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Abstract

Let W be a nonnegative random variable with expectation 1. For all $r \geqslant 2$, we consider the total mass Z_r^∞ of the associated Mandelbrot multiplicative cascade in the r-ary tree. For all $n \geqslant 1$, we also consider the total mass Z_r^n of the measure at height n in the r-ary tree. Liu, Rio, Rouault [11, 12, 18] established large deviation results for $(Z_r^n)_{r\geqslant 2}$ for all $n \in [\![1, \infty[\![$ (resp. for $n = \infty$) in case W has an everywhere finite cumulant generating function Λ_W (resp. W is bounded). Here, we extend these results to the case where Λ_W is only finite on a neighborhood of zero. And we establish all deviation results (moderate, large, and very large deviations). It is noticeable that we obtain nonconvex rate functions. Moreover, our proof of upper bounds of deviations for $(Z_r^\infty)_{r\geqslant 2}$ rely on the moment bound instead of the standard Chernoff bound.

Keywords: Mandelbrot multiplicative cascades ; large deviation principles ; heavy tails ; moment bound ; combinatorial problems.

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Contents

1 Introduction

Multiplicative cascades were introduced by Mandelbrot in [13, 14] in order to analyze some problems of turbulence precisely. For $r \in \mathbb{N}^* \setminus \{1\}$, a multiplicative cascade is a random measure μ_r^∞ on the unit interval, defined as a limit of measures μ_r^n on the sub-σ-algebra generated by the r-adic intervals of level $n \in \mathbb{N}^*$. It is natural to study $Z_r^n := \mu_r^n([0,1])$ the total mass of the measure μ_r^n. It can be described as follows. Let W be a nonnegative random variable such that $\mathbb{E}[W] = 1$. Now, let $(W_{i_1,\dots,i_n})_{n\geqslant 1, 1\leqslant i_1,\dots,i_n\leqslant r}$ be a family of independent and identically distributed random variables distributed as W indexed by all finite sequences of integers between 1 and r. Then we define, for all $n \in \mathbb{N}^*$,

$$Z_r^n := \frac{1}{r^n} \sum_{1\leqslant i_1,\dots,i_n\leqslant r} W_{i_1} W_{i_1,i_2} \cdots W_{i_1,\dots,i_n} \tag{1}$$

and we set $Z_r^0 := 1$.
For fixed r, the properties of Z_r^n were studied in several works. First, Kahane and Peyrière [10] showed that $(Z_r^n)_{n\geqslant 1}$ is a nonnegative martingale with expectation 1 and that the properties of the limit Z_r^∞ were characterized by the behavior of the moments of W and the quantity $\mathbb{E}[W \log(W)]$. In particular, $\mathbb{E}[Z_r^\infty] \leqslant 1$ and it is proved in [10, Théorème 1] that $\mathbb{E}[Z_r^\infty] = 1$ if and only if $\mathbb{E}[W \log(W)] < \log(r)$. Moreover the distribution of Z_r^∞ is solution of the distributional equation

$$Z \overset{(d)}{=} \frac{1}{r} \sum_{i=1}^{r} W_i Z_i \tag{2}$$

where the Z_i's are independent copies of Z and are independent of $(W_i)_{1\leqslant i\leqslant r}$. This equation has been studied by Durrett and Liggett in [6] and by Guivarc'h in [8], and is closely related

to implicit renewal theory (see [7]). Besides, multifractal dimensions of μ_r^∞ were studied by Holley and Waymire in [9] and by Barral [2].
Now for $n \in \mathbb{N}^* \cup \{\infty\}$, the asymptotic behavior of $(Z_r^n)_{r\geqslant 2}$ when r goes to infinity was studied by Liu, Rio, and Rouault in [11, 12, 18]. In [12], the authors studied $(Z_r^\infty)_{r\geqslant 2}$ and obtained the law of large numbers and the central limit theorem under minimal assumptions. They also provided large deviation results under strong assumptions on the tail of the variable W. Let

$$\Lambda_W(t) := \log \mathbb{E}[e^{tW}] \in (-\infty, \infty] \tag{3}$$

be the log-Laplace transform of W, and let

$$\Lambda_W^*(x) := \sup_{t\in\mathbb{R}}[tx - \Lambda_W(t)] \in [0, \infty] \tag{4}$$

be the Fenchel-Legendre transform of Λ_W. The following result is proved in [11, 12].

Theorem 1 (light-tailed – large deviations)**.** *Let $n \in \mathbb{N}^* \cup \{\infty\}$. Assume that one of the two following statements is true:*

1. *(finite tree) $n \in \mathbb{N}^*$ and $\Lambda_W < \infty$ everywhere;*
2. *(infinite tree) $n = \infty$ and $\operatorname{ess\,sup}(W) < \infty$.*

Then, the sequence $(Z_r^n)_{r\geqslant 2}$ satisfies a large deviation principle at speed $(r)_{r\geqslant 2}$ with rate function Λ_W^*.

It is worth noticing that, for unbounded random variables W, even when the Laplace transform was finite everywhere, the large deviation principle for $n = \infty$ (infinite tree) was unknown.
In view of Lemma 18 below, Theorem 1 follows from the estimation of left and right deviations. Since the variables Z_r^n are nonnegative, left deviations easily follow from Gärtner-Ellis theorem in great generality, as shown in [12].

Proposition 2 (left large deviations)**.** *Let $n \in \mathbb{N}^* \cup \{\infty\}$. If $n = \infty$, we assume that $\mathbb{E}[W \log(W)] < \infty$. For all $a \in [0, 1]$,*

$$\frac{1}{r} \log \mathbb{P}(Z_r^n \leqslant a) \xrightarrow[r\to\infty]{} -\Lambda_W^*(a).$$

However, the study of right deviations is more delicate. The case $\operatorname{ess\,sup}(W) < \infty$ is the case where the variables Z_r^n are bounded above (by $\operatorname{ess\,sup}(W)$) and Gärtner-Ellis theorem applies. However, if $\operatorname{ess\,sup}(W) = \infty$, even if $\Lambda_W < \infty$, $\mathbb{E}[e^{tZ_r^n}]$ may be infinite for all $t > 0$ and Gärtner-Ellis theorem cannot provide the result in that case.
In this work, we extend these results to the case where Λ_W might only be finite on $(-\infty, t_0)$ or $(-\infty, t_0]$ for some $t_0 \in (0, \infty]$ and we obtain, for both finite and infinite trees, moderate, large, and very large deviation principles (see Theorems 3 and 4). In [11], the key argument to obtain the large deviation result in the finite tree and with $\Lambda_W < \infty$ everywhere (Theorem

1 with assumption 1) is to truncate the variable and use exponential approximation. In that case, the rate function is Λ_W^*: the large deviations are the same as those of the first level of the tree. In the case where $\Lambda_W(t) = \infty$ for $t \geqslant t_0$, the rate function is not Λ_W^* as soon as $n \in (\mathbb{N}^* \setminus \{1\}) \cup \{\infty\}$, so this argument seems hopeless. Moreover, the Laplace transforms of the random variables Z_r^n are infinite on $(0, \infty)$, hence the exponential Markov's inequality cannot either be used here to prove the upper bounds. Note that, in the case where $\Lambda_W < \infty$ everywhere, we do not know if the truncation argument works. Here, we use different techniques to bypass these problems. In the case of the finite tree, we decompose the event of deviations and control each term; this technique is standard to obtain upper bounds of large deviations for heavy-tailed random variables (see e.g., [15, 16]). In the case of the infinite tree, we bound the moments of Z_r^∞. For moderate and very large deviations, the upper bound follows immediately from the moment Markov's inequality. As for large deviations, we prove that (see Corollary 13)

$$\limsup_{r\to\infty} \frac{1}{r} \log \mathbb{E}[(Z_r^\infty)^{\eta r}] = O(\eta^2)$$

which is the key argument to prove that the rate function is non degenerated and is indeed the limit of the rate functions in finite trees (compare items 2 of Theorems 3 and 4).
The paper is organized as follows. The main Theorems for both finite and infinite trees are presented in Section 2. Section 3 is dedicated to the proof of the large deviation results for finite trees whereas the proofs for the infinite tree are given in Section 4.

2 Main results

From now on, it is always assumed that there exists $c \in (0, \infty]$ such that

$$\frac{1}{w} \log \mathbb{P}(W \geqslant w) \xrightarrow[w\to\infty]{} -c. \tag{5}$$

Since we want to obtain large deviation principles, it is natural from the proofs to assume that $\log \mathbb{P}(W \geqslant rw)/r$ converges for all $w \geqslant 0$. Then the limit $\psi(w)$ is either linear in w or infinite, since $\psi(w) = w\psi(1)$ for all nonnegative rational number w and ψ is nonincreasing. Remark that, by Proposition 5, item 1 below, $c = \infty$ if and only if $\Lambda_W < \infty$ everywhere. If $c \in (0, \infty)$, then Λ_W is finite on a neighborhood of zero. Examples where $c \in (0, \infty)$ include the cases where the law of W is the exponential distribution of mean 1 ($c = 1$), any gamma distribution of mean 1 for instance.

Theorem 3 (finite tree)**.**

1. Moderate deviations. *For all $\alpha \in (0, 1/2)$, for all $n \in \mathbb{N}^*$, the sequence $(r^\alpha(Z_r^n - 1))_{r\geqslant 2}$ satisfies a large deviation principle at speed $(r^{1-2\alpha})_{r\geqslant 2}$ with rate function*

$$J(a) := \frac{a^2}{2\,\mathrm{Var}(W)}.$$

2. Large deviations. *For any $n \in \mathbb{N}^*$, the sequence $(Z_r^n)_{r\geqslant 2}$ satisfies a large deviation principle at speed $(r)_{r\geqslant 2}$ with rate function I^n defined by induction by $I^1 = \Lambda_W^*$ and, for all $a \in \mathbb{R}$,*

$$I^n(a) := \inf\left\{cw + I^{n-1}(z) + \Lambda_W^*(s)\ ;\ w \geqslant 0,\ z \geqslant 0,\ s \geqslant 0,\ wz + s = a\right\}. \qquad (6)$$

3. Very large deviations. *For all $\alpha > 0$, for all $n \in \mathbb{N}^*$, the sequence $(r^{-\alpha} Z_r^n)_{r\geqslant 2}$ satisfies a large deviation principle at speed $(r^{1+\alpha/n})_{r\geqslant 2}$ with rate function $a \mapsto \infty \mathbb{1}_{a<0} + cna^{1/n}\mathbb{1}_{a\geqslant 0}$.*

Here are some remarks concerning the very large deviations (Theorem 3, item 2). It follows from the proof that the left deviations are in fact at speed $(r)_{r\geqslant 2}$, with rate function $a \mapsto \infty \mathbb{1}_{a<0} - \log(p)\mathbb{1}_{a=0}$. Concerning the right deviations, for $c = \infty$, the rate function at speed $(r^{1+\alpha/n})_{r\geqslant 2}$ is degenerate, and the informative speed should be $o(r^{1+\alpha/n})$. For instance, if $\log \mathbb{P}(W \geqslant w) \sim -c'w^\tau$ with $\tau > 1$, then the speed may depend on (c', τ), but this result is out of the scope of this paper.
Concerning the large deviations (Theorem 3, item 2), according to Proposition 2, that, for all $n \in \mathbb{N}^*$,

$$\forall a \leqslant 1 \quad I^n(a) = \Lambda_W^*(a).$$

Moreover, if $c = \infty$, then, for all $n \in \mathbb{N}^*$, $I^n = \Lambda_W^*$. This case (equivalent to $\Lambda_W < \infty$ everywhere; see Proposition 5, item 1) was studied in [11] and the proof relies on exponential approximation. Here we provide another proof that encompasses all the cases $c \in (0, \infty]$. Moreover, some properties of the functions I^n are given in Propositions 6 and 8. Notably, we show that $I^n(a) \sim cna^{1/n}$ as $a \to \infty$, so the rate function of the very large deviations coincides with the asymptotics of the rate function of the large deviations. Also, we prove that $(I^n)_{n\geqslant 1}$ is a decreasing sequence of functions. In particular, we may introduce

$$I^\infty := \lim_{n\to\infty} \downarrow I^n,$$

which appears to be the rate function of the large deviation principle in the infinite tree, as stated below.

Theorem 4 (infinite tree)**.**

1. Moderate deviations. *For all $\alpha \in (0, 1/2)$, the sequence $(r^\alpha(Z_r^\infty - 1))_{r\geqslant 2}$ satisfies a large deviation principle at speed $(r^{1-2\alpha})_{r\geqslant 2}$ with rate function J.*

2. Large deviations. *The sequence $(Z_r^\infty)_{r\geqslant 2}$ satisfies a large deviation principle at speed $(r)_{r\geqslant 2}$ with rate function I^∞.*

3. Very large deviations. *For all $\alpha > 0$, the sequence $(r^{-\alpha} Z_r^\infty)_{r\geqslant 2}$ satisfies a large deviation principle at speed $(r\log(r))_{r\geqslant 2}$ with rate function $a \mapsto \infty \mathbb{1}_{a<0} + c\alpha e \mathbb{1}_{a>0}$.*

Remark. Here are some remarks concerning the very large deviations in Theorem 4.

1. It follows from the proof that the left deviations are in fact at speed $(r)_{r\geqslant 2}$, with rate function $a \mapsto \infty \mathbb{1}_{a<0} - \log(p)\mathbb{1}_{a=0}$.

2. Concerning the right deviations, for $c = \infty$, the rate function at speed $(r\log(r))_{r\geqslant 2}$ is degenerate, and the informative speed should be $o(r\log(r))$. For instance, if $\log \mathbb{P}(W \geqslant w) \sim -c'w^\tau$ with $\tau > 1$, then the speed may depend on (c', τ), but this result is out of the scope of this paper.

3. As for $c \in (0, \infty)$, the rate function at speed $(r\log(r))_{r\geqslant 2}$ is a positive constant on $(0, \infty)$. We expect that, for all $\varepsilon > 0$, $\mathcal{L}\left(r^{-\alpha}Z_r \mid r^{-\alpha}Z_r \geqslant \varepsilon\right)$ satisfies a large deviation at some speed $o(r\log(r))$ and we conjecture that the speed is in fact $(r)_{r\geqslant 2}$, by approximating the event $\{r^{-\alpha}Z_r \approx ea/(e-1)\}$ by the event $\{W_{1^1} \approx rea^{1/\log(r^\alpha)}, \dots, W_{1^n} \approx rea^{1/\log(r^\alpha)}\}$. To get such a result, we should compute the second order in the asymptotics of large deviations, i.e.

$$\mathbb{P}(r^{-\alpha}Z_r \approx a) = \exp(-r\log(r^\alpha)ce + rK(a) + o(r)).$$

The proofs of the large deviation principles derive from left and right deviation estimates and the standard argument that is recalled in appendix (Lemma 18). Table 1 below summarizes the pre-existing results and our contribution:

	$c = \infty$ $\operatorname{ess\,sup}(W) < \infty$	$c = \infty$ $\operatorname{ess\,sup}(W) = \infty$	$c \in (0, \infty)$ $\operatorname{ess\,sup}(W) = \infty$
$n \in \mathbb{N}^*$	[11, Theorem 6.2 (a)]		Theorem 3
$n = \infty$	[12, Theorem 1.4]	Theorem 4	

Table 1: Summary of previous and new results. Theorems 3 and 4 encompass the other results on the same line.

