r^\infty$  is solution of the distributional equation

$$Z \stackrel{(d)}{=} \frac{1}{r} \sum_{i=1}^r W_i Z_i \quad (2)$$

where the  $Z_i$ 's are independent copies of  $Z$ , also independent of  $(W_i)_{1 \leq i \leq r}$ . This equation has been studied by Durrett and Liggett [7] and by Guivarc'h [10], and is closely related to implicit renewal theory (see [9]).

Now for  $n \in \llbracket 1, \infty \rrbracket^1$ , the asymptotic behavior of  $(Z_r^n)_{r \geq 2}$  when  $r$  goes to infinity was studied by Liu, Rio, and Rouault [15, 16, 22]. In [15, 16], the authors studied  $(Z_r^n)_{r \geq 2}$  and obtained the law of large numbers and the central limit theorem under minimal assumptions. They also provided large deviation results under strong assumptions on the tail of the variable  $W$ . Let

$$\Lambda_W(t) := \log \mathbb{E}[e^{tW}] \in (-\infty, \infty]$$

be the cumulant generating function of  $W$  and let

$$\Lambda_W^*(x) := \sup_{t \in \mathbb{R}} [tx - \Lambda_W(t)] \in [0, \infty]$$

be the Fenchel-Legendre transform of  $\Lambda_W$ . The following result is proved in [15, 16].

**Theorem 1** (very light tail – all trees – large deviations). *Let  $n \in \llbracket 1, \infty \rrbracket$ . Assume that one of the two following statements is true:*

1. (finite tree)  $n \in \llbracket 1, \infty \llbracket$  and  $\Lambda_W < \infty$  everywhere;
2. (infinite tree)  $n = \infty$  and  $\text{ess sup}(W) < \infty$ .

*Then, the sequence  $(Z_r^n)_{r \geq 2}$  satisfies a large deviation principle at speed  $(r)_{r \geq 2}$  with rate function  $\Lambda_W^*$ .*

It is worth noticing that, for unbounded random variables  $W$ , even when the cumulant generating function  $\Lambda_W$  is finite everywhere, we did not find the large deviation principle for  $n = \infty$  (infinite tree) in the literature. The left deviations, stated in the following proposition, essentially follow from Gärtner-Ellis theorem without any assumption on  $W$ .

---

<sup>1</sup>If  $(a, b) \in \mathbb{R}^2$ ,  $\llbracket a, b \rrbracket := \{n \in \mathbb{Z} ; a \leq n \leq b\}$  and, similarly, we define  $\llbracket a, b \llbracket$ , etc.

**Proposition 2** (all tails – all trees – left large deviations). *Let  $n \in \llbracket 1, \infty \rrbracket$ . If  $n = \infty$ , we assume that  $\mathbb{E}[W \log(W)] < \infty$ . For all  $a \in [0, 1]$ ,*

$$-\frac{1}{r} \log \mathbb{P}(Z_r^n \leq a) \xrightarrow[r \rightarrow \infty]{} \Lambda_W^*(a).$$

The result is proved in [16, Theorem 1.4] for  $n = \infty$  and  $a \in (0, 1)$ . It is also a consequence of [15, Theorem 6.2] for  $n \in \llbracket 1, \infty \rrbracket$  at every point of continuity  $a$  of  $\Lambda_W^*$ . There is no difficulty to extend the result as stated above (and we discuss this extension in Sections 3.2 and 4.3). The study of right deviations is more delicate. The case  $\text{ess sup}(W) < \infty$  is the case where the variables  $Z_r^n$  are bounded above (by  $\text{ess sup}(W)$ ) and Gärtner-Ellis theorem applies. However, if  $\text{ess sup}(W) = \infty$ , even if  $\Lambda_W < \infty$  everywhere,  $\mathbb{E}[e^{tZ_r^n}]$  may be infinite for all  $t > 0$  and Gärtner-Ellis theorem cannot provide the result in that case.

In this work, we extend these results to the case where  $\Lambda_W$  is allowed to take infinite values: not only do we complete Theorem 1, extending item 2 to the case where  $\Lambda_W < \infty$  everywhere, but also we study the case where  $\text{sup}(\text{dom}(\Lambda_W)) \in [0, \infty)$  and obtain results for  $W$  with an exponential or a stretched exponential tail. Besides, in each case, we obtain, for both finite and infinite trees, deviation estimates of all orders (see Theorems 4, 5, 7, 8, and 9). In [15], the key argument to obtain the large deviation result in the finite tree and with  $\Lambda_W < \infty$  everywhere (Theorem 1 with assumption 1) is to truncate the variable and use exponential approximation. In that case, the rate function is  $\Lambda_W^*$ : the large deviations are the same as those of the first level of the tree. In the case  $\Lambda_W$  admits infinite values, the rate function is not  $\Lambda_W^*$  anymore as soon as  $n \in \llbracket 2, \infty \rrbracket$ , so this argument seems hopeless. Moreover, the cumulant generating functions of the random variables  $Z_r^n$  are infinite on  $(0, \infty)$ , hence the exponential Markov's inequality cannot either be used here to prove the upper bounds. Note that, in the case where  $\Lambda_W < \infty$  everywhere, we do not know if the truncation argument works. Here, we use different techniques to bypass these problems. In the case of the finite tree, we decompose the event of deviations and control each term; this technique is standard to obtain upper bounds of large deviations for heavy-tailed random variables (see e.g., [19, 20]). In the case of the infinite tree, bounding the moments of  $Z_r^\infty$  appears to be the corner stone of our approach. For instance, when  $W$  has an exponential tail, the upper bound of moderate and very large deviations follows immediately from the moment Markov's inequality; as for the large deviations, we prove that (see Corollary 24)

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{E}[(Z_r^\infty)^{\eta r}] = O(\eta^2),$$

which is the key argument to prove that the rate function is non degenerated and is indeed the limit of the rate functions in finite trees (compare items 1 of Theorems 5 and 7).

The paper is organized as follows. The main theorems for both finite and infinite trees are stated in Section 2. Section 3 is dedicated to the proof of the deviation results for finite trees whereas the proofs for the infinite tree are given in Section 4. Finally, Section 5 contains the proof of Theorem 6, which precises the very large deviation estimates in the finite trees when  $W$  has a very light tail.

## 2 Main results

Proposition 2 easily generalizes to left deviations of all orders. The left very large deviations are trivial, since  $\mathbb{P}(Z_r^n - 1 \leq r^\alpha a) = 0$  for all  $\alpha > 0$ ,  $a < 0$ , and  $r$  large enough. As for the left moderate deviations, they are given in the following proposition.

**Proposition 3** (all tails – all trees – left moderate deviations). *Assume that  $\mathbb{E}[W^2] < \infty$ . Then, for all  $n \in \llbracket 1, \infty \rrbracket$ ,  $\alpha \in (-1/2, 0)$ , and  $a \leq 0$ ,*

$$-\frac{1}{r^{1+2\alpha}} \log \mathbb{P}(Z_r^n - 1 \leq r^\alpha a) \xrightarrow[r \rightarrow \infty]{} J(a) := \frac{a^2}{2 \text{Var}(W)}.$$

The main purpose of the paper is to study the right deviations of  $(Z_r^n - 1)_{r \geq 2}$ , which strongly depend on the right tail of  $W$ . First, we obtain the moderate deviations for **light-tailed** random variables  $W$ , i.e. satisfying  $\Lambda_W < \infty$  on some neighborhood of 0.

**Theorem 4** (light tails – all trees – right moderate deviations). *Assume that  $\Lambda_W < \infty$  on some neighborhood of 0. Then, for all  $n \in \llbracket 1, \infty \rrbracket$ ,  $\alpha \in (-1/2, 0)$ , and  $a \geq 0$ ,*

$$-\frac{1}{r^{1+2\alpha}} \log \mathbb{P}(Z_r^n - 1 \geq r^\alpha a) \xrightarrow[r \rightarrow \infty]{} J(a).$$

Together with Proposition 3 and the monotonicity properties of the rate function  $J$ , Theorem 4 implies a large deviation principle on  $\mathbb{R}_+$ : for  $\alpha \in (-1/2, 0)$ ,  $(r^{-\alpha}(Z_r^n - 1))_{r \geq 2}$  satisfies a large deviation principle at speed  $(r^{1+2\alpha})_{r \geq 2}$  with rate function  $J$ . Also note that, when  $\sup(\text{dom}(\Lambda_W)) < \infty$ , it is easy to prove that

$$\forall n \in \llbracket 2, \infty \rrbracket \quad \forall r \geq 2 \quad \forall t > 0 \quad \mathbb{E}[e^{tZ_r^n}] = \infty,$$

hence our proofs cannot be direct consequences of Gärtner–Ellis theorem. A few days before the submission of this paper, one of the authors talked to Q. Liu about the present work and Q. Liu brought to our attention that he had studied the moderate deviations, using similar tools as in [15]: he obtained the same result as us in the finite tree, but his approach only applies to bounded  $W$  in the infinite tree (his results are written in an unpublished paper [8]).

As for the other deviation results, it appears that obtaining matching lower and upper bounds relies on a finer estimate of the tail of  $W$ . In the sequel, we thus assume that there exists  $(\tau, c) \in (0, \infty) \times (0, \infty) \cup \{(1, \infty)\}$  such that

$$(\mathbf{A}_{\tau, c}) \quad \frac{1}{w^\tau} \log \mathbb{P}(W \geq w) \xrightarrow{w \rightarrow \infty} -c.$$

Since it is a general result that

$$\sup(\text{dom}(\Lambda_W)) = - \limsup_{w \rightarrow \infty} \frac{1}{w} \log \mathbb{P}(X \geq w) \quad (3)$$

(see, e.g., [3, Section 9.1.1]), the case  $(\tau, c) = (1, \infty)$  is precisely the case where  $\Lambda_W < \infty$  everywhere ( $W$  with a **very light tail**<sup>2</sup>), which is considered in Theorem 1, item 1. It includes the case where the variable  $W$  is **bounded**, as assumed in Theorem 1, item 2, and the case where  $(\tau, c) \in (1, \infty) \times (0, \infty)$  ( $W$  with a **compressed exponential tail**) considered in Theorem 6, precisising the very large deviations in this case. The early motivation of this paper was to examine the case  $(\tau, c) \in \{1\} \times (0, \infty)$  where  $W$  has an **exponential tail**. It appears that we have obtained deviations of all orders in the general case  $(\tau, c) \in \{1\} \times (0, \infty]$ , essentially corresponding to light-tailed random variables  $W$ , with a unified proof. Whereas the moderate deviations in this case were stated above (see Proposition 3 and Theorem 4), the large and very large deviation results can be found in Sections 2.1 (finite tree) and 2.2 (infinite tree). Eventually, our techniques can be extended to the case  $(\tau, c) \in (0, 1) \times (0, \infty)$  corresponding to  $W$  with a **stretched exponential tail**: such  $W$  has a heavier tail than exponential and notably satisfy  $\Lambda_W = \infty$  on  $(0, \infty)$ . All our results in this latter case are stated in Section 2.3.

## 2.1 Light tails — Finite tree

In this section, it is always assumed that  $(\mathbf{A}_{\tau, c})$  holds with  $(\tau, c) \in \{1\} \times (0, \infty]$ . Examples where  $c \in (0, \infty)$  include the cases where the law of  $W$  is the exponential distribution of mean 1 ( $c = 1$ ) or any gamma distribution of mean 1. The following theorem describes all regimes of deviations in the finite trees. Note that, in the whole text, we use the convention  $\infty \cdot 0 = 0$ .

**Theorem 5** (light tails – finite tree – large and very large right deviations). *Assume that  $(\mathbf{A}_{\tau, c})$  holds with  $(\tau, c) \in \{1\} \times (0, \infty]$ . Let  $n \in \llbracket 1, \infty \rrbracket$ .*

1. Large deviations. For all  $a \geq 1$ ,

$$-\frac{1}{r} \log \mathbb{P}(Z_r^n \geq a) \xrightarrow{r \rightarrow \infty} I^n(a)$$

where the functions  $I^n$  are defined by induction by  $I^1 = \Lambda_W^*$  and, for all  $a \geq 0$ ,

$$I^n(a) := \inf \{ cw + I^{n-1}(z) + \Lambda_W^*(s) ; w \geq 0, z \geq 0, s \geq 0, wz + s = a \}. \quad (4)$$

2. Very large deviations. For all  $\alpha \in (0, \infty)$  and  $a \geq 0$ ,

$$-\frac{1}{r^{1+\alpha/n}} \log \mathbb{P}(Z_r^n \geq r^\alpha a) \xrightarrow{r \rightarrow \infty} K^n(a) := cna^{1/n}.$$

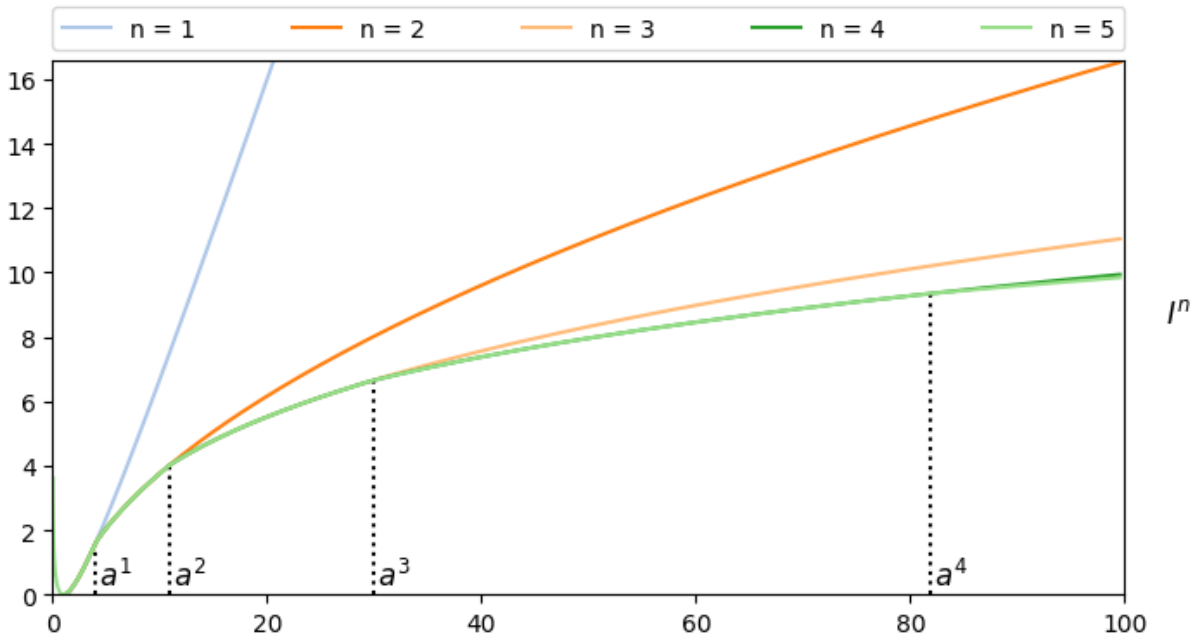
Theorem 5 deserves some comments. First of all, together with Proposition 2 and the monotonicity properties of the rate functions, Theorem 5 implies large deviation principles on  $\mathbb{R}_+$ : (1)  $(Z_r^n)_{r \geq 2}$  satisfies a large deviation principle at speed  $(r)_{r \geq 2}$  with rate function  $I^n$  (notice that, on  $[0, 1]$ ,  $I^n = \Lambda_W^*$ ; see Proposition 11); (2)

<sup>2</sup>In the literature, this case is known as “over-exponential”, “super-exponential”, or “sub-exponential”. To avoid confusion, we opt for the (non standard) term “very light tail”.

for  $\alpha \in (0, \infty)$ ,  $(r^{-\alpha} Z_r^n)_{r \geq 2}$  satisfies a large deviation principle at speed  $(r^{1+\alpha/n})_{r \geq 2}$  with rate function  $K^n$ . Unsurprisingly, just like for the simple random walk  $(Z_r^1)_{r \geq 2}$ , the rate function  $K^n$  appearing in the very large deviations depends only on the tail of  $W$ , through the constant  $c$  (similarly, the rate function  $J$  in the moderate deviations depends only on the variance  $\text{Var}(W)$  of  $W$ ; see Theorem 4), whereas the rate function  $I^n$  appearing in the large deviations depends on the entire law of  $W$ , through  $\Lambda_W$ . The large deviations (item 1) reveal a very interesting behavior. Contrary to the moderate deviations, the cases  $c = \infty$  and  $c \in (0, \infty)$  are not similar anymore. One can easily prove, by induction in (4), that

$$c = \infty \implies \forall n \in \llbracket 1, \infty \llbracket \quad I^n = \Lambda_W^* \quad (5)$$

(see Proposition 12): this case was studied in [15] and the proof there relies on exponential approximation. Here we provide another proof that encompasses all the cases  $c \in (0, \infty]$ . The first notable fact is that, when  $c \in (0, \infty)$  and  $n \geq 2$ , the rate function  $I^n$  is not  $\Lambda_W^*$  anymore, as follows from the asymptotics given in Proposition 17. (Notice, by the way, that the rate function  $K^n$  of the very large deviations coincides with the asymptotics of the rate function  $I^n$  of the large deviations.) Figure ?? shows some functions  $I^n$  when  $W \sim \mathcal{E}(1)$  and properties of the functions  $I^n$  are gathered in Section 3.2.1. Notably, we prove that  $(I^n)_{n \geq 1}$  is



a nonincreasing sequence of functions (Proposition 13). In particular, we may introduce

$$I^\infty := \lim_{n \rightarrow \infty} \downarrow I^n, \quad (6)$$

which appears to be the rate function of the large deviation principle in the infinite tree, as stated in Theorem 7 below. Furthermore, as Figure ?? shows, one proves (see Proposition 18) that there are successive “splitting points”  $a^n$ , for  $n \geq 1$ , defined by

$$a^n := \inf \{ a \geq 1 ; I^{n+1}(a) < I^n(a) \}. \quad (7)$$

Here is a nice interpretation of this fact. For any  $n \in \llbracket 1, \infty \llbracket$ , the large deviations in the tree of height  $n$  are explained by (i.e. have the same first order log-probability asymptotics as) the deviations of a subtree, as illustrated in Figure 1: the larger the deviation requested, the higher the subtree deviating. Let us detail what happens in the case  $n = 4$ . When  $a \in (1, a^1]$  (see Figure 1a),  $I^4(a) = I^1(a) = \Lambda_W^*(a)$ , so the deviation is explained by the deviation of the first level of the tree (in red), each black subtree being typically of order 1. When  $a \in (a^1, a^2]$  (see Figure 1b),  $I^4(a) = I^2(a)$  and the deviation is explained by the first two levels of the tree. Looking deeper into the definition of  $I^2(a) = \inf \{ cw + I^1(z) + \Lambda_W^*(s) ; w \geq 0, z \geq 0, s \geq 0, wz + s = a \}$ , one sees that this deviation of the first two levels is explained by: (1) the deviation of one edge in the first level (squiggly red edge), contributing to the term  $cw$  in the definition of  $I^2(a)$ ; (2) the deviation of the one-level subtree below (straight red edges below), contributing to the term  $I^1(z)$ ; (3) the deviation of the sum of the other edges in the first level (top straight red edges), contributing to the term  $\Lambda_W^*(s)$ . When  $a \in (a^2, a^3]$  (see Figure 1c),  $I^4(a) = I^3(a) = \inf \{ cw + I^2(z) + \Lambda_W^*(s) ; w \geq 0, z \geq 0, s \geq 0, wz + s = a \}$  and the deviation is explained by the first three levels of the tree, with the same deeper interpretation. Note that the term  $I^2(z)$

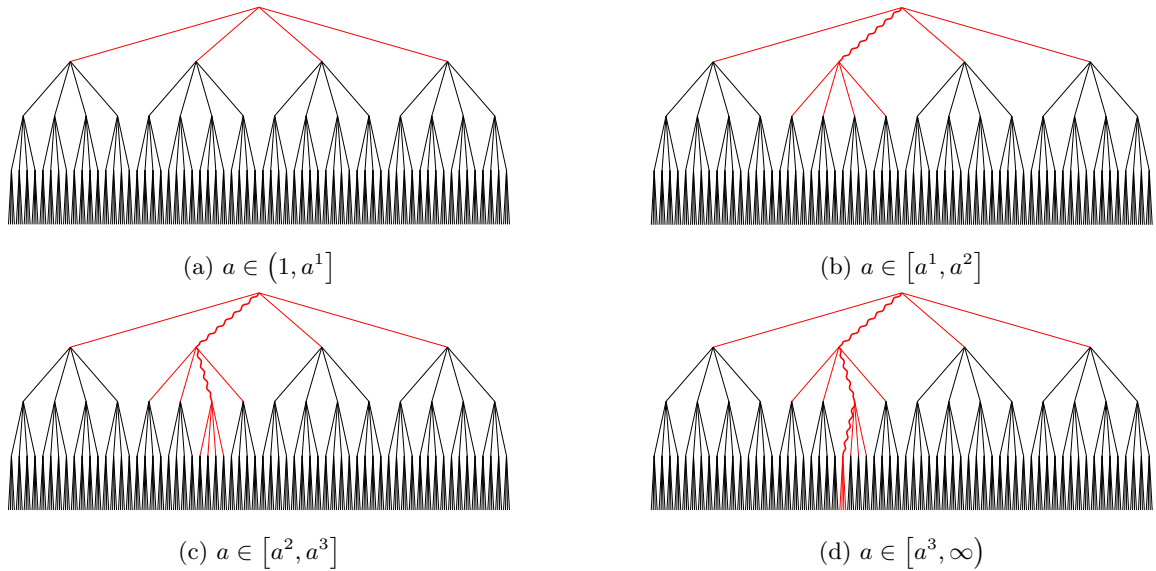


Figure 1: Typical event explaining  $I^4(a)$ . One observes a longer and longer “haired spine” forced to deviate.

corresponds to the deviation of the subtree below the top squiggly red edge: by induction, the picture for it is as in Figure 1b. The same applies to  $a \geq a^3$  (see Figure 1d). The edges explaining the deviations form what we call a “haired spine”: the squiggly edges form a spine, each of those being of order  $r$ , and the straight red edges attached to the spine deviate as a whole in each level.

Let us notice a last fact on the functions  $I^n$ . Even if they are continuous (see Proposition 17), they are not differentiable at splitting points  $a^n$ , in general. See for instance Figure 2 in the case  $W \sim \mathcal{E}(1)$ . Here is the explanation

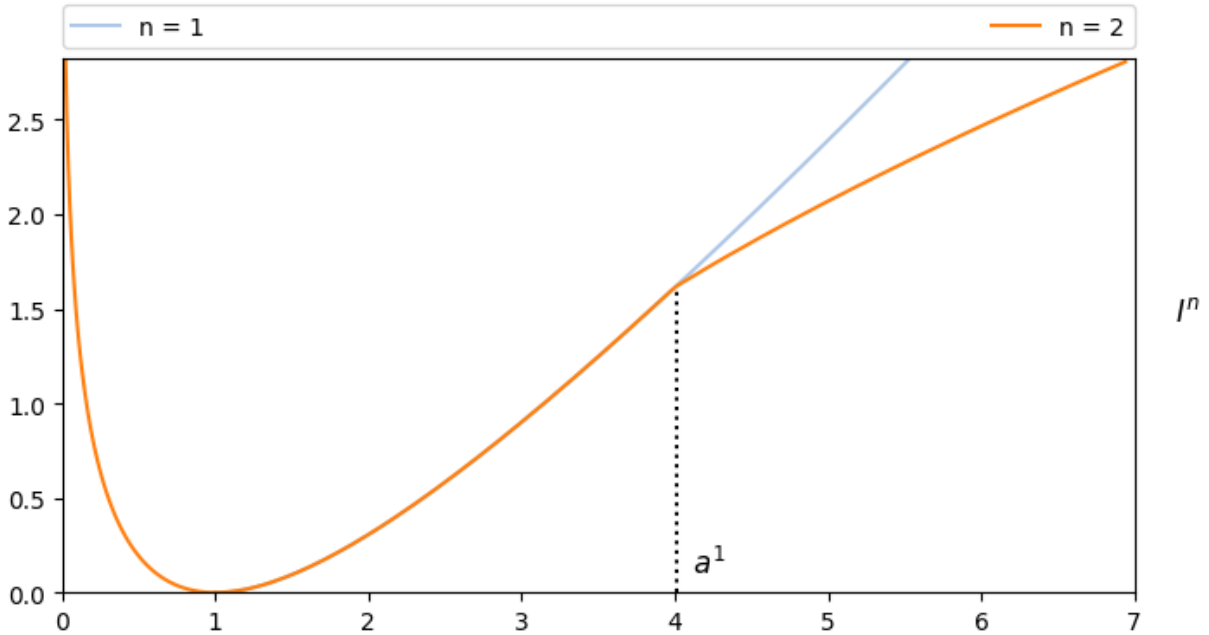


Figure 2: Non differentiability of  $I^2$  at splitting point  $a^1$  for  $W \sim \mathcal{E}(1)$ .

of what we observe on Figure 2. Remember that  $I^2(a) = \inf \{G^1(w, z, s) ; w \geq 0, z \geq 0, s \geq 0, wz + s = a\}$  where  $G^1(w, z, s) = cw + I^1(z) + \Lambda_W^*(s)$ . One observes a transition at  $a = a^1 = 4$  for the point of minimum: if  $a < a^1$ , the minimum is attained at  $(w, z, s) = (0, 1, a)$ ; if  $a = a^1$ , it is attained at two points,  $(w, z, s) = (0, 1, a)$  and  $(w, z, s) = (1, 2, 2)$ , the second being critical; if  $a > a^1$ , it is attained only at the critical point (see Figure 3). In particular, at  $a = a^1$ , two kinds of events explain the deviation (phase transition): either the first level of the tree deviates globally (as in Figure 1a); or some edge in the first level is of order  $r$ , the one-level tree below of order  $2r$ , and the sum of the other edges in the first level of order  $2r$  (as in Figure 1b).

As for the very large deviations (item 2), first notice that the deviation is explained by the deviation of all edges along a (“bare”) spine of maximal length (each edge being of order  $(r^\alpha a)^{1/n}$ ). Also, the speed depends on  $n$ ,

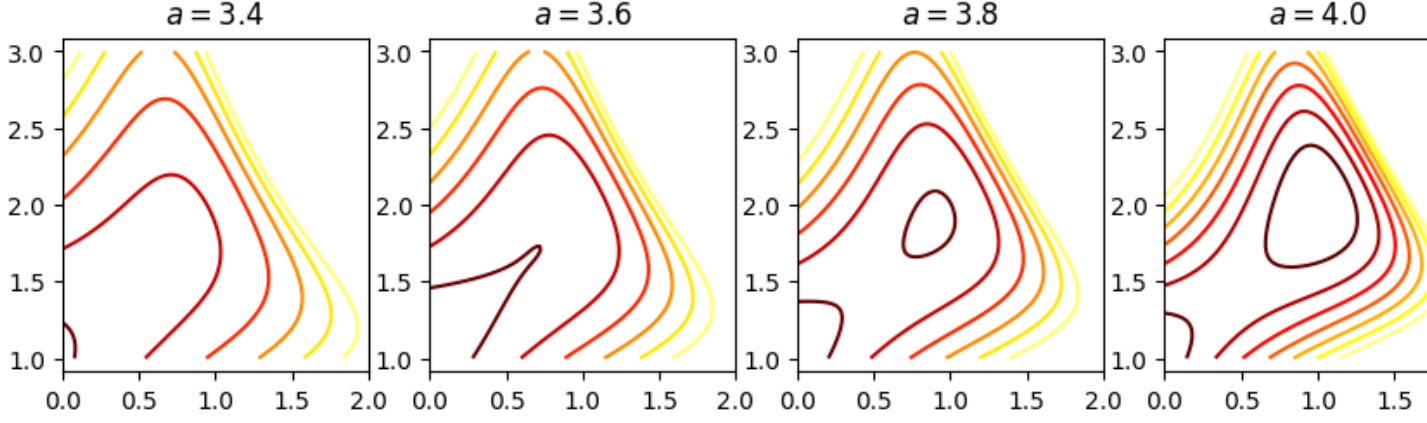


Figure 3: Level lines of  $G^1(w, z, a - wz) = cw + I^1(z) + \Lambda_W^*(a - wz)$  for different values of  $a$  (here  $W \sim \mathcal{E}(1)$ ).

which was not the case for the moderate and large deviations. Besides, for  $c = \infty$ , the rate function at speed  $(r^{1+\alpha/n})_{r \geq 2}$  is  $\infty \mathbb{1}_{a>0}$ , so the informative speed should be  $o(r^{1+\alpha/n})$ . It appears that this speed depends on the precise decay of the tail of  $W$ . In the following theorem, we make explicit this speed and compute the rate function in the particular case of assumption  $(\mathbf{A}_{\tau, c})$  with  $(\tau, c) \in (1, \infty) \times (0, \infty)$  (this subcase of  $(\tau, c) = (1, \infty)$  corresponds to compressed exponential tails). For all  $a \in [0, \infty)$  and  $n \in \llbracket 1, \infty \llbracket$ , we set  $K_\tau^n(a) := cna^{\tau/n}$ .

