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# First Order Logic and Twin-Width in Tournaments

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## Abstract

We characterise the classes of tournaments with tractable first-order model checking. For every hereditary class of tournaments  $\mathcal{T}$ , first-order model checking either is fixed parameter tractable, or is AW[\*]-hard. This dichotomy coincides with the fact that  $\mathcal{T}$  has either bounded or unbounded twin-width, and that the growth of  $\mathcal{T}$  is either at most exponential or at least factorial. From the model-theoretic point of view, we show that NIP classes of tournaments coincide with bounded twin-width. Twin-width is also characterised by three infinite families of obstructions:  $\mathcal{T}$  has bounded twin-width if and only if it excludes at least one tournament from each family. This generalises results of Bonnet et al. on ordered graphs.

The key for these results is a polynomial time algorithm which takes as input a tournament  $T$  and computes a linear order  $<$  on  $V(T)$  such that the twin-width of the birelation  $(T, <)$  is at most some function of the twin-width of  $T$ . Since approximating twin-width can be done in FPT time for an ordered structure  $(T, <)$ , this provides a FPT approximation of twin-width for tournaments.

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## 1 Introduction

Tournaments can represent the outcome of a ranking of candidates, which need not be a total order. E.g., in the Condorcet voting paradox, three referees whose preference lists are  $(A, B, C)$ ,  $(B, C, A)$ , and  $(C, A, B)$ , lead to a cycle  $A \leftarrow B \leftarrow C \leftarrow A$  in the preference relation. Classical algorithmic problems arise from trying to choose a subset of winners: the DOMINATING SET (DS) problem asks for a subset  $D$  which is preferred to any other candidate, i.e. for any  $y \notin D$ , there is some  $x \in D$  which is preferred to  $y$ ; and the FEEDBACK VERTEX SET (FVS) problem asks to build a preference order by ignoring a subset of candidates.

These problems can be parameterized by the size  $k$  of the desired solution. A problem is *fixed parameter tractable* (FPT) if it admits an algorithm running in time  $O(f(k) \cdot n^c)$ , for some function  $f$  and constant  $c$ . It is known that FVS is FPT for tournaments [18], whereas DS is unlikely to be FPT. However general tournaments may not be representative of usual instances: for example, majority voting tournaments with a fixed number  $r$  of referees form a very restricted class. A cornerstone paper by Alon et al. [2], based on Vapnik-Chervonenkis dimension, shows that  $k$ -DS is trivially FPT on  $r$ -majority tournaments, because the size of a minimum dominating set is bounded by  $f(r)$ . This exemplifies how difficult problems can become easy on restricted classes, here bounded VC-dimension.



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To put these questions in a much broader perspective, remark that the previous problems can be expressed in first-order logic (FO). A  $k$ -DS is described by the formula

$$\exists x_1, x_2, \dots, x_k. \forall y. (y = x_1) \vee (y \rightarrow x_1) \vee \dots \vee (y = x_k) \vee (y \rightarrow x_k).$$

That  $k$ -FVS is also expressible in first-order logic is only true in tournaments, and not in general graphs. It is based on the simple remark that a tournament is acyclic if and only if it is transitive, i.e. it has no directed 3-cycle, which is easily expressed in FO. Thus  $k$ -DS and  $k$ -FVS are instances of the FO MODEL CHECKING (or FOMC) problem: given as input a tournament  $T$  and a first-order formula  $\phi$ , does  $T$  satisfies  $\phi$ ? FO model checking is difficult on the class of all graphs [12], and using back-and-forth FO encodings, one can show that it is just as hard on tournaments. We investigate which subclasses of tournaments admit an FPT algorithm for FO model checking.

### 1.1 Main results

We prove a dichotomy: in any class  $\mathcal{T}$  of tournaments (closed under subtournaments), FOMC is either FPT or AW[\*]-complete. The key of this dichotomy is *twin-width* (tw), a complexity parameter introduced by Bonnet et al. [7]: FOMC in  $\mathcal{T}$  is FPT if  $\mathcal{T}$  has bounded twin-width, and AW[\*]-complete otherwise. This dichotomy coincides with a model theoretic characterisation: the class  $\mathcal{T}$  has bounded twin-width if and only if it is NIP, meaning that arbitrary graphs cannot be described from tournaments in  $\mathcal{T}$  through a fixed FO formula. This equivalence of twin-width and NIP, called *delineation*, was conjectured for tournaments in [4]. The equivalence between NIP and FPT FO model checking also confirms the *nowhere FO dense* conjecture of Gajarský et al. [15] for tournaments.

Furthermore, the dichotomy for FO model checking coincides with a gap in the growth function of the class  $\mathcal{T}$ , i.e. the number of tournaments of  $\mathcal{T}$  on  $n$  vertices up to isomorphism. If  $\mathcal{T}$  has bounded twin-width, then its growth is at most  $2^{O(n)}$ , whereas it is at least  $(\lfloor n/2 \rfloor - 1)!$  when twin-width is unbounded. This exponential/factorial gap generalises the Marcus-Tardos theorem on permutations avoiding a fixed pattern [19]. It may also be compared to results of Boudabbous and Pouzet [9] which show that hereditary classes of tournaments have growth either at most polynomial or at least exponential.

► **Theorem 1.1.** *Let  $\mathcal{T}$  be a hereditary class of tournaments. Under the assumption  $\text{FPT} \neq \text{AW}[*]$ , the following are equivalent:*

1.  $\mathcal{T}$  has bounded twin-width,
2. FO model checking in  $\mathcal{T}$  is FPT,
3. FO model checking in  $\mathcal{T}$  is not AW[\*]-complete,
4.  $\mathcal{T}$  does not FO interpret the class of all graphs,
5.  $\mathcal{T}$  is monadically NIP, i.e. does not FO transduce all graphs,
6. the growth of  $\mathcal{T}$  is at most  $c^n$  for some constant  $c$ ,
7. the growth of  $\mathcal{T}$  is less than  $(\lfloor \frac{n}{2} \rfloor - 1)!$ .

These equivalences are completed by three minimal classes of obstructions, characterising twin-width by excluded substructures. These obstructions encode arbitrary permutations.

► **Theorem 1.2.** *There are three hereditary classes  $\mathcal{F}_=, \mathcal{F}_\leq, \mathcal{F}_\geq$  such that a hereditary class  $\mathcal{T}$  of tournaments has unbounded twin-width if and only if one of  $\mathcal{F}_=, \mathcal{F}_\leq, \mathcal{F}_\geq$  is a subclass of  $\mathcal{T}$ .*

Finally, we show that there is a fixed parameter tractable algorithm which approximates twin-width of tournaments up to some function.

► **Theorem 1.3.** *There is a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  and an FPT algorithm, which given as input a tournament  $T$ , produces either a witness  $\text{tww}(T) \leq f(k)$ , or a witness that  $\text{tww}(T) \geq k$ .*

These results can be generalised to oriented graphs with bounded independence number, and to relational structures consisting of a tournament augmented by arbitrary binary relations, see the full version of this paper.

