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DIMENSION GAP FOR THE LIMIT SETS OF ANOSOV REPRESENTATIONS

FRANÇOIS LEDRAPPIER AND PABLO LESSA

ABSTRACT. We consider a representation of a finitely generated $\text{CAT}(-K)$ group Γ in $\text{SL}(d, \mathbb{R})$ that is Zariski dense and k -Anosov for at least two values of k . We exhibit a gap for the Minkowski dimension of minimal sets for the action of Γ on flags spaces.

1. INTRODUCTION

1.1. Main results. Let (Γ, \mathcal{S}) be a finitely generated group with a symmetric set of generators \mathcal{S} . Fix $K > 0$. The finitely generated group (Γ, \mathcal{S}) is said to be $\text{CAT}(-K)$ if the Cayley graph has the $\text{CAT}(-K)$ property for the word metric, i.e. geodesic triangles are thinner than in the hyperbolic space of constant curvature $-K$ (see [GdlH90]).

For $d \geq 2$, consider P a subset of $\{1, \dots, d-1\}$. A representation $\rho : \Gamma \rightarrow \text{SL}(d, \mathbb{R})$ is called a P -Anosov representation if there exists $c > 0$ such that

$$\frac{s_p(\rho(\gamma))}{s_{p+1}(\rho(\gamma))} > c \exp(c|\gamma|),$$

for all $\gamma \in \Gamma$ and $p \in P$. Here $|\gamma|$ denotes word length in Γ with respect to some fixed finite symmetric generating set \mathcal{S} and, for $1 \leq i \leq d$, $s_i(\gamma)$ denotes the i -th largest singular value of $\gamma \in \text{SL}(d, \mathbb{R})$ with respect to the usual inner product on \mathbb{R}^d . A representation is called *Borel-Anosov* if it is $\{1, \dots, d-1\}$ -Anosov.

Anosov representations have been introduced by F. Labourie ([Lab06]) and have been the subject of many studies (cf. in particular [GGKW17], [KLP16], [KLP17], [KLP18] and [BPS19]). It is widely accepted that non-trivial Anosov representations present the right analog of convex cocompact representations in higher rank. Many properties are suggested by this analogy. In another direction, we present in this note a phenomenon which is purely higher rank: there is a dimension gap for minimal invariant subsets for the action of $\rho(\Gamma)$ on the spaces of flags.

Let Q be a non-empty subset of $\{1, \dots, d-1\}$. We consider the space \mathcal{F}_Q of partial flags with signature Q , endowed with a rotationally invariant Riemannian metric. Let us recall the definition of the Minkowski dimension of a set $\Lambda \subset \mathcal{F}_Q$:

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for $\varepsilon > 0$, let $N(\Lambda, \varepsilon)$ be the covering number of the set Λ by balls of radius ε in the metric of \mathcal{F}_Q . The *Minkowski dimension* $\dim_M(\Lambda)$ is defined by

$$\dim_M(\Lambda) := \limsup_{\varepsilon \rightarrow 0} \frac{\log N(\Lambda, \varepsilon)}{\log(1/\varepsilon)}.$$

Our result is

Theorem 1.1 (Dimension gap for minimal sets of Anosov representations). *With the above notations, let (Γ, S) be a $CAT(-K)$ finitely generated group for some $K > 0$ and Λ be a minimal invariant set for the action of $\rho(\Gamma)$ on \mathcal{F}_Q . Assume that $\rho(\Gamma)$ is Zariski dense in $SL(d, \mathbb{R})$. Then, with $M := \#(Q \cap P)$,*

$$\dim_M(\Lambda) \leq \dim(\mathcal{F}_Q) - \frac{M-1}{2}.$$

In the case $d = 2$ the bound given by the above theorem is trivial. And, indeed, uniform discrete subgroups of $SL(2, \mathbb{R})$ have full limit set. A dimension gap for the limit set of classical Schottky groups was proved in [Doy88]. We note however, that there exist uniform discrete (and thus convex co-compact) subgroups of isometries of rank one symmetric spaces and thus there is no dimension gap for convex cocompact representations in rank one. On the other hand, there are many conditions that ensure that the limit set is a Lipschitz submanifold of the flag space (see the discussions in [PSW19]). Our result is easy in those cases.

If ρ is P -Anosov then it is also $P \cup P^*$ -Anosov for $P^* = \{d-p : p \in P\}$. Theorem 1.1 gives a non-trivial upper bound for the dimension of any minimal invariant set in \mathcal{F}_Q for all Q with $\#(Q \cap (P \cup P^*)) \geq 2$. In particular, as soon as the representation is k -Anosov for some $k \neq d/2$ and Zariski dense, then the codimension of the limit set in the full flag space is at least $1/2$.

There are a few simple reductions which allow us to state our core result. Firstly, we observe that the bundle $\pi_{Q, Q \cap P} : \mathcal{F}_Q \rightarrow \mathcal{F}_{Q \cap P}$ is smooth, so that for any closed $\Lambda \subset \mathcal{F}_{Q \cap P}$, we have

$$\dim_M((\pi_{Q, Q \cap P})^{-1}(\Lambda)) = \dim_M(\Lambda) + \dim \mathcal{F}_Q - \dim \mathcal{F}_{Q \cap P}.$$

So, it suffices to show theorem 1.1 for the case $Q = P$. In that case, the unique minimal invariant subset of \mathcal{F}_P is the limit set Λ_ρ .

Furthermore, we may assume that Γ is non-elementary since the limit set of an Anosov representation of an elementary (i.e. virtually cyclic) hyperbolic group is finite. Taking all this into account, we only have to prove

Theorem 1.2 (Dimension gap for limit sets of Anosov representations). *Let (Γ, S) be a non-elementary $CAT(-K)$ finitely generated group for some $K > 0$, $P \subset \{1, \dots, d-1\}$ with $M := \#P \geq 2$, ρ a P -Anosov representation of Γ in $SL(d, \mathbb{R})$ with $\rho(\Gamma)$ Zariski dense in $SL(d, \mathbb{R})$ and $\Lambda_\rho \subset \mathcal{F}_P$ the limit set of the representation ρ . Then,*

$$\dim_M(\Lambda_\rho) \leq \dim(\mathcal{F}_P) - \frac{M-1}{2}.$$

It seems natural to conjecture that a dimension gap will exist between the limit set and the corresponding space of flags, for any Zariski dense representation into

a reductive algebraic Lie group of rank $r \geq 2$ which is Anosov with respect to at least two simple roots.

Let $g \in \mathrm{SL}(d, \mathbb{R})$. The numbers $\lambda_i(g) := \lim_n \frac{1}{n} \log s_i(g^n)$ are the logarithms of the moduli of the eigenvalues of g . Apply the Oseledets theorem in the trivial case of one matrix $g \in \mathrm{SL}(d, \mathbb{R})$. The Oseledets decomposition is obtained from the Jordan decomposition by grouping together the eigenspaces corresponding to the eigenvalues with the same modulus. These moduli are the $\exp \lambda_i(g)$.

In our setting, if $\gamma \in \Gamma$, for all $p \in P$, $\lambda_p(\rho(\gamma)) - \lambda_{p+1}(\rho(\gamma)) \geq c\|\gamma\|$, where $\|\gamma\| := \lim_n \frac{1}{n} |\gamma^n|$. For all $\gamma \in \Gamma$, we obtain a coarser decomposition by grouping together, for $1 \leq k \leq M + 1$, the Oseledets subspaces associated with $\lambda_i(\rho(\gamma))$ with $\lambda_{p_{k-1}}(\rho(\gamma)) > \lambda_i(\rho(\gamma)) \geq \lambda_{p_k}(\rho(\gamma))$.¹ This associates to every $\gamma \in \Gamma$ an element $\eta(\rho(\gamma))$ of the space X_P of decompositions of $\mathbb{R}^d = E_1 \oplus \dots \oplus E_{M+1}$, with $\dim E_k = p_k - p_{k-1}$, namely

$$\eta(\rho(\gamma)) = E_1(\rho(\gamma)) \oplus \dots \oplus E_{M+1}(\rho(\gamma))$$

with the property that for all $k = 1, \dots, M + 1$, the space $E_k(\rho(\gamma))$ is generated by the eigenspaces of $\rho(\gamma)$ such that the modulus $\exp \lambda$ of the eigenvalue satisfies

$$\exp \lambda_{p_{k-1}}(\rho(\gamma)) > \exp \lambda \geq \exp \lambda_{p_k}(\rho(\gamma)).$$

Corollary 1.3. *Let (Γ, S) be a non-elementary $\mathrm{CAT}(-K)$ finitely generated group for some $K > 0$, $P \subset \{1, \dots, d - 1\}$ with $M := \#P$. With the above notations, if ρ is a P -Anosov, Zariski dense representation of Γ in $\mathrm{SL}(d, \mathbb{R})$, set $\Omega_\rho \subset X_P$ the closure of the set $\{\eta(\rho(\gamma)); \gamma \in \Gamma\}$. Then,*

$$\dim_M(\Omega_\rho) \leq \dim X_P - M + 1.$$

Indeed, corollary 1.3 is trivial for $d = 2$ and for $M = 1$. For $d \geq 3$ and $M \geq 2$, we show in section 2.3 why corollary 1.3 follows from theorem 1.2.

Remark 1.4. *In the case when the representation is Borel-Anosov, theorem 1.1, theorem 1.2 and corollary 1.3 hold for a general non-elementary finitely generated Gromov hyperbolic group Γ .*

The first non-trivial case for our results is when $d = 3$ and $P = \{1, 2\}$. So, let ρ be a Borel-Anosov representation of the $\mathrm{CAT}(-K)$ finitely generated non-elementary group (Γ, S) in $\mathrm{SL}(3, \mathbb{R})$ and we can consider the limit set Λ_ρ of the action of $\rho(\Gamma)$ on the space \mathcal{F} of complete flags in \mathbb{R}^3 . Besides, all $\gamma \in \Gamma$ admit three distinct eigenspaces written $(E_1(\rho(\gamma)), E_2(\rho(\gamma)), E_3(\rho(\gamma)))$ in the order of the absolute values of the eigenvalues. Let Ω_ρ denote the closure of the $\{(E_1(\rho(\gamma)), E_2(\rho(\gamma)), E_3(\rho(\gamma))), \gamma \in \Gamma\}$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. We obtain, from theorem 1.2 and corollary 1.3,

Corollary 1.5.

$$\dim_M(\Lambda_\rho) \leq \frac{5}{2} < 3 = \dim(\mathcal{F}), \quad \dim_M(\Omega_\rho) \leq 5 < 6 = \dim(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2).$$

¹With the convention that $p_0 = 0$, $\lambda_{p_0} = +\infty$ and $p_{M+1} := d$.