**Theorem 6** (compressed exponential tails – finite tree – very large deviations). *Assume that  $(\mathbf{A}_{\tau, c})$  holds with  $(\tau, c) \in (1, \infty) \times (0, \infty)$ . For all  $n \in \llbracket 1, \infty \llbracket$ ,  $\alpha > 0$ , and  $a > 0$ ,*

$$-\log \mathbb{P}(Z_r^n \geq r^\alpha a) \sim \begin{cases} r^{\frac{k+1}{2} + \frac{\alpha\tau}{k}} K_\tau^k(a) & \text{if } 1 \leq k < n, \frac{k(k-1)}{2} < \alpha\tau < \frac{k(k+1)}{2}, \text{ and } \alpha\tau < k(\tau - \frac{k+1}{2}) \\ r^{\frac{k(k+1)}{2n} + \frac{(n-k+\alpha)\tau}{n}} K_\tau^n(a) & \text{if } 1 \leq k < n, k \leq \tau < k+1, \text{ and } \alpha\tau > k(\tau - \frac{k+1}{2}) \\ r^{\frac{n+1}{2} + \frac{\alpha\tau}{n}} K_\tau^n(a) & \text{if } n \leq \tau \text{ and } \alpha\tau > \frac{n(n-1)}{2}. \end{cases}$$

The results of Theorems 5 and 6 are summarized in Figure 6 in the domain  $(\tau, \alpha) \in [1, \infty) \times (-1/2, \infty)$ .

## 2.2 Light tails — Infinite tree

We are now ready to state deviation results for the total mass  $(Z_r^\infty)_{r \geq 2}$  of the Mandelbrot cascades. As the following theorem shows, the large deviations can be seen as the natural extrapolation of the ones in the finite tree. The very large deviations need a special treatment since the speed in the finite trees depends on the height of the tree.

**Theorem 7** (light tails – infinite tree – all right deviations). *Assume that  $(\mathbf{A}_{\tau, c})$  holds with  $(\tau, c) \in \{1\} \times (0, \infty]$ .*

1. Large deviations. *Recalling the definition of  $I^\infty$  in (6), for all  $a \geq 1$ ,*

$$-\frac{1}{r} \log \mathbb{P}(Z_r^\infty \geq ra) \xrightarrow[r \rightarrow \infty]{} I^\infty(a).$$

2. Very large deviations. *For all  $\alpha > 0$  and  $a \geq 0$ ,*

$$-\frac{1}{r \log(r)} \log \mathbb{P}(Z_r^\infty \geq r^\alpha a) \xrightarrow[r \rightarrow \infty]{} ce\alpha \mathbb{1}_{a>0}.$$

Note that the rate function  $I^\infty$  in the large deviations (item 1) depends on the distribution of  $W$  (through  $\Lambda_W$ ), and notably on  $c$ . When  $c = \infty$ ,  $I^\infty = \Lambda^*$ , as follows from (5) and (6). This extends the already known Theorem 1, item 2 for bounded random variables  $W$  to the case of all very-light-tailed random variables  $W$ . Table 1 below summarizes pre-existing results and our contribution concerning the case of large deviations ( $\alpha = 0$ ).

When  $c \in (0, \infty)$ , we may add some comments to those of Section 2.1 about the splitting points  $a^n$ . Not only is the sequence  $(a^n)_{n \geq 1}$  increasing, but it diverges to infinity (see Proposition 18). As a consequence, for all  $a \geq 1$ , there exists  $n \in \llbracket 1, \infty \llbracket$  such that  $I^\infty(a) = I^n(a)$  (any  $n$  such that  $a \leq a^n$ ) and the deviation is explained by a subtree of finite height  $n$ , more precisely by a finite length “haired spine” (as described in Section 2.1).

Concerning the very large deviations (item 2), as in Theorem 5, item 2, in the case  $c = \infty$ , the rate function at speed  $(r \log(r))_{r \geq 2}$  is  $\infty \mathbb{1}_{(0, \infty)}$ , hence degenerated, and the informative speed should be  $o(r \log(r))$ . Moreover, for  $c \in (0, \infty)$ , the rate function at speed  $(r \log(r))_{r \geq 2}$  is a finite positive constant on  $(0, \infty)$ . We expect

	$c = \infty$ $\text{ess sup}(W) < \infty$	$c = \infty$ $\text{ess sup}(W) = \infty$	$c \in (0, \infty)$ $\text{ess sup}(W) = \infty$
$n \in \llbracket 1, \infty \llbracket$	[15, Theorem 6.2 (a)]		Theorem 5, item 1
$n = \infty$	[16, Theorem 1.4]	Theorem 7, item 1	

Table 1: Summary of previous and new results of large deviations ( $\alpha = 0$ ). Theorems 5 and 7 encompass the other results on the same line.

that, for all  $\varepsilon > 0$ ,  $\mathcal{L}(r^{-\alpha}Z_r \mid r^{-\alpha}Z_r \geq \varepsilon)$  satisfies a large deviation principle at some speed  $o(r \log(r))$  and we conjecture that the speed is in fact  $(r)_{r \geq 2}$ , by approximating the event  $\{Z_r \approx r^\alpha ea / (e - 1)\}$  by the event  $\{W_{1^1} \approx rea^{1/\log(r^\alpha)}, \dots, W_{1^n} \approx rea^{1/\log(r^\alpha)}\}$ . To get such a result, one could compute the second order in the asymptotics of the very large deviations, i.e.

$$\mathbb{P}(Z_r \approx r^\alpha a) = \exp(-r \log(r^\alpha) ce + rL(a) + o(r)),$$

but it is out of the scope of this paper.

### 2.3 Extension to the case of stretched exponential tails

What about the case  $(\tau, c) \in (0, 1) \times (0, \infty)$  in  $(\mathbf{A}_{\tau, c})$ , i.e. where  $W$  has a stretched exponential tail? Our proofs adapt to this case and we have the following results. Set  $\alpha_\tau := (\tau - 1)/(2 - \tau)$ . For all  $\alpha \in (-1/2, 0]$ , set

$$v_\tau(\alpha) := \begin{cases} 1 + 2\alpha & \text{if } \alpha \in (-1/2, \alpha_\tau] \\ \tau(1 + \alpha) & \text{if } \alpha \in (\alpha_\tau, 0]. \end{cases}$$

For all  $a \in [0, \infty)$ , remind that  $J(a) = \frac{a^2}{2\text{var}(W)}$  and, for all  $n \geq 1$ ,  $K_\tau^n(a) = cna^{\tau/n}$ . For all  $n \geq 1$ , we define a function  $I_\tau^n$  by induction by: for all  $a \geq 0$ ,  $I_\tau^1(a) = K_\tau^1(a) = ca^\tau$  and

$$I_\tau^n(a) := \inf \{cw^\tau + I_\tau^{n-1}(z) ; w \geq 0, z \geq 0, w(1+z) = a\}. \quad (8)$$

**Theorem 8** (stretched exponential tails – finite tree – all right deviations). *Assume that  $(\mathbf{A}_{\tau, c})$  holds with  $(\tau, c) \in (0, 1) \times (0, \infty)$ . For all  $n \in \llbracket 1, \infty \llbracket$ ,  $\alpha \in (-1/2, \infty)$ , and  $a \in (0, \infty)$ ,*

$$-\frac{1}{r^{v_\tau(\alpha)}} \log \mathbb{P}(Z_r^n - 1 \geq r^\alpha a) \xrightarrow{r \rightarrow \infty} \begin{cases} J(a) & \text{if } \alpha \in (-1/2, \alpha_\tau) \\ \inf \{K_\tau^1(w) + J(s) \mid w \geq 0, s \geq 0, w + s = a\} & \text{if } \alpha = \alpha_\tau \\ K_\tau^1(a) & \text{if } \alpha \in (\alpha_\tau, 0) \\ I_\tau^n(a) & \text{if } \alpha = 0 \end{cases}$$

and, if  $\alpha \in (0, \infty)$ ,

$$-\frac{1}{r^{\tau(1+\alpha/n)}} \log \mathbb{P}(Z_r^n - 1 \geq r^\alpha a) \xrightarrow{r \rightarrow \infty} K_\tau^n(a).$$

Theorem 8 is very similar to Theorem 5. Indeed, there are essentially three regimes: (1) when  $\alpha < 0$ , the deviation is explained by the deviation of the first level of the tree, with either a Gaussian behavior (with  $J$ ), or a big jump behavior (with  $K_\tau^1$ ), or the transition between the two (see [5]); (2) when  $\alpha > 0$ , the deviation is explained by the deviation of all edges along a spine of maximal length; (3) at  $\alpha = 0$ , there is a transition between the two previous regimes: the larger the deviation requested, the higher the subtree deviating. Concerning the latter case  $\alpha = 0$ , the graphs of the functions  $I_\tau^n$  are similar to those of the functions  $I^n$  (except for the behavior at the origin); see Figure 4. As for the explanation of the deviations, the situation is somewhat simpler than the one of the exponential case: here, only the edges along an increasing (“bare”) spine are deviating (see Figure 5). This stems from the fact that we are here in the big jump domain, as mentioned above. Also notice that, for  $\alpha \in (\alpha_\tau, \infty)$ , the speeds of right and left deviations differ. All our results about the finite tree (Theorems 4, 5, 6, and 8) are summarized in Figure 6.

In the infinite tree, at least for  $\alpha \in (-1/2, 0]$ , we obtain the limit of the rate function in the finite trees, i.e. as  $n \rightarrow \infty$  in the previous theorem. Let us define  $I_\tau^\infty := \lim_{n \rightarrow \infty} \downarrow I_\tau^n$  (see Proposition 19).

**Theorem 9** (stretched exponential tails – infinite tree – all right deviations). *Assume that  $(\mathbf{A}_{\tau, c})$  holds with  $(\tau, c) \in (0, 1) \times (0, \infty)$ . For all  $\alpha \in (-1/2, \infty)$  and  $a \in [0, \infty)$ ,*

$$-\frac{1}{r^{v_\tau(\alpha)}} \log \mathbb{P}(Z_r^\infty - 1 \geq r^\alpha a) \xrightarrow{r \rightarrow \infty} \begin{cases} J(a) & \text{if } \alpha \in (-1/2, \alpha_\tau) \\ \inf \{K_\tau^1(w) + J(s) \mid w \geq 0, s \geq 0, w + s = a\} & \text{if } \alpha = \alpha_\tau \\ K_\tau^1(a) & \text{if } \alpha \in (\alpha_\tau, 0) \\ I_\tau^\infty(a) & \text{if } \alpha = 0 \end{cases}$$

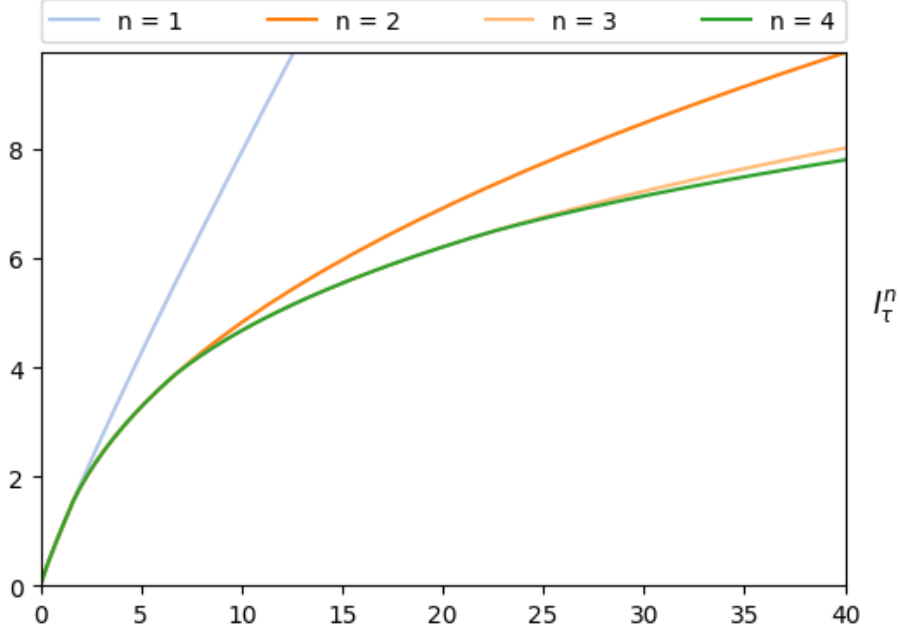


Figure 4: Graphs of  $I_r^n$  for  $n \in [1, 4]$  and  $c = 0.9$ .

and, if  $\alpha \in (0, \infty)$ ,

$$-\frac{1}{r^\tau \log(r)} \log \mathbb{P}(Z_r^\infty - 1 \geq r^\alpha a) \xrightarrow{r \rightarrow \infty} ce\tau\alpha \mathbb{1}_{a>0}.$$

### 3 Finite tree

By (1), we have, for all  $n \in [1, \infty[$ ,

$$Z_r^n = \frac{1}{r} \sum_{i=1}^r W_i Z_{r,i}^{n-1}, \quad (9)$$

where

$$Z_{r,i}^{n-1} := \frac{1}{r^{n-1}} \sum_{1 \leq i_2, \dots, i_n \leq r} W_{i,i_2} \cdots W_{i,i_2, \dots, i_n}$$

is distributed as  $Z_r^{n-1}$ . All proofs in this section rely on this induction formula.

#### 3.1 Proof of moderate deviations

The result for  $n = 1$  stems from the moderate deviation principle for a sum of independent and identically distributed random variables (see [6, Theorem 3.7.1]). Now, let  $n \geq 2$  be such that  $(r^{-\alpha}(Z_r^{n-1} - 1))_{r \geq 2}$  satisfies a large deviation principle at speed  $(r^{1+2\alpha})_{r \geq 2}$  with rate function  $J: a \mapsto \frac{a^2}{2 \text{Var}(W)}$ .

##### 3.1.1 Left deviations — Proposition 3

Let us prove that, for all  $t \leq 0$ ,

$$\Lambda_r^n(t) := \frac{1}{r^{1+2\alpha}} \log \mathbb{E} \left[ e^{tr^{1+\alpha}(Z_r^n - 1)} \right] \xrightarrow{r \rightarrow \infty} \frac{t^2 \text{Var}(W)}{2}. \quad (10)$$

Then, applying a unilateral version of Gärtner-Ellis theorem (see [21] or [5, Theorem 4]) and knowing that  $J$  is continuous and has all points in  $(-\infty, 0)$  exposed, we get, for all  $a \leq 0$ ,

$$\frac{1}{r^{1+2\alpha}} \log \mathbb{P}(Z_r^n - 1 \leq r^\alpha a) \xrightarrow{r \rightarrow \infty} -J(a).$$

Let  $t \leq 0$ . Using (9), independence and identity of distributions, we have

$$\Lambda_r^n(t) = r^{-2\alpha-1} \log \mathbb{E} \left[ e^{tr^\alpha \sum_{i=1}^r (W_i Z_{r,i}^{n-1} - 1)} \right] = r^{-2\alpha} \log \mathbb{E} \left[ e^{tr^\alpha (W Z_r^{n-1} - 1)} \right].$$

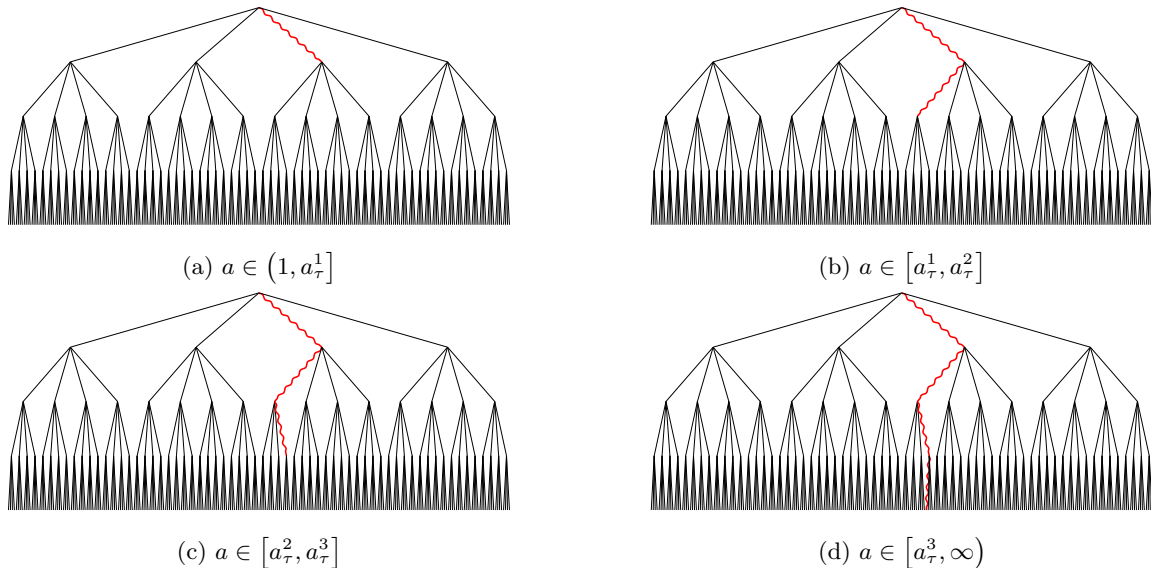


Figure 5: Typical event explaining  $I_\tau^4(a)$ . One observes a longer and longer spine forced to deviate.

Using the Taylor expansion  $e^x = 1 + x + x^2 e^{\theta(x)}/2$  with  $\theta(x) \leq \max(x, 0)$  and the fact that  $\mathbb{E}[W Z_r^{n-1}] = \mathbb{E}[W] \mathbb{E}[Z_r^{n-1}] = 1$ , we get

$$\Lambda_r^n(t) = r^{-2\alpha} \log \left( 1 + \frac{t^2}{2r^{-2\alpha}} \mathbb{E}[(W Z_r^{n-1} - 1)^2 e^{\theta(tr^\alpha(W Z_r^{n-1} - 1))}] \right) \xrightarrow{r \rightarrow \infty} \frac{t^2}{2} \mathbb{E}[(W - 1)^2].$$

Indeed,

$$\begin{aligned} & \left| \mathbb{E}[(W Z_r^{n-1} - 1)^2 e^{\theta(tr^\alpha(W Z_r^{n-1} - 1))}] - \mathbb{E}[(W - 1)^2] \right| \\ & \leq \left| \mathbb{E}[(W Z_r^{n-1} - 1)^2 (e^{\theta(tr^\alpha(W Z_r^{n-1} - 1))} - 1)] \right| + \left| \mathbb{E}[(W Z_r^{n-1} - 1)^2 - (W - 1)^2] \right| \\ & \leq \left| \mathbb{E}[(W Z_r^{n-1} - 1)^2] \right| (e^{-tr^\alpha} - 1) + \left| \mathbb{E}[(W Z_r^{n-1} - 1)^2 - (W - 1)^2] \right| \xrightarrow{r \rightarrow \infty} 0, \end{aligned}$$

since

$$\mathbb{E}[(Z_r^{n-1})^2] = \mathbb{E} \left[ \left( \frac{1}{r} \sum_{i=1}^r W_i Z_{r,i}^{n-2} \right)^2 \right] = \frac{1}{r^2} (r \mathbb{E}[W^2] \mathbb{E}[(Z_r^{n-2})^2] + r(r-1) \mathbb{E}[W]^2 \mathbb{E}[Z_r^{n-2}]^2) \xrightarrow{r \rightarrow \infty} 1.$$

**Remark 10.** Obviously, the previous estimate generalizes and the multinomial expansion yields, for all  $n \in \llbracket 1, \infty \rrbracket$ ,

$$\mathbb{E}[(Z_r^n)^h] \xrightarrow{r \rightarrow \infty} 1$$

as soon as  $\mathbb{E}[W^h] < \infty$ . Controlling all moments of  $Z_r^n$  appears to be crucial in the study of the right deviations for  $n = \infty$  (see Section 4).

### 3.1.2 Right deviations — Theorem 4 and Theorem 8

Let  $\alpha \in (-1/2, 0)$ . For  $n = 1$ , recall that the result is well known: it is the moderate deviation principle for a sum of independent and identically distributed real random variables. The light-tailed case can be found in [6, Theorem 3.7.1]; and the stretched exponential case ( $\tau < 1$  and  $c \in (0, \infty)$ ) can be found in [5]. Let us denote by  $L_{\tau, \alpha}$  the rate function of  $((Z_r^1 - 1)/r^\alpha)_{r \geq 2}$ , i.e., for all  $a > 0$ ,

$$L_{\tau, \alpha}(a) := \begin{cases} J(a) & \text{if } \alpha \in (-1/2, \alpha_\tau) \\ \inf \{ K_\tau^1(w) + J(s) \mid w \geq 0, s \geq 0, w + s = a \} & \text{if } \alpha = \alpha_\tau \\ K_\tau^1(a) & \text{if } \alpha \in (\alpha_\tau, 0). \end{cases}$$

Note that this definition encompasses the light-tailed case, when  $\tau = 1$ , since  $\alpha_1 = 0$ , and so  $L_{1, \alpha} = J$ .

For  $n \geq 2$ , let us prove that  $((Z_r^n - 1)/r^\alpha)_{r \geq 2}$  and  $((Z_r^1 - 1)/r^\alpha)_{r \geq 2}$  are exponentially equivalent at speed  $(r^{v_\tau(\alpha)})_{r \geq 2}$ , whence  $((Z_r^n - 1)/r^\alpha)_{r \geq 2}$  satisfies the same large deviation principle as  $((Z_r^1 - 1)/r^\alpha)_{r \geq 2}$  at speed  $(r^{v_\tau(\alpha)})_{r \geq 2}$ . We want to prove that, for all  $a > 0$ ,

$$\frac{1}{r^{v_\tau(\alpha)}} \log \mathbb{P}(|Z_r^n - Z_r^1| \geq r^\alpha a) \xrightarrow{r \rightarrow \infty} -\infty.$$



First, we prove that, for all  $a > 0$ ,

$$\frac{1}{r^{v_\tau(\alpha)}} \log \mathbb{P}(Z_r^n - Z_r^1 \leq -r^\alpha a) \xrightarrow{r \rightarrow \infty} -\infty.$$

This stems from Chernoff's bound, the fact that  $v_\tau(\alpha) \leq 1 + 2\alpha$ , and the fact that, for all  $t < 0$ ,

$$\frac{1}{r^{1+2\alpha}} \log \mathbb{E} \left[ e^{tr^{1+\alpha}(Z_r^n - Z_r^1)} \right] \xrightarrow{r \rightarrow \infty} 0.$$

Following the same lines as in Section 3.1.1, this latter convergence relies on the fact that  $\mathbb{E}[(Z_r^{n-1} - 1)^2] \rightarrow 0$  as  $r \rightarrow \infty$ , which is a consequence of the fact that  $\mathbb{E}[Z_r^{n-1}] = 1$  and  $\mathbb{E}[(Z_r^{n-1})^2] \rightarrow 1$ .

Second, we prove that, for all  $a > 0$  and  $M > 0$ ,

$$\limsup_{r \rightarrow \infty} \frac{1}{r^{v_\tau(\alpha)}} \log \mathbb{P}(Z_r^n - Z_r^1 \geq r^\alpha a) \leq -M. \quad (11)$$

Let  $\varepsilon > 0$ . Note that

$$\mathbb{P}(Z_r^n - Z_r^1 \geq r^\alpha a) = \mathbb{P} \left( \sum_{i=1}^r W_i(Z_{r,i}^{n-1} - 1) \geq r^{1+\alpha} a \right) = \sum_{q=0}^r P_{n,r,q}$$

with

$$P_{n,r,q} := \mathbb{P} \left( \sum_{i=1}^r W_i(Z_{r,i}^{n-1} - 1) \geq r^{1+\alpha} a, \forall i \in \llbracket 1, q \rrbracket \ Z_{r,i}^{n-1} - 1 \geq r^\alpha \varepsilon, \forall i \in \llbracket q+1, r \rrbracket \ Z_{r,i}^{n-1} - 1 < r^\alpha \varepsilon \right).$$

For  $q_0 \in \llbracket 0, r \rrbracket$  and  $\delta \in (0, L_{\tau,\alpha}(\varepsilon))$ , one has by induction

$$\sum_{q=q_0}^r \binom{r}{q} P_{n,r,q} \leq \sum_{q=q_0}^r \binom{r}{q} \mathbb{P}(Z_r^{n-1} - 1 \geq r^\alpha \varepsilon)^q \leq \sum_{q=q_0}^r (re^{-r^{v_\tau(\alpha)} \delta})^q \leq \frac{(re^{-r^{v_\tau(\alpha)} \delta})^{q_0}}{1 - re^{-r^{v_\tau(\alpha)} \delta}},$$

as soon as  $r$  is large enough. Hence,

$$\limsup_{r \rightarrow \infty} \frac{1}{r^{v_\tau(\alpha)}} \log \sum_{q=q_0}^r \binom{r}{q} P_{n,r,q} \leq -\delta q_0 \leq -M, \quad (12)$$

as soon as  $q_0$  is large enough. For  $q \in \llbracket 0, q_0 - 1 \rrbracket$ , one has, for all  $y > 0$ ,

$$\frac{1}{r^{v_\tau(\alpha)}} \log \mathbb{P}(W(Z_r^{n-1} - 1) \geq r^{1+\alpha} y) \xrightarrow{r \rightarrow \infty} -\infty.$$

This follows from the union bound

$$\mathbb{P}(W(Z_r^{n-1} - 1) \geq r^{1+\alpha} y) \leq \mathbb{P}(W \geq r^\beta y) + \mathbb{P}(Z_r^{n-1} - 1 \geq r^\gamma)$$

with

$$\beta = \begin{cases} \frac{3+2\alpha}{2+\tau} & \text{if } \alpha \leq 2\alpha_\tau \\ 1 + \frac{\alpha}{2} & \text{if } \alpha > 2\alpha_\tau, \end{cases}$$

and  $\gamma = 1 + \alpha - \beta$ , and the fact that  $\beta\tau = v_\tau(\gamma) > v_\tau(\alpha)$  (note that, in the case  $W$  is light-tailed, by (3),  $\limsup r^{-\beta} \log \mathbb{P}(W \geq r^\beta y) \leq -cy$  with  $c = \sup(\text{dom}(\Lambda_W)) > 0$ ). Finally, for all  $s > 0$ ,

$$\frac{1}{r^{v_\tau(\alpha)}} \log \mathbb{P} \left( \sum_{i=q+1}^r W_i \varepsilon \geq rs \right) \xrightarrow{r \rightarrow \infty} -\infty,$$

since  $v_\tau(\alpha) < v_\tau(1)$ . So, applying the contraction principle,

$$\frac{1}{r^{v_\tau(\alpha)}} \log P_{n,r,q} \leq \frac{1}{r^{v_\tau(\alpha)}} \log \mathbb{P} \left( \sum_{i=1}^q W_i(Z_{r,i}^{n-1} - 1) + \sum_{i=q+1}^r W_i r^\alpha \varepsilon \geq r^{1+\alpha} a \right) \xrightarrow{r \rightarrow \infty} -\infty. \quad (13)$$

Combining (12), (13), and the principle of the largest term (see [6, Lemma 1.2.15]), (11) is proved.

## 3.2 Proof of large deviations

### 3.2.1 About the functions $I^n$

Remember the definition of  $I^n$  given in (4).

**Proposition 11.** *For all  $(\tau, c) \in \{1\} \times (0, \infty]$  and  $n \in \llbracket 1, \infty \llbracket$ , we have  $I^n = \Lambda_W^*$  on  $[0, 1]$ .*

*Proof of Proposition 11.* For all  $n \geq 2$ , let us define the function

$$G^{n-1}(w, z, s) := cw + I^{n-1}(z) + \Lambda_W^*(s). \quad (14)$$

Let  $a \in [0, 1]$ . Obviously,  $I^n(a) \leq G^{n-1}(0, 1, a) = \Lambda_W^*(a)$ . If  $(w, z, s) \in [0, \infty)^3$  satisfies  $wz + s = a$ , then  $s \leq a$  and so  $G^{n-1}(w, z, s) \geq G^{n-1}(0, 1, a)$ , since  $\Lambda_W^*$  is nonincreasing on  $[0, 1]$ . Hence,  $I^n(a) \geq G^{n-1}(0, 1, a) = \Lambda_W^*(a)$ .  $\square$

The case  $(\tau, c) = (1, \infty)$  is very simple, as the following Proposition shows.

**Proposition 12.** *If  $(\tau, c) = (1, \infty)$ , then, for all  $n \in \llbracket 1, \infty \llbracket$ , we have  $I^n = \Lambda_W^*$  on  $[0, \infty)$ .*

*Proof of Proposition 12.* A straightforward induction yields the result, observing that, in (4),  $w$  has to be taken equal to zero to ensure  $cw < \infty$ .  $\square$

Until the end of this section, we assume that  $(\tau, c) \in \{1\} \times (0, \infty)$  and provide properties of  $I^n$  in this case.