3 Finite tree

3.1 About the functions I^n

The following estimates will be useful in the study of the functions I^n.

Proposition 5. *Recall the definition of c in (5), Λ_W in (3), and Λ_W^* in (4).*

1. $c = \sup(\operatorname{dom}(\Lambda_W))$;

2. *If $c \in (0, \infty)$, then $\Lambda_W^*(w) \sim cw$ as $w \to \infty$.*

Proof of Proposition 5.

1. Let $m = \sup(\mathrm{dom}(\Lambda_W))$. First, let $t < c' < c$. By (5), there exists $w_0 > 0$ such that, for all $w \geqslant w_0$,
$$\mathbb{P}(W \geqslant w) \leqslant e^{-c'w}.$$
Since $W \geqslant 0$, one has
$$\mathbb{E}[e^{tW}] = \int_{u=0}^{\infty} \mathbb{P}(e^{tW} \geqslant u)du = \int_{u=0}^{\infty} \mathbb{P}(W \geqslant \log(u)/t)du \leqslant e^{tw_0} + \int_{u=e^{tw_0}}^{\infty} u^{-c'/t}du < \infty.$$
Thus $t \leqslant m$, so $c \leqslant m$. Second, let $t < m$. By Markov's inequality,
$$\frac{1}{w}\log \mathbb{P}(W \geqslant w) \leqslant -t + \frac{1}{w}\log \mathbb{E}[e^{tW}].$$
Taking the limit superior as $w \to \infty$ leads to $-c \leqslant -t$ whence $c \geqslant m$.

2. First, for all $\varepsilon > 0$,
$$\Lambda_W^*(w) = \sup_{t\in\mathbb{R}}[tw - \Lambda_W(t)] \geqslant (c-\varepsilon)w - \Lambda_W(c-\varepsilon),$$
so, since $\Lambda_W(c-\varepsilon) < \infty$ by item 1, $\liminf_{w\to\infty} \Lambda_W^*(w)/w \geqslant c - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\liminf_{w\to\infty} \Lambda_W^*(w)/w \geqslant c$.
Secondly, for $w \geqslant 1$, by item 1,
$$\Lambda_W^*(w) = \sup_{0\leqslant t\leqslant c}[tw - \Lambda_W(t)] \leqslant cw - \inf_{0\leqslant t\leqslant c}\Lambda_W(t) = cw,$$
so $\limsup_{w\to\infty} \Lambda_W^*(w)/w \leqslant c$. □

Now we turn to the properties of the functions I^n.

Proposition 6.

1. *$I^n(1) = \Lambda_W^*(1) = 0$.*
2. *For all $n \in \mathbb{N}^*$, the function I^n is nondecreasing on $[1,\infty)$. Moreover, if $c \in (0,\infty]$, then the function I^n is increasing on $[1,\infty) \cap \{I^n < \infty\}$.*
3. *If $c \in (0,\infty]$, then, for all $a > 1$, $I^n(a) > 0$.*
4. *The sequence of functions $(I^n)_{n\geqslant 1}$ is nonincreasing.*
5. *For $a \geqslant 1$,*
$$I^n(a) = \inf\left\{cw + I^{n-1}(z) + \Lambda_W^*(s)\ ;\ w \geqslant 0,\ z \in [1,a]\,,\ s \in [1,a]\,,\ wz + s = a\right\}. \tag{7}$$

Proof of Proposition 6.

1. Obvious.

2. Let us define the function

$$G^{n-1}(w,z,s) := cw + I^{n-1}(z) + \Lambda_W^*(s). \tag{8}$$

Let $a_1 > a_0 \geqslant 1$ and $(w,z,s) \in (\mathbb{R}_+)^3$ be such that $wz+s = a_1$. Either $s \geqslant a_0$, then $G^{n-1}(w,z,s) \geqslant G^{n-1}(0,z,a_0)$, because Λ_W^* is nondecreasing on $[1,\infty)$; or $s < a_0$, then $G^{n-1}(w,z,s) \geqslant G^{n-1}((a_0-s)/z,z,s)$. Hence $I^n(a_1) \geqslant I^n(a_0)$. If $c \in (0,\infty)$, then the preceding argument holds with strict inequalities, using the fact that Λ_W^* (resp. $w \mapsto cw$) is increasing on $[1,\infty)$ (resp. on $[0,\infty)$), so I^n is increasing on $[1,\infty)$. If $c = \infty$, then $I^n = \Lambda_W^*$ is increasing on $[1,\infty) \cap \{\Lambda_W^* < \infty\}$.

3. It is an immediate consequence of items 1 and 2.

4. First, we prove that $I^2 \leqslant I^1$. It is enough to take $(w,z,s) = (0,1,a)$ in (6) to get

$$I^2(a) \leqslant \Lambda_W^*(a) = I^1(a)$$

by item 1 and $I^1 = \Lambda^*$. We conclude by induction:

$$I^{n+1}(a) \leqslant \inf\left\{cw + I^{n-1}(z) + \Lambda_W^*(s)\ ;\ w \geqslant 0,\ z \geqslant 0,\ s \geqslant 0,\ wz+s = a\right\} = I^n(a).$$

5. For all $(w,z,s) \in (\mathbb{R}_+)^3$, $G^{n-1}(w,z,s) \geqslant G^{n-1}(w,\max(1,z),\max(1,s))$. So, for all $a \geqslant 1$,

$$\begin{aligned} I^n(a) &= \inf\left\{cw + I^{n-1}(z) + \Lambda_W^*(s)\ ;\ w \geqslant 0,\ z \geqslant 1,\ s \geqslant 1,\ wz+s \geqslant a\right\} \\ &= \inf\left\{cw + I^{n-1}(z) + \Lambda_W^*(s)\ ;\ w \geqslant 0,\ z \geqslant 1,\ s \geqslant 1,\ wz+s = a\right\}, \end{aligned}$$

proceeding as in the proof of item 2. Hence, since $wz \geqslant 0$ and $wz+s = a$ imply $s \leqslant a$,

$$\begin{aligned} I^n(a) &= \inf\left\{cw + I^{n-1}(z) + \Lambda_W^*(s)\ ;\ w \geqslant 0,\ z \geqslant 1,\ s \in [1,a],\ wz+s = a\right\} \\ &= \inf\left\{cw + I^{n-1}(z) + \Lambda_W^*(s)\ ;\ w \geqslant 0,\ z \in [1,a],\ s \in [1,a],\ wz+s = a\right\}, \end{aligned}$$

where the latter equality is obvious for $c = 0$ and follows, for $c \in (0,\infty]$, from the fact that $I^{n-1}(z) > I^{n-1}(a) \geqslant I^n(a)$ for all $z > a$ (by items 2 and 4), and $cw + \Lambda_W^*(s) > 0$ for all $(w,s) \in [0,\infty) \times [1,\infty) \setminus \{(0,1)\}$. □

For all $a \geqslant 1$ and $z \in [1,a]$, let

$$h(a,z) := \frac{ca}{z} - \sup_{1 \leqslant s \leqslant a}\left[\frac{cs}{z} - \Lambda_W^*(s)\right] \tag{9}$$

so that, using (7) (Proposition 1, item 5),

$$I^n(a) = \inf_{1 \leqslant z \leqslant a}\left[I^{n-1}(z) + h(a,z)\right]. \tag{10}$$

Let also

$$a^* := \inf\left\{s \geqslant 0\ ;\ \Lambda_W'(s) = a\right\}$$

be the convex conjugate of a through Λ_W. Note that, if $a \leqslant \Lambda_W'(c^-)$, then a^* is the unique solution in s of $\Lambda_W'(s) = a$; and if $a > \Lambda_W'(c^-)$, then $a^* = \infty$.

Lemma 7. *For all $a \geqslant 1$ and $z \in [1, a]$,*

$$h(a,z) = \begin{cases} \Lambda_W^*(a) & \text{if } z \leqslant c/a^* \\ \frac{ca}{z} - \Lambda_W\left(\frac{c}{z}\right) & \text{if } z \geqslant c/a^*. \end{cases}$$

and the function h is continuous.

Proof of Lemma 7. An easy computation of Legendre transform leads to

$$\begin{aligned} h(a,z) &= \frac{ca}{z} - \sup_{1 \leqslant s \leqslant a} \left[\frac{cs}{z} - \Lambda_W^*(s)\right] \\ &= \begin{cases} \frac{ca}{z} - \left(\Lambda_W(a^*) + \left(\frac{c}{z} - a^*\right)a\right) & \text{if } c/z \geqslant a^* \\ \frac{ca}{z} - \Lambda_W\left(\frac{c}{z}\right) & \text{if } c/z \leqslant a^*. \end{cases} \\ &= \begin{cases} \Lambda_W^*(a) & \text{if } z \leqslant c/a^* \\ \frac{ca}{z} - \Lambda_W\left(\frac{c}{z}\right) & \text{if } z \geqslant c/a^*. \end{cases} \end{aligned}$$

Let $A \geqslant 1$. Since $H\colon (a,z,w) \mapsto cw + \Lambda_W^*(a - wz)$ is uniformly continuous on the compact set $\left\{(a,z,w)\ ;\ 1 \leqslant a \leqslant A,\ 1 \leqslant z \leqslant a,\ 0 \leqslant w \leqslant \frac{a-1}{z}\right\}$ and $(a,z) \mapsto (a-1)/z$ is continuous on $D(A) := \{(a,z)\ ;\ 1 \leqslant a \leqslant A,\ 1 \leqslant z \leqslant a\}$, h is continuous on $D(A)$. □

Proposition 8.

1. *For all $n \in \mathbb{N}^*$, the function I^n is continuous.*
2. *For all $n \in \mathbb{N}^*$, $I^n(a) \sim cna^{1/n}$ as $a \to \infty$.*

Proof of Proposition 8.

1. Let $a_0 \geqslant 1$ and $\varepsilon > 0$. Since the function I^n is nondecreasing by Proposition 6, item 2, it suffices to prove that there exists $a_1 > a_0$ such that $I^n(a_1) \leqslant I^n(a_0) + \varepsilon$. Since h is uniformly continuous on the compact set $D(a_0 + 1)$ where $D(a)$ has been defined int the proof of Lemma 7, there exists $a_1 \in (a_0, a_0 + 1)$ such that, for all $z \in [1, a_0]$, $h(a_1, z) < h(a_0, z) + \varepsilon$. Then, by (10),

$$\begin{aligned} I^n(a_1) &= \inf_{1 \leqslant z \leqslant a_1} \left[I^{n-1}(z) + h(a_1, z)\right] \\ &\leqslant \inf_{1 \leqslant z \leqslant a_0} \left[I^{n-1}(z) + h(a_1, z)\right] \\ &\leqslant \inf_{1 \leqslant z \leqslant a_0} \left[I^{n-1}(z) + h(a_0, z)\right] + \varepsilon \\ &= I^n(a_0) + \varepsilon. \end{aligned}$$

2. We proceed by induction. The result holds for $n = 1$ by Proposition 5, item 2 and $I^1 = \Lambda^*$. Let $n \in \mathbb{N}^* \setminus \{1\}$. First, notice that, for all $a \geqslant 1$ and $z \in [1, a]$, $h(a,z) \leqslant ca/z$

(it suffices to take $s = 1$ in the supremum in (9)) so that

$$\begin{aligned} I^{n+1}(a) &= \inf_{1 \leqslant z \leqslant a} \Big[I^n(z) + h(a,z) \Big] \\ &\leqslant \inf_{1 \leqslant z \leqslant a} \Big[I^n(z) + \frac{ca}{z} \Big] \\ &\leqslant I^n\big(a^{n/(n+1)}\big) + ca^{1/(n+1)} \\ &\sim c(n+1)a^{1/(n+1)} \end{aligned}$$

by (10), taking $1 \leqslant z = a^{n/(n+1)} \leqslant a$, and by induction. Let us turn to the minoration. For all $\varepsilon > 0$, there exists $A_\varepsilon > 0$ such that

$$\begin{aligned} I^{n+1}(a) &= \inf_{1 \leqslant z \leqslant a} \Big[I^n(z) + h(a,z) \Big] \\ &\geqslant c \inf_{1 \leqslant z \leqslant a} \Big[nz^{1/n}(1-\varepsilon) + \frac{a}{z} - \sup_{1 \leqslant s \leqslant a} \Big(s\Big(\frac{1}{z} - (1-\varepsilon) \Big) \Big) \Big] - A_\varepsilon \\ &\geqslant c \inf_{1 \leqslant z \leqslant a} \Big[nz^{1/n}(1-\varepsilon) + a \min\Big(\frac{1}{z}, 1-\varepsilon \Big) \Big] - A_\varepsilon \\ &= c \min((n+1)a^{1/(n+1)}(1-\varepsilon)^{n/(n+1)}, (n+a)(1-\varepsilon)) - A_\varepsilon \\ &\sim c(n+1)a^{1/(n+1)}(1-\varepsilon)^{n/(n+1)} \end{aligned}$$

using (10), by induction, and as $a \to \infty$. Since ε is arbitrary, we get the result. □

For all $n \in \mathbb{N}^*$, let us define

$$a_n := \inf \big\{ a \geqslant 1 \ ; \ I^{n+1}(a) < I^n(a) \big\}. \tag{11}$$

Proposition 9. *The sequence $(a_n)_{n \geqslant 1}$ is increasing and diverges to infinity.*

Proof of Proposition 9. Let $n \in \mathbb{N}^* \setminus \{1\}$. For all $a \in [1, a_{n-1}]$,

$$I^{n+1}(a) = \inf_{1 \leqslant z \leqslant a} \Big[I^n(z) + h(a,z) \Big] = \inf_{1 \leqslant z \leqslant a} \Big[I^{n-1}(z) + h(a,z) \Big] = I^n(a)$$

by (10). Thus $a_{n-1} \leqslant a_n$. Now, assume by contradiction that the sequence $(a_n)_{n \geqslant 1}$ is upper bounded by some $A > 0$. Let $a_W := \inf \{ a \geqslant 1 \ ; \ aa^* \geqslant c \}$. For all $a \geqslant a_W$ and $z \in [1, a]$,

$$h(a,z) \geqslant h(a,a) = c - \Lambda_W(c/a) \geqslant c - \Lambda_W(c/a_W) = \Lambda_W^*(a_W) =: \rho > 0.$$

Since the function I^n is uniformly continuous on $[1, A]$ (by Proposition 8, item 1), there exists $\eta(A) \in (0,1)$ such that, for all $(a,z) \in [1, A+1]^2$ with $z \in [a - \eta(A), a]$, $I^n(z) > I^n(a) - \rho$ (note that it implies $\eta(A) < a_W - 1$). Then, for all $a \in [a_W, A+1]$ and $z \in [a - \eta(A), a]$,

$$I^n(z) + h(a,z) > (I^n(a) - \rho) + \rho = I^n(a) \geqslant I^{n+1}(a).$$

Hence, for all $a \leqslant a_{n-1} + \eta(A)$,

$$\begin{aligned} I^{n+1}(a) &= \inf_{1 \leqslant z \leqslant a} \Big[I^n(z) + h(a,z) \Big] && \text{(by (10))} \\ &= \inf_{1 \leqslant z \leqslant a - \eta(A)} \Big[I^n(z) + h(a,z) \Big] \\ &= \inf_{1 \leqslant z \leqslant a - \eta(A)} \Big[I^{n-1}(z) + h(a,z) \Big] && \text{(since } z \leqslant a_{n-1}\text{)} \\ &\geqslant I^n(a), \end{aligned}$$

hence $a \leqslant a_n$. So $a_{n-1} + \eta(A) \leqslant a_n$: this is in contradiction with the fact that, for all $n \in \mathbb{N}^*$, $a_n \leqslant A$, and the conclusion of the proposition follows. □

3.2 Proof of Theorem 3, item 1 (moderate deviations)

We proceed by induction over $n \in \mathbb{N}^*$. The result for $n = 1$ stems from the moderate deviation principle for a sum of independent and identically distributed random variables (see [5, Theorem 3.7.1]). Now, let $n \in \mathbb{N}^* \setminus \{1\}$ be such that $(r^\alpha(Z_r^{n-1} - 1))_{r \geqslant 2}$ satisfies a large deviation principle at speed $(r^{1-2\alpha})_{r \geqslant 2}$ with rate function J.