## 1.2 Overview of the proof

A fundamental idea regarding twin-width is that upper bounds on twin-width can be witnessed by orders on vertices which exclude grid-like structures in the adjacency matrix. This appears in the founding works of Guillemot and Marx [17] and Bonnet et al. [7], and the relation between twin-width and orders has been deeply explored in [6]. However it is difficult to witness *lower bounds* on twin-width with this approach: one needs to somehow prove that *all* orders contain grids. To this purpose, we want to construct in any tournament  $T$  an order  $<$  which, if  $T$  has small twin-width, is a witness of this fact, i.e.  $\text{tww}(T, <) \leq f(\text{tww}(T))$ .

A tentative approach to obtain such an order is to describe it in FO logic. Indeed, FO transductions preserve twin-width up to some function [7, Theorem 39]. Thus, if  $\Phi$  is a transduction which on any tournament  $T$  gives some order  $<$ , then  $\text{tww}(T, <) \leq f(\text{tww}(T))$  as desired. With a few additional requirements, such as  $<$  being efficiently computable, it would be straightforward to obtain our results from the case of ordered graphs [6]. However this approach is impossible: to transduce a total order on the iterated lexicographic product of the 3-cycle with itself, one needs a first-order formula with size increasing in the number of iterations [3]. Remark that this counter-example has twin-width 1.

Instead, our approach is the following: we design a candidate total order  $<$  on  $T$ , computable in polynomial time. If the bi-relation  $(T, <)$  has small twin-width, we are done. On the other hand, if  $(T, <)$  has large twin-width, then its adjacency matrix w.r.t.  $<$  must contain a large high-rank grid by [6]. We then extract a subtournament  $T' \subset T$  which still has a substantial (but logarithmically smaller) high-rank grid, and in which  $<$  is roughly described by a FO transduction. This is enough to witness that  $T$  has large twin-width. Using Ramsey arguments, we extract from  $T'$  an obstruction  $\mathcal{F}_=$ ,  $\mathcal{F}_{\leq}$ , or  $\mathcal{F}_{\geq}$ . The construction of the order is remarkably simple: we consider a binary search tree (BST), i.e. a tree in which the left, resp. right, branch of a node  $x$  consists only of in-, resp. out-neighbours of  $x$ . The order  $<$  is the left-to-right order on nodes of the tree. To summarize, the crucial property is

► **Lemma 1.4.** *There is a function  $f$  such that for any tournament  $T$  and BST order  $<$  on  $T$ ,  $\text{tww}(T, <) \leq f(\text{tww}(T))$ .*

Lemma 1.4 implies Theorem 1.3: to approximate the twin-width of  $T$ , it suffices to compute any BST order, which takes polynomial time, and then apply the approximation algorithm for ordered structures [6, Theorem 2], which is FPT. This last algorithm produces either a contraction sequence (which is valid for  $(T, <)$  and a fortiori for  $T$ ), or a witness that  $(T, <)$  has large twin-width, which in turn implies that  $T$  has large twin-width by Lemma 1.4.

Our main technical result is about extracting the obstructions  $\mathcal{F}_=, \mathcal{F}_{\leq}, \mathcal{F}_{\geq}$ .

► **Theorem 1.5.** *Let  $\mathcal{T}$  be a hereditary class of tournaments with the property that there are tournaments  $T \in \mathcal{T}$  and BST orders  $<$  such that  $\text{tww}(T, <)$  is arbitrarily large. Then  $\mathcal{T}$  contains one of the classes  $\mathcal{F}_=, \mathcal{F}_{\leq}, \mathcal{F}_{\geq}$  as a subclass.*

Finally, the classes  $\mathcal{F}_=, \mathcal{F}_{\leq}, \mathcal{F}_{\geq}$  are complex in all the senses considered by Theorem 1.1.

► **Theorem 1.6.** *For each  $R \in \{=, \leq, \geq\}$ , the class  $\mathcal{F}_R$*

1. *has unbounded twin-width;*
2. *contains at least  $(\lfloor \frac{n}{2} \rfloor - 1)!$  tournaments on  $n$  vertices counted up to isomorphism;*
3. *contains at least  $(\lfloor \frac{n}{2} \rfloor - 1)! \cdot n!$  tournaments on vertex set  $\{1, \dots, n\}$  counted up to equality;*
4. *efficiently interprets the class of all graphs;*
5. *and has  $AW[*]$ -hard FO model checking problem.*

Theorems 1.5 and 1.6 together imply Theorem 1.2. They also imply Lemma 1.4 when applied to the class of tournaments with twin-width at most  $k$ : this class cannot contain any of  $\mathcal{F}_=$ ,  $\mathcal{F}_{\leq}$ ,  $\mathcal{F}_{\geq}$ , hence its tournaments must still have bounded twin-width when paired with BST orders. Finally, Theorems 1.2 and 1.6 directly imply that if  $\mathcal{T}$  is a hereditary class with unbounded twin-width, then  $\mathcal{T}$  satisfies none of the conditions of Theorem 1.1. The remaining implications of Theorem 1.1 – that is, when  $\mathcal{T}$  has bounded twin-width, all other conditions hold – follow from known results on twin-width. By [7, Theorem 1], FO model checking has an FPT algorithm when a witness of bounded twin-width is given. Combined with Theorem 1.3, this gives an FPT algorithm for classes of tournaments with bounded twin-width. By [7, Theorem 39], a class of structures with bounded twin-width cannot transduce all graphs. Finally, by [8, Corollary 7.3], a class of structures with bounded twin-width contains at most  $c^n$  structures on  $n$  vertices up to isomorphism, for some constant  $c$ .

### 1.3 Context and related parameters

It is interesting to compare twin-width to other classical complexity measures for tournaments. Bounded twin-width implies bounded VC-dimension, since classes with unbounded VC-dimension contain all possible bipartite subgraphs, which is against single-exponential growth. *Cutwidth* was introduced by Chudnovsky, Fradkin and Seymour [11] to study tournament immersions. Bounded cutwidth is certified by a vertex ordering which can be shown to exclude grids, hence it is also a witness of bounded twin-width. Another parameter, closely related to subdivisions in tournaments, is *pathwidth*, studied by Fradkin and Seymour [14]. Bounded pathwidth of tournaments implies bounded cliquewidth, which in turn also implies bounded twin-width, see [7]. Thus, we have the following chain of inclusions (if a parameter is bounded, all the ones listed after are also bounded): cutwidth, pathwidth, cliquewidth, twin-width, and VC-dimension. For more on the subject, see Fomin and Pilipczuk [13, 21].

Regarding the binary search tree method for ordering tournaments, it corresponds to the KWIKSORT algorithm of Ailon, Charikar and Newman for approximating the minimum feedback arc set [1]. A difference is that their result requires the BST to be randomly chosen, whereas arbitrary BST provide approximations of twin-width.