**Proposition 13.** *Assume that  $(\tau, c) \in \{1\} \times (0, \infty)$ . The sequence of functions  $(I^n)_{n \geq 1}$  is nonincreasing.*

*Proof of Proposition 13.* First, we prove that  $I^2 \leq I^1$ . It is enough to take  $(w, z, s) = (0, 1, a)$  in (4) to get

$$I^2(a) \leq \Lambda_W^*(a) = I^1(a)$$

since  $I^1(1) = 0$  and  $\Lambda_W^* = I^1$  by definition. We conclude by induction:

$$I^{n+1}(a) \leq \inf \{cw + I^{n-1}(z) + \Lambda_W^*(s) ; w \geq 0, z \geq 0, s \geq 0, wz + s = a\} = I^n(a). \quad \square$$

**Proposition 14.** *Assume that  $(\tau, c) \in \{1\} \times (0, \infty)$ .*

1. *For all  $n \geq 1$ , the function  $I^n$  is increasing on  $[1, \infty)$ .*
2. *For all  $n \geq 2$ , for all  $a \geq 1$ ,*

$$I^n(a) = \inf \{cw + I^{n-1}(z) + \Lambda_W^*(s) ; w \geq 0, z \in [1, a], s \in [1, a], wz + s = a\}. \quad (15)$$

**Remark 15.** *When  $(\tau, c) = (1, \infty)$ ,  $I^n = \Lambda_W^*$  is increasing on the interval  $\{\Lambda_W^* < \infty\}$ , which is bounded in case  $W$  is bounded.*

*Proof of Proposition 14.* By Proposition 11, on  $[0, 1]$ ,  $I^{n-1} = \Lambda_W^*$  is nonincreasing so that we have, for all  $a \geq 1$ ,

$$I^n(a) = \inf \{cw + I^{n-1}(z) + \Lambda_W^*(s) ; w \geq 0, z \geq 1, s \in [1, a], wz + s = a\}.$$

since  $wz \geq 0$  and  $wz + s = a$  imply  $s \leq a$ . The proof of (15) is closely linked to the fact that  $I^{n-1}$  is increasing on  $[1, \infty)$ , preventing the  $(w, z)$  in the minimization problem from escaping to  $(0, \infty)$ . Let us prove the result by induction. Obviously,  $I^1 = \Lambda_W^*$  is increasing on  $[1, \infty)$ . Now let  $n \geq 2$  and assume that  $I^{n-1}$  is increasing on  $[1, \infty)$ .

1. First, we prove (15). The result is obvious for  $a = 1$  (because  $I^n(1) = 0 = G^{n-1}(0, 1, 1)$  where  $G^{n-1}$  has been defined in (14)). Assume that  $a > 1$ . It is enough to prove that

$$I^n(a) < \inf \{cw + I^{n-1}(z) + \Lambda_W^*(s) ; w \geq 0, z > a, s \in [1, a], wz + s = a\}.$$

This stems from the fact that (here we use  $I^{n-1}$  nondecreasing on  $[1, \infty)$  and  $I^{n-1} \geq I^n$ , by Proposition 13)

$$\begin{aligned} G^{n-1}(w, z, s) &\geq \begin{cases} I^{n-1}(a) + \frac{c(a-1)}{2(a+1)} & \text{if } z \in (a, a+1] \text{ and } s \leq \frac{1+a}{2} \\ I^{n-1}(a) + \Lambda_W^*\left(\frac{1+a}{2}\right) & \text{if } z \in (a, a+1] \text{ and } s > \frac{1+a}{2} \\ I^{n-1}(a+1) & \text{if } z > a+1 \end{cases} \\ &\geq I^n(a) + \begin{cases} \frac{c(a-1)}{2(a+1)} & \text{if } z \in (a, a+1] \text{ and } s \leq \frac{1+a}{2} \\ \Lambda_W^*\left(\frac{1+a}{2}\right) & \text{if } z \in (a, a+1] \text{ and } s > \frac{1+a}{2} \\ I^{n-1}(a+1) - I^{n-1}(a) & \text{if } z > a+1, \end{cases} \end{aligned}$$

and the fact that  $\Lambda_W^*$  and  $I^{n-1}$  are increasing on  $[1, \infty)$ .

2. Finally, we prove that  $I^n$  is increasing on  $[1, \infty)$ . Let  $a_1 > a_0 \geq 1$  and  $(w_1, z_1, s_1) \in [0, \infty) \times [1, a_1]^2$  be such that  $w_1 z_1 + s_1 = a_1$ . Let  $(w_0, z_0, s_0) = (((a_0 - s_1) \vee 0)/z_1, z_1, s_1 \wedge a_0) \in [0, \infty) \times [1, \infty)^2$  (with the convention  $0/0 = 0$ ), so that  $w_0 z_0 + s_0 = a_0$ . Then,

$$\begin{aligned} G^{n-1}(w_1, z_1, s_1) - I^n(a_0) &\geq G^{n-1}(w_1, z_1, s_1) - G^{n-1}(w_0, z_0, s_0) \\ &= \frac{c}{z_1} (a_1 - s_1 - (a_0 - s_1) \vee 0) + \Lambda_W^*(s_1) - \Lambda_W^*(s_1 \wedge a_0) \\ &\geq \begin{cases} \frac{c}{a_1} \frac{a_1 - a_0}{2} & \text{if } s_1 \leq (a_0 + a_1)/2 \\ \Lambda_W^*(\frac{a_0 + a_1}{2}) - \Lambda_W^*(a_0) & \text{if } s_1 > (a_0 + a_1)/2. \end{cases} \end{aligned}$$

The latter expression being positive and independent of  $(w_1, z_1, s_1)$ , taking the infimum in  $(w_1, z_1, s_1) \in [0, \infty) \times [1, a_1]^2$ , we get  $I^n(a_1) - I^n(a_0) > 0$ .  $\square$

For all  $a \geq 1$  and  $z \in [1, a]$ , let

$$h(a, z) := \inf_{1 \leq s \leq a} \left[ \frac{c(a-s)}{z} + \Lambda_W^*(s) \right]$$

so that, using (15),

$$I^n(a) = \inf_{1 \leq z \leq a} [I^{n-1}(z) + h(a, z)]. \quad (16)$$

Let also

$$a^* := \inf \{s \geq 0 ; \Lambda'_W(s) = a\} \quad (17)$$

be the convex conjugate of  $a$  through  $\Lambda_W$ . Note that, if  $a \leq \Lambda'_W(c^-)$ , then  $a^*$  is the unique solution in  $s$  of  $\Lambda'_W(s) = a$ ; and if  $a > \Lambda'_W(c^-)$ , then  $a^* = \infty$ .

**Lemma 16.** Assume that  $(\tau, c) \in \{1\} \times (0, \infty)$ . For all  $a \geq 1$  and  $z \in [1, a]$ ,

$$h(a, z) = \begin{cases} \Lambda_W^*(a) & \text{if } z \leq c/a^* \\ \frac{ca}{z} - \Lambda_W\left(\frac{c}{z}\right) & \text{if } z \geq c/a^*. \end{cases}$$

In particular, the function  $h$  is continuous.

*Proof of Lemma 16.* An easy computation of Legendre transform leads to

$$\begin{aligned} h(a, z) &= \frac{ca}{z} - \sup_{1 \leq s \leq a} \left[ \frac{cs}{z} - \Lambda_W^*(s) \right] \\ &= \begin{cases} \frac{ca}{z} - (\Lambda_W(a^*) + (\frac{c}{z} - a^*)a) & \text{if } c/z \geq a^* \\ \frac{ca}{z} - \Lambda_W\left(\frac{c}{z}\right) & \text{if } c/z \leq a^*. \end{cases} \\ &= \begin{cases} \Lambda_W^*(a) & \text{if } z \leq c/a^* \\ \frac{ca}{z} - \Lambda_W\left(\frac{c}{z}\right) & \text{if } z \geq c/a^*. \end{cases} \end{aligned}$$

Let  $A \geq 1$ . Since the function  $H: (a, z, w) \mapsto cw + \Lambda_W^*(a - wz)$  is uniformly continuous on the compact set  $\{(a, z, w) ; 1 \leq a \leq A, 1 \leq z \leq a, 0 \leq w \leq \frac{a-1}{z}\}$  and  $(a, z) \mapsto (a-1)/z$  is continuous on

$$D(A) := \{(a, z) ; 1 \leq a \leq A, 1 \leq z \leq a\}, \quad (18)$$

$h$  is continuous on  $D(A)$ .  $\square$

**Proposition 17.** Assume that  $(\tau, c) \in \{1\} \times (0, \infty)$ . Let  $n \geq 1$ .

1. The function  $I^n$  is continuous.
2. For all  $a \geq 1$ ,  $I^n(a) \leq cna^{1/n}$ .
3. As  $a \rightarrow \infty$ ,  $I^n(a) \sim cna^{1/n}$ .

*Proof of Proposition 17.*

1. Let  $a_0 \geq 1$  and  $\varepsilon > 0$ . Since the function  $I^n$  is nondecreasing by Proposition 14, item 1, it suffices to prove that there exists  $a_1 > a_0$  such that  $I^n(a_1) \leq I^n(a_0) + \varepsilon$ . By Lemma 16,  $h$  is uniformly continuous on the compact set  $D(a_0 + 1)$ , where  $D(a)$  has been defined in (18). So, there exists  $a_1 \in (a_0, a_0 + 1)$  such that, for all  $z \in [1, a_0]$ ,  $h(a_1, z) < h(a_0, z) + \varepsilon$ . Then, starting from (16),

$$I^n(a_1) \leq \inf_{1 \leq z \leq a_0} [I^{n-1}(z) + h(a_1, z)] \leq \inf_{1 \leq z \leq a_0} [I^{n-1}(z) + h(a_0, z)] + \varepsilon = I^n(a_0) + \varepsilon.$$

2. Let us prove the result by induction over  $n$ . First, for all  $a \geq 1$ , by (3),

$$I^1(a) = \Lambda_W^*(a) = \sup_{0 \leq t \leq c} [ta - \Lambda_W(t)] \leq ca - \inf_{0 \leq t \leq c} \Lambda_W(t) = ca.$$

By induction, taking  $(w, z, s) = ((a-1)^{1/n}, (a-1)^{(n-1)/n}, 1)$  in (4), we get

$$I^n(a) \leq cn(a-1)^{1/n} \leq cna^{1/n}.$$

3. We proceed again by induction over  $n$ . For all  $\varepsilon > 0$ , for all  $a \in \mathbb{R}$ , for all  $t_0 < c$ ,

$$I^1(a) = \Lambda_W^*(a) = \sup_{t \in \mathbb{R}} [ta - \Lambda_W(t)] \geq t_0 a - \Lambda_W(t_0),$$

so, since  $\Lambda_W(t_0) < \infty$  by (3),  $\liminf_{a \rightarrow \infty} I^1(a)/a \geq t_0$ . Since  $t_0 < c$  is arbitrary, we get  $\liminf_{a \rightarrow \infty} I^1(a)/a \geq c$ . Together with item 2, this yields the result. Now, assume  $n \geq 2$ . For all  $\varepsilon > 0$ , there exists  $A_\varepsilon > 0$  such that

$$\begin{aligned} I^{n+1}(a) &= \inf_{1 \leq z \leq a} [I^n(z) + h(a, z)] \\ &\geq c \inf_{1 \leq z \leq a} \left[ nz^{1/n}(1-\varepsilon) + \frac{a}{z} - \sup_{1 \leq s \leq a} \left( s \left( \frac{1}{z} - (1-\varepsilon) \right) \right) \right] - A_\varepsilon \\ &\geq c \inf_{1 \leq z \leq a} \left[ nz^{1/n}(1-\varepsilon) + a \min \left( \frac{1}{z}, 1-\varepsilon \right) \right] - A_\varepsilon \\ &= c \min((n+1)a^{1/(n+1)}(1-\varepsilon)^{n/(n+1)}, (n+a)(1-\varepsilon)) - A_\varepsilon \\ &\sim c(n+1)a^{1/(n+1)}(1-\varepsilon)^{n/(n+1)} \end{aligned}$$

using (16), by induction, and as  $a \rightarrow \infty$ . Since  $\varepsilon$  is arbitrary, we get a minoration. Together with item 2, we obtain the result.  $\square$

**Proposition 18.** Assume that  $(\tau, c) \in \{1\} \times (0, \infty)$ . The sequence  $(a^n)_{n \geq 0}$  defined by

$$a^0 := 1 \quad \text{and} \quad a^n := \inf \{a \geq 1 ; I^{n+1}(a) < I^n(a)\}$$

is increasing and diverges to infinity.

*Proof of Proposition 18.* Let  $n \geq 2$ . For all  $a \in [1, a^{n-1}]$ ,

$$I^{n+1}(a) = \inf_{1 \leq z \leq a} [I^n(z) + h(a, z)] = \inf_{1 \leq z \leq a} [I^{n-1}(z) + h(a, z)] = I^n(a)$$

by (16). Thus  $a^{n-1} \leq a^n$ . Now, assume by contradiction that the sequence  $(a^n)_{n \geq 1}$  is upper bounded by some  $A > 0$ . Let  $a_W := \inf \{a \geq 1 ; aa^* \geq c\}$  where  $a^*$  was defined in (17). Note that  $a_W \leq a^1$ . For all  $a \geq a_W$  and  $z \in [1, a]$ ,

$$h(a, z) \geq h(a, a) = c - \Lambda_W(c/a) \geq c - \Lambda_W(c/a_W) = \Lambda_W^*(a_W) =: \rho > 0.$$

Since the function  $I^n$  is uniformly continuous on  $[1, A]$  (by Proposition 17, item 1), there exists  $\eta(A) \in (0, 1)$  such that, for all  $(a, z) \in [1, A+1]^2$  with  $z \in [a - \eta(A), a]$ ,  $I^n(z) \geq I^n(a) - \frac{\rho}{2}$  (note that it implies  $\eta(A) < a_W - 1$ ). Then, for all  $a \in [a_W, A+1]$  and  $z \in [a - \eta(A), a]$ ,

$$I^n(z) + h(a, z) \geq \left( I^n(a) - \frac{\rho}{2} \right) + \rho = I^n(a) + \frac{\rho}{2} \geq I^{n+1}(a) + \frac{\rho}{2}.$$

Hence, if  $a \leq a^{n-1} + \eta(A)$ ,

$$I^{n+1}(a) = \inf_{1 \leq z \leq a} [I^n(z) + h(a, z)] = \inf_{1 \leq z \leq a - \eta(A)} [I^n(z) + h(a, z)] = \inf_{1 \leq z \leq a - \eta(A)} [I^{n-1}(z) + h(a, z)] \geq I^n(a),$$

(in the third equality, we used  $z \leq a^{n-1}$ ) hence  $a \leq a^n$ . So  $a^{n-1} + \eta(A) \leq a^n$ : this is in contradiction with the fact that, for all  $n \geq 1$ ,  $a^n \leq A$ . The same argument with  $A = a^{n-1}$  shows that  $a^{n-1} < a^n$  (note that  $a^0 = 1 < a_W \leq a^1$ ), and the conclusion of the proposition follows.  $\square$

### 3.2.2 Left deviations — Proposition 2

As mentioned after Proposition 2, this is essentially proved in [15, 16]. To complete the proof when  $a$  is a point of continuity of  $\Lambda_W^*$  or if  $\Lambda_W^*(a) = \infty$ , one may either adapt the proof in [16, Section 6] for  $n \in \llbracket 1, \infty \llbracket$ : this can be done by showing that, for all  $t \leq 0$ ,

$$\Lambda_r^n(t) := \frac{1}{r} \log \mathbb{E}[e^{trZ_r^n}] = \log \mathbb{E}[e^{tWZ_r^{n-1}}] \xrightarrow{r \rightarrow \infty} \Lambda_W(t),$$

which stems from the fact that  $Z_r^{n-1} \rightarrow 1$  almost surely as  $r \rightarrow \infty$  (see [15, Theorem 3.8 (a)]) and from the dominated convergence theorem. Or one may use [15, Theorem 6.2 (a)] and apply the contraction principle for the total mass  $\nu \mapsto \nu(1)$  which is continuous with respect to the  $\tau$ -topology.

The only possible point in  $[0, 1]$  where  $\Lambda_W^*$  is not continuous is the minimum  $\underline{w}$  of the support of the law of  $W$ , in the case where  $W$  has an atom at  $\underline{w}$ : in this case, the proof [16, Equation (6.4) sqq.] adapts easily.

### 3.2.3 Right deviations for $\tau = 1$ — Theorem 5, item 1

Let us turn to the large deviations on the right. Again, we proceed by induction. The case  $n = 1$  stems from Cramér's theorem. In particular  $I^1 = \Lambda_W^*$ . Now let  $n \geq 2$  and assume that  $(Z_r^{n-1})_{r \geq 2}$  satisfies a large deviation principle at speed  $(r)_{r \geq 2}$  with rate function  $I^{n-1}$ . Let  $a > 1$  and  $\varepsilon > 0$ . We have

$$\begin{aligned} \frac{1}{r} \log \mathbb{P}(Z_r^n \geq a) &\geq \frac{1}{r} \log \mathbb{P}(W_1 Z_{r,1}^{n-1} + (W_2 + \dots + W_r)(1 - \varepsilon) \geq ra, Z_{r,2}^{n-1}, \dots, Z_{r,r}^{n-1} \geq 1 - \varepsilon) \\ &= \frac{1}{r} \log \mathbb{P}(W_1 Z_{r,1}^{n-1} + (W_2 + \dots + W_r)(1 - \varepsilon) \geq ra) + \frac{r-1}{r} \log \mathbb{P}(Z_r^{n-1} \geq 1 - \varepsilon). \end{aligned}$$

Knowing the left deviations of  $Z_r^{n-1}$ , the second term goes to 0 as  $r \rightarrow \infty$ . Applying the contraction principle to the continuous map  $(w, z, s) \mapsto wz + (1 - \varepsilon)s$  and  $(W_1/r, Z_{r,1}^{n-1}, (W_2 + \dots + W_r)/r)_{r \geq 2}$ , we get

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P}(Z_r^n \geq a) &\geq -\inf \{cw + I^{n-1}(z) + \Lambda_W^*(s) ; w \geq 0, z \geq 0, s \geq 0, wz + (1 - \varepsilon)s > a\} \\ &\geq -\inf \{cw + I^{n-1}(z) + \Lambda_W^*(s) ; w \geq 0, z \in [1, a], s \in [1, a], wz + (1 - \varepsilon)s = a + \varepsilon\} \\ &= -\inf \left\{ \frac{c(a + \varepsilon - s(1 - \varepsilon))}{z} + I^{n-1}(z) + \Lambda_W^*(s) ; z \in [1, a], s \in [1, a] \right\}. \end{aligned}$$

Hence, optimizing in  $\varepsilon > 0$ ,

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P}(Z_r^n \geq a) &\geq -\inf_{\varepsilon > 0} \inf_{\substack{1 \leq z \leq a \\ 1 \leq s \leq a}} \left[ \frac{c(a + \varepsilon - s(1 - \varepsilon))}{z} + I^{n-1}(z) + \Lambda_W^*(s) \right] \\ &= -\inf_{\substack{1 \leq z \leq a \\ 1 \leq s \leq a}} \left[ \frac{c(a - s)}{z} + I^{n-1}(z) + \Lambda_W^*(s) \right] \\ &= -I^n(a), \end{aligned}$$

where the latter equality is (15).

Now let us establish the upper bound. Let us fix  $M \in (0, \infty)$ . Let  $\varepsilon \in (0, a - 1)$ . For all  $r \geq 2$  and for  $q \in \llbracket 0, r \rrbracket$ , we define

$$P_{n,r,q} := \mathbb{P}(Z_r^n \geq a, Z_{r,1}^{n-1}, \dots, Z_{r,q}^{n-1} \geq 1 + \varepsilon, Z_{r,q+1}^{n-1}, \dots, Z_{r,r}^{n-1} < 1 + \varepsilon),$$

so that

$$\mathbb{P}(Z_r^n \geq a) = \sum_{q=0}^r \binom{r}{q} P_{n,r,q}.$$

For  $q_0 \in \llbracket 0, r \rrbracket$  and  $\delta \in (0, I^{n-1}(1 + \varepsilon))$  (note that  $I^{n-1}(1 + \varepsilon) > 0$  stems from: either Proposition 12 if  $c = \infty$ ; or  $I^{n-1}(1) = 0$  and by Proposition 14, item 1, if  $c \in (0, \infty)$ ), one has by induction

$$\sum_{q=q_0}^r \binom{r}{q} P_{n,r,q} \leq \sum_{q=q_0}^r \binom{r}{q} \mathbb{P}(Z_r^{n-1} \geq 1 + \varepsilon)^q \leq \sum_{q=q_0}^r (re^{-r\delta})^q \leq \frac{(re^{-r\delta})^{q_0}}{1 - re^{-r\delta}} \leq e^{-rM}, \quad (19)$$

as soon as  $q_0\delta > M$  and  $r$  is large enough. Now, remembering that  $\varepsilon \in (0, a - 1)$ ,

$$\frac{1}{r} \log P_{n,r,0} \leq \frac{1}{r} \log \mathbb{P}\left(\sum_{i=1}^r W_i \geq \frac{ra}{1 + \varepsilon}\right) \xrightarrow{r \rightarrow \infty} -\Lambda_W^*\left(\frac{a}{1 + \varepsilon}\right). \quad (20)$$

Finally, let  $q \in \llbracket 1, q_0 - 1 \rrbracket$ . Write

$$\begin{aligned} \frac{1}{r} \log P_{n,r,q} &\leq \frac{1}{r} \log \mathbb{P} \left( W_1 Z_{r,1}^{n-1} + \cdots + W_q Z_{r,q}^{n-1} + (W_{q+1} + \cdots + W_r) (1 + \varepsilon) \geq ra, \right. \\ &\quad \left. Z_{r,1}^{n-1}, \dots, Z_{r,q}^{n-1} \geq 1 + \varepsilon, Z_{r,q+1}^{n-1}, \dots, Z_{r,r}^{n-1} < 1 + \varepsilon \right) \\ &\leq \frac{1}{r} \log \mathbb{P} \left( W_1 Z_{r,1}^{n-1} + \cdots + W_q Z_{r,q}^{n-1} + (W_{q+1} + \cdots + W_r) (1 + \varepsilon) \geq ra \right). \end{aligned}$$

Applying the contraction principle to the continuous map  $(y_1, \dots, y_q, s) \mapsto y_1 + \cdots + y_q + s(1 + \varepsilon)$  and  $((W_1 Z_{r,1}^{n-1}, \dots, W_q Z_{r,q}^{n-1}, W_{q+1} + \cdots + W_r)/r)_{r \geq 2}$ , we get

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{1}{r} \log P_{n,r,q} &\leq -\inf \{ g^{n-1}(y_1) + \cdots + g^{n-1}(y_q) + \Lambda_W^*(s) ; \\ &\quad y_1, \dots, y_q \geq 0, s \geq 0, y_1 + \cdots + y_q + (1 + \varepsilon)s \geq a \} \\ &= -\inf \{ g^{n-1}(y) + \Lambda_W^*(s) ; y \geq 0, s \geq 0, y + (1 + \varepsilon)s \geq a \} \\ &=: -B_{n,\varepsilon}, \end{aligned} \tag{21}$$

where  $g^{n-1}$  is the rate function of  $(WZ_r^{n-1}/r)_{r \geq 2}$  at speed  $(r)_{r \geq 2}$ , that is the concave function defined by

$$g^{n-1}(y) := \inf \{ cw + I^{n-1}(z) ; w \geq 0, z \geq 0, wz = y \} = \inf \left\{ \frac{cy}{z} + I^{n-1}(z) ; z \geq 1 \right\}.$$

Applying the principle of the largest term to (19), (20), and (21) leads to

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P}(Z_r^n \geq a) \leq -\min \left( M, \Lambda_W^* \left( \frac{a}{1 + \varepsilon} \right), B_{n,\varepsilon} \right).$$

Since  $\Lambda_W^*$  is left continuous at  $a$  and  $\varepsilon \mapsto B_{n,\varepsilon}$  is nonincreasing, we may let  $\varepsilon \rightarrow 0$  and  $M \rightarrow \infty$  to obtain

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P}(Z_r^n \geq a) \leq -\min \left( \Lambda_W^*(a), \lim_{\varepsilon \rightarrow 0} B_{n,\varepsilon} \right). \tag{22}$$

If  $c = \infty$ ,

$$\lim_{\varepsilon \rightarrow 0} B_{n,\varepsilon} \geq \lim_{\varepsilon \rightarrow 0} \Lambda_W^* \left( \frac{a}{1 + \varepsilon} \right) = \Lambda_W^*(a) = I^n(a)$$

(recall that, when  $c = \infty$ ,  $I^n = \Lambda_W^*$  on  $\mathbb{R}$ ) and the upper bound is proved.

Assume now that  $c \in (0, \infty)$ . Following the same lines as in the proof of Proposition 14, item 2, we obtain that

$$B_{n,\varepsilon} = \inf \{ cw + I^{n-1}(z) + \Lambda_W^*(s) ; w \geq 0, z \in [1, a], s \in [1, a/(1 + \varepsilon)], wz + (1 + \varepsilon)s = a \}.$$

Then,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} B_{n,\varepsilon} &= \sup_{0 < \varepsilon < a-1} \inf_{\substack{1 \leq z \leq a, \\ 1 \leq s \leq \frac{a}{1+\varepsilon}}} \left[ \frac{c(a-s)}{z} + I^{n-1}(z) + \Lambda_W^*(s) - \frac{c\varepsilon s}{z} \right] \\ &\geq \sup_{0 < \varepsilon < a-1} \inf_{\substack{1 \leq z \leq a, \\ 1 \leq s \leq a}} \left[ \frac{c(a-s)}{z} + I^{n-1}(z) + \Lambda_W^*(s) - c\varepsilon a \right] = \inf_{\substack{1 \leq z \leq a, \\ 1 \leq s \leq a}} \left[ \frac{c(a-s)}{z} + I^{n-1}(z) + \Lambda_W^*(s) \right] = I^n(a) \end{aligned} \tag{23}$$

by (15), and the upper bound follows from (22), (23), and the fact that  $\Lambda_W^* \geq I^n$ .

### 3.2.4 Right deviations for $\tau \in (0, 1)$ — Theorem 8

First, we claim that all the propositions concerning  $I^n$  which are established in Section 3.2.1 have their counterpart for  $I_\tau^n$  defined in (8) and the proofs adapt straightforwardly.

**Proposition 19.** *Assume that  $(\tau, c) \in (0, 1) \times (0, \infty)$ .*

1. *The sequence of functions  $(I_\tau^n)_{n \geq 1}$  is nonincreasing.*
2. *For all  $n \geq 1$ , the function  $I_\tau^n$  is increasing on  $[0, \infty)$ .*
3. *For all  $n \geq 2$ , for all  $a \geq 0$ ,  $I_\tau^n(a) = \inf \{ cw^\tau + I_\tau^{n-1}(z) ; w \geq 0, z \in [0, a], w(1+z) = a \}$ .*
4. *For all  $n \geq 1$ , the function  $I_\tau^n$  is continuous.*

5. For all  $n \geq 1$  and  $a \geq 1$ ,  $I_\tau^n(a) \leq cna^{\tau/n}$ .

6. For all  $n \geq 1$ , as  $a \rightarrow \infty$ ,  $I_\tau^n(a) \sim cna^{\tau/n}$ .

7. The sequence  $(a_\tau^n)_{n \geq 1}$  defined by  $a_\tau^0 = 0$  and, for all  $n \geq 1$ ,  $a_\tau^n := \inf \{a \geq 1 ; I_\tau^{n+1}(a) < I_\tau^n(a)\}$  is increasing and diverges to infinity.

We are now able to adapt Section 3.2. For the lower bound, we use the fact that

$$\mathbb{P}(Z_r^n \geq 1+a) \geq \mathbb{P}\left(\frac{W_1 Z_{r,1}^{n-1}}{r} \geq a, \frac{1}{r} \sum_{i=2}^r W_i Z_{r,i}^{n-1} \geq 1\right) = \mathbb{P}\left(\frac{W_1 Z_{r,1}^{n-1}}{r} \geq a\right) \mathbb{P}\left(\frac{1}{r} \sum_{i=2}^r W_i Z_{r,i}^{n-1} \geq 1\right).$$

Hence, by contraction and the fact that the last probability converges to 1/2 (as a consequence of [15, Theorem 4.4 (a)] and the fact that  $W_1 Z_{r,1}/r \rightarrow 0$  in probability as  $r \rightarrow \infty$ ), we get

$$\liminf_{r \rightarrow \infty} \frac{1}{r^\tau} \log \mathbb{P}(Z_r^n - 1 \geq 1) \geq -\inf \{cw^\tau + I_\tau^{n-1}(z-1) ; wz \geq a\} = I_\tau^n(a).$$

As for the upper bound, we use a decomposition in terms of

$$P_{n,r,q} := \mathbb{P}(Z_r^n - 1 \geq a, Z_{r,1}^{n-1}, \dots, Z_{r,q}^{n-1} \geq 1 + \varepsilon, Z_{r,q+1}^{n-1}, \dots, Z_{r,r}^{n-1} < 1 + \varepsilon).$$

Then, (19) becomes, as soon as  $q_0 I_\tau^{n-1}(\varepsilon) > M$ ,

$$\sum_{q=q_0}^r \binom{r}{q} P_{n,r,q} \leq e^{-r^\tau M} \quad (24)$$

for  $r$  large enough and (20) becomes, using [5, Theorem 2],

$$\limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log P_{n,r,0} \leq -c \left(\frac{1+a}{1+\varepsilon} - 1\right)^\tau = K_\tau^1 \left(\frac{1+a}{1+\varepsilon} - 1\right). \quad (25)$$

Finally, for  $q \in \llbracket 1, q_0 - 1 \rrbracket$ , the proof relies on the same contraction principle argument with

$$\limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log \mathbb{P}\left(\frac{1}{r}(W Z_r^{n-1} - 1) \geq y\right) \leq -\inf \{cw^\tau + I_\tau^{n-1}(z-1) ; wz = y\} = -I_\tau^n(y)$$

and (21) becomes

$$\begin{aligned} & \limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log P_{n,r,q} \\ & \leq -\inf \{I_\tau^n(y_1) + \dots + I_\tau^n(y_q) + K_\tau^1(s-1) ; y_1, \dots, y_q \geq 0, s \geq 1, y_1 + \dots + y_q + (1+\varepsilon)s \geq a\} \\ & = -\min\left(I_\tau^n(a-\varepsilon), K_\tau^1\left(\frac{1+a}{1+\varepsilon} - 1\right)\right) \end{aligned} \quad (26)$$

using the concavity in  $(y_1, \dots, y_q, s)$  (note that  $I_\tau^n(a) = \inf \{(a/z)^\tau + I_\tau^{n-1}(z-1) ; z \geq 1\}$  is concave as an infimum of concave functions). Putting (24), (25), and (26) together and letting  $\varepsilon \rightarrow 0$  yield the result (note that  $K_\tau^1 = I_\tau^1 \geq I_\tau^n$ ).

