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Numerical and Theoretical Study of Patterns in a Chemotaxis Model

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PDE of Keller-Segel type: Cross-diffusion and logistic reaction

We are interested in the system (cf. [3, 5]):

$$\begin{cases} \partial_t u = \Delta(\gamma(v)u) + \sigma u(1-u), & x \in (0, 1), t > 0 \\ \partial_t v = \Delta v + u - v, & x \in (0, 1), t > 0 \\ \partial_x u(x, t) = 0, \quad \partial_x v(x, t) = 0, & x \in \{0, 1\}, t > 0 \end{cases} \quad (1)$$

Here u and v represent the density of cells and of a chemical, which is produced by the cells. We assume $\sigma \geq 0$ and that $\gamma > 0$ is decreasing, to model that the cells are attracted to the chemical.

As a first step to better understand the long-term behavior of solutions, we study the stationary system associated with (1). **We will study and look for non trivial stationary states.** The approach used is a computer assisted proof method applied to a fixed point problem. This guarantees the existence of solutions to (1) and gives an explicit numerical approximation with a quantified error.

Use of Fourier Series

The Neumann boundary conditions lead us to consider solutions in the form:

$$u(x) = u_0 + 2 \sum_{k=1}^{+\infty} u_k \cos(k\pi x) \quad \text{and} \quad v(x) = v_0 + 2 \sum_{k=1}^{+\infty} v_k \cos(k\pi x).$$

Let's pose $\mathbf{u} = (u_0, u_1, \dots, u_k, \dots) = (\mathcal{F}(u)_n)_{n \in \mathbb{N}}$, idem $\mathbf{v} = (\mathcal{F}(v)_n)_{n \in \mathbb{N}}$ and $\gamma(\mathbf{v}) = (\mathcal{F}(\gamma \circ v)_n)_{n \in \mathbb{N}}$.

Let $v > 1$ and $\ell_v^1 = \left\{ \mathbf{u} \mid \|\mathbf{u}\|_v := |u_0| + 2 \sum_{k=1}^{+\infty} |u_k| v^k < +\infty \right\}$.

Provided with the discrete convolution $*$, ℓ_v^1 is a Banach algebra.

Through ℓ_v^1 , the stationary equation associated to (1) can be written as $F(\mathbf{u}, \mathbf{v}) = (\mathbf{0}, \mathbf{0})$.

We consider $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ an approximate solution with a finite number of Fourier modes to the root-finding problem above. Starting from it, we justify the existence of a true solution in ℓ_v^1 in a quantified neighborhood of $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$.

The Newton-Kantorovitch Theorem

Considering a numerical approximation $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$, $v > 1$, $r^* > 0$ and A an approximated inverse of $DF(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ injective, combining approximated terms and estimated terms.

Assume there exist constants Y , Z_1 , and Z_2 satisfying:

$$\|AF(\bar{\mathbf{u}}, \bar{\mathbf{v}})\|_v \leq Y \quad (2a)$$

$$\|I - ADF(\bar{\mathbf{u}}, \bar{\mathbf{v}})\|_v \leq Z_1 \quad (2b)$$

$$\|AD^2F(\mathbf{u}, \mathbf{v})\|_v \leq Z_2, \quad \forall (\mathbf{u}, \mathbf{v}) \in \mathcal{B}_v((\bar{\mathbf{u}}, \bar{\mathbf{v}}), r^*) \quad (2c)$$

and

$$Z_1 < 1 \quad (3a)$$

$$2YZ_2 < (1 - Z_1)^2. \quad (3b)$$

Then for any r satisfying

$$\frac{1 - Z_1 - \sqrt{(1 - Z_1)^2 - 2YZ_2}}{Z_2} \leq r < \min(r^*, \frac{1 - Z_1}{Z_2}), \quad (4)$$

there is a unique fixed-point $(\mathbf{u}^*, \mathbf{v}^*)$ of $I - AF$ in $\mathcal{B}_v((\bar{\mathbf{u}}, \bar{\mathbf{v}}), r)$ the closed ball of radius r centered on $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ in $\ell_v^1 \times \ell_v^1$.