Left deviations — Set, for $s < 0$,

$$\Lambda_r^n(s) = \frac{1}{r^{1-2\alpha}} \log \mathbb{E}\left[e^{s r^{1-\alpha}(Z_r^n - 1)} \right].$$

We will prove that

$$\Lambda_r^n(s) \xrightarrow[r \to \infty]{} \frac{s^2 \operatorname{Var}(W)}{2}. \tag{12}$$

Then, applying a unilateral version of the Gärtner-Ellis theorem (see [17] or [4, Theorem 10]), we get

$$\frac{1}{r^{1-2\alpha}} \log \mathbb{P}(r^\alpha(Z_r^n - 1) \leqslant -a) \xrightarrow[r \to \infty]{} -\frac{a^2}{2 \operatorname{Var}(W)}.$$

By (1), we have

$$Z_r^n = \frac{1}{r} \sum_{i=1}^{r} W_i Z_{r,i}^{n-1}, \tag{13}$$

where

$$Z_{r,i}^{n-1} := \frac{1}{r^{n-1}} \sum_{1 \leqslant i_2, \ldots, i_n \leqslant r} W_{i,i_2} \cdots W_{i,i_2,\ldots,i_n}$$

is distributed as Z_r^{n-1}. By independence and identity of distributions, we have

$$\Lambda_r^n(s) = r^{2\alpha - 1} \log \mathbb{E}\left[e^{s r^{-\alpha} \sum_{i=1}^r (W_i Z_{r,i}^{n-1} - 1)} \right] = r^{2\alpha} \log \mathbb{E}\left[e^{s r^{-\alpha}(W Z_r^{n-1} - 1)} \right].$$

Using the Taylor expansion $e^x = 1 + x + x^2 e^{\theta(x)}/2$ with $\theta(x) \leqslant \max(x, 0)$ and the fact that $\mathbb{E}[WZ_r^n] = \mathbb{E}[W]\mathbb{E}[Z_r^n] = 1$, we get

$$\Lambda_r^n(s) = r^{2\alpha} \log\Big(1 + \frac{s^2}{2r^{2\alpha}} \mathbb{E}\big[(WZ_r^n - 1)^2 e^{\theta(sr^{-\alpha}(WZ_r^n - 1))}\big]\Big) \xrightarrow[r\to\infty]{} \frac{s^2}{2}\mathbb{E}[(W-1)^2],$$

which is (12). Indeed,

$$\begin{aligned}
&\Big|\mathbb{E}\big[(WZ_r^n - 1)^2 e^{\theta(sr^{-\alpha}(WZ_r^n-1))}\big] - \mathbb{E}[(W-1)^2]\Big| \\
&\quad\leqslant \Big|\mathbb{E}\big[(WZ_r^n - 1)^2\big(e^{\theta(sr^{-\alpha}(WZ_r^n-1))} - 1\big)\big]\Big| + \big|\mathbb{E}\big[(WZ_r^n - 1)^2 - (W-1)^2\big]\big| \\
&\quad\leqslant \big|\mathbb{E}\big[(WZ_r^n - 1)^2\big]\big|\,\big(e^{-sr^{-\alpha}} - 1\big) + \big|\mathbb{E}\big[(WZ_r^n - 1)^2 - (W-1)^2\big]\big| \xrightarrow[r\to\infty]{} 0,
\end{aligned}$$

since

$$\mathbb{E}[(Z_r^{n-1})^2] = \mathbb{E}\left[\left(\frac{1}{r}\sum_{i=1}^r W_i Z_{r,i}^{n-2}\right)^2\right] = \frac{1}{r^2}\big(r\mathbb{E}[W^2]\mathbb{E}[(Z_r^{n-2})^2] + r(r-1)\mathbb{E}[W]^2\mathbb{E}[Z_r^{n-2}]^2\big) \xrightarrow[r\to\infty]{} 1.$$

Remark. Obviously, the previous estimate generalizes and the multinomial expansion yields

$$\forall h \in \mathbb{N} \quad \mathbb{E}[(Z_r^{n-1})^h] \xrightarrow[r\to\infty]{} 1. \tag{14}$$

Controlling the moments of Z_r^n appears to be crucial in the study of the right deviations for $n = \infty$ (see Section 4.1). Here, mimicking the proof of the large deviations, we resort to induction to obtain the deviations in the finite tree.

Right deviations — Let us turn to the right deviations. Let $a \geqslant 0$. Using (13), one gets

$$\mathbb{P}(r^\alpha(Z_r^n - 1) \geqslant a) = \mathbb{P}\left(\sum_{i=1}^r (W_i Z_{r,i}^{n-1} - 1) \geqslant ar^{1-\alpha}\right).$$

Let us prove the lower bound. For all $\varepsilon > 0$,

$$\begin{aligned}
&\frac{1}{r^{1-2\alpha}} \log \mathbb{P}\left(\sum_{i=1}^r (W_i Z_{r,i}^{n-1} - 1) \geqslant ar^{1-\alpha}\right) \\
&\qquad\geqslant \frac{1}{r^{1-2\alpha}} \log \mathbb{P}\left(\sum_{i=1}^r (W_i Z_{r,i}^{n-1} - 1) \geqslant ar^{1-\alpha},\ \forall i \in [\![1, r]\!]\ \ Z_{r,i}^{n-1} \geqslant 1 - \varepsilon r^{-\alpha}\right) \\
&\qquad\geqslant \frac{1}{r^{1-2\alpha}} \log \mathbb{P}\left(\sum_{i=1}^r (W_i(1 - \varepsilon r^{-\alpha}) - 1) \geqslant ar^{1-\alpha}\right) + r^{2\alpha} \log \mathbb{P}(Z_r^{n-1} \geqslant 1 - \varepsilon r^{-\alpha}).
\end{aligned}$$

By induction (left deviations for Z_r^{n-1}),

$$\frac{1}{r^{1-2\alpha}} \log \mathbb{P}(Z_r^{n-1} - 1 < -\varepsilon r^{-\alpha}) \xrightarrow[r\to\infty]{} -J(-\varepsilon) < 0,$$

hence

$$r^{2\alpha}\log\mathbb{P}(Z_r^{n-1}\geqslant 1-\varepsilon r^{-\alpha})\xrightarrow[r\to\infty]{}0.$$

Now, since $\mathbb{E}[e^{sW}]<\infty$ for all $s<c$ (Proposition 5, item 1), we can proceed as in the proof of [5, Theorem 3.1.1] to apply the unilateral version of Gärtner-Ellis theorem and get

$$\frac{1}{r^{1-2\alpha}}\log\mathbb{P}\bigg(\sum_{i=1}^{r}(W_i(1-\varepsilon r^{-\alpha})-1)\geqslant ar^{1-\alpha}\bigg)\xrightarrow[r\to\infty]{}-J(a+\varepsilon). \tag{15}$$

Letting $\varepsilon\to 0$, one finally gets

$$\liminf_{r\to\infty}\frac{1}{r^{1-2\alpha}}\log\mathbb{P}(r^{\alpha}(Z_r^n-1)\geqslant a)\geqslant -J(a). \tag{16}$$

Let us turn to the upper bound. Let $\varepsilon>0$. For all $r\in\mathbb{N}^*\setminus\{1\}$ and for all $q\in[\![0,r]\!]$, we define

$$P_{n,r,q}=\mathbb{P}\bigg(\sum_{i=1}^{r}(W_iZ_{r,i}^{n-1}-1)\geqslant ar^{1-\alpha},\ \forall i\in[\![1,q]\!]\ \ Z_{r,i}^{n-1}-1\geqslant\varepsilon r^{-\alpha},\\ \forall i\in[\![q+1,r]\!]\ \ Z_{r,i}^{n-1}-1<\varepsilon r^{-\alpha}\bigg).$$

First, for $q_0\in[\![0,r]\!]$ and $\varepsilon'\in(0,\varepsilon)$, one has by induction

$$\begin{aligned}\sum_{q=q_0}^{r}\binom{r}{q}P_{n,r,q}&\leqslant\sum_{q=q_0}^{r}\binom{r}{q}\mathbb{P}(Z_r^{n-1}-1\geqslant\varepsilon r^{-\alpha})^q\\&\leqslant\sum_{q=q_0}^{r}\Big(re^{-r^{1-2\alpha}J(\varepsilon')}\Big)^q\\&\leqslant\frac{\Big(re^{-r^{1-2\alpha}J(\varepsilon')}\Big)^{q_0}}{1-re^{-r^{1-2\alpha}J(\varepsilon')}}\\&\leqslant e^{-r^{1-2\alpha}J(a)},\end{aligned} \tag{17}$$

as soon as $q_0J(\varepsilon')>J(a)$ and r is large enough. Second, for $q\in[\![0,q_0-1]\!]$, one has

$$\frac{1}{r^{1-2\alpha}}\log(P_{n,r,q})\leqslant\frac{1}{r^{1-2\alpha}}\log\mathbb{P}\bigg(\sum_{i=1}^{q}(W_iZ_{r,i}^{n-1}-1)+\sum_{i=q+1}^{r}(W_i(1+\varepsilon r^{-\alpha})-1)\geqslant ar^{1-\alpha}\bigg).$$

Now, for all $b\geqslant 0$, considering whether $r^{\alpha/2}(Z_r^{n-1}-1)$ is larger than 1 or not,

$$\begin{aligned}\mathbb{P}(WZ_r^{n-1}-1\geqslant br^{1-\alpha})&=\mathbb{P}\bigg(\frac{W}{r^{1-\alpha/2}}r^{\alpha/2}(Z_r^{n-1}-1)+\frac{W-1}{r^{1-\alpha}}\geqslant b\bigg)\\&\leqslant\mathbb{P}\big(r^{\alpha/2}(Z_r^{n-1}-1)\geqslant 1\big)+\mathbb{P}\bigg(\frac{W}{r^{1-\alpha}}\geqslant\frac{b+r^{-(1-\alpha)}}{1+r^{-\alpha/2}}\bigg),\end{aligned}$$

whence, by induction, using the large deviation principle for $(r^{\alpha/2}(Z_r^{n-1}-1))_{r\geqslant 2}$, and assumption (5) on the tail of W, we get

$$\frac{1}{r^{1-2\alpha}}\log\mathbb{P}(WZ_r^{n-1}-1\geqslant br^{1-\alpha})\xrightarrow[r\to\infty]{}\begin{cases}0 & \text{if } b=0\\ -\infty & \text{otherwise.}\end{cases} \tag{18}$$

Moreover, applying the unilateral version of Gärtner-Ellis theorem, we get

$$\frac{1}{r^{1-2\alpha}}\log\mathbb{P}\left(\sum_{i=q+1}^{r}(W_i(1+\varepsilon r^{-\alpha})-1)\geqslant br^{1-\alpha}\right)\xrightarrow[r\to\infty]{}-J(a-\varepsilon). \tag{19}$$

So, applying the contraction principle to the continuous map $(y_1,\dots,y_q,s)\mapsto y_1+\cdots+y_q+s$ and

$$\left(\frac{1}{r^{1-\alpha}}\left(W_1Z_{r,1}^{n-1}-1,\dots,W_qZ_{r,q}^{n-1}-1,\sum_{i=q+1}^{r}(W_i(1+\varepsilon r^{-\alpha})-1)\right)\right)_{r\geqslant 2},$$

and using (18) and (19), one gets

$$\limsup_{r\to\infty}\frac{1}{r^{1-2\alpha}}\log(P_{n,r,q})\leqslant -J(a-\varepsilon). \tag{20}$$

Applying the principle of the largest term to (17) and (20) and letting $\varepsilon\to 0$, one gets the desired upper bound:

$$\limsup_{r\to\infty}\frac{1}{r^{1-2\alpha}}\log\mathbb{P}(r^{\alpha}(Z_r^n-1)\geqslant a)\leqslant -J(a). \tag{21}$$

Finally, the large deviation principle stems from (16) and (21) and the fact that J is decreasing on $(-\infty,0]$ and increasing on $[0,\infty)$.

3.3 Proof of Theorem 3, item 2 (large deviations)

The large deviations on the left for $(Z_r^n)_{r\geqslant 2}$ are given in Proposition 2. Let us turn to the large deviations on the right. We proceed by induction. The case $n=1$ stems from Cramér's Theorem. In particular $I^1=\Lambda_W^*$. Now let $n\in\mathbb{N}^*\setminus\{1\}$ and assume that $(Z_r^{n-1})_{r\geqslant 2}$ satisfies a large deviation principle at speed $(r)_{r\geqslant 2}$ with rate function I^{n-1}. Let $a>1$ and $\varepsilon>0$. Applying the contraction principle to the continuous map $(w,z,s)\mapsto wz+(1-\varepsilon)s$

and $((W_1, Z_{r,1}^{n-1}, W_2 + \cdots + W_r)/r)_{r\geqslant 2}$, we get

$$\begin{aligned}
&\frac{1}{r}\log\mathbb{P}\left(Z_r^n \geqslant a\right)\\
&\quad\geqslant \frac{1}{r}\log\mathbb{P}\left(W_1 Z_{r,1}^{n-1} + (W_2+\cdots+W_r)(1-\varepsilon) \geqslant ra,\ Z_{r,2}^{n-1},\ldots,Z_{r,r}^{n-1} \geqslant 1-\varepsilon\right)\\
&\quad= \frac{1}{r}\log\mathbb{P}\left(W_1 Z_{r,1}^{n-1} + (W_2+\cdots+W_r)(1-\varepsilon) \geqslant ra\right) + \frac{r-1}{r}\log\mathbb{P}\left(Z_r^{n-1}\geqslant 1-\varepsilon\right)\\
&\quad\xrightarrow[r\to\infty]{} -\inf\left\{cw + I^{n-1}(z) + \Lambda_W^*(s)\ ;\ w\geqslant 0,\ z\geqslant 0,\ s\geqslant 0,\ wz+(1-\varepsilon)s\geqslant a\right\}\\
&\quad\geqslant -\inf\left\{cw + I^{n-1}(z) + \Lambda_W^*(s)\ ;\ w\geqslant 0,\ z\in[1,a],\ s\in[1,a],\ wz+(1-\varepsilon)s = a\right\}\\
&\quad= -\inf\left\{\frac{c(a-s(1-\varepsilon))}{z} + I^{n-1}(z) + \Lambda_W^*(s)\ ;\ z\in[1,a],\ s\in[1,a]\right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\liminf_{r\to\infty}\frac{1}{r}\log\mathbb{P}\left(Z_r^n\geqslant a\right)\\
&\qquad\geqslant -\inf_{\varepsilon>0}\inf_{\substack{1\leqslant z\leqslant a\\ 1\leqslant s\leqslant a}}\left[\frac{c(a-s(1-\varepsilon))}{z} + I^{n-1}(z) + \Lambda_W^*(s)\right]\\
&\qquad= -\inf_{\substack{1\leqslant z\leqslant a\\ 1\leqslant s\leqslant a}}\left[\frac{c(a-s)}{z} + I^{n-1}(z) + \Lambda_W^*(s) + \inf_{\varepsilon>0}\frac{cs\varepsilon}{z}\right]\\
&\qquad= -\inf_{\substack{1\leqslant z\leqslant a\\ 1\leqslant s\leqslant a}}\left[\frac{c(a-s)}{z} + I^{n-1}(z) + \Lambda_W^*(s)\right]\\
&\qquad= -\inf\left\{cw + I^{n-1}(z) + \Lambda_W^*(s)\ ;\ w\geqslant 0,\ z\in[1,a],\ s\in[1,a],\ wz+s = a\right\}\\
&\qquad= -I^n(a),
\end{aligned}$$

by (7).

