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# Numerical and Theoretical Study of Patterns in a Chemotaxis Model

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## PDE of Keller-Segel type: Cross-diffusion and logistic reaction

We are interested in the system (cf. [3, 5]):

$$\begin{cases} \partial_t u = \Delta(\gamma(v)u) + \sigma u(1-u), & x \in (0,1), t > 0 \\ \partial_t v = \Delta v + u - v, & x \in (0,1), t > 0 \\ \partial_x u(x,t) = 0, \quad \partial_x v(x,t) = 0, & x \in \{0,1\}, t > 0 \end{cases} \quad (1)$$

Here  $u$  and  $v$  represent the density of cells and of a chemical, which is produced by the cells. We assume  $\sigma \geq 0$  and that  $\gamma > 0$  is decreasing, to model that the cells are attracted to the chemical.

As a first step to better understand the long-term behavior of solutions, we study the stationary system associated with (1). **We will study and look for non trivial stationary states.** The approach used is a computer assisted proof method applied to a fixed point problem. This guarantees the existence of solutions to (1) and gives an explicit numerical approximation with a quantified error.

## Use of Fourier Series

The Neumann boundary conditions lead us to consider solutions in the form:

$$u(x) = u_0 + 2 \sum_{k=1}^{+\infty} u_k \cos(k\pi x) \quad \text{and} \quad v(x) = v_0 + 2 \sum_{k=1}^{+\infty} v_k \cos(k\pi x).$$

Let's pose  $\mathbf{u} = (u_0, u_1, \dots, u_k, \dots) = (\mathcal{F}(u)_n)_{n \in \mathbb{N}}$ , idem  $\mathbf{v} = (\mathcal{F}(v)_n)_{n \in \mathbb{N}}$  and  $\gamma(\mathbf{v}) = (\mathcal{F}(\gamma \circ v)_n)_{n \in \mathbb{N}}$ .

Let  $v > 1$  and  $\ell_v^1 = \left\{ \mathbf{u} \mid \|\mathbf{u}\|_v := |u_0| + 2 \sum_{k=1}^{+\infty} |u_k| v^k < +\infty \right\}$ .

Provided with the discrete convolution  $*$ ,  $\ell_v^1$  is a Banach algebra.

Through  $\ell_v^1$ , the stationary equation associated to (1) can be written as  $F(\mathbf{u}, \mathbf{v}) = (\mathbf{0}, \mathbf{0})$ .

We consider  $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$  an approximate solution with a finite number of Fourier modes to the root-finding problem above. Starting from it, we justify the existence of a true solution in  $\ell_v^1$  in a quantified neighborhood of  $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ .

## The Newton-Kantorovitch Theorem

Considering a numerical approximation  $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ ,  $v > 1$ ,  $r^* > 0$  and  $A$  an approximated inverse of  $DF(\bar{\mathbf{u}}, \bar{\mathbf{v}})$  injective, combining approximated terms and estimated terms.

Assume there exist constants  $Y$ ,  $Z_1$ , and  $Z_2$  satisfying:

$$\|AF(\bar{\mathbf{u}}, \bar{\mathbf{v}})\|_v \leq Y \quad (2a)$$

$$\|I - ADF(\bar{\mathbf{u}}, \bar{\mathbf{v}})\|_v \leq Z_1 \quad (2b)$$

$$\|AD^2F(\mathbf{u}, \mathbf{v})\|_v \leq Z_2, \quad \forall (\mathbf{u}, \mathbf{v}) \in \mathcal{B}_v((\bar{\mathbf{u}}, \bar{\mathbf{v}}), r^*) \quad (2c)$$

and

$$Z_1 < 1 \quad (3a)$$

$$2YZ_2 < (1 - Z_1)^2. \quad (3b)$$

Then for any  $r$  satisfying

$$\frac{1 - Z_1 - \sqrt{(1 - Z_1)^2 - 2YZ_2}}{Z_2} \leq r < \min(r^*, \frac{1 - Z_1}{Z_2}), \quad (4)$$

there is a unique fixed-point  $(\mathbf{u}^*, \mathbf{v}^*)$  of  $I - AF$  in  $\mathcal{B}_v((\bar{\mathbf{u}}, \bar{\mathbf{v}}), r)$  the closed ball of radius  $r$  centered on  $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$  in  $\ell_v^1 \times \ell_v^1$ .