Let Γ be a surface group. For the Hitchin component, the dimension of the projective limit set is 1 (see [Lab06]) and our result is not new. It is new and unexpected, as far as the authors are aware, for the Zariski dense representations that lie in the Barbot component, i.e. the same connected component as the Teichmüller times Identity representations.

1.2. Strategy of the proof of theorem 1.2.

1.2.1. *Random walk entropy.* Let \mathcal{M} be the set of probability measures μ on Γ with countable support and with finite first moment, meaning

$$\sum_{\gamma \in \Gamma} \mu(\gamma) |\gamma| < +\infty,$$

such that the semi-group Γ_μ generated by the support of μ is non-elementary (i.e. contains two independent loxodromic elements).

It is well known that if $\mu \in \mathcal{M}$ then μ has finite Shannon entropy, meaning

$$H(\mu) = - \sum_{\gamma \in \Gamma} \mu(\gamma) \log(\mu(\gamma)) < +\infty.$$

The random walk entropy h_μ of $\mu \in \mathcal{M}$ is defined by

$$(1) \quad h_\mu = \lim_{n \rightarrow +\infty} \frac{1}{n} H(\mu^{*n}),$$

where μ^{*n} is the n -fold convolution of μ with itself.

1.2.2. *Lyapunov exponents.* Denote \mathfrak{a}^+ the cone

$$\mathfrak{a}^+ := \{a \in \mathbb{R}^d : a_1 \geq \dots \geq a_d, \sum a_i = 0\}.$$

For $g \in \mathrm{SL}(d, \mathbb{R})$, $\log S(g) \in \mathfrak{a}^+$, where

$$\log S(g) := (\log s_1(g), \dots, \log s_d(g)).$$

The Lyapunov exponents $\{\lambda_i(\rho_*\mu)\} \in \mathfrak{a}^+$ induced by ρ , are defined by

$$(2) \quad \lambda(\rho_*\mu) := \{\lambda_i(\rho_*\mu)\} = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{\gamma \in \Gamma} \mu^{*n}(\gamma) \log(S(\rho(\gamma))).$$

The limits exist by Fekete lemma applied to $\{\sum_{j \leq i} \sum_{\gamma \in \Gamma} \mu^{*n}(\gamma) \log(s_j(\rho(\gamma)))\}_{n \in \mathbb{N}}$.

1.2.3. *Lyapunov and Falconer dimension.* We say a pair $(i, j), 0 < i < j \leq d$ is separated by $p \in P$ if $i \leq p < j$. For each $p \in P$ let S_p be the set of pairs separated by p , and define $S(P) = \bigcup_{p \in P} S_p$. On the cone \mathfrak{a}^+ we define the roots

$$\alpha_{i,j}(a) = a_i - a_j,$$

for $i < j$ and notice that they are non-negative on \mathfrak{a}^+ . Lyapunov and Falconer dimensions are special values of Lyapunov and Falconer functionals on \mathfrak{a}^+ .

We define for $h \geq 0$ the Lyapunov functional on \mathfrak{a}^+ by

$$L_h^P(a) = \text{maximum of } \sum_{(i,j) \in S(P)} r_{i,j}$$

subject to $0 \leq r_{i,j} \leq 1$ and $\sum_{(i,j) \in S(P)} r_{i,j} \alpha_{i,j}(a) \leq h$.

For $r \geq 0$ we also define the Falconer functional on \mathfrak{a}^+ by

$$F_r^P(a) = \text{minimum of } \sum_{(i,j) \in S(P)} r_{i,j} \alpha_{i,j}(a)$$

subject to $0 \leq r_{i,j} \leq 1$ and $\sum_{(i,j) \in S(P)} r_{i,j} \geq r$.

Lemma 1.6 (Duality between Falconer and Lyapunov functionals). *Let $a \in \mathfrak{a}^+$ be such that $\alpha_{i,j}(a) > 0$ for all $(i,j) \in S(P)$. Then $r \mapsto F_r^P(a)$ is an increasing homeomorphism from $[0, D]$ to $[0, F_D(a)]$ where $D = \#S(P)$, and $h \mapsto L_h^P(a)$ is its inverse.*

In particular, for all $r, h \geq 0$ one has $F_r^P(a) \leq h$ if and only if $L_h^P(a) \geq r$.

Following [DO80] and [KY79], we define the Lyapunov dimension of $\rho_*\mu$ on \mathcal{F}_P by

$$\dim_{LY}^P(\rho_*\mu) := L_{h_\mu}(\lambda(\rho_*\mu)).$$

In the spirit of [Fal88], we define the Falconer dimension $\dim_F^P(\rho)$ of ρ relative to \mathcal{F}_P as the critical parameter $r \geq 0$ for convergence of the series

$$\Phi_\rho^P(r) := \sum_{\gamma \in \Gamma} \exp(-F_r^P(\rho(\gamma))).$$

1.2.4. *Proof of theorem 1.2.* The proof of theorem 1.2 is a consequence of the following results.

Theorem 1.7 (Upper bound on Lyapunov dimension). *For all $\mu \in \mathcal{M}$, one has*

$$\dim_{LY}^P(\rho_*\mu) \leq \dim(\mathcal{F}_P) - \frac{M-1}{2}.$$

Theorem 1.8 (Inequality between Falconer dimension and Lyapunov dimension). *Assume that (Γ, S) is a non-elementary $CAT(-K)$ finitely generated group for some $K > 0$ and that the representation $\rho(\Gamma)$ is Zariski dense dans $SL(d, \mathbb{R})$. Then,*

$$\dim_F^P(\rho) \leq \sup_{\mu \in \mathcal{M}} \dim_{LY}^P(\rho_*\mu)$$

Theorem 1.9 (Inequality between Minkowski and Falconer dimension). *One has*

$$\dim_M(\Lambda_\rho) \leq \dim_F^P(\rho).$$

The key observation behind theorem 1.7 is that the entropy h_μ is realized as the Furstenberg entropy of the action of $\rho(\Gamma)$ on the Grassmannian $\text{Gr}(p, \mathbb{R}^d)$ for any $p \in P$. This follows from the P -Anosov property and identification of the Poisson boundary and geometric boundary for μ -random walks on Γ ([Kai00]).

The inequality in theorem 1.7 is then a consequence of the inequality between Furstenberg entropy and the sum of the relevant exponents (see [LL23b] and section 3).

Theorem 1.8 is the converse of a general fact about discrete groups in $\mathrm{SL}(d, \mathbb{R})$. Let Γ be a finitely generated group, $\rho : \gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$ a Zariski dense representation such that $\rho(\Gamma)$ is discrete and P be a non-trivial subset of $\{1, \dots, d-1\}$. Observe that our definition of $\dim_F^P(\rho)$ does not depend on the fact that ρ has the P -Anosov property. Similarly, if μ is a probability measure on Γ with finite first moment, our definition of $\dim_{LY}^P(\rho_*\mu)$ depends only on P , on the entropy h_μ and on the exponents $\lambda_i(\rho_*\mu), i = 1, \dots, d$. Then,

Proposition 1.10. *Let Γ be a finitely generated group, $\rho : \gamma \rightarrow \mathrm{SL}(d, \mathbb{R})$ a Zariski dense representation such that $\rho(\Gamma)$ is discrete and P be a non-trivial subset of $\{1, \dots, d-1\}$. Let μ a probability measure on G such that $\sum_{g \in G} \log \|g\| \mu(g) < +\infty$. We have*

$$\dim_{LY}^P(\rho_*\mu) \leq \dim_F^P(\rho).$$

Proposition 1.10 is proven in section 2.2.2.

When the conclusion of theorem 1.8 holds, its conjunction with proposition 1.10 can be seen as a variational principle for the Falconer dimension:

$$(3) \quad \dim_F^P(\rho) = \sup_{\mu \in \mathcal{M}} \dim_{LY}^P(\rho_*\mu).$$

The proof of theorem 1.8 will appear in a paper by Yuxiang Jiao, Jialun Li, Wenyu Pan and Disheng Xu ([JLPX23]) in the case when the representation is Borel-Anosov. Remark 1.4 follows. In [JLPX23], given $\varepsilon > 0$, the authors construct a free semi-group in Γ that is rich enough that the uniform probability measure μ_ε on the generators satisfies $\dim_{LY}^P(\rho_*\mu_\varepsilon) \geq \dim_F^P(\rho) - \varepsilon$. The complete dominated splitting is used to control the almost additivity of the singular values. Here, we consider a general subset P and we assume that the representation $\rho(\Gamma)$ is Zariski dense in $\mathrm{SL}(d, \mathbb{R})$. We were not able to prove a similar almost additivity in the case of a partial dominated splitting. Instead, following [Qui02b], we construct a suitable Patterson-Sullivan measure on \mathcal{F}_P with the property that if ν is the stationary measure on \mathcal{F}_P of some random walk $\mu \in \mathcal{M}$, then, necessarily, $\dim_{LY}^P(\rho_*\mu) = \dim_F^P(\rho)$.

Thanks to a construction by Chris Connell and Roman Muchnik ([CM07]), such a probability measure μ exists if (Γ, S) is $\mathrm{CAT}(-K)$ for some $K > 0$. More general conditions ensuring the existence of such a measure μ are announced by [CEPR].

Finally, the proof of theorem 1.9 is done in section 5. Similarly to other previous works (see [Zha97], [BCH10], [PSW21], [GMT19], [FS20]), it rests on covering the limit set in $\partial\Gamma$ by shadows and counting the covering numbers of the images of these shadows in Λ_ρ .

1.3. Related facts. For convex cocompact groups in $\mathrm{SO}(1, n)$, Dennis Sullivan proved in [Sul79] that the Hausdorff dimension and the Minkowski dimension of the limit set coincide with the Poincaré exponent of the group (which is the Falconer

dimension in that case). This result can be extended to more general cases if one uses a conformally invariant metric on the boundary (see [Lin04], [DK22]). The problem is more delicate for the rotation invariant Riemannian metric (see e.g. [Duf17] for the $SU(1, n)$ case).

