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Optimal identification experiment design with robust output amplitude constraints

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Abstract

In this paper, we consider optimal experiment design with amplitude constraints for the case of a multisine excitation. The output amplitude constraint is robustified using an uncertainty region for the unknown true system. The main contribution of this paper is to treat the robust amplitude constraint without the classical gridding approximation. In particular, we provide an LMI optimization problem to verify whether the output constraint is respected for all systems in the uncertainty region. This LMI formulation can be combined with the accuracy constraint in the optimal experiment design problem. Like each optimal experiment design problem with amplitude constraints, the resulting optimization problem is non-convex, but we provide two approaches to efficiently initialize the algorithm used to solve this optimization problem. The efficiency of the proposed methodology is tested using a numerical simulation.

Key words: Optimal experiment design, Amplitude constraints, System identification

1 Introduction

In this paper, we consider optimal identification experiment design when the constraints on the input and the output signals are not expressed in term of power, but in term of time-domain amplitude.

Optimal identification experiment design has been extensively studied for prediction error identification [18]. In most of the contributions (see, e.g., [18,16,8,14,1]), the constraints on the input and the output signals are expressed as power constraints. One of the main reasons for this is that both the inverse of the covariance matrix of the estimate $\hat{\theta}_N$ of the parameter vector θ_0 of the true system (which is generally the measure of the model accuracy that is used in the accuracy constraint of the optimal experiment design problem) and the power of the input and output signals (which is a measure of the cost of the identification) are affine functions of the power spectrum of the excitation signal. This property allows one to formulate the optimal experiment design problem as a convex optimization problem (see, e.g., [16,8,14]).

The main issue with this approach is that, in many applications (see, e.g., [22] for an example), the constraints are not formulated as constraints on the power of the input and output signals, but as constraints on the amplitude of the time-domain sequence of these signals. Designing optimal identification experiments with amplitude constraints is thus a very important problem in practice. Unlike in the power constraint case, such optimal experiment design problem boils down to a non-convex optimization problem. The paper [21] indeed shows that it can be reformulated as an optimization problem involving both LMI constraints and a rank constraint which makes the optimization problem non-convex. In the literature, different approaches have been proposed to address this non-convex optimization problem. An iterative approach involving different non-convex optimization problems is proposed in [20] (see also [24]) while the accuracy objective and the respect of the amplitude constraints are treated separately in [10]. Other contributions use relaxations to approximate the original non-convex optimization problem by a convex one. A first order approximation is used in [27]. In [21], the non-convex rank constraint is neglected and a randomization procedure is subsequently devised to improve the results. A similar approach is used in [9,17]. Other relaxations are proposed in [11,12].

Since the output obviously depends on the unknown true parameter vector θ_0 (i.e., on the true system $G(\theta_0)$), we here consider an optimal experiment design problem with a *robust* output amplitude constraint i.e., the output amplitude constraint must be satisfied for the outputs of all systems $G(\theta)$ with θ in an uncertainty re-

gion U_{init} containing the unknown θ_0 [25]. Since $\theta_0 \in U_{init}$, such a formulation ensures that the output of the unknown true system $G(\theta_0)$ will respect the amplitude constraint. Robust amplitude constraints (and other types of robust constraints in optimal experiment design) are classically handled via a gridding approximation (see [16,8,25,21]): the uncertainty set U_{init} (containing an infinite number of elements) is replaced by a number n of grid points of this uncertainty set U_{init} and the robust output amplitude constraint is replaced by n constraints (one for each grid point) in the optimal experiment design problem. Since θ_0 will most probably not be among the grid points, this gridding approach entails an approximation i.e., the output of the unknown true system $G(\theta_0)$ is not guaranteed to satisfy the amplitude constraint. The main contribution of this paper is to develop a methodology (inspired from robustness analysis) to handle the robust output amplitude constraint without the approximation introduced by the gridding of U_{init}. This approach will be based on the philosophy we introduced in previous contributions to handle robust output *power* constraints without approximation (see [7] for multisine excitation and see [6] for filtered white noise excitation).

As mentioned above, we will develop a methodology inspired from robustness analysis to handle robust output *amplitude* constraints. Since the (robust) output *amplitude* constraint can easily be translated into a (robust) output *power* constraint for filtered white noise excitation, we will only consider the case of multisine excitation in this paper. More precisely, we will develop an LMI optimization problem allowing, for a given multisine excitation, to verify whether the output amplitude constraint is satisfied for the outputs of all systems $G(\theta)$ with $\theta \in U_{init}$. Since amplitude constraints must be respected at each time instant, we treat, in this LMI formulation, the time similarly as the parameter vector θ i.e., as an uncertain parameter varying in a set. This prevents the use of a gridding of the time axis (which would introduce a similar approximation as the gridding of U_{init}). Using the same philosophy, we also develop an LMI optimization problem to verify whether the input amplitude constraint is verified at each time instant. These LMI formulations for the input and output amplitude constraints can be combined with the accuracy constraint in the optimal experiment design problem. The resulting optimization problem obviously remains a non-convex optimization problem, but we provide two approaches to efficiently initialize the numerical algorithm used to solve this optimization problem.

Notations. Continuous-time signals will be denoted $x(t)$ where $t \in \mathbb{R}$ is the time index while discrete-time signals will be denoted $x[n]$ where n is the sample number. The variable s is the Laplace variable while z will denote both the Z-transform variable and the shift operator. We use j to represent $\sqrt{-1}$. For a complex number a (i.e., $a \in \mathbb{C}$), $|a|$, $\angle a$ and $Re(a)$ will denote, respectively, its modulus, its argument and its real part. For a real number a (i.e., $a \in \mathbb{R}$), $|a|$ is the abolute value of a. For a matrix A, $rank(A)$ is the rank of A, $\lambda_{min}(A)$ (resp. $\lambda_{max}(A)$) is the smallest (resp. largest) eigenvalue of A while A^T (resp. A^*) is its transpose (resp. conjugate transpose). The matrix

$$\begin{pmatrix} X_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & X_n \end{pmatrix}$$

will be denoted $diag(X_1, ..., X_n)$ if the elements X_i $(i = 1, ...n)$ are scalar quantities, while it will be denoted $bdiag(X_1, ..., X_n)$ if the elements X_i $(i = 1, ...n)$ are matrices. In addition, I_n represents the identity matrix of dimension $n \times n$ and $\otimes$, the Kronecker product.

2 Problem statement

2.1 Identification experiment and experiment design problem

We consider the identification of a discrete-time model of a stable single-input single-output true system which can be described by a continuous-time transfer function $G_0(s)$ with input u and output y. The discrete-time data for the identification of the discrete-time model of $G_0(s)$ will be gathered in open loop with a sampling rate T_s. The continuous-time true system will be excited by the following (continuous-time) multisine excitation signal:

$$u(t, A) = \sum_{i=1}^{L} (a_{i,s}\ sin(\omega_i t) + a_{i,c}\ cos(\omega_i t)) \qquad (1)$$

with $A = (a_{1,s}, a_{1,c}, a_{2,s}, a_{2,c}, ..., a_{L,c})^T \in \mathbb{R}^{2L}$. The fundamental frequency of this multisine will be denoted ω_0. Each of the L frequencies ω_i $(i = 1, ..., L)$ thus satisfies the following relation:

$$\omega_i = \alpha_i\ \omega_0 \qquad (2)$$

for an integer $\alpha_i \neq 0$ $(i = 1, ..., L)$. Since the discrete-time data will be gathered with a sampling rate T_s, we will suppose that $\omega_L < \frac{\pi}{T_s}$. In the optimal experiment design problem that will be introduced in the sequel, the L frequencies ω_i $(i = 1, ..., L)$ are fixed a-priori by the user and the amplitudes $a_{i,s}$ and $a_{i,c}$ $(i = 1, ..., L)$ will therefore be the decision variables of the optimal experiment design problem [1]. These amplitudes (which are also the coefficients of the trigonometric Fourier series of $u(t, A)$) are gathered in the amplitude vector $A \in \mathbb{R}^{2L}$ defined above.

In the sequel, we will address (robust) output amplitude constraints based on a steady-state expression of

[1] Note that optimizing $a_{i,s}$ and $a_{i,c}$ $(i = 1, ..., L)$ in (1) is the same as optimizing b_i and ϕ_i $(i = 1, ..., L)$ in $\sum_{i=1}^{L} b_i\ sin(\omega_i t + \phi_i)$.

the output. To ensure as much as possible that the output constraint is also satisfied during transient, the excitation signal (1) will be slowly ramped up [2]. By doing this, we prevent as much as possible that the output exceeds its steady-state expression during transient. For reference purpose, we will assume that this ramp up process is performed when $t < 0$ and that the system is at steady state at $t = 0$. At that moment, we start collecting discrete-time input-output data at a sampling rate T_s applying an anti-aliasing filter to both the input and the output. This leads to a set of discrete-time data $Z^N = \{y[n],\ u[n] \mid n = 0, ..., N-1\}$ which are related as follows

$$y[n] = G(z, \theta_0)u[n] + v[n] \tag{3}$$

where $G(z, \theta_0)$ is a stable discrete-time transfer function satisfying $G(e^{j\omega T_s}, \theta_0) = G_0(j\omega)$ for $\omega \in [0,\ \frac{\pi}{T_s}]$ (and therefore also for all ω_i $(i = 1, ..., L)$ in (1)) and where $v[n]$ is a discrete-time signal representing the disturbance acting on the system. This disturbance is modeled as a time series $v[n] = H(z, \theta_0)e[n]$ with $e[n]$ a white noise of variance σ_e^2 and $H(z, \theta_0)$ a stable, inversely stable and monic transfer function [18]. The transfer functions $G(z, \theta_0)$ and $H(z, \theta_0)$ are parameterized with an unknown parameter vector $\theta_0 \in \mathbb{R}^k$ in a given model structure $G(z, \theta)$ and $H(z, \theta)$. The typical model structures (BJ, ARX, ARMAX, ...) are all rational in the vector θ [18]. In particular, for these model structures, $G(z, \theta)$ can be written as $(Z_N(z)\theta)/(1 + Z_D(z)\theta)$ where $Z_N(z)$ and $Z_D(z)$ are row vectors of transfer functions [5,7].

Under the instrumentation described above, the discrete-time signal $u[n]$ in (3) is given by $u[n] = u(t = nT_s, A)$ $(n = 0, ..., N-1)$ while $y[n]-v[n] = G(z, \theta_0)u[n]$ $(n = 0, ..., N-1)$ is given by $\breve{y}(t = nT_s, A, \theta_0)$ with

$$\begin{aligned}\breve{y}(t, A, \theta) = &\sum_{i=1}^{L} g(\omega_i, \theta)... \\ &... \left(a_{i,s}\ sin(\omega_i t + \phi(\omega_i, \theta)) + a_{i,c}\ cos(\omega_i t + \phi(\omega_i, \theta))\right)\end{aligned} \tag{4}$$

where $g(\omega_i, \theta) = |G(e^{j\omega_i T_s}, \theta)|$ and $\phi(\omega_i, \theta) = \angle G(e^{j\omega_i T_s}, \theta)$. Note also that the continuous-time signal $\breve{y}(t, A, \theta_0)$ represents the noise-free output of the continuous-time system for $t \geq 0$.

Let us now specify precisely the amplitude constraints that have to be respected when designing the multisine excitation $u(t, A)$.