### 1.4 Organisation of the paper

Section 2 summarises our definitions and notations. In section 3 the classes  $\mathcal{F}_=$ ,  $\mathcal{F}_{\leq}$ ,  $\mathcal{F}_{\geq}$  of obstructions to twin-width are defined, and we prove Theorem 1.6. Section 4 defines binary search trees, the associated orders, and some related notions. We then prove a crucial lemma which, from a partition into intervals of a BST order, extracts some FO definable substructure. Section 5 proves Lemma 1.4 using the former lemma, combined with results of [6]. See the extended version of this paper [16] for the full proof of Theorem 1.5, which builds on that of Lemma 1.4.

## 2 Preliminaries

This section summarizes the notions and notations used in this work. For  $n \in \mathbb{N}$ , we denote by  $[n]$  the interval of integers  $\{1, \dots, n\}$ .

### 2.1 Tournaments, relational structures

A *tournament*  $T$  consists of a set of vertices  $V(T)$ , and for each  $u \neq v \in V(T)$ , an arc oriented either  $u \rightarrow v$  or  $v \rightarrow u$  (but not both). If  $x \in V(T)$ , then  $N^+(x) = \{y \mid x \rightarrow y\}$  and  $N^-(x) = \{y \mid y \rightarrow x\}$  are the *in-* and *out-neighbourhood* respectively. A tournament is *transitive* if it contains no directed cycle, in which case it defines a total order on its vertices. We call *chain* a subset  $X \subset V(D)$  which induces a transitive tournament.

Relational structures generalise graphs and tournaments. A relational *signature* is a finite set  $\Sigma$  of *relation symbols*  $R$ , each with an arity  $r \in \mathbb{N}$ . A  $\Sigma$ -structure consists of a domain  $A$  (vertices), and for each symbol  $R \in \Sigma$  of arity  $r$ , an interpretation  $R^S \subseteq A^r$  (hyperedges). E.g., tournaments and graphs are structures over a signature with a single binary relation. We restrict ourselves to *binary* structures, i.e. where all relation symbols have arity 2. An *ordered structure*  $S$  is a structure over a relation  $\Sigma$  with a special symbol  $<$ , whose interpretation  $<^S$  is a total order on the domain of  $S$ .

If  $S$  is a structure with domain  $A$  and  $B \subseteq A$ , the substructure  $S[B]$  induced by  $B$  has domain  $B$ , and interprets each relation  $R$  as the restriction of  $R^S$  to  $B$ . All classes of structures considered here are *hereditary*, i.e. closed under induced substructures.

### 2.2 Matrices

A matrix is a map  $M : R \times C \rightarrow \Gamma$ , where  $R, C$  are the *ordered sets* of rows and columns of the matrix, and  $\Gamma$  and its alphabet (usually,  $\Gamma = \{0, 1\}$ ). A submatrix of  $M$  is the restriction of  $M$  to some subsets of rows and columns. A *division*  $\mathcal{D}$  of  $M$  consist of partitions  $\mathcal{R}, \mathcal{C}$  of the rows and columns respectively into *intervals*. It is a  $k$ -division if the partitions have  $k$  parts each. A *cell* of the division is the submatrix induced by  $X \times Y$  for some  $X \in \mathcal{R}, Y \in \mathcal{C}$ . A  $k$ -grid in a 0,1-matrix is a division in which every cell contains a “1”.

For a tournament  $T$  and a total order  $<$  on  $V(T)$ , the adjacency matrix  $A_{(T, <)}$  has  $V(T)$  ordered by  $<$  as rows and columns, and contains a “1” at position  $(u, v)$  if and only if  $u \rightarrow v$ . This generalises to binary structures over any signature  $\Sigma$ , with  $\{0, 1\}^\Sigma$  as alphabet.

### 2.3 Orders, Quasi-orders

A *quasi-order*  $\preceq$  is a reflexive and transitive binary relation. The associated equivalence relation is  $x \sim y$  iff  $x \preceq y \wedge y \preceq x$ . The strict component of the quasi-order is  $x \prec y$  iff  $x \preceq y$  and  $y \not\preceq x$ . The quasi-order is *total* if for all  $x, y$ , either  $x \preceq y$  or  $y \preceq x$ . An *interval* of a quasi-order  $\preceq$  is a set of the form  $\{z \mid x \preceq z \preceq y\}$  for some  $x, y$ , called *endpoints*. An interval is a union of equivalence classes of  $\sim$ . Two subsets  $X, Y$  are *overlapping* if there exist  $x_1, x_2 \in X$  and  $y_1, y_2 \in Y$  such that  $x_1 \preceq y_1$  and  $x_2 \succeq y_2$ . Equivalently,  $X, Y$  are non-overlapping iff there are disjoint intervals  $I_X, I_Y$  such that  $X \subseteq I_X$  and  $Y \subseteq I_Y$ .

### 2.4 Permutations

We denote by  $\mathfrak{S}_n$  the group of permutations on  $n$  elements. The *permutation matrix*  $M_\sigma$  has a “1” at position  $(i, j)$  if and only if  $j = \sigma(i)$ . A permutation  $\tau$  is a *pattern* of  $\sigma$  if  $M_\tau$  is a submatrix of  $M_\sigma$ . We say that  $\sigma$  contains a  $k$ -grid if  $M_\sigma$  contains a  $k$ -grid. When this is the case, any permutation in  $\mathfrak{S}_k$  is a pattern of  $\sigma$ . For example, the permutation  $\sigma$  on  $k^2$  elements defined by  $\sigma(ki + j + 1) = kj + i + 1$  for any  $0 \leq i, j < k$  contains a  $k$ -grid.

A permutation can be represented as a *bi-order*, i.e. the superposition of two total orders. Precisely, for  $\sigma \in \mathfrak{S}_n$ , the structure  $\mathcal{O}_\sigma$  has domain  $[n]$ , and has for relations the natural order  $<$ , and the permuted order  $<_\sigma$  defined as  $i <_\sigma j$  if and only if  $\sigma(i) < \sigma(j)$ . Any bi-order is isomorphic to some  $\mathcal{O}_\sigma$ . Remark that  $\tau$  is a pattern of  $\sigma$  if and only if  $\mathcal{O}_\tau$  is isomorphic to an induced substructure of  $\mathcal{O}_\sigma$ . We write  $\mathcal{O}_\mathfrak{S}$  for the class of all finite bi-orders.

## 2.5 Twin-width

Twin-width, denoted e.g.  $\text{tw}(G)$ , is a complexity parameter defined on graphs, and more generally on binary structures. We refer the reader to [7] for the definition, based on contraction sequences – it will not be used in this work. Instead, we rely on the following characterisation by grid-like structures in adjacency matrices. Recall that a division of a matrix is a partition of rows and columns into intervals. We say that a matrix is *k-diverse* if it contains at least  $k$  different rows and  $k$  different columns – which is equivalent to having rank at least  $k'$  up to single-exponential bounds. Then, a *rank-k division* is a  $k$ -division in which every cell is  $k$ -diverse. Bonnet et al. proved

► **Theorem 2.1** ([6, Theorem 2]). *There are functions  $f, g$  such that for any graph (or binary structure)  $G$  and any order  $<$  on  $V(G)$ ,*

- *if  $\text{tw}(G, <) \geq f(k)$  then the matrix  $A_{(G, <)}$  has a rank- $k$  division, and*
- *if the matrix  $A_{(G, <)}$  has a rank- $g(k)$  division, then  $\text{tw}(G, <) \geq k$ .*

*Furthermore there is an FPT algorithm which given  $G$ ,  $<$ , and  $k$ , finds either a rank- $k$  division in  $A_{(G, <)}$  or a contraction sequence of width  $f(k)$  for  $(G, <)$ .*

## 2.6 First-order logic

Recall from the introduction that we are interesting in FO MODEL CHECKING: given as input a structure  $S$  and a first-order formula  $\phi$ , test if  $S \models \phi$ . We consider the complexity of this problem parametrized by the size  $|\phi|$ . In general, this problem is  $\text{AW}[*]$ -complete.

