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The asymptotics of the optimal holomorphic extensions of holomorphic jets along submanifolds

Siarhei Finski

Abstract. For a complex submanifold in a complex manifold, we consider the operator which for a given holomorphic jet of a vector bundle along the submanifold associates the L^2 -optimal holomorphic extension of it to the ambient manifold. When the vector bundle is given by big tensor powers of a positive line bundle, we give an asymptotic formula for this extension operator.

Résumé. Pour un sous-variété complexe dans un variété complexe, nous considérons l'opérateur qui, pour un jet holomorphe d'un fibré vectoriel le long du sous-variété, associe l'extension holomorphe optimale L^2 de ce jet au variété ambiant. Lorsque le fibré vectoriel est donné par les grandes puissances tensorielles d'un fibré en droites positives, nous donnons une formule asymptotique pour cet opérateur d'extension.

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1 Introduction

The main goal of this paper is to study the asymptotics of the L^2 -optimal holomorphic extensions of holomorphic jets associated to high tensor powers of a positive line bundle along a submanifold.

More precisely, we fix two (not necessarily compact) complex manifolds X, Y , of dimensions n and m respectively. We fix also a complex embedding $\iota : Y \rightarrow X$, a positive line bundle (L, h^L) over X and an arbitrary Hermitian vector bundle (F, h^F) over X . In particular, we assume that for the curvature R^L of the Chern connection on (L, h^L) , the closed real $(1, 1)$ -differential form

$$\omega := \frac{\sqrt{-1}}{2\pi} R^L \quad (1.1)$$

is positive. We denote by g^{TX} the Riemannian metric on X induced by ω as follows

$$g^{TX}(\cdot, \cdot) := \omega(\cdot, J\cdot), \quad (1.2)$$

where $J : TX \rightarrow TX$ is the complex structure on X . We denote by g^{TY} the induced metric on Y .

We assume throughout the whole article that the triple (X, Y, g^{TX}) , and the Hermitian vector bundles (L, h^L) , (F, h^F) , are of bounded geometry in the sense of Definitions 2.4, 2.5.

This means that we assume uniform lower bounds $r_X, r_Y > 0$ on the injectivity radii of X, Y , the existence of the geodesic tubular neighborhood of Y of uniform size $r_\perp > 0$ in X , and some uniform bounds on related curvatures and the second fundamental form of the embedding.

Now, we fix some positive (with respect to the orientation given by the complex structure) volume forms dv_X, dv_Y on X and Y . For smooth sections f, f' of $L^p \otimes F$, $p \in \mathbb{N}$, over X , we define the L^2 -scalar product using the pointwise scalar product $\langle \cdot, \cdot \rangle_h$, induced by h^L and h^F , by

$$\langle f, f' \rangle_{L^2(X, L^p \otimes F)} := \int_X \langle f(x), f'(x) \rangle_h dv_X(x). \quad (1.3)$$

We denote by $H_{(2)}^0(X, L^p \otimes F)$ the vector space of L^2 -holomorphic sections of $L^p \otimes F$ over X .

Given a continuous smoothing linear operator $K : L^2(X, L^p \otimes F) \rightarrow L^2(X, L^p \otimes F)$, the Schwartz kernel theorem guarantees the existence of the Schwartz kernel, $K(x_1, x_2) \in (L^p \otimes F)_{x_1} \otimes (L^p \otimes F)_{x_2}^*$; $x_1, x_2 \in X$, evaluated with respect to dv_X , i.e.

$$(Ks)(x_1) = \int_X K(x_1, x_2) \cdot s(x_2) dv_X(x_2), \quad s \in L^2(X, L^p \otimes F). \quad (1.4)$$

Similarly, we define the Schwartz kernels $K_1(y, x), K_2(x, y)$, $x \in X, y \in Y$, for smoothing operators $K_1 : L^2(X, L^p \otimes F) \rightarrow L^2(Y, \iota^*(L^p \otimes F))$, $K_2 : L^2(Y, \iota^*(L^p \otimes F)) \rightarrow L^2(X, L^p \otimes F)$ with respect to the volume forms dv_X and dv_Y respectively.