Amplitude constraints. *Consider the time interval $T_P = [0,\ \frac{2\pi}{\omega_0}]$ corresponding to a period of the multisines $u(t, A)$ (see (1)) and $\breve{y}(t, A, \theta_0)$ (see (4)). The multisines $u(t, A)$ and $\breve{y}(t, A, \theta_0)$ must satisfy*

$$-\ \bar{\mathbf{u}}_{\mathbf{max}} \leq u(t, A) \leq \bar{\mathbf{u}}_{\mathbf{max}}\ \forall t \in T_P \tag{5}$$

$$-\ \bar{\mathbf{y}}_{\mathbf{max}} \leq \breve{y}(t, A, \theta_0) \leq \bar{\mathbf{y}}_{\mathbf{max}}\ \forall t \in T_P \tag{6}$$

where $\bar{\mathbf{u}}_{\mathbf{max}} > 0$ and $\bar{\mathbf{y}}_{\mathbf{max}} > 0$ are two user-chosen constants. In (5) and (6), we can restrict attention to $t \in T_P$ since $u(t, A)$ and $\breve{y}(t, A, \theta_0)$ are both periodic signals having a fundamental period equal to $\frac{2\pi}{\omega_0}$. ■

Using the data set Z^N, the prediction error estimate $\hat{\theta}_N$ of θ_0 [18] is given by

$$\begin{aligned}\hat{\theta}_N &= \arg\min_\theta \tfrac{1}{N} \sum_{n=0}^{N-1} \epsilon^2[n, \theta] \\ \text{with } \epsilon[n, \theta] &= H^{-1}(z, \theta)\left(y[n] - G(z, \theta)u[n]\right).\end{aligned} \tag{7}$$

Assuming that, for the chosen L in (1), the data Z^N are informative [18], the estimate $\hat{\theta}_N$ is (asymptotically) normally distributed around θ_0 with a covariance matrix $P_\theta > 0$ that can be estimated from the data [18]. We can also determine an expression of the inverse of the covariance matrix as a function of the true parameter vector θ_0 and of the square of the amplitudes $a_{i,s}$ and $a_{i,c}$ $(i = 1, ..., L)$ i.e.,

$$P_\theta^{-1}(A, \theta_0) = \left(\sum_{i=1}^{L} M(\omega_i, \theta_0)\left(a_{i,s}^2 + a_{i,c}^2\right)\right) + M_v(\theta_0) \tag{8}$$

for matrices $M(\omega_i, \theta_0) \geq 0$ $(i = 1, ..., L)$ and $M_v(\theta_0) \geq 0$ of known expressions and which are both proportional to the number of data N (see, e.g., [24]). The covariance matrix $P_\theta > 0$ is a measure of the modeling error $\hat{\theta}_N - \theta_0$ and its inverse therefore a measure of the model accuracy.

In this paper, our objective will be to design the amplitude vector A defining the excitation signal (1) as the one solving the following (non-convex) optimization problem for a given matrix $R_{adm} \geq 0$:

$$\begin{aligned}&\arg\max_{\xi, A}\ \xi \\ &s.t.\ P_\theta^{-1}(A, \theta_0) \geq \xi\ R_{adm} \text{ and (5)-(6).}\end{aligned} \tag{9}$$

Let us discuss this formulation more precisely. If you choose $R_{adm} = I_k$, (9) is an E-optimal experiment design problem whose objective is to determine the amplitude vector A of the multisine excitation (1) which leads to an estimate $\hat{\theta}_N$ of θ_0 with the smallest [3] covariance matrix $P_\theta(A, \theta_0)$ while guaranteeing that the amplitude constraints (5) and (6) are respected. Since P_θ is inversely proportional to N, it is clear that higher accuracy for $\hat{\theta}_N$ can be achieved under the amplitude constraints (5) and (6) if N increases (i.e., if the experiment duration increases). In this sense, the optimization problem (9) also allows to determine the minimal

[2] This can, e.g., be achieved by multiplying (1) by a function $\gamma(t)$ which slowly grows from zero to one.

[3] The smallest covariance matrix P_θ is here defined as the one with the smallest value of $\lambda_{max}(P_\theta)$ (or equivalently the largest value of $\lambda_{min}(P_\theta^{-1})$).

experiment duration N for which it is possible to find an amplitude vector A that respects (5) and (6) and ensures $P_\theta^{-1}(A,\theta_0) \geq R_{adm}$ for some matrix R_{adm} representing the desired accuracy (see, e.g., [24] for the details on how this can be done based on (9)).

Remark. Instead of (1), the excitation applied to the continuous-time system may also be the output of a Zero Order Hold to which a discrete-time multisine $u[n]$ is applied. In (3), $u[n]$ is thus here also given by $\sum_{i=1}^{L}(a_{i,s}\ sin(\omega_i n T_s) + a_{i,c}\ cos(\omega_i n T_s))$. Due to the property of the Zero Order Hold, we could thus keep (5) for the constraint on the continuous-time input or replace it by $|u[n]| \leq \bar{\mathbf{u}}_{\mathbf{max}}$ for $n = 0, ..., \frac{2\pi}{\omega_0 T_s}$. As far as the output amplitude is concerned, we can still say that, in steady state, $G(z,\theta_0)u[n]$ is given by $\breve{y}(t = nT_s, A, \theta_0)$ with the expression (4). However, for $t \neq nT_s$, the noise-free output of the continuous-time system will slightly differ from $\breve{y}(t, A, \theta_0)$ (this difference will be small if the gain of the continuous transfer function G_0 is sufficiently small for $\omega \geq \frac{\pi}{T_s}$). Consequently, we can still use the results of this paper (i.e., results aiming at satisfying (6)) for this alternative instrumentation, but we will then not have a formal guarantee that the output constraint is satisfied for $t \neq nT_s$. Note that this was also the case in the approaches of, e.g., [20,21].

Remark. Since $\bar{\mathbf{y}}_{\mathbf{max}}$ bounds the noise-free output, if we wish to bound the noisy output of the system, $\bar{\mathbf{y}}_{\mathbf{max}}$ must be chosen considering the (maximal) amplitude of the disturbance acting on the system (that can be determined using an experiment with $u(t, A) = 0$).

2.2 Robustification of the output amplitude constraint (6)

Since θ_0 is unknown, we cannot use the output amplitude constraint (6) as such. We will instead use a robustified version of this output amplitude constraint. As in, e.g., [7], this robustified version of the constraint will be based on an initial uncertainty region U_{init} for the unknown true parameter vector θ_0. This initial uncertainty region U_{init} can be obtained via an initial identification experiment that has delivered an initial estimate $\hat{\theta}_{init}$ of θ_0 with covariance matrix $P_{\theta,init}$. Consequently, the following ellipsoid U_{init} is a η%-confidence region for the unknown true parameter vector θ_0:

$$U_{init} = \left\{\theta \in \mathbb{R}^k \mid (\theta - \hat{\theta}_{init})^T P_{\theta,init}^{-1}(\theta - \hat{\theta}_{init}) \leq \chi\right\} \tag{10}$$

with χ such that $Pr(\chi^2(k) \leq \chi) = \eta$ (say 95 %). For the sequel, it is important to note that U_{init} can be equivalently rewritten as

$$U_{init} = \{\bar{\theta} \in \mathbb{R}^k \mid \bar{\theta}^T\bar{\theta} \leq 1\} \tag{11}$$

if we perform the change of variable

$$\bar{\theta} \triangleq R(\theta - \hat{\theta}_{init}) \tag{12}$$

with $\frac{P_{\theta,init}^{-1}}{\chi} = R^T R$. Let us also denote by $G(z,\bar{\theta})$ the parametrization $G(z,\theta)$ where θ has been replaced by $\hat{\theta}_{init}+R^{-1}\bar{\theta}$. Similarly, let us define $\breve{y}(t, A, \bar{\theta})$ as in (4) but with $G(z,\theta)$ replaced by $G(z,\bar{\theta})$. We can then robustify the output amplitude constraint (6) as follows:

$$-\bar{\mathbf{y}}_{\mathbf{max}} \leq \breve{y}(t, A, \bar{\theta}) \leq \bar{\mathbf{y}}_{\mathbf{max}}\ \forall t \in T_P\ and\ \forall \bar{\theta} \in U_{init} \tag{13}$$

For future reference, let us introduce the following quantities:

$$\mathbf{u}(A) \triangleq \max_{t \in T_P} |u(t, A)| \tag{14}$$

$$\breve{\mathbf{y}}_{\mathbf{wc}}(A) \triangleq \max_{t \in T_P} \max_{\bar{\theta} \in U_{init}} |\breve{y}(t, A, \bar{\theta})|. \tag{15}$$

Using these notions, (5) and (13) are respectively equivalent to

$$\mathbf{u}(A) \leq \bar{\mathbf{u}}_{\mathbf{max}}, \tag{16}$$

$$\breve{\mathbf{y}}_{\mathbf{wc}}(A) \leq \bar{\mathbf{y}}_{\mathbf{max}}. \tag{17}$$

The robustified version of the optimal experiment design problem (9) is thus

$$\begin{aligned} &\arg\max_{\xi, A}\ \xi \\ &s.t.\ P_\theta^{-1}(A, \hat{\theta}_{init}) \geq \xi\ R_{adm} \text{ and (16) and (17).} \end{aligned} \tag{18}$$

Note that the accuracy constraint $P_\theta^{-1} \geq \xi\ R_{adm}$ could also be robustified using U_{init} and the results in [7]. However, since the focus of this paper is on the amplitude constraints, we have decided to replace the unknown θ_0 by its initial estimate $\hat{\theta}_{init}$ to make the accuracy constraint tractable.

Before proceeding, let us make the following assumption on the initial uncertainty U_{init}.

Assumption 1 *The uncertainty U_{init} defined in (10) contains the unknown parameter vector θ_0. Moreover, U_{init} is small enough to guarantee that, like $G(z,\theta_0)$, $G(z,\theta)$ is a stable transfer function for all $\theta \in U_{init}$.* ■

The first part of Assumption 1 is there to ensure that (17) effectively implies (6). Note that, in practice, this guarantee will only hold with a probability of η%. The second part of the assumption guarantees that (4) can be written for each $\theta \in U_{init}$. We can easily verify whether a given U_{init} satisfies this part of Assumption 1 using the results in [5].

3 Computation of $\mathbf{u}(A)$ and $\breve{\mathbf{y}}_{\mathbf{wc}}(A)$

3.1 Introduction

In order to address the optimal experiment design problem (18), we will need to be able to evaluate the quantities $\mathbf{u}(A)$ and $\breve{\mathbf{y}}_{\mathbf{wc}}(A)$ appearing in the constraints of (18). Note that, besides their importance for optimal experiment design, these quantities are also crucial

if we wish to perform an identification experiment using an excitation (1) with an user-chosen amplitude vector A. These quantities will indeed allow the user to verify whether the chosen A satisfies the (robust) amplitude constraints (16) and (17).

The quantity $\mathbf{u}(A)$ is the solution of the following optimization problem:

$$\begin{aligned} \mathbf{u}(A) &= \arg\min_{u_{max}} u_{max} \\ s.t. \quad & -u_{max} \leq u(t,A) \leq u_{max} \ \forall t \in T_P. \end{aligned} \tag{19}$$

Similarly, the quantity $\breve{\mathbf{y}}_{\mathbf{wc}}(A)$ is the solution of the following optimization problem:

$$\begin{aligned} \breve{\mathbf{y}}_{\mathbf{wc}}(A) &= \arg\min_{y_{max}} y_{max} \\ s.t. \quad & -y_{max} \leq \breve{y}(t,A,\bar{\theta}) \leq y_{max} \ \forall t \in T_P, \forall \bar{\theta} \in U_{init}. \end{aligned} \tag{20}$$

In the sequel, we will show that, using robustness analysis tools, we can exactly compute, for any given A, the solution $\mathbf{u}(A)$ of (19). We will also show that we can derive a (tight) upper bound $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A)$ for the solution $\breve{\mathbf{y}}_{\mathbf{wc}}(A)$ of (20). Note that, if $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A) \leq \bar{\mathbf{y}}_{\mathbf{max}}$, we have also that $\breve{\mathbf{y}}_{\mathbf{wc}}(A) \leq \bar{\mathbf{y}}_{\mathbf{max}}$. Consequently, the quantity $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A)$ can be used together with $\mathbf{u}(A)$ to check whether a given excitation $u(t,A)$ can be applied without risk to the considered system G_0.