► **Theorem 2.2** ([12]). *FO MODEL CHECKING is  $\text{AW}[*]$ -complete on the class of all graphs.*

On the other hand, FO model checking is FPT for classes of structures with bounded twin-width, as long as a witness of twin-width is given.

► **Theorem 2.3** ([7, Theorem 1]). *Given a binary structure  $S$  on  $n$  vertices, a contraction sequence of width  $k$  for  $S$ , and a FO formula  $\phi$ , one can test if  $S \models \phi$  in time  $f(k, \phi) \cdot n$ .*

*Interpretations* are transformations of structures described using logical formulæ. For two relational signatures  $\Sigma, \Delta$ , a FO interpretation  $\Phi$  from  $\Sigma$  to  $\Delta$  consists of, for each relation  $R \in \Delta$  of arity  $r$ , a FO formula  $\phi_R(x_1, \dots, x_r)$  over the language  $\Sigma$ , and one last formula  $\phi_{\text{dom}}(x)$  again over  $\Sigma$ . If  $S$  is a  $\Sigma$ -structure, the result  $\Phi(S)$  is obtained by

- choosing the same domain as  $S$ ,
- interpreting  $R \in \Delta$  as  $\{(v_1, \dots, v_r) \mid S \models \phi_R(v_1, \dots, v_r)\}$ , the tuples satisfying  $\phi_R$ ,
- and finally taking the substructure induced by  $\{v \mid S \models \phi_{\text{dom}}(v)\}$ .

For instance, the square of a graph  $G$  has the same vertices as  $G$ , with an edge  $xy$  iff the distance of  $x$  and  $y$  in  $G$  is at most 2. This is a FO interpretation with edges defined by

$$\phi(x, y) = E(x, y) \vee (\exists z. E(x, z) \wedge E(z, y))$$

where  $E(x, x)$  denotes adjacency. The domain formula just “true” since we do not wish to delete vertices in this case. FO interpretations are closed under composition.

*Transductions* generalise interpretation with a non-deterministic coloring step. Let  $\Sigma^+$  be the signature obtained by adding  $r$  new *unary* relations  $C_1, \dots, C_r$  to  $\Sigma$ . If  $S$  is a  $\Sigma$ -structure, we denote by  $S^+$  the set of  $\Sigma^+$ -structures obtained from  $S$  by choosing an arbitrary interpretation of each  $C_i$  as a subset of  $V(S)$ . Now a FO transduction  $\Phi : \Sigma \rightarrow \Delta$  is described by the choice of  $\Sigma^+$  augmenting  $\Sigma$  with unary relations, and a FO interpretation  $\Phi_I$  from  $\Sigma^+$  to  $\Delta$ . The result of  $\Phi$  is the set of  $\Delta$ -structures  $\Phi(S) = \{\Phi_I(T) \mid T \in S^+\}$ . That is, the interpretation of the unary relations  $C_1, \dots, C_r$  on  $S$  are chosen non-deterministically, and then  $\Phi_I$  is applied.<sup>1</sup> Like interpretations, transductions can be composed.

Given classes  $\mathcal{C}, \mathcal{D}$  of structures, we say that  $\mathcal{C}$  *interprets* (resp. *transduces*)  $\mathcal{D}$  if there is a FO interpretation (resp. transduction)  $\Phi$  such that  $\Phi(\mathcal{C}) \supseteq \mathcal{D}$ . We furthermore say that  $\mathcal{C}$  *efficiently interprets*  $\mathcal{D}$  if there is also an algorithm which given as input  $D \in \mathcal{D}$ , finds in polynomial time some  $C \in \mathcal{C}$  such that  $\Phi(C) = D$ . It is straightforward to show that this additional condition gives an FPT reduction for model checking.

► **Lemma 2.4.** *If  $\mathcal{C}$  efficiently interprets  $\mathcal{D}$ , then there is an FPT reduction from FO MODEL CHECKING on  $\mathcal{D}$  to FO MODEL CHECKING on  $\mathcal{C}$ .*

Recall that  $\mathcal{O}_{\mathfrak{S}}$  denotes the class of bi-orders, which are encodings of permutations. The following is a folklore result, see e.g. [6, Lemma 10] for a very similar claim.

► **Lemma 2.5.** *The class  $\mathcal{O}_{\mathfrak{S}}$  of bi-orders efficiently interprets the class of all graphs.*

Thus, using Lemma 2.4 and Theorem 2.2, FO MODEL CHECKING on  $\mathcal{O}_{\mathfrak{S}}$  is AW[\*]-complete.

FO transductions also preserve twin-width, up to some function.

► **Theorem 2.6** ([7, Theorem 39]). *If  $\mathcal{S}$  is a class of binary structures with bounded twin-width and  $\Phi$  is a FO transduction defined on  $\mathcal{S}$ , then  $\Phi(\mathcal{S})$  also has bounded twin-width.*

A class of structures  $\mathcal{S}$  is said to be *monadically NIP* if  $\mathcal{S}$  does not transduce the class of all graphs. Theorem 2.6 implies that classes with bounded twin-width are monadically NIP. The weaker notion of (non-monadically) NIP also exists, however Braunfeld and Laskowski recently proved that NIP and monadically NIP are equivalent for hereditary classes [10].

## 2.7 Enumeration

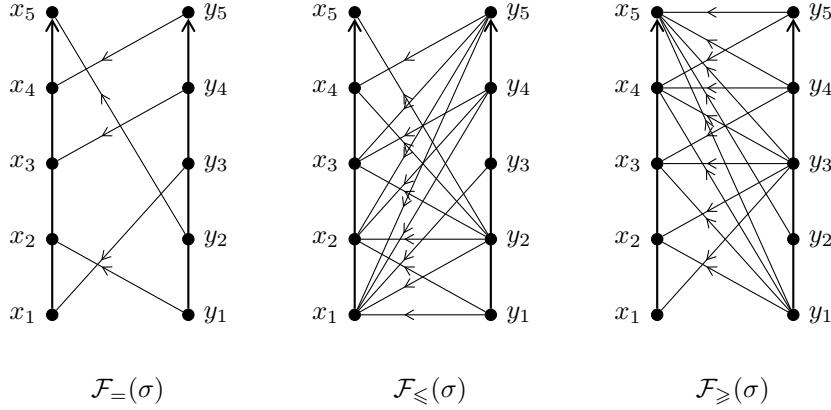
A class  $\mathcal{S}$  of graphs (or binary relational structures) is *small* if there exists  $c$  such that  $\mathcal{S}$  contains at most  $c^n \cdot n!$  structures on the vertex set  $[n]$ . For instance, the class of trees is small, and more generally proper minor closed classes of graphs are small as shown by Norine et al. [20]. This was further generalised to classes of bounded twin-width by Bonnet et al.

