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# $\mathsf{L}^2$-HYPOCOERCIVITY FOR NON-EQUILIBRIUM KINETIC EQUATIONS

HELGE DIETERT

Abstract. The recent work [11] developed a general framework to show hypocoercivity for a stationary Gibbs state and allowed spatial degeneracy, confining potentials and boundary conditions. In this work, we show that the explicit energy approach in the weighted $\mathsf{L}^2$ space works for general non-equilibrium steady states and that it can be adapted to cases with weaker confinement leading to algebraic decay.

## 1. Introduction

1.1. **Motivation.** The work [11] studies the hypocoercivity for general linear kinetic equations allowing spatial degeneracy, confining potentials and boundary conditions, but focuses on Gibbs states where the transport and collision operator vanish separately for the stationary state.

The basic idea (see also the exposition [12]) is to use the transport to transfer the information in a good region with dissipation which is ensured by a geometric control condition generalising the condition found in Bernard and Salvarani [2] and Han-Kwan and Léautaud [19]. In the good region with dissipation, we can reduce the required control to the spatial density which is then controlled by a construction using the Bogovskiĭ operator inspired from [1] and similar to the works [6, 7, 10].

In this work, we are interested in the decay towards non-equilibrium stationary states which are created by a non-isothermal collision operator. Here the stationary state is not explicit and recent works on the decay rate have been achieved by Doeblin and Harris theorem, see e.g. Bernou [3] for the non-isothermal problem and Yoldaş [21] for a general review on the Harris theorem in hypocoercive equations. The energy estimate approach of Dolbeault, Mouhot and Schmeiser [13] has been extended to some non-explicit states in, e.g., [5, 16].

The aim of this work is to show that the approach with the Bogovoskiĭ operator covers these more general cases in a natural way leading to a robust and quantitative method. Moreover, we will show that it naturally covers weak confinement forces. As the treatment of spatial degeneracy and boundary conditions can be adapted from [11], we focus here on the decay without these difficulties for better readability.

1.2. **Considered problem.** Assume the evolution of a kinetic density $f = f(t, x, v)$ for $x, v \in \mathbb{R}^d$, $d \in \mathbb{N}$, given by the linear evolution

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (Gf) = Lf, \tag{1.1}$$

where $G = G(x, v)$ is an arbitrary smooth vector field modelling an external force and $L$ is a collision operator. Modelling a non-isothermal background, we assume a function $M = M(x, v)$ such that for every $x \in \mathbb{R}^d$, the function $v \mapsto M(x, v)$ is a probability distribution to which the collision operator $L$ drives the density $f$.

The aim is now to understand the decay behaviour of (1.1) in view of the degenerate dissipation as $L$ only changes the velocity distribution.

For our discussion, we take either a Fokker-Planck like collision operator $L = L_{\mathrm{FP}}$ defined by

$$L_{\mathrm{FP}} f := \nabla_v \left( M \cdot \nabla_v \left( \frac{f}{M} \right) \right) \tag{1.2}$$

or a BGK operator $L = L_{\mathrm{BGK}}$ defined by

$$L_{\mathrm{BGK}} f := \langle f \rangle M - f, \tag{1.3}$$

where we introduced the general definition $\langle \cdot \rangle$ for the spatial density, i.e.

$$\langle g \rangle (t,x) := \int_{v \in \mathbb{R}^d} g(t,x,v) \, \mathrm{d}v. \tag{1.4}$$

The first assumption is the existence of a stationary state $f_\infty$.

**Hypothesis 1** (Stationary state)**.** Assume a stationary state $f_\infty = f_\infty(x,v) \geq 0$ such that

$$v \cdot \nabla_x f_\infty + \nabla_v \cdot (G f_\infty) = L f_\infty \tag{1.5}$$

which is normalised as

$$\int_{x,v \in \mathbb{R}^d} f_\infty \, \mathrm{d}x \, \mathrm{d}v = 1 \tag{1.6}$$

and satisfies

$$\inf_x \inf_{|v| \leq 1} \frac{f_\infty(x,v)}{\langle f_\infty \rangle (x)} =: c_\infty > 0. \tag{1.7}$$

If $L = L_{\mathrm{FP}}$, assume also that

$$\sup_x \sup_{|v| \leq 1} \frac{|\nabla_v M|}{M} =: c_{\mathrm{MFP}} < \infty \tag{1.8}$$

and if $L = L_{\mathrm{BGK}}$ assume that

$$\sup_x \int_{|v| \leq 1} M^2 \, \frac{\langle f_\infty \rangle}{f_\infty} \, \mathrm{d}v =: c_{\mathrm{MBGK}} < \infty. \tag{1.9}$$

**Remark 1.1.** The stationary state can be found by several strategies. A first approach is by a perturbative argument of an equilibrium state with an explicit solution. Another source of non-equilibrium states in kinetic theory is obtained through fixed-point arguments like in Evans and Menegaki [15]. Yet another approach is given through the Krein-Rutman theory, see [17].

**Remark 1.2.** The conditions (1.7), (1.8) and (1.9) are concerned with velocities $|v| \leq 1$. The condition (1.7) is used to gain a control on the spatial density $\langle f \rangle$ and (1.8) and (1.9), respectively, are used to bound error terms.

The relative entropy is a natural distance towards a stationary state and in the linearised setting this motivates the use of the weighted $\mathsf{L}^2$ space $\mathcal{L}^2_\infty(\mathbb{R}^{2d})$ defined by the norm

$$\|g\|^2_{\mathcal{L}^2_\infty(\mathbb{R}^{2d})} := \int_{x,v \in \mathbb{R}^d} |g(x,v)|^2 \, \frac{\mathrm{d}x \, \mathrm{d}v}{f_\infty} \tag{1.10}$$

for a function $g : \mathbb{R}^{2d} \to \mathbb{R}$. If we consider a function $g = g(t,x,v)$ over a time interval $[0,T]$, we similarly use the weighted space $\mathcal{L}^2_\infty([0,T] \times \mathbb{R}^{2d})$ defined by

$$\|g\|^2_{\mathcal{L}^2_\infty([0,T] \times \mathbb{R}^{2d})} := \int_0^T \int_{x,v \in \mathbb{R}^d} |g(t,x,v)|^2 \, \frac{\mathrm{d}x \, \mathrm{d}v}{f_\infty} \mathrm{d}t. \tag{1.11}$$

Henceforth, we will always work in these weighted spaces.

The proof for exponential decay is based on a local coercivity in the velocity variable and a spatial coercivity for the density which is similar to the spatial diffusion in the fluid limit. The obtained decay rate is then constructive and depends on the coercivity of these two components.