3.2 Phasors and LFT representations

Let us introduce phasor notations and transform the amplitude vector $A = (a_{1,s}, a_{1,c}, a_{2,s}, a_{2,c}, ..., a_{L,c})^T$ of dimension $2L$ into the complex vector $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_L)^T$ of dimension L. The i^{th} entry $\mathcal{A}_i$ of $\mathcal{A}$ is defined as

$$\mathcal{A}_i = a_{i,c} - j\ a_{i,s} \quad (i = 1, ..., L). \tag{21}$$

Let us also define $\mathcal{U}(t)$ as $\mathcal{U}(t) = (e^{j\omega_1 t}, e^{j\omega_2 t}, ..., e^{j\omega_L t})^T$. We have then

$$u(t,A) = Re(\mathcal{A}^T \mathcal{U}(t)). \tag{22}$$

For the sequel, it is important to observe that $u(t,A)$ depends on t via the phasors $e^{j\omega_i t}$ $(i = 1, ..., L)$. Let us introduce the complex variable

$$\tau \triangleq e^{j\omega_0 t} \tag{23}$$

with ω_0 the fundamental frequency of $u(t,A)$. Using (2), it is clear that, for each ω_i, $e^{j\omega_i t} = \tau^{\alpha_i}$. Consequently, instead of a function of t, we can write $u(t,A)$ as a function $u(\tau,A)$ of τ which has the property $u(\tau,A) = u(t,A)$ when (23) holds. This function $u(\tau,A)$ is given by

$$u(\tau,A) = Re(\mathcal{A}^T \mathcal{U}(\tau)) \tag{24}$$

with $\mathcal{U}(\tau) = (\tau^{\alpha_1}, \tau^{\alpha_2}, ..., \tau^{\alpha_L})^T$. When t takes all values in T_P, the complex variable τ defined in (23) takes all values in the set

$$\mathcal{T} = \{\tau \in \mathbb{C} \mid \tau^* \tau = 1\}. \tag{25}$$

Consequently, the constraint in (19) is equivalent to

$$-u_{max} \leq Re(\mathcal{A}^T \mathcal{U}(\tau)) \leq u_{max} \ \forall \tau \in \mathcal{T}. \tag{26}$$

It is also clear that the only quantity dependent on τ in (26) i.e., $\mathcal{U}(\tau) = (\tau^{\alpha_1}, \tau^{\alpha_2}, ..., \tau^{\alpha_L})^T$ can be rewritten as the following LFT in τI_{α_L}:

$$\begin{aligned} & p_u = (\tau I_{\alpha_L})\ q_u \\ & \begin{pmatrix} q_u \\ \mathcal{U}(\tau) \end{pmatrix} = \underbrace{\left(\begin{array}{c|c} M_{11,\mathcal{U}} & M_{12,\mathcal{U}} \\ \hline M_{21,\mathcal{U}} & M_{22,\mathcal{U}} \end{array}\right)}_{M_{\mathcal{U}}} \begin{pmatrix} p_u \\ 1 \end{pmatrix} \end{aligned} \tag{27}$$

with $M_{22,\mathcal{U}} = (0, ..., 0)^T$, $M_{12,\mathcal{U}} = (1, 0, ..., 0)^T$, $M_{21,\mathcal{U}}$ a matrix whose entries are equal to zero except the entries (i, α_i) for $i = 1, ..., L$ which are equal to one, and with $M_{11,\mathcal{U}}$ a matrix of dimension $\alpha_L \times \alpha_L$ given by

$$M_{11,\mathcal{U}} = \begin{pmatrix} 0 & 0 \\ I_{\alpha_{L-1}} & 0 \end{pmatrix}.$$

Let us now define the quantity $\breve{y}(\tau, A, \bar{\theta})$ as follows:

$$\breve{y}(\tau, A, \bar{\theta}) = Re(\mathcal{A}^T \mathcal{Y}(\tau, \bar{\theta})) \tag{28}$$

with

$$\mathcal{Y}(\tau, \bar{\theta}) = diag(G(e^{j\omega_1 T_s}, \bar{\theta}), ..., G(e^{j\omega_L T_s}, \bar{\theta}))\ \mathcal{U}(\tau). \tag{29}$$

It is clear that $\breve{y}(\tau, A, \bar{\theta}) = \breve{y}(t, A, \bar{\theta})$ when (23) holds. Consequently, the constraint in (20) is equivalent to

$$-y_{max} \leq Re(\mathcal{A}^T \mathcal{Y}(\tau, \bar{\theta})) \leq y_{max} \ \forall \tau \in \mathcal{T},\ \forall \bar{\theta} \in U_{init}. \tag{30}$$

Moreover, recall that the model structure $G(z, \theta)$ can be written as $Z_N(z)\theta/(1+Z_D(z)\theta)$ (see Section 2.1) and recall also the linear change of variable (12). Consequently, $diag(G(e^{j\omega_1 T_s}, \bar{\theta}), ..., G(e^{j\omega_L T_s}, \bar{\theta}))$ is an LFT in $I_L \otimes \bar{\theta}$ (see, e.g., [7]). Using this fact and the fact that $\mathcal{U}(\tau)$ is an LFT in τI_{α_L} (see (27)), it is possible to find vectors of signals p_τ, $p_{\bar{\theta}}$, q_τ, $q_{\bar{\theta}}$ and a matrix $M_{\mathcal{Y}}$ such that $\mathcal{Y}(\tau, \bar{\theta})$ can be expressed as

$$\begin{aligned} & \begin{pmatrix} p_\tau \\ p_{\bar{\theta}} \end{pmatrix} = bdiag\left(\tau I_{\alpha_L},\ I_L \otimes \bar{\theta}\right) \begin{pmatrix} q_\tau \\ q_{\bar{\theta}} \end{pmatrix} \\ & \begin{pmatrix} q_\tau \\ q_{\bar{\theta}} \\ \mathcal{Y}(\tau, \bar{\theta}) \end{pmatrix} = \underbrace{\left(\begin{array}{cc|c} M^{\tau\tau}_{11,\mathcal{Y}} & M^{\tau\bar{\theta}}_{11,\mathcal{Y}} & M^{\tau}_{12,\mathcal{Y}} \\ M^{\bar{\theta}\tau}_{11,\mathcal{Y}} & M^{\bar{\theta}\bar{\theta}}_{11,\mathcal{Y}} & M^{\bar{\theta}}_{12,\mathcal{Y}} \\ \hline M^{\tau}_{21,\mathcal{Y}} & M^{\bar{\theta}}_{21,\mathcal{Y}} & M_{22,\mathcal{Y}} \end{array}\right)}_{M_{\mathcal{Y}}} \begin{pmatrix} p_\tau \\ p_{\bar{\theta}} \\ 1 \end{pmatrix} \end{aligned} \tag{31}$$

In the sequel, we will use the symbol $\star$ to introduce shorthand notations of LFTs i.e., the shorthand notation of (27) is $\mathcal{U}(\tau) = (\tau I_{\alpha_L}) \star M_{\mathcal{U}}$ and the one of (31) is $\mathcal{Y}(\tau, \bar{\theta}) = \left(bdiag\left(\tau I_{\alpha_L}, I_L \otimes \bar{\theta}\right)\right) \star M_{\mathcal{Y}}$.

3.3 Multipliers

In the previous subsection, we have seen that the constraints appearing in (19) and in (20) can be equivalently rewritten as (26) and (30), respectively. These constraints can be both considered as robust constraints since they entail the verification of inequalities for all $\tau \in \mathcal{T}$ in the case of (26) and for all $\tau \in \mathcal{T}$ and for all $\bar{\theta} \in U_{init}$ in the case of (30). Recall also that, in these inequalities, the quantities $\mathcal{U}(\tau)$ and $\mathcal{Y}(\tau, \bar{\theta})$ (i.e., the only quantities depending on τ and/or $\bar{\theta}$) are LFTs, respectively, in τI_{α_L} and in $bdiag\left(\tau I_{\alpha_L},\ I_L \otimes \bar{\theta}\right)$. In order to find tractable versions of (26) and (30), we will need to find appropriate descriptions of the uncertainty τI_{α_L} with $\tau \in \mathcal{T}$ and of the uncertainty $I_L \otimes \bar{\theta}$ with $\bar{\theta} \in U_{init}$.

These appropriate descriptions are the so-called sets of multipliers for these two uncertainties (we remind the expressions of these sets in Propositions 1 and 2). In a nutshell, a set of multipliers for a given uncertainty $p(t) = \Delta q(t)$ with $\Delta \in \mathbf{\Delta}$ is an explicit and affine parametrization of the quadratic constraint satisfied by the graphs of the signals p and q when $\Delta \in \mathbf{\Delta}$ [26,13,23].

Proposition 1 *Consider the set $\mathcal{T}$ defined in* (25) *and an arbitrary scalar α. Then,*

$$\begin{pmatrix} \tau I_\alpha \\ I_\alpha \end{pmatrix}^* \Pi_\alpha \begin{pmatrix} \tau I_\alpha \\ I_\alpha \end{pmatrix} = 0 \quad \forall \tau \in \mathcal{T} \tag{32}$$

when $\Pi_\alpha = bdiag(S, -S)$ with S any Hermitian matrix of dimension $\alpha \times \alpha$. The set of matrices Π_α having this structure will be denoted $\mathbf{\Pi}_\alpha$ in the sequel.

Proof. For any $\Pi_\alpha = bdiag(S, -S)$, the quadratic expression in (32) is equal to $(\tau^*\tau - 1)S$ and this quantity is indeed equal to zero for all $\tau \in \mathcal{T}$. ■

Proposition 2 ([2]) *Consider the set $U_{init} = \{\bar{\theta} \in \mathbb{R}^k \mid \bar{\theta}^T\bar{\theta} \leq 1\}$ and an arbitrary scalar L. Then,*

$$\begin{pmatrix} I_L \otimes \bar{\theta} \\ I_L \end{pmatrix}^T \Sigma \begin{pmatrix} I_L \otimes \bar{\theta} \\ I_L \end{pmatrix} \geq 0 \quad \forall \bar{\theta} \in U_{init} \tag{33}$$

when Σ has the following structure

$$\Sigma = \left(\begin{array}{c|c} -Q \otimes I_k + \tilde{B} + j\ \tilde{D} & P^T - j\ Z^T \\ \hline P + j\ Z & Q \end{array}\right) \tag{34}$$

where Q is any positive semi-definite Hermitian matrix of dimension $L \times L$ and $\tilde{B}$, $\tilde{D}$, P, Z real matrices having the structure given in Appendix A. The set of matrices Σ having the structure (34) *will be denoted $\mathbf{\Sigma}$ in the sequel.* ■

In the sequel, we will also require the following result which is a particular case of the generalized KYP lemma [15].

Lemma 1 ([15]) *Consider a matrix $\mathcal{F}(\tau)$ which depends on a complex scalar variable τ that lies in the set $\mathcal{T}$ defined in* (25)*. Assume that $\mathcal{F}(\tau)$ can be written as an LFT in τI_α and that this LFT is minimal (i.e., we cannot write $\mathcal{F}(\tau)$ as an LFT in $\tau I_{\tilde{\alpha}}$ with $\tilde{\alpha} < \alpha$). We thus have:*

$$\mathcal{F}(\tau) = (\tau I_\alpha) \star \left(\begin{array}{c|c} \mathcal{F}_{11} & \mathcal{F}_{12} \\ \hline \mathcal{F}_{21} & \mathcal{F}_{22} \end{array}\right)$$

where $\mathcal{F}_{11}$ is a matrix of dimension $\alpha \times \alpha$. Consider finally an Hermitian matrix Ξ of appropriate dimension. Then, $\mathcal{F}^*(\tau)\Xi\mathcal{F}(\tau) \leq 0$ for all $\tau \in \mathcal{T}$ is equivalent to the existence of $\Pi_\alpha \in \mathbf{\Pi}_\alpha$ (see Proposition 1) such that:

$$\begin{pmatrix} I & 0 \\ \mathcal{F}_{11} & \mathcal{F}_{12} \end{pmatrix}^* \Pi_\alpha \begin{pmatrix} I & 0 \\ \mathcal{F}_{11} & \mathcal{F}_{12} \end{pmatrix} + \begin{pmatrix} \mathcal{F}_{21} & \mathcal{F}_{22} \end{pmatrix}^* \Xi \begin{pmatrix} \mathcal{F}_{21} & \mathcal{F}_{22} \end{pmatrix} \leq 0$$

3.4 LMI optimization problems

Using the LFTs and the sets of multipliers introduced in the previous subsections, we can now derive LMI optimization problems allowing to compute $\mathbf{u}(A)$ (see Proposition 3) and an upper bound $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A)$ for $\breve{\mathbf{y}}_{\mathbf{wc}}(A)$ (see Proposition 4).