Following the pioneer work of Falconer ([Fal88]), there are many results comparing Hausdorff dimension and Falconer dimension for affine IFS. The variational principle (3) for IFS can be found in [Fal88] and [K04]. Under the form (3), it is due to Ian D. Morris and Pablo Shmerkin ([MS19]) in the case of dimension 2, Ian D. Morris and Çağrı Sert ([MS21]) in general. The properties of stationary measures for random walks on groups of matrices are very parallel.

Given theorems 1.8 and 1.9, a natural step to prove $\dim_H(\Lambda_\rho) = \dim_M(\Lambda_\rho) = \dim_F^P(\rho)$ for P -Anosov representations is finding necessary conditions on a given probability μ on Γ and a $\rho_*(\mu)$ -stationary probability measure ν on \mathcal{F}_P for having

$$(4) \quad \dim_{loc} \nu = \dim_{LY}^P(\rho_*\mu).$$

Here, $\dim_{loc} \nu$ is the ν -a.e. constant value of $\lim_{\varepsilon \rightarrow 0} \frac{\log \nu(B(x, \varepsilon))}{\log \varepsilon}$ (cf. [LL23b]). Relation (4) for IFS is a famous problem, with many deep contributions (most relevant for us are [Fal88], [JPS07], [Hoc14], [BHR19], [HR19], [FS22], [Rap22], [MS21]). For random walks on discrete subgroups of $SL(d, \mathbb{R})$, relation (4) is classical when $d = 2$ (see [HS17] for the general non-discrete case). [JLPX23] prove relation (4) for $d = 3$ when the representation ρ is Borel-Anosov and $\rho(\Gamma)$ is Zariski dense. They also observe that relation (4) may be wrong when the Zariski closure of $\rho(\Gamma)$ is conjugated to $SL(2, \mathbb{R})$. See [DS22] for the case of $SU(1, 2)$.

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2. RANDOM WALK ENTROPY

Let Γ be a non-elementary finitely generated word hyperbolic group, $P \subset \{1, \dots, d-1\}$ with $M := \#P \geq 2$, ρ a P -Anosov representation of Γ in $SL(d, \mathbb{R})$. We let \mathcal{M} denote the set of probability measures μ on Γ with finite first moment

$$\sum_{\gamma} |\gamma| \mu(\gamma) < +\infty,$$

and such that the semi-group Γ_μ generated by the support of μ contains two independent loxodromic elements.

In this section, we prove that the random walk entropy h_μ given by (1) coincides with the Furstenberg entropy (see below) of the stationary measure on the Gromov boundary of Γ and its images on adequate flag spaces. We also relate the Falconer dimension, and the random walk entropy to the growth indicator defined by [Qui02a]. To conclude the section, we prove corollary 1.3.

2.1. Furstenberg entropy. Recall that a probability measure ν on a compact space X on which Γ -acts continuously is said to be μ -stationary where μ is a probability measure on Γ , if and only if

$$\nu = \sum_{\gamma \in \Gamma} \mu(\gamma) \gamma_* \nu.$$

The Furstenberg entropy of a μ -stationary measure is defined as

$$\kappa(\mu, \nu) = \sum_{\gamma \in \Gamma} \mu(\gamma) \int_X \log \left(\frac{d\gamma_* \nu}{d\nu}(\gamma x) \right) d\nu(x),$$

or $+\infty$ if the Radon-Nikodym derivative in the integral does not exist for some γ in the support of μ .

In this subsection we show that for $\mu \in \mathcal{M}$, the Furstenberg entropy of the natural μ -stationary measures on the Gromov boundary $\partial\Gamma$, the Grassmannian manifolds of p -dimensional subspaces of \mathbb{R}^d for $p \in P$, and the space of partial flags \mathcal{F}_P all coincide with the random walk entropy h_μ .

This particular feature of Anosov representations is useful since the Furstenberg entropy is what occurs in the dimension formulas of [LL23b].

2.1.1. Stationary measure on the Gromov boundary $\partial\Gamma$.

Theorem 2.1. *For each $\mu \in \mathcal{M}$ there exists a unique μ -stationary measure ν_μ on the Gromov boundary $\partial\Gamma$ of Γ , and its Furstenberg entropy is given by $\kappa(\mu, \nu_\mu) = h_\mu$.*

Proof. If $\gamma_{-1}, \dots, \gamma_{-n}, \dots$ are i.i.d. random elements in Γ with common distribution μ . Then by [Kai00, Theorem 7.6] there exists a random limit point

$$X = \lim_{n \rightarrow +\infty} \gamma_{-1} \cdots \gamma_{-n} \in \partial\Gamma,$$

almost surely, and the distribution ν_μ of X is the unique μ -stationary measure on $\partial\Gamma$.

Furthermore, $(\partial\Gamma, \nu_\mu)$ is isomorphic to the Poisson boundary of (Γ, μ) by [Kai00, Theorem 7.7].

The Furstenberg entropy $\kappa(\mu, \nu_\mu)$ is equal to the difference between h_μ and the entropy of the random walk $\gamma_1, \gamma_1 \gamma_2, \dots$ conditioned on X (see [KV83, Corollary 2]). However, because $(\partial\Gamma, \nu)$ is the Poisson boundary this conditional entropy is zero ([Kai00, Theorem 4.3 and 4.5]) and therefore $h_\mu = \kappa(\mu, \nu_\mu)$ as claimed. \square

2.1.2. Boundary maps. Let $p \in P$ and $\gamma \in \Gamma$ be such that $s_p(\rho(\gamma)) > s_{p+1}(\rho(\gamma))$.

There exists a unique p -dimensional subspace $\xi^p(\gamma) \subset \mathbb{R}^d$ on which p -dimensional volume is most contracted by $\rho(\gamma)^{-1}$.

We let $\xi(\gamma) = (\{0\} \subset \xi^{p_1}(\gamma) \subset \dots \subset \xi^{p_M}(\gamma)) \subset \mathbb{R}^d \in \mathcal{F}_P$ where $P = \{p_1 < \dots < p_M\}$, whenever all $\xi^p(\gamma)$ are well defined.

Since ρ is P -Anosov this is the case outside of a finite subset of Γ .

We refer to [GdlH90, Chapitre 7] for basic properties of the Gromov compactification of Γ . We briefly recall that Γ with its word metric is a proper geodesic metric space.

The group Γ is word hyperbolic, so there exists $\delta > 0$ such that for every geodesic triangle each side is contained in the δ -neighborhood of the other two.

The Gromov boundary $\partial\Gamma$ is the set of equivalence classes of word geodesic rays in Γ , where two rays are equivalent if they are at bounded Hausdorff distance.

A basis of neighborhoods of a point $x \in \partial\Gamma$ in the Gromov compactification $\bar{\Gamma} = \Gamma \cup \partial\Gamma$ is defined by taking for each $C > 0$ the set

$$N(x, C) = \{x\} \cup \{y \in \bar{\Gamma} : \min_n |\alpha_n| > C \text{ for all geodesics } \alpha \text{ joining } x \text{ and } y\}.$$

The Gromov product may be used to give this compactification a class of Hölder equivalent metrics (see below).

Theorem 2.2. *The map ξ defined above extends Hölder continuously to the Gromov boundary $\partial\Gamma$. Furthermore, $\xi^p : \partial\Gamma \rightarrow \text{Gr}(p, \mathbb{R}^d)$ is injective for each $p \in P \setminus \{0, d\}$.*

Proof. For each $p \in P$, the extension of ξ^p to a continuous mapping from $\partial\Gamma$ to $\text{Gr}(p, \mathbb{R}^d)$ (the *Cartan property*) is due to [GGKW17] (see [Can], Proposition 30.3). Notice that if $p \in P$ then ρ is also $(d-p)$ -Anosov. From [BPS19, Proposition 4.9] one has that if $x, y \in \partial\Gamma$ are distinct then $\xi^p(x) \oplus \xi^{d-p}(y) = \mathbb{R}^d$. Since $\xi^{\min(p, d-p)}(x) \subset \xi^{\max(p, d-p)}(x)$, this implies, for $p \neq 0, d$, that $\xi^p(x) \neq \xi^p(y)$ so that ξ^p is injective as claimed.

The proof of the Hölder property follows from the proof of proposition 30.3 in [Can] by keeping track of the distance $\text{dist}(\xi^p(x), \xi^p(y))$, for $x, y \in \partial\Gamma$ and $p \in P$. Recall that the *Gromov product* $(x|y)$ defined, for $x, y \in \Gamma$, by

$$(x|y) := \frac{1}{2}(|x| + |y| - |x^{-1}y|)$$

extends to $\Gamma \cup \partial\Gamma$. M. Gromov defines a metric d on $\partial\Gamma$ such that (cf e.g. [CDP90], Ch. 11) there exist constants $c_0 > 0, C_0 > 1$, with

$$C_0^{-1} \exp(-c_0(x|y)) \leq d(x, y) \leq C_0 \exp(-c_0(x|y)).$$

Let $x, y \in \partial\Gamma$ and choose a geodesic $[x, y]$ from x to y . Let α be the point in the geodesic $[x, y]$ with the smallest word length. By the triangle inequality,

$$|\alpha| \geq (x|y).$$

Moreover, if we write $[x, y] = \{\alpha_n\}_{n \in \mathbb{Z}}$ with $\alpha_0 = \alpha$, then there is D such that

$$|\alpha_n| \geq (x|y) + D^{-1}|n| - D$$

By [Can], lemma 29.3, there is a constant C' such that, for all $\gamma \in \Gamma$ and s in the generating set \mathcal{S} , we have

$$\text{dist}(\xi^p(\rho(\gamma s)), \xi^p(\rho(\gamma))) \leq C' \frac{\sigma_{p+1}(\rho(\gamma))}{\sigma_p(\rho(\gamma))}.$$

Therefore, if $p \in P$, for all $\gamma \in \Gamma$ and s in the generating set \mathcal{S} ,

$$\text{dist}(\xi^p(\rho(\gamma s)), \xi^p(\rho(\gamma))) \leq C' c^{-1} \exp(-c|\gamma|).$$

It follows that

$$\text{dist}(\xi^p(\alpha), \xi^p(x)) \leq [C' c^{-1} \exp(cD) \sum_{n=0}^{\infty} \exp(-cD^{-1}n)] \exp(-c(x|y)).$$

The estimate is similar for $\text{dist}(\xi^p(\alpha), \xi^p(y))$. We have a constant D' such that

$$\text{dist}(\xi^p(x), \xi^p(y)) \leq \text{dist}(\xi^p(\alpha), \xi^p(x)) + \text{dist}(\xi^p(\alpha), \xi^p(y)) \leq D' d(x, y)^{c/c_0}.$$

□

2.1.3. Dynamical stationary measures. Fix $\mu \in \mathcal{M}$, let $\gamma_{-1}, \dots, \gamma_{-n}, \dots$ be i.i.d. random elements of Γ with common distribution μ , $\chi_1 > \dots > \chi_N$ be the distinct Lyapunov exponents of μ , and d_1, \dots, d_N their multiplicities. Let $A = \{d_1, d_1 + d_2, \dots, d_1 + \dots + d_{N-1}\}$. By Oseledec's theorem for each $a \in A$ there is a random limit

$$U_a = \lim_{n \rightarrow +\infty} \xi^a(\gamma_{-1} \cdots \gamma_{-n}),$$

almost surely.