Now let us establish the upper bound. Let $\varepsilon > 0$. For all $r\in\mathbb{N}^*\setminus\{1\}$ and for $q\in[\![0,r]\!]$, we define

$$P_{n,r,q} := \mathbb{P}\left(Z_r^n\geqslant a,\ Z_{r,1}^{n-1},\ldots,Z_{r,q}^{n-1}\geqslant 1+\varepsilon,\ Z_{r,q+1}^{n-1},\ldots,Z_{r,r}^{n-1} < 1+\varepsilon\right),$$

so that

$$\mathbb{P}\left(Z_r^n\geqslant a\right) = \sum_{q=0}^{r}\binom{r}{q}P_{n,r,q}.$$

For $q_0 \in [\![0, r]\!]$ and $\varepsilon' \in (0, \varepsilon)$, one has by induction

$$\begin{aligned}\sum_{q=q_0}^{r} \binom{r}{q} P_{n,r,q} &\leqslant \sum_{q=q_0}^{r} \binom{r}{q} \mathbb{P}(Z_r^{n-1} \geqslant 1+\varepsilon)^q \\ &\leqslant \sum_{q=q_0}^{r} \left(re^{-rI^{n-1}(1+\varepsilon')}\right)^q \\ &\leqslant \frac{\left(re^{-rI^{n-1}(1+\varepsilon')}\right)^{q_0}}{1-re^{-rI^{n-1}(1+\varepsilon')}} \\ &\leqslant e^{-rI^n(a)}, \end{aligned} \tag{22}$$

as soon as $q_0 I^{n-1}(1+\varepsilon') > I^n(a)$ (we use Proposition 6, item 3) and r is large enough. Now,

$$\lim_{r\to\infty} \frac{1}{r} \log P_{n,r,0} \leqslant \lim_{r\to\infty} \frac{1}{r} \log \mathbb{P}\Big(\sum_{i=1}^{r} W_i \geqslant \frac{ra}{1+\varepsilon}\Big) = -\Lambda_W^*\Big(\frac{a}{1+\varepsilon}\Big) \leqslant -I^n\Big(\frac{a}{1+\varepsilon}\Big). \tag{23}$$

Finally, let $q \in [\![1, q_0 - 1]\!]$. Applying the contraction principle to the continuous map $(y_1, \dots, y_q, s) \mapsto y_1 + \cdots + y_q + s(1+\varepsilon)$ and $((W_1 Z_{r,1}^{n-1}, \dots, W_q Z_{r,q}^{n-1}, W_{q+1} + \cdots + W_r)/r)_{r\geqslant 2}$, we get

$$\begin{aligned}\frac{1}{r} \log P_{n,r,q} &\leqslant \frac{1}{r} \log \mathbb{P}\left(W_1 Z_{r,1}^{n-1} + \cdots + W_q Z_{r,q}^{n-1} + (W_{q+1} + \cdots + W_r)(1+\varepsilon) \geqslant ra,\right. \\ &\qquad \left. Z_{r,1}^{n-1}, \dots, Z_{r,q}^{n-1} \geqslant 1+\varepsilon,\ Z_{r,q+1}^{n-1}, \dots, Z_{r,r}^{n-1} < 1+\varepsilon\right) \\ &\leqslant \frac{1}{r} \log \mathbb{P}\left(W_1 Z_{r,1}^{n-1} + \cdots + W_q Z_{r,q}^{n-1} + (W_{q+1} + \cdots + W_r)(1+\varepsilon) \geqslant ra\right) \\ &\xrightarrow[r\to\infty]{} -\inf\{g^{n-1}(y_1) + \cdots + g^{n-1}(y_q) + \Lambda_W^*(s)\ ; \\ &\qquad y_1,\ \dots,\ y_q \geqslant 0,\ s \geqslant 0,\ y_1 + \cdots + y_q + (1+\varepsilon)s \geqslant a\} \\ &= -\inf\left\{g^{n-1}(y) + \Lambda_W^*(s)\ ;\ y \geqslant 0,\ s \geqslant 0,\ y + (1+\varepsilon)s \geqslant a\right\} \\ &=: -B_\varepsilon, \end{aligned}$$

where g^{n-1} is the rate function of $(WZ_r^{n-1}/r)_{r\geqslant 2}$ at speed $(r)_{r\geqslant 2}$, that is the concave function defined by

$$g^{n-1}(y) := \inf\left\{cw + I^{n-1}(z)\ ;\ w \geqslant 0,\ z > 0,\ wz = y\right\} = \inf\left\{\frac{cy}{z} + I^{n-1}(z)\ ;\ z > 0\right\}.$$

Following the same lines as in the proof of Proposition 6, item 5, we obtain that, for $\varepsilon \leqslant a-1$,

$$B_\varepsilon = \inf\left\{cw + I^{n-1}(z) + \Lambda_W^*(s)\ ;\ w \geqslant 0,\ z \in [1, a],\ s \in [1, a/(1+\varepsilon)],\ wz + (1+\varepsilon)s = a\right\}.$$

Hence,

$$\limsup_{r\to\infty} \frac{1}{r} \log P_{n,r,q} \leqslant -\sup_{0<\varepsilon\leqslant a-1} B_\varepsilon.$$

If $c = \infty$, then, since Λ_W^* is left continuous at a,

$$\limsup_{r\to\infty} \frac{1}{r} \log P_{n,r,q} \leqslant -\sup_{\varepsilon>0} \Lambda_W^*\Big(\frac{a}{1+\varepsilon}\Big) = -\Lambda_W^*(a) = -I^n(a).$$

Assume now that $c < \infty$. Then,

$$\begin{aligned}
\limsup_{r\to\infty} \frac{1}{r} \log P_{n,r,q} &\leqslant -\sup_{0<\varepsilon\leqslant a-1} \inf_{\substack{1\leqslant z\leqslant a,\\ 1\leqslant s\leqslant \frac{a}{1+\varepsilon}}} \Big[\frac{c(a-s)}{z} + I^{n-1}(z) + \Lambda_W^*(s) - \frac{c\varepsilon s}{z}\Big] \\
&\leqslant -\sup_{0<\varepsilon\leqslant a-1} \inf_{\substack{1\leqslant z\leqslant a\\ 1\leqslant s\leqslant a}} \Big[\frac{c(a-s)}{z} + I^{n-1}(z) + \Lambda_W^*(s) - c\varepsilon a\Big] \\
&= -\inf_{\substack{1\leqslant z\leqslant a\\ 1\leqslant s\leqslant a}} \Big[\frac{c(a-s)}{z} + I^{n-1}(z) + \Lambda_W^*(s)\Big] \\
&= -\inf\big\{cw + I^{n-1}(z) + \Lambda_W^*(s)\ ;\ w \geqslant 0,\ z\in[1,a]\,,\ s\in[1,a]\,,\ wz+s=a\big\} \\
&= -I^n(a) \qquad (24)
\end{aligned}$$

by (7). We conclude by applying the principle of the largest term to (22), (23), and (24).

3.4 Proof of Theorem 3, item 3 (very large deviations)

Left deviations — For $a < 0$, $\mathbb{P}(Z_r^n \leqslant r^\alpha a) = 0$. Assume now that $a = 0$. Let $p := \mathbb{P}(W = 0)$ and, for all $r \in \mathbb{N}^* \setminus \{1\}$ and $n \in \mathbb{N}^*$, let $q_r^n := \mathbb{P}(Z_r^n \leqslant 0) = \mathbb{P}(Z_r^n = 0)$. Obviously, $q_r^1 = p^r$ and, using (13),

$$q_r^n = \mathbb{P}(Z_r^n = 0) = \mathbb{P}(WZ_r^{n-1} = 0)^r = (1 - \mathbb{P}(W \neq 0)\mathbb{P}(Z_r^{n-1} \neq 0))^r = (p + (1-p)q_r^{n-1})^r.$$

If $p = 0$, then $q_r^n = 0$ for all $r \in \mathbb{N}^* \setminus \{1\}$ and $n \in \mathbb{N}^*$. Otherwise, $0 < p < 1$ (recall that $\mathbb{E}[W] = 1$ implies $p < 1$), $r^{-1}\log(q_r^1) = \log(p)$ and one proves by induction that

$$\frac{1}{r}\log(q_r^n) = \log(p + (1-p)q_r^{n-1}) \xrightarrow[r\to\infty]{} \log(p).$$

So,

$$-\frac{1}{r^{1+\alpha/n}} \log \mathbb{P}(Z_r^n \leqslant r^\alpha a) \xrightarrow[r\to\infty]{} \begin{cases} \infty & \text{if } a < 0 \text{ or } (a = 0 \text{ and } p = 0) \\ 0 & \text{if } (a = 0 \text{ and } p > 0). \end{cases}$$

Right deviations — The case $a = 0$ is obvious. Let $a > 0$. Let us prove the minoration. For all $k \in \mathbb{N}^*$, we introduce the notation 1^k to represent the word $1, 1, \ldots, 1$ of length k. One has

$$\begin{aligned}
\frac{1}{r^{1+\alpha/n}} \log \mathbb{P}(Z_r^n \geqslant r^\alpha a) &\geqslant \frac{1}{r^{1+\alpha/n}} \log \mathbb{P}\Big(\frac{W_{1^1}\cdots W_{1^n}}{r^n} \geqslant r^\alpha a\Big) \\
&\geqslant \frac{n}{r^{1+\alpha/n}} \log \mathbb{P}\Big(W \geqslant r(r^\alpha a)^{1/n}\Big) \\
&\xrightarrow[r\to\infty]{} -cna^{1/n}.
\end{aligned}$$

As for the majoration, we proceed by induction. The case $n = 1$ corresponds to the very large deviations for i.i.d. random variables. Since (5) is satisfied, we may apply [3, Proposition 1.1 and Remark 1.1] to obtain

$$\log \mathbb{P}(Z_r^1 \geqslant r^\alpha a) \sim -r\Lambda_W^*(r^\alpha a) \sim -cr^{1+\alpha}a,$$

as $r \to \infty$ where the last estimate stems from Proposition 5, item 2. Then we follow the same lines as in the upper bound for the large and moderate deviations. Let $\varepsilon > 0$. For all $r \in \mathbb{N}^* \setminus \{1\}$ and for $q \in [\![0, r]\!]$, we define

$$P_{n,r,q} := \mathbb{P}\left(Z_r^n \geqslant r^\alpha a,\ Z_{r,1}^{n-1}, \dots, Z_{r,q}^{n-1} \geqslant r^{\alpha(n-1)/n}\varepsilon,\ Z_{r,q+1}^{n-1}, \dots, Z_{r,r}^{n-1} < r^{\alpha(n-1)/n}\varepsilon\right),$$

so that

$$\mathbb{P}\left(Z_r^n \geqslant r^\alpha a\right) = \sum_{q=0}^{r} \binom{r}{q} P_{n,r,q}.$$

For $q_0 \in [\![0, r]\!]$ and $c' \in (0, c)$, one has by induction

$$\begin{aligned}
\sum_{q=q_0}^{r} \binom{r}{q} P_{n,r,q} &\leqslant \sum_{q=q_0}^{r} \binom{r}{q} \mathbb{P}(Z_r^{n-1} \geqslant r^{\alpha(n-1)/n}\varepsilon)^q \\
&\leqslant \sum_{q=q_0}^{r} \left(re^{-r^{1+\alpha/n}c'(n-1)\varepsilon^{1/(n-1)}}\right)^q \\
&\leqslant \frac{\left(re^{-r^{1+\alpha/n}c'(n-1)\varepsilon^{1/(n-1)}}\right)^{q_0}}{1 - re^{-r^{1+\alpha/n}c'(n-1)\varepsilon^{1/(n-1)}}} \\
&\leqslant e^{-r^{1+\alpha/n}c'na^{1/n}},
\end{aligned}$$

as soon as $(n-1)q_0\varepsilon^{1/(n-1)} > na^{1/n}$ and r is large enough. Hence

$$\limsup_{r\to\infty} \frac{1}{r^{1+\alpha/n}} \log \sum_{q=q_0}^{r} \binom{r}{q} P_{n,r,q} \leqslant -cna^{1/n}. \tag{25}$$

Now,

$$P_{n,r,0} = \mathbb{P}\left(\sum_{i=1}^{r} W_i Z_{r,i}^{n-1} \geqslant r^{1+\alpha}a,\ Z_{r,1}^{n-1}, \dots, Z_{r,r}^{n-1} < r^{\alpha(n-1)/n}\varepsilon\right) \leqslant \mathbb{P}\left(\sum_{i=1}^{r} W_i \geqslant r^{1+\alpha/n}\frac{a}{\varepsilon}\right).$$

Using the exponential Chebyshev inequality, for all $t \in (0, c)$, one has

$$\begin{aligned}
\limsup_{r\to\infty} \frac{1}{r^{1+\alpha/n}} \log \mathbb{P}\left(\sum_{i=1}^{r} W_i \geqslant r^{1+\alpha/n}\frac{a}{\varepsilon}\right) &\leqslant \limsup_{r\to\infty} \frac{1}{r^{1+\alpha/n}}(-tr^{1+\alpha/n}a/\varepsilon + r\Lambda_W(t)) \\
&= -\frac{ta}{\varepsilon}.
\end{aligned} \tag{26}$$

Hence, for $\varepsilon > 0$ small enough, one gets

$$\limsup_{r\to\infty} \frac{1}{r^{1+\alpha/n}} \log P_{n,r,0} \leqslant -cna^{1/n}.$$

Finally, let $q \in [\![1, q_0 - 1]\!]$. One has

$$P_{n,r,q} \leqslant \mathbb{P}\left(\sum_{i=1}^{q} W_i Z_{r,i}^{n-1} + \sum_{i=q+1}^{r} W_i r^{\alpha(n-1)/n} \varepsilon \geqslant r^{1+\alpha} a \right).$$

We want to apply the contraction principle to the continuous map $(y_1, \dots, y_q, s) \mapsto y_1 + \cdots + y_q + s$ and

$$\left(\frac{1}{r^{1+\alpha}} \left(W_1 Z_{r,1}^{n-1}, \dots, W_q Z_{r,q}^{n-1}, \sum_{i=q+1}^{r} W_i r^{\alpha(n-1)/n} \varepsilon \right) \right)_{r\geqslant 2}.$$