Proposition 3 *Consider a given amplitude vector A and its complex expression $\mathcal{A}$ (see* (21)*). Consider also the LFT expression* (27) *for $\mathcal{U}(\tau)$ and the set of multipliers $\mathbf{\Pi}_{\boldsymbol{\alpha}_\mathbf{L}}$ corresponding to τI_{α_L} (see Proposition 1 with $\alpha = \alpha_L$). Then, $\mathbf{u}(A)$ (see* (14)*) is the solution u^{opt}_{max} of the following LMI optimization problem having as decision variables a real scalar $u_{max} \geq 0$ and two matrices $\Pi^1_{\alpha_L}$ and $\Pi^2_{\alpha_L}$ in $\mathbf{\Pi}_{\boldsymbol{\alpha}_\mathbf{L}}$:*

$$\arg\ \min\ u_{max}\ s.t.$$

$$\mathcal{V}^*_{u,1}\Pi^1_{\alpha_L}\mathcal{V}_{u,1} + \mathcal{V}^*_{u,2}\begin{pmatrix} 0 & 0.5 \\ 0.5 & -u_{max} \end{pmatrix}\mathcal{V}_{u,2} \leq 0 \tag{35}$$

$$0 \leq -\mathcal{V}^*_{u,1}\Pi^2_{\alpha_L}\mathcal{V}_{u,1} + \mathcal{V}^*_{u,2}\begin{pmatrix} 0 & 0.5 \\ 0.5 & u_{max} \end{pmatrix}\mathcal{V}_{u,2} \tag{36}$$

with

$$\mathcal{V}_{u,1} = \begin{pmatrix} I & 0 \\ M_{11,\mathcal{U}} & M_{12,\mathcal{U}} \end{pmatrix} \quad \mathcal{V}_{u,2} = \begin{pmatrix} \mathcal{A}^T M_{21,\mathcal{U}} & \mathcal{A}^T M_{22,\mathcal{U}} \\ 0 & 1 \end{pmatrix}.$$

Proof. Let us first prove that the existence of $\Pi^1_{\alpha_L} \in \mathbf{\Pi}_{\boldsymbol{\alpha}_\mathbf{L}}$ such that (35) holds is equivalent to $Re(\mathcal{A}^T\mathcal{U}(\tau)) \leq u_{max}\ \forall \tau \in \mathcal{T}$. For this purpose, note that $Re(\mathcal{A}^T\mathcal{U}(\tau)) \leq u_{max}$ can be rewritten as

$$\bar{\mathcal{U}}^*(\tau)\begin{pmatrix}0 & 0.5\\ 0.5 & -u_{max}\end{pmatrix}\bar{\mathcal{U}}(\tau)\le 0 \;\; with \bar{\mathcal{U}}(\tau)=\begin{pmatrix}\mathcal{A}^T\mathcal{U}(\tau)\\ 1\end{pmatrix}.$$

Note also that, using (27), we have that

$$\bar{\mathcal{U}}(\tau)=(\tau I_{\alpha_L})\star\left(\begin{array}{c|c} M_{11,\mathcal{U}} & M_{12,\mathcal{U}}\\ \hline \mathcal{A}^T M_{21,\mathcal{U}} & \mathcal{A}^T M_{22,\mathcal{U}}\\ \hline 0 & 1\end{array}\right).$$

That the existence of $\Pi^1_{\alpha_L}\in\mathbf{\Pi}_{\alpha_\mathbf{L}}$ such that (35) holds is equivalent to $Re(\mathcal{A}^T\mathcal{U}(\tau))\le u_{max}\ \forall\tau\in\mathcal{T}$ then follows from Lemma 1 (the LFT of $\bar{\mathcal{U}}(\tau)$ is minimal since (27) is minimal). Using a similar reasoning and Lemma 1, we can also prove that the existence of $\Pi^2_{\alpha_L}\in\mathbf{\Pi}_{\alpha_\mathbf{L}}$ such that (36) holds is equivalent to $-u_{max}\le Re(\mathcal{A}^T\mathcal{U}(\tau))$ $\forall\tau\in\mathcal{T}$. The proposition is then proven since it is shown, in Section 3, that $\mathbf{u}(A)$ is the solution u^{opt}_{max} of the optimization problem consisting in minimizing u_{max} under the constraint (26). ■

Proposition 4 *Consider a given amplitude vector A and its complex expression $\mathcal{A}$ (see (21)). Consider the LFT expression (31) for $\mathcal{Y}(\tau,\bar{\theta})$, the set of multipliers $\mathbf{\Pi}_{\alpha_\mathbf{L}}$ corresponding to τI_{α_L} (see Proposition 1 with $\alpha=\alpha_L$) and the set of multipliers $\mathbf{\Sigma}$ corresponding to $I_L\otimes\bar{\theta}$ (see Proposition 2). Then, an upper bound for $\check{\mathbf{y}}_{\mathbf{wc}}(A)$ (see (15)) is the solution y^{opt}_{max} of the following LMI optimization problem having as decision variables a real scalar $y_{max}\ge 0$, two matrices $\Pi^1_{\alpha_L}$, $\Pi^2_{\alpha_L}$ in $\mathbf{\Pi}_{\alpha_\mathbf{L}}$ and two matrices Σ^1, Σ^2 in $\mathbf{\Sigma}$:*

$$\arg\ \min\ y_{max}\ s.t.$$

$$\mathcal{V}^*_{y,1}\Pi^1_{\alpha_L}\mathcal{V}_{y,1}+\mathcal{V}^*_{y,2}\Sigma^1\mathcal{V}_{y,2}+\mathcal{V}^*_{y,3}\begin{pmatrix}0 & 0.5\\ 0.5 & -y_{max}\end{pmatrix}\mathcal{V}_{y,3}\le 0 \quad (37)$$

$$0\le -\mathcal{V}^*_{y,1}\Pi^2_{\alpha_L}\mathcal{V}_{y,1}-\mathcal{V}^*_{y,2}\Sigma^2\mathcal{V}_{y,2}+\mathcal{V}^*_{y,3}\begin{pmatrix}0 & 0.5\\ 0.5 & y_{max}\end{pmatrix}\mathcal{V}_{y,3} \quad (38)$$

with

$$\mathcal{V}_{y,1}=\begin{pmatrix}I & 0 & 0\\ M^{\tau\tau}_{11,\mathcal{Y}} & M^{\tau\bar{\theta}}_{11,\mathcal{Y}} & M^{\tau}_{12,\mathcal{Y}}\end{pmatrix}\quad \mathcal{V}_{y,2}=\begin{pmatrix}0 & I & 0\\ M^{\bar{\theta}\tau}_{11,\mathcal{Y}} & M^{\bar{\theta}\bar{\theta}}_{11,\mathcal{Y}} & M^{\bar{\theta}}_{12,\mathcal{Y}}\end{pmatrix}$$

$$\mathcal{V}_{y,3}=\begin{pmatrix}\mathcal{A}^T M^{\tau}_{21,\mathcal{Y}} & \mathcal{A}^T M^{\bar{\theta}}_{21,\mathcal{Y}} & \mathcal{A}^T M_{22,\mathcal{Y}}\\ 0 & 0 & 1\end{pmatrix}.$$

Proof. Let us first prove that (37) and (38) imply (30). For this purpose, let us consider the signals p_τ, $p_{\bar{\theta}}$, q_τ and $q_{\bar{\theta}}$ in the LFT (31) for a fixed $\tau\in\mathcal{T}$ and for a fixed $\bar{\theta}\in U_{init}$. Let us then pre- and post-multiply with $(p^*_\tau,p^*_{\bar{\theta}},1)$ and $(p^T_\tau,p^T_{\bar{\theta}},1)^T$ the LMI constraints (37) and (38). Using (31), this yields

$$g^*_\tau\ \Pi^1_{\alpha_L}\ g_\tau+g^*_{\bar{\theta}}\ \Sigma^1\ g_{\bar{\theta}}+Re(\mathcal{A}^T\mathcal{Y}(\tau,\bar{\theta}))\le y_{max} \quad (39)$$

$$-y_{max}\le -g^*_\tau\ \Pi^2_{\alpha_L}\ g_\tau-g^*_{\bar{\theta}}\ \Sigma^2\ g_{\bar{\theta}}+Re(\mathcal{A}^T\mathcal{Y}(\tau,\bar{\theta})) \quad (40)$$

with $g_\tau=(p^T_\tau,q^T_\tau)^T$ and $g_{\bar{\theta}}=(p^T_{\bar{\theta}},q^T_{\bar{\theta}})^T$. The above reasoning can be done for any value of $\tau\in\mathcal{T}$ and for any value of $\bar{\theta}\in U_{init}$. In other words, for the multipliers $\Pi^1_{\alpha_L}$, $\Pi^2_{\alpha_L}$, Σ^1 and Σ^2 found by the optimization problem, (39) and (40) hold true for all $\tau\in\mathcal{T}$ and for all $\bar{\theta}\in U_{init}$. Observe also that, because of (31),

$$g_\tau=\begin{pmatrix}\tau I_{\alpha_L}\\ I_{\alpha_L}\end{pmatrix}q_\tau\ \ and\ \ g_{\bar{\theta}}=\begin{pmatrix}I_L\otimes\bar{\theta}\\ I_L\end{pmatrix}q_{\bar{\theta}}$$

Consequently, due to Propositions 1 and 2, we have that $g^*_\tau\Pi^1_{\alpha_L}g_\tau+g^*_{\bar{\theta}}\Sigma^1 g_{\bar{\theta}}$ in (39) is positive for all $\tau\in\mathcal{T}$ and for all $\bar{\theta}\in U_{init}$. Similarly, $-g^*_\tau\Pi^2_{\alpha_L}g_\tau-g^*_{\bar{\theta}}\Sigma^2 g_{\bar{\theta}}$ in (40) is negative for all $\tau\in\mathcal{T}$ and for all $\bar{\theta}\in U_{init}$. We have thus proven that (37) and (38) imply (30). The proposition is then proven since it is shown, in Section 3, that $\check{\mathbf{y}}_{\mathbf{wc}}(A)$ is the solution y^{opt}_{max} of the optimization problem consisting in minimizing y_{max} under the constraint (30). ■

Proposition 4 makes use of the LFT (31) for $\mathcal{Y}(\tau,\bar{\theta})$ and the sets of multipliers for the two uncertainties appearing in this LFT. The sets of multipliers given in Propositions 1 and 2 describe well the two uncertainties when they are considered separately. However, since these two uncertainties are here combined, we will generally have a tighter upper bound for $\check{\mathbf{y}}_{\mathbf{wc}}(A)$ (see, e.g., [3]) if the multiplier Σ linked to the second uncertainty $\bar{\theta}$ is made dependent on τ i.e., the first uncertainty. The following proposition gives a τ-dependent version of Σ. This proposition is an extension of our earlier results in [6].

Proposition 5 *Consider the set of multipliers $\mathbf{\Sigma}$ linked to $I_L\otimes\bar{\theta}$ with $\bar{\theta}\in U_{init}$ (see Proposition 2). Consider also the variable τ defined in (23) and that varies in the set $\mathcal{T}$ (see (25)). Using τ and some user-chosen integer $b\ge 1$, define the vector $\mathcal{B}(\tau)=(1,\tau,\tau^2,...,\tau^b)^T$ and the matrix $\Psi(\tau)$:*

$$\Psi(\tau)=\begin{pmatrix}\mathcal{B}(\tau)\otimes I_{kL} & 0\\ 0 & \mathcal{B}(\tau)\otimes I_L\\ I_{kL} & 0\\ 0 & I_L\end{pmatrix}.$$

Then, for each $\tau\in\mathcal{T}$, the τ-dependent matrices $\Psi^*(\tau)\mathcal{P}\Psi(\tau)$ with $\mathcal{P}$ having the structure described below are all elements Σ of $\mathbf{\Sigma}$. The matrix $\mathcal{P}$ has the following structure:

$$\mathcal{P} = \left[\begin{array}{cc} \left[\begin{array}{cc} -\Lambda \otimes I_k & 0 \\ 0 & \Lambda \end{array}\right] & \mathcal{P}_{21}^T \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{array}\right] \text{ with } \Lambda = \begin{pmatrix} Q_0 & \cdots & Q_b \\ \vdots & 0 & 0 \\ Q_b^* & 0 & 0 \end{pmatrix}$$

$$\mathcal{P}_{21} = \left(\begin{array}{c|c} \left(\tilde{B}_0, ..., \tilde{B}_b\right) + \left(0, \tilde{D}_1, ..., \tilde{D}_b\right) \left(\begin{array}{c} P_0 \\ ... \\ P_b \end{array}\right)^T - \left(\begin{array}{c} 0 \\ Z_1 \\ Z_b \end{array}\right)^T & \\ \hline \left(0, Z_1, ..., Z_b\right) + \left(P_0, ..., P_b\right) & 0 \end{array}\right)$$

$$\mathcal{P}_{22} = \begin{pmatrix} j\tilde{D}_0 & -jZ_0^T \\ jZ_0 & 0 \end{pmatrix}.$$

The real matrices $\tilde{B}_i$, $\tilde{D}_i$, P_i and Z_i $(i = 0, ..., b)$ of the parametrization of $\mathcal{P}$ can take any values provided that they have, respectively, the same structure as the real matrices $\tilde{B}$, $\tilde{D}$, P and Z in (34)*. Finally, provided that $(\mathcal{B}^*(\tau) \otimes I_L)\Lambda(\mathcal{B}(\tau) \otimes I_L) \geq 0$ for all $\tau \in \mathcal{T}$, the matrix Q_0 can be any Hermitian matrix of dimension $L \times L$ and Q_i $(i = 1, ..., b)$ any complex matrices of dimension $L \times L$.*

When Q_i, $\tilde{B}_i$, $\tilde{D}_i$, P_i and Z_i are all equal to zero for $i = 1, ..., b$, $\Psi^*(\tau)\mathcal{P}\Psi(\tau)$ becomes τ-independent and is equal to (34) *with Q, $\tilde{B}$, $\tilde{D}$, P and Z replaced, respectively, by Q_0, $2\tilde{B}_0$, $\tilde{D}_0$, $2P_0$ and Z_0.*