► **Theorem 2.7** ([5, Theorem 2.4]). *Classes of structures with bounded twin-width are small.*

## 3 Forbidden classes of tournaments

This section defines the three minimal classes  $\mathcal{F}_{=}$ ,  $\mathcal{F}_{\leq}$ , and  $\mathcal{F}_{\geq}$  of obstructions to twin-width in tournaments. Each of them corresponds to some encoding of the class of all permutations. For  $R \in \{=, \leq, \geq\}$  and any permutation  $\sigma$ , we will define a tournament  $\mathcal{F}_R(\sigma)$ . The class  $\mathcal{F}_R$  is the hereditary closure of all  $\mathcal{F}_R(\sigma)$ .

<sup>1</sup> Additional operations such as duplication of vertices are often allowed in transductions, but these will not be needed in this work.



■ **Figure 1** The three classes of obstructions to twin-width in tournaments. For readability, edges oriented from some  $x_i$  to some  $y_j$  have been omitted. Each class consists of some encoding of the class of all permutations, represented here with the permutation  $\sigma = 31452$ .

Let  $R \in \{=, \leq, \geq\}$ , and let  $\sigma \in \mathfrak{S}_n$  be a permutation on  $n$  elements. The tournament  $\mathcal{F}_R(\sigma)$  consists of  $2n$  vertices, called  $x_1, \dots, x_n, y_1, \dots, y_n$ . Let  $X = \{x_1, \dots, x_n\}$  and  $Y = \{y_1, \dots, y_n\}$ . Each of  $X, Y$  is a chain under the natural order, i.e. there is an edge from  $x_i$  to  $x_j$ , resp. from  $y_i$  to  $y_j$ , if and only if  $i < j$ . The edges between  $X$  and  $Y$  encode  $\sigma$  in a way specified by the relation  $R$ : there is an edge oriented from  $y_j$  to  $x_i$  if and only if  $i R \sigma^{-1}(j)$ . See Figure 1 for an example.

Thus in  $\mathcal{F}_=(\sigma)$  the edges oriented from  $Y$  to  $X$  form a matching which encodes  $\sigma$ . In  $\mathcal{F}_{\leq}(\sigma)$  and  $\mathcal{F}_{\geq}(\sigma)$ , these edges form a half-graph which orders  $X$  and  $Y$  by inclusion of neighbourhoods, so that the order on  $X$  is the natural one, and the order on  $Y$  encodes  $\sigma$ . Precisely, in  $\mathcal{F}_{\geq}(\sigma)$ , for any  $i, j \in [n]$ , we have

$$(N^-(x_i) \cap Y) \subseteq (N^-(x_j) \cap Y) \iff i \leq j \quad (1)$$

$$\text{and } (N^-(y_i) \cap X) \subseteq (N^-(y_j) \cap X) \iff \sigma^{-1}(i) \leq \sigma^{-1}(j), \quad (2)$$

while in  $\mathcal{F}_{\leq}(\sigma)$ , the direction of inclusions is reversed.

► **Lemma 3.1.** *For each  $R \in \{=, \leq, \geq\}$ , the class  $\mathcal{F}_R$  efficiently interprets the class  $\mathcal{O}_{\mathfrak{S}}$  of bi-orders. Precisely, there is an interpretation  $\Phi_R$ , and for any permutation  $\sigma \in \mathfrak{S}_n$ ,  $n \geq 2$ , there is a  $\sigma' \in \mathfrak{S}_{n+1}$  computable in polynomial time such that  $\mathcal{O}_{\sigma} = \Phi_R(\mathcal{F}_R(\sigma'))$ .*

**Proof.** We will first show that  $\mathcal{F}_R(\sigma)$  transduces  $\mathcal{O}_{\sigma}$ , and then how to remove the coloring step of the transduction by slightly extending  $\sigma$ .

Let  $X = \{x_1, \dots, x_n\}$  and  $Y = \{y_1, \dots, y_n\}$  be as in the definition of  $\mathcal{F}_R(\sigma)$ . The transduction uses coloring to guess the set  $X$ . It then defines two total orders on  $Y$ , which together describe  $\sigma$ . The first ordering is given by the direction of edges inside  $Y$ . The second depends on  $R$ :

- If  $R$  is  $=$ , edges oriented from  $Y$  to  $X$  are a perfect matching. The direction of edges in  $X$ , interpreted through this matching, defines the second order on  $Y$ .
- If  $R$  is  $\geq$  or  $\leq$ , the second order is inclusion, respectively inverse inclusion, of neighbourhoods intersected with  $X$ , see (2).

With the knowledge of which subset is  $X$ , each of these orders is clearly definable with a first-order formula. Finally, the transduction deletes vertices of  $X$ , leaving only  $Y$  and the two orders which encode  $\sigma$ .

Let us now show how to deterministically define the partition  $X, Y$ , at the cost of extending  $\sigma$  with one fixed value. Here, we assume  $n \geq 2$ .

- If  $R$  is  $=$ , define  $\sigma' \in \mathfrak{S}_{n+1}$  by  $\sigma'(n+1) = n+1$  and  $\sigma'(i) = \sigma(i)$  for any  $i \leq n$ . Then, in  $\mathcal{F}_=(\sigma')$ , the unique vertex with out-degree 1 is  $y_{n+1}$ . Its out-neighbour is  $x_{n+1}$ , which verifies  $X = N^-(x_{n+1}) \cup \{x_{n+1}\} \setminus \{y_{n+1}\}$ .
- If  $R$  is  $\leq$ , define  $\sigma'(1) = n+1$  and  $\sigma'(i+1) = \sigma(i)$ . Then  $y_{n+1}$  is the unique vertex with out-degree 1, and its out-neighbour is  $x_1$ , which satisfies  $X = N^+(x_1) \cup \{x_1\}$ .
- If  $R$  is  $\geq$ , we once again define  $\sigma'(1) = n+1$  and  $\sigma'(i+1) = \sigma(i)$ . Then  $x_1$  has in-degree 1, and its in-neighbour is  $y_{n+1}$ . The only other vertex which may have in-degree 1 is  $y_1$ , and this happens if and only if  $\sigma'(2) = 1$ . When this is the case, the direction of the edge  $x_1 \rightarrow y_1$  still allows to distinguish  $x_1$  in FO logic. Then, having defined  $x_1$ , we obtain  $y_{n+1}$  as its in-neighbour, which satisfies  $X = N^+(y_{n+1})$ .

In all three cases,  $\mathcal{F}_R(\sigma')$  contains two extra vertices compared to  $\mathcal{F}_R(\sigma)$ . These extra vertices can be uniquely identified in first-order logic, and can then be used to define  $X$ . Combined with the previous transduction, this gives an interpretation of  $\mathcal{O}_\sigma$  in  $\mathcal{F}_R(\sigma')$ . ◀

We can now prove that the classes  $\mathcal{F}_R$  are complex.