The collection U of U_a for $a \in A$ is a random element in the space of flags of signature A . Its distribution, and the projections of its distribution to coarser partial flag spaces are what were called dynamical stationary measures in [LL23b]. In particular since $\lambda_p(\rho_*\mu) > \lambda_{p+1}(\rho_*\mu)$ for all $p \in P$ we have that $\xi_*\nu_\mu$ is the dynamical stationary measure on \mathcal{F}_P , and $\xi_*^p\nu_\mu$ is the dynamical stationary measure on the Grassmannian of p -dimensional subspaces of \mathbb{R}^d for each $p \in P$.

2.1.4. Furstenberg entropy.

Theorem 2.3. *For each $\mu \in \mathcal{M}$ one has*

$$h_\mu = \kappa(\mu, \xi_*\nu_\mu) = \kappa(\mu, \xi_*^p\nu_\mu),$$

for all $p \in P, p \neq 0, d$.

Proof. This is immediate from injectivity of the maps ξ^p given by theorem 2.2. □

2.2. Growth indicator function.

2.2.1. Growth function and Falconer dimension. Given a subcone $\mathcal{C} \subset \mathfrak{a}^+$ let $\Gamma_{\mathcal{C}}$ denote the set of elements with $\log S(\rho(\gamma)) \in \mathcal{C}$.

Given $a \in \mathfrak{a}^+$ we define the growth indicator at a by

$$(5) \quad \psi_\Gamma(a) = \|a\| \inf_{\mathcal{C}} \limsup_{T \rightarrow \infty} \frac{1}{T} \log \#\{\gamma \in \Gamma, \log S(\rho(\gamma)) \in \mathcal{C}, \|\log S(\rho(\gamma))\| \leq T\}$$

where the infimum is over open cones containing a .

Theorem 2.4 (Quint, [Qui02a]). *Assume that $\rho(\Gamma)$ is Zariski dense. The growth indicator function ψ_Γ is concave and $\{\psi_\Gamma > 0\}$ is the interior of the limit cone defined by*

$$\mathcal{L}_\Gamma = \lim_{t \rightarrow +\infty} \frac{1}{t} \log S(\rho(\Gamma)),$$

where the limit is taken in the Hausdorff topology on closed subsets of \mathfrak{a}^+ and $\log S(\rho(\Gamma))$ is the collection of all $\log S(\rho(\gamma))$, $\gamma \in \Gamma$.

Recall that we defined the Falconer dimension $\dim_F^P(\rho)$ as the critical value of the series $r \mapsto \sum_{\gamma \in \Gamma} \exp(-F_r^P(\log(S(\rho(\gamma)))))$.

Lemma 2.5. *The Falconer dimension $\dim_F^P(\rho)$ is given by*

$$\begin{aligned} \dim_F^P(\rho) &= \inf\{r > 0 : \psi_\Gamma(a) < F_r^P(a) \text{ for all } a \in \mathcal{L}_\Gamma\} \\ &= \sup\{r > 0 : \psi_\Gamma(a) \geq F_r^P(a) \text{ for some } a \in \mathcal{L}_\Gamma\} \end{aligned}$$

2.2.2. *Proof of proposition 1.10.* The following inequality is essentially due to Guivarc'h.

Lemma 2.6 (Fundamental inequality). *For all $\mu \in \mathcal{M}$ one has*

$$h_\mu \leq \psi_\Gamma(\lambda(\rho_*\mu)).$$

Proof. Indeed, for all $\varepsilon > 0$ and n large enough, there are at least $\exp(n(h_\mu - \varepsilon))$ elements γ of Γ with $\log(S(\rho(\gamma)))$ close to $n\lambda(\rho_*\mu)$. \square

Recall that, given $\mu \in \mathcal{M}$ we defined the Lyapunov dimension by

$$\dim_{LY}^P(\rho_*\mu) = L_{h_\mu}(\lambda(\rho_*\mu)).$$

Proposition 1.10 is the following statement

$$\sup_{\mu \in \mathcal{M}} \dim_{LY}^P(\rho_*\mu) \leq \dim_F^P(\rho).$$

Proof. Fix $\mu \in \mathcal{M}$ and let $r_0 = \dim_{LY}^P(\rho_*\mu) = L_{h_\mu}(\lambda(\rho_*\mu))$. From the duality between Lyapunov and Falconer functionals we have $F_{r_0}(\lambda_\mu) = h_\mu$.

We have $h_\mu \leq \psi_\Gamma(\lambda(\rho_*\mu))$ by the fundamental inequality, i.e. $F_{r_0}(\lambda(\rho_*\mu)) \leq \psi_\Gamma(\lambda(\rho_*\mu))$. Hence, $r_0 \leq \dim_F^P(\rho)$ by lemma 2.5. \square

2.3. Eigenspace splitting. In this subsection, we show that theorem 1.2 implies corollary 1.3.

Consider, for $\gamma \in \Gamma$, the P -decomposition $\eta(\rho(\gamma)) = E_1(\rho(\gamma)) \oplus \dots \oplus E_{M-1}(\rho(\gamma))$. By the P -Anosov property, the angles between $E_1(\rho(\gamma)) \oplus \dots \oplus E_k(\rho(\gamma))$ and $E_{k+1}(\rho(\gamma)) \oplus \dots \oplus E_{M-1}(\rho(\gamma))$ are uniformly bounded from below by some positive number for all $1 \leq k \leq M-2$ and for all γ but a finite number.

Recall that we denote, for all γ but a finite number, $\xi(\gamma) = \{0\} \subset \xi^{p_1}(\gamma) \subset \dots \subset \xi^{p_{M-2}}(\gamma) \subset \mathbb{R}^d \in \mathcal{F}_P$. In the same way, $\xi(\gamma^{-1}) = \{0\} \subset \xi^{d-p_{M-2}}(\gamma^{-1}) \subset \dots \subset \xi^{d-p_1}(\gamma^{-1}) \subset \mathbb{R}^d \in \mathcal{F}_{P^*}$. It follows from [GGKW17], lemma 2.26² that the unstable flag $f(\rho(\gamma)) \in \mathcal{F}_P$,

$$f(\rho(\gamma)) := \{0\} \subset E_1(\rho(\gamma)) \subset \dots \subset E_1(\rho(\gamma)) \oplus \dots \oplus E_{M-2}(\rho(\gamma)) \subset \mathbb{R}^d$$

and the stable flag $f'(\rho(\gamma)) \in \mathcal{F}_{P^*}$,

$$f'(\rho(\gamma)) := \{0\} \subset E_{M-1}(\rho(\gamma)) \subset \dots \subset E_2(\rho(\gamma)) \oplus \dots \oplus E_{M-1}(\rho(\gamma)) \subset \mathbb{R}^d$$

are given by

$$f(\rho(\gamma)) = \lim_{n \rightarrow +\infty} \xi(\gamma^n), \quad f'(\rho(\gamma)) = \lim_{n \rightarrow -\infty} \xi(\gamma^n).$$

In particular, $f(\rho(\gamma)) \in \Lambda_\rho \subset \mathcal{F}_P$, and $f'(\rho(\gamma)) \in \Lambda_\rho^*$, where Λ_ρ^* is the limit set associated to the representation ρ in \mathcal{F}_{P^*} . Thus, $(f, f') \in (\Lambda_\rho \times \Lambda_\rho^*)' \subset (\mathcal{F}_P \times \mathcal{F}_{P^*})'$, where $'$ indicates that the pairs of flags are in general position.

The set of pairs $(f, f') \in (\mathcal{F}_P \times \mathcal{F}_{P^*})'$ such that the angles between the opposite partial spaces are bounded from below by some positive number form a compact subset of $(\mathcal{F}_P \times \mathcal{F}_{P^*})'$ and the mapping that associates to (f, f') the underlying decomposition is uniformly Lipschitz on that compact set. For almost all $\gamma \in \Gamma$, the decomposition $\eta(\rho(\gamma))$ is obtained by this mapping from $(f(\rho(\gamma)) \times f'(\rho(\gamma)))$. Therefore the closure Ω_ρ of the set of splittings $\{\eta(\rho(\gamma)), \gamma \in \Gamma\}$ is the image of a compact subset of $(\Lambda_\rho \times \Lambda_\rho^*)'$ by a Lipschitz mapping and its Minkowski dimension is at most the Minkowski dimension of $\Lambda_\rho \times \Lambda_\rho^*$. Since the Minkowski dimension of a product is the sum of the Minkowski dimensions, corollary 1.3 follows from theorem 1.2 applied to the representation ρ which is both P -Anosov and P^* -Anosov.

3. ENTROPY GAP AND PROOF OF THEOREM 1.7

In this section we write λ_i for $\lambda_i(\rho_*\mu)$. The purpose of this section is to show that for all $\mu \in \mathcal{M}$, the random walk entropy h_μ is far from the sum

$$\sum_{(i,j) \in S(P)} \lambda_i - \lambda_j = F_{\#S(P)}^P(\lambda(\rho_*\mu)),$$

which would be needed to maximize the value of the Lyapunov dimension $\dim_{LY}^P(\rho_*\mu)$.

This is the main estimate needed for the proof of theorem 1.7.