Moreover,  $(\mathbf{u}^*, \mathbf{v}^*)$  is a solution of  $F(\mathbf{u}, \mathbf{v}) = (\mathbf{0}, \mathbf{0})$ . The functions  $(u^*, v^*)$  described by Fourier's sequences  $(\mathbf{u}^*, \mathbf{v}^*)$  are solutions of (1).

## How to choose A

According to [1], we choose

$$A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} = \begin{pmatrix} \mathbf{w}^{11} \Delta^{-1} & \mathbf{w}^{12} \Delta^{-1} \\ \mathbf{w}^{21} \Delta^{-1} & \mathbf{w}^{22} \Delta^{-1} \end{pmatrix}, \quad (5)$$

with

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \approx \begin{pmatrix} \mathbf{w}^{11} & \mathbf{w}^{12} \\ \mathbf{w}^{21} & \mathbf{w}^{22} \end{pmatrix} * \begin{pmatrix} \gamma(\bar{\mathbf{v}}) & \gamma'(\bar{\mathbf{v}}) * \bar{\mathbf{u}} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

and  $A^{ij}$  are from the inverse of a finite dimensional projection of  $DF(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ .

Here, the choice of  $A$  is balanced between **what can be known by approximation** (the information given by the computer) and **what must be known by estimation** (the information provided by the mathematician).

## How to manage $\gamma$

In the literature [1, 2, 4, 5] we are interested in several types of  $\gamma$ . We want to manage all these cases. In order to obtain (2), we need to answer the following questions:

- Given an expression of  $\gamma$ , can we find a "good" approximation of  $\gamma(\bar{\mathbf{v}})$ ,  $\bar{\mathbf{a}}$ , in  $\ell_v^1$  for any  $\bar{\mathbf{v}}$  finite?
- Given  $\bar{\mathbf{a}}$ , can we estimate the error  $\|\gamma(\bar{\mathbf{v}}) - \bar{\mathbf{a}}\|_v$ ? Can we bound  $\|\gamma(\mathbf{v})\|_v$  for  $\mathbf{v} \in \mathcal{B}_v(\bar{\mathbf{v}}, r)$ ?

Analytical $\gamma$	$\bar{\mathbf{a}} \approx \gamma(\bar{\mathbf{v}})$	Error estimated	Local bound
$P(X) = \sum_{k=0}^K p_k X^k$	$\sum_{k=0}^K p_k \bar{\mathbf{v}}^{*k}$	0	$ P (\ \bar{\mathbf{v}}\ _v + r) := \sum_{k=0}^K  p_k (\ \bar{\mathbf{v}}\ _v + r)^k$
$\exp(\alpha \cdot)$	$\sum_{k=0}^K \frac{\alpha^k}{k!} \bar{\mathbf{v}}^{*k}$	$ \alpha ^K \frac{\ \bar{\mathbf{v}}\ _v^K}{K!} \exp( \alpha  \ \bar{\mathbf{v}}\ _v)$	$\exp( \alpha (\ \bar{\mathbf{v}}\ _v + r))$
$x \mapsto \frac{1}{g(x)}$	$\bar{\mathbf{a}}$ s.t. $\bar{\mathbf{a}} * g(\bar{\mathbf{v}}) \approx \mathbf{1}$	$\frac{\ \bar{\mathbf{a}} * g(\bar{\mathbf{v}}) - \mathbf{1}\ _v}{1 - \ \bar{\mathbf{a}} * g(\bar{\mathbf{v}}) - \mathbf{1}\ _v}$	$\frac{\ \bar{\mathbf{a}}\ _v}{1 - \ \bar{\mathbf{a}}\ _v  g' (\ \bar{\mathbf{v}}\ _v + r) r - \ \bar{\mathbf{a}} * g(\bar{\mathbf{v}}) - \mathbf{1}\ _v}$

With such tools, we can manage any product and division of power series!

## Method to get theorems of existence of solution

### Methods to get numerical approximations

- Linearization around a trivial homogeneous state, study of instability
- Numerical continuation based on Bifurcation Theory
- Something you know it's a good approximation

### Algorithm to check numerical approximation

1. Given a point  $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$  finite.
2. Build the object  $A$
3. Compute  $Y$ ,  $Z_1$ ,  $Z_2$  from (2a-c)
4. Are (3a-b), (4) satisfied? Conclude.