Proof. See Appendix B. ∎

In the sequel, we will use the τ-dependent multiplier $\Psi^*(\tau)\mathcal{P}\Psi(\tau)$ instead of Σ in the constraints of the LMI optimization problem. The property given at the end of Proposition 5 illustrates that the τ-independent multiplier Σ is a special case of the τ-dependent multiplier $\Psi^*(\tau)\mathcal{P}\Psi(\tau)$. We therefore expect that, in the vast majority of the cases, the upper bound obtained with this τ-dependent multiplier will be tighter than (or at least equal to) the upper bound obtained with Proposition 4. In order to formulate the LMI optimization problem involving the τ-dependent multiplier, we note that the (minimal) LFT expression of $\mathcal{B}(\tau)$ can be deduced similarly as the one of $\mathcal{U}(\tau)$ (see Section 3.2) and that the (minimal) LFT expressions of $\mathcal{B}(\tau) \otimes I_L$ and of $\Psi(\tau)$ (that will be required in the sequel) can be easily deduced from the one of $\mathcal{B}(\tau)$:

$$\mathcal{B}(\tau) \otimes I_L = (\tau I_{bL}) \star \left(\begin{array}{c|c} \mathcal{B}_{11} & \mathcal{B}_{12} \\ \hline \mathcal{B}_{21} & \mathcal{B}_{22} \end{array}\right) \tag{41}$$

$$\Psi(\tau) = \left(\tau I_{bL(k+1)}\right) \star \left(\begin{array}{c|c} \Psi_{11} & \Psi_{12} \\ \hline \Psi_{21} & \Psi_{22} \end{array}\right). \tag{42}$$

where $\mathcal{B}_{11}$ (resp. Ψ_{11}) is a matrix of dimension $bL \times bL$ (resp. $bL(k+1) \times bL(k+1)$). In order to address the dependence on the uncertainty τ of the multiplier $\Psi^*(\tau)\mathcal{P}\Psi(\tau)$, we will need to combine the LFT (42) of $\Psi(\tau)$ and the LFT (31) of $\mathcal{Y}(\tau, \bar{\theta})$ (i.e., the LFT that is used in Proposition 4). In particular, we will need to construct the LFT of $(\mathcal{Y}_\Psi^T(\tau, \bar{\theta}),\ \mathcal{Y}^T(\tau, \bar{\theta}))^T$ with

$$\mathcal{Y}_\Psi(\tau, \bar{\theta}) = \Psi(\tau) g_{\bar{\theta}}$$

where $g_{\bar{\theta}} = (p_{\bar{\theta}}^T, q_{\bar{\theta}}^T)^T$ with $p_{\bar{\theta}}$ and $q_{\bar{\theta}}$ the internal signals in the LFT (31). That we will need this particular LFT is linked to the fact that, in (39), the multiplier Σ is used in combination with $g_{\bar{\theta}}$. In order to derive this LFT, let us first observe that (31) gives

$$g_{\bar{\theta}} = \underbrace{\begin{pmatrix} 0 & I & 0 \\ M_{11,\mathcal{Y}}^{\bar{\theta}\tau} & M_{11,\mathcal{Y}}^{\bar{\theta}\bar{\theta}} & M_{12,\mathcal{Y}}^{\bar{\theta}} \end{pmatrix}}_{=\mathcal{G}} \begin{pmatrix} p_\tau \\ p_{\bar{\theta}} \\ 1 \end{pmatrix}.$$

Then, denoting p_ψ and q_ψ the internal signals of the LFT (42) (i.e., $p_\psi = \left(\tau I_{bL(k+1)}\right) q_\psi$), we obtain

$$\begin{pmatrix} p_\psi \\ p_\tau \\ p_{\bar{\theta}} \end{pmatrix} = bdiag\left(\tau I_{bL(k+1)},\ \tau I_{\alpha_L},\ I_L \otimes \bar{\theta}\right) \begin{pmatrix} q_\psi \\ q_\tau \\ q_{\bar{\theta}} \end{pmatrix}$$

$$\begin{pmatrix} q_\psi \\ q_\tau \\ q_{\bar{\theta}} \\ \mathcal{Y}_\Psi(\tau, \bar{\theta}) \\ \mathcal{Y}(\tau, \bar{\theta}) \end{pmatrix} = \begin{pmatrix} \mathcal{M}_1 \\ \begin{pmatrix} 0 & M_{11,\mathcal{Y}}^{\tau\tau} & M_{11,\mathcal{Y}}^{\tau\bar{\theta}} & M_{12,\mathcal{Y}}^{\tau} \end{pmatrix} \\ \begin{pmatrix} 0 & M_{11,\mathcal{Y}}^{\bar{\theta}\tau} & M_{11,\mathcal{Y}}^{\bar{\theta}\bar{\theta}} & M_{12,\mathcal{Y}}^{\bar{\theta}} \end{pmatrix} \\ \mathcal{M}_2 \\ \begin{pmatrix} 0 & M_{21,\mathcal{Y}}^{\tau} & M_{21,\mathcal{Y}}^{\bar{\theta}} & M_{22,\mathcal{Y}} \end{pmatrix} \end{pmatrix} \begin{pmatrix} p_\psi \\ p_\tau \\ p_{\bar{\theta}} \\ 1 \end{pmatrix} \tag{43}$$

$$\text{with } \mathcal{M}_1 = \left(\Psi_{11} \middle| \Psi_{12}\mathcal{G}\right) \text{ and } \mathcal{M}_2 = \left(\Psi_{21} \middle| \Psi_{22}\mathcal{G}\right).$$

We have now all the ingredients to improve the upper bound for $\breve{\mathbf{y}}_{\mathbf{wc}}(A)$ using the τ-dependent multiplier $\Psi^*(\tau)\mathcal{P}\Psi(\tau)$.

Proposition 6 *Consider a given amplitude vector A and its complex expression $\mathcal{A}$ (see* (21)*). Consider also the vector $\mathcal{B}(\tau)$ for a given $b \geq 1$ and the corresponding matrix $\Psi(\tau)$ (see Proposition 5). Consider finally the LFTs* (41) *and* (43)*. Define $\alpha_{tot} = \alpha_L + bL(k+1)$. Then, an upper bound for $\breve{\mathbf{y}}_{\mathbf{wc}}(A)$ (see* (15)*) is the solution y_{max}^{opt} of the following LMI optimization problem having as decision variables a real scalar $y_{max} \geq 0$, two matrices $\Pi_{\alpha_{tot}}^1$, $\Pi_{\alpha_{tot}}^2$ in $\mathbf{\Pi_{\alpha_{tot}}}$ (see Proposition 1 with $\alpha = \alpha_{tot}$), two matrices $\mathcal{P}^1$ and $\mathcal{P}^2$ having the structure of the matrix $\mathcal{P}$ in Proposition 5 and finally two matrices*

Π^1_{bL}, Π^2_{bL} *in* $\mathbf{\Pi_{bL}}$ *(see Proposition 1 with* $\alpha = bL$*):*

$$\arg\ \min\ y_{max}\ s.t.$$

$$\mathcal{V}_1^*\Pi^1_{\alpha_{tot}}\mathcal{V}_1 + \mathcal{M}_2^*\ \mathcal{P}^1\mathcal{M}_2 + \ \mathcal{V}_2^*\begin{pmatrix} 0 & 0.5 \\ 0.5 & -y_{max}\end{pmatrix}\mathcal{V}_2 \leq 0 \tag{44}$$

$$0 \leq -\mathcal{V}_1^*\Pi^2_{\alpha_{tot}}\mathcal{V}_1 - \mathcal{M}_2^*\mathcal{P}^2\mathcal{M}_2 + \ \mathcal{V}_2^*\begin{pmatrix} 0 & 0.5 \\ 0.5 & y_{max}\end{pmatrix}\mathcal{V}_2 \tag{45}$$

$$0 \leq -\mathcal{V}_3^*\Pi^1_{bL}\mathcal{V}_3 + \mathcal{V}_4^*\Lambda^1\mathcal{V}_4 \tag{46}$$

$$0 \leq -\mathcal{V}_3^*\Pi^2_{bL}\mathcal{V}_3 + \mathcal{V}_4^*\Lambda^2\mathcal{V}_4 \tag{47}$$

where Λ^1 *(resp.* Λ^2*) is the element* Λ *(see Proposition 5) of the matrix* $\mathcal{P}^1$ *(resp.* $\mathcal{P}^2$*) and where*

$$\mathcal{V}_1 = \begin{pmatrix} \begin{pmatrix} I & 0 & 0 & 0\end{pmatrix} \\ \begin{pmatrix} 0 & I & 0 & 0\end{pmatrix} \\ \mathcal{M}_1 \\ \begin{pmatrix} 0 & M^{\tau\tau}_{11,\mathcal{Y}} & M^{\tau\bar\theta}_{11,\mathcal{Y}} & M^{\tau}_{12,\mathcal{Y}}\end{pmatrix}\end{pmatrix}$$

$$\mathcal{V}_2 = \begin{pmatrix} 0 & \mathcal{A}^T M^{\tau}_{21,\mathcal{Y}} & \mathcal{A}^T M^{\bar\theta}_{21,\mathcal{Y}} & \mathcal{A}^T M_{22,\mathcal{Y}} \\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$\mathcal{V}_3 = \begin{pmatrix} I & 0 \\ \mathcal{B}_{11} & \mathcal{B}_{12}\end{pmatrix} \ \ and \ \ \mathcal{V}_4 = \begin{pmatrix} \mathcal{B}_{21} & \mathcal{B}_{22}\end{pmatrix}.$$

Proof. Let us first note that, due to Lemma 1, (46) is equivalent to $(\mathcal{B}^*(\tau)\otimes I_L)\Lambda^1(\mathcal{B}(\tau)\otimes I_L) \geq 0$ for all $\tau \in \mathcal{T}$ which is a requirement of the parametrization of Proposition 5. The LMI (47) ensures the same property for Λ^2. Let us then prove that (44) and (45) imply (30). For this purpose, let us consider (43) for a given $\tau \in \mathcal{T}$ and for a given $\bar\theta \in U_{init}$ and let us consider the corresponding signals p_ψ, p_τ, $p_{\bar\theta}$, q_ψ, q_τ and $q_{\bar\theta}$. Let us then pre- and post-multiply with $(p^*_\psi, p^*_\tau, p^*_{\bar\theta}, 1)$ and $(p^T_\psi, p^T_\tau, p^T_{\bar\theta}, 1)^T$ the LMI constraints (44) and (45). Using (43) and the fact that $\mathcal{Y}_\Psi(\tau,\bar\theta) = \Psi(\tau)g_{\bar\theta}$ ($g_{\bar\theta} = (p^T_{\bar\theta}, q^T_{\bar\theta})^T$), this yields

$$g^*_{\psi\tau}\ \Pi^1_{\alpha_{tot}}\ g_{\psi\tau} + g^*_{\bar\theta}\ \Psi^*(\tau)\mathcal{P}^1\Psi(\tau)\ g_{\bar\theta} + Re(\mathcal{A}^T\mathcal{Y}(\tau,\bar\theta)) \leq y_{max} \tag{48}$$

$$-y_{max} \leq -g^*_{\psi\tau}\ \Pi^2_{\alpha_{tot}}\ g_{\psi\tau} - g^*_{\bar\theta}\ \Psi^*(\tau)\mathcal{P}^2\Psi(\tau)\ g_{\bar\theta} + Re(\mathcal{A}^T\mathcal{Y}(\tau,\bar\theta)) \tag{49}$$

with $g_{\psi\tau} = (p^T_\psi, p^T_\tau, q^T_\psi, q^T_\tau)^T$. The above reasoning can be done for any value of $\tau \in \mathcal{T}$ and for any value of $\bar\theta \in U_{init}$. In other words, for the multipliers $\Pi^1_{\alpha_{tot}}$, $\Pi^2_{\alpha_{tot}}$, $\mathcal{P}^1$ and $\mathcal{P}^2$ found by the optimization problem, (48) and (49) hold true for all $\tau \in \mathcal{T}$ and for all $\bar\theta \in U_{init}$. Observe also that, because of (43),

$$g_{\psi\tau} = \begin{pmatrix} \tau I_{\alpha_{tot}} \\ I_{\alpha_{tot}}\end{pmatrix} q_{\psi\tau} \ \ and \ \ g_{\bar\theta} = \begin{pmatrix} I_L\otimes\bar\theta \\ I_L\end{pmatrix} q_{\bar\theta}$$

with $q_{\psi\tau} = (q^T_\psi, q^T_\tau)^T$. Consequently, due to Propositions 1, 2 and 5, we have that $g^*_{\psi\tau}\Pi^1_{\alpha_{tot}}g_{\psi\tau} + g^*_{\bar\theta}\Psi^*(\tau)\mathcal{P}^1\Psi(\tau)g_{\bar\theta}$ in (48) is positive for all $\tau \in \mathcal{T}$ and for all $\bar\theta \in U_{init}$. Similarly, $-g^*_{\psi\tau}\Pi^2_{\alpha_{tot}}g_{\psi\tau} - g^*_{\bar\theta}\Psi^*(\tau)\mathcal{P}^2\Psi(\tau)g_{\bar\theta}$ in (49) is negative for all $\tau \in \mathcal{T}$ and for all $\bar\theta \in U_{init}$. We have thus proven that (44) and (45) imply (30). The proposition is then proven since it is shown, in Section 3, that $\breve{\mathbf{y}}_{\mathbf{wc}}(A)$ is the solution y^{opt}_{max} of the optimization problem consisting in minimizing y_{max} under the constraint (30). ■

As already mentioned, Proposition 6 will in the vast majority of the cases yield a tighter upper bound for $\breve{\mathbf{y}}_{\mathbf{wc}}(A)$ than Proposition 4 and, if necessary, the conservatism can be further reduced by increasing the value of b. However, it is also clear that the LMI problem of Proposition 6 is more complex than the one of Proposition 4 and this complexity increases with the value of b. Using Proposition 6 will thus take more computation time than using Proposition 4.