Lemma 3.1 (Main lemma). *For each $\mu \in \mathcal{M}$ it holds that*

$$h_\mu + \frac{M-1}{2} (\lambda_1 - \lambda_d) \leq \sum_{(i,j) \in S(P)} \lambda_i - \lambda_j.$$

²The argument in the non-hyperbolic case goes back to David Ruelle's proof of Oseledets theorem ([Rue79], cf. [Led84] proposition 3.2 or [Sim15], pages 141–142 for details).

3.1. Proof of theorem 1.7 assuming lemma 3.1. Given $\mu \in \mathcal{M}$ by definition we have, setting $b_{i,j} := 1 - a_{i,j}$,

$$(6) \quad \dim_{LY}^P(\rho_*\mu) = \#S(P) - \min \sum_{(i,j) \in S(P)} b_{i,j},$$

where the minimum is over all choices of $b_{i,j}$ satisfying $0 \leq b_{i,j} \leq 1$ and

$$\sum_{(i,j) \in S(P)} b_{i,j}(\lambda_i - \lambda_j) \geq -h_\mu + \sum_{(i,j) \in S(P)} (\lambda_i - \lambda_j).$$

Since $\lambda_1 - \lambda_d$ is the largest possible difference $\lambda_i - \lambda_j$, it follows that

$$(7) \quad \dim_{LY}^P(\rho_*\mu) \leq \#S(P) - \frac{\left(\sum_{(i,j) \in S(P)} \lambda_i - \lambda_j \right) - h_\mu}{\lambda_1 - \lambda_d}.$$

Letting $P = \{p_1 < \dots < p_M\}$ one has

$$\#S(P) = p_1(d - p_1) + (p_2 - p_1)(d - p_2) + \dots + (p_M - p_{M-1})(d - p_M) = \dim(\mathcal{F}_P).$$

Substituting this into the inequality (7), and using lemma 3.1 we obtain

$$\dim_{LY}^P(\rho_*\mu) \leq \dim(\mathcal{F}_P) - \frac{M-1}{2},$$

as claimed.

3.2. Upper bounds for Furstenberg entropy. In view of theorem 2.3 it will be useful to bound the Furstenberg entropy of the stationary measures $\xi_*^p \nu_\mu$ for each $p \in P$.

Lemma 3.2. *For each $\mu \in \mathcal{M}$ and $p \in P$ one has*

$$h_\mu \leq \sum_{(i,j) \in S_p} \lambda_i - \lambda_j.$$

Proof. Let $\chi_1 > \dots > \chi_N$ be the distinct Lyapunov exponents and d_1, \dots, d_N their multiplicities. For each $l = 1, \dots, N$ let $A_l = \{i : \lambda_i = \chi_l\}$.

Fix $p \in P$, because ρ is p -Anosov one has $\lambda_p > \lambda_{p+1}$ and therefore there exists k such that $d_1 + \dots + d_k = p$.

From [LL23b, Theorem 2.1] the dynamical stationary measure $\xi_*^p \nu_\mu$ on the Grassmannian of p -dimensional subspaces has Furstenberg entropy bounded by

$$\kappa(\mu, \xi_*^p \nu_\mu) \leq \sum_{l \leq k < m} d_l d_m (\chi_l - \chi_m).$$

Noticing that $\#A_l = d_l$ we obtain

$$\begin{aligned} \sum_{l \leq k < m} d_l d_m (\chi_l - \chi_m) &= \sum_{l \leq k < m} \sum_{i \in A_l, j \in A_m} \lambda_i - \lambda_j \\ &= \sum_{i \leq p < j} \lambda_i - \lambda_j \\ &= \sum_{(i,j) \in S_p} \lambda_i - \lambda_j, \end{aligned}$$

which concludes the proof. \square

3.3. Proof of lemma 3.1. Let $M = \#P$ and for $A \subset P$ let $S(A)$ be the set of (i, j) such that $1 \leq i \leq a < j \leq d$ for some $a \in A$.

We write

$$\begin{aligned} M \left(\sum_{(i,j) \in S(P)} \lambda_i - \lambda_j \right) - \sum_{p \in P} \sum_{(i,j) \in S_p} \lambda_i - \lambda_j \\ = \sum_{k=1}^{d-1} \left(M a_k(P) - \sum_{p \in P} b_k(p) \right) \alpha_k, \end{aligned}$$

where $a_k(A)$ is the number of $(i, j) \in S(A)$ with $i \leq k < j$, and $b_k(p) = a_k(\{p\})$.

Since by lemma 3.2

$$M h_\mu \leq \sum_{p \in P} \sum_{(i,j) \in S_p} \lambda_i - \lambda_j,$$

it suffices to show that for each $k = 1, \dots, d-1$ one has

$$M a_k(P) - \sum_{p \in P} b_k(p) \geq \frac{M(M-1)}{2}.$$

For this purpose we fix k and enumerate $P = \{p_1, \dots, p_M\}$ in such a way that

$$|p_1 - k| \geq |p_2 - k| \geq \dots \geq |p_M - k|,$$

and set $A_i = \{p_1, \dots, p_i\}$ for $i = 1, \dots, M$.

The desired lower bound will follow from the following two claims:

- (1) For each $i = 1, \dots, M$ one has $a_k(A_i) \geq b_k(p_i)$.
- (2) For each $i = 1, \dots, M-1$ one has $a_k(A_{i+1}) \geq a_k(A_i) + 1$.

Claim (1) is trivial since $\{p_i\} \subset A_i$.

To establish claim (2) we first suppose that $p_{i+1} \leq k$. If this is the case then $(p_{i+1}, k+1) \in S(A_{i+1})$. If $(p_{i+1}, k+1) \in S(A_i)$ there would have to be some $j \leq i$ with $p_{i+1} < p_j < k+1$ contradicting the choice of enumeration of P . Hence $a_k(A_{i+1}) \geq a_k(A_i) + 1$ in this case. Similarly, if $p_{i+1} > k$ then $(k, p_{i+1}) \in S(A_{i+1}) \setminus S(A_i)$ and claim (2) follows.

To conclude the proof we observe that from claim (1) we have $a_k(A_1) - b_k(p_1) \geq 0 = \frac{1 \times 0}{2}$.

Using claims (1) and (2) we show inductively for $i = 1, \dots, M - 1$ that

$$\begin{aligned} (i+1)a_k(A_{i+1}) - \sum_{j=1}^{i+1} b_k(p_j) &= a_k(A_{i+1}) - b_k(p_{i+1}) + ia_k(A_{i+1}) - \sum_{j=1}^i b_k(p_j) \\ &\geq i + ia_k(A_i) - \sum_{j=1}^i b_k(p_j) \\ &\geq i + \frac{i(i-1)}{2} = \frac{(i+1)i}{2}, \end{aligned}$$

which concludes the proof setting $i = M - 1$.

4. PROOF OF THEOREM 1.8

4.1. Patterson-Sullivan measures and types. Recall theorem 2.4 and the definition of the indicator growth function ψ (see (5)).

Let $\beta : \mathfrak{a} \rightarrow \mathbb{R}$ be linear and strictly positive on the interior of \mathcal{L} , and let δ_β be the critical value of r for the convergence of the series

$$\sum_{\gamma \in \Gamma} \exp(-r\beta(\log S(\rho(\gamma)))).$$

Then,

$$\delta_\beta = \inf\{r > 0, r\beta > \psi\},$$

and furthermore $\delta_\beta\beta$ is tangent to ψ along a unique ray \mathbb{R}_+a_β .

For $\gamma \in \Gamma, t \in \partial\Gamma$, we denote $i(\gamma, t) \in \mathfrak{a}$ the value of the Iwasawa cocycle associated to $\rho(\gamma)$ and the flag $\xi(t)$, namely:

$$i(\gamma, t) = (a_1, \dots, a_d) \text{ with } \sum_{j=1}^i a_j = \log \left| \det \rho(\gamma) \Big|_{\xi^i(t)} \right| \text{ for } i = 1, \dots, d.$$

Theorem 4.1. [Qui02b] *[Higher rank Patterson-Sullivan measures] Assume $\rho(\Gamma)$ is Zariski dense dans $\mathrm{SL}(d, \mathbb{R})$. Then, given $\beta : \mathfrak{a} \rightarrow \mathbb{R}$ linear and positive on the interior of \mathcal{L} , there exists a probability measure ν_β on $\partial\Gamma$ such that, for all $\gamma \in \Gamma, t \in \partial\Gamma$,*

$$\frac{d\gamma_*\nu_\beta}{d\nu_\beta}(\gamma t) = \exp(\delta_\beta\beta(i(\gamma, t))).$$

For $a \in \mathfrak{a}^+$, order the differences $\zeta_k(a) := \alpha_{i_k, j_k}(a)$, for all $(i_k, j_k) \in S(P)$, in such a way that $0 \leq \zeta_1(a) \leq \dots \leq \zeta_{\#S(P)}(a)$. We define the *type* $\mathcal{T}(a)$ the corresponding order on the pairs $(i, j) \in S(P)$. There is a finite set of possible types. For example, for $d = 3$, the type is given by the sign of $(a_1 - a_2) - (a_2 - a_3) = -3a_2$. Similarly, for a matrix $\gamma \in \Gamma$, the type $\mathcal{T}(\gamma)$ is the type of the singular values $\log S(\rho(\gamma)) = s_1(\rho(\gamma)) \geq \dots \geq s_d(\rho(\gamma))$ and for a measure $\mu \in \mathcal{M}(\Gamma)$, the type $\mathcal{T}(\mu)$ is the type of the exponents $\lambda(\rho_*\mu) = \lambda_1(\rho_*\mu) \geq \dots \geq \lambda_d(\rho_*\mu)$.

Let $q \in \mathbb{N}, q - 1 < \dim_F^P(\rho) \leq q$. For a type \mathcal{T} , define a linear functional $\varphi_{\mathcal{T}}$ on \mathfrak{a} by

$$(8) \quad \varphi_{\mathcal{T}}(a) := \left(\sum_{k=1}^{q-1} \zeta_k(a) + (\dim_F^P(\rho) - q + 1)\zeta_q(a) \right),$$

where the order of the indices $k, 1 \leq k \leq S(P)$, corresponds to the type \mathcal{T} .