**Hypothesis 2** (Local coercivity)**.** Assume that there exists $\lambda_1 > 0$ such that for all $x \in \mathbb{R}^d$ and all $g = g(v)$ it holds for the case $L = L_{\mathrm{FP}}$ that

$$\int_{v\in\mathbb{R}^d} \left| g - \langle g\rangle \frac{f_\infty}{\langle f_\infty\rangle} \right|^2 \frac{\mathrm{d}v}{f_\infty} \le \lambda_1 \int_{v\in\mathbb{R}^d} \left| \nabla_v \left( \frac{g}{f_\infty} \right) \right|^2 f_\infty \,\mathrm{d}v \tag{1.12}$$

and for the case $L = L_{\mathrm{BGK}}$ that

$$\begin{aligned} &\int_{v\in\mathbb{R}^d} \left| g - \langle g\rangle \frac{f_\infty}{\langle f_\infty\rangle} \right|^2 \frac{\mathrm{d}v}{f_\infty} \\ &\le \frac{\lambda_1}{2} \int_{v,v_*\in\mathbb{R}^d} \left( \frac{g(v)}{f_\infty(x,v)} - \frac{g(v_*)}{f_\infty(x,v_*)} \right)^2 f_\infty(x,v) M(x,v_*) \,\mathrm{d}v \,\mathrm{d}v_*. \end{aligned} \tag{1.13}$$

**Hypothesis 3** (Spatial coercivity)**.** Define the weight $w = w(x)$ by

$$w(x) = 1 + \frac{\int_{|v|\le 1} f_\infty |G(x,v)|^2 \,\mathrm{d}v}{\langle f_\infty\rangle} + \left( \int_{|v|\le 1} \left| \nabla_x \left( \frac{f_\infty}{\langle f_\infty\rangle} \right) \right| \mathrm{d}v \right)^2 + \left| \frac{\nabla_x \langle f_\infty\rangle}{\langle f_\infty\rangle} \right|^2. \tag{1.14}$$

Then assume that there exist a time $T > 0$ and a constant $\lambda_2$ such that for every $g = g(t,x)$, $t \in [0,T]$, $x \in \mathbb{R}^d$, there exists a vector field $\mathbf{F} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{1+d}$ satisfying

$$\begin{cases} \nabla_{t,x} \cdot \mathbf{F}(t,x) = g(t,x) & \text{in } (t,x) \in [0,T] \times \mathbb{R}^d, \\ \mathbf{F}(0,x) = \mathbf{F}(T,x) = 0 & \text{on } x \in \mathbb{R}^d \end{cases} \tag{1.15}$$

and

$$\int_{t,x\in[0,T]\times\mathbb{R}^d} \left[ |\mathbf{F}|^2 w + |\nabla \mathbf{F}|^2 \right] \frac{\mathrm{d}t\,\mathrm{d}x}{\langle f_\infty\rangle} \le \lambda_2 \int_{t,x\in[0,T]\times\mathbb{R}^d} |g|^2 \frac{\mathrm{d}t\,\mathrm{d}x}{\langle f_\infty\rangle}. \tag{1.16}$$

**Remark 1.3.** Under mild regularity assumptions on $\langle f_\infty\rangle$, the construction in [11, Section 2] shows that **(H3)** is implied by a suitable Poincaré inequality, see Appendix A. In the case that $w \approx 1 + |\nabla \log\langle f_\infty\rangle|^2$, the required Poincaré inequality becomes

$$\int |h|^2 \, w \, \langle f_\infty\rangle \,\mathrm{d}t\,\mathrm{d}x \lesssim \int |\nabla h|^2 \, \langle f_\infty\rangle \,\mathrm{d}t\,\mathrm{d}x \tag{1.17}$$

for all functions $h$ with $\int h \langle f_\infty\rangle = 0$.

Under these assumption, we show exponential decay towards the equilibrium distribution. As the overall mass is conserved, this is equivalent to saying that a perturbation with zero averages converges to zero.

**Theorem 1.** *Assume* **(H1)**, **(H2)**, **(H3)**. *Then there exist constructive constants* $\lambda, C > 0$ *such that a solution* $f = f(t,x,v)$ *of* (1.1) *with initial data* $f_{\mathrm{in}} = f_{\mathrm{in}}(x,v)$ *of zero average* $\int_{x,v} f_{\mathrm{in}} \,\mathrm{d}x\,\mathrm{d}v = 0$ *satisfies for all* $t > 0$

$$\|f\| \le C \mathrm{e}^{-\lambda t} \|f_{\mathrm{in}}\|. \tag{1.18}$$

**Remark 1.4** (Gibbs states)**.** The result also covers the traditional Gibbs states, where $M$ is a Gaussian with constant temperature, say $M(x,v) = (2\pi)^{-d/2}\mathrm{e}^{-v^2/2}$ and $G = -\nabla_x \phi$ for a potential $\phi$. Then $f_\infty = Z^{-1}\mathrm{e}^{-v^2-\phi}$ is a stationary state where $Z$ is a normalisation constant. In this case, the required Poincaré inequality is

$$\int |h|^2(1+|\nabla\phi|^2)\mathrm{e}^{-\phi}\,\mathrm{d}t\,\mathrm{d}x \lesssim \int |\nabla h|^2\mathrm{e}^{-\phi}\,\mathrm{d}t\,\mathrm{d}x, \tag{1.19}$$

which is classical in this setting and the gained weight $|\nabla\phi|^2$ is well-known.

In this setting the transport and collision operator vanish separately which Bouin, Dolbeault and Ziviani [4] calls factorised Gibbs state. As in this work, they show decay depending on a local coercivity and a spatial dispersion corresponding to (1.19).

**Remark 1.5.** The strategy can incorporate boundary conditions and more general collision operators, see [11] for such results. By using a transfer along the transport, also spatial degenerate settings can be controlled.

In the case of a weak confinement, the required local coercivity **(H2)** or the spatial coercivity **(H3)** fail for large velocities and large positions, respectively. Using a uniform moment bound, the weakened coercivity can be interpolated to obtain polynomial decay, see [4] for a review for Gibbs states.

Here we will show that the method can cover these cases as well. We will demonstrate this by considering the case of a missing spatial confinement, where we loose a weight in the spatial coercivity and assume a uniform moment bound.