▶ **Theorem 1.6.** *For each  $R \in \{=, \leq, \geq\}$ , the class  $\mathcal{F}_R$*

1. *has unbounded twin-width;*
2. *contains at least  $(\lfloor \frac{n}{2} \rfloor - 1)!$  tournaments on  $n$  vertices counted up to isomorphism;*
3. *contains at least  $(\lfloor \frac{n}{2} \rfloor - 1)! \cdot n!$  tournaments on vertex set  $\{1, \dots, n\}$  counted up to equality;*
4. *efficiently interprets the class of all graphs;*
5. *and has  $AW[*]$ -hard FO model checking problem.*

**Proof.** Item 4 is straightforward by Lemmas 2.5 and 3.1, since efficient interpretations can be composed. By Lemma 2.4 and Theorem 2.2, this in turn implies Item 5. Item 3 implies Item 2 by a simple counting argument: in an isomorphism class, there are at most  $n!$  choices of labelling of vertices with  $\{1, \dots, n\}$  (less if there are automorphisms). Furthermore, each of Items 3 and 4 implies Item 1, by Theorem 2.7 and Theorem 2.6 respectively. Thus it only remains to prove Item 3.

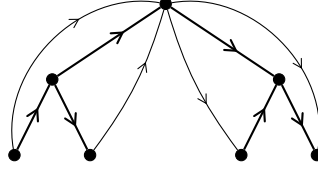
By Lemma 3.1, for any permutation  $\sigma \in \mathfrak{S}_n$  there is some  $\mathcal{F}_R(\sigma')$  on  $2n+2$  vertices such that  $\Phi_R(\mathcal{F}_R(\sigma')) = \sigma$ , where  $\Phi_R$  is a fixed interpretation. Since interpretations preserve isomorphism, it follows that there are at least  $n!$  non-isomorphic tournaments on  $2n+2$  vertices in  $\mathcal{F}_R$ . Furthermore, the arguments of Lemma 3.1, it is easy to show that these  $\mathcal{F}_R(\sigma')$  have no non-trivial automorphism. Thus, there are exactly  $(2n+2)!$  distinct labellings of  $\mathcal{F}_R(\sigma')$  with  $\{1, \dots, 2n+2\}$ . In total, this gives  $(2n+2)! \cdot n!$  distinct graphs on vertices  $\{1, \dots, 2n+2\}$  in  $\mathcal{F}_R$ , proving Item 3. ◀

Thus the classes  $\mathcal{F}_=, \mathcal{F}_\leq, \mathcal{F}_\geq$  are obstructions to fixed parameter tractability of FO model checking and twin-width. The rest of the paper shows that they are the only obstructions. One may also verify that all three are minimal, i.e. none of them is contained in another.

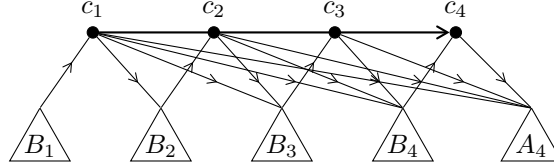
## 4 Binary search tree orders

This section presents the good order for twin-width in tournaments. It is based on *binary search trees* (BST), which we define in a tournament  $T$  as a rooted ordered binary tree  $S$  (meaning that each node has a left and right child, either of which may be missing), whose nodes are the vertices of  $T$ , and such that for any  $x \in S$

- the left child of  $x$  (if any) and its descendants are in  $N_T^-(x)$ , and
- the right child of  $x$  (if any) and its descendants are in  $N_T^+(x)$ , see Figure 2.



■ **Figure 2** A binary search tree in a tournament. The direction of omitted edges is not constrained.



■ **Figure 3** Example of construction of the quasi-order  $\preceq_C^+$ . The quasi-order is from left to right, and the triangles are equivalence classes. The direction of omitted edges (from  $B_i$  to  $B_j \cup \{c_j\}$  for  $i < j$ ) is not constrained. For  $\preceq_C^-$ , the direction of all edges would be reversed.

The *order* associated to  $S$ , denoted  $<_S$ , is the left-to-right order, i.e. the one which places a node  $x$  after its left child and its descendants, but before its right child and its descendants. Such an order is called a *BST order*.

Remark that because  $T$  is only a tournament and not an order as in a standard BST, there is no restriction on the direction of edges between the left and right subtrees of  $x$ . On the other hand, if  $x$  is an ancestor of  $y$ , then there is an edge oriented from  $x$  to  $y$  if and only if  $x <_S y$ . Thus we have

► **Remark 4.1.** In a tournament  $T$ , any branch  $B$  of a BST  $S$  forms a chain which coincides with  $<_S$ . That is, for  $x, y \in B$ , the edge in  $T$  is oriented from  $x$  to  $y$  if and only if  $x <_S y$ .

We will now define *chain quasi-orders*, which are FO definable quasi-orders with which we will approximate BST orders. Let  $C$  be a chain in  $T$ . Its *chain quasi-order*  $\preceq_C^+$  is defined as follows. Enumerate the vertices of  $C$  as  $c_1, \dots, c_k$  so that edges are oriented from  $c_i$  to  $c_j$  when  $i < j$ . Define  $A_i = \bigcap_{j \leq i} N^+(c_j)$ , and  $B_i = A_{i-1} \cap N^-(c_i)$ . Then each of  $B_1, \dots, B_k$  and  $A_k$  is an equivalence class of  $\preceq_C^+$ , and the classes are ordered as

$$B_1 \prec_C^+ c_1 \prec_C^+ B_2 \prec_C^+ c_2 \prec_C^+ \dots B_k \prec_C^+ c_k \prec_C^+ A_k,$$

see Figure 3. This can be seen as the left-to-right order of a partial BST consisting only of a single branch  $c_1, \dots, c_k$ , with  $c_1$  as root and  $c_k$  as leaf. It is also a coarsening of the lexicographic order: the latter would refine the order inside each class  $B_i$  using  $c_{i+1}, \dots, c_k$ .

The dual quasi-order  $\preceq_C^-$  is defined in the same way, but reversing the direction of all edges. Thus, we now enumerate  $C$  so that edges are from  $c_i$  to  $c_j$  when  $i > j$ , while  $A_i = \bigcap_{j \leq i} N^-(c_i)$  and  $B_i = A_{i-1} \cap N^+(c_i)$ . The rest of the definition is the same.

► **Lemma 4.2.** *There is a first-order transduction  $\Phi$  which non-deterministically computes any chain quasi-order. That is, for any tournament  $T$  and chain quasi-order  $\preceq_C^o$ ,  $(T, \preceq_C^o) \in \Phi(T)$ .*

**Proof.** The transduction first guesses  $C$  and  $o$ , and obtains the order within  $C$  from the edges of  $T$ . It is then simple to express the definition of  $\preceq_C^o$  in first-order logic. ◀

We now prove our main technical lemma on BSTs, which shows that BST orders can to some extent be approximated by chain quasi-orders.