All the work is here!

## Some results and several theorems for $\gamma(x) = \frac{1}{1+x^9}$

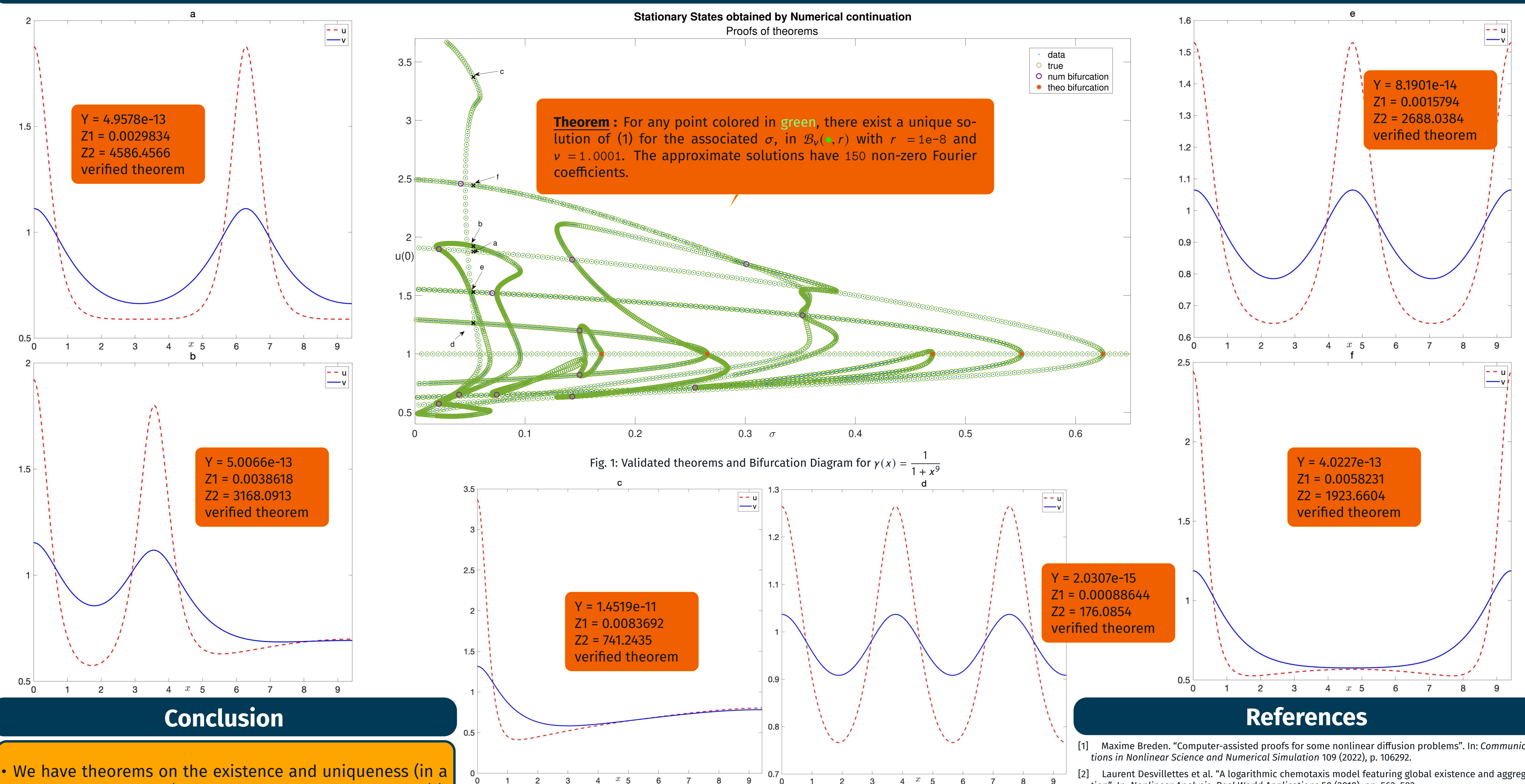


Fig. 2: Some Stationary solutions for  $\sigma = 0.053$

## Conclusion

- We have theorems on the existence and uniqueness (in a known neighborhood) of non trivial stationary states of (1).
- We have developed (and are developing) a technique to manage non-polynomial terms.

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