**Hypothesis 3W** (Weak spatial coercivity)**.** With the weight $w$ from (1.14) assume that there exist a time $T>0$, constants $k>\ell>0$ and $\lambda_2, C_k>0$ such that for every $g=g(t,x)$, $t\in[0,T]$, $x\in\mathbb{R}^d$ there exists a vector field $\mathbf{F}:[0,T]\times\mathbb{R}^d\to\mathbb{R}^{1+d}$ satisfying

$$\begin{cases} \nabla_{t,x}\cdot\mathbf{F}(t,x) = g(t,x) & \text{in } (t,x)\in[0,T]\times\mathbb{R}^d, \\ \mathbf{F}(0,x)=\mathbf{F}(T,x)=0 & \text{on } x\in\mathbb{R}^d \end{cases} \tag{1.20}$$

and

$$\int_{t,x\in[0,T]\times\mathbb{R}^d} \left[|\mathbf{F}|^2 w + |\nabla\mathbf{F}|^2\right]\frac{\mathrm{d}t\,\mathrm{d}x}{\langle f_\infty\rangle} \le \lambda_2 \int_{t,x\in[0,T]\times\mathbb{R}^d} |g|^2(1+|x|^2)^\ell \frac{\mathrm{d}t\,\mathrm{d}x}{\langle f_\infty\rangle}. \tag{1.21}$$

Moreover, assume that we have a uniform moment bound of the spatial density of the solution as

$$\sup_{t\ge 0}\int_{x\in\mathbb{R}^d} \langle f\rangle^2(1+|x|^2)^k\,\mathrm{d}x \le C_k. \tag{1.22}$$

The relation of such a weighted Bogosvkiĭ estimate to the classical weighted Poincaŕe estimates is discussed in Appendix A.

**Theorem 2** (Polynomial decay)**.** *Assume* **(H1)**, **(H2)**, **(H3W)**. *Then there exist constructive constants $\lambda, C>0$ such that a solution $f=f(t,x,v)$ of (1.1) with initial data $f_{\mathrm{in}}=f_{\mathrm{in}}(x,v)$ of zero average $\int_{x,v} f_{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v = 0$ satisfies for all $t>0$*

$$\|f\| \le \frac{C}{(1+t)^{\frac{k}{2\ell}}}\|f_{\mathrm{in}}\|. \tag{1.23}$$

**Remark 1.6.** A similar statement can be obtained for a weaker coercivity of the velocity variable $v$.

1.3. **Literature.** The literature on hypocoercivity is enormous by now so that we cannot give a complete overview. An slightly old overview is given in the book by Villani [20] and a newer introduction can, e.g., be found in the thesis by Evans [14]. An overview of the convergence results for Gibbs states is in [4].

The use of the Bogovskiĭ operator is classical in the study of fluid dynamics, see e.g. the textbook [18]. However, there it is studied without a weight which was introduced in [11]. An overview of the used Poincaré inequalities in kinetic theory can, e.g., be found in the introduction of [4].

The field of hypocoercivity is still very active and, in particular, recently there has been several studies for general states using the Harris theorem, e.g., [3, 8, 9, 21] and using the energy estimates of Dolbeault, Mouhot and Schmeiser [13] in, e.g., [5, 16].

## 2. Proof of exponential decay

In this section, we will prove our main result Theorem 1. The evolution of (1.1) is given by a semigroup with generator $A$ defined as

$$Af = -v \cdot \nabla_x f - \nabla_v \cdot (Gf) + Lf. \tag{2.1}$$

The existence of such a semigroup is standard and the semigroup has as a core smooth functions. Therefore, it suffices to show the a priori estimates for the claimed result.

Over the function space $\mathcal{L}^2_\infty$ the adjoint operator $A^*$ is given by

$$A^* = v \cdot \nabla_x - f_\infty^{-1} v \cdot \nabla_x f_\infty + G \cdot \nabla_v - f_\infty^{-1} G \cdot \nabla_v f_\infty + L^*, \tag{2.2}$$

where the adjoint for the $L_{\mathrm{FP}}$ is

$$L^*_{\mathrm{FP}} f = \frac{f_\infty}{M} \nabla_v \cdot \left( M \nabla_v \left( \frac{f}{f_\infty} \right) \right) \tag{2.3}$$

and for $L_{\mathrm{BGK}}$

$$L^*_{\mathrm{BGK}} f = f_\infty \langle f \frac{M}{f_\infty} \rangle - f. \tag{2.4}$$

We can then split $A$ into the symmetric part $A_s = (A + A^*)/2$ and the antisymmetric part $A_a = (A - A^*)/2$. Using that $f_\infty$ is a stationary solution, i.e. (1.5) from **(H1)**, yields

$$A_s = -\frac{1}{2} f_\infty^{-1} L f_\infty + \frac{1}{2}(L + L^*). \tag{2.5}$$

2.1. **Dissipation.** For the evolution, we define the dissipation $\mathcal{D}$ as

$$\mathcal{D} := -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|f\|^2_{\mathcal{L}^2_\infty} = -\langle f, Af \rangle .$$

The antisymmetric part $A_a$ vanishes in the dissipation. From the symmetric part, we find for $L = L_{\mathrm{FP}}$ that

$$\begin{aligned}\mathcal{D} &= \frac{1}{2}\int_{x,v\in\mathbb{R}^d}\left[\frac{f^2}{f_\infty^2}L_{\mathrm{FP}}f_\infty - \frac{f}{f_\infty}L_{\mathrm{FP}}f - \frac{f}{f_\infty}L_{\mathrm{FP}}^*f\right]\mathrm{d}x\,\mathrm{d}v\\ &= \int_{x,v\in\mathbb{R}^d}\left[-\frac{f}{f_\infty}\nabla_v\left(\frac{f}{f_\infty}\right)\cdot M\nabla_v\left(\frac{f_\infty}{M}\right) + \nabla_v\left(\frac{f}{f_\infty}\right)\cdot M\nabla_v\left(\frac{f}{M}\right)\right]\mathrm{d}x\,\mathrm{d}v\\ &= \int_{x,v\in\mathbb{R}^d}\left|\nabla_v\left(\frac{f}{f_\infty}\right)\right|^2 f_\infty\mathrm{d}x\,\mathrm{d}v\end{aligned} \tag{2.6}$$

and for $L = L_{\mathrm{BGK}}$ that

$$\begin{aligned}\mathcal{D} &= \frac{1}{2}\int_{x,v\in\mathbb{R}^d}\left[\frac{f^2}{f_\infty^2}L_{\mathrm{BGK}}f_\infty - \frac{f}{f_\infty}L_{\mathrm{BGK}}f - \frac{f}{f_\infty}L_{\mathrm{BGK}}^*f\right]\mathrm{d}x\,\mathrm{d}v\\ &= \frac{1}{2}\int_{x,v\in\mathbb{R}^d}\left[\frac{f^2}{f_\infty} - f\frac{M}{f_\infty}\langle f\rangle - \langle f\frac{M}{f_\infty}\rangle f f^2\frac{M\langle f_\infty\rangle}{f_\infty^2}\right]\mathrm{d}x\,\mathrm{d}v\\ &= \frac{1}{2}\int_{x,v,v_*\in\mathbb{R}^d}\left(\frac{f(x,v)}{f_\infty(x,v)} - \frac{f(x,v_*)}{f_\infty(x,v_*)}\right)^2 f_\infty(x,v)M(x,v_*)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}v_*.\end{aligned} \tag{2.7}$$