For a type \mathcal{T} , define the restricted function $\Phi_{\rho, \mathcal{T}}^P(r)$ for $r > 0$ by, for q' such that $q' < r \leq q' + 1$,

$$\Phi_{\rho, \mathcal{T}}^P(r) := \sum_{\gamma \in \Gamma, \mathcal{T}(\gamma) = \mathcal{T}} \exp - \left(\sum_{k=1}^{q'-1} \zeta_k(\rho(\gamma)) + (r - q' + 1)\zeta_{q'}(\rho(\gamma)) \right),$$

where the order of the indices $k, 1 \leq k \leq S(P)$, corresponds to the type \mathcal{T} .

Proposition 4.2. *There exists a type \mathcal{T}_0 such that the function $r \mapsto \Phi_{\rho, \mathcal{T}_0}^P(r)$ has $\dim_F^P(\rho)$ as critical value, the functional $\varphi := \varphi_{\mathcal{T}_0}$ is positive on the interior of \mathcal{L} and $\delta_{\varphi} = 1$.*

Proof. Write, for $a \in \mathfrak{a}^+, r \in \mathbb{R}_+, q'$ such that $q' < r \leq q' + 1$ and a type \mathcal{T} , $\varphi_{r, \mathcal{T}}(a) = \left(\sum_{k=1}^{q'-1} \zeta_k(a) + (r - q' + 1)\zeta_{q'}(a) \right)$, where $\zeta_k(a) = \alpha_{i_k, j_k}(a)$ and the pairs (i_k, j_k) are ordered according to \mathcal{T} .

Write also $\psi_r(a) := \left(\sum_{k=1}^{q'-1} \vartheta_k(a) + (r - q' + 1)\vartheta_{q'}(a) \right)$, where $\vartheta_k(a) = \alpha_{i_k, j_k}(a)$ are non-decreasing in k . Since we start choosing the ϑ 's in non-decreasing order we have, for all r, a, \mathcal{T} , $\psi_r(a) \leq \varphi_{r, \mathcal{T}}(a)$ and therefore,

$$\Phi_{\rho}^P(r) = \sum_{\gamma \in \Gamma} \exp(-\psi_r(\log S(\rho(\gamma)))) \geq \max_{\mathcal{T}} \Phi_{\rho, \mathcal{T}}^P(r).$$

In particular, if $r > \dim_F^P(\rho)$, and \mathcal{T} is any type, $\Phi_{\rho, \mathcal{T}}^P(r) < +\infty$. On the other hand, there is a type \mathcal{T}_0 and $r_0, r_0 < \dim_F^P(\rho)$, such that for $r_0 < r < \dim_F^P(\rho)$,

$$\Phi_{\rho, \mathcal{T}_0}^P(r) = \max_{\mathcal{T}} \Phi_{\rho, \mathcal{T}}^P(r) \geq \frac{1}{\#S(P)!} \Phi_{\rho}^P(r)$$

and therefore the series $\Phi_{\rho, \mathcal{T}_0}^P(r)$ diverges.

So, $\dim_F^P(\rho)$ is the critical value of $\Phi_{\rho, \mathcal{T}_0}^P = \sum_{\gamma \in \Gamma} \exp(-\varphi_{r, \mathcal{T}_0}(\log S(\rho(\gamma))))$. Choose this type as \mathcal{T}_0 and consider the corresponding $\varphi = \varphi_{\dim_F^P(\rho), \mathcal{T}_0}$. We have $\delta_{\varphi} = 1$ since

$$(9) \quad \limsup_{T \rightarrow \infty} \frac{1}{T} \log \#\{\gamma \in \Gamma, \varphi(\log S(\rho(\gamma))) \leq T\} = 1.$$

It remains to show that the functional φ is positive on the interior of \mathcal{L} . This follows from theorem 2.4 and

Lemma 4.3. *We have, for any a in the interior of \mathfrak{a}^+ , $\psi(a) \leq \varphi(a)$.*

Proof. Using the Euclidean scalar product to identify \mathfrak{a} and \mathfrak{a}^* , there is $a_0 \in \mathfrak{a}$ such that

$$\varphi(a) = \langle a_0, a \rangle.$$

For \mathcal{C} a cone in \mathfrak{a} , let $\alpha(\mathcal{C})$ be the cosine of the angle between \mathcal{C} and a_0 :

$$\alpha(\mathcal{C}) := \sup_{a \in \mathcal{C}} \frac{\langle a_0, a \rangle}{\|a_0\| \|a\|} \leq 1.$$

Then for $\gamma \in \Gamma$ such that $\log S(\rho(\gamma)) \in \mathcal{C}$, $\|\log S(\rho(\gamma))\| \geq \frac{\varphi(\log S(\rho(\gamma)))}{\alpha(\mathcal{C})\|a_0\|}$, so that we have

$$\begin{aligned} \tau_{\mathcal{C}}(\Gamma) &:= \limsup_{T \rightarrow \infty} \frac{1}{T} \log \#\{\gamma \in \Gamma, \log S(\rho(\gamma)) \in \mathcal{C}, \|\log S(\rho(\gamma))\| \leq T\} \\ &\leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log \#\{\gamma \in \Gamma, \varphi(\log S(\rho(\gamma))) \leq \alpha(\mathcal{C})\|a_0\|T\} \\ &= \alpha(\mathcal{C})\|a_0\| \limsup_{T \rightarrow \infty} \frac{1}{\alpha(\mathcal{C})\|a_0\|T} \log \#\{\gamma \in \Gamma, \varphi(\log S(\rho(\gamma))) \leq \alpha(\mathcal{C})\|a_0\|T\} \\ &= \alpha(\mathcal{C})\|a_0\|, \end{aligned}$$

where we used (9) for the last equality. Recall the definition (5) of ψ . It is true that

$$\psi_{\Gamma}(a) = \|a\| \inf_{\mathcal{C} \ni a} \tau_{\mathcal{C}}(\Gamma) \leq \|a\| \|a_0\| \inf_{\mathcal{C} \ni a} \alpha(\mathcal{C}) = \langle a_0, a \rangle = \varphi(a).$$

□

□

Corollary 4.4. *There exists a probability measure ν_0 on $\partial\Gamma$ such that*

$$(10) \quad \frac{d\gamma_*\nu_0}{d\nu_0}(\gamma t) = \exp(\varphi(i(\gamma, t))),$$

where $\varphi = \varphi_{\mathcal{T}_0}$ is given by formula (8) and \mathcal{T}_0 by proposition 4.2.

Proof. Apply theorem 4.1 with $\beta = \varphi$. □

4.2. The measure ν_0 is stationary. We now use the $\text{CAT}(-K)$ property to find a probability measure $\mu_0 \in \mathcal{M}$ such that ν_0 is the unique stationary measure for the action of (Γ, μ_0) on $\partial\Gamma$.

Proposition 4.5. *Assume (Γ, S) is a $\text{CAT}(-K)$ finitely generated group for some $K > 0$ and consider the measure ν_0 from corollary 4.4. Then, there exists a probability measure μ_0 on Γ , with $\sum_{\gamma \in \Gamma} |\gamma| \mu_0(\gamma) < +\infty$, such that ν_0 is the unique μ_0 -stationary measure on $\partial\Gamma$.*

Proof. Firstly, we have

Lemma 4.6. *For all $\gamma \in \Gamma$, the function $t \mapsto \varphi(i(\gamma, t))$ is Hölder continuous on $\partial\Gamma$.*

Proof. By theorem 2.2, $t \mapsto \xi(t)$ is Hölder continuous from $\partial\Gamma$ to \mathcal{F}_P . Moreover, since the representation ρ is P -Anosov, the mapping $f \mapsto i(\gamma, f)$ is Hölder continuous on $\xi(\partial\Gamma)$ for all $\gamma \in \Gamma$. Indeed, since the splitting is dominated, $\xi(\partial\Gamma)$ is a compact subset of \mathcal{F}_P and $f \mapsto i(\gamma, f)$ is smooth. Finally, the function φ on \mathfrak{a} is linear. \square

Let Γ be a finitely generated Gromov-hyperbolic group and $\varphi(\gamma, t)$ a Hölder continuous cocycle on $\partial\Gamma$. We consider a probability measure on $\partial\Gamma$ that satisfies the relation (10). We want to find a probability μ_0 on Γ , with $\sum_{\gamma \in \Gamma} |\gamma| \mu_0(\gamma) < +\infty$, such that we have the stationarity relation

$$\nu_0 = \sum_{\gamma} \mu_0(\gamma) \gamma_* \nu_0.$$

The delicate part is to ensure that $\sum_{\gamma \in \Gamma} |\gamma| \mu_0(\gamma) < +\infty$. Under the hypothesis that (Γ, S) is a $\text{CAT}(-K)$ group for some $K > 0$, it is proven by C. Connell and R. Muchnik in [CM07].³ This achieves the proof of the proposition. \square

We claim that theorem 1.8 follows.

Indeed, recall that, from theorem 2.1 the entropy h_{μ_0} satisfies, by (10),

$$\begin{aligned} h_{\mu_0} &= \kappa(\mu_0, \nu_0) = \sum_{\gamma \in \Gamma} \mu_0(\gamma) \int_{\partial\Gamma} \log \left(\frac{d\gamma_* \nu_0}{d\nu_0}(\gamma t) \right) d\nu_0(t) \\ &= \sum_{\gamma \in \Gamma} \mu_0(\gamma) \int_{\partial\Gamma} \varphi(i(\gamma, t)) d\nu_0(t). \end{aligned}$$

Using the linearity of φ , the stationarity of ν_0 and the first moment of μ_0 , we get

$$h_{\mu_0} = \varphi \left(\int_{\mathcal{F}_P} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\gamma} \mu_0^{*n}(\gamma) i(\rho(\gamma), t) d\nu_0(t) \right).$$

The limit in the above RHS is $\lambda(\rho_* \mu_0)$ by a theorem of Furstenberg's (see e.g. [LL23a], theorem 4.4). It follows that

$$h_{\mu_0} = \varphi(\lambda(\rho_* \mu_0)).$$

So, if we compute a lower estimate for $\dim_{LY}^P(\rho_* \mu_0) = L_{h_{\mu_0}}(\lambda(\rho_* \mu_0))$ by choosing $r_{i,j} = 1$ for the first $q-1$ pairs $(i, j) \in S(P)$ in the \mathcal{T}_0 order, $r_{i,j} = \dim_F(\rho) - q + 1$ for the q th pair in the \mathcal{T}_0 order and 0 for the other pairs, we obtain

$$(11) \quad \dim_{LY}^P(\rho_* \mu_0) \geq \dim_F^P(\rho).$$

³Actually, the hypothesis of [CM07] is that Γ acts cocompactly by isometries on a $\text{CAT}(-K)$ space. The action of Γ on its Cayley graph is by isometries and cocompact. The first moment property is then proven on page 488 of [CM07].