We denote below by TX and TY the (real) tangent bundles of X, Y . We identify the holomorphic parts $T^{1,0}X, T^{1,0}Y$ of the corresponding complexifications with the associated holomorphic tangent bundles. We denote by g^N the metric on the normal bundle N of Y in X induced by g^{TX} . We introduce similarly the complex vector bundle $N^{1,0}$ and endow it with the holomorphic structure given by $T^{1,0}X/T^{1,0}Y$. From g^{TX} , we induce the Hermitian structure on $N^{1,0}$ and denote it by an abuse of notation by g^N . Let $P^N : TX|_Y \rightarrow N$, $P^Y : TX|_Y \rightarrow TY$, be the orthogonal projections induced by g^{TX} . Clearly, $\nabla^N := P^N \nabla^{TX}|_Y$ defines a connection on N .

The Hermitian product g^N on $N^{1,0}$ induces Hermitian product on $\text{Sym}^k(N^{1,0})^*$, $k \in \mathbb{N}$, in the usual way, cf. (1.24). Then, similarly to (1.3), using dv_Y , we define the L^2 -product $\langle \cdot, \cdot \rangle_{k, L^2(Y, L^p \otimes F)}$ on $H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$.

We denote by \mathcal{J}_Y the ideal sheaf of holomorphic germs on X , which vanish on Y . For $k \in \mathbb{N}$, we endow the space $H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k)$ with the L^2 -metric induced by the natural inclusion $H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k) \hookrightarrow H_{(2)}^0(X, L^p \otimes F)$.

We assume that for the Riemannian volume forms $dv_{g^{TX}}, dv_{g^{TY}}$ of $(X, g^{TX}), (Y, g^{TY})$, for any $r \in \mathbb{N}$, there is $C_r > 0$, such that over X and Y , the following bounds hold

$$\left\| \frac{dv_{g^{TX}}}{dv_X} \right\|_{\mathcal{C}^r(X)}, \left\| \frac{dv_X}{dv_{g^{TX}}} \right\|_{\mathcal{C}^r(X)}, \left\| \frac{dv_{g^{TY}}}{dv_Y} \right\|_{\mathcal{C}^r(Y)}, \left\| \frac{dv_Y}{dv_{g^{TY}}} \right\|_{\mathcal{C}^r(Y)} \leq C_r. \quad (1.5)$$

From Weierstrass division theorem, cf. (4.8), we see that for any $k \in \mathbb{N}$, a k -jet associated to a section from $H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k)$ is holomorphic. Moreover, in Theorem 4.3, we prove that for any $k \in \mathbb{N}$, there is $p_0 \in \mathbb{N}$, depending only on k and the triple (X, Y, g^{TX}) , such that for any $p \geq p_0$, the k -jets have bounded L^2 -norm. In other words, we prove that the operator

$$\text{Res}_{k,p} : H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k) \rightarrow H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F)), \quad (1.6)$$

given by the following identity

$$\text{Res}_{k,p}(f) = (\nabla^k f)|_Y, \quad (1.7)$$

is well-defined, where ∇ is some connection on $L^p \otimes F$ (due to the vanishing condition of f on Y , the definition doesn't depend on the choice of the connection ∇).

By extending Ohsawa-Takegoshi theorem for holomorphic jets, in Theorem 4.4, we prove that for any $k \in \mathbb{N}$, there is $p_0 \in \mathbb{N}$, such that for any $p \geq p_0$, the operator (1.6) is surjective. Then, for the Bergman projector $B_{k,p}^Y$, given by the orthogonal projection from the space of L^2 -sections $L^2(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$ to $H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$, the extension operator

$$E_{k,p} : L^2(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F)) \rightarrow H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k), \quad (1.8)$$

is defined by $E_{k,p}(g) = f$, where f satisfies $\text{Res}_{k,p}(f) = B_{k,p}^Y g$ and has the minimal L^2 -norm among the sections from $H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k)$ with this property (since $H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^{k+1})$ is closed inside of $H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k)$, such f is unique). Ohsawa-Takegoshi extension theorem for holomorphic jets in our setting means precisely that the operator $E_{k,p}$ is well-defined and bounded for p big enough. Study of such operators is motivated by the previous works of Popovici [32], Demailly [14] and Cao-Demailly-Matsumura [6] on the extension of holomorphic jets in the classical setting (i.e. $p = 0$), and the works of Ohsawa-Takegoshi [30], [29], Manivel [27] and Demailly [12, §13], on the extensions of holomorphic sections (again for $p = 0$).