### 3.3 Proof of very large deviations — Theorem 5, item 2, and Theorem 8

The case  $a = 0$  is obvious. Let  $a > 0$ . Let us prove the lower bound. For all  $k \geq 1$ , we introduce the notation  $1^k$  to represent the word  $1, 1, \dots, 1$  of length  $k$ . One has

$$\begin{aligned} \frac{1}{r^{\tau(1+\alpha/n)}} \log \mathbb{P}(Z_r^n \geq r^\alpha a) & \geq \frac{1}{r^{\tau(1+\alpha/n)}} \log \mathbb{P}\left(\frac{W_{1^1} \cdots W_{1^n}}{r^n} \geq r^\alpha a\right) \\ & \geq \frac{n}{r^{\tau(1+\alpha/n)}} \log \mathbb{P}(W \geq r(r^\alpha a)^{1/n}) \xrightarrow{r \rightarrow \infty} -cna^{\tau/n}. \end{aligned}$$

As for the upper bound, we proceed by induction. For  $n = 1$ , one has

$$\frac{1}{r^{\tau(1+\alpha)}} \log \mathbb{P}(Z_r^1 \geq r^\alpha a) \xrightarrow{r \rightarrow \infty} -ca^\tau. \quad (27)$$

Indeed, if  $\tau < 1$ , it stems from [5, Theorem 2]. And, if  $\tau = 1$ , the upper bound follows from Chernoff's inequality ( $\log \mathbb{P}(Z_r^1 \geq r^\alpha a) \leq r\Lambda_W^*(r^\alpha a)$ ) and Proposition 17, item 3; and the lower bound from the fact that  $\log \mathbb{P}(Z_r^1 \geq r^\alpha a) \geq r \log \mathbb{P}(W \geq r^\alpha a)$ .

Then, for  $n \geq 2$ , we follow the same lines as in the upper bound for the large and moderate deviations. Let  $\varepsilon > 0$ . For all  $r \geq 2$  and for  $q \in \llbracket 0, r \rrbracket$ , we define

$$P_{n,r,q} := \mathbb{P} \left( Z_r^n \geq r^\alpha a, Z_{r,1}^{n-1}, \dots, Z_{r,q}^{n-1} \geq r^{\alpha(n-1)/n} \varepsilon, Z_{r,q+1}^{n-1}, \dots, Z_{r,r}^{n-1} < r^{\alpha(n-1)/n} \varepsilon \right),$$

so that

$$\mathbb{P} (Z_r^n \geq r^\alpha a) = \sum_{q=0}^r \binom{r}{q} P_{n,r,q}.$$

For  $q_0 \in \llbracket 0, r \rrbracket$  and  $c' \in (0, c)$ , one has by induction

$$\begin{aligned} \sum_{q=q_0}^r \binom{r}{q} P_{n,r,q} &\leq \sum_{q=q_0}^r \binom{r}{q} \mathbb{P} (Z_r^{n-1} \geq r^{\alpha(n-1)/n} \varepsilon)^q \\ &\leq \sum_{q=q_0}^r (r e^{-r^\tau(1+\alpha/n) c' (n-1) \varepsilon^{\tau/(n-1)}})^q \\ &\leq \frac{(r e^{-r^\tau(1+\alpha/n) c' (n-1) \varepsilon^{\tau/(n-1)}})^{q_0}}{1 - r e^{-r^\tau(1+\alpha/n) c' (n-1) \varepsilon^{\tau/(n-1)}}} \\ &\leq e^{-r^\tau(1+\alpha/n) c' n a^{\tau/n}}, \end{aligned}$$

as soon as  $(n-1)q_0 \varepsilon^{\tau/(n-1)} > n a^{\tau/n}$  and  $r$  is large enough. Hence, since  $c' \in (0, c)$  is arbitrary,

$$\limsup_{r \rightarrow \infty} \frac{1}{r^\tau(1+\alpha/n)} \log \sum_{q=q_0}^r \binom{r}{q} P_{n,r,q} \leq -c n a^{\tau/n}. \quad (28)$$

Now,

$$\begin{aligned} \frac{1}{r^\tau(1+\alpha/n)} \log P_{n,r,0} &= \frac{1}{r^\tau(1+\alpha/n)} \log \mathbb{P} \left( \sum_{i=1}^r W_i Z_{r,i}^{n-1} \geq r^{1+\alpha} a, Z_{r,1}^{n-1}, \dots, Z_{r,r}^{n-1} < r^{\alpha(n-1)/n} \varepsilon \right) \\ &\leq \frac{1}{r^\tau(1+\alpha/n)} \log \mathbb{P} \left( \sum_{i=1}^r W_i \geq r^{1+\alpha/n} \frac{a}{\varepsilon} \right) \xrightarrow{r \rightarrow \infty} -c \left( \frac{a}{\varepsilon} \right)^\tau, \end{aligned} \quad (29)$$

by (27). Hence, for  $\varepsilon > 0$  small enough, one gets

$$\limsup_{r \rightarrow \infty} \frac{1}{r^\tau(1+\alpha/n)} \log P_{n,r,0} \leq -c n a^{\tau/n}. \quad (30)$$

Finally, let  $q \in \llbracket 1, q_0 - 1 \rrbracket$ . One has

$$P_{n,r,q} \leq \mathbb{P} \left( \sum_{i=1}^q W_i Z_{r,i}^{n-1} + \sum_{i=q+1}^r W_i r^{\alpha(n-1)/n} \varepsilon \geq r^{1+\alpha} a \right).$$

We want to apply the contraction principle to the continuous map  $(y_1, \dots, y_q, s) \mapsto y_1 + \dots + y_q + s$  and

$$\left( \frac{1}{r^{1+\alpha}} \left( W_1 Z_{r,1}^{n-1}, \dots, W_q Z_{r,q}^{n-1}, \sum_{i=q+1}^r W_i r^{\alpha(n-1)/n} \varepsilon \right) \right)_{r \geq 2}.$$

As for the first  $q$  variables, applying the contraction principle to the continuous map  $(w, z) \mapsto wz$  and  $(W/r^{1+\alpha/n}, Z_r^{n-1}/r^{\alpha(n-1)/n})_{r \geq 2}$ , we get by induction, for all  $y > 0$ ,

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{1}{r^\tau(1+\alpha/n)} \log \mathbb{P} (W Z_r^{n-1} \geq r^{1+\alpha} y) &\leq -\inf \left\{ c w^\tau + (n-1) c z^{\tau/(n-1)} ; w > 0, z > 0, w z = y \right\} \\ &= -c \inf \left\{ \left( \frac{y}{z} \right)^\tau + (n-1) z^{\tau/(n-1)} ; z > 0 \right\} \\ &= -c n y^{\tau/n} \end{aligned} \quad (31)$$

(the infimum is attained at  $z = y^{(n-1)/n}$ ). Hence, applying the contraction principle and using (29) and (31), we get

$$\limsup_{r \rightarrow \infty} \frac{1}{r^\tau(1+\alpha/n)} \log P_{n,r,q} \leq -c \inf \left\{ n (y_1^{\tau/n} + \dots + y_q^{\tau/n}) + \left( \frac{s}{\varepsilon} \right)^\tau ; y_1 + \dots + y_q + s = a \right\} = -c n a^{\tau/n} \quad (32)$$

as soon as  $\varepsilon$  is small enough. Finally, applying the principle of the largest term to (28), (30), and (32), one gets

$$\limsup_{r \rightarrow \infty} \frac{1}{r^\tau(1+\alpha/n)} \log \mathbb{P} (Z_r^n \geq r^\alpha a) \leq -c n a^{\tau/n}.$$

## 4 Infinite tree

The heart of the proofs in the finite tree is the recursion formula (9) for  $Z_r^n$  which allows to proceed by induction. However, in the infinite tree, the analogue of (9) is the fixed point equation (2) and there is no more recursion. From now on, we set  $Z_r := Z_r^\infty$  for all  $r \geq 2$ . By [12, Théorèmes 1 et 3], if  $\text{ess sup}(W) = \infty$  and  $r > \exp \mathbb{E}[W \log(W)]$ , then

$$\forall t > 0 \quad \mathbb{E}[e^{tZ_r}] = \infty.$$

Therefore, under assumption  $(\mathbf{A}_{\tau,c})$  and  $\text{ess sup}(W) = \infty$  (see Table 1), the standard exponential Markov inequality cannot be used to prove upper bounds. Here, we bypass this problem by bounding above the moments of  $Z_r$ .

### 4.1 Upper bounds for the moments of $Z_r^\infty$

Let us turn to the control of the moments of  $Z_r$ . Using the fact that positive martingales converge almost surely, we may let  $n \rightarrow \infty$  in (9) to get

$$Z_r = \frac{1}{r} \sum_{i=1}^r W_i Z_{r,i} \quad \text{a.s.}$$

where the random variables  $Z_{r,1}, \dots, Z_{r,r}, W_1, \dots, W_r$  are independent, the  $W_i$  (resp.  $Z_{r,i}$ ) having the same distribution as  $W$  (resp.  $Z_r$ ). Obviously, this equation implies (2). It is proved in [12, Théorème 1] that, for

$$r > \exp \mathbb{E}[W \log(W)],$$

$Z_r$  is the unique solution  $Z$  of (2) such that  $\mathbb{E}[Z] = 1$ . Moreover, by [12, Théorème 2],  $\mathbb{E}[Z_r^h] < \infty$  if and only if  $\mathbb{E}[W^h] < r^{h-1}$ , which is equivalent to  $h < \chi(r)$  for

$$\chi(r) := \sup \{h \in [1, \infty) ; \mathbb{E}[W^h] < r^{h-1}\}.$$

In the sequel, we always consider values of  $r$  which satisfy  $r > \exp \mathbb{E}[W \log(W)]$ . Moreover, we assume that there exists  $(\tau, c) \in (0, 1] \times (0, \infty) \cup \{(1, \infty)\}$  such that

$$(\tilde{\mathbf{A}}_{\tau,c}) \quad \limsup_{w \rightarrow \infty} \frac{1}{w^\tau} \log \mathbb{P}(W \geq w) = -c.$$

which is slightly more general than  $(\mathbf{A}_{\tau,c})$ .

**Proposition 20** (Moments of  $W$ ). *Let  $W$  be a random variable satisfying  $(\tilde{\mathbf{A}}_{\tau,c})$  with  $(\tau, c) \in (0, 1] \times (0, \infty) \cup \{(1, \infty)\}$ .*

1. *One has*

$$\limsup_{h \rightarrow \infty} \frac{\tau}{h} \log \frac{\mathbb{E}[W^h]}{\Gamma(h/\tau)} \leq -\log(c).$$

2. *One has*

$$\liminf_{r \rightarrow \infty} \frac{\chi(r)}{r^\tau} \geq \tau c e.$$

3. *For all  $\eta \in (0, ce)$ , for all  $r$  large enough,*

$$\sup_{h \in [2, \eta r^\tau]} \frac{\mathbb{E}[W^h]}{r^{h-1}} = \frac{\mathbb{E}[W^2]}{r}.$$

4. *Let  $W_1, \dots, W_r$  be independent copies of  $W$ . There exists  $C > 0$  such that, for all  $h \geq 1$ ,*

$$\mathbb{E}[(W_1 + \dots + W_r)^h]^{1/h} \leq C \max(r, h^{1/\tau}).$$

**Remark 21.** *Under  $(\mathbf{A}_{\tau,c})$  with  $(\tau, c) \in (0, 1] \times (0, \infty) \cup \{(1, \infty)\}$ , one even has*

$$\frac{\tau}{h} \log \frac{\mathbb{E}[W^h]}{\Gamma(h/\tau)} \xrightarrow{h \rightarrow \infty} -\log(c) \quad \text{and} \quad \frac{\chi(r)}{r^\tau} \xrightarrow{r \rightarrow \infty} \tau c e.$$

*Proof of Proposition 20.*

1. Let  $\tilde{c} \in (0, c)$  and let  $A \geq 1$  be such that, for all  $w \geq A$ ,

$$\mathbb{P}(W \geq w) \leq e^{-\tilde{c}w^\tau}.$$

Denoting  $k := \lceil h/\tau \rceil$ , one has

$$\mathbb{E}[W^h] \leq A + \int_A^\infty e^{-\tilde{c}t^{\tau/h}} dt \leq A + \int_A^\infty e^{-\tilde{c}t^{1/k}} dt = A + k \int_{A^{1/k}}^\infty e^{-\tilde{c}w} w^{k-1} dw.$$

Then, integrating by parts,

$$\mathbb{E}[W^h] \leq A + \frac{k}{\tilde{c}} A^{1-\frac{1}{k}} e^{-\tilde{c}A^{1/k}} + \frac{k(k-1)}{\tilde{c}} \int_{A^{1/k}}^\infty e^{-\tilde{c}w} w^{k-2} dw.$$

By induction, we get

$$\mathbb{E}[W^h] \leq A + \frac{k!}{\tilde{c}^k} e^{-\tilde{c}A^{1/k}} \sum_{l=0}^{k-1} \frac{(\tilde{c}A^{1/k})^l}{l!} \leq A + \frac{k!}{\tilde{c}^k}.$$

Letting  $h \rightarrow \infty$  and then  $\tilde{c} \rightarrow c$  yields the upper bound

$$\limsup_{h \rightarrow \infty} \frac{\tau}{h} \log \frac{\mathbb{E}[W^h]}{\Gamma(h/\tau)} \leq -\log(c).$$

2. Let  $\eta \in (0, ce)$ . It suffices to prove that, for all  $r$  large enough,

$$\frac{\mathbb{E}[W^{\lfloor \eta r^\tau \rfloor}]}{r^{\eta \tau r^\tau - 1}} < 1.$$

Let  $\tilde{c} \in (0, c)$  be such that  $\eta < \tilde{c}e$ . Using item 1 and Stirling bounds (see [1, Section 3])

$$\forall h \in (0, \infty) \quad \left(\frac{h}{e}\right)^h \sqrt{\frac{2\pi}{h}} \leq \Gamma(h) \leq \left(\frac{h}{e}\right)^h \sqrt{\frac{2\pi}{h}} e^{\frac{1}{12h}}, \quad (33)$$

we get, for  $r$  large enough,

$$\frac{\mathbb{E}[W^{\eta \tau r^\tau}]}{r^{\eta \tau r^\tau - 1}} \leq \frac{\Gamma(\eta \tau r^\tau)}{r^{\eta \tau r^\tau - 1} \tilde{c}^{\eta \tau r^\tau}} \leq \left(\frac{\eta}{\tilde{c}e}\right)^{\eta \tau r^\tau} r \sqrt{\frac{2\pi}{\eta \tau r^\tau}} e^{\frac{1}{12\eta \tau r^\tau}} < 1. \quad (34)$$

3. Let  $\eta \in (0, ce)$ . Since the function  $h \mapsto \mathbb{E}[W^h]/r^{h-1}$  is log-convex (see [12, p. 132]), if  $\eta r \geq 2$ , then

$$\sup_{h \in [2, \eta \tau r^\tau]} \frac{\mathbb{E}[W^h]}{r^{h-1}} = \max \left\{ \frac{\mathbb{E}[W^2]}{r}, \frac{\mathbb{E}[W^{\eta \tau r^\tau}]}{r^{\eta \tau r^\tau - 1}} \right\}.$$

The conclusion stems from the fact that (34) yields besides

$$\frac{\mathbb{E}[W^{\eta \tau r^\tau}]}{r^{\eta \tau r^\tau - 1}} = o\left(\frac{1}{r}\right).$$

4. We apply Corollary 1 in [14] to get

$$\mathbb{E}[(W_1 + \dots + W_r)^h]^{1/h} \leq 2e^2 \sup_{1 \leq s \leq h} \left\{ \frac{h}{s} \left(\frac{r}{h}\right)^{1/s} \mathbb{E}[W^s]^{1/s} \right\}.$$

Now by item 1 and using Stirling's estimates, there exists  $D > 0$  such that  $\mathbb{E}[W^h]^{1/s} \leq Ds^{1/\tau}$ , whence

$$\mathbb{E}[(W_1 + \dots + W_r)^h]^{1/h} \leq 2e^2 D h \sup_{1 \leq s \leq h} \left\{ s^{1/\tau-1} \left(\frac{r}{h}\right)^{1/s} \right\}.$$

An easy computation shows that the function  $s \mapsto s^{1/\tau-1} \left(\frac{r}{h}\right)^{1/s}$  attains its supremum at  $s = 1$  or at  $s = h$  whence

$$\mathbb{E}[(W_1 + \dots + W_r)^h]^{1/h} \leq 2e^2 D \max \left( r, h^{1/\tau} \left(\frac{r}{h}\right)^{1/h} \right).$$

Finally, we prove that, for all  $r \geq 1$  and  $h \geq 1$ ,  $f(r, h) := h^{1/\tau} \left(\frac{r}{h}\right)^{1/h} \leq e^{1/(\tau e)} \max(r, h^{1/\tau})$ , whence the result with  $C = 2e^{2+1/(\tau e)} D$ . Indeed, for  $h \geq r^\tau$ , the expression  $\left(\frac{r}{h}\right)^{1/h}$  attains its maximum at  $h = r^\tau$ , the maximum value being  $r^{(1-\tau)/r^\tau} \leq e^{(1-\tau)/(\tau e)}$ ; and for  $1 \leq h \leq r^\tau$ ,  $f(r, h) \leq h^{1/\tau} r^{1/h} \leq r^{1+1/r^\tau} \leq r e^{1/(\tau e)}$ .  $\square$

Now, it is proved in [12, Equation (13)] that, for all  $r > \exp \mathbb{E}[W \log(W)]$ , we have the following recursion formula:

$$\forall h \in \llbracket 2, \chi(r) \rrbracket \quad \mathbb{E}[Z_r^h] = \frac{h!}{r^h \left(1 - \frac{\mathbb{E}[W^h]}{r^{h-1}}\right)} \sum_{\substack{h_1 + \dots + h_r = h, \\ 0 \leq h_j \leq h-1}} \prod_{j=1}^r \frac{\mathbb{E}[W^{h_j}] \mathbb{E}[Z_r^{h_j}]}{h_j!}. \quad (35)$$

Let us turn to bounds on the moments of  $Z_r$ . First, we consider the small moments.

**Proposition 22.** *There exists  $C > 0$  such that, for all  $\delta > 0$ , there exists  $\eta_0 \in (0, 1 \wedge ce)$  such that, for all  $r$  large enough,*

$$\forall h \in \llbracket 1, \eta_0 \tau r^\tau \rrbracket \quad \mathbb{E}[Z_r^h] \leq \exp \left\{ \frac{h^2}{2(r-h)} \left( \text{Var}(W) + \delta + \frac{C}{h} \right) \right\}.$$

*Proof of Proposition 22.* Let  $\delta > 0$ . Let us fix some  $\eta \in (0, 1 \wedge ce)$ . By Proposition 20, item 3, there exists  $r_0 > \exp \mathbb{E}[W \log(W)]$  such that

$$\forall r \geq r_0 \quad \sup_{h \in \llbracket 2, \eta \tau r^\tau \rrbracket} \frac{\mathbb{E}[W^h]}{r^{h-1}} = \frac{\mathbb{E}[W^2]}{r} \leq \frac{1}{2}. \quad (36)$$

In particular, for all  $r \geq r_0$ ,  $\eta \tau r^\tau < \chi(r)$ . Now, let us fix  $r \geq r_0$ . For all  $h \geq 1$ , denote by  $E(h)$  the statement:

$$\mathbb{E}[Z_r^h] \leq \exp \left\{ \frac{h^2}{2(r-h)} \left( \text{Var}(W) + \delta + \frac{C}{h} \right) \right\}$$

and let us prove  $E(h)$  by induction for  $h \in \llbracket 1, \eta \tau r^\tau \rrbracket$  (note that  $\eta < 1$  implies  $h < \eta \tau r^\tau < r$ ). Since  $\mathbb{E}[Z_r] = 1$ ,  $E(1)$  is obvious. Now, in case  $\eta \tau r^\tau \geq 2$ , let  $h \in \llbracket 2, \eta \tau r^\tau \rrbracket$  and assume that  $E(k)$  is satisfied for all  $k \in \llbracket 1, h-1 \rrbracket$ . From (35) and the fact that  $h \leq \eta \tau r^\tau < \chi(r)$ , one has

$$\begin{aligned} \mathbb{E}[Z_r^h] &= \frac{h!}{r^h \left(1 - \frac{\mathbb{E}[W^h]}{r^{h-1}}\right)} \sum_{\substack{m_0 + \dots + m_{h-1} = r, \\ m_1 + \dots + (h-1)m_{h-1} = h}} \binom{r}{m_0, \dots, m_{h-1}} \prod_{k=0}^{h-1} \left( \frac{\mathbb{E}[W^k] \mathbb{E}[Z_r^k]}{k!} \right)^{m_k} \\ &= \frac{h! r!}{r^h \left(1 - \frac{\mathbb{E}[W^h]}{r^{h-1}}\right)} \sum_{\substack{m_0 + \dots + m_{h-1} = r, \\ m_1 + \dots + (h-1)m_{h-1} = h}} \prod_{k=0}^{h-1} \frac{1}{m_k!} \left( \frac{\mathbb{E}[W^k] \mathbb{E}[Z_r^k]}{k!} \right)^{m_k}, \end{aligned} \quad (37)$$

where  $m_k$  stands for the number of  $h_j$  equal to  $k$  (multiplicity of  $k$  in the composition  $(h_1, \dots, h_r)$  of  $h$ ). Then, the latter sum above is equal to

$$\begin{aligned} &\sum_{\substack{m_0 + \dots + m_{h-1} = r, \\ m_1 + \dots + (h-1)m_{h-1} = h}} \frac{(r-h)^{m_0 - (r-h)}}{m_0!} \cdot \frac{h^{-(h-m_1)}}{m_1!} \prod_{k=2}^{h-1} \frac{1}{m_k!} \left( \frac{\mathbb{E}[W^k] \mathbb{E}[Z_r^k] h^k}{k! (r-h)^{k-1}} \right)^{m_k} \\ &\leq \sum_{\substack{m_0 + \dots + m_{h-1} = r, \\ m_1 + \dots + (h-1)m_{h-1} = h}} \frac{1}{(r-h)!} \cdot \frac{1}{h!} \prod_{k=2}^{h-1} \frac{1}{m_k!} \left( \frac{\mathbb{E}[W^k] \mathbb{E}[Z_r^k] h^k}{k! (r-h)^{k-1}} \right)^{m_k} \\ &\leq \frac{1}{(r-h)! h!} \sum_{m_2, \dots, m_{h-1} \geq 0} \prod_{k=2}^{h-1} \frac{1}{m_k!} \left( \frac{\mathbb{E}[W^k] \mathbb{E}[Z_r^k] h^k}{k! (r-h)^{k-1}} \right)^{m_k} \\ &= \frac{1}{(r-h)! h!} \exp \left\{ \sum_{k=2}^{h-1} \frac{\mathbb{E}[W^k] \mathbb{E}[Z_r^k] h^k}{k! (r-h)^{k-1}} \right\}. \end{aligned} \quad (38)$$

Besides, using Stirling bounds in (33) and  $\log(1+x) \leq x - x^2/2 + x^3/3$  for all  $x \geq 0$ , we have

$$\begin{aligned} \frac{r!}{(r-h)! r^h} &\leq \left( \frac{r}{r-h} \right)^{r-h+1/2} \exp \left\{ -h + \frac{1}{12r} \right\} \\ &\leq \exp \left\{ \left( r-h + \frac{1}{2} \right) \left( \frac{h}{r-h} - \frac{1}{2} \left( \frac{h}{r-h} \right)^2 + \frac{1}{3} \left( \frac{h}{r-h} \right)^3 \right) - h + \frac{1}{12r} \right\} \\ &\leq \exp \left\{ \frac{h^2}{2(r-h)} \left( -1 + \frac{1}{h} + \frac{2h}{3(r-h)} + \frac{h}{3(r-h)^2} + \frac{r-h}{6rh^2} \right) \right\} \\ &\leq \exp \left\{ \frac{h^2}{2(r-h)} \left( -1 + \frac{13}{12h} + \frac{h}{r-h} \right) \right\}. \end{aligned} \quad (39)$$

Combining (37), (38), (39), and (36) together with  $(1-x)^{-1} \leq \exp(2x)$  for  $x \in [0, 1/2]$ , we get

$$\begin{aligned} \mathbb{E}[Z_r^h] &\leq \exp \left\{ \frac{h^2}{2(r-h)} \left( -1 + \frac{13}{12h} + \frac{h}{r-h} \right) + \frac{2\mathbb{E}[W^2]}{r} + \sum_{k=2}^{h-1} \frac{\mathbb{E}[W^k]\mathbb{E}[Z_r^k]h^k}{k!(r-h)^{k-1}} \right\} \\ &\leq \exp \left\{ \frac{h^2}{2(r-h)} \left( -1 + \frac{C}{h} + \frac{h}{r-h} + 2 \sum_{k=2}^{h-1} \frac{\mathbb{E}[W^k]\mathbb{E}[Z_r^k]}{k!} \left( \frac{h}{r-h} \right)^{k-2} \right) \right\} \end{aligned} \quad (40)$$

with  $C = 2\mathbb{E}[W^2] + 13/12$ . Using Proposition 20, item 1, and Stirling's formula, there exists  $A > 0$  such that, for all  $k \geq 3$ ,

$$\frac{\mathbb{E}[W^k]}{k!} \leq k^{(k-2)(\frac{1}{\tau}-1)} A^{k-2}.$$

Hence, for  $3 \leq k \leq h-1 \leq \eta\tau r^\tau$ ,

$$\frac{\mathbb{E}[W^k]}{k!} \left( \frac{h}{r-h} \right)^{k-2} = \left( \frac{k^{\frac{1}{\tau}-1} h A}{r-h} \right)^{k-2} \leq \left( \frac{h^{\frac{1}{\tau}} A}{r-h} \right)^{k-2} \leq \left( \frac{(\eta\tau)^{\frac{1}{\tau}} A}{1-\eta} \right)^{k-2}.$$

Note that  $D(\tilde{\eta}) := \frac{(\tilde{\eta}\tau)^{\frac{1}{\tau}} A}{1-\tilde{\eta}}$  goes to zero as  $\tilde{\eta} \rightarrow 0$ . For all  $\tilde{\eta} \in (0, 1)$ , let

$$B(\tilde{\eta}) := \exp \left\{ \frac{\tilde{\eta}}{2(1-\tilde{\eta})} \left( \text{Var}(W) + \delta + \frac{C}{2} \right) \right\}.$$

Assume now that  $\eta$  is small enough that

$$\frac{\eta}{1-\eta} + \mathbb{E}[W^2](B(\eta)^2 - 1) + \frac{2B(\eta)^3 D(\eta)}{1-B(\eta)D(\eta)} < \delta. \quad (41)$$

Using (40) and noting that  $h/(r-h) \leq \eta/(1-\eta)$ , one gets

$$\begin{aligned} \mathbb{E}[Z_r^h] &\leq \exp \left\{ \frac{h^2}{2(r-h)} \left( -1 + \frac{C}{h} + \frac{\eta}{1-\eta} + \mathbb{E}[W^2]B(\eta)^2 + 2 \sum_{k=3}^{h-1} B(\eta)^k D(\eta)^{k-2} \right) \right\} \\ &\leq \exp \left\{ \frac{h^2}{2(r-h)} \left( \text{Var}(W) + \delta + \frac{C}{h} \right) \right\}, \end{aligned}$$

bounding above the finite geometric sum by the value of the infinite sum and using (41).  $\square$

The following estimate concerns the higher moments of  $Z_r$ .