5. COVERING BY BALLS AND PROOF OF THEOREM 1.9

Fix $\eta > 0$ and set $s := \dim_F^P(\rho) + \eta$. We may assume $s < \dim(\mathcal{F}_P)$, otherwise $\dim_M(\Lambda_\rho) \leq \dim_F^P(\rho) + \eta$ holds trivially. Since $s > \dim_F^P(\rho)$, the series $\Phi_\rho^P(s) := \sum_{\gamma \in \Gamma} \varphi_s^P(\rho(\gamma))$ converges. We are going to construct covers \mathcal{U} of Λ_ρ by balls of arbitrarily small radius ε with less than $\Phi_\rho^P(s)\varepsilon^{-s+o(\varepsilon)}$ elements. This shows that the covering number $N(\Lambda_\rho, \varepsilon)\varepsilon^{-s+o(\varepsilon)}$ is bounded from above by $\Phi_\rho^P(s)$ uniformly in ε . Therefore, $\dim_M(\Lambda_\rho) \leq s = \dim_F^P(\rho) + \eta$ for all positive η and theorem 1.9 follows.

5.1. Shadows and Anosov representations. Recall that Γ is a hyperbolic group and that we chose the generating set \mathcal{S} to be symmetric. The distance $d(\gamma, \gamma')$ on Γ is given by the word length of $\gamma^{-1}\gamma'$. A *geodesic* $\sigma = \{\gamma_n\}_{n \in \mathbb{Z}}$ in Γ is a sequence such that for all (i, j) , $d(\gamma_i, \gamma_j) = |j - i|$. Any point $x \in \partial\Gamma$ is the limit point of (at least) one geodesic ray $\sigma = \{\gamma_j\}_{j \geq 0}$ with $\gamma_0 = e$.

For a geodesic ray $\sigma = \{\gamma_j\}_{j \geq 0}$ with $\gamma_0 = e$, we call the R -shadow of the geodesic ray the image by ξ of the set of limit points of geodesic rays $\sigma' = \{\gamma'_j\}_{j \geq 0}$ satisfying $\gamma'_j = \gamma_j$ for $j \leq R$. By definition, the R -shadow of a geodesic ray is a subset of \mathcal{F}_P .

For $(i, j) \in S(P)$, $\gamma \in \Gamma$, write $\zeta_{i,j}(\gamma) := \alpha_{i,j}(\log S(\rho(\gamma)))$. The main step for the proof of theorem 1.9 is the following proposition:

Proposition 5.1. *For each geodesic ray $\sigma = \{\gamma_j\}_{j \geq 0}$ with $\gamma_0 = e$, all $\zeta > 0$, the number of balls of radius $\exp(-\zeta)$ in \mathcal{F}_P needed to cover the R -shadow of the geodesic ray is at most*

$$\exp \left(\sum_{(i,j) \in S(P)} [\zeta - \zeta_{i,j}(\gamma_R)]^+ + o(R) \right),$$

where, for a real ϖ , $\varpi^+ = \max\{\varpi, 0\}$.

5.2. Proof of Theorem 1.9 assuming proposition 5.1. Fix $\varepsilon > 0$ small. We need to cover Λ_ρ by well-chosen shadows and then cover these shadows by ε -balls.

For $\gamma \in \Gamma$, we write the $\zeta_{i,j}(\gamma)$, $(i, j) \in S(P)$, in nondecreasing order as $0 < \zeta_1(\gamma) \leq \zeta_2(\gamma) \leq \dots \leq \zeta_{\#S(P)}(\gamma)$. For $\eta > 0$, write $s := \dim_F^P(\rho) + \eta$. We may assume $s < \#S(P)$ and let q be a positive integer such that $q - 1 \leq s \leq q$. For any geodesic ray $\sigma = \{\gamma_j\}_{j \geq 0}$, we will use proposition 5.1 with $\zeta = \zeta_q(\gamma_R)$ to estimate the ε -covering number of its R -shadow.

Let $\sigma = \{\gamma_j\}_{j \geq 0}$ be a geodesic ray such that $\gamma_0(x) = e$. Then, the sequence $\{\zeta_q(\gamma_n)\}_{n \geq 0}$ diverges to infinity, has bounded gaps and there are $C, K > 0$ such that $\zeta_q(\gamma_{n+K}) - \zeta_q(\gamma_n) > C$ (see [GGKW17, theorem 1.3]). It follows that for any chosen geodesic ray σ , there is a well defined smallest index $n(\rho, \sigma)$ and $C > 1$ such that

$$(12) \quad \log(1/\varepsilon) \leq \zeta_q((\gamma_{n(\rho, \sigma)})) < \log(1/\varepsilon) + C.$$

By proposition 5.1 applied with $R = n(\rho, \sigma)$ and $\zeta = \zeta_q(\gamma_{n(\rho, \sigma)})$, we can cover the R -shadow of γ_R with less than

$$\exp \left(\sum_{(i,j) \in S(P)} [\zeta_q(\gamma_R) - \zeta_{i,j}(\gamma_R)]^+ + o(\log(1/\varepsilon)) \right)$$

balls of radius ε . We claim that

$$\sum_{(i,j) \in S(P)} [\zeta_q(\gamma_R) - \zeta_{i,j}(\gamma_R)]^+ \leq -\min_{(i,j) \in S(P)} \sum c_{i,j} \zeta_{i,j}(\gamma_R) + s \log(1/\varepsilon) + C,$$

where the minimum is over $0 \leq c_{i,j} \leq 1$ with $\sum c_{i,j} = s$.

Indeed, since we have ordered the values $\zeta_k(\gamma_R)$ in nondecreasing order, the above minimum is attained for

$$c_k = 1 \text{ for } k < q, \quad c_q = s - q + 1, \quad c_k = 0 \text{ for } k > q.$$

With that choice of c_k 's, we have

$$\sum_{(i,j) \in S(P)} [\zeta_q(\gamma_R) - \zeta_{i,j}(\gamma_R)]^+ = - \left(\sum_k c_k \zeta_k(\gamma_R) + c_q \zeta_q(\gamma_R) \right) + s \zeta_q(\gamma_R)$$

and the claim follows from (12).

Cover now Λ_ρ by $n(\rho, \sigma)$ -shadows of distinct $\gamma_{n(\rho, \sigma)}$. As announced, this proves that

$$N(\Lambda_\rho, \varepsilon) \leq \sum_{\gamma \in \Gamma} \varphi_s^P(\rho(\gamma)) \varepsilon^{-s+o(\varepsilon)} = \Phi_\rho^P(s) \varepsilon^{-s+o(\varepsilon)}.$$

5.3. Geometry of \mathcal{F}_P . We recall the description in [LL23b] of the geometric structure of the successive Lipschitz foliations by Euclidean spaces on \mathcal{F}_P . Write $P = \{p_1, \dots, p_M\}$. Recall that, by convention, $p_{M+1} = d$.

Recall that a topology on $\{1, \dots, M+1\}$ is called admissible if the subsets $\{i, i+1, \dots, M+1\}$ are open. An admissible topology is described by its atoms $T(i)$, where $T(i)$ is the smallest open set containing $\{i\}$. We write T_0 for the topology with atoms $T(i) = \{i, i+1, \dots, M+1\}$, T_P for the topology with atoms $T(i) = \{i\}$. An admissible topology T is finer than another one T' (denoted $T \prec T'$) if any T' -open set is T -open. By definition, any admissible topology is finer than T_0 .

Given an admissible topology T , we define the (weighted) configuration space \mathcal{X}_T (with weights $p_1, p_2 - p_1, \dots, d - p_M$) as the space of sequences $(x_I)_{I \in T}$ such that

- (1) x_I is a $\sum_{i \in I} (p_i - p_{i-1})$ -dimensional subspace of \mathbb{R}^d for each $I \in T$,
- (2) $x_{I \cup J} = x_I + x_J$ for all $I, J \in T$, and
- (3) $x_{I \cap J} = x_I \cap x_J$ for all $I, J \in T$.

Each configuration space \mathcal{X}_T is endowed with the distance corresponding to its natural embedding in the product of Grassmannian manifolds.

For $T \prec T'$, there is a natural projection $\pi_{T, T'} : \mathcal{X}_T \rightarrow \mathcal{X}_{T'}$. The space \mathcal{X}_{T_P} is identified with the pairs in $(\mathcal{F}_P, \mathcal{F}_{P^*})$ in general position. In particular, given $y \in \mathcal{F}_{P^*}$ the projection π_{T_P, T_0} is of the form $\pi^y \times \text{Id}$, where π^y is the natural

projection from the set of flags in \mathcal{F}_P in general position with y to a point. The fibers $(\pi^y)^{-1}(y)$ are $\#S(P)$ -dimensional open subsets of \mathcal{F}_P .

We say a sequence of subspaces $V = (V_1, \dots, V_{M+1})$ is a splitting compatible with $y \in \mathcal{F}_{P^*}$ if for all $i, 1 \leq i \leq M+1$,

$$(13) \quad y_{\{j:j \geq i\}} = \bigoplus_{\{j:j \geq i\}} V_j.$$

Notice that in particular this implies $\dim(V_i) = p_i - p_{i-1}$ for all i and $\mathbb{R}^d = \bigoplus_{i=1}^{M+1} V_i$.

In [LL23b] lemma 4.1, we show that setting

$$(14) \quad V_i(y) = y_{\{j:j \geq i\} \cap (y_{\{j:j > i\}})^\perp},$$

yields a compatible splitting for each $y \in \mathcal{F}_{P^*}$ that we call the perpendicular splitting compatible with y .