One of the main goals of this article is to find an explicit asymptotic expansion of $E_{k,p}$, as $p \rightarrow \infty$. To describe our results precisely, we need to fix some further notation.

For $y \in Y$, $Z_N \in N_y$, let $\mathbb{R} \ni t \mapsto \exp_y^X(tZ_N) \in X$ be the geodesic in X in direction Z_N , where we identified N_y as an orthogonal complement of $T_y Y$ in $T_y X$. Bounded geometry condition means, in particular, that this map induces a diffeomorphism of r_\perp -neighborhood of the zero section in N with a tubular neighborhood U of Y in X . From now on, we use this identification implicitly. Of course, $(y, 0)$, $y \in Y$, then corresponds to Y .

We denote by $\pi_0 : U \rightarrow Y$ the natural projection $(y, Z_N) \mapsto y$. Over U , we identify L, F to $\pi_0^*(L|_Y), \pi_0^*(F|_Y)$ by the parallel transport with respect to the respective Chern connections along the geodesic $[0, 1] \ni t \mapsto (y, tZ_N) \in X$, $|Z_N| < r_\perp$. We also define the function κ_N as follows

$$dv_X = \kappa_N dv_Y \wedge dv_N, \quad (1.9)$$

where dv_N is the relative Riemannian volume form on (N, g^N) . Of course, we have $\kappa_N|_Y = 1$ if

$$dv_X = dv_{g^{TX}}, \quad dv_Y = dv_{g^{TY}}. \quad (1.10)$$

We fix a smooth function $\rho : [0, +\infty[\rightarrow [0, 1]$, satisfying

$$\rho(x) = \begin{cases} 1, & \text{for } x < \frac{1}{4}, \\ 0, & \text{for } x > \frac{1}{2}. \end{cases} \quad (1.11)$$

We fix $g \in \mathcal{C}^\infty(Y, \text{Sym}^k N^* \otimes \iota^*(L^p \otimes F))$, $k \in \mathbb{N}$, and define using the above isomorphisms over U the section $\{g\} \in \mathcal{C}^\infty(X, L^p \otimes F)$ as follows

$$\{g\}(y, Z_N) := \rho\left(\frac{|Z_N|}{r_\perp}\right) \cdot \frac{Z_N^{\otimes k}}{k!} \cdot g(y), \quad (1.12)$$

where the norm $|Z_N|$, is taken with respect to g^N . Away from U , we extend $\{g\}$ by zero.

We define the operator $E_{k,p}^0 : L^2(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F)) \rightarrow L^2(X, L^p \otimes F)$ as follows. For $g \in L^2(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$, we let $(E_{k,p}^0 g)(x) = 0$, $x \notin U$, and in U , we put

$$(E_{k,p}^0 g)(y, Z_N) = \{B_{k,p}^Y g\}(y, Z_N) \cdot \exp\left(-p \frac{\pi}{2} |Z_N|^2\right). \quad (1.13)$$

Now, for $g \in H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$, the section $E_{k,p}^0 g$ satisfies $(\nabla^k E_{k,p}^0 g)|_Y = g$, but $E_{k,p}^0 g$ is not holomorphic over X , unless $g = 0$. Nevertheless, $E_{k,p}^0 g$ can be used to approximate very well the holomorphic section $E_{k,p} g$. More precisely, we have the following result.