Moreover, $(\mathbf{u}^*, \mathbf{v}^*)$ is a solution of $F(\mathbf{u}, \mathbf{v}) = (\mathbf{0}, \mathbf{0})$. The functions (u^*, v^*) described by Fourier's sequences $(\mathbf{u}^*, \mathbf{v}^*)$ are solutions of (1).

How to choose A

According to [1], we choose

$$A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} = \begin{pmatrix} \mathbf{w}^{11} \Delta^{-1} & \mathbf{w}^{12} \Delta^{-1} \\ \mathbf{w}^{21} \Delta^{-1} & \mathbf{w}^{22} \Delta^{-1} \end{pmatrix}, \quad (5)$$

with

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \approx \begin{pmatrix} \mathbf{w}^{11} & \mathbf{w}^{12} \\ \mathbf{w}^{21} & \mathbf{w}^{22} \end{pmatrix} * \begin{pmatrix} \gamma(\bar{\mathbf{v}}) & \gamma'(\bar{\mathbf{v}}) * \bar{\mathbf{u}} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

and A^{ij} are from the inverse of a finite dimensional projection of $DF(\bar{\mathbf{u}}, \bar{\mathbf{v}})$.

Here, the choice of A is balanced between **what can be known by approximation** (the information given by the computer) and **what must be known by estimation** (the information provided by the mathematician).

How to manage γ

In the literature [1, 2, 4, 5] we are interested in several types of γ . We want to manage all these cases. In order to obtain (2), we need to answer the following questions:

- Given an expression of γ , can we find a "good" approximation of $\gamma(\bar{\mathbf{v}})$, $\bar{\mathbf{a}}$, in ℓ_v^1 for any $\bar{\mathbf{v}}$ finite?
- Given $\bar{\mathbf{a}}$, can we estimate the error $\|\gamma(\bar{\mathbf{v}}) - \bar{\mathbf{a}}\|_v$? Can we bound $\|\gamma(\mathbf{v})\|_v$ for $\mathbf{v} \in \mathcal{B}_v(\bar{\mathbf{v}}, r)$?

Analytical γ	$\bar{\mathbf{a}} \approx \gamma(\bar{\mathbf{v}})$	Error estimated	Local bound
$P(X) = \sum_{k=0}^K p_k X^k$	$\sum_{k=0}^K p_k \bar{\mathbf{v}}^{*k}$	0	$ P (\ \bar{\mathbf{v}}\ _v + r) := \sum_{k=0}^K p_k (\ \bar{\mathbf{v}}\ _v + r)^k$
$\exp(\alpha \cdot)$	$\sum_{k=0}^K \frac{\alpha^k}{k!} \bar{\mathbf{v}}^{*k}$	$ \alpha ^K \frac{\ \bar{\mathbf{v}}\ _v^K}{K!} \exp(\alpha \ \bar{\mathbf{v}}\ _v)$	$\exp(\alpha (\ \bar{\mathbf{v}}\ _v + r))$
$x \mapsto \frac{1}{g(x)}$	$\bar{\mathbf{a}}$ s.t. $\bar{\mathbf{a}} * g(\bar{\mathbf{v}}) \approx \mathbf{1}$	$\frac{\ \bar{\mathbf{a}} * g(\bar{\mathbf{v}}) - \mathbf{1}\ _v}{1 - \ \bar{\mathbf{a}} * g(\bar{\mathbf{v}}) - \mathbf{1}\ _v}$	$\frac{\ \bar{\mathbf{a}}\ _v}{1 - \ \bar{\mathbf{a}}\ _v g' (\ \bar{\mathbf{v}}\ _v + r) r - \ \bar{\mathbf{a}} * g(\bar{\mathbf{v}}) - \mathbf{1}\ _v}$

With such tools, we can manage any product and division of power series!

Method to get theorems of existence of solution

Methods to get numerical approximations

- Linearization around a trivial homogeneous state, study of instability
- Numerical continuation based on Bifurcation Theory
- Something you know it's a good approximation

Algorithm to check numerical approximation

- Given a point $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ finite.
- Build the object A
- Compute Y , Z_1 , Z_2 from (2a-c)
- Are (3a-b), (4) satisfied? Conclude.

All the work is here!