As for the first q variables, applying the contraction principle to the continuous map $(w, z) \mapsto wz$ and $(W/r^{1+\alpha/n}, Z_r^{n-1}/r^{\alpha(n-1)/n})_{r\geqslant 2}$, we get by induction

$$\begin{aligned} \frac{1}{r^{1+\alpha/n}} \log \mathbb{P}(W Z_r^{n-1} \geqslant r^{1+\alpha} y) &\xrightarrow[r\to\infty]{} - \inf \left\{ cw + (n-1)cz^{1/(n-1)} \ ; \ w \geqslant 0, \ z > 0, \ wz = y \right\} \\ &= - \inf \left\{ \frac{cy}{z} + (n-1)cz^{1/(n-1)} \ ; \ z > 0 \right\} \\ &= -cny^{1/n} \end{aligned}$$

(the infimum is attained at $z = y^{(n-1)/n}$). Hence, applying the contraction principle and (26), we get

$$\frac{1}{r^{1+\alpha/n}} \log P_{n,r,q} \leqslant -c \inf \left\{ n(y_1^{1/n} + \cdots + y_q^{1/n}) + \frac{s}{\varepsilon}; \ y_1 + \cdots + y_q + s = a \right\} = -cna^{1/n}. \tag{27}$$

Applying the principle of the largest term to (25), (26), and (27), one gets

$$\limsup_{r\to\infty} \frac{1}{r^{1+\alpha/n}} \log \mathbb{P}(Z_r^n \geqslant r^\alpha a) \leqslant -cna^{1/n}.$$

4 Infinite tree

The heart of the proofs in the finite tree is the recursion formula (13) for Z_r^n which allows to proceed by induction. However, in the infinite tree, the analogue of (13) is the fixed point equation (2) and there is no more recursion. From now on, we set $Z_r := Z_r^\infty$ for all $r \in \mathbb{N}^* \setminus \{1\}$. By [10, Théorèmes 1 et 3], if $\operatorname{ess\,sup}(W) = \infty$ and $r > \exp \mathbb{E}[W \log(W)]$, then

$$\forall t > 0 \quad \mathbb{E}[e^{tZ_r}] = \infty.$$

Therefore, under assumption (5) and $\operatorname{ess\,sup}(W) = \infty$ (see Table 1), the standard exponential Markov inequality cannot be used to prove the upper bounds. Here, we bypass this problem by bounding the moments of Z_r. For the moderate and very large deviations, we optimize the bound given by Markov inequality over the moments to derive the exact upper bound. As for the large deviations, we prove that (see Corollary 13, item 3)

$$\limsup_{r\to\infty} \frac{1}{r} \log \mathbb{E}[Z_r^{\eta r}] = O(\eta^2)$$

which is the key argument to prove that the rate function is non degenerate and is indeed the limit of the rate functions in the finite trees (see Theorem 3, item 2).

4.1 Upper bounds for the moments of Z_r

Let us turn to the control of the moments of Z_r. Using the fact that positive martingales converge almost surely, we may let $n \to \infty$ in (13) to get

$$Z_r = \frac{1}{r} \sum_{i=1}^{r} W_i Z_{r,i} \quad \text{a.s.}$$

where the random variables $Z_{r,1}$, ..., $Z_{r,r}$, W_1, ..., W_r are independent, the W_i (resp. $Z_{r,i}$) having the same distribution as W (resp. Z_r). It is proved in [10, Théorème 1] that, for

$$r > \exp \mathbb{E}[W \log(W)], \tag{28}$$

Z_r is the unique solution Z of (2) such that $\mathbb{E}[Z] = 1$. Moreover, under (28), by [10, Théorème 2], $\mathbb{E}[Z_r^h] < \infty$ if and only if $\mathbb{E}[W^h] < r^{h-1}$, which is equivalent to $h < \chi(r)$ for

$$\chi(r) := \sup \left\{ h \in [1, \infty) \ ; \ \mathbb{E}[W^h] < r^{h-1} \right\}.$$

In the sequel, we always consider values of r which satisfy (28). Moreover, for all $(a, b) \in \mathbb{R}^2$, we denote by $[\![a, b]\!]$ the set of integers between a and b, i.e. $\{h \in \mathbb{Z} \mid a \leqslant h \leqslant b\}$. Similarly, with obvious definition, we may also use the variants $[\![a, b[\![$, etc.

Proposition 10 (Moments of W)**.** *Assume* (5)*.*

1. *One has*
$$\frac{1}{h} \log \frac{\mathbb{E}[W^h]}{h!} \xrightarrow[h\to\infty]{} -\log(c).$$
2. *One has*
$$\frac{\chi(r)}{r} \xrightarrow[r\to\infty]{} ce.$$
3. *For all $\eta \in (0, ce)$, for all r large enough,*
$$\sup_{h \in [\![2, \eta r]\!]} \frac{\mathbb{E}[W^h]}{r^{h-1}} = \frac{\mathbb{E}[W^2]}{r}.$$

Proof of Proposition 10.

1. Let $0 < \varepsilon' < \varepsilon < c$ and let A be such that, for all $w \geqslant A$,
$$\mathbb{P}(W > w) \leqslant e^{-(c-\varepsilon')w}.$$
Then, integrating by parts,
$$\begin{aligned}\mathbb{E}[W^h] &\leqslant A + \int_A^\infty e^{-(c-\varepsilon')t^{1/h}} dt = A + h\int_{A^{1/h}}^\infty e^{-(c-\varepsilon')w} w^{h-1} dw \\ &= A + \frac{hA^{(h-1)/h}}{c-\varepsilon'} e^{-(c-\varepsilon')A^{1/h}} + \frac{h(h-1)}{c-\varepsilon'} \int_{A^{1/h}}^\infty e^{-(c-\varepsilon')w} w^{h-2} dw.\end{aligned}$$
By induction, we get
$$\mathbb{E}[W^h] \leqslant A + \frac{h!}{(c-\varepsilon')^h} e^{-(c-\varepsilon')A^{1/h}} \sum_{l=0}^{h-1} \frac{((c-\varepsilon')A^{1/h})^l}{l!} \leqslant A + \frac{h!}{(c-\varepsilon')^h} = o\left(\frac{h!}{(c-\varepsilon)^h}\right)$$
as $h \to \infty$, whence the result. Similarly, the lower bound stems from
$$\mathbb{E}[W^h] \geqslant \int_A^\infty e^{-(c+\varepsilon)t^{1/h}} dt.$$

2. Let $\eta > 0$. It suffices to prove that, for all r large enough,
$$\frac{\mathbb{E}[W^{\lfloor \eta r \rfloor}]}{r^{\lfloor \eta r \rfloor - 1}} \begin{cases} < 1 & \text{for } \eta < ce \\ > 1 & \text{for } \eta > ce. \end{cases}$$
If $\eta < ce$, let $\varepsilon > 0$ be such that $\eta < (c-\varepsilon)e$. Using item 1 and Stirling bounds (see [1, Section 3])
$$\forall h \in (0, \infty) \quad \left(\frac{h}{e}\right)^h \sqrt{2\pi h} \leqslant \Gamma(h+1) \leqslant \left(\frac{h}{e}\right)^h \sqrt{2\pi h} e^{\frac{1}{12h}}, \tag{29}$$
we get, for r large enough,
$$\frac{\mathbb{E}[W^{\lfloor \eta r \rfloor}]}{r^{\lfloor \eta r \rfloor - 1}} \leqslant \frac{\lfloor \eta r \rfloor !}{r^{\lfloor \eta r \rfloor - 1}(c-\varepsilon)^{\lfloor \eta r \rfloor}} \leqslant \left(\frac{\lfloor \eta r \rfloor}{(c-\varepsilon)er}\right)^{\lfloor \eta r \rfloor} r\sqrt{2\pi \lfloor \eta r \rfloor}\, e^{\frac{1}{12\lfloor \eta r \rfloor}} < 1. \tag{30}$$
Same argument for $\eta > ce$.

3. Let $\eta \in (0, ce)$. Since the function $h \mapsto \mathbb{E}[W^h]/r^{h-1}$ is log-convex (see [10, p. 132]), if $\eta r \geqslant 2$, then
$$\sup_{h \in [\![2, \eta r]\!]} \frac{\mathbb{E}[W^h]}{r^{h-1}} = \max\left\{\frac{\mathbb{E}[W^2]}{r}, \frac{\mathbb{E}[W^{\lfloor \eta r \rfloor}]}{r^{\lfloor \eta r \rfloor - 1}}\right\}.$$
Let $\varepsilon > 0$ be such that $\eta < (c-\varepsilon)e$. The conclusion stems from the fact that (30) yields besides
$$\frac{\mathbb{E}[W^{\lfloor \eta r \rfloor}]}{r^{\lfloor \eta r \rfloor - 1}} = o\left(\frac{1}{r}\right). \qquad \square$$

Now, it is proved in [10, Equation (13)] that, for all $r > \exp \mathbb{E}[W \log(W)]$, we have the following recursion formula:

$$\forall h \in [\![2, \chi(r)[\![\quad \mathbb{E}[Z_r^h] = \frac{h!}{r^h\big(1 - \frac{\mathbb{E}[W^h]}{r^{h-1}}\big)} \sum_{\substack{h_1+\cdots+h_r=h,\\ 0\leqslant h_j\leqslant h-1}} \prod_{j=1}^{r} \frac{\mathbb{E}[W^{h_j}]\mathbb{E}[Z_r^{h_j}]}{h_j!}. \tag{31}$$

Let us turn to bounds on the moments of Z_r. First, we consider the small moments.

Proposition 11. *For all $\delta > 0$, there exists $\eta \in (0, 1 \wedge ce)$ such that, for all r large enough,*

$$\forall h \in [\![1, \eta r]\!] \quad \mathbb{E}[Z_r^h] \leqslant \exp\left\{\frac{h^2}{2(r-h)}\left(\mathrm{Var}(W) + \delta + \frac{C}{h}\right)\right\},$$

with $C = 2\mathbb{E}[W^2] + 13/12$.

Proof of Proposition 11. Let $\delta > 0$. Let us fix some $\eta \in (0, 1 \wedge ce)$. By Proposition 10, item 3, there exists $r_0 > \exp \mathbb{E}[W \log(W)]$ such that

$$\forall r \geqslant r_0 \qquad \sup_{h\in[\![2,\eta r]\!]} \frac{\mathbb{E}[W^h]}{r^{h-1}} = \frac{\mathbb{E}[W^2]}{r} \leqslant \frac{1}{2}. \tag{32}$$

In particular, for all $r \geqslant r_0$, $\eta r < \chi(r)$. Now, let us fix $r \geqslant r_0$. For all $h \geqslant 1$, denote by $E(h)$ the statement:

$$\mathbb{E}[Z_r^h] \leqslant \exp\left\{\frac{h^2}{2(r-h)}\left(\mathrm{Var}(W) + \delta + \frac{C}{h}\right)\right\}$$

and let us prove $E(h)$ by induction for $h \in [\![1, \eta r]\!]$. Since $\mathbb{E}[Z_r] = 1$, $E(1)$ is obvious. Now, in case $\eta r \geqslant 2$, let $h \in [\![2, \eta r]\!]$ and assume that $E(k)$ is satisfied for all $k \in [\![1, h-1]\!]$. From (31) and the fact that $h \leqslant \eta r < \chi(r)$, one has

$$\begin{aligned}
\mathbb{E}[Z_r^h] &= \frac{h!}{r^h\big(1 - \frac{\mathbb{E}[W^h]}{r^{h-1}}\big)} \sum_{\substack{m_0+\cdots+m_{h-1}=r,\\ m_1+\cdots+(h-1)m_{h-1}=h}} \binom{r}{m_0, \ldots, m_{h-1}} \prod_{k=0}^{h-1} \left(\frac{\mathbb{E}[W^k]\mathbb{E}[Z_r^k]}{k!}\right)^{m_k} \\
&= \frac{h!r!}{r^h\big(1 - \frac{\mathbb{E}[W^h]}{r^{h-1}}\big)} \sum_{\substack{m_0+\cdots+m_{h-1}=r,\\ m_1+\cdots+(h-1)m_{h-1}=h}} \prod_{k=0}^{h-1} \frac{1}{m_k!} \left(\frac{\mathbb{E}[W^k]\mathbb{E}[Z_r^k]}{k!}\right)^{m_k},
\end{aligned} \tag{33}$$

where m_k stands for the number of h_j equal to k (multiplicity of k in the composition

$(h_1, \dots, h_r)$ of h). Then, the latter sum above is equal to

$$\begin{aligned}
&\sum_{\substack{m_0+\cdots+m_{h-1}=r,\\ m_1+\cdots+(h-1)m_{h-1}=h}} \frac{(r-h)^{m_0-(r-h)}}{m_0!} \cdot \frac{h^{-(h-m_1)}}{m_1!} \prod_{k=2}^{h-1} \frac{1}{m_k!} \left(\frac{\mathbb{E}[W^k]\mathbb{E}[Z_r^k]h^k}{k!(r-h)^{k-1}} \right)^{m_k} \\
&\leqslant \sum_{\substack{m_0+\cdots+m_{h-1}=r,\\ m_1+\cdots+(h-1)m_{h-1}=h}} \frac{1}{(r-h)!} \cdot \frac{1}{h!} \prod_{k=2}^{h-1} \frac{1}{m_k!} \left(\frac{\mathbb{E}[W^k]\mathbb{E}[Z_r^k]h^k}{k!(r-h)^{k-1}} \right)^{m_k} \\
&\leqslant \frac{1}{(r-h)!h!} \sum_{m_2,\dots,m_{h-1}\geqslant 0} \prod_{k=2}^{h-1} \frac{1}{m_k!} \left(\frac{\mathbb{E}[W^k]\mathbb{E}[Z_r^k]h^k}{k!(r-h)^{k-1}} \right)^{m_k} \\
&= \frac{1}{(r-h)!h!} \exp\left\{ \sum_{k=2}^{h-1} \frac{\mathbb{E}[W^k]\mathbb{E}[Z_r^k]h^k}{k!(r-h)^{k-1}} \right\}. \qquad (34)
\end{aligned}$$

Besides, using Stirling bounds (29) and $\log(1+x) \leqslant x - x^2/2 + x^3/3$ for all $x \geqslant 0$, we have

$$\begin{aligned}
\frac{r!}{(r-h)!r^h} &\leqslant \left(\frac{r}{r-h} \right)^{r-h+1/2} \exp\left\{ -h + \frac{1}{12r} \right\} \\
&\leqslant \exp\left\{ \left(r-h+\frac{1}{2} \right) \left(\frac{h}{r-h} - \frac{1}{2}\left(\frac{h}{r-h} \right)^2 + \frac{1}{3}\left(\frac{h}{r-h} \right)^3 \right) - h + \frac{1}{12r} \right\} \\
&\leqslant \exp\left\{ \frac{h^2}{2(r-h)} \left(-1 + \frac{1}{h} + \frac{2h}{3(r-h)} + \frac{h}{3(r-h)^2} + \frac{r-h}{6rh^2} \right) \right\} \\
&\leqslant \exp\left\{ \frac{h^2}{2(r-h)} \left(-1 + \frac{13}{12h} + \frac{h}{r-h} \right) \right\}. \qquad (35)
\end{aligned}$$