Theorem 1.1. *For any $k \in \mathbb{N}$, there are $C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, we have*

$$\|E_{k,p} - E_{k,p}^0\| \leq \frac{C}{p^{\frac{n-m+k+1}{2}}}. \quad (1.14)$$

where $\|\cdot\|$ is the operator norm. Also, as $p \rightarrow \infty$, we have

$$\|E_{k,p}^0\| \sim \frac{1}{p^{\frac{n-m+k}{2}}} \cdot \sup_{y \in Y} \kappa_N^{\frac{1}{2}}(y) \cdot \frac{1}{\sqrt{k!} \cdot (2\pi)^k}. \quad (1.15)$$

Moreover, under assumption (1.10), in (1.14), one can replace $p^{-\frac{n-m+k+1}{2}}$ by an asymptotically better estimate if and only if Y is a totally geodesic submanifold of (X, g^{TX}) , i.e. the second fundamental form, see (2.2), vanishes.

Remark 1.2. a) For $k = 0$, this result was proved in [16, Theorem 1.1]. The proof we present here is different even for $k = 0$. It is still based a lot on the ideas and the techniques from [16] and [17].

b) The boundness of κ_N follows from the bounded geometry condition, see [16, Section §2.1]. In particular, the right-hand side of (1.15) is finite.

c) Our result refines a theorem of Randriambololona [33, Théorème 3.1.10], stating in the compact case that for any $\epsilon > 0$, $k \in \mathbb{N}$, there is $p_1 \in \mathbb{N}^*$, such that $\|E_{k,p}\| \leq \exp(\epsilon p)$ for $p \geq p_1$.

Let us now describe our second result, establishing a relation between two natural metrics on the space of holomorphic jets. For this, define the map

$$\begin{aligned} \text{Jet}_{k,p} : H_{(2)}^0(X, L^p \otimes F) / H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^{k+1}) \\ \rightarrow \bigoplus_{r=0}^k H_{(2)}^0(Y, \text{Sym}^r(N^{1,0})^* \otimes \iota^*(L^p \otimes F)), \end{aligned} \quad (1.16)$$

as follows. We let $\text{Jet}_{k,p}(f) = (g_0, \dots, g_k)$ if and only if for any $r \in \mathbb{N}$, $r \leq k$, the following holds

$$f - \sum_{l=0}^r E_{l,p} g_l \in H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^{r+1}). \quad (1.17)$$

Alternatively, we let $g_0 = f|_Y$, and inductively define

$$g_r := \text{Res}_{r,p} \left(f - \sum_{l=0}^{r-1} E_{l,p} g_l \right). \quad (1.18)$$

As a direct consequence of the Ohsawa-Takegoshi extension theorem for holomorphic jets, cf. Theorem 4.4, for any $k \in \mathbb{N}$, there is $p_1 \in \mathbb{N}$, such that the map $\text{Jet}_{k,p}$ is an isomorphism for $p \geq p_1$. As we explain next, this statement can be refined in the metric setting.

More precisely, consider the scalar product $\langle \cdot, \cdot \rangle_{\text{Jet}_{k,p}}$ on the space $\bigoplus_{r=0}^k H_{(2)}^0(Y, \text{Sym}^r(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$, defined as follows

$$\langle \cdot, \cdot \rangle_{\text{Jet}_{k,p}} := \sum_{r=0}^k \frac{1}{r!} \cdot \frac{1}{(2\pi)^r} \frac{1}{p^{n-m+r}} \cdot \langle \cdot, \cdot \rangle_{r, L^2(Y, L^p \otimes F)} \quad (1.19)$$

Let us now consider the scalar product on $H_{(2)}^0(X, L^p \otimes F) / H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^{k+1})$ induced by the L^2 -scalar product on $H_{(2)}^0(X, L^p \otimes F)$, which we denote by an abuse of notation by $\langle \cdot, \cdot \rangle_{L^2(X, L^p \otimes F)}$.