2.2. **Decay criterion.** By the computation of the dissipation, we see that the dissipation is non-negative so that we have a contraction semigroup. For exponential decay, it then suffices to show that for a fixed time $T > 0$ there exists a constant $C$ such that any solution satisfies

$$\int_0^T \|f\|_{\mathcal{L}^2_\infty}^2\,\mathrm{d}t \le C\int_0^T \mathcal{D}\,\mathrm{d}t. \tag{2.8}$$

Indeed if this is true, we can use that the evolution is contractive to find

$$T\|f(T)\|_{\mathcal{L}^2_\infty}^2 \le C\int_0^T \mathcal{D}\,\mathrm{d}t = C(\|f_{\mathrm{in}}\|_{\mathcal{L}^2_\infty}^2 - \|f(T)\|_{\mathcal{L}^2_\infty}^2)$$

so that

$$\|f(T)\|_{\mathcal{L}^2_\infty}^2 \le \left(1+\frac{T}{C}\right)^{-1}\|f_{\mathrm{in}}\|_{\mathcal{L}^2_\infty}^2.$$

This shows the contraction over the time $T$ and thus exponential decay. Hence it only remains to prove (2.8).

2.3. **Reduction to spatial density.** In order to prove (2.8), we first use the local coercivity to reduce the problem to the local density. We find that

$$\begin{aligned}\int_0^T \|f\|_{\mathcal{L}^2_\infty}^2\,\mathrm{d}t &\le \int_0^T\int_{x,v\in\mathbb{R}^d} 2\left[\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle}\right)^2 + \left(f - \langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle}\right)^2\right]\frac{\mathrm{d}x\,\mathrm{d}v}{f_\infty}\mathrm{d}t\\ &\le 2\int_0^T\int_{x\in\mathbb{R}^d}\langle f\rangle^2\frac{\mathrm{d}x\,\mathrm{d}t}{\langle f_\infty\rangle} + 2\lambda_1\int_0^T\mathcal{D}\,\mathrm{d}t,\end{aligned}$$

where we used **(H2)** in the second inequality. Hence it remains to prove that

$$\int_0^T\int_{x\in\mathbb{R}^d}\langle f\rangle^2\frac{\mathrm{d}x\,\mathrm{d}t}{\langle f_\infty\rangle} \lesssim \int_0^T\mathcal{D}\,\mathrm{d}t. \tag{2.9}$$

2.4. **Control of spatial density.** In order to control the spatial density $\langle f\rangle$, we use that the dissipation gives a control on the gradient $\nabla_{t,x}$ of $\langle f\rangle$. Denoting the time with index 0, we will show that there exist fields $(K_i)_{i=0,\dots,d}$ and $(J_{ij})_{i,j=0,\dots,d}$ such that for $i=0,\dots,d$

$$\partial_i\left(\frac{\langle f\rangle}{\langle f_\infty\rangle}\right)=K_i+\sum_{j=0}^{d}\partial_j J_{ij} \tag{2.10}$$

and the fields are bounded for $i=0,\dots,d$ as (recall the weight $w$ from (1.14) from **(H3)**)

$$\int_0^T\int_{x\in\mathbb{R}^d}|K_i|^2\frac{\langle f_\infty\rangle}{w}\,\mathrm{d}x\,\mathrm{d}t\lesssim\int_0^T\mathcal{D}\,\mathrm{d}t \tag{2.11}$$

and for $i,j=0,\dots,d$

$$\int_0^T\int_{x\in\mathbb{R}^d}|J_{ij}|^2\langle f_\infty\rangle\,\mathrm{d}x\,\mathrm{d}t\lesssim\int_0^T\mathcal{D}\,\mathrm{d}t. \tag{2.12}$$

The proof is based on a simple lemma, where $B_1$ denotes the unit ball in $\mathbb{R}^d$.

**Lemma 3.** *Suppose $f_\infty$ satisfies (1.7) from* **(H1)** *and take the weight $w$ of (1.14) from* **(H3)**. *Then there exist functions $\psi_i=\psi_i(x,v)$ for $i=0,\dots,d$ with $\operatorname{supp}\psi\subset\mathbb{R}^d\times B_1$ satisfying the bound*

$$\sup_{x,v}\left(|\psi|+|\nabla_v\psi|+|\nabla_v^2\psi|+\frac{|\nabla_x\psi|}{\sqrt{w}}\right)\le C$$

*for a constant $C$ and the relation*

$$\int_{v\in\mathbb{R}^d}\frac{f_\infty}{\langle f_\infty\rangle}\psi_i\,\mathrm{d}v=\delta_{i0}$$

*and for $j=1,\dots,d$*

$$\int_{v\in\mathbb{R}^d}\frac{f_\infty}{\langle f_\infty\rangle}\psi_i\,v_j\,\mathrm{d}v=\delta_{ij}.$$

*Proof.* Let $\phi$ be a smooth cutoff to $B_1$ and use the notation $v_k=1$ if $k=0$. Then take the basis functions $e_k(v)=v_k\chi(v)$ for $k=0,\dots,d$ and consider for a fixed point $x$ in space the matrix

$$M_{ij}=\int_{v\in\mathbb{R}^d}\frac{f_\infty}{\langle f_\infty\rangle}v_ie_j\,\mathrm{d}v=\int_{v\in\mathbb{R}^d}\frac{f_\infty}{\langle f_\infty\rangle}v_iv_j\,\chi(v)\,\mathrm{d}v.$$

The matrix $M$ is symmetric and we find the required functions as

$$\psi_i=M_{ij}^{-1}e_j.$$

By the assumed lower bound (1.7) from **(H1)** we have a uniform lower bound on the eigenvalues of the matrix $M$. Hence $M^{-1}$ is uniformly bounded and the bounds on $\psi$, $\nabla_v\psi$ and $\nabla_v^2\psi$ follow. From (1.14) from **(H3)**, we find that $|\nabla_xM|\lesssim\sqrt{w}$ which implies the claimed bound for $\nabla_x\psi$. □

In order to obtain (2.10) for $i=0,\dots,d$ we note by linearity that

$$\begin{aligned}&\int_{v\in\mathbb{R}^d}(\partial_t-A_a)\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle}\right)\frac{\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v\\&=\int_{v\in\mathbb{R}^d}(\partial_t-A_a)\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle}-f\right)\frac{\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v+\int_{v\in\mathbb{R}^d}(\partial_t-A_a)f\,\frac{\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v.\end{aligned} \tag{2.13}$$

In order to compute the LHS of (2.13), first note that

$$A_a f = -v\cdot\nabla_x f - \nabla_v\cdot(Gf) + \frac{1}{2}f_\infty^{-1}\left(v\cdot\nabla_x f_\infty + \nabla_v\cdot(Gf_\infty)\right)f + \frac{1}{2}(Lf - L^*f).$$