Some results and several theorems for $\gamma(x) = \frac{1}{1+x^9}$

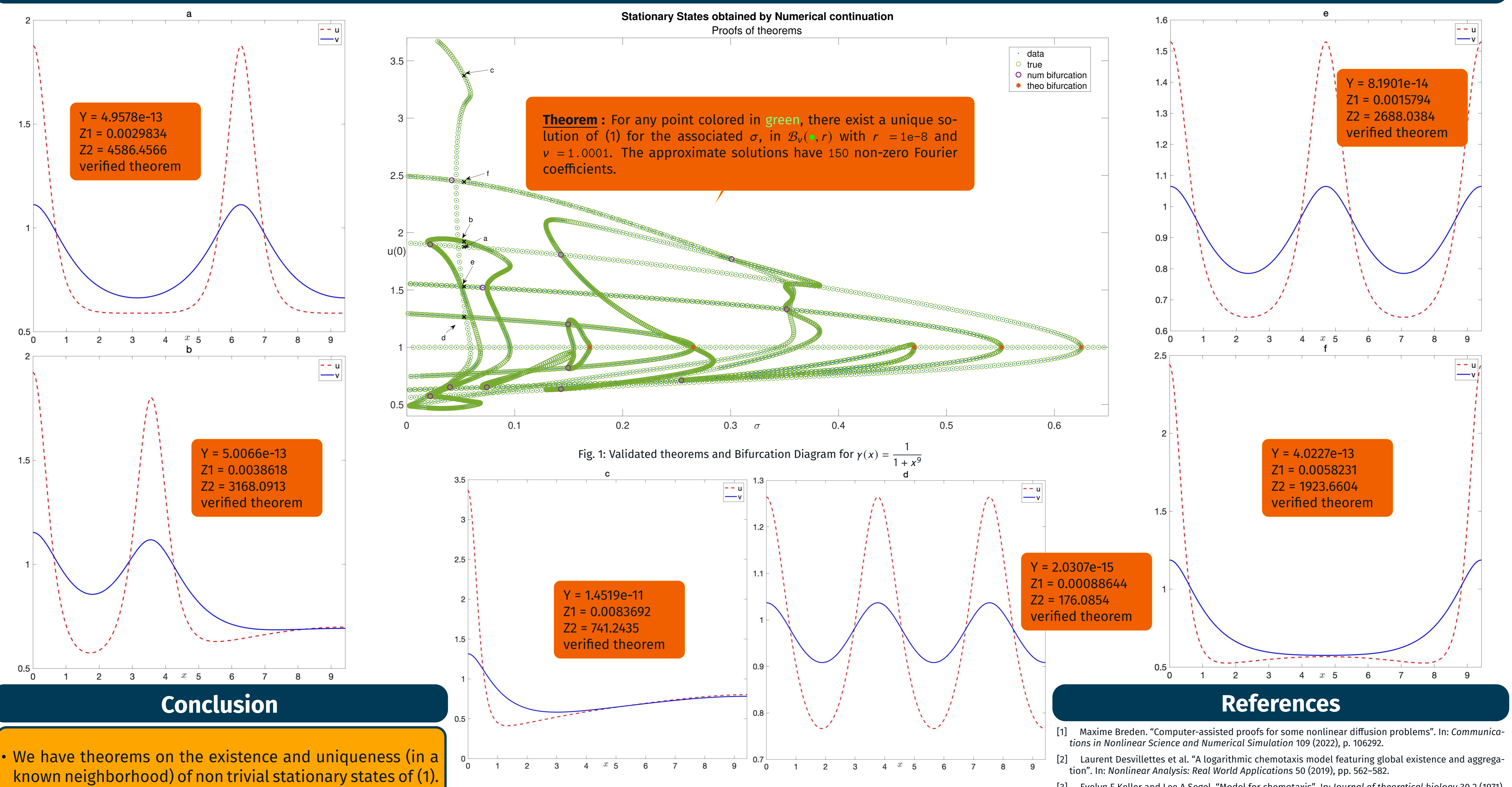


Fig. 2: Some Stationary solutions for $\sigma = 0.053$

Conclusion

- We have theorems on the existence and uniqueness (in a known neighborhood) of non trivial stationary states of (1).
- We have developed (and are developing) a technique to manage non-polynomial terms.

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