Theorem 1.3. *Under the assumption (1.10), the map $\text{Jet}_{k,p}$ is an asymptotic isometry with respect to the above scalar products. More precisely, for any $k \in \mathbb{N}$, there are $C > 0$, $p_1 \in \mathbb{N}$, such that for any $p \geq p_1$, for the norms $\| \cdot \|_{L^2(X, L^p \otimes F)}$, $\| \cdot \|_{\text{Jet}_{k,p}}$ on $\bigoplus_{r=0}^k H_{(2)}^0(Y, \text{Sym}^r(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$ associated to the scalar products defined above, we have*

$$1 - \frac{C}{p} \leq \frac{\| \cdot \|_{\text{Jet}_{k,p}}}{\| \cdot \|_{L^2(X, L^p \otimes F)}} \leq 1 + \frac{C}{p}. \quad (1.20)$$

Remark 1.4. a) An analogue of Theorem 1.3 in the setting of geometric quantization was established by Ma-Zhang [26, Theorem 0.10]. Authors proved that there is a similar asymptotic relation between the L^2 -metric on the space of holomorphic sections invariant under the Hamiltonian action of a compact connected Lie group and the L^2 -metric on the corresponding symplectic reduction.

b) For $k = 0$, a refinement of Theorem 1.3 was established in [17, Theorem 1.8].

The proofs of Theorems 1.1, 1.3 are done in Section 5, essentially through establishing the exponential bound for the Schwartz kernel of the extension operator in Theorem 5.1.a) and the asymptotic expansion of this Schwartz kernel in Theorems 5.3.

These results seem to be new even when $Y = \{x\}$, where $x \in X$ is a fixed point. In this case, they give some results on the asymptotic expansion of *higher order peak sections*, previously studied by Tian [38]. See Section 5.3 for more details.

As the last application of our techniques, we prove the asymptotic expansion of the Schwartz kernel of the logarithmic Bergman kernel of order $k \in \mathbb{N}$. More precisely, let $B_p^{X,kY}$ be the orthogonal projection from $L^2(X, L^p \otimes F)$ to $H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k)$. We call it the *logarithmic Bergman kernel of order k* . The following theorem shows that away from Y , $B_p^{X,kY}$ is asymptotically close to B_p^X , and in a tubular neighborhood around Y , it can be expressed as a product of the Bergman kernel B_p^Y and the logarithmic Bergman kernel of the model space, as described in Section 3.1.

Theorem 1.5. *For any $k \in \mathbb{N}$, there are $c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $r \in \mathbb{N}$, there is $C > 0$, such that for any $p \geq p_1$, $x_1, x_2 \in X$, the following estimate holds*

$$\begin{aligned} & \left| (B_p^{X,kY} - B_p^X)(x_1, x_2) \right|_{\mathcal{C}^r} \\ & \leq Cp^{n+\frac{r}{2}} \exp \left(-c\sqrt{p} \cdot (\text{dist}(x_1, x_2) + \text{dist}(x_2, Y) + \text{dist}(x_1, Y)) \right), \end{aligned} \quad (1.21)$$

where the pointwise \mathcal{C}^r -norm at a point $(x_1, x_2) \in X \times X$ is the sum of the norms induced by h^L, h^F and g^{TX} , evaluated at (x_1, x_2) , of the derivatives up to order r with respect to the connection induced by the Chern connections on L, F and the Levi-Civita connection on TX . Moreover, in the tubular neighborhood around Y , under the identification as described before (1.9), the following holds. For any $k \in \mathbb{N}$, there are $\epsilon, c, Q > 0$, $p_1 \in \mathbb{N}^*$, such that for any $r \in \mathbb{N}$, there is $C > 0$, such that for any $y_1, y_2 \in Y$, $Z_1, Z_2 \in \mathbb{R}^{2(n-m)}$, $|Z_1|, |Z_2| < \epsilon$, $p \geq p_1$, we have