► **Lemma 4.3.** *Let  $T$  be a tournament and  $S$  be a BST with order  $<_S$ . There is a function  $f(k) = 2^{O(k)}$  independent of  $T$  and  $S$  such that for any family  $\mathcal{P}$  of at least  $f(k)$  disjoint intervals of  $<_S$ , there is a chain  $C$  in  $T$ , an orientation  $o \in \{+, -\}$  and a subfamily  $\mathcal{P}' \subset \mathcal{P}$  such that  $|\mathcal{P}'| \geq k$  and such that the intervals of  $\mathcal{P}'$  are non-overlapping for  $\preceq_C^o$ .*

*Furthermore,  $C$ ,  $o$ , and  $\mathcal{P}'$  can be computed in linear time.*

**Proof.** Let  $T$  be a tournament,  $S$  a BST of  $T$  and  $<_S$  the corresponding order. Let  $\mathcal{P}$  be a family of at least  $f(k)$  disjoint intervals of  $<_S$ , where  $f(k) = 2^{O(k)}$  will be determined later.

We choose a branch  $B = b_0, \dots, b_p$  of  $S$  by the following process. First  $b_0$  is the root of  $S$ . For each (yet to be determined)  $b_i$ , let  $S_i$  be the subtree of  $S$  rooted in  $b_i$ , and define the weight  $w_i$  to be the number of classes of  $\mathcal{P}$  intersected by  $S_i$ . Then  $b_{i+1}$  is chosen to be the child of  $b_i$  which maximizes  $w_{i+1}$ . This choice ensures that

$$2w_{i+1} + 1 \geq w_i. \quad (3)$$

For each  $i < p$ , let  $d_i$  be the child of  $b_i$  other than  $b_{i+1}$  (sometimes  $d_i$  does not exist), and let  $D_i$  be the subtree of  $S$  rooted at  $d_i$  ( $D_i$  is empty if  $d_i$  does not exist). Furthermore, let  $L, R$  be the sets of vertices which are before, resp. after the leaf  $b_p$  in the order  $<_S$ . For any  $0 \leq i \leq j \leq p$ , let

$$L_{i,j} \stackrel{\text{def}}{=} \bigcup_{\substack{i \leq \ell < j \\ b_\ell \in L}} \{b_\ell\} \cup D_\ell, \quad \text{and} \quad R_{i,j} \stackrel{\text{def}}{=} \bigcup_{\substack{i \leq \ell < j \\ b_\ell \in R}} \{b_\ell\} \cup D_\ell.$$

Roughly speaking,  $L_{i,j}$ , resp.  $R_{i,j}$  consists of subtrees branching out of  $B$  on the left, resp. right, between  $b_i$  and  $b_j$ .

► **Claim 4.4.** For any  $i, j$ , the subtree  $S_i$  is partitioned into  $L_{i,j} <_S S_j <_S R_{i,j}$ .

**Proof.** Clearly  $L_{i,j}, S_j, R_{i,j}$  partition  $S_i$ . Furthermore, if  $\ell < j$  and  $b_\ell \in L$ , then  $b_\ell <_S S_j$ , and in turn  $D_\ell <_S b_\ell$ . This proves  $L_{i,j} <_S S_j$ , and symmetrically  $S_j <_S R_{i,j}$ . ◁

► **Claim 4.5.** For  $0 \leq i < j \leq p$ , if  $w_i \geq w_j + 3$ , then there is a part  $P \in \mathcal{P}$  such that  $P \subset L_{i,j}$  or  $P \subset R_{i,j}$ .

**Proof.** At least three parts of  $\mathcal{P}$  intersect  $S_i$  but not  $S_j$ . Since these three parts and  $S_i$  are all intervals of  $<_S$ , one of these parts, say  $P$ , is contained in  $S_i$ . Thus  $P$  is a subset of  $S_i$  but does not intersect  $S_j$ , which by Claim 4.4 implies  $P \subset L_{i,j}$  or  $P \subset R_{i,j}$ . ◁

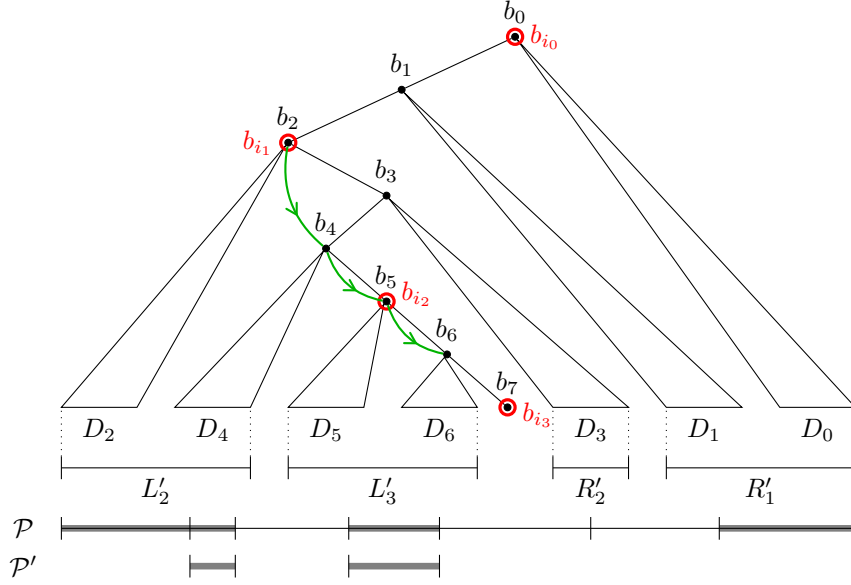
Construct a sequence  $i_0 < \dots < i_{2k}$  of indices in  $\{0, \dots, p\}$  inductively by taking  $i_0 = 0$ , and choosing  $i_{\ell+1}$  minimal such that  $w_{i_{\ell+1}} \leq w_{i_\ell} - 3$ . Using (3) and the minimality of  $i_{\ell+1}$ , we obtain for all  $\ell$  that  $2w_{i_{\ell+1}} + 1 \geq w_{i_{\ell+1}-1} > w_{i_\ell} - 3$ , hence

$$2w_{i_{\ell+1}} + 3 \geq w_{i_\ell}. \quad (4)$$

We can now define  $f$  by  $f(0) = 1$  and  $f(k+1) = 4f(k) + 9$ . By assumption,  $w_0 = |\mathcal{P}| \geq f(k)$ , and it follows from (4) that the construction of  $i_\ell$  can be carried out up to  $i_{2k}$ .

Define  $L'_\ell = L_{i_{\ell-1}, i_\ell}$ , and similarly  $R'_\ell = R_{i_{\ell-1}, i_\ell}$ , see Figure 4. By Claim 4.5, for any  $\ell \in [2k]$ , either  $L'_\ell$  or  $R'_\ell$  contains a part of  $\mathcal{P}$ . Thus, either there are at least  $k$  distinct  $L'_\ell$  containing a part of  $\mathcal{P}$ , or there are at least  $k$  distinct  $R'_\ell$  containing a part of  $\mathcal{P}$ . Assume the former case without loss of generality. We will now forget the vertices which are not in  $L$ .