**Proposition 23.** For all  $\eta \in (0, ce)$ ,

$$\limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log \mathbb{E}[Z_r^{\eta\tau r^\tau}] < \infty.$$

*Proof of Proposition 23.* Let  $r > \exp \mathbb{E}[W \log(W)]$ . Let  $\eta \in (0, ce)$ . Let  $c'$  and  $c''$  be real numbers such that  $\eta/e < c'' < c' < c$ . As a consequence of Proposition 20, item 1, there exists  $D \geq 1$  such that, for all  $h \geq 1$ ,

$$\mathbb{E}[W^h] \leq D \left( \frac{h}{\tau c' e} \right)^{h/\tau}. \quad (42)$$

Let  $\theta \in (1/2, 1)$ . For all  $h \in \llbracket 2, \eta\tau r^\tau \rrbracket$ , we write the Euclidean division  $h = q \lfloor \theta h \rfloor + s$ . Using (35), (42), and the convexity of  $(h_1, \dots, h_r) \mapsto \log \prod_{j=1}^r \mathbb{E}[Z_r^{h_j}]$ , we get

$$\begin{aligned} \mathbb{E}[Z_r^h] &= \frac{h!}{r^h \left(1 - \frac{\mathbb{E}[W^h]}{r^{h-1}}\right)} \sum_{\substack{h_1 + \dots + h_r = h, \\ 0 \leq h_j \leq h-1}} \prod_{j=1}^r \mathbb{E}[Z_r^{h_j}] \prod_{j=1}^r \frac{\mathbb{E}[W^{h_j}]}{h_j!} \\ &\leq \frac{h!}{r^h \left(1 - \frac{\mathbb{E}[W^h]}{r^{h-1}}\right)} \left( \sum_{\forall j, h_j < \theta h} \mathbb{E}[Z_r^{\lfloor \theta h \rfloor}]^q \mathbb{E}[Z_r^s] \prod_{j=1}^r \frac{\mathbb{E}[W^{h_j}]}{h_j!} + \sum_{k=\lceil \theta h \rceil}^{h-1} \sum_{\exists j, h_j=k} \mathbb{E}[Z_r^{h-1}] \prod_{j=1}^r \frac{\mathbb{E}[W^{h_j}]}{h_j!} \right) \\ &\leq \frac{h!}{r^h \left(1 - \frac{\mathbb{E}[W^h]}{r^{h-1}}\right)} \left( \mathbb{E}[Z_r^{\theta h}]^\beta \frac{1}{h!} \mathbb{E}[(W_1 + \dots + W_r)^h] \right. \\ &\quad \left. + \mathbb{E}[Z_r^{h-1}]^r \sum_{k=\lceil \theta h \rceil}^{h-1} \frac{\mathbb{E}[W^k]}{k!} \frac{1}{(h-k)!} \mathbb{E}[(W_2 + \dots + W_r)^{h-k}] \right) \end{aligned} \quad (43)$$

with

$$\beta := \frac{1}{\theta - \frac{1}{2}} + 1 \geq \frac{1}{\theta - \frac{1}{h}} + 1 \geq \left\lfloor \frac{h}{\lfloor \theta h \rfloor} \right\rfloor + 1 = q + 1,$$

and using the fact that  $h \in [1, \infty) \mapsto \mathbb{E}[Z_r^h]$  is nondecreasing and greater or equal to 1.

Our strategy is to bound the moments of  $Z_r$  by backward induction, finally relying on the fact that the small moments are well bounded. Indeed, by Proposition 22, there exists  $\eta_0 \in (0, 1 \wedge ce)$  such that

$$\limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log \mathbb{E}[Z_r^{\eta_0 \tau r^\tau}] < \infty.$$

Let us consider the first term in (43). By Proposition 20, items 3 and 4, there exists  $C_1 > 0$  and  $r_0 \geq 2$  such that, for all  $r \geq r_0$  and  $h \in \llbracket 2, \eta \tau r^\tau \rrbracket$ ,

$$\frac{1}{r^h \left(1 - \frac{\mathbb{E}[W_r^h]}{r^{h-1}}\right)} \mathbb{E}[(W_1 + \dots + W_r)^h] \leq e^{C_1 r^\tau}. \quad (44)$$

As for the second term in (43), let us estimate

$$f(h, k) := \frac{1}{r^h} \binom{h}{k} \mathbb{E}[W^k] \mathbb{E}[(W_2 + \dots + W_r)^{h-k}]. \quad (45)$$

First, using Stirling's bounds,

$$\binom{h}{k} \leq \frac{h^h}{k^k (h-k)^{h-k}} e^{1/(12h)} \sqrt{\frac{h}{2\pi k(h-k)}} \leq \frac{h^h}{k^k (h-k)^{h-k}}$$

as soon as  $h \geq 2$  and  $1 \leq k \leq h-1$ . Using this bound together with (42) and Proposition 20, item 4, we get

$$\begin{aligned} f(h, k) &\leq \frac{1}{r^h} \frac{h^h}{k^k (h-k)^{h-k}} D\left(\frac{k}{\tau c' e}\right)^{k/\tau} (C \max(r, h^{1/\tau}))^{h-k} \\ &\leq \left(\frac{h}{r^\tau \tau c' e}\right)^{k/\tau} \left(\frac{k}{h}\right)^{k(1/\tau-1)} \left(1 - \frac{k}{h}\right)^{-(h-k)} D\left(C \max\left(1, \left(\frac{h}{r^\tau}\right)^{1/\tau}\right)\right)^{h-k}. \end{aligned}$$

Now, for  $k \leq h \leq \eta \tau r^\tau$ ,

$$f(h, k) \leq \left(\frac{\eta}{c' e}\right)^{k/\tau} \left(1 - \frac{k}{h}\right)^{-(h-k)} \tilde{C}^{h-k}$$

where  $\tilde{C} := DC \max(1, (\eta \tau)^{1/\tau})$ . Now, there exists  $\theta \in (1/2, 1)$  such that, if  $k/h \geq \theta$ , then  $(\tilde{C}(1 - k/h))^{1-k/h} \leq (c'/c'')^{1/\tau}$ , leading to

$$f(h, k) \leq \left(\frac{\eta}{c' e}\right)^{h/\tau}. \quad (46)$$

Now, putting together (43), (44), (45), and (46), we get, for all  $h \in \llbracket 2, \eta \tau r^\tau \rrbracket$ ,

$$\mathbb{E}[Z_r^h] \leq \mathbb{E}[Z_r^{\theta h}]^\beta e^{C_1 r^\tau} + \mathbb{E}[Z_r^{h-1}] \frac{r(1-\theta)h}{1 - \frac{\mathbb{E}[W_r^h]}{r^{h-1}}} \left(\frac{\eta}{c' e}\right)^{h/\tau}.$$

Hence, for all  $r$  large enough, for all  $h \in \llbracket \eta_0 \tau r^\tau, \eta \tau r^\tau \rrbracket$ ,

$$\begin{aligned} \mathbb{E}[Z_r^h] &\leq e^{C_1 r^\tau} \mathbb{E}[Z_r^{\theta h}]^\beta + \frac{1}{2} \mathbb{E}[Z_r^{h-1}] \\ &\leq e^{C_1 r^\tau} \mathbb{E}[Z_r^{\theta h}]^\beta + \frac{1}{2} \left( e^{C_1 r^\tau} \mathbb{E}[Z_r^{\theta(h-1)}]^\beta + \frac{1}{2} \mathbb{E}[Z_r^{h-2}] \right) \\ &\leq e^{C_1 r^\tau} \mathbb{E}[Z_r^{\theta h}]^\beta \sum_{k=0}^{h - \lfloor \theta h \rfloor - 1} \frac{1}{2^k} + \frac{1}{2^{h - \lfloor \theta h \rfloor}} \mathbb{E}[Z_r^{\theta h}], \end{aligned}$$

by induction and the monotonicity of  $h \mapsto \mathbb{E}[Z_r^{\theta h}]$ . Since  $\beta \geq 1$ ,  $C_1 \geq 0$ , and  $\mathbb{E}[Z_r^{\theta h}] \geq 1$ , we obtain

$$\mathbb{E}[Z_r^h] \leq e^{C_1 r^\tau} \mathbb{E}[Z_r^{\theta h}]^\beta \sum_{k=0}^{h - \lfloor \theta h \rfloor} \frac{1}{2^k} \leq 2e^{C_1 r^\tau} \mathbb{E}[Z_r^{\theta h}]^\beta. \quad (47)$$

Let  $K := \min \{k \geq 1 ; \theta^k \eta \leq \eta_0\}$ . Applying (47) recursively, we get

$$\begin{aligned} \log \mathbb{E}[Z_r^{\eta r^\tau}] &\leq \log(2) + C_1 r^\tau + \beta \log \mathbb{E}[Z_r^{\theta \eta r^\tau}] \\ &\leq (\log(2) + C_1 r^\tau)(1 + \beta) + \beta^2 \log \mathbb{E}[Z_r^{\theta^2 \eta r^\tau}] \\ &\leq (\log(2) + C_1 r^\tau) \sum_{k=0}^{K-1} \beta^k + \beta^K \log \mathbb{E}[Z_r^{\theta^K \eta r^\tau}]. \end{aligned}$$

Finally,

$$\limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log \mathbb{E}[Z_r^{\eta r^\tau}] \leq C_1 \frac{\beta^K - 1}{\beta - 1} + \beta^K \limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log \mathbb{E}[Z_r^{\eta_0 r^\tau}] < \infty. \quad \square$$

Propositions 22 and 23 immediately entail the following result.

**Corollary 24.**

1. For all  $\gamma \in [0, \frac{1}{2} \wedge \tau)$  and  $\zeta > 0$ ,

$$\lim_{r \rightarrow \infty} \mathbb{E}[Z_r^{\zeta r^\gamma}] = 1.$$

2. For all  $\gamma \in [\frac{1}{2} \wedge \tau, \tau)$  and  $\zeta > 0$ ,

$$\limsup_{r \rightarrow \infty} \frac{1}{r^{2\alpha-1}} \log \mathbb{E}[Z_r^{\zeta r^\gamma}] \leq \frac{\zeta^2 \text{Var}(W)}{2}.$$

3. For all  $\eta \in (0, ce)$ ,

$$\kappa(\eta) := \limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log \mathbb{E}[Z_r^{\eta r^\tau}] < \infty$$

and  $\kappa(\eta) = O(\eta^2)$  as  $\eta \rightarrow 0$  (if  $\tau < 1$ ,  $\kappa$  even vanishes on a neighborhood of zero).

## 4.2 Proof of moderate deviations

### 4.2.1 Left deviations — Proposition 3

The proof of Section 3.1.1 adapts straightforwardly to  $Z_r$  instead of  $Z_r^n$ , noting that (2) implies

$$\Lambda_r(t) := \frac{1}{r^{1+2\alpha}} \log \mathbb{E}[e^{tr^{1+\alpha}(Z_r-1)}] = \frac{1}{r^{1+2\alpha}} \log \mathbb{E} \left[ \exp \left( \frac{tr^{1+\alpha}}{r} \sum_{i=1}^r (W_i Z_{r,i} - 1) \right) \right] = r^{-2\alpha} \log \mathbb{E}[e^{tr^\alpha(W Z_r - 1)}].$$

### 4.2.2 Right deviations for $W$ light-tailed — Theorem 4

Let  $\alpha \in (-1/2, 0)$  and  $a \geq 0$ . Using the fixed point equation (2), one gets

$$\mathbb{P}(Z_r - 1 \geq r^\alpha a) = \mathbb{P} \left( \sum_{i=1}^r (W_i Z_{r,i} - 1) \geq r^{1+\alpha} a \right).$$

Let us prove the lower bound. For all  $\varepsilon > 0$ ,

$$\begin{aligned} &\frac{1}{r^{1+2\alpha}} \log \mathbb{P} \left( \sum_{i=1}^r (W_i Z_{r,i} - 1) \geq r^{1+\alpha} a \right) \\ &\geq \frac{1}{r^{1+2\alpha}} \log \mathbb{P} \left( \sum_{i=1}^r (W_i Z_{r,i} - 1) \geq r^{1+\alpha} a, \forall i \in \llbracket 1, r \rrbracket \ Z_{r,i} \geq 1 - r^\alpha \varepsilon \right) \\ &\geq \frac{1}{r^{1+2\alpha}} \log \mathbb{P} \left( \sum_{i=1}^r (W_i (1 - r^\alpha \varepsilon) - 1) \geq r^{1+\alpha} a \right) + r^{-2\alpha} \log \mathbb{P}(Z_r \geq 1 - r^\alpha \varepsilon). \end{aligned}$$

Since  $\Lambda_W$  is finite on a neighborhood of 0, we can apply Gärtner-Ellis theorem and get

$$\frac{1}{r^{1+2\alpha}} \log \mathbb{P} \left( \sum_{i=1}^r (W_i (1 - r^\alpha \varepsilon) - 1) \geq r^{1+\alpha} a \right) \xrightarrow{r \rightarrow \infty} -J(a + \varepsilon). \quad (48)$$

The left deviations for  $Z_r$  yield

$$\frac{1}{r^{1+2\alpha}} \log \mathbb{P}(Z_r - 1 < -r^\alpha \varepsilon) \xrightarrow{r \rightarrow \infty} -J(-\varepsilon) < 0,$$

hence

$$r^{-2\alpha} \log \mathbb{P}(Z_r \geq 1 - r^\alpha \varepsilon) \xrightarrow{r \rightarrow \infty} 0.$$

Letting  $\varepsilon \rightarrow 0$ , one finally gets

$$\liminf_{r \rightarrow \infty} \frac{1}{r^{1+2\alpha}} \log \mathbb{P}(Z_r - 1 \geq r^\alpha a) \geq -J(a).$$

Now, let us turn to the upper bound. By assumption,  $m := \sup(\text{dom}(\Lambda_W)) \in (0, \infty]$ ; therefore, using (3),  $(\tilde{\mathbf{A}}_{\tau, c})$  is satisfied with  $(\tau, c) = (1, m)$ . Using Markov's inequality and Corollary 24, item 2, for any  $\zeta > 0$ ,

$$\limsup_{r \rightarrow \infty} \frac{1}{r^{1+2\alpha}} \log \mathbb{P}(Z_r - 1 \geq r^\alpha a) \leq \limsup_{r \rightarrow \infty} \left( \frac{1}{r^{1+2\alpha}} \log \mathbb{E}[Z_r^{\zeta r^{1+\alpha}}] - \zeta r^{-\alpha} \log(1 + r^\alpha a) \right) \leq \frac{\zeta^2 \text{Var}(W)}{2} - \zeta a.$$

Optimizing in  $\zeta$ , we conclude to the required upper bound.

### 4.2.3 Right deviations for $\tau \in (0, 1)$ — Theorem 9

We are to study the deviations of  $Z_r - 1 = \frac{1}{r} \sum_{i=1}^r (W_i Z_{r,i} - 1)$  which can be seen as a mean of centered variables in a triangular array. We resort to [13] to establish that these deviations behave like those of  $Z_r^1 - 1 = \frac{1}{r} \sum_{i=1}^r (W_i - 1)$ . Hence, we will check that assumptions (H1), (H2'), and (H2+) in [13] hold, with  $n = r$ ,  $N_n = r$ ,  $Y_n = W Z_r - 1$ ,  $\varepsilon = 1 - \tau$ ,  $q = c$ ,  $\sigma^2 = \text{Var}(W)$ , and  $\alpha$  there equal to  $1 + \alpha$  here. First,

$$\mathbb{E}[(W Z_r - 1)^2] = \mathbb{E}[W^2] \mathbb{E}[Z_r^2] - 1 \xrightarrow{r \rightarrow \infty} \text{Var}(W),$$

by Corollary 24, item 1; hence (H2') is satisfied. As for (H2+), applying Hölder's inequality and then expanding, we get

$$\mathbb{E}[|W Z_r - 1|^3] \leq \mathbb{E}[(W Z_r - 1)^4]^{3/4} \xrightarrow{r \rightarrow \infty} (\mathbb{E}[W^4] - 4\mathbb{E}[W^3] + 6\mathbb{E}[W^2] - 3)^{3/4} = o(r^{-\alpha}).$$

Finally, let us turn to (H1). Take any sequence  $(y_r)_{r \geq 2}$  such that there exists  $C > 0$  such that  $C^{-1} r^{(1+\alpha)(1-\tau)} \leq y_r \leq C r^{1+\alpha}$ . First,

$$\frac{1}{y_r^\tau} \log \mathbb{P}(W Z_r - 1 \geq y_r) \geq \frac{1}{y_r^\tau} \log \mathbb{P}(W \geq y_r + 1) + \frac{1}{y_r^\tau} \log \mathbb{P}(Z_r \geq 1) \xrightarrow{r \rightarrow \infty} -c,$$

by  $(\mathbf{A}_{\tau, c})$  and the fact that  $\mathbb{P}(Z_r \geq 1) \rightarrow 1/2$  as  $r \rightarrow \infty$ , by the central limit theorem stated in [16, Theorem 1.2]. As for the upper bound, let  $h_r = c\tau y_r^\tau$  and apply Markov's inequality:

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{1}{y_r^\tau} \log \mathbb{P}(W Z_r - 1 \geq y_r) &\leq \limsup_{r \rightarrow \infty} \frac{1}{y_r^\tau} \log \mathbb{P}(W Z_r \geq y_r) \\ &\leq \limsup_{r \rightarrow \infty} \frac{1}{y_r^\tau} \left( \log \mathbb{E}[W^{h_r}] - h_r \log(y_r) \right) + \limsup_{r \rightarrow \infty} \frac{1}{y_r^\tau} \log \mathbb{E}[Z_r^{h_r}] \\ &\leq -c + \limsup_{r \rightarrow \infty} \frac{c^2 \tau^2 y_r^\tau}{r}, \end{aligned}$$

using Proposition 20, item 1, and Proposition 22 (note that  $h_r = o(r^\tau)$ , since  $\alpha < 0$ ). Eventually,  $y_r^\tau \leq C r^{(1+\alpha)\tau} = o(r)$ , so the last limit superior in the display is 0. Applying Theorems 2.1, 2.2, and 2.3 of [13], we get the result.

**Remark 25.** *In view of the counterpart in the finite tree (see Section 3.1.2), it seems natural to ask whether there is exponential equivalence between  $(r^\alpha(Z_r - 1))_{r \geq 2}$  and  $(r^\alpha(Z_r^1 - 1))_{r \geq 2}$ . Using our control of the moments of  $Z_r$  (see Section 4.1), we can prove this exponential equivalence for  $\alpha < \tau - 1$ , but we do not know how to prove it for  $\alpha \in (-1/2, 0) \cap [\tau - 1, 0)$ .*

## 4.3 Proof of large deviations

### 4.3.1 Left deviations — Proposition 2

As mentioned after Proposition 2, the result is essentially a consequence of [15, 16]. It remains to examine the case when  $a = \underline{w} = 0$ , where  $\underline{w}$  is the minimum of the support of the law of  $W$ : the proof [16, Equation (6.4) sqq.] adapts easily.

### 4.3.2 Right deviations for $\tau = 1$ — Theorem 7, item 1

Remind that, by (6),

$$I^\infty = \lim_{n \rightarrow \infty} \downarrow I^n.$$

**Proposition 26** (Lower bound). *For all  $a \geq 1$ ,  $\liminf_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P}(Z_r \geq a) \geq -I^\infty(a)$ .*

*Proof of Proposition 26.* Let  $a \geq 1$  and assume that  $I^\infty(a) < \infty$  (otherwise the result is trivial). Remind that, for all  $k \geq 0$ ,  $1^k$  represents the word  $1, 1, \dots, 1$  of length  $k$ . For all  $r \geq 2$ ,  $n \geq 1$ ,  $k \geq 0$ , and  $i \in \llbracket 1, r \rrbracket$ , let

$$Z_{r,1^k,i}^n := \frac{1}{r^n} \sum_{1 \leq i_1, \dots, i_n \leq r} W_{1^k,i,i_1} W_{1^k,i,i_1,i_2} \cdots W_{1^k,i,i_1,\dots,i_n} \quad (49)$$

and let  $Z_{r,1^k,i}$  be the limit of the martingale  $(Z_{r,1^k,i}^n)_{n \geq 1}$  (by convention,  $Z_{r,1^0,i} = Z_{r,i}$ ). Note that all random variables  $W_{1^k,i}$  (with  $k \geq 0$  and  $1 \leq i \leq r$ ) and  $Z_{r,1^k,i}$  (with  $k \geq 0$  and  $2 \leq i \leq r$ ) are independent. Now let  $n \geq 1$ ,  $\varepsilon > 0$ , and consider a tuple  $((s_1, w_1), (s_{1,1}, w_{1,1}), \dots, (s_{1^{n-2},1}, w_{1^{n-2},1}), z) \in (\mathbb{R}^2)^{n-1} \times \mathbb{R}$  such that

$$s_1(1 - \varepsilon) + w_1 \left( s_{1,1}(1 - \varepsilon) + \cdots + w_{1^{n-3},1} \left( s_{1^{n-2},1}(1 - \varepsilon) + w_{1^{n-2},1} z(1 - \varepsilon) \right) \right) \geq a. \quad (50)$$

Then

$$\begin{aligned} \mathbb{P}(Z_r \geq a) &\geq \mathbb{P} \left( \bigcap_{k \in \llbracket 0, n-2 \rrbracket} \left\{ W_{1^k,1} \geq w_{1^k,1} r, \sum_{i=2}^r W_{1^k,i} \geq s_{1^k,1} r, \forall i \in \llbracket 2, r \rrbracket \right\} Z_{r,1^k,i} \geq 1 - \varepsilon \right) \\ &\quad \cap \left\{ \sum_{i=1}^r W_{1^{n-1},i} \geq \frac{zr}{1 - \varepsilon}, \forall i \in \llbracket 1, r \rrbracket \right\} Z_{r,1^{n-1},i} \geq 1 - \varepsilon \Big\} \\ &\geq \mathbb{P} \left( \bigcap_{k \in \llbracket 0, n-2 \rrbracket} \left\{ W_{1^k,1} \geq w_{1^k,1} r, \sum_{i=2}^r W_{1^k,i} \geq s_{1^k,1} r \right\} \cap \left\{ \sum_{i=1}^r W_{1^{n-1},i} \geq \frac{zr}{1 - \varepsilon} \right\} \right) \\ &\quad \times \mathbb{P}(Z_r \geq 1 - \varepsilon)^{n(r-1)+1}. \end{aligned}$$

Remind that  $\mathbb{P}(Z_r \geq 1 - \varepsilon) \rightarrow 1$  as  $r \rightarrow \infty$ , so by independence,

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P}(Z_r \geq a) &\geq \liminf_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P} \left( \bigcap_{k \in \llbracket 0, n-2 \rrbracket} \left\{ W_{1^k,1} \geq w_{1^k,1} r, \sum_{i=2}^r W_{1^k,i} \geq s_{1^k,1} r \right\} \cap \left\{ \sum_{i=1}^r W_{1^{n-1},i} \geq \frac{zr}{1 - \varepsilon} \right\} \right) \\ &= -(cw_1 + \Lambda_W^*(s_1) + cw_{1,1} + \Lambda_W^*(s_{1,1}) + \cdots + cw_{1^{n-2},1} + \Lambda_W^*(s_{1^{n-2},1}) + \Lambda_W^*(z)). \end{aligned}$$

Optimizing in  $((s_1, w_1), (s_{1,1}, w_{1,1}), \dots, (s_{1^{n-2},1}, w_{1^{n-2},1}), z) \in (\mathbb{R}^2)^{n-1} \times \mathbb{R}$  satisfying (50) and using (4), we get

$$\liminf_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P}(Z_r \geq a) \geq -I^n \left( \frac{a}{1 - \varepsilon} \right).$$

If  $I^n$  is continuous (from the right) at  $a$ , then, taking the limit as  $\varepsilon \rightarrow 0$  and then as  $n \rightarrow \infty$ , we conclude that

$$\liminf_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P}(Z_r \geq a) \geq - \lim_{n \rightarrow \infty} I^n(a) = -I^\infty(a).$$

As follows from Proposition 12 and Proposition 17, item 1, the only case where  $I^n$  is not continuous (from the right) at  $a$  is the case when  $W$  is bounded and  $a$  is the maximum of the support of the law of  $W$ . This case is dealt with in [16, Equation (6.4) sqq.].  $\square$

**Proposition 27** (Upper bound). *For all  $a \geq 1$ ,  $\limsup_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P}(Z_r \geq a) \leq -I^\infty(a)$ .*

Before proving Proposition 27, let us introduce the notation

$$\bar{I}^\infty(a) := - \limsup_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P}(Z_r \geq a).$$

**Lemma 28.** *For all  $a > 1$ ,  $\bar{I}^\infty(a) > 0$ .*

*Proof of Lemma 28.* Using Markov's inequality and Corollary 24, item 3, for all  $a > 1$ , for all  $\eta \in (0, ce)$ ,

$$-\bar{I}^\infty(a) = \limsup_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P}(Z_r \geq a) \leq \limsup_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{E}[Z_r^{\eta r}] - \eta \log(a) \leq \kappa(\eta) - \eta \log(a), \quad (51)$$

Hence  $\bar{I}^\infty(a) > 0$ , since  $\kappa(\eta) = O(\eta^2)$  as  $\eta \rightarrow 0$ , and Lemma 28 is proved.  $\square$

To prove the upper bound, we follow the same lines as in the proof of the upper bound of large deviations for  $Z_r^n$  (see Section 3.2). Let  $a > 1$  and  $\varepsilon \in (0, a - 1)$ . For all  $r \geq 2$  and for  $q \in \llbracket 0, r \rrbracket$ , we define

$$P_{r,q} := \mathbb{P}(Z_r \geq a, Z_{r,1}, \dots, Z_{r,q} \geq 1 + \varepsilon, Z_{r,q+1}, \dots, Z_{r,r} < 1 + \varepsilon),$$

so that

$$\mathbb{P}(Z_r \geq a) = \sum_{q=0}^r \binom{r}{q} P_{r,q}.$$

For all  $q_0 \in \llbracket 0, r \rrbracket$  and  $\delta \in (0, \bar{I}^\infty(1 + \varepsilon))$  (recall that  $\bar{I}^\infty(1 + \varepsilon) > 0$  by Lemma 28),

$$\sum_{q=q_0}^r \binom{r}{q} P_{r,q} \leq \sum_{q=q_0}^r \binom{r}{q} \mathbb{P}(Z_r \geq 1 + \varepsilon)^q \leq \sum_{q=q_0}^r (re^{-r\delta})^q.$$

Then, for all  $M \in (0, \infty)$ , choosing  $q_0$  such that  $q_0\delta > M$ , we have, for all  $r$  large enough,

$$\sum_{q=q_0}^r \binom{r}{q} P_{r,q} \leq \frac{(re^{-r\delta})^{q_0}}{1 - re^{-r\delta}} \leq e^{-rM}. \quad (52)$$

Besides, remembering that  $\varepsilon \in (0, a - 1)$ ,

$$\lim_{r \rightarrow \infty} \frac{1}{r} \log P_{r,0} \leq \lim_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P}\left(\sum_{i=1}^r W_i \geq \frac{ra}{1 + \varepsilon}\right) = -\Lambda_W^*\left(\frac{a}{1 + \varepsilon}\right). \quad (53)$$

Finally, let  $q \in \llbracket 1, q_0 - 1 \rrbracket$ . Write

$$\begin{aligned} \frac{1}{r} \log P_{r,q} &\leq \frac{1}{r} \log \mathbb{P}(W_1 Z_{r,1} + \dots + W_q Z_{r,q} + (W_{q+1} + \dots + W_r)(1 + \varepsilon) \geq ra, \\ &\quad Z_{r,1}, \dots, Z_{r,q} \geq 1 + \varepsilon, Z_{r,q+1}, \dots, Z_{r,r} < 1 + \varepsilon) \\ &\leq \frac{1}{r} \log \mathbb{P}(W_1 Z_{r,1} + \dots + W_q Z_{r,q} + (W_{q+1} + \dots + W_r)(1 + \varepsilon) \geq ra). \end{aligned}$$

Using the upper bound in the contraction principle for the continuous map  $(y_1, \dots, y_q, s) \mapsto y_1 + \dots + y_q + (1 + \varepsilon)s$  and  $((W_1 Z_{r,1}, \dots, W_q Z_{r,q}, W_{q+1} + \dots + W_r)/r)_{r \geq 2}$ , we get

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{1}{r} \log P_{r,q} &\leq -\inf \left\{ g(y_1) + \dots + g(y_q) + \Lambda_W^*(s) ; y_1, \dots, y_q \geq 0, s \geq 0, y_1 + \dots + y_q + (1 + \varepsilon)s \geq a \right\} \\ &= -\inf \left\{ g(y) + \Lambda_W^*(s) ; y \geq 0, s \geq 0, y + (1 + \varepsilon)s \geq a \right\} \\ &=: -B_\varepsilon, \end{aligned} \quad (54)$$

where  $g$  is the concave function defined by

$$g(y) := \inf \left\{ cw + \bar{I}^\infty(z) ; w \geq 0, z \geq 0, wz = y \right\} = \inf \left\{ \frac{cy}{z} + \bar{I}^\infty(z) ; z \geq 1 \right\}$$

and satisfying the upper bound of large deviations for  $(W Z_r/r)_{r \geq 2}$  at speed  $(r)_{r \geq 2}$ . Applying the principle of the largest term to (52), (53), and (54) leads to

$$-\bar{I}^\infty(a) = \limsup_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P}(Z_r \geq a) \leq -\min \left( M, \Lambda_W^*\left(\frac{a}{1 + \varepsilon}\right), B_\varepsilon \right).$$

Since  $\Lambda_W^*$  is left continuous at  $a$  and  $\varepsilon \mapsto B_\varepsilon$  is nonincreasing, we may let  $\varepsilon \rightarrow 0$  and  $M \rightarrow \infty$  and obtain

$$\bar{I}^\infty(a) \geq \min \left( \Lambda_W^*(a), \lim_{\varepsilon \rightarrow 0} B_\varepsilon \right). \quad (55)$$

*Proof of Proposition 27 for  $c = \infty$ .* In this case,

$$\lim_{\varepsilon \rightarrow 0} B_\varepsilon \geq \lim_{\varepsilon \rightarrow 0} \Lambda_W^* \left( \frac{a}{1+\varepsilon} \right) = \Lambda_W^*(a) = I^\infty(a).$$

Combining this inequality with (55), we conclude that

$$\bar{I}^\infty \geq \Lambda_W^* = I^\infty,$$

so Proposition 27 is proved for  $c = \infty$ . □

Now we turn to the proof of Proposition 27 in the case  $c \in (0, \infty)$ .