Combining (33), (34), (35), and (32) together with $(1-x)^{-1} \leqslant \exp(2x)$ for $x \in [0, 1/2]$, we get

$$\begin{aligned}
\mathbb{E}[Z_r^h] &\leqslant \exp\left\{ \frac{h^2}{2(r-h)} \left(-1 + \frac{13}{12h} + \frac{h}{r-h} \right) + \frac{2\mathbb{E}[W^2]}{r} + \sum_{k=2}^{h-1} \frac{\mathbb{E}[W^k]\mathbb{E}[Z_r^k]h^k}{k!(r-h)^{k-1}} \right\} \\
&\leqslant \exp\left\{ \frac{h^2}{2(r-h)} \left(-1 + \frac{C}{h} + \frac{h}{r-h} + 2\sum_{k=2}^{h-1} \frac{\mathbb{E}[W^k]\mathbb{E}[Z_r^k]}{k!} \left(\frac{h}{r-h} \right)^{k-2} \right) \right\} \qquad (36)
\end{aligned}$$

with $C = 2\mathbb{E}[W^2] + 13/12$. By Proposition 10, item 1, there exists $A > 0$ such that, for all $k \in \mathbb{N}$,

$$\frac{\mathbb{E}[W^k]}{k!} \leqslant A^k.$$

For all $\tilde{\eta} \in (0, 1)$, let

$$B(\tilde{\eta}) := \exp\left\{ \frac{\tilde{\eta}}{2(1-\tilde{\eta})} \left(\mathrm{Var}(W) + \delta + \frac{C}{2} \right) \right\}.$$

Assume now that η is small enough that

$$\frac{\eta}{1-\eta} + \mathbb{E}[W^2](B(\eta)^2 - 1) + 2A^2 B(\eta)^2 \frac{AB(\eta)\frac{\eta}{1-\eta}}{1 - AB(\eta)\frac{\eta}{1-\eta}} < \delta. \tag{37}$$

Using (36) and noting that $h/(r-h) \leqslant \eta/(1-\eta)$, one gets

$$\begin{aligned}\mathbb{E}[Z_r^h] &\leqslant \exp\left\{\frac{h^2}{2(r-h)}\left(-1 + \frac{C}{h} + \frac{\eta}{1-\eta} + \mathbb{E}[W^2]B(\eta)^2 + 2\sum_{k=3}^{h-1} A^k B(\eta)^k \left(\frac{\eta}{1-\eta}\right)^{k-2}\right)\right\} \\ &\leqslant \exp\left\{\frac{h^2}{2(r-h)}\left(\mathrm{Var}(W) + \delta + \frac{C}{h}\right)\right\},\end{aligned}$$

bounding above the finite geometric sum by the value of the infinite sum, and using (37). □

The following estimate concerns the higher moments of Z_r.

Proposition 12. *For all $\eta \in (0, ce)$,*

$$\limsup_{r\to\infty} \frac{1}{r} \log \mathbb{E}[Z_r^{\eta r}] < \infty.$$

Proof of Proposition 12. Let $r > \exp \mathbb{E}[W \log(W)]$. Let $\eta \in (0, ce)$. Let $\varepsilon \in (0, c)$ be such that $\eta < (c-\varepsilon)e$. As a consequence of Proposition 10, item 1, there exists $D \geqslant 1$ such that, for all $h \geqslant 1$,

$$\frac{\mathbb{E}[W^h]}{h!} \leqslant \frac{D}{(c-\varepsilon)^h}. \tag{38}$$

Let $\theta \in (1/2, 1)$. For all $h \geqslant 2$, we write the Euclidean division $h = q\lfloor \theta h \rfloor + s$. Using (31), (38), and the convexity of $(h_1, \ldots, h_r) \mapsto \log \prod_{j=1}^r \mathbb{E}[Z_r^{h_j}]$, we get

$$\mathbb{E}[Z_r^h] \leqslant \frac{h!}{r^h\left(1 - \frac{\mathbb{E}[W^h]}{r^{h-1}}\right)}\left(\frac{D^h}{(c-\varepsilon)^h} \sum_{0 \leqslant h_j \leqslant \theta h} \mathbb{E}[Z_r^{\lfloor \theta h \rfloor}]^q \mathbb{E}[Z_r^s] + \frac{D^{(1-\theta)h+1}}{(c-\varepsilon)^h} \sum_{\exists h_j > \theta h} \mathbb{E}[Z_r^{h-1}]\mathbb{E}[Z_r]\right).$$

The number of terms in the first sum is less than the total number of compositions of h in r parts, i.e.

$$\binom{r+h-1}{r-1} \leqslant \binom{r+h}{r},$$

and similarly the number of terms in the second sum is less than

$$r\binom{r+h-\lceil \theta h \rceil - 2}{r-2} \leqslant r\binom{r+(1-\theta)h}{r}.$$

(where, for all $x \in \mathbb{R}$, $\binom{x}{n} = x(x-1)\cdots(x-n+1)/n!$). Besides, noting

$$\beta := \frac{1}{\theta - \frac{1}{2}} + 1 \geqslant \frac{1}{\theta - \frac{1}{h}} + 1 \geqslant \left\lfloor \frac{h}{\lfloor \theta h \rfloor} \right\rfloor + 1 = q + 1,$$

and using the fact that $h \in [1,\infty) \mapsto \mathbb{E}[Z_r^h]$ is nondecreasing and greater or equal to 1,

$$\mathbb{E}[Z_r^h] \leqslant \frac{h!}{r^h\big(1-\frac{\mathbb{E}[W^h]}{r^{h-1}}\big)}\left(\frac{D^h}{(c-\varepsilon)^h}\binom{r+h}{r}\mathbb{E}[Z_r^{\theta h}]^\beta + \frac{D^{(1-\theta)h+1}}{(c-\varepsilon)^h} r\binom{r+(1-\theta)h}{r}\mathbb{E}[Z_r^{h-1}]\right). \tag{39}$$

Our strategy is to bound the moments of Z_r by backward induction, finally relying on the fact that the small moments are well bounded. Indeed, by Proposition 11, there exists $\eta_1 \in (0, 1 \wedge ce)$ such that

$$\limsup_{r\to\infty} \frac{1}{r}\log \mathbb{E}[Z_r^{\eta_1 r}] < \infty.$$

Let $\eta_0 = \eta_1/4$. We claim that, for all $\gamma \in (0,1]$, there exists $C(\gamma) \in \mathbb{R}$ such that, for all r large enough,

$$\forall h \in [\![\eta_0 r, \eta r]\!] \quad \frac{h!}{r^h\big(1-\frac{\mathbb{E}[W^h]}{r^{h-1}}\big)}\binom{r+\gamma h}{r}\frac{D^{\gamma h}}{(c-\varepsilon)^h} \leqslant e^{C(\gamma) r}. \tag{40}$$

and $C(\gamma) < 0$ for γ small enough. Indeed, using (29) and applying Proposition 10, item 3,

$$\begin{aligned}
&\frac{1}{r}\log\left(\frac{h!}{r^h\big(1-\frac{\mathbb{E}[W^h]}{r^{h-1}}\big)}\binom{r+\gamma h}{r}\frac{D^{\gamma h}}{(c-\varepsilon)^h}\right)\\
&\quad\leqslant \frac{h}{r}\log\left(\frac{h}{(c-\varepsilon)er}\right) + \frac{h}{r}\gamma\log(D) + \left(1+\frac{\gamma h}{r}+\frac{1}{2r}\right)\log\left(1+\frac{\gamma h}{r}\right) - \frac{\gamma h}{r}\log\left(\frac{\gamma h}{r}\right)\\
&\qquad\qquad - \frac{1}{2r}\log(\gamma) + \frac{2\mathbb{E}[W^2]}{r^2} + \frac{1}{12rh} + \frac{1}{12r(r+\gamma h)}\\
&\quad\leqslant \eta_0\log\left(\frac{\eta}{(c-\varepsilon)e}\right) + \gamma\eta\log(D) + (1+\gamma\eta)\log(1+\gamma\eta) + \min(e^{-1}, -\gamma\eta\log(\gamma\eta)) + \gamma,
\end{aligned}$$

for all r large enough and the conclusion follows, noticing that the latter quantity is negative for small $\gamma > 0$.

From now on, we assume that $\theta \in (1/2, 1)$ is such that $C(1-\theta) < 0$. Let also $C_1 := C(1)$. Hence, starting from (39) and using (40), we get, for all r large enough, for all $h \in [\![\eta_1 r, \eta r]\!]$,

$$\begin{aligned}
\mathbb{E}[Z_r^h] &\leqslant e^{C_1 r}\mathbb{E}[Z_r^{\theta h}]^\beta + \frac{1}{2}\mathbb{E}[Z_r^{h-1}]\\
&\leqslant e^{C_1 r}\mathbb{E}[Z_r^{\theta h}]^\beta + \frac{1}{2}\Big(e^{C_1 r}\mathbb{E}[Z_r^{\theta(h-1)}]^\beta + \frac{1}{2}\mathbb{E}[Z_r^{h-2}]\Big)\\
&\leqslant e^{C_1 r}\mathbb{E}[Z_r^{\theta h}]^\beta \sum_{k=0}^{h-\lfloor\theta h\rfloor-1}\frac{1}{2^k} + \frac{1}{2^{h-\lfloor\theta h\rfloor}}\mathbb{E}[Z_r^{\theta h}],
\end{aligned}$$

by induction (note that the smallest moment to which we apply the induction argument is of order $\theta(\lfloor\theta h\rfloor + 1) \geqslant \theta^2 h \geqslant h/4 \geqslant \eta_0 r$, so (40) may be used) and the monotonicity of

$h \mapsto \mathbb{E}[Z_r^{\theta h}]$. Since $\beta \geqslant 1$, $C_1 \geqslant 0$, and $\mathbb{E}[Z_r^{\theta h}] \geqslant 1$, we obtain

$$\mathbb{E}[Z_r^h] \leqslant e^{C_1 r}\mathbb{E}[Z_r^{\theta h}]^\beta \sum_{k=0}^{h-\lfloor \theta h \rfloor} \frac{1}{2^k} \leqslant 2e^{C_1 r}\mathbb{E}[Z_r^{\theta h}]^\beta. \tag{41}$$

Let $K := \min\left\{k \geqslant 1 \;\middle|\; \theta^k \eta \leqslant \eta_1\right\}$. Applying (41) recursively, we get

$$\begin{aligned}
\log \mathbb{E}[Z_r^{\eta r}] &\leqslant \log(2) + C_1 r + \beta \log \mathbb{E}[Z_r^{\theta \eta r}] \\
&\leqslant (\log(2) + C_1 r)(1+\beta) + \beta^2 \log \mathbb{E}[Z_r^{\theta^2 \eta r}] \\
&\leqslant (\log(2) + C_1 r) \sum_{k=0}^{K-1} \beta^k + \beta^K \log \mathbb{E}[Z_r^{\theta^K \eta r}].
\end{aligned} \tag{42}$$

Finally,

$$\limsup_{r\to\infty} \frac{1}{r} \log \mathbb{E}[Z_r^{\eta r}] \leqslant C_1 \frac{\beta^K - 1}{\beta - 1} + \beta^K \limsup_{r\to\infty} \frac{1}{r} \log \mathbb{E}[Z_r^{\eta_1 r}] < \infty. \qquad \square$$

Propositions 11 and 12 immediately entail the following result.

Corollary 13.

1. *For all $\alpha \in [0, 1/2)$ and $\zeta > 0$,*

$$\lim_{r\to\infty} \frac{1}{r^{2\alpha-1}} \log \mathbb{E}[Z_r^{\zeta r^\alpha}] = 1.$$

2. *For all $\alpha \in [1/2, 1)$ and $\zeta > 0$,*

$$\limsup_{r\to\infty} \frac{1}{r^{2\alpha-1}} \log \mathbb{E}[Z_r^{\zeta r^\alpha}] \leqslant \frac{\zeta^2 \operatorname{Var}(W)}{2}.$$

3. *For all $\eta \in (0, ce)$,*

$$\kappa(\eta) := \limsup_{r\to\infty} \frac{1}{r} \log \mathbb{E}[Z_r^{\eta r}] < \infty$$

and $\kappa(\eta) = O(\eta^2)$ as $\eta \to 0$.

4.2 Proof of Theorem 4, item 1 (moderate deviations)

Left deviations — Let $\alpha \in (0, 1/2)$ and $a > 0$. We want to estimate $\mathbb{P}(r^\alpha(Z_r - 1) \leqslant -a)$. For all $t \leqslant 0$, we consider

$$\begin{aligned}
\Lambda_r(s) &:= \frac{1}{r^{1-2\alpha}} \log \mathbb{E}[e^{sr^{1-\alpha}(Z_r - 1)}] \\
&= \frac{1}{r^{1-2\alpha}} \log \mathbb{E}\left[\exp\left(r^{1-\alpha} s \frac{1}{r} \sum_{i=1}^{r} (W_i Z_{r,i} - 1)\right)\right] \\
&= r^{2\alpha} \log \mathbb{E}[e^{sr^{-\alpha}(WZ_r - 1)}].
\end{aligned}$$

Using the Taylor expansion $e^x = 1 + x + x^2 e^{\theta(x)}/2$ with $\theta(x) \leqslant \max(x, 0)$ and the fact that $\mathbb{E}[WZ_r] = \mathbb{E}[W]\mathbb{E}[Z_r] = 1$, we get

$$\Lambda_r(s) = r^{2\alpha} \log\Big(1 + \frac{s^2}{2r^{2\alpha}} \mathbb{E}\big[(WZ_r - 1)^2 e^{\theta(sr^{-\alpha}(WZ_r - 1))}\big]\Big) \xrightarrow[r\to\infty]{} \frac{s^2}{2}\mathbb{E}[(W-1)^2].$$

Indeed,

$$\begin{aligned}
&\Big|\mathbb{E}\big[(WZ_r - 1)^2 e^{\theta(sr^{-\alpha}(WZ_r-1))}\big] - \mathbb{E}[(W-1)^2]\Big| \\
&\quad\leqslant \Big|\mathbb{E}\big[(WZ_r-1)^2\big(e^{\theta(sr^{-\alpha}(WZ_r-1))} - 1\big)\big]\Big| + \big|\mathbb{E}\big[(WZ_r-1)^2 - (W-1)^2\big]\big| \\
&\quad\leqslant \big|\mathbb{E}\big[(WZ_r-1)^2\big]\big|\,\big(e^{-sr^{-\alpha}} - 1\big) + \big|\mathbb{E}\big[(WZ_r-1)^2 - (W-1)^2\big]\big| \xrightarrow[r\to\infty]{} 0
\end{aligned}$$

by Corollary 13, item 1. Finally, the unilateral version of Gärtner-Ellis theorem applies and gives

$$\frac{1}{r^{1-2\alpha}} \log \mathbb{P}(r^\alpha(Z_r - 1) \leqslant -a) \xrightarrow[r\to\infty]{} -\frac{a^2}{2\operatorname{Var}(W)}.$$

Right deviations — Let $a \geqslant 0$. Using (2), one gets

$$\mathbb{P}(r^\alpha(Z_r - 1) \geqslant a) = \mathbb{P}\Big(\sum_{i=1}^r (W_i Z_{r,i} - 1) \geqslant ar^{1-\alpha}\Big).$$

Let us prove the lower bound. For all $\varepsilon > 0$,

$$\begin{aligned}
&\frac{1}{r^{1-2\alpha}} \log \mathbb{P}\Big(\sum_{i=1}^r (W_i Z_{r,i} - 1) \geqslant ar^{1-\alpha}\Big) \\
&\qquad\geqslant \frac{1}{r^{1-2\alpha}} \log \mathbb{P}\Big(\sum_{i=1}^r (W_i Z_{r,i} - 1) \geqslant ar^{1-\alpha},\ \forall i \in [\![1, r]\!]\ \ Z_{r,i} \geqslant 1 - \varepsilon r^{-\alpha}\Big) \\
&\qquad\geqslant \frac{1}{r^{1-2\alpha}} \log \mathbb{P}\Big(\sum_{i=1}^r (W_i(1 - \varepsilon r^{-\alpha}) - 1) \geqslant ar^{1-\alpha}\Big) + r^{2\alpha} \log \mathbb{P}(Z_r \geqslant 1 - \varepsilon r^{-\alpha}).
\end{aligned}$$

The left deviations for Z_r yield

$$\frac{1}{r^{1-2\alpha}} \log \mathbb{P}(Z_r - 1 < -\varepsilon r^{-\alpha}) \xrightarrow[r\to\infty]{} -J(-\varepsilon) < 0,$$

hence

$$r^{2\alpha} \log \mathbb{P}(Z_r \geqslant 1 - \varepsilon r^{-\alpha}) \xrightarrow[r\to\infty]{} 0.$$

Using (15) and letting $\varepsilon \to 0$, one finally gets

$$\liminf_{r\to\infty} \frac{1}{r^{1-2\alpha}} \log \mathbb{P}(r^\alpha(Z_r - 1) \geqslant a) \geqslant -J(a).$$

Now, let us turn to the upper bound. Using Markov's inequality and Corollary 13, item 2, for any $\zeta > 0$,

$$\begin{aligned}\limsup_{r\to\infty} \frac{1}{r^{1-2\alpha}} \log \mathbb{P}\big(Z_r - 1 \geqslant ar^{-\alpha}\big) &\leqslant \limsup_{r\to\infty} \left(\frac{1}{r^{1-2\alpha}} \log \mathbb{E}[Z_r^{\zeta r^{1-\alpha}}] - \zeta r^{\alpha} \log(1 + ar^{-\alpha}) \right) \\ &\leqslant \frac{\zeta^2 \operatorname{Var}(W)}{2} - \zeta a.\end{aligned}$$

Otimizing in ζ, we conclude to the required upper bound.