Remark. The conservatism of a given upper bound $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A)$ for $\breve{\mathbf{y}}_{\mathbf{wc}}(A)$ (computed either via Proposition 4 or via Proposition 6) can be evaluated by comparing this upper bound with a lower bound for $\breve{\mathbf{y}}_{\mathbf{wc}}(A)$. This lower bound can be computed by choosing a number of grid points in U_{init} and by determining, among those grid points $\bar\theta_i$, the one (i.e., $\bar\theta_{wc}$) leading to the largest value of $\max_{t\in T_{grid}}|\breve{y}(t,A,\bar\theta_i)|$ (T_{grid} is a fine grid of T_P). The lower bound for $\breve{\mathbf{y}}_{\mathbf{wc}}(A)$ is then given by $\max_{t\in T_{grid}}|\breve{y}(t,A,\bar\theta_{wc})|$.

4 Optimal experiment design with robust amplitude constraints

4.1 Integration of the LMI formulations for the amplitude constraints

Let us now come back to the robust optimal experiment design problem (18) whose objective is the design of an optimal multisine excitation. This optimization problem contains the constraint (17) where the quantity $\breve{\mathbf{y}}_{\mathbf{wc}}(A)$ cannot be exactly computed. We therefore replace (17) by the constraint $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A) \leq \bar{\mathbf{y}}_{\mathbf{max}}$ that implies (17). This yields the following tractable optimal experiment design problem:

$$\begin{aligned} &\arg\max_{\xi,A}\ \xi \\ &s.t.\ P_\theta^{-1}(A,\hat\theta_{init}) \geq \xi\ R_{adm} \\ &\text{and } \mathbf{u}(A) \leq \bar{\mathbf{u}}_{\mathbf{max}} \text{ and } \breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A) \leq \bar{\mathbf{y}}_{\mathbf{max}}\end{aligned} \tag{50}$$

where $\mathbf{u}(A)$ can be computed via Proposition 3 and $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A)$ via Proposition 4 or via Proposition 6. The optimization problem (50) is a constrained [4] non-convex

[4] The first constraint of (50) can be removed by considering as objective function $1/\lambda_{max}\left(P_\theta(A,\hat\theta_{init})\ R_{adm}\right)$.

optimization problem which can, e.g., be addressed with a SQP algorithm (this algorithm is, e.g., implemented in the Matlab function *fmincon*). Since (50) is non-convex, the SQP algorithm is not guaranteed to yield the global optimum and the performance of this algorithm will depend on its initialization (that must be close to the global optimum). In Section 5, we will propose two approaches to initialize the SQP algorithm for (50). These two approaches will yield an amplitude vector A_{app} that approaches the solution of the optimization problem (50) (or the original optimization problem (18)).

4.2 Unconstrained optimization problem equivalent to (50)

In this section, we will show that (50) can be transformed into an unconstrained optimization problem. For this purpose, let us first determine the solution of (50) in the direction of a given vector A_u.

Proposition 7 *Let us consider the optimization problem (50) where we restrict attention to amplitude vectors A that can be written as $A = \kappa A_u$ with a positive real scalar $\kappa > 0$ and with a given vector A_u (e.g., a normalized vector such that $A_u^T A_u = 1$). Then, the solutions $A_{opt}(A_u)$ and $\xi_{opt}(A_u)$ of this particular optimization problem are given by*

$$A_{opt}(A_u) = \kappa_{opt}\left(A_u\right)\ A_u \tag{51}$$

$$\xi_{opt}(A_u) = \frac{1}{\lambda_{max}\left(P_\theta(\kappa_{opt}\left(A_u\right)\ A_u, \hat{\theta}_{init})\ R_{adm}\right)} \tag{52}$$

with $\kappa_{opt}(A)$ a function defined as follows for any amplitude vector A:

$$\kappa_{opt}(A) = min\left(\frac{\bar{\mathbf{u}}_{\mathbf{max}}}{\mathbf{u}(A)}, \frac{\bar{\mathbf{y}}_{\mathbf{max}}}{\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A)}\right) \tag{53}$$

where $min(x, y)$ is equal to x when $x \leq y$ and is equal to y otherwise.

Proof. Let us consider two amplitude vectors $A_1 = \kappa_1 A_u$ and $A_2 = \kappa_2 A_u$ with κ_1 and κ_2 two scalars such that $\kappa_1 > \kappa_2 > 0$. For these two amplitude vectors, we have that $P_\theta^{-1}(A_1, \hat{\theta}_{init}) > P_\theta^{-1}(A_2, \hat{\theta}_{init})$. This means that the optimal value of the amplitude vector $A = \kappa A_u$ will be characterized by the largest value of $\kappa > 0$ such that $\mathbf{u}(\kappa A_u) \leq \bar{\mathbf{u}}_{\mathbf{max}}$ and $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(\kappa A_u) \leq \bar{\mathbf{y}}_{\mathbf{max}}$. Since $\mathbf{u}(\kappa A_u) = \kappa \mathbf{u}(A_u)$ and $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(\kappa A_u) = \kappa \breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A_u)$ (see Appendix C), this largest value of κ is clearly equal to $\kappa_{opt}(A_u)$ (see (53)). The expressions (51) and (52) then follow directly. ∎

As indicated in the proof of Proposition 7, $\kappa_{opt}(A_u)$ (see (53)) is the largest value of $\kappa > 0$ such that both $\mathbf{u}(\kappa A_u) \leq \bar{\mathbf{u}}_{\mathbf{max}}$ and $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(\kappa A_u) \leq \bar{\mathbf{y}}_{\mathbf{max}}$. Consequently, the scaled vector $A_{opt}(A_u)$ (see (51)) satisfies either $\mathbf{u}(A_{opt}(A_u)) = \bar{\mathbf{u}}_{\mathbf{max}}$ or $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A_{opt}(A_u)) = \bar{\mathbf{y}}_{\mathbf{max}}$.

Proposition 7 gives the solution of (50) in a given direction A_u. In Proposition 8, we will see that the determination of the optimal direction can be formulated as an unconstrained optimization problem. For this purpose, we need to parametrize the set of all possible directions. Let us for this purpose introduce the vector $\varphi = (\varphi_1, \varphi_2, ..., \varphi_{2L-1})^T$ containing $2L-1$ angles and let us define, based on φ, the following unit vector $A_u(\varphi) \in \mathbb{R}^{2L}$:

$$A_u(\varphi) = \begin{pmatrix} cos(\varphi_1) \\ sin(\varphi_1)cos(\varphi_2) \\ ... \\ sin(\varphi_1)...sin(\varphi_{2L-2})cos(\varphi_{2L-1}) \\ sin(\varphi_1)...sin(\varphi_{2L-2})sin(\varphi_{2L-1}) \end{pmatrix}. \tag{54}$$

The vector φ corresponds to the spherical coordinates of the vector $A_u(\varphi)$ and we have that the set $\{A_u(\varphi) \mid \varphi \in \mathbb{R}^{2L-1}\}$ is equal [5] to $\{A_u \in \mathbb{R}^{2L} \mid A_u^T A_u = 1\}$ [4].

Using this parametrization and Proposition 7, we can now propose the following equivalent formulation of (50).

Proposition 8 *Consider the function $\kappa_{opt}(A)$ defined based on the computable quantities $\mathbf{u}(A)$ and $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A)$ (see (53)) and the parametrization $A_u(\varphi)$ of the normalized amplitude vectors A_u (see (54)). Then, the solutions A_{opt} and ξ_{opt} of the optimization problem (50) are given by*

$$A_{opt} = \kappa_{opt}\left(A_u(\varphi_{opt})\right)\ A_u(\varphi_{opt}) \tag{55}$$

$$\xi_{opt} = \frac{1}{\lambda_{max}\left(P_\theta(A_{opt}, \hat{\theta}_{init})\ R_{adm}\right)} \tag{56}$$

where φ_{opt} is the solution of the following unconstrained optimization problem:

$$\arg \min_{\varphi \in \mathbb{R}^{2L-1}} \lambda_{max}\left(P_\theta(\kappa_{opt}(A_u(\varphi))\ A_u(\varphi), \hat{\theta}_{init})\ R_{adm}\right). \tag{57}$$

Moreover, the optimum A_{opt} given in (55) satisfies either $\mathbf{u}(A_{opt}) = \bar{\mathbf{u}}_{\mathbf{max}}$ or $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A_{opt}) = \bar{\mathbf{y}}_{\mathbf{max}}$.

Proof. Each amplitude vector A is entirely characterized by its direction $A_u(\varphi)$ (parametrized by the vector φ) and its norm κ i.e., $A(\kappa, \varphi) = \kappa A_u(\varphi)$. Recall that Proposition 7 gives the optimal value of κ for each direction $A_u(\varphi)$. It remains thus to determine the optimal direction (or equivalently the optimal value of φ) i.e., the value of φ yielding the maximal value of $\xi_{opt}(A_u(\varphi))$ (see (52)). This is exactly the objective of (57). The expressions (55) and (56) then follow from an application of Proposition 7 for $A_u = A_u(\varphi_{opt})$ and the last statement of the proposition is a consequence of the definition (53) of $\kappa_{opt}(A)$ ∎

[5] It is in fact sufficient to restrict the angles φ_i ($i = 1, ..., 2L-2$) to the interval $[0,\ \pi]$ and the angle φ_{2L-1} to the interval $[0,\ 2\pi]$.

The unconstrained non-linear optimization problem (57) can be addressed using a classical gradient-descent algorithm (such algorithm is, e.g., implemented in the Matlab function *fminunc*). Since (57) is also non-convex, the gradient-based algorithm is not guaranteed to yield the global optimum and the performance of this algorithm will depend on its initialization (that must be close to the global optimum). Like for (50), we can here also use the procedures of Section 5 for this initialization.

In the numerical simulations that we have performed with Matlab (see Section 6), we have observed that the gradient-based algorithm (Matlab function *fminunc*) used to solve (57) and the SQP algorithm (Matlab function *fmincon*) used to solve (50) yield the same optimum when initialized at the same point, but the gradient-based algorithm yields this optimum in less computation time.

Remark. Let us denote by φ_{lm} the (local or global) minimum to which the gradient-based algorithm used to solve (57) converges and let denote by A_{lm} the corresponding amplitude vector i.e., $A_{lm} = \kappa_{opt}\left(A_u(\varphi_{lm})\right) A_u(\varphi_{lm})$ (see (55)). Due to (53), the constraints (16) and (17) are satisfied at A_{lm} and we have that either $\mathbf{u}(A_{lm}) = \bar{\mathbf{u}}_{\mathbf{max}}$ or $\check{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A_{lm}) = \bar{\mathbf{y}}_{\mathbf{max}}$. Note that, due to Assumption 1, the constraint (6) is also satisfied for this amplitude vector A_{lm}.

5 Two approaches to initialize the algorithm used to solve the non-convex optimal experiment design problem

5.1 Introduction

In order to initialize the algorithm used to solve the non-convex optimal experiment design problem, we propose two approaches yielding amplitude vectors A_{app} close to the solution of the optimal experiment design problem (50) (or equivalently close to the solution of (18) from which (50) follows).