**Lemma 29.** *If  $c \in (0, \infty)$ , then, for all  $a > 1$ ,*

$$\bar{I}^\infty(a) = \inf \left\{ cw + \bar{I}^\infty(z) + \Lambda_W^*(s) ; w \geq 0, z \in [1, a], s \in [1, a], wz + s = a \right\}.$$

*Proof of Lemma 29.* First, we claim that, for all  $\varepsilon \in (0, a-1)$ ,

$$B_\varepsilon = \inf \left\{ cw + \bar{I}^\infty(z) + \Lambda_W^*(s) ; w \geq 0, z \in [1, a], s \in [1, a/(1+\varepsilon)], wz + (1+\varepsilon)s = a \right\}.$$

Indeed, let us see how we may adapt the proof of Proposition 14, item 2. Since  $c \in (0, \infty)$ , we know that  $\bar{I}^\infty(a) \leq I^\infty(a) < \infty$ . Hence, by (51), there exists  $z_0(a)$  such that  $\bar{I}^\infty(z_0(a)) > \bar{I}^\infty(a)$ . So, introducing  $G(w, z, s) = cw + \bar{I}^\infty(z) + \Lambda_W^*(s)$ , we have

$$G(w, z, s) \geq \bar{I}^\infty(a) + \begin{cases} \frac{c(a/(1+\varepsilon)-1)}{2z_0(a)} & \text{if } z \in (a, z_0(a)] \text{ and } s \leq \frac{1+a/(1+\varepsilon)}{2} \\ \Lambda_W^* \left( \frac{1+a/(1+\varepsilon)}{2} \right) & \text{if } z \in (a, z_0(a)] \text{ and } s > \frac{1+a/(1+\varepsilon)}{2} \\ \bar{I}^\infty(z_0(a)) - \bar{I}^\infty(a) & \text{if } z > z_0(a), \end{cases}$$

and the claim follows. Now we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} B_\varepsilon &= \sup_{0 < \varepsilon < a-1} \inf_{\substack{1 \leq z \leq a, \\ 1 \leq s \leq \frac{a}{1+\varepsilon}}} \left[ \frac{c(a-s)}{z} + \bar{I}^\infty(z) + \Lambda_W^*(s) - \frac{c\varepsilon s}{z} \right] \\ &\geq \sup_{0 < \varepsilon < a-1} \inf_{\substack{1 \leq z \leq a, \\ 1 \leq s \leq a}} \left[ \frac{c(a-s)}{z} + \bar{I}^\infty(z) + \Lambda_W^*(s) - c\varepsilon a \right] \\ &= \inf \left\{ cw + \bar{I}^\infty(z) + \Lambda_W^*(s) ; w \geq 0, z \in [1, a], s \in [1, a], wz + s = a \right\}. \end{aligned}$$

Noting that the latter infimum is bounded above by  $\Lambda_W^*(a)$  (taking  $(w, z, s) = (0, 1, a)$ ) and combining this inequality with (55) yields

$$\bar{I}^\infty(a) \geq \inf \left\{ cw + \bar{I}^\infty(z) + \Lambda_W^*(s) ; w \geq 0, z \in [1, a], s \in [1, a], wz + s = a \right\}.$$

Second, by definition and since  $\mathbb{P}(Z_r \geq 1-\varepsilon) \rightarrow 1$  as  $r \rightarrow \infty$  (by the left deviations proved at the beginning of the section), one has, for all  $\varepsilon > 0$ ,  $z \geq 1$ , and  $s \in [1, a]$ ,

$$\begin{aligned} -\bar{I}^\infty(a) &= \limsup_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P}(Z_r \geq a) \\ &\geq \limsup_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P}(W_1 Z_{r,1} + (W_2 + \dots + W_r)(1-\varepsilon) \geq ra) + \frac{r-1}{r} \log \mathbb{P}(Z_r \geq 1-\varepsilon) \\ &\geq \limsup_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{P} \left( \frac{W_1}{r} \geq \frac{a-s(1-\varepsilon)}{z}, Z_{r,1} \geq z, \frac{1}{r}(W_2 + \dots + W_r) \geq s \right) \\ &= - \left( \frac{c(a-s)}{z} + \bar{I}^\infty(z) + \Lambda_W^*(s) + \frac{c\varepsilon s}{z} \right), \end{aligned}$$

from which we deduce that

$$\begin{aligned} \bar{I}^\infty(a) &\leq \inf_{\varepsilon > 0} \inf_{\substack{1 \leq z \leq a, \\ 1 \leq s \leq \frac{a}{1-\varepsilon}}} \left[ \frac{c(a-s)}{z} + \bar{I}^\infty(z) + \Lambda_W^*(s) + \frac{c\varepsilon s}{z} \right] \\ &= \inf \left\{ cw + \bar{I}^\infty(z) + \Lambda_W^*(s) ; w \geq 0, z \in [1, a], s \in [1, a], wz + s = a \right\}. \end{aligned} \quad \square$$

**Lemma 30.** *If  $c \in (0, \infty)$ , then the nondecreasing function  $\bar{I}^\infty$  is continuous.*

*Proof of Lemma 30.* Let  $a_0 \geq 1$  and  $\varepsilon > 0$ . Since  $\bar{I}^\infty$  is nondecreasing, it suffices to prove that there exists  $a_1 > a_0$  such that  $\bar{I}^\infty(a_1) \leq \bar{I}^\infty(a_0) + \varepsilon$ . This follows the same lines as in the proof of Proposition 17, item 1.  $\square$

For all  $a \geq 1$ , we may define

$$z(a) := \inf \operatorname{argmin} \left\{ \bar{I}^\infty(z) + h(a, z) ; z \in [1, a] \right\}.$$

Recall that  $a_W := \inf \{a \geq 1 ; aa^* \geq c\}$ , where  $a^*$  was defined in (17). Now if  $a \in [1, a_W]$  and  $z \in [1, a]$ , then  $z \leq c/a^*$ , so  $h(a, z) = \Lambda_W^*(a)$ , whence  $z(a) = 1$  and

$$\bar{I}^\infty(a) = \Lambda_W^*(a) \geq I^\infty(a). \quad (56)$$

**Lemma 31.** *For all  $A > a_W$ , there exists  $\eta(A) > 0$  such that, for all  $a \in [a_W, A]$ ,  $z(a) < a - \eta(A)$ .*

*Proof of Lemma 31.* For all  $a \geq a_W$  and  $z \in [1, a]$ ,

$$h(a, z) \geq h(a, a) = c - \Lambda_W(c/a) \geq c - \Lambda_W(c/a_W) = \Lambda_W^*(a_W) =: \rho > 0.$$

Now let  $A > a_W$ . Since the function  $\bar{I}^\infty$  is uniformly continuous on  $[1, A]$ , there exists  $\eta(A) > 0$  such that, for all  $(a, z) \in [1, A]^2$ , with  $a - \eta(A) \leq z \leq a$ ,  $\bar{I}^\infty(z) > \bar{I}^\infty(a) - \rho$  (note that it implies  $\eta(A) < a_W - 1$ ). Then, for all  $a \in [a_W, A]$  and  $z \in [a - \eta(A), a]$ ,

$$\bar{I}^\infty(z) + h(a, z) > (\bar{I}^\infty(a) - \rho) + \rho = \bar{I}^\infty(a),$$

whence the conclusion of the lemma.  $\square$

*Proof of Proposition 27 for  $c \in (0, \infty)$ .* Let  $E := \left\{ a \geq 1 ; \forall z \in [1, a] \bar{I}^\infty(z) \geq I^\infty(z) \right\}$  and let  $a_\infty = \sup(E)$ . By (56), we know that  $a_\infty \geq a_W$ . The aim is to prove that  $a_\infty = \infty$ . Assume that  $a_\infty \in [a_W, \infty)$ . Let  $A = a_\infty + 1 > a_W$  and  $\eta(A)$  be defined as in Lemma 31. Let  $a$  be such that  $a_\infty < a \leq a_\infty + \min(1, \eta(A))$ . One has

$$\bar{I}^\infty(a) = \inf_{1 \leq z < a - \eta(A)} \left[ \bar{I}^\infty(z) + h(a, z) \right] \geq \inf_{1 \leq z < a - \eta(A)} \left[ I^\infty(z) + h(a, z) \right] \geq \inf_{1 \leq z \leq a} \left[ I^\infty(z) + h(a, z) \right] = I^\infty(a),$$

so  $a \in E$  which is a contradiction, and we conclude that  $a_\infty = \infty$ .  $\square$

**Proposition 32.** *As  $a \rightarrow \infty$ ,  $I^\infty(a) \sim ce \log(a)$ .*

*Proof of Proposition 32.* It follows from Proposition 17, item 2, that, for all  $a \geq e$ ,

$$I^\infty(a) \leq I^{\lfloor \log(a) \rfloor}(a) \leq c \lfloor \log(a) \rfloor \exp\left(\frac{\log(a)}{\lfloor \log(a) \rfloor}\right) \sim ce \log(a).$$

Besides, by (51), for all  $\eta \in (0, ce)$ ,  $\liminf_{a \rightarrow \infty} \frac{I^\infty(a)}{\log(a)} = \liminf_{a \rightarrow \infty} \frac{\bar{I}^\infty(a)}{\log(a)} \geq \eta$ , whence the conclusion letting  $\eta \rightarrow ce$ .  $\square$

### 4.3.3 Right deviations for $\tau \in (0, 1)$ — Theorem 9

Remind that, by definition,

$$I_\tau^\infty = \lim_{n \rightarrow \infty} \downarrow I_\tau^n.$$

Let us prove the lower bound. Recall the notation introduced at the beginning of the proof of Proposition 26. Let  $a \geq 1$ ,  $n \geq 1$ ,  $\varepsilon > 0$ , and consider  $(w_1, w_{1,1}, \dots, w_{1^{n-2},1}, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$  such that

$$(1 - \varepsilon) + w_1((1 - \varepsilon) + \dots + w_{1^{n-3},1}((1 - \varepsilon) + w_{1^{n-2},1}z(1 - \varepsilon))) \geq a. \quad (57)$$

Then

$$\begin{aligned} \mathbb{P}(Z_r \geq a) &\geq \mathbb{P}\left( \bigcap_{k \in [0, n-2]} \left\{ W_{1^k,1} \geq w_{1^k,1}r, \sum_{i=2}^r W_{1^k,i} \geq r, \forall i \in [2, r] Z_{r,1^k,i} \geq 1 - \varepsilon \right\} \right. \\ &\quad \left. \cap \left\{ \sum_{i=1}^r W_{1^{n-1},i} \geq \frac{zr}{1 - \varepsilon}, \forall i \in [1, r] Z_{r,1^{n-1},i} \geq 1 - \varepsilon \right\} \right) \\ &\geq \mathbb{P}\left( \bigcap_{k \in [0, n-2]} \left\{ W_{1^k,1} \geq w_{1^k,1}r, \sum_{i=2}^r W_{1^k,i} \geq r \right\} \cap \left\{ \sum_{i=1}^r W_{1^{n-1},i} \geq \frac{zr}{1 - \varepsilon} \right\} \right) \\ &\quad \times \mathbb{P}(Z_r \geq 1 - \varepsilon)^{n(r-1)+1}. \end{aligned}$$

Remind that  $\mathbb{P}(Z_r \geq 1 - \varepsilon) \rightarrow 1$  as  $r \rightarrow \infty$ , so by independence,

$$\begin{aligned} & -\liminf_{r \rightarrow \infty} \frac{1}{r^\tau} \log \mathbb{P}(Z_r \geq a) \\ & \leq -\liminf_{r \rightarrow \infty} \frac{1}{r^\tau} \log \mathbb{P}\left(\bigcap_{k \in \llbracket 0, n-2 \rrbracket} \left\{W_{1^k, 1} \geq w_{1^k, 1} r, \sum_{i=2}^r W_{1^k, i} \geq r\right\} \cap \left\{\sum_{i=1}^r W_{1^{n-1}, i} \geq \frac{zr}{1-\varepsilon}\right\}\right) \\ & = cw_1^\tau + cw_{1,1}^\tau + \cdots + cw_{1^{n-2}, 1}^\tau + I_\tau^1(z). \end{aligned}$$

Optimizing in  $(w_1, w_{1,1}, \dots, w_{1^{n-2}, 1}, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$  satisfying (57) and using (8), we get

$$-\liminf_{r \rightarrow \infty} \frac{1}{r^\tau} \log \mathbb{P}(Z_r \geq a) \leq I_\tau^n\left(\frac{a}{1-\varepsilon}\right).$$

Since  $I_\tau^n$  is continuous (see Proposition 19, item 4), taking the limit as  $\varepsilon \rightarrow 0$  and then as  $n \rightarrow \infty$ , we conclude that

$$\liminf_{r \rightarrow \infty} \frac{1}{r^\tau} \log \mathbb{P}(Z_r \geq a) \geq -\lim_{n \rightarrow \infty} I_\tau^n(a) = -I_\tau^\infty(a).$$

As for the upper bound, let us define, for all  $a \geq 0$ ,

$$\bar{I}_\tau^\infty(a) := -\limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log \mathbb{P}(Z_r - 1 \geq a).$$

Note that, for all  $a > 0$ , applying Markov's inequality and Corollary 24, item 3,

$$\bar{I}_\tau^\infty(a) = -\limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log \mathbb{P}(Z_r \geq 1 + a) \geq -\limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log \mathbb{E}[Z_r^{\eta_0 \tau r^\tau}] + \eta_0 \tau \log(1 + a) = \eta_0 \tau \log(1 + a). \quad (58)$$

Now, let  $a \geq 0$  and  $\varepsilon > 0$ . We use the decomposition

$$P_{r,q} := \mathbb{P}(Z_r - 1 \geq a, Z_{r,1}, \dots, Z_{r,q} \geq 1 + \varepsilon, Z_{r,q+1}, \dots, Z_{r,r} < 1 + \varepsilon).$$

The terms  $P_{r,q}$  for  $q$  large do not contribute as

$$\limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log \sum_{q=q_0}^r \binom{r}{q} P_{r,q} \leq \limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log \sum_{q=q_0}^r \binom{r}{q} \mathbb{P}(Z_r \geq 1 + \varepsilon)^q = -q_0 \bar{I}_\tau^\infty(\varepsilon) < -\bar{I}_\tau^\infty(a) \quad (59)$$

as soon as  $q_0 > \bar{I}_\tau^\infty(a) / \bar{I}_\tau^\infty(\varepsilon)$  (note that (58) implies that  $\bar{I}_\tau^\infty > 0$  on  $(0, \infty)$ ). Moreover, using [5, Theorem 2], (20) becomes

$$\limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log P_{r,0} \leq -c \left(\frac{1+a}{1+\varepsilon} - 1\right)^\tau = K_\tau^1 \left(\frac{1+a}{1+\varepsilon} - 1\right). \quad (60)$$

Finally, for  $q \in \llbracket 1, q_0 - 1 \rrbracket$ , the proof relies on the same contraction principle argument. From

$$P_{r,q} \leq \mathbb{P}\left((W_1 Z_{r,1} - 1) + \cdots + (W_q Z_{r,q} - 1) + (W_{q+1}(1 + \varepsilon) - 1) + \cdots + (W_r(1 + \varepsilon) - 1) \geq a\right),$$

we are lead to

$$\limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log P_{r,q} \leq -\inf\{g(y_1) + \cdots + g(y_q) + K_\tau^1(s - 1); y_1, \dots, y_q \geq 0, s \geq 1, y_1 + \cdots + y_q + (1 + \varepsilon)s \geq a\}$$

where

$$g(y) = \inf\left\{cw^\tau + \bar{I}_\tau^\infty(z - 1); w \geq 0, z \geq 1, wz = y\right\} = \inf\left\{c\left(\frac{y}{z}\right)^\tau + \bar{I}_\tau^\infty(z - 1); z \geq 1\right\}$$

is a concave function, as an infimum of concave functions. Hence,

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log P_{r,q} & \leq -\inf\{g(y) + c(s - 1)^\tau; y \geq 0, s \geq 1, y + (1 + \varepsilon)s \geq a\} \\ & = -\inf\left\{cw^\tau + \bar{I}_\tau^\infty(z - 1) + c(s - 1)^\tau; w \geq 0, z \geq 1, s \geq 1, wz + (1 + \varepsilon)s \geq a\right\} \\ & = -\min\left(K_\tau^1\left(\frac{a - \varepsilon}{1 + \varepsilon}\right), \inf\left\{cw^\tau + \bar{I}_\tau^\infty(z - 1); w \geq 0, z \geq 1, wz \geq a - \varepsilon\right\}\right), \quad (61) \end{aligned}$$

using the concavity in  $(w, s)$ . Putting together (59), (60), (61), and letting  $\varepsilon \rightarrow 0$ , we get the following result (adapting the proof of Lemma 29).

**Lemma 33.** For all  $a \geq 0$ ,

$$\bar{I}_\tau^\infty(a) = \inf \left\{ cw^\tau + \bar{I}_\tau^\infty(z) ; w \geq 0, z \in [0, a], w(1+z) \geq a \right\}.$$

Then, (56) adapts as follows.

**Lemma 34.** There exists  $a_0 > 0$  such that  $\bar{I}_\tau^\infty = I_\tau^\infty$  on  $[0, a_0]$ .

*Proof of Lemma 34.* We already know by the lower bound that  $\bar{I}_\tau^\infty \leq I_\tau^\infty$  everywhere. Moreover, an easy study of the functions  $I_\tau^n$  shows that  $I_\tau^\infty = K_\tau^1$  on some interval  $[0, b_1]$  with  $b_1 > 0$  (adapt Proposition 18). So, we only need to prove that there exists  $a_0 > 0$  such that  $\bar{I}_\tau^\infty \geq K_\tau^1$  on  $[0, a_0]$ . By (58), we have  $\bar{I}_\tau^\infty(z) \geq \eta_0 \tau z/2$  for  $z \in [0, 1]$  (since  $\log(1+z) \geq z/2$ ). Hence, using Lemma 33,

$$\bar{I}_\tau^\infty(a) \geq \inf \left\{ cw^\tau + \frac{\eta_0 \tau z}{2} ; w \geq 0, z \in [0, a], w(1+z) \geq a \right\} = ca^\tau$$

as soon as  $a \leq (\eta_0/(2c))^{1/\tau} =: a_0$ . □

**Lemma 35.**  $\bar{I}_\tau^\infty = I_\tau^\infty$  on  $[0, \infty)$ .

*Proof of Lemma 35.* Lemmas 30 and 31, and the end of the proof of Proposition 27 adapt easily, with function  $h$  replaced by

$$h_\tau(a, z) = c \left( \frac{a}{1+z} \right)^\tau$$

which is continuous on  $D := \{(a, z) ; a \geq 0, z \in [0, a]\}$  and, moreover, bounded below by  $\rho_\tau := c(a_{\tau,1}/(1+a_{\tau,1}))^\tau > 0$  for all  $a \geq a_{\tau,1}$  and  $z \in [0, a]$ . □

**Proposition 36.** As  $a \rightarrow \infty$ ,  $I_\tau^\infty(a) \sim ce\tau \log(a)$ .

*Proof of Proposition 32.* It follows from Proposition 19, item 5, that, for all  $a \geq e$ ,

$$I_\tau^\infty(a) \leq I_\tau^{\lfloor \tau \log(a) \rfloor}(a) \leq c \lfloor \tau \log(a) \rfloor \exp\left(\frac{\tau \log(a)}{\lfloor \tau \log(a) \rfloor}\right) \sim ce\tau \log(a).$$

Besides, by Corollary 24, item 3, for all  $\eta \in (0, ce)$ ,  $\liminf_{a \rightarrow \infty} \frac{I_\tau^\infty(a)}{\log(a)} \geq \eta$ , whence the conclusion letting  $\eta \rightarrow ce$ . □

#### 4.4 Proof of very large deviations — Theorem 7, item 2, and Theorem 9

The case  $a = 0$  is obvious. Assume  $a > 0$ . Let us prove the upper bound. Recall the definition of  $Z_{r,1^n}$  given after (49). One has

$$\log \mathbb{P}(Z_r \geq r^\alpha a) \geq \log \mathbb{P}\left(\frac{W_1^{1^1} \cdots W_1^{1^n}}{r^n} \geq r^\alpha a, Z_{r,1^n} \geq 1\right) \geq \log \mathbb{P}\left(\frac{W_1^{1^1} \cdots W_1^{1^n}}{r^n} \geq r^\alpha a\right) + \log \mathbb{P}(Z_r \geq 1).$$

By [16, Theorem 1.2],

$$\frac{1}{r^\tau \log(r)} \log \mathbb{P}(Z_r \geq 1) \xrightarrow{r \rightarrow \infty} 0.$$

Now, taking  $n = n_r := \lfloor \tau \alpha \log(r) \rfloor$ , we get

$$\frac{1}{r^\tau \log(r)} \log \mathbb{P}\left(\frac{W_1^{1^1} \cdots W_1^{1^{n_r}}}{r^{n_r}} \geq r^\alpha a\right) \geq \frac{n_r}{r^\tau \log(r)} \log \mathbb{P}(W \geq r(r^\alpha a)^{1/n_r}) \xrightarrow{r \rightarrow \infty} -ce\tau \alpha.$$

**Remark 37.** Recall that, in the finite tree,  $\log \mathbb{P}(Z_r^n \geq r^\alpha a) \sim -cr^\tau n(r^\alpha a)^{\tau/n}$ . The minimum of  $n(r^\alpha a)^{\tau/n}$  is attained at  $n = \tau \log(r^\alpha a)$ . Hence the choice of  $n_r$  above.

As for the upper bound, by Markov's inequality and Corollary 24, item 3, for all  $\eta \in (0, ce)$ , there exists  $\kappa(\eta) \in (0, \infty)$  such that, for all  $r \geq 1$ ,

$$\log \mathbb{P}(Z_r \geq r^\alpha a) \leq \log \mathbb{E}[Z_r^{\eta r^\tau}] - \eta \tau r^\tau \log(r^\alpha a) \leq \kappa(\eta) r^\tau - \eta \tau r^\tau \alpha \log(r) - \eta \tau r^\tau \log(a) \sim -r^\tau \log(r) \eta \tau \alpha,$$

as  $r \rightarrow \infty$ , and we conclude letting  $\eta \rightarrow ce$ .

## 5 Proof of very large deviations for $\tau \in (1, \infty)$ — Theorem 6

Here, we assume that  $(\mathbf{A}_{\tau, c})$  holds with  $(\tau, c) \in (1, \infty) \times (0, \infty)$ . The proof proceeds by induction and essentially relies on Laplace's method. Because there are many different cases to deal with, we only expose the step from  $n = 1$  to  $n = 2$  which gives all the ideas. Here is a sketch of proof. As soon as  $n \leq \tau$ , we can apply Gärtner–Ellis theorem to get

$$\log \mathbb{P}(Z_r^n \geq r^\alpha a) \sim -r \Lambda_{W Z_r^{n-1}}^*(r^\alpha a).$$

To do this, we have to compute the asymptotic as  $r \rightarrow \infty$  of

$$\Lambda_{r^{-\alpha} Z_r^n}(t) = \frac{1}{r^{v_r(\alpha)}} \log \mathbb{E}[e^{r^{v_r(\alpha)} t Z_r^n / r^\alpha}] = \frac{1}{r^{v_r(\alpha)-1}} \log \mathbb{E}[e^{r^{v_r(\alpha)-\alpha-1} t W Z_r^{n-1}}].$$

This relies on Varadhan's lemma, that requires to know a priori that this sequence is bounded from above, and eventually relies on an upper bound for  $\mathbb{P}(W Z_r^{n-1} \geq r^{1+\alpha-v_r(\alpha)} a)$  uniform in  $a$ . Due to the fact that many different regimes appear, we are lead to manage different cases separately for which we obtain automatically sharp upper bounds. Therefore, we prefer another equivalent exposition of the proof, by showing separately the lower bound and the upper bound. For  $n > \tau$ , the Laplace transform of  $Z_r^n$  is infinite on  $(0, \infty)$ , so neither Gärtner–Ellis theorem nor the Chernoff bound may be used directly. We proceed by truncating the variables, decomposing

$$\mathbb{P}(Z_r^n \geq y) = P_{n,r}(y) + R_{n,r}(y)$$

where

$$P_{n,r}(y) := \mathbb{P}\left(\sum_{i=1}^r W_i Z_{r,i}^{n-1} \geq ry, \forall i \in [1, r] \ W_i Z_{r,i}^{n-1} < ry\right)$$

and

$$R_{n,r}(y) := \mathbb{P}\left(\sum_{i=1}^r W_i Z_{r,i}^{n-1} \geq ry, \exists i \in [1, r] \ W_i Z_{r,i}^{n-1} \geq ry\right) \leq n \mathbb{P}(W Z_r^{n-1} \geq ry).$$

This latter inequality allows to upper bound  $R_{n,r}(y)$  properly. As for  $P_{n,r}(y)$ , one uses

$$\log(P_{n,r}(y)) \leq -r \Lambda_{W Z_r^{n-1} | W Z_r^{n-1} \leq ry}^*(y).$$

### 5.1 Properties of $Z_r^1$

**Proposition 38.** *Assume that  $(\tau, c) \in (1, \infty) \times (0, \infty)$ .*

1. As  $t \rightarrow \infty$ ,  $\Lambda_W(t) \sim t^{\frac{\tau}{\tau-1}} (c\tau)^{-\frac{1}{\tau-1}} (1 - \frac{1}{\tau})$ .
2. As  $w \rightarrow \infty$ ,  $\Lambda_W^*(w) \sim cw^\tau$ .