Moreover, as the evolution preserves the overall mass $L^*f_\infty = 0$. As $L$ is only acting in the $v$ variable, we find

$$(L-L^*)\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle}\right) = \frac{\langle f\rangle}{\langle f_\infty\rangle}(L-L^*)f_\infty$$

so that

$$\begin{aligned}
&\int_{v\in\mathbb{R}^d}(\partial_t - A_a)\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle}\right)\frac{\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v\\
&= \int_{v\in\mathbb{R}^d}(\partial_t + v\cdot\nabla_x)\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle}\right)\frac{\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v + \int_{v\in\mathbb{R}^d}\nabla_v\left(G\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle}\right)\frac{\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v\\
&\quad - \int_{v\in\mathbb{R}^d}\frac{\langle f\rangle}{\langle f_\infty\rangle}\left(v\cdot\nabla_x f_\infty + \nabla_v\cdot(Gf_\infty)\right)\frac{\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v\\
&= \int_{v\in\mathbb{R}^d}(\partial_t + v\cdot\nabla_x)\left(\frac{\langle f\rangle}{\langle f_\infty\rangle}\right)f_\infty\frac{\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v\\
&= \partial_i\left(\frac{\langle f\rangle}{\langle f_\infty\rangle}\right),
\end{aligned}$$

where we used the property of $\psi_i$ from Lemma 3 in the last equality.

For the first term of the RHS of (2.13), we find

$$\begin{aligned}
&\int_{v\in\mathbb{R}^d}(\partial_t - A_a)\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle} - f\right)\frac{\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v\\
&= \int(\partial_t + v\cdot\nabla_x)\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle} - f\right)\frac{\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v + \int\nabla_v\cdot\left(G\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle} - f\right)\right)\frac{\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v\\
&\quad - \frac{1}{2}\int f_\infty^{-1}\left(v\cdot\nabla_x f_\infty + \nabla_v\cdot(Gf_\infty)\right)\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle} - f\right)\frac{\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v\\
&\quad - \frac{1}{2}\int(L-L^*)\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle} - f\right)\frac{\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v\\
&= \partial_t\int\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle} - f\right)\frac{\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v + \nabla_x\cdot\int\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle} - f\right)\frac{v\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v\\
&\quad - \int\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle} - f\right)v\cdot\nabla_x\left(\frac{\psi_i}{\langle f_\infty\rangle}\right)\mathrm{d}v - \int\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle} - f\right)\frac{G\cdot\nabla_v\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v\\
&\quad - \frac{1}{2}\int(f_\infty^{-1}Lf_\infty + L - L^*)\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle} - f\right)\frac{\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v.
\end{aligned}$$

For the second term on the RHS of (2.13), we use that $(\partial_t - A_a)f = A_s f$ so that

$$\int_{v\in\mathbb{R}^d}(\partial_t - A_a)f\,\frac{\psi_i}{\langle f\rangle}\,\mathrm{d}v = \frac{1}{2}\int(-f_\infty^{-1}Lf_\infty + L + L^*)f\,\frac{\psi_i}{\langle f_\infty\rangle}.$$

Hence we find a representation (2.10) with (using again the notation $v_j = 1$ for $j = 0$)

$$J_{ij} = \int\left(\langle f\rangle\frac{f_\infty}{\langle f_\infty\rangle} - f\right)\frac{v_j\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v \tag{2.14}$$

and

$$
\begin{aligned}
K_i = &- \int \left( \langle f \rangle \frac{f_\infty}{\langle f_\infty \rangle} - f \right) v \cdot \nabla_x \left( \frac{\psi_i}{\langle f_\infty \rangle} \right) \mathrm{d}v \\
&- \int \left( \langle f \rangle \frac{f_\infty}{\langle f_\infty \rangle} - f \right) \frac{G \cdot \nabla_v \psi_i}{\langle f_\infty \rangle} \mathrm{d}v \\
&- \frac{1}{2} \int (f_\infty^{-1} L f_\infty + L - L^*) \left( \langle f \rangle \frac{f_\infty}{\langle f_\infty \rangle} - f \right) \frac{\psi_i}{\langle f_\infty \rangle} \mathrm{d}v \\
&+ \frac{1}{2} \int (-f_\infty^{-1} L f_\infty + L + L^*) f \frac{\psi_i}{\langle f_\infty \rangle} \mathrm{d}v.
\end{aligned} \tag{2.15}
$$

We can bound $J_{ij}$ as

$$
|J_{ij}|^2 \langle f_\infty \rangle \leq \left( \int \left( \langle f \rangle \frac{f_\infty}{\langle f_\infty \rangle} - f \right)^2 \frac{\mathrm{d}v}{f_\infty} \right) \left( \int \frac{|v_j \psi_i|^2}{\langle f_\infty \rangle^2} f_\infty \, \mathrm{d}v \, \langle f_\infty \rangle \right).
$$

The second bracket is uniformly bounded so that the local coercivity **(H2)** implies the bound (2.12).

For the first term of $K_i$ we find the bound

$$
\begin{aligned}
&\left| \int \left( \langle f \rangle \frac{f_\infty}{\langle f_\infty \rangle} - f \right) v \cdot \nabla_x \left( \frac{\psi_i}{\langle f_\infty \rangle} \right) \mathrm{d}v \right|^2 \frac{\langle f_\infty \rangle}{w} \\
&\leq \left( \int \left( \langle f \rangle \frac{f_\infty}{\langle f_\infty \rangle} - f \right)^2 \frac{\mathrm{d}v}{f_\infty} \right) \left( \int |v|^2 \left| \nabla_x \left( \frac{\psi_i}{\langle f_\infty \rangle} \right) \right|^2 f_\infty \, \mathrm{d}v \, \frac{\langle f_\infty \rangle}{w} \right).
\end{aligned}
$$

By the control of $\nabla_x \psi$ from Lemma 3 and the definition of $w$, the local coercivity again yields the claimed bound of (2.11) for this term.

The second term of $K_i$ is bounded similarly using the definition of $w$.

In the case $L = L_{\mathrm{FP}}$, we find for the third term that

$$
\begin{aligned}
&\int (f_\infty^{-1} L f_\infty + L - L^*) \left( \langle f \rangle \frac{f_\infty}{\langle f_\infty \rangle} - f \right) \frac{\psi_i}{\langle f_\infty \rangle} \mathrm{d}v \\
&= 2 \int \nabla_v \left( f_\infty^{-1} \left( \langle f \rangle \frac{f_\infty}{\langle f_\infty \rangle} - f \right) \right) f_\infty \frac{\nabla_v \psi}{\langle f_\infty \rangle} \\
&\quad + 2 \int f_\infty^{-1} \left( \langle f \rangle \frac{f_\infty}{\langle f_\infty \rangle} - f \right) \frac{f_\infty}{M} \frac{\nabla_v (M \nabla \psi)}{\langle f_\infty \rangle}.
\end{aligned}
$$

For the first term note that $\nabla_v \left( f_\infty^{-1} \left( \langle f \rangle \frac{f_\infty}{\langle f_\infty \rangle} - f \right) \right) = \nabla_v (f / f_\infty)$ which is controlled by the dissipation in this case. For the second term use (1.8). Hence we find the required bound of (2.11) for this term.

