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▶ To cite this version:

Maxime Payan, Maxime Breden. Numerical and Theoretical Study of Patterns in a Chemotaxis Model. SIAM Conference on Applications of Dynamical Systems (DS23), May 2023, Portland (OR), United States. hal-04247634

HAL Id: hal-04247634

https://hal.science/hal-04247634

Submitted on 18 Oct 2023

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NUMERICAL AND THEORETICAL STUDY OF PATTERNS IN A CHEMOTAXIS MODEL

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PDE of Keller-Segel type: Cross-diffusion and logistic reaction

We are interested in the system (cf. [3, 5]):

$$\begin{cases} \partial_t u = \Delta(\gamma(v)u) + \sigma u(1-u), & x \in (0,1), \ t > 0 \\ \partial_t v = \Delta v + & u - v, & x \in (0,1), \ t > 0 \\ \partial_x u(x,t) = 0, & \partial_x v(x,t) = 0, \ x \in \{0,1\}, \ t > 0 \end{cases}$$
(1)

Here u and v represent the density of cells and of a chemical, which is produced by the cells. We assume $\sigma \geq 0$ and that $\gamma > 0$ is decreasing, to model that the cells are attracted to the chemical.

As a first step to better understand the long-term behavior of solutions, we study the stationary system associated with (1). We will study and look for non trivial stationary states. The approach used is a computer assisted proof method applied to a fixed point problem. This guarantees the existence of solutions to (1) and gives an explicit numerical approximation with a quantified error.

Use of Fourier Series

The Neumann boundary conditions lead us to consider solutions in the form:

$$u(x) = u_0 + 2\sum_{k=1}^{+\infty} u_k \cos(k\pi x)$$
 and $v(x) = v_0 + 2\sum_{k=1}^{+\infty} v_k \cos(k\pi x)$.

Let's pose $\mathbf{u} = (u_0, u_1, \dots, u_k, \dots) = (\mathcal{F}(u)_n)_{n \in \mathbb{N}}$, idem $\mathbf{v} = (\mathcal{F}(v)_n)_{n \in \mathbb{N}}$ and $\gamma(\mathbf{v}) = (\mathcal{F}(\gamma \circ v)_n)_{n \in \mathbb{N}}$. Let v > 1 and $\ell_v^1 = \left\{ \mathbf{u} \mid ||\mathbf{u}||_v := |u_0| + 2 \sum_{k=1}^{+\infty} |u_k| v^k < +\infty \right\}.$

Provided with the discrete convolution *, ℓ_{ν}^{1} is a Banach algebra.

Through ℓ_{ν}^{1} , the stationary equation associated to (1) can be written as $F(\mathbf{u},\mathbf{v})=(\mathbf{0},\mathbf{0})$.

We consider $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ an approximate solution with a finite number of Fourier modes to the root-finding problem above. Starting from it, we justify the existence of a true solution in ℓ_{ν}^{1} in a quantified neighborhood of $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$.

The Newton-Kantorovitch Theorem

Considering a numerical approximation $(\bar{\bf u}, \bar{\bf v})$, v > 1, $r^* > 0$ and A an approximated inverse of $DF(\bar{\bf u}, \bar{\bf v})$ injective, combining approximated terms and estimated terms.

Assume there exist constants Y, Z_1 , and Z_2 satisfying:

$$||AF(\bar{\mathbf{u}},\bar{\mathbf{v}})||_{\nu} \le Y$$

$$||I - ADF(\bar{\mathbf{u}},\bar{\mathbf{v}})||_{\nu} \le Z_{1}$$
(2a)
(2b)

$$||AD^{2}F(\mathbf{u},\mathbf{v})||_{\nu} \leq Z_{2}, \quad \forall (\mathbf{u},\mathbf{v}) \in \mathcal{B}_{\nu}((\bar{\mathbf{u}},\bar{\mathbf{v}}),r^{*})$$
(2c)

and

$$Z_1 < 1 \tag{3a}$$

$$2YZ_2 < (1 - Z_1)^2. {(3b)}$$

Then for any r satisfying

$$\frac{1 - Z_1 - \sqrt{(1 - Z_1)^2 - 2YZ_2}}{Z_2} \le r < \min(r^*, \frac{1 - Z_1}{Z_2}), \tag{4}$$

there is a unique fixed-point $(\mathbf{u}^*, \mathbf{v}^*)$ of I - AF in $\mathcal{B}_{\nu}((\bar{\mathbf{u}}, \bar{\mathbf{v}}), r)$ the closed ball of radius r centered on $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ in $\ell_{\nu}^{1} \times \ell_{\nu}^{1}$.