4.3 Proof of Theorem 4, item 2 (large deviations)

Remind that

$$I^{\infty} = \lim_{n\to\infty} \downarrow I^n.$$

In particular, for all $a \leqslant 1$, $I^{\infty}(a) = \Lambda_W^*(a)$ and the large deviations on the left for $(Z_r)_{r\geqslant 2}$ are given in Proposition 2. Let us turn to the large deviations on the right. Let us introduce, for all $a \geqslant 1$,

$$-\underline{I}^{\infty}(a) := \liminf_{r\to\infty} \frac{1}{r} \log \mathbb{P}(Z_r \geqslant a) \leqslant \limsup_{r\to\infty} \frac{1}{r} \log \mathbb{P}(Z_r \geqslant a) =: -\overline{I}^{\infty}(a). \tag{43}$$

Proposition 14 (Lower bound)**.** *For all $a \geqslant 1$, $\underline{I}^{\infty}(a) \leqslant I^{\infty}(a)$.*

Proof of Proposition 14. Remind that, for all $k \in \mathbb{N}^*$, 1^k represents the word $1, 1, \ldots, 1$ of length k. For all $r \in \mathbb{N}^* \setminus \{1\}$, $n \in \mathbb{N}^*$, $k \in \mathbb{N}$, and $i \in [\![1, r]\!]$, let

$$Z^n_{r,1^k,i} := \frac{1}{r^n} \sum_{1\leqslant i_1,\ldots,i_n\leqslant r} W_{1^k,i,i_1} W_{1^k,i,i_1,i_2} \cdots W_{1^k,i,i_1,\ldots,i_n} \tag{44}$$

and let $Z_{r,1^k,i}$ be the limit of the martingale $(Z^n_{r,1^k,i})_{n\geqslant 1}$ (by convention, $Z_{r,1^0,i} = Z_{r,i}$). Note that $(W_{1^k,i})_{k\geqslant 0, 1\leqslant i\leqslant r} \bullet (Z_{r,1^k,i})_{k\geqslant 0, 2\leqslant i\leqslant r}$ is a family of independent random variables. Now let $a \geqslant 1$, $n \in \mathbb{N}^*$, $\varepsilon > 0$, and consider $((s_1, w_1), (s_{1,1}, w_{1,1}), \ldots, (s_{1^{n-2},1}, w_{1^{n-2},1}), z) \in (\mathbb{R}^2)^{n-1} \times \mathbb{R}$ such that

$$s_1(1-\varepsilon) + w_1\Big(s_{1,1}(1-\varepsilon) + \cdots + w_{1^{n-3},1}\Big(s_{1^{n-2},1}(1-\varepsilon) + w_{1^{n-2},1} z(1-\varepsilon)\Big)\Big) \geqslant a. \tag{45}$$

Then

$$\begin{aligned}
\frac{1}{r}\log\mathbb{P}(Z_r \geqslant a) \geqslant \frac{1}{r}\log\mathbb{P}\Big(&W_1 \geqslant w_1 r,\ \sum_{i=2}^{r} W_i \geqslant s_1 r,\ \forall i \in [\![2,r]\!]\ \ Z_{r,i} \geqslant 1-\varepsilon,\\
&W_{1,1} \geqslant w_{1,1} r, \sum_{i=2}^{r} W_{1,i} \geqslant s_{1,1} r,\ \forall i \in [\![2,r]\!]\ \ Z_{r,1,i} \geqslant 1-\varepsilon,\\
&\cdots\\
&W_{1^{n-2},1} \geqslant w_{1^{n-2},1} r,\ \sum_{i=2}^{r} W_{1^{n-2},i} \geqslant s_{1^{n-2},1} r,\ \forall i \in [\![2,r]\!]\ \ Z_{r,1^{n-2},i} \geqslant 1-\varepsilon,\\
&\sum_{i=1}^{r} W_{1^{n-1},i} \geqslant \frac{zr}{1-\varepsilon},\ \forall i \in [\![1,r]\!]\ \ Z_{r,1^{n-1},i} \geqslant 1-\varepsilon\Big)\\
\geqslant \frac{1}{r}\log\mathbb{P}\Big(&W_1 \geqslant w_1 r,\ \sum_{i=2}^{r} W_i \geqslant s_1 r,\ W_{1,1} \geqslant w_{1,1} r,\ \sum_{i=2}^{r} W_{1,i} \geqslant s_{1,1} r,\ \cdots\\
&W_{1^{n-2},1} \geqslant w_{1^{n-2},1} r,\ \sum_{i=2}^{r} W_{1^{n-2},i} \geqslant s_{1^{n-2},1} r,\ \sum_{i=1}^{r} W_{1^{n-1},i} \geqslant zr\Big)\\
&+ \frac{n(r-1)+1}{r}\log\mathbb{P}(Z_r \geqslant 1-\varepsilon).
\end{aligned}$$

Remind that $\mathbb{P}(Z_r \geqslant 1-\varepsilon) \to 1$ as $r \to \infty$, so by independence,

$$\begin{aligned}
-\underline{I}^{\infty}(a) &= \liminf_{r\to\infty} \frac{1}{r}\log\mathbb{P}(Z_r \geqslant a)\\
&\geqslant \liminf_{r\to\infty} \frac{1}{r}\log\mathbb{P}\Big(W_1 \geqslant w_1 r,\ \sum_{i=2}^{r} W_i \geqslant s_1 r,\ W_{1,1} \geqslant w_{1,1} r,\ \sum_{i=2}^{r} W_{1,i} \geqslant s_{1,1} r,\\
&\qquad \cdots,\ W_{1^{n-2},1} \geqslant w_{1^{n-2},1} r,\ \sum_{i=2}^{r} W_{1^{n-2},i} \geqslant s_{1^{n-2},1} r,\\
&\qquad \sum_{i=1}^{r} W_{1^{n-1},i} \geqslant \frac{zr}{1-\varepsilon}\Big)\\
&= -\big(cw_1 + \Lambda_W^*(s_1) + cw_{1,1} + \Lambda_W^*(s_{1,1}) + \cdots + cw_{1^{n-2},1} + \Lambda_W^*(s_{1^{n-2},1}) + \Lambda_W^*(z)\big).
\end{aligned}$$

Optimizing in $((s_1,w_1),(s_{1,1},w_{1,1}),\ldots,(s_{1^{n-2},1},w_{1^{n-2},1}),z) \in (\mathbb{R}^2)^{n-1}\times\mathbb{R}$ satisfying (45) and using the induction relation of Theorem 2, we get

$$-\underline{I}^{\infty}(a) \geqslant -I^n\Big(\frac{a}{1-\varepsilon}\Big).$$

Taking the limit as $\varepsilon \to 0$ and using the continuity of I^n (see Proposition 8, item 1), and then taking the limit as $n \to \infty$, we conclude that

$$\underline{I}^{\infty}(a) \leqslant \lim_{n\to\infty} I^n(a) = I^{\infty}(a). \qquad \square$$

Now, we turn to the upper bound, i.e. $\overline{I}^{\infty} \geqslant I^{\infty}$.

Proposition 15. *Let $a \geqslant 1$.*

1. *If $c = \infty$, then*
$$\overline{I}^{\infty}(a) = \Lambda_W^*(a) = I^{\infty}(a).$$
2. *If $c < \infty$,*
$$\overline{I}^{\infty}(a) = \inf\left\{cw + \overline{I}^{\infty}(z) + \Lambda_W^*(s) \ ; \ w \geqslant 0, \ z \in [1,a], \ s \in [1,a], \ wz + s = a\right\}.$$

Proof of Proposition 15. Let $a \geqslant 1$. We follow the same lines as in Section 3.3 for Z_r^n. Let $\varepsilon > 0$. For all $r \in \mathbb{N}^* \setminus \{1\}$ and for $q \in [\![0, r]\!]$, we define

$$P_{r,q} := \mathbb{P}\left(Z_r \geqslant a, \ Z_{r,1}, \dots, Z_{r,q} \geqslant 1+\varepsilon, \ Z_{r,q+1}, \dots, Z_{r,r} < 1+\varepsilon\right),$$

so that

$$\mathbb{P}\left(Z_r \geqslant a\right) = \sum_{q=0}^{r} \binom{r}{q} P_{r,q}.$$

Let $q_0 \in [\![0, r]\!]$ and $\varepsilon' \in (0, \varepsilon)$. By the definition of $\overline{I}^{\infty}$,

$$\begin{aligned}
\sum_{q=q_0}^{r} \binom{r}{q} P_{r,q} &\leqslant \sum_{q=q_0}^{r} \binom{r}{q} \mathbb{P}(Z_r \geqslant 1+\varepsilon)^q \\
&\leqslant \sum_{q=q_0}^{r} \left(re^{-r\overline{I}^{\infty}(1+\varepsilon')}\right)^q \\
&\leqslant \frac{\left(re^{-r\overline{I}^{\infty}(1+\varepsilon')}\right)^{q_0}}{1 - re^{-r\overline{I}^{\infty}(1+\varepsilon')}}.
\end{aligned}$$

By Markov's inequality and Corollary 13, item 3, for all $z > 1$, for all $\eta \in (0, ce)$,

$$-\overline{I}^{\infty}(z) = \limsup_{r\to\infty} \frac{1}{r} \log \mathbb{P}(Z_r \geqslant z) \leqslant \limsup_{r\to\infty} \frac{1}{r} \log \mathbb{E}[Z_r^{\eta r}] - \eta \log(z) \leqslant \kappa(\eta) - \eta \log(z). \tag{46}$$

Hence $\overline{I}^{\infty}(z) > 0$, since $\kappa(\eta) = O(\eta^2)$ as $\eta \to 0$. Therefore, we may choose q_0 such that $q_0 \overline{I}^{\infty}(1+\varepsilon') > \overline{I}^{\infty}(a)$ and

$$\sum_{q=q_0}^{r} \binom{r}{q} P_{r,q} \leqslant e^{-r\overline{I}^{\infty}(a)}$$

as soon as r is large enough. Now,

$$\lim_{r\to\infty} \frac{1}{r} \log P_{r,0} \leqslant \lim_{r\to\infty} \frac{1}{r} \log \mathbb{P}\Big(\sum_{i=1}^{r} W_i \geqslant \frac{ra}{1+\varepsilon}\Big) = -\Lambda_W^*\Big(\frac{a}{1+\varepsilon}\Big).$$

Finally, let $q \in [\![1, q_0 - 1]\!]$. Using the upper bound in the contraction principle for the continuous map $(y_1, \dots, y_q, s) \mapsto y_1 + \cdots + y_q + (1+\varepsilon)s$ and $((W_1 Z_{r,1}, \dots, W_q Z_{r,q}, W_{q+1} + \cdots + W_r)/r)_{r \geqslant 2}$, we get

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{1}{r} \log P_{r,q} \\
&\quad \leqslant \limsup_{r\to\infty} \frac{1}{r} \log \mathbb{P}\left(W_1 Z_{r,1} + \cdots + W_q Z_{r,q} + (W_{q+1} + \cdots + W_r)(1+\varepsilon) \geqslant ra,\right. \\
&\qquad\qquad \left. Z_{r,1}, \dots, Z_{r,q} \geqslant 1+\varepsilon,\ Z_{r,q+1}, \dots, Z_{r,r} < 1+\varepsilon\right) \\
&\quad \leqslant \limsup_{r\to\infty} \frac{1}{r} \log \mathbb{P}\left(W_1 Z_{r,1} + \cdots + W_q Z_{r,q} + (W_{q+1} + \cdots + W_r)(1+\varepsilon) \geqslant ra\right) \\
&\quad \leqslant -\inf\Big\{ g(y_1) + \cdots + g(y_q) + \Lambda_W^*(s)\ ; \\
&\qquad\qquad y_1,\ \dots,\ y_q \geqslant 0,\ s \geqslant 0,\ y_1 + \cdots + y_q + (1+\varepsilon)s \geqslant a \Big\} \\
&\quad = -\inf\left\{ g(y) + \Lambda_W^*(s)\ ;\ y \geqslant 0,\ s \geqslant 0,\ y + (1+\varepsilon)s \geqslant a \right\},
\end{aligned}$$

where g is the concave function defined by

$$g(y) := \inf\left\{ cw + \overline{I}^{\infty}(z)\ ;\ w \geqslant 0,\ z > 0,\ wz = y \right\} = \inf\left\{ \frac{cy}{z} + \overline{I}^{\infty}(z)\ ;\ z > 0 \right\}$$

and satisfying the upper bound of large deviations for $(WZ_r/r)_{r\geqslant 2}$ at speed $(r)_{r\geqslant 2}$. Then, proceeding as in the proof of Theorem 3, item 2, we obtain

$$\overline{I}^{\infty}(a) \geqslant \sup_{0<\varepsilon\leqslant a-1} \inf\left\{ cw + \overline{I}^{\infty}(z) + \Lambda_W^*(s)\ ;\ w \geqslant 0,\ z \in [1,a],\ s \in \left[1, \frac{a}{1+\varepsilon}\right],\ wz + (1+\varepsilon)s = a \right\}.$$

If $c = \infty$, then, since Λ_W^* is left continuous at a,

$$\overline{I}^{\infty}(a) \geqslant \sup_{\varepsilon>0} \Lambda_W^*\left(\frac{a}{1+\varepsilon}\right) = \Lambda_W^*(a) = I^{\infty}(a).$$

Combining this inequality with (43) and Proposition 15, item 1 is proved.