In the case $L = L_{\mathrm{BGK}}$, we find for the third term that

$$
\begin{aligned}
&\int (f_\infty^{-1} L f_\infty + L - L^*) \left( \langle f \rangle \frac{f_\infty}{\langle f_\infty \rangle} - f \right) \frac{\psi_i}{\langle f_\infty \rangle} \mathrm{d}v \\
&= \int f_\infty^{-1} (\langle f_\infty \rangle M - f_\infty) \left( \langle f \rangle \frac{f_\infty}{\langle f_\infty \rangle} - f \right) \frac{\psi_i}{\langle f_\infty \rangle} \mathrm{d}v \\
&\quad - \int f_\infty \langle \frac{M}{f_\infty} \left( \langle f \rangle \frac{f_\infty}{\langle f_\infty \rangle} - f \right) \rangle \frac{\psi_i}{\langle f_\infty \rangle} \mathrm{d}v.
\end{aligned}
$$

The bound for the first term follows form **(H2)** and (1.9). The last term can be rewritten as

$$\begin{aligned}&\int f_\infty \langle \frac{M}{f_\infty}\left(\langle f\rangle \frac{f_\infty}{\langle f_\infty\rangle} - f\right)\rangle \frac{\psi_i}{\langle f_\infty\rangle}\,\mathrm{d}v\\ &= \left(\int_{v,v_*} \frac{M(v)f_\infty(x,v_*)}{\langle f_\infty\rangle}\left[\frac{f(x,v_*)}{f_\infty(x,v_*)} - \frac{f(x,v)}{f_\infty(x,v)}\right]\right)\left(\int \frac{f_\infty}{\langle f_\infty\rangle}\psi_i \mathrm{d}v\right)\end{aligned}$$

so that we find the claimed bound with the dissipation.

For the fourth term, note in the case $L = L_{\mathrm{FP}}$ that

$$\int (-f_\infty^{-1} L f_\infty + L + L^*) f \frac{\psi}{\langle f_\infty\rangle}\,\mathrm{d}v = -2\int \nabla\left(\frac{f}{f_\infty}\right) f_\infty \frac{\nabla\psi}{\langle f_\infty\rangle}\,\mathrm{d}v$$

which yields the claimed bound. In the case $L = L_{\mathrm{BGK}}$ we find

$$\begin{aligned}&\int (-f_\infty^{-1} L f_\infty + L + L^*) f \frac{\psi}{\langle f_\infty\rangle}\,\mathrm{d}v\\ &= \int_{v,v_*}\left[\frac{f(x,v_*)}{f_\infty(x,v_*)} - \frac{f(x,v)}{f_\infty(x,v)}\right]\left[M(x,v)f_\infty(x,v_*) + M(x,v_*)f_\infty(x,v)\right]\frac{\psi_i(v)}{\langle f_\infty\rangle}\,\mathrm{d}v\,\mathrm{d}v_*\end{aligned}$$

which again yields the claimed bound by the dissipation.

2.5. **Conclusion.** As we assume zero overall mass, we find that

$$\int_{[0,T]\times\mathbb{R}^d} \langle f\rangle\,\mathrm{d}t\,\mathrm{d}x = 0$$

so that we can apply **(H3)** to find $\mathbf{F}$ such that

$$\nabla_{t,x}\cdot\mathbf{F} = \langle f\rangle$$

and

$$\int_{t,x\in[0,T]\times\mathbb{R}^d} \left[|\mathbf{F}|^2 w + |\nabla\mathbf{F}|^2\right]\frac{\mathrm{d}t\,\mathrm{d}x}{\langle f_\infty\rangle} \le \lambda_2 \|\langle f\rangle\|^2_{\mathcal{L}^2_\infty([0,T]\times\mathbb{R}^d)}, \tag{2.16}$$

where we use the analogous weighted norm over $t$ and $x$ as

$$\|\langle f\rangle\|^2_{\mathcal{L}^2_\infty([0,T]\times\mathbb{R}^d)} = \int_{t,x\in[0,T]\times\mathbb{R}^d} \langle f\rangle^2 \frac{\mathrm{d}t\,\mathrm{d}x}{\langle f_\infty\rangle}.$$

Using (2.10) we therefore find

$$\begin{aligned}\|\langle f\rangle\|^2_{\mathcal{L}^2_\infty([0,T]\times\mathbb{R}^d)} &= \int_{[0,T]\times\mathbb{R}^d} \frac{\langle f\rangle}{\langle f_\infty\rangle}\nabla_{t,x}\cdot\mathbf{F}\,\mathrm{d}t\,\mathrm{d}x\\ &= -\int_{[0,T]\times\mathbb{R}^d} (K_i + \partial_j J_{ij}) F_i\,\mathrm{d}t\,\mathrm{d}x\\ &= -\int_{[0,T]\times\mathbb{R}^d} K_i\, F_i\,\mathrm{d}t\,\mathrm{d}x + \int_{[0,T]\times\mathbb{R}^d} J_{ij}\partial_j F_i\,\mathrm{d}t\,\mathrm{d}x\\ &\le \left(\int |K|^2 \frac{\langle f_\infty\rangle}{w}\right)^{1/2}\left(\int |\mathbf{F}|^2 \frac{w}{\langle f_\infty\rangle}\right)^{1/2}\\ &\quad + \left(\int |J|^2 \langle f_\infty\rangle\right)^{1/2}\left(\int |\nabla\mathbf{F}|^2 \frac{1}{\langle f_\infty\rangle}\right)^{1/2}.\end{aligned}$$

Using the bound (2.16) for $\mathbf{F}$ and (2.11) for $K$ and (2.12) for $J$ we therefore find

$$\|\langle f\rangle\|^2_{\mathcal{L}^2_\infty} \lesssim \int_0^T \mathcal{D}\,\mathrm{d}t.$$

This was the remaining bound to be proven (2.9) for the exponential decay. Hence the proof of exponential decay (Theorem 1) is finished.