Given $y \in \mathcal{F}_{P^*}$ and V a splitting compatible with y , we denote by $\text{Nil}(V)$ the space of linear mapping $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that

$$(15) \quad f(V_{M+1}) = \{0\} \text{ and } f(V_i) \subset \bigoplus_{j:j > i} V_j,$$

for $i = 1, \dots, M$. We have $\dim \text{Nil}(V) = \#S(P)$. Given $y \in \mathcal{F}_{P^*}$ and a compatible splitting V we define a mapping

$$\varphi_V : \text{Nil}(V) \rightarrow \mathcal{F}^y$$

by setting

$$(16) \quad \varphi_V(f)_I = (\text{Id} + f) \left(\bigoplus_{i \in I} V_i(y) \right),$$

for all $I \in T_P$, where $\text{Id} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the identity mapping.

For each $y \in \mathcal{F}_{P^*}$ we consider the perpendicular compatible splitting $V(y) = (V_1(y), \dots, V_{M+1}(y))$ and we define

$$(17) \quad \mathcal{V} = \{(y, f) : y \in \mathcal{F}_{P^*}, f \in \text{Nil}(V)\}.$$

This is a vector bundle with base \mathcal{F}_{P^*} given by the projection onto the first coordinate. It is a sub-bundle of the product $\mathcal{F}_{P^*} \times \text{Hom}(\mathbb{R}^d, \mathbb{R}^d)$. We endow it with the metric given by the sum of the distance in \mathcal{F}_{P^*} and the Hilbert-Schmidt norm on $\text{Hom}(\mathbb{R}^d, \mathbb{R}^d)$ associated with the Euclidean structures on V_i, V_j . We have

Theorem 5.2. *The mapping $\varphi : \mathcal{V} \rightarrow \mathcal{F}_P$ defined by*

$$\varphi(y, f) = \varphi_{V(y)}(f),$$

is a locally bilipschitz homeomorphism.

Proof. This follows from [LL23b] theorem 2.4 and its proof ([LL23b], section 4). \square

Theorem 5.3. *Let $y \in \mathcal{F}_{P^*}$ and V, W be two splittings compatible with y . Then, the mapping*

$$\varphi_W^{-1} \circ \varphi_V : \text{Nil}(V) \rightarrow \text{Nil}(W),$$

is affine.

Proof. See [LL23b], lemma 5.8. □

Let $g \in SL(d, \mathbb{R})$. We note that if $V = (V_1, \dots, V_{M+1})$ is a splitting compatible with y then $g^{-1}V = (g^{-1}V_1, \dots, g^{-1}V_{M+1})$ is a splitting compatible with $g^{-1}y$. For the coordinates given by these two splittings the action of g is linear between the corresponding fibers:

Theorem 5.4 (Linearizing coordinates). *For each $y \in \mathcal{F}_{P^*}$ and each adapted splitting V one has*

$$\varphi_V^{-1} \circ g \circ \varphi_{g^{-1}V}(f) = gfg^{-1},$$

for all $g \in G$ and all $f \in \text{Nil}(g^{-1}V)$.

In particular, $\varphi_V^{-1} \circ g \circ \varphi_{g^{-1}V} : \text{Nil}(g^{-1}V) \rightarrow \text{Nil}(V)$ is linear and the action of $SL(d, \mathbb{R})$ on \mathcal{X}_{T_P} is affine on each fiber.

Proof. See [LL23b] lemma 6.2 and corollary 6.3. □

5.4. Proof of proposition 5.1. Let $\gamma \neq e \in \Gamma$. Recall that we defined $\xi^p(\gamma) \subset \mathbb{R}^d$ as the unique p -dimensional subspace on which p -dimensional volume is most contracted by $\rho(\gamma)^{-1}$. Since the representation ρ satisfies the P^* -Anosov condition, we can associate to the matrix $\rho(\gamma)$ the flag

$$\xi^*(\gamma) := (\{0\}, \xi^{d-p_M}(\gamma^{-1}), \dots, \xi^{d-p_1}(\gamma^{-1}), \mathbb{R}^d) \in \mathcal{F}_{P^*}.$$

From the definitions we obtain:

Lemma 5.5. *The sequence of subspaces $V = (V_1, \dots, V_{M+1})$ is a splitting compatible with $y = \xi^*(\gamma_R)$ where V_j is the sum of the eigenspaces of $\sqrt{\rho(\gamma_R)(\rho(\gamma_R))^t}$ corresponding to the singular values s_i for $p_{j-1} < i \leq p_j$.*

Let $y = \xi^*(\gamma_R)$. Using the coordinates on the space \mathcal{F}_P^y given by theorem 5.2 with the splitting compatible with y given by lemma 5.5, we can write

Lemma 5.6. *Let $y = \xi^*(\gamma_R)$ and V the splitting compatible with y given by lemma 5.5. Write $\varphi_V : \text{Nil}_{T_P, T_0}(V) \rightarrow \mathcal{X}_{T_P, T_0}^y = \mathcal{F}_P^y$ for the coordinate mapping given by (16). Let x belong to the R -shadow of the geodesic ray $\sigma = \{\gamma_j\}_{0 \leq j \leq R}$. Then, there exists $K, \tau > 0$ such that for $R \geq K$,*

$$\|\varphi_V^{-1}(\gamma_R^{-1}x)\| < \tau.$$

Proof. Since x belong to the R -shadow of the geodesic ray $\sigma = \{\gamma_j\}_{0 \leq j \leq R}$ there is a geodesic ray $\sigma'(x) =: \{\gamma'_j\}_{j \geq 0}$ such that $\gamma'_0 = e, \gamma'_R = \gamma_R$ and $\xi(\sigma') = x$. Applying γ_R^{-1} , there is a geodesic ray $\sigma''(x) = \gamma_R^{-1}\sigma'(x) =: \{\gamma''_j\}_{j \geq 0}$ such that

$\gamma_0'' = \gamma_R^{-1}$, $\gamma_R'' = e$ and $\xi(\sigma'') = \gamma_R^{-1}x$. By the proof of theorem 2.2, if R is large enough, there is τ_0 such that

$$\text{dist}(\xi(V), \gamma_R^{-1}x) \leq \tau_0,$$

where $\xi(V) \in \mathcal{F}_P$ is given by $\xi(V) := 0 \subset V_1 \subset V_1 \oplus V_2 \subset \dots \subset \mathbb{R}^d$. Using theorems 5.2 and 5.3, the lemma follows. \square

We can now prove proposition 5.1: let $\sigma = \{\gamma_j\}_{j \geq 0}$ be a geodesic ray with $\gamma_0 = e$. Let V the splitting compatible with $\xi^*(\gamma_R)$ given by lemma 5.5. By lemma 5.6, if $R > K$, the R -shadow of σ is contained in $\varphi_{\rho(\gamma_R)V}^{-1} \circ \rho(\gamma_R) \circ \varphi_V(B(0, \tau))$, where $B(0, \tau)$ is the ball of radius τ in $\text{Nil}_{T_P, T_0}(V)$. Proposition 5.1 amounts to the following

Lemma 5.7. *With the preceding notations, the image of $B(0, \tau)$ by $\varphi_{\rho(\gamma_R)V}^{-1} \circ \rho(\gamma_R) \circ \varphi_V$ is an ellipsoid with axes $\tau \exp(\zeta_{i,j}(\gamma_R))$, for all $(i, j) \in S(P)$.*

Proof. By theorem 5.4,

$$\varphi_{\rho(\gamma_R)V}^{-1} \circ \rho(\gamma_R) \circ \varphi_V(f) = \rho(\gamma_R) f \rho(\gamma_R)^{-1}.$$

Write $\text{Nil}_{T_P, T_0}(V)$ as $\bigoplus_{1 \leq i < j \leq M+1} \text{Hom}(V_i, V_j)$ and $f \in B(0, \tau)$ as $f = \{f_{i,j}\}_{1 \leq i < j \leq M+1}$, with all $|f_{i,j}| < \tau$. The matrix $\rho(\gamma_R)$ is block diagonal, made of matrices $g^i \in \text{Hom}(V_i, V_i)$. Therefore, $\rho(\gamma_R) f \rho(\gamma_R)^{-1} = \{g^j f_{i,j} (g^i)^{-1}, 1 \leq i < j \leq M+1\}$.

There is an orthonormal basis of V_i , namely $\{v_1, \dots, v_{p_i - p_{i-1}}\}$, such that $(g^i)^{-1}v_\ell, \ell = 1, \dots, p_i - p_{i-1}$ form an orthogonal system with $\|(g^i)^{-1}v_\ell\| = (s_{i-1+\ell}(\rho(\gamma_R)))^{-1}$. Similarly, there is an orthonormal basis of V_j , namely $\{u_1, \dots, u_{p_j - p_{j-1}}\}$, such that $g^j u_k, k = 1, \dots, p_j - p_{j-1}$ form an orthogonal system with $\|g^j u_k\| = s_{j-1+k}(\rho(\gamma_R))$.

For $1 \leq \ell \leq p_i - p_{i-1}, 1 \leq k \leq p_j - p_{j-1}$, write $f^{\ell,k}$ for the element of $\text{Hom}(V_i, V_j)$ that sends $(g^i)^{-1}v_\ell / \|(g^i)^{-1}v_\ell\|$ to u_k and the orthogonal space $((g^i)^{-1}v_\ell)^\perp$ to 0. For all $\ell, k, 0 < \ell \leq p_i - p_{i-1}, 0 < k \leq p_j - p_{j-1}$, the $f^{\ell,k}$ form an orthogonal basis of $\text{Hom}(V_i, V_j)$ such that the $\varphi_{\rho(\gamma_R)V}^{-1} \circ \rho(\gamma_R) \circ \varphi_V(f^{\ell,k})$ are orthogonal with norm $(s_{i-1+\ell}(\rho(\gamma_R)))^{-1} s_{j-1+k}(\rho(\gamma_R)) = \exp(-\zeta_{i-1+k, j-1+\ell}(\gamma_R))$. The lemma follows by putting all the $\text{Hom}(V_i, V_j)$ together. \square

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FRANÇOIS LEDRAPPIER, SORBONNE UNIVERSITÉ, UMR 8001, LPSM, BOÎTE COURRIER 158, 4, PLACE JUSSIEU, 75252 PARIS CEDEX 05, FRANCE,

Email address: `fledrapp@nd.edu`

PABLO LESSA, CMAT, FACULTAD DE CIENCIAS, IGUÁ 4225, 11400 MONTEVIDEO, URUGUAY

Email address: `lessa@cmat.edu.uy`