5.2 Semi-definite relaxation inspired from [21]

In [21], an experiment design problem with amplitude constraints is considered for the case where the excitation signal is parametrized via the N elements $u[n]$ $(n = 0, ..., N-1)$ of the excitation sequence. Let us adapt the framework of [21] to the (more compact) parametrization of the excitation signal $u(t, A)$ based on the amplitude vector A of dimension $2L$ (which is generally much smaller than N). The optimization problem (18) can be reformulated as follows:

$$\arg\max_{\xi, A} \ \xi$$

$$s.t. \left(\sum_{i=1}^{L} M(\omega_i, \hat{\theta}_{init}) \left(a_{i,s}^2 + a_{i,c}^2\right)\right) + M_v(\hat{\theta}_{init}) \geq \xi \ R_{adm} \tag{58}$$

$$and \ \mathcal{Z}_u^T(t) \ AA^T \ \mathcal{Z}_u(t) \leq \bar{\mathbf{u}}_{\mathbf{max}}^2 \quad \forall t \in T_P$$

$$and \ \mathcal{Z}_y^T(t, \bar{\theta}) \ AA^T \ \mathcal{Z}_y(t, \bar{\theta}) \leq \bar{\mathbf{y}}_{\mathbf{max}}^2 \quad \forall t \in T_P \ and \ \forall \bar{\theta} \in U_{init}$$

where $\mathcal{Z}_u(t) = (sin(\omega_1 t), cos(\omega_1 t), sin(\omega_2 t), ..., cos(\omega_L t))^T$ and

$$\mathcal{Z}_y(t, \bar{\theta}) = \begin{pmatrix} g(\omega_1, \bar{\theta}) \ sin(\omega_1 t + \phi(\omega_1, \bar{\theta})) \\ g(\omega_1, \bar{\theta}) \ cos(\omega_1 t + \phi(\omega_1, \bar{\theta})) \\ g(\omega_2, \bar{\theta}) \ sin(\omega_2 t + \phi(\omega_2, \bar{\theta})) \\ ... \\ g(\omega_L, \bar{\theta}) \ cos(\omega_L t + \phi(\omega_L, \bar{\theta})) \end{pmatrix}.$$

In order to deal with the amplitude constraints that have to be satisfied for all $t \in T_P$ in (58), we will here for simplicity use a gridding approximation i.e., we will only impose the amplitude constraints at the time instants in a fine grid T_{grid} of T_P. A similar gridding approximation will be used (as proposed in [21]) to address the fact that the output amplitude constraint has to be satisfied for all $\bar{\theta} \in U_{init}$. For this purpose, we will consider a set Θ_{grid} containing a number of grid points $\bar{\theta}_i \in U_{init}$.

Let us now notice as in [21] that, in the optimization problem (58), the matrix $X = AA^T$ is a positive semi-definite matrix with $rank(X) = 1$. Let us also notice that $a_{i,s}^2$ and $a_{i,c}^2$ (variables appearing in the expression of P_θ) are also entries of the matrix X. Using this fact and the approximations introduced in the previous paragraph, (58) can be reformulated as follows:

$$\arg\max_{\xi, X = X^T \geq 0} \ \xi \tag{59}$$

$$s.t. \sum_{i=1}^{L} M(\omega_i, \hat{\theta}_{init}) \left(X_{2i-1,2i-1} + X_{2i,2i}\right) + M_v(\hat{\theta}_{init}) \geq \xi R_{adm} \tag{60}$$

$$and \ \mathcal{Z}_u^T(t) \ X \ \mathcal{Z}_u(t) \leq \bar{\mathbf{u}}_{\mathbf{max}}^2 \quad \forall t \in T_{grid} \tag{61}$$

$$and \ \mathcal{Z}_y^T(t, \bar{\theta}) \ X \ \mathcal{Z}_y(t, \bar{\theta}) \leq \bar{\mathbf{y}}_{\mathbf{max}}^2 \quad \forall t \in T_{grid} \ and \ \forall \bar{\theta} \in \Theta_{grid} \tag{62}$$

$$and \ rank(X) = 1. \tag{63}$$

In (60), we use the notation $X_{i,j}$ for the entry (i, j) of the matrix X. Similarly as was observed in [21], the optimization problem (59)-(63) would be an LMI optimization problem in the absence of the rank constraint (63). The relaxation proposed in [21,19] is to solve the optimization problem by neglecting the rank constraint (63) i.e., to solve the optimization problem (59)-(62). As shown in [19], the optimal solution X_{opt} of this optimization problem allows to derive, via a randomization procedure, an amplitude vector A_{app} that approaches the solution of the original optimization problem (59)-(63). We will here slightly modify this

randomization procedure to compensate for the approximations introduced by the use of T_{grid} and Θ_{grid}. This will be achieved by the use of the computable quantities $\mathbf{u}(A)$ and $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A)$ (see Section 3).

The randomization procedure is as follows: we generate a number n_{grid} of possible A according to a normal distribution $\mathcal{N}(0, X_{opt})$ as proposed in [19]. We then apply Proposition 7 to each of these A i.e., for each A, we determine, in the direction of A, the solution of the optimization problem (50) (i.e., the non-convex optimization problem of which A_{app} has to be close to the solution and that is formulated without gridding approximations). This procedure yields n_{grid} values $A_{scaled} = \kappa_{opt}(A)A$ and n_{grid} values of $\xi_{opt}(A)$ (see (52)). The amplitude vector A_{app} approaching the solution of (50) is then chosen as the value of A_{scaled} corresponding to the highest value of $\xi_{opt}(A)$.

Due to the use of Proposition 7 in the randomization procedure, we have the guarantee that the constraints (16) and (17) are satisfied for this amplitude vector A_{app} and also that either $\mathbf{u}(A_{app}) = \bar{\mathbf{u}}_{\mathbf{max}}$ or $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A_{app}) = \bar{\mathbf{y}}_{\mathbf{max}}$.

5.3 Use of power constraints instead of amplitude constraints

The second approach consists in solving an optimal experiment design problem with power constraints instead of amplitude constraints. The considered power constraints are $\mathcal{P}(u(t,A)) \leq \bar{\mathbf{u}}^2_{\mathbf{max}}$ and $\mathcal{P}(\breve{y}(t,A,\bar{\theta})) < \bar{\mathbf{y}}^2_{\mathbf{max}}\ \forall \bar{\theta} \in U_{init}$ where $\mathcal{P}(x(t))$ denotes the power of the signal $x(t)$. We therefore consider the following optimal experiment design problem:

$$\begin{aligned} &\arg\max_{\xi,C}\ \xi \\ s.t.\ &\sum_{i=1}^{L} M(\omega_i,\hat{\theta}_N)\ c_i + \ M_v(\hat{\theta}_N) \geq \xi\ R_{adm}\ and \\ &\frac{1}{2}\sum_{i=1}^{L} c_i \leq \bar{\mathbf{u}}^2_{\mathbf{max}} \\ &\frac{1}{2}\sum_{i=1}^{L} c_i\ |G(\mathrm{e}^{j\omega_i T_s},\bar{\theta})|^2 \leq \bar{\mathbf{y}}^2_{\mathbf{max}} \quad \forall \bar{\theta} \in U_{init}. \end{aligned} \tag{64}$$

where $c_i = a^2_{i,s} + a^2_{i,c}$ $(i = 1,...,L)$ and $C = (c_1, c_2, ..., c_L)^T$. Using the results in [7], this optimization problem can be transformed into an LMI optimization problem yielding an optimal solution $C_{opt} = (c_1^{opt}, c_2^{opt}, ..., c_L^{opt})^T$. We can here also use a randomization procedure to determine, based on c_i^{opt} $(i = 1,...,L)$, an amplitude vector A_{app} approaching the solution of the original optimal experiment design problem (50). Let us for this purpose generate a number n_{grid} of possible A satisfying $a^2_{i,s} + a^2_{i,c} = c_i^{opt}$ $(i = 1,...,L)$. Then, as in the previous subsection, we apply Proposition 7 to each of these A. This procedure yields n_{grid} values $A_{scaled} = \kappa_{opt}(A)A$ and n_{grid} values of $\xi_{opt}(A)$ (see (52)). The amplitude vector A_{app} approaching the solution of (50) is then here also chosen as the value of A_{scaled} corresponding to the highest value of $\xi_{opt}(A)$.

Like in the previous subsection, the use of Proposition 7 in the randomization procedure gives the guarantee that the constraints (16) and (17) are satisfied for this amplitude vector A_{app} and also that either $\mathbf{u}(A_{app}) = \bar{\mathbf{u}}_{\mathbf{max}}$ or $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A_{app}) = \bar{\mathbf{y}}_{\mathbf{max}}$.

6 Numerical illustration

We consider an OE true system (3) with:

$$G(z,\theta_0) = \frac{\theta_{0,1}z^{-1}+\theta_{0,2}z^{-2}}{1+\theta_{0,3}z^{-1}+\theta_{0,4}z^{-2}}$$
$$H(z,\theta_0) = 1$$

where $\theta_0 = (\theta_{0,1},\theta_{0,2},\theta_{0,3},\theta_{0,4})^T = (0.8988, 0.1034, -0.9723, 0.8385)^T$ and $\sigma_e^2 = 1$. The sampling rate is $T_s = 1\ s$. The amplitude constraint thresholds are $\bar{\mathbf{u}}_{\mathbf{max}} = \bar{\mathbf{y}}_{\mathbf{max}} = 1$. The initial uncertainty region U_{init} in (10) is characterized by $\hat{\theta}_{init} = (0.8\ 0.01\ -0.9854\ 0.8187)^T$, $\chi = 9.49$ and

$$P^{-1}_{\theta,init} = 10^3 \begin{pmatrix} 0.3150 & 0.1885 & -0.4652 & 0.2692 \\ 0.1885 & 0.3150 & -0.9327 & -0.4652 \\ -0.4652 & -0.9327 & 4.1346 & 2.4499 \\ 0.2692 & -0.4652 & 2.4499 & 4.1346 \end{pmatrix}$$

We wish to design a multisine excitation $u(t,A)$ (see (1)) of fundamental frequency $\omega_0 = 0.1\pi\ rad/s$ (the period is thus 20 s) and containing $L = 3$ frequencies i.e., $\omega_1 = \omega_0$, $\omega_2 = 3\omega_0$ and $\omega_3 = 5\omega_0$. To design the optimal amplitude vector A of $u(t,A)$, we consider the optimal experiment design problem (50) corresponding to an identification experiment of duration $N = 1000$ and where R_{adm} is chosen equal to I_4. Moreover, for the first part of this numerical example, we will use Proposition 4 to compute the upper bound $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A)$ for $\breve{\mathbf{y}}_{\mathbf{wc}}(A)$ (which appears in the constraints of (50)). Since $R_{adm} = I_4$, the objective function ξ of (50) is equal to $\lambda_{min}(P_\theta^{-1}(A,\hat{\theta}_{init}))$ which is clearly a measure of the accuracy of the identified parameter vector $\hat{\theta}_N$.

Since, in this example, determining the optimal multisine with (57) takes twice less time than with (50), we will from now on consider the equivalent formulation (57) of (50). The gradient-based algorithm (implemented in the Matlab function *fminunc*) has been here used to solve (57). An amplitude vector A_{lm} can easily be determined from the solution ϕ_{lm} of this algorithm (see the remark at the end of Section 4). For all the initializations discussed in the next paragraph, the gradient-based algorithm yielded amplitude vectors A_{lm} for which the objective function of the optimal experiment design problem [6] i.e., $\lambda_{min}(P_\theta^{-1}(A_{lm},\hat{\theta}_{init}))$ is equal to 187.87. These amplitude vectors A_{lm} are thus equivalent from

[6] We here consider the objective function of (50). The objective function of (57) is its inverse.

the point-of-view of the considered optimal experiment design problem. This equivalence can be explained by the fact that the corresponding multisines $u(t, A_{lm})$ are just time shifted versions of each other. In the sequel, we will therefore only consider one of these A_{lm} i.e., $(-0.0688, 0.2212, -0.0662, -0.0120, -0.5075, 0.4621)^T$.

As mentioned in the previous paragraph, we have considered different initializations for the gradient-based algorithm. For this purpose, we have considered the procedures of Sections 5.2 and 5.3 where, in the randomization procedure, the upper bound $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A)$ for $\breve{\mathbf{y}}_{\mathbf{wc}}(A)$ is also computed via Proposition 4. Note furthermore that, in our implementation of the procedure of Section 5.2, T_{grid} contains 201 linearly-spaced points in T_P and Θ_{grid} only contains $\hat{\theta}_{init}$ for simplicity. For both approaches, we have run the randomization procedure twice i.e., once with $n_{grid} = 100$ and once with $n_{grid} = 2000$. This has led to four amplitude vectors A_{app}. The values of the objective function $\lambda_{min}(P_\theta^{-1}(A_{app}, \hat{\theta}_{init}))$ for these four A_{app} are given in Table 1.