## 3. Weak spatial confinement

In this section, we prove Theorem 2. The proof commences as for the exponential decay in Section 2. It runs through unchanged until the last step, where, by the loss of weight, we only find the control

$$\|\langle f\rangle\|^2_{\mathcal{L}^2_\infty([0,T]\times\mathbb{R}^d)} \lesssim \left(\int_0^T \mathcal{D}\,\mathrm{d}t\right)^{1/2} \|\langle f\rangle(1+x^2)^{\ell/2}\|_{\mathcal{L}^2_\infty([0,T]\times\mathbb{R}^d)}$$

By interpolating $\|\langle f\rangle(1+x^2)^{\ell/2}\|_{\mathcal{L}^2_\infty([0,T]\times\mathbb{R}^d)}$ between $\|\langle f\rangle\|_{\mathcal{L}^2_\infty([0,T]\times\mathbb{R}^d)}$ and $\|\langle f\rangle(1+x^2)^{k/2}\|_{\mathcal{L}^2_\infty([0,T]\times\mathbb{R}^d)}$, which we assume to be bounded in **(H3W)**, we find

$$\|\langle f\rangle\|^{2(1+a)}_{\mathcal{L}^2_\infty([0,T]\times\mathbb{R}^d)} \lesssim \int_0^T \mathcal{D}\,\mathrm{d}t,$$

where $a=\ell/k$.

Combining the estimates, we therefore find after the time $T$ that

$$\epsilon \min\left(\frac{\|f\|^2_{\mathcal{L}^2_\infty([0,T]\times\mathbb{R}^{2d})}}{T}, \left(\frac{\|f\|^2_{\mathcal{L}^2_\infty([0,T]\times\mathbb{R}^{2d})}}{T}\right)^{1+a}\right) \le \int_0^T \mathcal{D}\,\mathrm{d}t,$$

where we may assume that $\epsilon<1$. Then we find exponential decay as before as long as $\|f\|_{\mathcal{L}^2_\infty}\ge 1$ which is faster than the claimed algebraic decay.

Hence it remains to show the algebraic decay when $\|f\|_{\mathcal{L}^2_\infty}\le 1$. For this, let us denote the values at times $t=nT$, $n\in\mathbb{N}$, as

$$Y_n = \|f(t=nT)\|_{\mathcal{L}^2_\infty}$$

where we restrict to $Y_n\le 1$. Then the above estimate shows that

$$\epsilon Y_{n+1}^{2(1+a)} \le Y_n^2 - Y_{n+1}^2.$$

This inequality implies elementary that

$$Y_{n+1}^2 \le Y_n^2 - \epsilon 2^{-2(1+a)} Y_n^{2(1+a)}. \tag{3.1}$$

To see this, one can assume that we find a bound with $Y_{n+1}\ge Y_n/2$ and then replace $Y_{n+1}^{2(1+a)}$ by $2^{-2(1+a)}Y_n$. Then the assumption can be verified for the assumed bound.

Then (3.1) implies that

$$\frac{1}{Y_{n+1}^{2a}} \ge \frac{1}{Y_n^{2a}}\left(1-\epsilon 2^{-2(1+a)}Y_n^{2a}\right)^{-2a}$$

as $\left(1-\epsilon 2^{-2(1+a)}Y_n^{2a}\right)^{-2a} \ge 1+2a\epsilon 2^{-2(1+a)}Y_n^{2a}$ so that we have found

$$\frac{1}{Y_{n+1}^{2a}} \ge \frac{1}{Y_n^{2a}} + 2a\epsilon 2^{-2(1+a)}.$$

This implies that

$$Y_n \lesssim (1+n)^{-\frac{1}{2a}},$$

which is the claimed algebraic decay.

## Appendix A. Relation of Bogovskii inequality and Poincaré inequality

In this appendix, we discuss the relation of the existence of a suitable Poincaré inequality and the spatial assumption **(H3)** and **(H3W)**, respectively. For details and further discussion, we refer to [11].

We collect the basic construction in the following lemma, which incorporates possible weighted weaker forms.

**Lemma 4.** *Suppose a domain $\mathsf{U} \subset \mathbb{R}^d$, $d \in \mathbb{N}$, with a nice boundary and a potential $\Phi \in C^1(\mathbb{R}^d)$ satisfying $|\nabla^2 \Phi| \lesssim 1 + |\nabla \Phi|$, a weight $W \geq 0$ and constant $C_P > 0$ such that for all $h : \mathbb{R}^d \to \mathbb{R}$ with $\int_{\mathsf{U}} h \, \mathrm{e}^{-\Phi} = 0$ it holds that*

$$\int_{\mathsf{U}} |h|^2 \, W \, \mathrm{e}^{-\Phi} \leq C_P \int_{\mathsf{U}} |\nabla h|^2 \, \mathrm{e}^{-\Phi}.$$

*Then there exists a constant $C_B$ so that for every $g : \mathsf{U} \to \mathbb{R}$ with $\int_{\mathsf{U}} g = 0$, there exists a vector field $\mathbf{F} : \mathsf{U} \to \mathbb{R}^d$ such that*

$$\begin{cases} \nabla \cdot \mathbf{F} = g & \text{in } \mathsf{U} \\ \mathbf{F} = 0 & \text{on } \partial \mathsf{U} \end{cases}$$

*and*

$$\int_{\mathsf{U}} |\mathbf{F}|^2 \, \mathrm{e}^{\Phi} \leq C_B \int_{\mathsf{U}} |g|^2 \, \frac{\mathrm{e}^{\Phi}}{W}$$

*and*

$$\int_{\mathsf{U}} |\nabla \mathbf{F}|^2 \, \frac{\mathrm{e}^{\Phi}}{1 + |\nabla \Phi|^2} \leq C_B \int_{\mathsf{U}} |g|^2 \, \left( \frac{1}{W} + \frac{1}{1 + |\nabla \Phi|^2} \right) \mathrm{e}^{\Phi}.$$

For a discussion of possible boundaries, we refer to [11]. In this work, we only apply it to $\mathsf{U} = [0, T] \times \mathbb{R}^d$ where the boundary at $t = 0$ and $t = T$. For the details, we again refer to [11] and just sketch the main arguments with the more general weights here.

*Proof sketch.* We first solve the elliptic problem

$$\begin{cases} \nabla \cdot (\mathrm{e}^{-\Phi} \nabla h) = g & \text{in } \mathsf{U} \\ \mathbf{n} \cdot (\mathrm{e}^{-\Phi} \nabla h) = 0 & \text{on } \partial \mathsf{U} \end{cases}$$

for $h$ with $\int_{\mathsf{U}} h \, \mathrm{e}^{-\Phi} = 0$. By the assumed Poincaré inequality, we find a solution and defining $\mathbf{F}_0 = \mathrm{e}^{-\Phi} \nabla h$ it yields the bound

$$\int |\mathbf{F}_0|^2 \mathrm{e}^{\Phi} \lesssim \int |g|^2 \, \frac{\mathrm{e}^{\Phi}}{W}$$

Moreover, $\nabla \cdot \mathbf{F}_0 = g$.

We then find a covering $(B_k)_k$ of the domain $\mathsf{U}$ with a corresponding partition of unity $(\theta_k)_k$ where each component is of diameter comparable to $(1 + |\nabla \Phi|^2)^{-1/2}$ so that in each component $\mathrm{e}^{\Phi}$ is like a constant in the sense that the weight $\mathrm{e}^{\Phi}$ is only varying by a uniformly bounded factor. Moreover, we can ensure that $|\nabla \theta_k| \lesssim (1 + |\nabla \Phi|^2)^{1/2}$.