*Proof of Proposition 38.* The proof of item 2 can be found in [4]. As for item 1, let  $c' \in (0, c)$ . Let  $w_0 > 0$  be such that, for all  $w \geq w_0$ ,  $\mathbb{P}(W \geq w) \leq e^{-c'w^\tau}$ . Then, for all  $t > 0$ ,

$$\begin{aligned} \mathbb{E}[e^{tW}] &= 1 + \int_0^{w_0} t e^{tw} \mathbb{P}(W \geq w) dw + \int_{w_0}^\infty t e^{tw} \mathbb{P}(W \geq w) dw \\ &\leq 1 + \int_0^{w_0} t e^{tw} dw + \int_{w_0}^\infty t e^{tw-c'w^\tau} dw = e^{tw_0} + t^{\frac{\tau}{\tau-1}} \int_{u=w_0 t^{-\frac{\tau}{\tau-1}}}^\infty e^{t^{\frac{\tau}{\tau-1}}(u-c'u^\tau)} du. \end{aligned}$$

The maximum of the function  $f(u) = u - c'u^\tau$  is attained at  $u_0 = (c'\tau)^{-\frac{1}{\tau-1}}$  and is equal to  $f(u_0) = u_0(1 - \frac{1}{\tau})$ . Hence, applying Laplace's method, one gets,

$$\limsup_{t \rightarrow \infty} t^{-\frac{\tau}{\tau-1}} \Lambda_W(t) \leq (c'\tau)^{-\frac{1}{\tau-1}} \left(1 - \frac{1}{\tau}\right)$$

and the upper bound follows, letting  $c' \rightarrow c$ . Following the same lines and using the fact that, for all  $c' > c$ , there exists  $w_1 > 0$  such that, for all  $w \geq w_1$ ,  $\mathbb{P}(W \geq w) \geq e^{-c'w^\tau}$ , we get the lower bound.  $\square$

**Proposition 39.** *For all  $\alpha > 0$  and  $a > 0$ ,  $\log \mathbb{P}(Z_r^1 \geq r^\alpha a) \sim -r^{1+\alpha\tau} ca^\tau$  as  $r \rightarrow \infty$ .*

*Proof of Proposition 39.* Gärtner–Ellis theorem entails that, for all  $\alpha \geq 0$ ,  $\log \mathbb{P}(Z_r^1 \geq r^\alpha a) \sim -r \Lambda^*(r^\alpha a)$  and the result follows from Proposition 38, item 2.  $\square$

**Proposition 40** (Uniform upper bound). *For all  $z > 1$ ,  $\log \mathbb{P}(Z_r^1 \geq z) \leq -r \Lambda^*(z)$ .*

*Proof of Proposition 40.* It is a standard consequence of Markov's inequality (see [6, Equation (2.2.12)]).  $\square$

## 5.2 Asymptotics of $WZ_r^1$

**Proposition 41.** For all  $\alpha > 0$  and  $a > 0$ ,

$$\log \mathbb{P}(WZ_r^1 \geq r^\alpha a) \sim - \begin{cases} r^{\alpha\tau} ca^\tau & \text{if } \alpha < \frac{1}{\tau} \\ r \inf_{\substack{bd=a \\ \frac{1+\alpha\tau}{2}}} (cb^\tau + \Lambda_W^*(d)) & \text{if } \alpha = \frac{1}{\tau} \\ r^{\frac{1+\alpha\tau}{2}} 2ca^{\frac{\tau}{2}} & \text{if } \alpha > \frac{1}{\tau}. \end{cases}$$

*Proof of Proposition 41.* The result easily stems from the contraction principle. Estimating  $\mathbb{P}(W \geq r^\beta b)$  and  $\mathbb{P}(Z_r^1 \geq r^\delta d)$  for  $(\beta, \delta, b, d) \in [0, \infty)^4$  thanks to assumption  $(\mathbf{A}_{\tau, \mathbf{c}})$ , Theorem 1, item 1, and Proposition 39, we conclude that: (1) if  $\alpha < \frac{1}{\tau}$ , we shall take  $(\beta, \delta, b, d) = (\alpha, 0, a, 1)$ ; (2) if  $\alpha = \frac{1}{\tau}$ , we shall take  $(\beta, \delta) = (\alpha, 0)$ , hence the result; (3) if  $\alpha > \frac{1}{\tau}$ , we shall take  $(\beta, \delta) = (\frac{1}{2}(\alpha + \frac{1}{\tau}), \frac{1}{2}(\alpha - \frac{1}{\tau}))$ , hence the speed is  $r^{\beta\tau} = r^{\frac{1+\alpha\tau}{2}}$  and the rate function is  $\inf \{cb^\tau + cd^\tau ; bd = a\} = 2ca^{\frac{\tau}{2}}$ .  $\square$

**Proposition 42** (Uniform upper bound). For all  $\delta \in (0, 1)$ , there exists  $(r_0, y_0, C_1, C_2) \in (0, \infty)^4$  such that, for all  $r \geq r_0$  and  $y \geq y_0$ ,

$$\begin{aligned} \log \mathbb{P}(WZ_r^1 \geq y) &\leq -(1 - \delta) \times \begin{cases} cy^\tau & \text{if } y \leq C_1 r^{\frac{1}{\tau}} \\ r \inf_{bd=yr^{-1/\tau}} (cb^\tau + \Lambda_W^*(d)) & \text{if } C_1 r^{\frac{1}{\tau}} < y < C_2 r^{\frac{1}{\tau}} \\ 2cr^{\frac{1}{2}} y^{\frac{\tau}{2}} & \text{if } C_2 r^{\frac{1}{\tau}} \leq y \end{cases} \\ &\leq -(1 - \delta)r \inf_{bd=yr^{-1/\tau}} (cb^\tau + \Lambda_W^*(d)). \end{aligned}$$

*Proof of Proposition 42.* Remember that, for all  $z > 1$ ,  $\mathbb{P}(Z_r^1 \geq z) \leq e^{-r\Lambda^*(z)}$ , by Proposition 40. For all  $\tilde{c} < c$ , let  $w(\tilde{c}) > 0$  be such that, for all  $w \geq w(\tilde{c})$ ,  $\mathbb{P}(W \geq w) \leq e^{-\tilde{c}w^\tau}$ . Let  $\eta > 1$  and  $c', c'' > 0$  be such that  $(1 - \delta)c < c' = c''\eta^{-\tau} < c'' < c$ . Let  $M \geq 1$ . Notice that

$$\{(w, z) \in [0, \infty)^2 ; wz \geq 1\} \subset \{w \geq \eta^{-1}\} \cup \{z \geq \eta^M\} \cup \bigcup_{m=1}^{M-1} \{w \geq \eta^{-m-1}, z \geq \eta^m\}.$$

So, for all  $y > 0$ ,

$$\mathbb{P}(WZ_r^1 \geq y) \leq \mathbb{P}(W \geq y\eta^{-1}) + \mathbb{P}(Z_r^1 \geq \eta^M) + \sum_{m=1}^{M-1} \mathbb{P}(W \geq y\eta^{-m-1})\mathbb{P}(Z_r^1 \geq \eta^m).$$

Now, if  $y\eta^{-M} \geq w(c'')$ , then

$$\mathbb{P}(WZ_r^1 \geq y) \leq e^{-c''\eta^{-\tau}y^\tau} + e^{-r\Lambda_W^*(\eta^M)} + \sum_{m=1}^{M-1} e^{-[c''(y\eta^{-m-1})^\tau + r\Lambda_W^*(\eta^m)]}.$$

Bounding from above each of the  $M + 1$  terms in the latter sum by the maximum of them, we get

$$\begin{aligned} \log \mathbb{P}(WZ_r^1 \geq y) &\leq \log(M + 1) - \min(r\Lambda_W^*(\eta^M), \min_{0 \leq m \leq M-1} [c''(y\eta^{-m-1})^\tau + r\Lambda_W^*(\eta^m)]) \\ &= \log(M + 1) - \min(r\Lambda_W^*(\eta^M), \min_{0 \leq m \leq M-1} [c'(y\eta^{-m})^\tau + r\Lambda_W^*(\eta^m)]). \end{aligned}$$

- To handle small values of  $y$ , we may take  $M = 1$ :

$$\log \mathbb{P}(WZ_r^1 \geq y) \leq \log(2) - \min(r\Lambda_W^*(\eta), c'y^\tau).$$

If  $yr^{-1/\tau} \leq (\Lambda_W^*(\eta)/c)^{1/\tau} =: C_1$ , then, a fortiori,  $yr^{-1/\tau} \leq (\Lambda_W^*(\eta)/c')^{1/\tau}$  and

$$\min(r\Lambda_W^*(\eta), c'y^\tau) = c'y^\tau.$$

Finally, there exists  $y_0 \geq \eta w(c'')$  (independent of  $r$ ) such that, if  $y_0 \leq y \leq C_1 r^{1/\tau}$ , then

$$\log \mathbb{P}(WZ_r^1 \geq y) \leq -(1 - \delta)cy^\tau.$$

- Otherwise, we have

$$\begin{aligned} \log \mathbb{P}(WZ_r^1 \geq y) &\leq \log(M + 1) + c'(y\eta^{-M})^\tau - \min_{0 \leq m \leq M} [c'(y\eta^{-m})^\tau + r\Lambda_W^*(\eta^m)] \\ &\leq \log(M + 1) + c'(y\eta^{-M})^\tau - \inf_{\substack{bd=y \\ d \geq 1}} [c'b^\tau + r\Lambda_W^*(d)]. \end{aligned}$$

We take  $M$  such that  $w(c'') \leq y\eta^{-M} < \eta w(c'')$ , i.e.  $M := \lceil \log(y/w(c''))/\log(\eta) \rceil$ . So, for  $y \geq y_1 > 1$ , there exists  $C > 0$  such that

$$\log \mathbb{P}(WZ_r^1 \geq y) \leq C \log \log(y) - \inf_{bd=y} [c'b^\tau + r\Lambda_W^*(d)].$$

1. To handle large values of  $y$ , we use Proposition 38, item 2, providing some  $w_1$  such that

$$\forall w \geq w_1 \quad \Lambda_W^*(w) \geq c'w^\tau.$$

If  $yr^{-1/\tau} \geq 4w_1^2 =: C_2$ , then

$$\inf_{bd=y} [c'b^\tau + r\Lambda_W^*(d)] \geq 2c'r^{\frac{1}{2}}y^{\frac{\tau}{2}}.$$

Indeed,

$$\inf_{\substack{bd=y \\ d \geq w_1}} [c'b^\tau + r\Lambda_W^*(d)] \geq c' \inf_{\substack{bd=y \\ d \geq w_1}} [b^\tau + rd^\tau] = 2c'r^{\frac{1}{2}}y^{\frac{\tau}{2}},$$

the infimum being attained at  $(b, d) = (y^{1/2}(r^{1/(2\tau)}), r^{-1/(2\tau)})$  and

$$\inf_{\substack{bd=y \\ d < w_1}} [c'b^\tau + r\Lambda_W^*(d)] \geq c'(y/w_1)^\tau \geq 2c'r^{\frac{1}{2}}y^{\frac{\tau}{2}}.$$

Finally, there exists  $r_0 \geq 1$  such that, for all  $r \geq r_0$  and  $y \geq C_2r^{1/\tau}$ ,

$$\log \mathbb{P}(WZ_r^1 \geq y) \leq -(1 - \delta)2cr^{\frac{1}{2}}y^{\frac{\tau}{2}}.$$

2. For  $y$  in the middle, i.e.  $y = r^{1/\tau}a$  with  $a \in [C_1, C_2]$ , we have

$$\frac{1}{r} \log \mathbb{P}(WZ_r^1 \geq y) \leq \frac{C \log \log(r^{\frac{1}{\tau}}C_2)}{r} + (c - c')C_2^\tau - \inf_{bd=a} [cb^\tau + \Lambda_W^*(d)].$$

The function  $\varphi: a \in [C_1, C_2] \mapsto \inf_{bd=a} [cb^\tau + \Lambda_W^*(d)]$  is bounded from below by  $\varphi(C_1) > 0$ . Choosing  $c'' < c$  and  $\eta > 1$  such that  $c' = c''\eta^{-\tau}$  satisfies  $(c - c')C_2^\tau < \delta\varphi(C_1)/2$ , and  $r_0 \geq 1$  such that, for all  $r \geq r_0$ ,  $\frac{C \log \log(r^{1/\tau}C_2)}{r} < \delta\varphi(C_1)/2$ , one gets, for all  $r \geq r_0$  and for all  $a \in [C_1, C_2]$ ,

$$\frac{1}{r} \log \mathbb{P}(WZ_r^1 \geq r^{\frac{1}{\tau}}a) \leq -(1 - \delta) \inf_{bd=a} [cb^\tau + \Lambda_W^*(d)].$$

Finally, for the last upper bound, obviously  $\inf_{bd=y} (cb^\tau + r\Lambda_W^*(d)) \leq cy^\tau$  (taking  $(b, d) = (y, 1)$ ). As for the third line, for all  $\tilde{c} > c$ , by the remark after Proposition 38, there exists  $w_2(\tilde{c})$  such that

$$\forall w \geq w_2(\tilde{c}) \quad \Lambda_W^*(w) \leq \tilde{c}w^\tau.$$

Hence,

$$\inf_{bd=y} (cb^\tau + r\Lambda_W^*(d)) \leq \inf_{\substack{bd=y \\ d \geq w_2(\tilde{c})}} (cb^\tau + r\Lambda_W^*(d)) \leq \inf_{\substack{bd=y \\ d \geq w_2(\tilde{c})}} (cb^\tau + r\tilde{c}d^\tau) \leq 2(c\tilde{c})^{1/2}r^{1/2}y^{\tau/2}.$$

Now, letting  $\tilde{c} \rightarrow c$ , we get

$$\inf_{bd=y} (cb^\tau + r\Lambda_W^*(d)) \leq 2cr^{1/2}y^{\tau/2}. \tag{62}$$

The proof is now complete.  $\square$

### 5.3 Proof for $Z_r^2$ when $\tau \geq 2$

**Proposition 43.** *Assume that  $\tau \geq 2$ . For all  $\alpha > 0$  and  $a > 0$ ,*

$$\log \mathbb{P}(Z_r^2 \geq r^\alpha a) \sim - \begin{cases} r^{1+\alpha\tau}ca^\tau & \text{if } 0 < \alpha < \frac{1}{\tau} \\ r^{\frac{3+\alpha\tau}{2}}2ca^{\frac{\tau}{2}} & \text{if } \alpha > \frac{1}{\tau}. \end{cases}$$

Moreover, defining, for  $a \geq 0$ ,  $T(a) := \inf_{bd=a} (cb^\tau + \Lambda_W^*(d))$ , one has, for  $\alpha = \frac{1}{\tau}$ ,

$$-T(a) \leq \liminf_{r \rightarrow \infty} \frac{1}{r^2} \log \mathbb{P}(Z_r^2 \geq r^{1/\tau}a) \leq \limsup_{r \rightarrow \infty} \frac{1}{r^2} \log \mathbb{P}(Z_r^2 \geq r^{1/\tau}a) \leq -T^{**}(a).$$

In particular, if  $T$  is convex and lower semicontinuous, then

$$-\log \mathbb{P}(Z_r^2 \geq r^{1/\tau}a) \sim r^2T(a).$$

**Remark 44.** *There are examples where  $T$  is nonconvex. For instance, one can show numerically that it is the case when  $W = B|Z|/\mathbb{E}[B|Z|]$  where  $B \sim \text{Ber}(p)$  for some  $p \in (0, 1)$  (for instance  $p = 0.1$ ) and  $Z \sim \mathcal{N}(0, 1)$ .*

*Proof of Proposition 43. Lower bound* — Notice that

$$\log \mathbb{P}(Z_r^2 \geq r^\alpha a) \geq r \log \mathbb{P}(WZ_r^1 \geq r^\alpha a) \sim - \begin{cases} r^{1+\alpha\tau} ca^\tau & \text{if } \alpha < \frac{1}{\tau} \\ r^2 \inf_{bd=a} (cb^\tau + \Lambda_W^*(d)) & \text{if } \alpha = \frac{1}{\tau} \\ r^{\frac{3+\alpha\tau}{2}} 2ca^{\frac{\tau}{2}} & \text{if } \alpha > \frac{1}{\tau}, \end{cases}$$

applying Proposition 41.

**Upper bound for  $\alpha \neq 1/\tau$**  — Markov's inequality yields, for all  $t = Kr^\gamma > 0$ ,

$$\log \mathbb{P}(Z_r^2 \geq r^\alpha a) \leq -tr^{1+\alpha}a + r \log \mathbb{E}[e^{tWZ_r^1}].$$

Integrating by parts, we get

$$\mathbb{E}[e^{tWZ_r^1}] = \int_0^\infty e^{ty} \mathbb{P}(WZ_r^1 \in dy) = 1 + \int_0^\infty te^{ty} \mathbb{P}(WZ_r^1 \geq y) dy.$$

Let  $\delta \in (0, 1)$  and let  $(y_0, r_0, C_1, C_2) \in (0, \infty)^4$  given by Proposition 42. Let  $c' = (1 - \delta)c$ .

- First,

$$I_1 := 1 + \int_0^{y_0} te^{ty} \mathbb{P}(WZ_r^1 \geq y) dy \leq 1 + \int_0^{y_0} te^{ty} dy = e^{ty_0}.$$

- Using Proposition 42,

$$I_2 := \int_{y_0}^{C_1 r^{1/\tau}} te^{ty} \mathbb{P}(WZ_r^1 \geq y) dy \leq \int_{y_0}^{C_1 r^{1/\tau}} te^{ty - c'y^\tau} dy =: \tilde{I}_2.$$

The concave function  $f_2: y \mapsto ty - c'y^\tau$  has a unique critical point  $y_2 := (\frac{t}{c'\tau})^{\frac{1}{\tau-1}}$  and

$$f_2(y_2) = \left(\frac{t}{c'\tau}\right)^{\frac{1}{\tau-1}} \left(1 - \frac{1}{\tau}\right).$$

Note that  $y_2 \geq y_0$  for  $r$  large enough and that  $y_2 \leq C_1 r^{\frac{1}{\tau}}$  if and only if  $t \leq c'\tau C_1^{\tau-1} r^{\frac{\tau-1}{\tau}}$ , which is true for  $r$  large enough if and only if  $\gamma < \frac{\tau-1}{\tau}$ , or  $\gamma = \frac{\tau-1}{\tau}$  and  $K \leq c'\tau C_1^{\tau-1}$ . In this case,

$$\log(\tilde{I}_2) \sim f_2(y_2),$$

otherwise

$$\log(\tilde{I}_2) \sim f_2(C_1 r^{\frac{1}{\tau}}) = tC_1 r^{\frac{1}{\tau}} - c'C_1^\tau r.$$

- Since the tail is nonincreasing,

$$I_3 := \int_{C_1 r^{1/\tau}}^{C_2 r^{1/\tau}} te^{ty} \mathbb{P}(WZ_r^1 \geq y) dy \leq \mathbb{P}(WZ_r^1 \geq C_1 r^{1/\tau}) \int_{C_1 r^{1/\tau}}^{C_2 r^{1/\tau}} te^{ty} dy \leq \mathbb{P}(WZ_r^1 \geq C_1 r^{1/\tau}) e^{C_2 t r^{1/\tau}} =: \tilde{I}_3.$$

Using Proposition 41, we get

$$\log(\tilde{I}_3) \sim -r \inf_{bd=a} (cb^\tau + \Lambda_W^*(d)) + C_2 K r^{\gamma+1/\tau}.$$

- Finally, using Proposition 42,

$$I_4 := \int_{C_2 r^{1/\tau}}^\infty te^{ty} \mathbb{P}(WZ_r^1 \geq y) dy \leq \int_{C_2 r^{1/\tau}}^\infty te^{ty - 2c'r^{1/2}y^{\tau/2}} dy =: \tilde{I}_4.$$

Let us find the maximum of the concave function  $f_4: y \mapsto ty - 2c'r^{1/2}y^{\tau/2}$ . If  $\tau = 2$ , this function is affine and the integral converges if and only if  $t < 2c'r^{1/2}$ , in which case

$$\log(\tilde{I}_4) \sim f_4(C_2 r^{1/2}) = -(2c'r^{1/2} - t)C_2 r^{1/2}.$$

If  $\tau > 2$ ,  $f_4$  has a unique critical point  $y_4 := (\frac{t}{c'\tau r^{1/2}})^{\frac{2}{\tau-2}}$  and

$$f_4(y_4) = \left(\frac{t}{c'\tau r^{1/2}}\right)^{\frac{2}{\tau-2}} \left(1 - \frac{2}{\tau}\right).$$

Now,  $y_4 \geq C_2 r^{\frac{1}{\tau}}$  if and only if  $t \geq c' \tau C_2^{\frac{\tau}{2}-1} r^{\frac{\tau-1}{\tau}}$ , which is true for  $r$  large enough if and only if  $\gamma > \frac{\tau-1}{\tau}$ , or  $\gamma = \frac{\tau-1}{\tau}$  and  $K \geq c' \tau C_2^{\frac{\tau}{2}-1}$ . In this case,

$$\log(\tilde{I}_4) \sim f_4(y_4),$$

otherwise,

$$\log(\tilde{I}_4) \sim f_4(C_2 r^{1/\tau}) = t C_2 r^{\frac{1}{\tau}} - 2c' C_2^{\frac{\tau}{2}} r.$$

To conclude : (1) if  $\alpha < \frac{1}{\tau}$ , we shall take  $\gamma = \alpha(\tau - 1) < \frac{\tau-1}{\tau}$  and  $K = c' \tau a^{\tau-1}$  and only  $\tilde{I}_2$  counts ; (2) if  $\alpha > \frac{1}{\tau}$ : either  $\tau > 2$  in which case we shall take  $\gamma = \frac{1+\alpha(\tau-2)}{2} > \frac{\tau-1}{\tau}$  and  $K = c' \tau a^{\frac{\tau}{2}-1}$  and only  $\tilde{I}_4$  counts; or  $\tau = 2$  in which case we shall take  $\gamma = \frac{1}{2}$  and  $K = 2c''$  with  $c'' < c'$  and  $\log \mathbb{E}[e^{tWZ_r^1}] = O(r)$ .

**Upper bound for  $\alpha = 1/\tau$**  — Markov's inequality yields, for all  $t = Kr^{1-1/\tau} > 0$ ,

$$\log \mathbb{P}(Z_r^2 \geq r^{1/\tau} a) \leq -tr^{1+1/\tau} a + r \log \mathbb{E}[e^{tWZ_r^1}].$$

Integrating by parts, we get

$$\mathbb{E}[e^{tWZ_r^1}] = \int_0^\infty e^{ty} \mathbb{P}(WZ_r^1 \in dy) = 1 + \int_0^\infty t e^{ty} \mathbb{P}(WZ_r^1 \geq y) dy.$$

Let  $\delta \in (0, 1)$  and let  $(y_0, r_0) \in (0, \infty)^2$  given by Proposition 42. Then, for  $r \geq r_0$ ,

$$\mathbb{E}[e^{tWZ_r^1}] \leq 1 + \int_0^{y_0} t e^{ty} dy + \int_{y_0}^\infty t e^{ty-(1-\delta)rT(yr^{-1/\tau})} dy \leq e^{Kr^{1-1/\tau}y_0} + Kr \int_0^\infty e^{r(Ku-(1-\delta)T(u))} du.$$

Hence

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{E}[e^{tWZ_r^1}] \leq \sup_{u \geq 0} (Ku - (1-\delta)T(u)) \leq (1-\delta)T^*\left(\frac{K}{1-\delta}\right).$$

Finally,

$$\limsup_{r \rightarrow \infty} \frac{1}{r^2} \log \mathbb{P}(Z_r^2 \geq r^{1/\tau} a) \leq -Ka + \limsup_{r \rightarrow \infty} \frac{1}{r} \log \mathbb{E}[e^{tWZ_r^1}] \leq -Ka + (1-\delta)T^*\left(\frac{K}{1-\delta}\right).$$

Optimizing in  $K > 0$ , we get

$$\limsup_{r \rightarrow \infty} \frac{1}{r^2} \log \mathbb{P}(Z_r^2 \geq r^{1/\tau} a) \leq -(1-\delta)T^{**}(a),$$

and the conclusion follows, letting  $\delta \rightarrow 0$ . □

To go on in the induction (for  $Z_r^3$ ), we need the following analogue of Proposition 40 for  $Z_r^2$ .

**Proposition 45** (Uniform upper bound). *Assume that  $\tau \geq 2$ . For all  $\varepsilon > 0$  and  $\delta \in (0, 1)$ , there exists  $(r_0, C, C_1, C_2) \in (0, \infty)^4$  such that, for all  $r \geq r_0$  and  $z \geq r^\varepsilon$ ,*

$$\log \mathbb{P}(Z_r^2 \geq z) \leq -(1-\delta) \times \begin{cases} cz^\tau & \text{if } r^\varepsilon \leq z \leq C_1 r^{\frac{1}{\tau}} \\ r^2 T^{**}(zr^{-\frac{1}{\tau}}) & \text{if } C_1 r^{\frac{1}{\tau}} < z < C_2 r^{\frac{1}{\tau}} \\ 2cr^{\frac{3}{2}} z^{\frac{\tau}{2}} & \text{if } C_2 r^{\frac{1}{\tau}} \leq z. \end{cases}$$

*Proof of Proposition 45.* Markov's inequality yields, for all  $t > 0$ ,

$$\log \mathbb{P}(Z_r^2 \geq z) \leq -trz + r \log \mathbb{E}[e^{tWZ_r^1}].$$

Integrating by parts, we get

$$\mathbb{E}[e^{tWZ_r^1}] = \int_0^\infty e^{ty} \mathbb{P}(WZ_r^1 \in dy) = 1 + \int_0^\infty t e^{ty} \mathbb{P}(WZ_r^1 \geq y) dy.$$

In the sequel, we decompose the latter integral as the sum of four integrals and get

$$\log \mathbb{P}(Z_r^2 \geq z) \leq r(-tz + \log(I_1 + I_2 + I_3 + I_4)) \leq r(-tz + \log(4) + \log(I_1) + \log(I_2) + \log(I_3) + \log(I_4)).$$

Let  $\delta \in (0, 1)$  and let  $(r_0, y_0, C_1, C_2) \in (0, \infty)^4$  given by Proposition 42.

• First,

$$I_1 := 1 + \int_0^{y_0} t e^{ty} \mathbb{P}(WZ_r^1 \geq y) dy \leq 1 + \int_0^{y_0} t e^{ty} dy = e^{ty_0}.$$

- Using Proposition 42,

$$I_2 := \int_{y_0}^{C_1 r^{1/\tau}} te^{ty} \mathbb{P}(WZ_r^1 \geq y) dy \leq \int_{y_0}^{C_1 r^{1/\tau}} te^{ty - (1-\delta)cy^\tau} dy.$$

We find the minimum of the convex function  $f_2: y \mapsto -ty + (1-\delta)cy^\tau$  and get

$$\log(I_2) \leq \log(C_1 r^{1/\tau} t) + \underbrace{\left( \frac{t^\tau}{(1-\delta)c\tau} \right)^{\frac{1}{\tau-1}} \left( 1 - \frac{1}{\tau} \right)}_{=: L_2}.$$

- Now, using Proposition 42 again,

$$I_3 := \int_{C_1 r^{1/\tau}}^{C_2 r^{1/\tau}} te^{ty} \mathbb{P}(WZ_r^1 \geq y) dy \leq \int_{C_1 r^{1/\tau}}^{C_2 r^{1/\tau}} te^{ty - (1-\delta)rT(yr^{-1/\tau})} dy.$$

Hence,

$$\log(I_3) \leq \log(C_2 r^{1/\tau} t) + \underbrace{\sup_{y \in [C_1 r^{1/\tau}, C_2 r^{1/\tau}]} (ty - (1-\delta)rT(yr^{-1/\tau}))}_{=: L_3}.$$

- Finally, using Proposition 42,

$$I_4 := \int_{C_2 r^{1/\tau}}^{\infty} te^{ty} \mathbb{P}(WZ_r^1 \geq y) dy \leq \int_{C_2 r^{1/\tau}}^{\infty} te^{ty - 2(1-\delta)cr^{1/2}y^{\tau/2}} dy.$$

Let us define the convex function  $f_4: y \mapsto -ty + 2(1-\delta)cr^{1/2}y^{\tau/2}$ . If  $\tau = 2$ ,  $f_4$  is an affine function and, for  $t < 2(1-\delta)cr^{1/2}$ ,

$$\log(I_4) \leq \log\left( \frac{t}{2(1-\delta)cr^{1/2} - t} \right) - (2(1-\delta)cr^{1/2} - t)C_2 r^{1/\tau}.$$

If  $\tau > 2$ , we set

$$y_{\square} := \left( \frac{t+1}{(1-\delta)cr^{1/2}\tau} \right)^{\frac{2}{\tau-2}} \vee (C_2 r^{1/\tau}),$$

so that, for all  $y \geq y_{\square}$ ,  $f_4'(y) \geq 1$  (and, moreover,  $f_4(y) \geq 0$ ). Then,  $I_4 \leq I_{41} + I_{42}$  with

$$I_{41} := \int_{C_2 r^{1/\tau}}^{y_{\square}} te^{-f_4(y)} dy \leq ty_{\square} e^{-f_4(y_4)}$$

where  $y_4 = (t/((1-\delta)cr^{1/2}\tau))^{\frac{2}{\tau-2}}$  is the point of minimum of the function  $f_4$ , and

$$I_{42} := \int_{y_{\square}}^{\infty} \frac{tf_4'(y)}{f_4'(y_{\square})} e^{-f_4(y)} dy \leq te^{-f_4(y_{\square})}.$$

Hence,

$$\log(I_{41}) \leq \log(ty_{\square}) - f_4(y_4)$$

and

$$\log(I_{42}) \leq \log(t) - f_4(y_{\square}) \leq \log(t) - f_4(y_4),$$

so

$$\begin{aligned} \log(I_4) &\leq \log(I_{41} + I_{42}) \leq \log(2) + \max(\log(I_{41}), \log(I_{42})) \\ &\leq \log(2) + \log(ty_{\square}) - f_4(y_4) \\ &= \log(2) + \log(ty_{\square}) + \underbrace{\left( \frac{t^{\frac{\tau}{2}}}{(1-\delta)c\tau r^{\frac{1}{2}}} \right)^{\frac{2}{\tau-2}} \left( 1 - \frac{2}{\tau} \right)}_{=: L_4}. \end{aligned}$$

First consider the case  $\tau = 2$ .

- (small  $z$ ) We introduce  $C'_1 \in (0, 1 - \delta)$  small enough so that  $\eta := -2cC'_1C_2 + (1 - \delta)T(C_1) > 0$ . Then, for  $z \leq C'_1r^{1/2}$  and  $t = 2cz$ , using the fact that  $T$  is nondecreasing on  $[0, \infty)$ ,

$$L_3 \leq tC_2r^{1/2} - (1 - \delta)rT(C_1) \leq -\eta r$$

and  $t \leq 2(1 - \delta)cr^{1/2} - 1$  as soon as  $r$  is large enough. Hence, absorbing everything in  $L_2$ , we get, for  $r$  large enough,

$$\log \mathbb{P}(Z_r^2 \geq z) \leq -r \left( tz - (1 + \delta) \frac{t^2}{4(1 - \delta)c} \right) = -r \left( 2 - \frac{1 + \delta}{1 - \delta} \right) cz^2,$$

and the conclusion follows for  $z \leq C'_1r^{1/2}$ , since  $\frac{1 + \delta}{1 - \delta} \rightarrow 1$  as  $\delta \rightarrow 0$ .

- (large  $z$ ) Here, we set  $t = 2(1 - 2\delta)cr^{1/2}$ , so  $L_2 \leq r$  and  $L_3 \leq 2cC_2r$ . Bounding above the logarithmic terms by  $r$  and choosing  $C'_2 > 0$  large enough that  $(2cC_2 + 2)/(2c(1 - \delta)C'_2) < \delta$ , we get

$$\begin{aligned} \log \mathbb{P}(Z_r^2 \geq z) &\leq -r(tz - (2cC_2 + 2)r) \\ &\leq -r^{3/2}2c(1 - 2\delta)z \left( 1 - \frac{2cC_2 + 2}{2c(1 - \delta)C'_2} \right) \leq -r^{3/2}2c(1 - 2\delta)z(1 - \delta), \end{aligned}$$

and the conclusion follows for  $z \geq C'_2r^{1/2}$ , since  $(1 - 2\delta)(1 - \delta) \rightarrow 1$  as  $\delta \rightarrow 0$ .