Let us turn to item 2 and assume that $c < \infty$. We have

$$\begin{aligned}
\overline{I}^{\infty}(a) &\geqslant \sup_{0<\varepsilon\leqslant a-1} \inf_{\substack{1\leqslant z\leqslant a,\\ 1\leqslant s\leqslant \frac{a}{1+\varepsilon}}} \left[\frac{c(a-s)}{z} + \overline{I}^{\infty}(z) + \Lambda_W^*(s) - \frac{c\varepsilon s}{z}\right] \\
&\geqslant \sup_{0<\varepsilon\leqslant a-1} \inf_{\substack{1\leqslant z\leqslant a,\\ 1\leqslant s\leqslant a}} \left[\frac{c(a-s)}{z} + \overline{I}^{\infty}(z) + \Lambda_W^*(s) - c\varepsilon a\right] \\
&= \inf_{\substack{1\leqslant z\leqslant a\\ 1\leqslant s\leqslant a}} \left[\frac{c(a-s)}{z} + \overline{I}^{\infty}(z) + \Lambda_W^*(s)\right] \\
&= \inf\left\{ cw + \overline{I}^{\infty}(z) + \Lambda_W^*(s)\ ;\ w \geqslant 0,\ z \in [1,a],\ s \in [1,a],\ wz + s = a \right\},
\end{aligned}$$

where the latter equality follows from the fact that $\overline{I}^{\infty}$ is nondecreasing on $[1,\infty)$ by the very definition in (43), and even $\overline{I}^{\infty}(z) > \overline{I}^{\infty}(a)$ for all z large enough by (46), and $cw+\Lambda_W^*(s) > 0$ for all $(w,s) \in [0,\infty) \times [1,\infty) \setminus \{(0,1)\}$.

Second, by definition and since $\mathbb{P}(Z_r \geqslant 1-\varepsilon) \to 1$ as $r \to \infty$, one has, for all $\varepsilon > 0$, $z \geqslant 1$, and $s \in [1,a]$,

$$\begin{aligned}
-\overline{I}^{\infty}(a) &= \limsup_{r\to\infty} \frac{1}{r} \log \mathbb{P}(Z_r \geqslant a) \\
&\geqslant \limsup_{r\to\infty} \frac{1}{r} \log \mathbb{P}(W_1 Z_{r,1} + (W_2+\cdots+W_r)(1-\varepsilon) \geqslant ra) + \frac{r-1}{r} \log \mathbb{P}(Z_r \geqslant 1-\varepsilon) \\
&\geqslant \limsup_{r\to\infty} \frac{1}{r} \log \mathbb{P}\Big(W_1 \geqslant \frac{r(a-s(1-\varepsilon))}{z},\ Z_{r,1} \geqslant z,\ \frac{1}{r}(W_2+\cdots+W_r) \geqslant s\Big) \\
&= -\Big(\frac{c(a-s)}{z} + \overline{I}^{\infty}(z) + \Lambda_W^*(s) + \frac{cs\varepsilon}{z}\Big)
\end{aligned}$$

from which we deduce that

$$\begin{aligned}
\overline{I}^{\infty}(a) &\leqslant \inf_{\varepsilon>0} \inf_{\substack{1\leqslant z\leqslant a \\ 1\leqslant s\leqslant \frac{a}{1-\varepsilon}}} \Big[\frac{c(a-s)}{z} + \overline{I}^{\infty}(z) + \Lambda_W^*(s) + \frac{cs\varepsilon}{z}\Big] \\
&\leqslant \inf_{\substack{1\leqslant z\leqslant a \\ 1\leqslant s\leqslant a}} \Big[\frac{c(a-s)}{z} + \overline{I}^{\infty}(z) + \Lambda_W^*(s)\Big] \\
&= \inf\big\{cw + \overline{I}^{\infty}(z) + \Lambda_W^*(s)\ ;\ w \geqslant 0,\ z \in [1,a]\,,\ s \in [1,a]\,,\ wz+s=a\big\}\,. \qquad \square
\end{aligned}$$

In view of the latter proposition, it remains to prove the inequality $\overline{I}^{\infty} \geqslant I^{\infty}$ in the case $c < \infty$.

Lemma 16. *The nondecreasing function $\overline{I}^{\infty}$ is continuous.*

Proof of Lemma 16. Let $a_0 \geqslant 1$ and $\varepsilon > 0$. Since $\overline{I}^{\infty}$ is nondecreasing, it suffices to prove that there exists $a_1 > a_0$ such that $\overline{I}^{\infty}(a_1) \leqslant \overline{I}^{\infty}(a_0) + \varepsilon$. This follows the same lines as in the proof of Proposition 8, item 1. $\square$

For all $a \geqslant 1$, we may define

$$z(a) := \inf \operatorname{argmin}\big\{\overline{I}^{\infty}(z) + h(a,z)\ ;\ z \in [1,a]\big\}$$

Recall that $a_W := \inf\{a \geqslant 1\ ;\ aa^* \geqslant c\}$. Now if $a \in [1,a_W]$ and $z \in [1,a]$, then $z \leqslant c/a^*$, so $h(a,z) = \Lambda_W^*(a)$, whence $z(a) = 1$ and

$$\overline{I}^{\infty}(a) = \Lambda_W^*(a) \geqslant I^{\infty}(a). \tag{47}$$

Lemma 17. *For all $A > a_W$, there exists $\eta(A) > 0$ such that, for all $a \in [a_W, A]$, $z(a) < a - \eta(A)$.*

Proof of Lemma 17. For all $a \geqslant a_W$ and $z \in [1, a]$,

$$h(a, z) \geqslant h(a, a) = c - \Lambda_W(c/a) \geqslant c - \Lambda_W(c/a_W) = \Lambda_W^*(a_W) =: \rho > 0.$$

Now let $A > a_W$. Since the function $\overline{I}^\infty$ is uniformly continuous on $[1, A]$, there exists $\eta(A) > 0$ such that, for all $(a, z) \in [1, A]^2$, with $a - \eta(A) \leqslant z \leqslant a$, $\overline{I}^\infty(z) > \overline{I}^\infty(a) - \rho$ (note that it implies $\eta(A) < a_W - 1$). Then, for all $a \in [a_W, A]$ and $z \in [a - \eta(A), a]$,

$$\overline{I}^\infty(z) + h(a, z) > (\overline{I}^\infty(a) - \rho) + \rho = \overline{I}^\infty(a),$$

whence the conclusion of the lemma. □

Let $E := \left\{a \geqslant 1 \ ; \ \forall z \in [1, a] \ \overline{I}^\infty(z) \geqslant I^\infty(z)\right\}$ and let $a_\infty = \sup(E)$. By (47), we know that $a_\infty \geqslant a_W$. The aim is to prove that $a_\infty = \infty$. Assume that $a_\infty \in [a_W, \infty)$. Let $A = a_\infty + 1 > a_W$ and $\eta(A)$ be defined as in Lemma 17. Let a be such that $a_\infty < a \leqslant a_\infty + \min(1, \eta(A))$. One has

$$\begin{aligned}\overline{I}^\infty(a) &= \inf_{1 \leqslant z < a - \eta(A)} \left[\overline{I}^\infty(z) + h(a, z)\right] \\ &= \inf_{1 \leqslant z < a - \eta(A)} \left[I^\infty(z) + h(a, z)\right] \geqslant \inf_{1 \leqslant z \leqslant a} \left[I^\infty(z) + h(a, z)\right] = I^\infty(a),\end{aligned}$$

so $a \in E$ which is a contradiction, and we conclude that $a_\infty = \infty$.

4.4 Proof of Theorem 4, item 3 (very large deviations)

Left deviations — For $a < 0$, $\mathbb{P}(Z_r \leqslant r^\alpha a) = 0$. Assume now that $a = 0$. Note that $\mathbb{P}(Z_r \leqslant 0) = \mathbb{P}(Z_r = 0)$. Let $p := \mathbb{P}(W = 0)$. If $p = 0$, then $\mathbb{P}(Z_r = 0) = 0$. Otherwise, $0 < p < 1$ (recall that $\mathbb{E}[W] = 1$ implies $p < 1$) and

$$\mathbb{P}(Z_r = 0) \geqslant \mathbb{P}(Z_r^1 = 0) = p^r.$$

Moreover, using (2),

$$\mathbb{P}(Z_r = 0) = \mathbb{P}(W Z_r = 0)^r = (1 - \mathbb{P}(W \neq 0)\mathbb{P}(Z_r \neq 0))^r = (p + (1 - p)\mathbb{P}(Z_r = 0))^r.$$

Then $\mathbb{P}(Z_r = 0)$ is a fixed point of the strictly convex function $f \colon x \in [0, 1] \mapsto (p + (1-p)x)^r$. For $r > \exp \mathbb{E}[W \log(W)]$, recalling that $\mathbb{P}(Z_r = 0) < 1$, f has exactly two fixed points: $\mathbb{P}(Z_r = 0)$ and 1. Let $\varepsilon > 0$ and $x_{r,\varepsilon} := p^{r(1-\varepsilon)}$. Then, for all r large enough, $f(x_{r,\varepsilon}) < x_{r,\varepsilon}$, whence $\mathbb{P}(Z_r = 0) < x_{r,\varepsilon}$. Then we conclude that

$$-\frac{1}{r \log(r)} \log \mathbb{P}(Z_r \leqslant r^\alpha a) \to \begin{cases} \infty & \text{for } a < 0 \text{ or } (a = 0 \text{ and } p = 0) \\ 0 & \text{for } (a = 0 \text{ and } p > 0). \end{cases}$$

Right deviations — The case $a = 0$ is obvious. Assume $a > 0$. Let us prove the minoration. Recall the definition of $Z_{r,1^n}$ given after Equation (44). One has

$$\begin{aligned}\frac{1}{r\log(r)}\log\mathbb{P}(Z_r \geqslant r^\alpha a) &\geqslant \frac{1}{r\log(r)}\log\mathbb{P}\Big(\frac{W_{1^1}\cdots W_{1^n}}{r^n} \geqslant r^\alpha a,\ Z_{r,1^n} \geqslant 1\Big)\\ &\geqslant \frac{1}{r\log(r)}\log\mathbb{P}\Big(\frac{W_{1^1}\cdots W_{1^n}}{r^n} \geqslant r^\alpha a\Big) + \frac{1}{r\log(r)}\log\mathbb{P}(Z_r \geqslant 1).\end{aligned}$$

By [12, Theorem 1.2],

$$\frac{1}{r\log(r)}\log\mathbb{P}(Z_r \geqslant 1) \xrightarrow[r\to\infty]{} 0.$$

Now, $n(r^\alpha a)^{1/n}$ is minimum for $n = n_r := \lfloor \log(r^\alpha a)\rfloor$. So

$$\frac{1}{r\log(r)}\log\mathbb{P}\Big(\frac{W_{1^1}\cdots W_{1^{n_r}}}{r^{n_r}} \geqslant r^\alpha a\Big) \geqslant \frac{n_r}{r\log(r)}\log\mathbb{P}\Big(W \geqslant r(r^\alpha a)^{1/n_r}\Big) \xrightarrow[r\to\infty]{} -c\alpha e.$$

As for the upper bound, by Markov's inequality and Corollary 13, item 3, for all $\eta \in (0, ce)$, there exists $\kappa(\eta) \in (0,\infty)$ such that, for all $r \geqslant 1$,

$$\begin{aligned}\log\mathbb{P}(Z_r \geqslant r^\alpha a) &\leqslant \log\mathbb{E}[Z_r^{\eta r}] - \eta r\log(r^\alpha a)\\ &\leqslant \kappa(\eta)r - \eta r\alpha\log(r) - \eta r\log(a)\\ &\sim -r\log(r)\eta\alpha,\end{aligned}$$

as $r \to \infty$ and we conclude letting $\eta \to ce$.

A Deviation estimates and large deviation principles

Lemma 18. *Let $(Y_r)_{r\geqslant 1}$ be a sequence of real-valued random variables and $(v_r)_{r\geqslant 1}$ be a sequence of positive numbers diverging to infinity. Assume that there exist $m \in \mathbb{R}$ and two functions $I_-\colon (-\infty, m] \to [0,\infty]$ and $I_+\colon [m,\infty) \to [0,\infty]$ such that I_- is decreasing (or infinite), I_+ is increasing (or infinite),*

$$\forall a \in (-\infty, m] \quad \frac{1}{v_r}\log\mathbb{P}(Y_r \leqslant a) \xrightarrow[r\to\infty]{} -I_-(a)$$

and

$$\forall a \in [m,\infty) \quad \frac{1}{v_r}\log\mathbb{P}(Y_r \geqslant a) \xrightarrow[r\to\infty]{} -I_+(a).$$

Then, the sequence $(Y_r)_{r\geqslant 1}$ satisfies a large deviation principle at speed $(v_r)_{r\geqslant 1}$ with rate function

$$I(a) = \begin{cases} I_-(a) & \text{if } a < m\\ \min\{I_-(m), I_+(m)\} & \text{if } a = m\\ I_+(a) & \text{if } a > m.\end{cases}$$

Proof of Lemma 18. Let F be a closed subset of $\mathbb{R}$. Let us introduce

$$a := \sup(F \cap (-\infty, m]) \quad \text{and} \quad b := \inf(F \cap [m, \infty)).$$

By convention, $\sup(\emptyset) = -\infty$, $\inf(\emptyset) = \infty$, $I_-(-\infty) = I_+(\infty) = \infty$. Then,

$$\begin{aligned}\limsup_{r\to\infty} \frac{1}{v_r} \log \mathbb{P}(Y_r \in F) &\leqslant \limsup_{r\to\infty} \frac{1}{v_r} \log \mathbb{P}(Y_r \in (-\infty, a] \cup [b, \infty)) \\ &\leqslant -\min\{I_-(a), I_+(b)\} \\ &= -\inf_{x\in F} I(x),\end{aligned}$$

using the fact that I_- is decreasing over $(-\infty, m]$ and I_+ is increasing over $[m, \infty)$. Now, let G be an open subset of $\mathbb{R}$. Let $x \in \mathbb{R}$ and $\varepsilon > 0$ be such that $(x - \varepsilon, x + \varepsilon) \subset G$. Assume that $x \geqslant m$ (the case $x \leqslant m$ is treated analogously). Now,

$$\begin{aligned}\liminf_{r\to\infty} \frac{1}{v_r} \log \mathbb{P}(Y_r \in G) &\geqslant \liminf_{r\to\infty} \frac{1}{v_r} \log \mathbb{P}(Y_r \in (x - \varepsilon, x + \varepsilon)) \\ &\geqslant \liminf_{r\to\infty} \frac{1}{v_r} \log \mathbb{P}(Y_r \in [x, x + \varepsilon)) \\ &\geqslant \liminf_{r\to\infty} \frac{1}{v_r} \log\big(\mathbb{P}(Y_r \geqslant x) - \mathbb{P}(Y_r \geqslant x + \varepsilon)\big) \\ &= -I_+(x)\end{aligned}$$

since

$$\lim_{r\to\infty} \frac{1}{v_r} \log \mathbb{P}(Y_r \geqslant x) = -I_+(x) > -I_+(x + \varepsilon) = \lim_{r\to\infty} \frac{1}{v_r} \log \mathbb{P}(Y_r \geqslant x + \varepsilon). \qquad \square$$

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