Define  $C = L \cap B$ . By Remark 4.1, this is a chain, whose order coincides with  $<_S$ . Furthermore, at any node  $x$  of  $C$ , the branch  $B$  does descend on the right side, since  $x <_S b_p$ . Thus, the order in  $C$  also coincides with the ancestor-descendent order of  $S$ . (Remark here



■ **Figure 4** Sketch of the proof of Lemma 4.3. In the upper half, the BST  $T$  with the extracted branch  $B$ ; circled in red, the extracted subsequence  $b_{i_\ell}$ ; in green arrows, the chain  $C = B \cap L = \{b_2, b_4, b_5, b_6\}$ . Below the tree, from top to bottom: the partition in  $L'_\ell$  and  $R'_\ell$ ; the initial family (here partition)  $\mathcal{P}$ , with the parts contained in some  $L'_\ell$  or  $R'_\ell$  highlighted; the final family  $\mathcal{P}'$ , obtained by selecting a part of  $\mathcal{P}$  inside each possible  $L'_\ell$ .

that if we were in  $R$  instead of  $L$ , the order of  $C$  would be the inverse of the ancestor-descendant order.) Now, if  $C$  is enumerated as  $c_0 <_S \dots <_S c_t$ , and  $C_i$  is the subtree branching out on the left of  $c_i$ , defined similarly to  $D_i$ , then the chain quasi-order  $\preceq_C^+$  restricted to  $L$  is exactly

$$C_0 \prec_C^+ c_0 \prec_C^+ C_1 \prec_C^+ c_1 \prec_C^+ \dots \prec_C^+ c_t$$

where each subtree  $C_i$  is an equivalence class. (In  $R$ , we would instead use  $\preceq_C^-$ .) From this description, we obtain that any  $L_{i,j}$  is an interval of  $\preceq_C^+$  restricted to  $L$ .

For each  $L'_\ell$ , select a part of  $\mathcal{P}$  included in  $L'_\ell$  if any, and define  $\mathcal{P}'$  as the collection of selected parts. Thus  $\mathcal{P}' \subset \mathcal{P}$ , and we know from the choice of the family  $\{L'_\ell\}_{\ell \in [2k]}$  that  $|\mathcal{P}'| \geq k$ . Furthermore, if  $X \neq Y$  are parts of  $\mathcal{P}'$ , there are  $s \neq t$  such that  $X \subseteq L'_s$  and  $Y \subseteq L'_t$ . Since each  $L'_\ell$  is an interval of  $(L, \preceq_C^+)$ , this implies that  $X$  and  $Y$  are non-overlapping for  $\preceq_C^+$ . Thus  $\mathcal{P}'$  satisfies all desired properties.

Finally, given the BST  $S$  and the family  $\mathcal{P}$ , it is routine to compute the weights  $w_i$  of all nodes in  $S$  by a bottom-up procedure; this only requires to compute the left-most and right-most parts of  $\mathcal{P}$  intersecting each subtree. From this, one can in linear time choose the branch  $B$ , the indices  $i_\ell$ , the better side  $L$  or  $R$ , and finally compute  $C$  and  $\mathcal{P}'$ . ◀

## 5 BST orders witness twin-width

In this section, we prove Lemma 1.4, i.e. that BST orders are good for twin-width. The proof heavily uses model-theoretic results from [6]. Due to space constraints, the combinatorial proof of the stronger result Theorem 1.5 is omitted, see the extended version of this paper [16].

If  $\mathcal{T}$  is a class of tournaments, we denote by  $\mathcal{T}^{BST}$  the class of ordered tournaments  $(T, <_S)$  where  $T \in \mathcal{T}$  and  $<_S$  is the order of some BST  $S$  on  $T$ . With this notation, Lemma 1.4 can be restated as

► **Lemma 5.1.** *If  $\mathcal{T}$  is a hereditary class of tournaments with bounded twin-width, then  $\mathcal{T}^{BST}$  also has bounded twin-width.*

**Proof.** Fix  $\mathcal{T}$  a class of tournaments with twin-width at most  $t$ , and assume by contradiction that  $\mathcal{T}^{BST}$  has unbounded twin-width. Then by Theorem 2.1, for any  $k$  there is some  $(T, <_S) \in \mathcal{T}^{BST}$  whose adjacency matrix contains a rank- $k$  division. That is, there are partitions  $A_1, \dots, A_k$  and  $B_1, \dots, B_k$  of  $V(T)$  into intervals of  $<_S$  such that the adjacency matrix of any  $A_i$  versus  $B_j$  is  $k$ -diverse.

If  $k$  is chosen to be  $k = f(\ell)$  where  $f$  is the function of Lemma 4.3, then we obtain two chain quasi-orders  $\preceq^A, \preceq^B$  in  $T$ , and subfamilies  $A_{i_1}, \dots, A_{i_\ell}$  and  $B_{j_1}, \dots, B_{j_\ell}$  which are non-overlapping for  $\preceq^A$  and  $\preceq^B$  respectively. We can in fact assume that  $A_{i_1}, \dots, A_{i_\ell}$  are disjoint intervals of  $\preceq^A$ , by replacing them by their closure

$$\bar{A}_{i_t} \stackrel{\text{def}}{=} \{x \mid \exists y, z \in A_{i_t}, y \preceq^A x \preceq^A z\}$$

Let  $T^+$  be the structure  $T$  augmented by the quasi-orders  $\preceq^A$  and  $\preceq^B$ . In  $T^+$ , each interval  $\bar{A}_{i_t}$  can be described by its two endpoints. Using the terminology of [6, section 9], this means that  $\bar{A}_{i_1}, \dots, \bar{A}_{i_\ell}$  is a *definable disjoint family*. Naturally, the same holds for  $\bar{B}_{j_1}, \dots, \bar{B}_{j_\ell}$ . Finally,  $A_i$  versus  $B_j$  being  $k$ -diverse is a very special case of the model-theoretic notion of  $A$  having  $k$  *distinct  $\Delta$ -types over  $B$* , when  $\Delta$  consists only of the formula “being adjacent”.

Let  $\mathcal{T}^+$  denote the class of tournaments in  $\mathcal{T}$  augmented by any two chain quasi-orders. We have just proved that for arbitrary large  $k, \ell$ , there are structures  $T^+ \in \mathcal{T}^+$  containing two families of  $\ell$  disjoint subsets  $(\bar{A}_{i_t})_{t \in [\ell]}$  and  $(\bar{B}_{j_t})_{t \in [\ell]}$  definable by a fixed formula, and such that each  $\bar{A}_{i_t}$  has  $k$  distinct  $\Delta$ -types over each  $\bar{B}_{j_t}$ . That is, the class  $\mathcal{T}^+$  is *unrestrained* in the sense of [6, Definition 50]. By [6, Theorem 54], it follows that  $\mathcal{T}^+$  is not monadically NIP, hence has unbounded twin-width. But it follows from Lemma 4.2 that  $\mathcal{T}^+$  is obtained from  $\mathcal{T}$  by a first-order transduction, contradicting that  $\mathcal{T}$  has bounded twin-width. ◀

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