- (middle  $z$ ) Here, we merge  $I_2$  and  $I_3$  and use the last upper bound in Proposition 42. We get, for all  $t \leq 2(1 - \delta)cr^{1/2} - 1$ ,

$$\log \mathbb{P}(Z_r^2 \geq z) \leq r \left( -tz + \log(3) + ty_0 + \log(C_2r^{1/2}t) + (1 - \delta)rT^* \left( \frac{t}{(1 - \delta)r^{1/2}} \right) + \log(t) \right).$$

Optimizing in  $t$ , one gets

$$\begin{aligned} \log \mathbb{P}(Z_r^2 \geq z) &\leq r \left( -r(1 - \delta) \sup_{t \leq 2(1 - \delta)cr^{1/2} - 1} \left( \frac{t}{(1 - \delta)r^{1/2}} \frac{z}{r^{1/2}} - T^* \left( \frac{t}{(1 - \delta)r^{1/2}} \right) \right) \right. \\ &\quad \left. + \log(3) + (2cr^{1/2})y_0 + \log(2cC_2r) + \log(2cr^{1/2}) \right) \\ &= r \left( -r(1 - \delta) \sup_{\tilde{t} \leq 2c - (1 - \delta)^{-1}r^{-1/2}} \left( \tilde{t}\tilde{z} - T^*(\tilde{t}) \right) + \log(3) + (2cr^{1/2})y_0 + \log(2cC_2r) + \log(2cr^{1/2}) \right), \end{aligned}$$

noting  $\tilde{z} = zr^{-1/2}$ . Note that  $T^*(\tilde{t}) = \infty$  for  $\tilde{t} > 2c$  (using the fact that, for all  $a$ ,  $T(a) \leq 2ca$  by (62)). Hence, for all  $\varepsilon > 0$ , there exists  $r_0 \geq 2$  such that, for all  $r \geq r_0$ , for all  $\tilde{z} \in [C'_1, C'_2]$ ,

$$\sup_{\tilde{t} \leq 2c - (1 - \delta)^{-1}r^{-1/2}} \left( \tilde{t}\tilde{z} - T^*(\tilde{t}) \right) \geq T^{**}(\tilde{z}) - \varepsilon.$$

Indeed, since pointwise convergence of convex functions implies uniform convergence on compact subsets, this amounts to show that, for all  $\tilde{z} \in [C'_1, C'_2]$ ,

$$\sup_{\tilde{t} < 2c - \eta} \left( \tilde{t}\tilde{z} - T^*(\tilde{t}) \right) \xrightarrow{\eta \rightarrow 0} \sup_{\tilde{t} \leq 2c} \left( \tilde{t}\tilde{z} - T^*(\tilde{t}) \right),$$

which stems from the continuity of  $\tilde{t} \mapsto \tilde{t}\tilde{z}$  and the fact that  $T^*$  is nondecreasing. Finally,  $T^{**}(C'_1) > 0$ . Indeed,

$$T(a) \geq \underline{T}(a) := \begin{cases} 0 & \text{for } a \in [0, C'_1/2) \\ T(C'_1/2) & \text{for } a \in [C'_1/2, a_0) \\ 2ca & \text{for } a \geq a_0 \end{cases}$$

for  $a_0 \geq C'_1$  large enough, whence  $T^{**}(C'_1) \geq \underline{T}^{**}(C'_1) = T(C'_1/2) \cdot C'_1/(2a_0 - C'_1) > 0$ .

Eventually, all terms may be absorbed in the term  $-r^2(1 - \delta)T^{**}(\tilde{z})$  up to an additional  $(1 - \delta)$  factor and the conclusion follows, since  $(1 - \delta)^2 \rightarrow 1$  as  $\delta \rightarrow 0$ .

Now we turn to the case  $\tau > 2$ .

- (small  $z$ ) We introduce  $C'_1 \in (0, 1 - \delta)$  small enough that  $\eta := -2c(C'_1)^{\tau-1}C_2 + (1 - \delta)T(C_1) > 0$ . Then, for  $z \leq C'_1r^{1/\tau}$  and  $t = \tau cz^{\tau-1}$ , proceeding as for  $L_3$  when  $\tau = 2$  and  $z$  small,

$$L_3 \leq tC_2r^{1/\tau} - (1 - \delta)rT(C_1) \leq -\eta r.$$

Furthermore,  $L_4$  is bounded above by  $L_2$  for  $z \leq C'_1 r^{1/\tau}$  and  $C'_1$  small enough (note that  $(t^\tau/r)^{\frac{1}{\tau-2}} \leq t^{\frac{\tau}{\tau-1}}$  is equivalent to  $t^{\frac{\tau}{\tau-1}} \leq r$ ). Hence, absorbing everything in  $L_2$ , we get, for  $r$  large enough,

$$\log \mathbb{P}(Z_r^2 \geq z) \leq -r \left( tz - (1+\delta) \left( \frac{t^\tau}{(1-\delta)c\tau} \right)^{\frac{1}{\tau-1}} \left( 1 - \frac{1}{\tau} \right) \right) = -rc\tau z^\tau \left( 1 - \frac{1+\delta}{(1-\delta)^{\frac{1}{\tau-1}}} \left( 1 - \frac{1}{\tau} \right) \right)$$

and the conclusion follows for  $z \leq C'_1 r^{1/2}$ , since  $\frac{1+\delta}{(1-\delta)^{\frac{1}{\tau-1}}} \rightarrow 1$  as  $\delta \rightarrow 0$ .

- (large  $z$ ) Here, we set  $t = c\tau r^{1/2} z^{\frac{\tau}{2}-1}$ , so  $L_2$  is bounded above by  $L_4$  for  $z \geq C'_2 r^{1/\tau}$  and  $C'_2$  large enough (remind that  $(t^\tau/r)^{\frac{1}{\tau-2}} \geq t^{\frac{\tau}{\tau-1}}$  is equivalent to  $t^{\frac{\tau}{\tau-1}} \geq r$ ); moreover,  $L_3 \leq tC_2 r^{1/\tau}$  is also bounded above by  $L_4$  for  $z \geq C'_2 r^{1/\tau}$  and  $C'_2$  large enough (note that  $tr^{1/\tau} \leq (t^\tau/r)^{\frac{1}{\tau-2}}$  is equivalent to  $r^{\frac{\tau-1}{\tau}} \leq t$ ). Bounding above the logarithmic terms by  $r$  and choosing  $C'_2 > 0$  large enough, we get

$$\log \mathbb{P}(Z_r^2 \geq z) \leq -r \left( tz - (1+\delta) \left( \frac{t^{\frac{\tau}{2}}}{(1-\delta)c\tau r^{\frac{1}{2}}} \right)^{\frac{2}{\tau-2}} \left( 1 - \frac{2}{\tau} \right) \right) = -r^{3/2} z^{\tau/2} c\tau \left( 1 - \frac{1+\delta}{(1-\delta)^{\frac{2}{\tau-2}}} \left( 1 - \frac{2}{\tau} \right) \right),$$

and the conclusion follows for  $z \geq C'_2 r^{1/2}$ , since  $(1+\delta)(1-\delta)^{-\frac{2}{\tau-2}} \rightarrow 1$  as  $\delta \rightarrow 0$ .

- (middle  $z$ ) Here, we merge  $I_2, I_3$ , and  $I_{41}$  and use the last upper bound in Proposition 42. We get, for all  $t > 0$ ,

$$\log \mathbb{P}(Z_r^2 \geq z) \leq r \left( -tz + \log(3) + ty_0 + \log(ty_0) + (1-\delta)rT^* \left( \frac{t}{(1-\delta)r^{1-1/\tau}} \right) + \log(t) \right).$$

And, for  $r$  large,  $t \geq r^{1/2}$ , and  $z \geq C'_1 r^{1/\tau}$ ,

$$\log \mathbb{P}(Z_r^2 \geq z) \leq r \left( -tz(1-\delta) + (1-\delta)rT^* \left( \frac{t}{(1-\delta)r^{1-1/\tau}} \right) \right).$$

Optimizing in  $t \geq 2r^{1/2}$ , one gets

$$\begin{aligned} \log \mathbb{P}(Z_r^2 \geq z) &\leq -r^2(1-\delta) \sup_{t \geq 2r^{1/2}} \left( \frac{t}{(1-\delta)r^{1-1/\tau}} \frac{(1-\delta)z}{r^{1/\tau}} - T^* \left( \frac{t}{(1-\delta)r^{1-1/\tau}} \right) \right) \\ &= -r^2(1-\delta) \sup_{\tilde{t} \geq r^{\frac{1}{\tau}-\frac{1}{2}}} \left( \tilde{t}\tilde{z} - T^*(\tilde{t}) \right), \end{aligned}$$

as soon as  $\delta \leq \frac{1}{2}$  and noting  $\tilde{z} = (1-\delta)zr^{-1/\tau}$ . For all  $\varepsilon > 0$ , there exists  $r_0 \geq 2$  such that, for all  $r \geq r_0$ , for all  $\tilde{z} \in (1-\delta)[C'_1, C'_2]$ ,

$$\sup_{\tilde{t} \geq r^{\frac{1}{\tau}-\frac{1}{2}}} (\tilde{t}\tilde{z} - T^*(\tilde{t})) \geq \sup_{\tilde{t} > 0} (\tilde{t}\tilde{z} - T^*(\tilde{t})) - \varepsilon = T^{**}(\tilde{z}) - \varepsilon,$$

since pointwise convergence of convex functions implies uniform convergence on compact subsets. Using the uniform continuity of  $T^{**}$  on  $[(1-\delta)C'_1, C'_2]$  (note that  $T^{**}$  is convex and  $T^{**} \leq T < \infty$  on  $[0, \infty)$ ), for  $\delta$  small enough, for all  $r$  and for all  $z \in r^{1/\tau}[C'_1, C'_2]$ ,  $T^{**}(\tilde{z}) = T^{**}((1-\delta)zr^{-1/\tau}) \geq T^{**}(zr^{-1/\tau}) - \varepsilon$ . Finally, we absorb the term  $2\varepsilon$  in  $T^{**}(zr^{-1/\tau})$  as before, because  $T^{**}(C'_1) > 0$ . Hence the result.  $\square$

## 5.4 Proof for $Z_r^2$ when $\tau < 2$

**Proposition 46.** *Assume that  $\tau \in (1, 2)$ . Then,*

$$-\log \mathbb{P}(Z_r^2 \geq r^\alpha a) \sim \begin{cases} r^{1+\alpha\tau} ca^\tau & \text{if } \alpha < \frac{\tau-1}{\tau} \\ r^\tau c \inf_{b+d=a} (2b^{\frac{\tau}{2}} + d^\tau) & \text{if } \alpha = \frac{\tau-1}{\tau} \\ r^{\frac{1+(1+\alpha)\tau}{2}} 2ca^{\frac{\tau}{2}} & \text{if } \alpha > \frac{\tau-1}{\tau}. \end{cases}$$

*Proof of Proposition 46. Lower bound for  $\alpha < \frac{\tau-1}{\tau}$  —* Let  $\varepsilon \in (0, 1)$ . We have

$$\log \mathbb{P}(Z_r^2 \geq r^\alpha a) \geq r \log \mathbb{P}(WZ_r^1 \geq r^\alpha a) \sim -r^{1+\alpha\tau} c \left( \frac{a}{1-\varepsilon} \right)^\tau,$$

using Proposition 41 and the fact that  $\alpha < \frac{1}{\tau}$ . We conclude letting  $\varepsilon \rightarrow 0$ .

**Lower bound for  $\alpha = \frac{\tau-1}{\tau}$**  — Let  $(b_0, d_0)$  be such that

$$2b_0^{\frac{\tau}{2}} + d_0^\tau = \inf_{b+d=a} (2b^{\frac{\tau}{2}} + d^\tau).$$

Then, using Proposition 41,

$$\begin{aligned} \frac{1}{r} \log \mathbb{P}(Z_r^2 \geq r^{1+\alpha} a) &\geq \frac{1}{r} \log \mathbb{P}(W_1 Z_{r,1}^1 \geq r^{1+\alpha} b_0, \forall i \in \llbracket 2, r \rrbracket \ W_i Z_{r,i}^1 \geq r^\alpha d_0) \\ &= \frac{1}{r} \log \mathbb{P}(W Z_r^1 \geq r^{1+\alpha} b_0) + \frac{r-1}{r} \log \mathbb{P}(W Z_r^1 \geq r^\alpha d_0) \xrightarrow{r \rightarrow \infty} 2b_0^{\frac{\tau}{2}} + d_0^\tau. \end{aligned}$$

**Lower bound for  $\alpha > \frac{\tau-1}{\tau}$**  — Applying Proposition 41, we get

$$\log \mathbb{P}(Z_r^2 \geq r^\alpha a) \geq \log \mathbb{P}(W Z_r^1 \geq r^{1+\alpha} a) \sim -r^{\frac{1+(1+\alpha)\tau}{2}} 2ca^{\frac{\tau}{2}}.$$

**Upper bound for  $\alpha < \frac{\tau-1}{\tau}$**  — Let us define, for all  $q \in \llbracket 0, r \rrbracket$ ,

$$P_{r,q} = \mathbb{P}\left(\sum_{i=1}^r W_i Z_{r,i}^1 \geq r^{1+\alpha} a, \forall i \in \llbracket 1, q \rrbracket \ W_i Z_{r,i}^1 \geq r^{1+\alpha} a, \forall i \in \llbracket q+1, r \rrbracket \ W_i Z_{r,i}^1 < r^{1+\alpha} a\right).$$

First, we have

$$\sum_{q=1}^r \binom{r}{q} P_{r,q} \leq \sum_{q=1}^r (r \mathbb{P}(W Z_r^1 \geq r^{1+\alpha} a))^q \leq \sum_{q=1}^r (r \exp(-r^{\frac{1+(1+\alpha)\tau}{2}} C))^q \leq \frac{\exp(-r^{\frac{1+(1+\alpha)\tau}{2}} C)}{1 - \exp(-r^{\frac{1+(1+\alpha)\tau}{2}} C)}.$$

Since  $\alpha < \frac{\tau-1}{\tau}$ , we have  $\frac{1+(1+\alpha)\tau}{2} > 1 + \alpha\tau$ , hence

$$\frac{1}{r^{1+\alpha\tau}} \log \sum_{q=1}^r \binom{r}{q} P_{r,q} \xrightarrow{r \rightarrow \infty} -\infty.$$

Secondly, let us turn to  $P_{r,0}$ . Let  $t > 0$ . Using the fact that  $\mathbb{1}_{x \geq 0} \leq e^x$ , we have

$$P_{r,0} = \mathbb{E}\left[\mathbb{1}_{\sum_{i=1}^r W_i Z_{r,i}^1 \geq r^{1+\alpha} a} \prod_{i=1}^r \mathbb{1}_{W_i Z_{r,i}^1 < r^{1+\alpha} a}\right] \leq e^{-tr^{1+\alpha} a} \mathbb{E}\left[e^{tW Z_r^1} \mathbb{1}_{W Z_r^1 < r^{1+\alpha} a}\right]^r.$$

Integrating by parts, we get

$$\mathbb{E}\left[e^{tW Z_r^1} \mathbb{1}_{W Z_r^1 < r^{1+\alpha} a}\right] = \int_0^{r^{1+\alpha} a} e^{ty} \mathbb{P}(W Z_r^1 \in dy) \leq 1 + \int_0^{r^{1+\alpha} a} t e^{ty} \mathbb{P}(W Z_r^1 \geq y) dy.$$

We choose  $t = Kr^\gamma$  with  $K > 0$  and  $\gamma = \alpha(\tau - 1)$  (so that  $1 + \alpha + \gamma = 1 + \alpha\tau$ ). Let  $\varepsilon \in (0, \alpha)$ .

• First,

$$I_1 := 1 + \int_0^{r^\varepsilon} t e^{ty} \mathbb{P}(W Z_r^1 \geq y) dy \leq e^{tr^\varepsilon} = e^{o(r^{\alpha+\gamma})}.$$

• Now, for  $c' \in (0, c)$ ,

$$I_2 := \int_{r^\varepsilon}^{C_1 r^{1/\tau}} t e^{ty} \mathbb{P}(W Z_r^1 \geq y) dy \leq \int_{r^\varepsilon}^{C_1 r^{1/\tau}} t e^{ty - c' y^\tau} dy =: \tilde{I}_2.$$

The concave function  $f_2: y \mapsto ty - c' y^\tau$  has a unique critical point  $y_2 := (\frac{t}{c'\tau})^{\frac{1}{\tau-1}}$  and

$$f_2(y_2) = \left(\frac{t^\tau}{c'\tau}\right)^{\frac{1}{\tau-1}} \left(1 - \frac{1}{\tau}\right).$$

Now,  $y_2 = \Theta(r^{\frac{\gamma}{\tau-1}}) = \Theta(r^\alpha) = o(r^{1/\tau})$ , since  $\alpha < \frac{\tau-1}{\tau} < \frac{1}{\tau}$ . So,

$$\log(\tilde{I}_2) \sim f_2(y_2) = K^{\frac{\tau}{\tau-1}} r^{\alpha\tau} \frac{1 - \frac{1}{\tau}}{(c'\tau)^{\frac{1}{\tau-1}}}$$

hence

$$\log(e^{-tr^{1+\alpha} a} \tilde{I}_2^r) \sim r^{1+\alpha\tau} \left(-Ka + K^{\frac{\tau}{\tau-1}} \frac{1 - \frac{1}{\tau}}{(c'\tau)^{\frac{1}{\tau-1}}}\right).$$

Taking  $K = c'\tau a^{\tau-1}$ , we get

$$\log(e^{-tr^{1+\alpha}a}\tilde{I}_2^r) \sim -r^{1+\alpha\tau}c'a^\tau.$$

• Besides,

$$I_3 := \int_{C_1r^{1/\tau}}^{C_2r^{1/\tau}} te^{ty}\mathbb{P}(WZ_r^1 \geq y)dy \leq \mathbb{P}(WZ_r^1 \geq C_1r^{1/\tau}) \int_{C_1r^{1/\tau}}^{C_2r^{1/\tau}} te^{ty}dy \leq e^{-c'C_1^\tau r + C_2tr^{1/\tau}}$$

and  $(-c'C_1^\tau r + C_2tr^{1/\tau})^+ = o(r^{\alpha+\gamma})$ .

• Finally,

$$I_4 := \int_{C_2r^{1/\tau}}^{r^{1+\alpha}a} te^{ty}\mathbb{P}(WZ_r^1 \geq y)dy \leq \int_{C_2r^{1/\tau}}^{r^{1+\alpha}a} te^{ty-r^{1/2}2c'y^{\tau/2}}dy =: \tilde{I}_4.$$

Since  $\tau < 2$ , we find the maximum of the convex function  $f_4: y \mapsto ty - r^{1/2}2c'y^{\tau/2}$ . We have

$$f_4(C_2r^{1/\tau}) = C_2tr^{1/\tau} - 2c'C_2^{\tau/2}r \sim -2c'C_2^{\tau/2}r$$

and

$$f_4(r^{1+\alpha}a) = Kar^{1+\alpha+\gamma} - 2c'r^{\frac{1+(1+\alpha)\tau}{2}}a^{\tau/2} \sim -2c'r^{\frac{1+(1+\alpha)\tau}{2}}a^{\tau/2} = o(r^{\alpha+\gamma}),$$

since  $\alpha < \frac{\tau-1}{\tau}$  implies  $\frac{1+(1+\alpha)\tau}{2} > 1 + \alpha\tau = 1 + \alpha + \gamma$ . Hence  $\log(\tilde{I}_4) = o(r^{\alpha+\gamma})$ .

**Upper bound for  $\alpha > \frac{\tau-1}{\tau}$**  — Following the same lines and using the same notation as in the proof for  $\alpha < \frac{\tau-1}{\tau}$ , we have

$$\sum_{q=1}^r \binom{r}{q} P_{r,q} \leq \frac{\exp(-r^{\frac{1+(1+\alpha)\tau}{2}}2c'a^{\tau/2})}{1 - \exp(-r^{\frac{1+(1+\alpha)\tau}{2}}2c'a^{\tau/2})} \leq \exp(-r^{\frac{1+(1+\alpha)\tau}{2}}2c''a^{\tau/2})$$

for  $0 < c'' < c' < c$ . Letting  $c'' \rightarrow c$  leads to a good upper bound for this term. Then, we turn to  $P_{r,0}$ . For  $t > 0$ , we still have

$$P_{r,0} \leq e^{-tr^{1+\alpha}a} \mathbb{E} \left[ e^{tWZ_r^1} \mathbb{1}_{WZ_r^1 < r^{1+\alpha}a} \right]^r.$$

We choose  $t = Kr^\gamma$  with  $K > 0$  and  $\gamma = \frac{(\tau-1)+\alpha(\tau-2)}{2}$  (so that  $1 + \alpha + \gamma = \frac{1+(1+\alpha)\tau}{2}$ ) and we want to show that

$$\log^+ \mathbb{E} \left[ e^{tWZ_r^1} \mathbb{1}_{WZ_r^1 < r^{1+\alpha}a} \right] = o(r^{\frac{(1+\alpha)\tau-1}{2}}) = o(r^{\alpha+\gamma}).$$

Let  $\varepsilon \in (0, \alpha \wedge \frac{1}{\tau}]$  and consider the same integrals as before.

• We still have

$$I_1 \leq e^{tr^\varepsilon} = e^{o(r^{\alpha+\gamma})}.$$

• As for  $I_2$ , we find that, as soon as  $\alpha \geq \frac{\tau-1}{2-\tau}$ ,  $y_2 = o(1)$  and

$$\log(\tilde{I}_2) \sim f_2(r^\varepsilon) = Kr^{\gamma+\varepsilon} - c'r^{\tau\varepsilon} \sim -c'r^{\tau\varepsilon} < 0.$$

If  $\alpha < \frac{\tau-1}{2-\tau}$ , we have  $y_2 = o(r^{\frac{1}{\tau}})$  (using  $\alpha > \frac{\tau-1}{\tau}$ ) and

$$\log(\tilde{I}_2) \sim f_2(y_2) = \left(\frac{K^\tau}{c'^\tau}\right)^{\frac{1}{\tau-1}} r^{\frac{\gamma\tau}{\tau-1}} \left(1 - \frac{1}{\tau}\right) = o(r^{\alpha+\gamma}).$$

- As before,  $I_3 \leq e^{-c'C_1^\tau r + C_2tr^{1/\tau}}$  and  $(-c'C_1^\tau r + C_2tr^{1/\tau})^+ = o(r^{\alpha+\gamma})$ .
- Finally, to bound above  $I_4$ , taking  $K < 2c'a^{\tau/2-1}$ ,

$$f_4(r^{1+\alpha}a) = Kar^{1+\alpha+\gamma} - 2c'r^{\frac{1+(1+\alpha)\tau}{2}}a^{\tau/2} = r^{1+\alpha+\gamma}(Ka - 2c'a^{\tau/2}) < 0.$$

Thus

$$\limsup_{r \rightarrow \infty} \frac{1}{r^{1+\alpha+\gamma}} \log(P_{r,0}) \leq -Ka$$

and we conclude letting  $K \rightarrow 2c'a^{\tau/2-1}$  and  $c' \rightarrow c$ .

**Upper bound for  $\alpha = \frac{\tau-1}{\tau}$**  — Let  $\varepsilon > 0$  (to be chosen later) and let us define, for all  $q \in \llbracket 0, r \rrbracket$ ,

$$\Pi_{r,q} := \mathbb{P} \left( \sum_{i=1}^r W_i Z_{r,i}^1 \geq r^{1+\alpha}a, \forall i \in \llbracket 1, q \rrbracket \quad W_i Z_{r,i}^1 \geq r^{1+\alpha}\varepsilon, \forall i \in \llbracket q+1, r \rrbracket \quad W_i Z_{r,i}^1 < r^{1+\alpha}\varepsilon \right).$$

First, we have, for  $c' \in (0, c)$ ,

$$\frac{1}{r^\tau} \log \Pi_{r,q} \leq \frac{1}{r^\tau} \log \mathbb{P}(WZ_r^1 \geq r^{1+\alpha} \varepsilon)^q \leq -r^{\frac{1+(1+\alpha)\tau}{2}-\tau} 2qc' \varepsilon^{\tau/2} = -2qc' \varepsilon^{\tau/2}$$

so that

$$\limsup_{r \rightarrow \infty} \frac{1}{r^{1+\alpha\tau}} \log \sum_{q=q_0}^r \binom{r}{q} \Pi_{r,q} \leq -ca^\tau.$$

as soon as  $q > a^\tau / (2\varepsilon^{\tau/2}) =: q_0$ .

Secondly, let us turn to  $\Pi_{r,0}$ . Let  $t > 0$ . Using the fact that  $\mathbb{1}_{x \geq 0} \leq e^x$ , we get

$$\Pi_{r,0} = \mathbb{E} \left[ \mathbb{1}_{\sum_{i=1}^r W_i Z_{r,i}^1 \geq r^{1+\alpha} a} \prod_{i=1}^r \mathbb{1}_{W_i Z_{r,i}^1 < r^{1+\alpha} \varepsilon} \right] \leq e^{-tr^{1+\alpha} a} \mathbb{E} \left[ e^{tWZ_r^1} \mathbb{1}_{WZ_r^1 < r^{1+\alpha} \varepsilon} \right]^r.$$

We choose  $t = Kr^{\alpha(\tau-1)}$  with  $K > 0$ . Following the very same lines as in the first two cases, we get:

- $\log(I_1) \leq ty_0 = o(r^{\tau-1})$ ;
- $y_2 < C_1 r^{1/\tau}$ , so

$$\log(\tilde{I}_2) \sim f_2(y_2) \leq \left( \frac{K^\tau}{\tau c'} \right)^{\frac{1}{\tau-1}} \left( 1 - \frac{1}{\tau} \right) r^{\tau-1}$$

hence

$$\log(e^{-tr^{1+\alpha} a} \tilde{I}_2^r) \leq r^\tau \left( -Ka + \left( \frac{K^\tau}{\tau c'} \right)^{\frac{1}{\tau-1}} \left( 1 - \frac{1}{\tau} \right) \right);$$

- $I_3 \leq e^{-c' C_1^\tau r + C_2 t r^{1/\tau}}$  and  $(-c' C_1^\tau r + C_2 t r^{1/\tau})^+ = o(r^{\tau-1})$ ;
- finally, as soon as  $K < 2c\varepsilon^{\tau/2-1}$ ,

$$f_4(r^{1+\alpha} \varepsilon) = (K\varepsilon - 2c'\varepsilon^{\tau/2})r^\tau < 0.$$

Since  $\frac{\tau}{2} - 1 < 0$ , we may choose  $\varepsilon > 0$  such that  $c'\tau a^{\tau-1} < 2c\varepsilon^{\tau/2-1}$ . Hence, for  $K = c'\tau a^{\tau-1}$ , we get

$$\limsup_{r \rightarrow \infty} \frac{1}{r^\tau} \log \Pi_{r,0} \leq -Ka + \left( \frac{K^\tau}{\tau c'} \right)^{\frac{1}{\tau-1}} \left( 1 - \frac{1}{\tau} \right) = -ca^\tau. \quad (63)$$

Thirdly,

$$\begin{aligned} \Pi_{r,1} &= \mathbb{P} \left( Z_r^2 \geq r^{1+\alpha} a, W_1 Z_{r,1}^1 \geq r^{1+\alpha} \varepsilon, \forall i \in [2, r] \ W_i Z_{r,i}^1 < r^{1+\alpha} \varepsilon \right) \\ &= \mathbb{P} \left( W_1 Z_{r,1}^1 + \sum_{i=2}^r W_i Z_{r,i}^1 \geq r^{1+\alpha} a, W_1 Z_{r,1}^1 \geq r^{1+\alpha} \varepsilon, \forall i \in [2, r] \ W_i Z_{r,i}^1 < r^{1+\alpha} \varepsilon \right) \\ &\leq \exp \left\{ -r^\tau c \inf_{b+d \geq a} (2b^{\tau/2} + d^\tau) \right\} \end{aligned}$$

using the contraction principle, Proposition 41, and the previous upper bound for  $\Pi_{r,0}$  in (63). The remaining terms  $\Pi_{r,q}$ , for  $q \in [2, q_0 - 1]$ , can be handled similarly using the contraction principle and the fact that the function  $b \mapsto b^{\tau/2}$  is concave since  $\tau < 2$ .  $\square$

**Proposition 47** (Uniform upper bound). *Assume that  $\tau < 2$ . For all  $\varepsilon > 0$  and  $\delta \in (0, 1)$ , there exists  $(r_0, C, C_1, C_2) \in (0, \infty)^4$  such that, for all  $r \geq r_0$  and  $z \geq r^\varepsilon$ ,*

$$\log \mathbb{P}(Z_r^2 \geq z) \leq -(1-\delta) \times \begin{cases} cz^\tau & \text{if } r^\varepsilon \leq z \leq C_1 r^{\frac{\tau-1}{\tau}} \\ r^\tau c \inf_{b+d=zr^{-(\tau-1)/\tau}} (2b^{\frac{\tau}{2}} + d^\tau) & \text{if } C_1 r^{\frac{\tau-1}{\tau}} < z < C_2 r^{\frac{\tau-1}{\tau}} \\ 2cr^{\frac{1+\tau}{2}} z^{\frac{\tau}{2}} & \text{if } C_2 r^{\frac{\tau-1}{\tau}} \leq z. \end{cases}$$

*Proof of Proposition 47.* Completely analogous to that of Proposition 45.  $\square$

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