On each component consider $g_k = \nabla \cdot (\theta_k \mathbf{F}_0)$ which satisfies $\int g_k = 0$. Hence we can find on each component $B_k$ a vector field $\mathbf{F}_k$ vanishing outside the component $B_k$ so that $\nabla \cdot \mathbf{F}_k = g_k$ and

$$\|\mathbf{F}_k\|_{\mathsf{L}^2(B_k)} \lesssim \|\theta_k \mathbf{F}_0\|_{\mathsf{L}^2(B_k)} \quad \text{and} \quad \|\nabla \mathbf{F}_k\|_{\mathsf{L}^2(B_k)} \lesssim \|g_k\|_{\mathsf{L}^2(B_k)}.$$

Then $\mathbf{F} = \sum_k \mathbf{F}_k$ is the sought vector field. □

For verifying **(H3)** as in Remark 1.3, we apply the above lemma with $(t, x) \in [0, T] \times \mathbb{R}^d$ and take $\Phi$ such that $\mathrm{e}^{\Phi}$ is comparable to $w\langle f_\infty \rangle$ and chose $W = w$.

In the case of a weaker spatial confinement, we take as before $\Phi$ and only assume for $\ell > 0$ the weaker Poincaré inequality

$$\int_{\mathsf{U}} |h|^2 \frac{w}{(1+|x|^2)^\ell} \mathrm{e}^{-\Phi} \lesssim \int_{\mathsf{U}} |\nabla h|^2 \mathrm{e}^{-\Phi}.$$

Then the previous lemma with $W = w(1+|x|^2)^\ell$ implies **(H3W)**.

## References


[1] D. Albritton, S. Armstrong, J.-C. Mourrat and M. Novack. *Variational methods for the kinetic Fokker-Planck equation*. 2019. arXiv: 1902.04037v2 [math.AP].

[2] Étienne Bernard and Francesco Salvarani. "On the exponential decay to equilibrium of the degenerate linear Boltzmann equation". English. In: *J. Funct. Anal.* 265.9 (2013), pp. 1934–1954. DOI: 10.1016/j.jfa.2013.06.012.

[3] Armand Bernou. *Asymptotic Behavior of Degenerate Linear Kinetic Equations with Non-Isothermal Boundary Conditions*. 2023. arXiv: 2308.01694v1 [math.AP].

[4] Emeric Bouin, Jean Dolbeault and Luca Ziviani. $\mathrm{L}^2$ *Hypocoercivity methods for kinetic Fokker-Planck equations with factorised Gibbs states*. 2023. arXiv: 2304.12040v2 [math.AP].

[5] Emeric Bouin, Franca Hoffmann and Clément Mouhot. "Exponential decay to equilibrium for a fiber lay-down process on a moving conveyor belt". English. In: *SIAM J. Math. Anal.* 49.4 (2017), pp. 3233–3251. DOI: 10.1137/16M1077490.

[6] Giovanni Brigati. *Time averages for kinetic Fokker-Planck equations*. 2021. arXiv: 2106.12801v3 [math.AP].

[7] Giovanni Brigati and Gabriel Stoltz. *How to construct decay rates for kinetic Fokker–Planck equations?* 2023. arXiv: 2302.14506v1 [math.AP].

[8] José A. Cañizo, Chuqi Cao, Josephine Evans and Havva Yoldaş. "Hypocoercivity of linear kinetic equations via Harris's theorem". English. In: *Kinet. Relat. Models* 13.1 (2020), pp. 97–128. DOI: 10.3934/krm.2020004.

[9] Chuqi Cao. "The kinetic Fokker-Planck equation with general force". English. In: *J. Evol. Equ.* 21.2 (2021), pp. 2293–2337. DOI: 10.1007/s00028-021-00684-4.

[10] Yu Cao, Jianfeng Lu and Lihan Wang. "On Explicit $L^2$-Convergence Rate Estimate for Underdamped Langevin Dynamics". In: *Archive for Rational Mechanics and Analysis* 247.5 (2023). DOI: 10.1007/s00205-023-01922-4. URL: http://dx.doi.org/10.1007/s00205-023-01922-4.

[11] Helge Dietert, Frédéric Hérau, Harsha Hutridurga and Clément Mouhot. *Quantitative Geometric Control in Linear Kinetic Theory*. 2022. arXiv: 2209.09340v1 [math.AP].

[12] Helge Dietert, Frédéric Hérau, Harsha Hutridurga and Clément Mouhot. *Trajectorial hypocoercivity and application to control theory*. 2022. arXiv: 2210.13893v1 [math.AP].

[13] Jean Dolbeault, Clément Mouhot and Christian Schmeiser. "Hypocoercivity for linear kinetic equations conserving mass". English. In: *Trans. Am. Math. Soc.* 367.6 (2015), pp. 3807–3828. DOI: 10.1090/S0002-9947-2015-06012-7.

[14] Josephine Evans. "Deterministic and Stochastic Approaches to Relaxation to Equilibrium for Particle Systems". PhD thesis. University of Cambridge, 2019. DOI: 10.17863/CAM.30238. URL: https://www.repository.cam.ac.uk/handle/1810/282876.

[15] Josephine Evans and Angeliki Menegaki. "Existence of a nonequilibrium steady state for the nonlinear BGK equation on an interval". English. In: *Pure Appl. Anal.* 3.1 (2021), pp. 223–252. DOI: 10.2140/paa.2021.3.223.

[16] Josephine Evans and Angeliki Menegaki. *Properties of Non-Equilibrium Steady States for the non-linear BGK equation on the torus*. 2023. arXiv: 2305.01297v1 [math.AP].

[17] Claudia Fonte Sanchez, Pierre Gabriel and Stéphane Mischler. "On the Krein-Rutman theorem and beyond". working paper or preprint. May 2023. URL: https://hal.science/hal-04093201.

[18] Giovanni P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems*. English. 2nd ed. New York, NY: Springer, 2011. DOI: 10.1007/978-0-387-09620-9.

[19] Daniel Han-Kwan and Matthieu Léautaud. "Geometric analysis of the linear Boltzmann equation. I: Trend to equilibrium". English. In: *Ann. PDE* 1.1 (2015). Id/No 3, p. 84. DOI: 10.1007/s40818-015-0003-z.

[20] Cédric Villani. *Hypocoercivity*. English. Vol. 950. Mem. Am. Math. Soc. Providence, RI: American Mathematical Society (AMS), 2009. DOI: 10.1090/S0065-9266-09-00567-5.

[21] Havva Yoldaş. "On quantitative hypocoercivity estimates based on Harris-type theorems". English. In: *J. Math. Phys.* 64.3 (2023). Id/No 031101, p. 28. DOI: 10.1063/5.0089698.

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