Table 1
Objective function $\lambda_{min}(P_\theta^{-1}(A_{app}, \hat{\theta}_{init}))$

Cases	$\lambda_{min}(P_\theta^{-1}(A_{app}, \hat{\theta}_{init}))$
Section 5.2 with $n_{grid} = 100$	159.59
Section 5.2 with $n_{grid} = 2000$	171.38
Section 5.3 with $n_{grid} = 100$	164.69
Section 5.3 with $n_{grid} = 2000$	168.33

With respect to the four A_{app} in Table 1, the amplitude vector A_{lm} obtained by solving (57) using the gradient-based algorithm improves the objective function with at least 10% since $\lambda_{min}(P_\theta^{-1}(A_{lm}, \hat{\theta}_{init})) = 187.87$. In this example, we therefore observe that the initialization procedure gives amplitude vectors A_{app} that are relatively close to A_{lm}, but also that it is useful to improve the result of this initialization procedure with (57). In this example, the robustness of the gradient-based algorithm used to solve (57) is important. To show this, we have also initialized this algorithm with a vector in the direction $A_u = (1/\sqrt{6})(1, 1, 1, 1, 1, 1)^T$. This direction is far from optimal since $\kappa_{opt}(A_u)A_u$ yields an objective function $\lambda_{min}(P_\theta^{-1}(\kappa_{opt}(A_u)A_u, \hat{\theta}_{init})) = 7.92$ which is much smaller than the values in Table 1. However, initialized with this A_u, the gradient-based algorithm also yields an amplitude vector A_{lm} for which $\lambda_{min}(P_\theta^{-1}(A_{lm}, \hat{\theta}_{init})) = 187.87$.

Let us now analyze a bit more the amplitude vector $A_{lm} = (-0.0688, 0.2212, -0.0662, -0.0120, -0.5075, 0.4621)^T$ delivered by the gradient-based algorithm used to solve (57). Using Proposition 3, we obtain $\mathbf{u}(A_{lm}) = 0.9385$ which is indeed the maximal value of $|u(t, A_{lm})|$ for $t \in T_P$. Using Proposition 4, we obtain [7] $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A_{lm}) = 1$. As expected (see the remark at the end of Section 4), for A_{lm}, one of the amplitude constraints is active (here the output amplitude constraint). Let us now investigate the conservatism of the upper bound for $\breve{\mathbf{y}}_{\mathbf{wc}}(A_{lm})$ computed with Proposition 4. For this purpose, we also compute a lower bound for $\breve{\mathbf{y}}_{\mathbf{wc}}(A_{lm})$ using the procedure given in the remark at the end of Section 3. This leads here to $0.986477 \leq \breve{\mathbf{y}}_{\mathbf{wc}}(A_{lm})$. The conservatism linked to the LMI procedure of Proposition 4 is thus less than 1.4 % (which is small). Let us nevertheless see if a tighter upper bound can be obtained via Proposition 6 with, e.g., $b = 1$. This is indeed the case since Proposition 6 with $b = 1$ yields [8] $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A_{lm}) = 0.986550$: the conservatism is now less than 0.007 % (and thus negligible).

We have tested Proposition 6 with $b = 1$ for a number of other A and the observed conservatism is always negligible. Consequently, one can now consider a version of the optimal experiment design problem (50) (or its equivalent (57)) where the upper bound for $\breve{\mathbf{y}}_{\mathbf{wc}}(A)$ is obtained using Proposition 6 ($b = 1$). Due to the reduced conservatism, this new version is closer to the original optimal experiment design problem (18). The solution of this modified optimal experiment design problem in the direction of A_{lm} (which was the best solution when Proposition 4 was used) can be determined via Proposition 7. In our case, this solution is $A_{lm,bis} = \kappa_{opt}(A_{lm})\ A_{lm}$ with $\kappa_{opt}(A_{lm}) = min(1/0.9385, 1/0.986550) \approx 1.014$. With $A_{lm,bis}$, we obtain $\lambda_{min}(P_\theta^{-1}(A_{lm,bis}, \hat{\theta}_{init})) = 193.02$, $\mathbf{u}(A_{lm,bis}) = 0.9507$ (via Proposition 3) and $\breve{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A_{lm,bis}) = 1$ (via Proposition 6 with $b = 1$). One can wonder whether another direction than the one of A_{lm} could be more optimal. For this purpose, we run the gradient-based algorithm to solve this new version of (57) with an initialization in the direction of A_{lm}. This does not improve the result. Consequently, if we restrict attention to the tools developed in this paper, $u(t, A_{lm,bis})$ seems the optimal multisine.

7 Conclusions

We have developed a methodology to tackle the problem of optimal experiment design with robust amplitude constraints. The developed framework allows to handle these robust amplitude constraints without approximation when the identification is performed in open loop and when the considered true system has one input and one output. In future work, we will extend the framework to closed-loop identification and to multivariable systems.

[7] Note that the maximal value of $|\breve{y}(t, A_{lm}, \hat{\theta}_{init})|$ for $t \in T_P$ is 0.7425 ($\hat{\theta}_{init}$ is the center of U_{init}).

[8] Note that solving the LMI problem of Proposition 6 with $b = 1$ takes 16 times more time than solving the one in Proposition 4. This is the reason why Proposition 4 was used in the earlier phase.

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A Structure of $\tilde{B}$, $\tilde{D}$, P, Z in Proposition 2

As shown in [2], the real matrices $\tilde{B}$ and $\tilde{D}$ in the expression of Σ have the following structures:

$$\tilde{B} = \begin{pmatrix} 0 & K_{12} & \dots & K_{1L} \\ -K_{12} & 0 & \dots & \vdots \\ \vdots & & \ddots & K_{(L-1)L} \\ -K_{1L} & \dots & -K_{(L-1)L} & 0 \end{pmatrix}$$

$$\tilde{D} = \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1L} \\ R_{12} & R_{22} & \dots & R_{2L} \\ \vdots & & \ddots & \vdots \\ R_{1L} & R_{2L} & \dots & R_{LL} \end{pmatrix}$$

with the constraints that all blocks K_{il} (resp. R_{il}) satisfy $K_{il} = -K_{il}^T \in \mathbb{R}^{k\times k}$ (resp. $R_{il} = -R_{il}^T \in \mathbb{R}^{k\times k}$). The matrix P (resp. Z) has a similar structure as $\tilde{B}$ (resp. $\tilde{D}$), but with the skew-symmetric blocks replaced by row vectors of dimension k [2].

B Proof of Proposition 5

The proof is rather straightforward. When we perform the product $\Psi^*(\tau)\mathcal{P}\Psi(\tau)$, we indeed obtain a matrix having the structure (34) for any value of $\tau \in \mathcal{T}$. More precisely, the matrix Q in (34) is given by $(\mathcal{B}^*(\tau)\otimes I_L)\Lambda(\mathcal{B}(\tau)\otimes I_L) = Q_0 + Q_1\tau + Q_1^*\tau^* + \dots + Q_b\tau + Q_b^*\tau^*$. It is clear that this matrix is Hermitian for all $\tau \in \mathcal{T}$.

The additional condition in the statement of the theorem ensures that this Hermitian matrix is also positive semi-definite for all $\tau \in \mathcal{T}$. When we perform the product $\Psi^*(\tau)\mathcal{P}\Psi(\tau)$, the matrix $j\tilde{D}$ in (34) is given by

$$j\tilde{D} = j\tilde{D}_0 + (0, \tilde{D}_1, ..., \tilde{D}_b)\ (\mathcal{B} \otimes I_{Lk}) + (\mathcal{B}^* \otimes I_{Lk}) \begin{pmatrix} 0 \\ \tilde{D}_1^T \\ ... \\ \tilde{D}_b^T \end{pmatrix}$$

The right hand side of this expression is equal to $j\tilde{D}_0 + \tilde{D}_1(\tau - \tau^*) + ... + \tilde{D}_b(\tau^b - (\tau^b)^*)$ since $\tilde{D}_i = -\tilde{D}_i^T$ $(i = 1, ...b)$. It is clear that this expression has indeed the desired structure since $\tau^i - (\tau^i)^*$ $(i = 1, ...b)$ is an imaginary number for all $\tau \in \mathcal{T}$. In $\Psi^*(\tau)\mathcal{P}\Psi(\tau)$, the matrix jZ in (34) is given by

$$jZ = jZ_0 + (0, Z_1, ..., Z_b)\ (\mathcal{B} \otimes I_{Lk}) - (\mathcal{B}^* \otimes I_L) \begin{pmatrix} 0 \\ Z_1 \\ ... \\ Z_b \end{pmatrix}$$

The right hand side of this expression is equal to $jZ_0 + Z_1(\tau - \tau^*) + ... + Z_b(\tau^b - (\tau^b)^*)$ which has thus also the desired structure. Following the same procedure, the matrix $\tilde{B}$ in (34) is given by $2\tilde{B}_0 + \tilde{B}_1(\tau + \tau^*) + ... + \tilde{B}_b(\tau^b + (\tau^b)^*)$ since $\tilde{B}_i = \tilde{B}_i^T$ $(i = 1, ...b)$ while the matrix P in (34) is given by $2P_0 + P_1(\tau + \tau^*) + ... + P_b(\tau^b + (\tau^b)^*)$. These expressions have the desired structure since $\tau^i + (\tau^i)^*$ $(i = 1, ...b)$ is a real number for all $\tau \in \mathcal{T}$. From the above expressions, the last statement of the proposition is straightforward.

Remark. In Proposition 1 of [6], we proposed a less general parametrization for $\mathcal{P}$ where the matrices Q_i were restricted to be real matrices, the factorizations of $j\tilde{D}$ was reduced to $\tilde{D}_1(\tau - \tau^*) + ... + \tilde{D}_b(\tau^b - (\tau^b)^*)$ and the one of jZ to $Z_1(\tau - \tau^*) + ... + Z_b(\tau^b - (\tau^b)^*)$.

C Last steps of the proof of Proposition 7

That $\mathbf{u}(\kappa A_u) = \kappa \mathbf{u}(A_u)$ follows from the definition (14) of $\mathbf{u}(A)$. That $\check{\mathbf{y}}_{\mathbf{wc}}(\kappa A_u) = \kappa \check{\mathbf{y}}_{\mathbf{wc}}(A_u)$ also follows from (15). That $\check{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(\kappa A_u) = \kappa \check{\mathbf{y}}^{\mathbf{ub}}_{\mathbf{wc}}(A_u)$ can be derived by considering the proof of Proposition 4 and, in particular, the inequalities (39) and (40). Note indeed that the existence of $\Pi^1_{\alpha_L}$, $\Pi^2_{\alpha_L}$, Σ^1 and Σ^2 such that (39) and (40) hold for all $\tau \in \mathcal{T}$ and for all $\bar{\theta} \in U_{init}$ is equivalent to the existence of $\Pi^3_{\alpha_L}$, $\Pi^4_{\alpha_L}$, Σ^3 and Σ^4 such that the following inequalities hold for all $\tau \in \mathcal{T}$ and for all $\bar{\theta} \in U_{init}$:

$$g_\tau^*\ \Pi^3_{\alpha_L}\ g_\tau + g_{\bar{\theta}}^*\ \Sigma^3\ g_{\bar{\theta}} + Re(\kappa\ \mathcal{A}^T \mathcal{Y}(\tau, \bar{\theta})) \leq \kappa\ y_{max}$$

$$-\kappa\ y_{max} \leq -g_\tau^*\ \Pi^4_{\alpha_L}\ g_\tau - g_{\bar{\theta}}^*\ \Sigma^4\ g_{\bar{\theta}} + Re(\kappa\ \mathcal{A}^T \mathcal{Y}(\tau, \bar{\theta})).$$

We can indeed choose $\Pi^3_{\alpha_L} = \kappa\Pi^1_{\alpha_L}$, $\Pi^4_{\alpha_L} = \kappa\Pi^2_{\alpha_L}$, $\Sigma^3 = \kappa\Sigma^1$ and $\Sigma^4 = \kappa\Sigma^2$. An element of $\mathbf{\Pi}_\alpha$ (resp. $\mathbf{\Sigma}$) multiplied by a positive scalar constant is indeed another element of $\mathbf{\Pi}_\alpha$ (resp. $\mathbf{\Sigma}$). A similar reasoning can also be applied to the inequalities (48) and (49) in the proof of Proposition 6.