Moreover, $(\mathbf{u}^*, \mathbf{v}^*)$ is a solution of $F(\mathbf{u}, \mathbf{v}) = (\mathbf{0}, \mathbf{0})$. The functions (u^*, v^*) described by Fourier's sequences $(\mathbf{u}^*, \mathbf{v}^*)$ are solutions of (1).

How to choose A

According to [1], we choose

$$A = \begin{pmatrix} A^{11} & A^{12} & A^{12} & W^{11}\Delta^{-1} & W^{12}\Delta^{-1} \\ \hline A^{21} & W^{21}\Delta^{-1} & W^{22}\Delta^{-1} \end{pmatrix},$$
 (5)

with

$$\left(\begin{array}{c|c} \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1} \end{array} \right) \approx \left(\begin{array}{c|c} \mathbf{w}^{11} & \mathbf{w}^{12} \\ \hline \mathbf{w}^{21} & \mathbf{w}^{22} \end{array} \right) * \left(\begin{array}{c|c} \gamma(\bar{\mathbf{v}}) & \gamma'(\bar{\mathbf{v}}) * \bar{\mathbf{u}} \\ \hline \mathbf{0} & \mathbf{1} \end{array} \right)$$

and A^{ij} are from the inverse of a finite dimensional projection of $DF(\bar{\mathbf{u}},\bar{\mathbf{v}})$.

Here, the choice of A is balanced between what can be known by approximation (the information given by the computer) and what must be known by estimation (the information provided by the mathematician).

How to manage γ

In the literature [1, 2, 4, 5] we are interested in several types of γ . We want to manage all these cases. In order to obtain (2), we need to answer the following questions:

- Given an expression of γ , can we find a "good" approximation of $\gamma(\bar{\mathbf{v}})$, $\bar{\mathbf{a}}$, in ℓ_{ν}^{1} for any $\bar{\mathbf{v}}$ finite?
- Given $\bar{\mathbf{a}}$, can we estimate the error $||\gamma(\bar{\mathbf{v}}) \bar{\mathbf{a}}||_{\nu}$? Can we bound $||\gamma(\mathbf{v})||_{\nu}$ for $\mathbf{v} \in \mathcal{B}_{\nu}(\bar{\mathbf{v}}, r)$?

Analytical γ	$ar{\mathbf{a}} pprox oldsymbol{\gamma}(ar{\mathbf{v}})$	Error estimated	Local bound
$P(X) = \sum_{k=0}^{K} p_k X^k$		0	$ P (\bar{\mathbf{v}} _{v} + r) := \sum_{k=0}^{K} p_{k} (\bar{\mathbf{v}} _{v} + r)^{k}$
$\exp(\alpha \cdot)$	$\sum_{k=0}^{K} \frac{\alpha^k}{k!} \bar{\mathbf{v}}^{*k}$	$ \alpha ^{K} \frac{ \bar{\mathbf{v}} _{\nu}^{K}}{K!} \exp(\alpha \bar{\mathbf{v}} _{\nu})$	$\exp(\alpha (\bar{\mathbf{v}} _{v}+r))$
$x \mapsto \frac{1}{g(x)}$	$\bar{\mathbf{a}}$ s.t. $\bar{\mathbf{a}}*g(\bar{\mathbf{v}})\approx 1$	$ \bar{\mathbf{a}} _{\nu} \frac{ \bar{\mathbf{a}}*g(\bar{\mathbf{v}}) - 1 _{\nu}}{1 - \bar{\mathbf{a}}*g(\bar{\mathbf{v}}) - 1 _{\nu}}$	$\frac{ \bar{\mathbf{a}} _{\nu}}{1 - \bar{\mathbf{a}} _{\nu} g' (\bar{\mathbf{v}} _{\nu} + r)r - \bar{\mathbf{a}}*g(\bar{\mathbf{v}}) - 1 _{\nu}}$

With such tools, we can manage any product and division of power series!

Method to get theorems of existence of solution

Methods to get numerical approximations

- Linearization around a trivial homogeneous state, study of instability
- Numerical continuation based on Bifurcation Theory
- Something you know it's a good approximation

Algorithm to check numerical approximation

- 1. Given a point $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ finite.
- 2. Build the object A

suppressed motility model". In: Physica D: Nonlinear Phenomena 402 (2020), p. 132259.

model". In: IMA Journal of Applied Mathematics 86.3 (2021), pp. 577-603.

[5] Zhi-An Wang and Xin Xu. "Steady states and pattern formation of the density-suppressed motility

- 3. Compute Y, Z_1 , Z_2 from (2a-c)
- 4. Are (3a-b), (4) satisfied? Conclude.

Some results and several theorems for $\gamma(x) = \frac{1}{1+x^9}$