$$\begin{aligned} & \left| B_p^{X,kY}((y_1, Z_1), (y_2, Z_2)) - p^{n-m} \cdot B_p^Y(y_1, y_2) \cdot \kappa_N^{-\frac{1}{2}}(y_1) \cdot \kappa_N^{-\frac{1}{2}}(y_2) \cdot \right. \\ & \cdot \exp \left(-\frac{\pi}{2}p(|Z_1|^2 + |Z_2|^2) \right) \cdot \sum_{l=k+1}^{\infty} \pi^l \cdot p^l \cdot \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=l}} \frac{(z_1 \bar{z}_2)^\beta}{\beta!} \Big|_{\mathcal{C}^r} \leq Cp^{n+k+\frac{r-1}{2}} \cdot |Z_1 Z_2|^k \cdot \\ & \cdot (1 + \sqrt{p}|Z_1| + \sqrt{p}|Z_2|)^Q \cdot \exp \left(-c\sqrt{p} \cdot (\text{dist}_X(y_1, y_2) + |Z_1 - Z_2|) \right), \end{aligned} \quad (1.22)$$

where the \mathcal{C}^r -norm is taken with respect to y_1, y_2, Z_1, Z_2 , and we used the multiindex notation as we explain in (1.23).

Remark 1.6. The study of the asymptotic expansion of $B_p^{X,kY}$ has recently received considerable attention for $k = [\epsilon p]$, where $\epsilon > 0$ is some small constant. See for example Ross-Thomas [36], Pokorny-Singer [31], Ross-Singer [35], Zelditch-Zhou [41], Coman-Marinescu [9]. We believe that our methods can be applied in this asymptotic regime as well.

Let us finally say a few words about the tools we use in this article. The proofs of Theorems 1.1, 1.3 rely on the exponential estimate for the Bergman kernel, cf. Ma-Marinescu [25], on the asymptotic expansion of the Bergman kernel due to Dai-Liu-Ma [10], and on some technical results about the algebras of operators with Taylor-type expansion of the Schwartz kernel, which are inspired by the work of Ma-Marinescu [24], cf. [22, §7].

As an important intermediate result, we establish the asymptotic version of Ohsawa-Takegoshi extension theorem for holomorphic jets. For this, we follow our strategy from [16], which treats holomorphic sections instead of jets, and which was itself inspired by Bismut-Lebeau [2] and Demailly [11]. Using this result, we prove the existence of a sequence of operators, which we call multiplicative defect, relating the extension operator and the adjoint of the restriction operator. We establish then that this sequence of operators forms a Toeplitz operator with weak exponential decay, which is a notion introduced in [17] (as a refinement of the notion of Toeplitz operators in the sense of [22, §7]). The asymptotic criteria for those operators, established in [17], relying on the analogous result of Ma-Marinescu [23, Theorem 4.9] for compact manifolds and Toeplitz operators in the sense of [22, §7], plays a foundational role in our approach.

Differently from the approach from [16], which was based on the spectral geometry and localisation techniques, the main results of this article are obtained through the use of the multiplicative

defect, and we argue by induction on the order of jets, $k \in \mathbb{N}$. The general strategy for dealing with semi-classical limits here is inspired by Bismut [1] and Bismut-Vasserot [3].

To conclude, we mention that recently there has been a surge of interest in Ohsawa-Takegoshi extension theorem for holomorphic jets, see for example Hosono [20], McNeal-Varolin [28], Rao-Zhang [34]. See also Cao-Păun [7] for an application of a version of extension theorem for jets to a conjecture of deformational invariance of plurigenera for Kähler families.

This article is organized as follows. In Section 2, we recall the bounded geometry assumptions, and the results about the convergence of the exponential integrals on such manifolds. In Section 3, we recall the kernel calculus which studies the composition rules of basic operators on the model vector space. We recall a notion of Toeplitz operators with weak exponential decay and an asymptotic criteria for those operators. In Section 4, we prove the existence of a sequence of operators, which we call multiplicative defect. Finally, in Section 5, by the use of the above results, we prove the exponential bounds for the extension operator, orthogonal Bergman kernel of order k , and study the asymptotic expansion of their Schwartz kernels. As a consequence of those studies, we establish the main results of this article.

Notations. For $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$, $B = (B_1, \dots, B_k) \in \mathbb{C}^k$, we write by

$$|\alpha| = \sum_{i=1}^k \alpha_i, \quad \alpha! = \prod_{i=1}^k \alpha_i!, \quad B^\alpha = \prod_{i=1}^k B_i^{\alpha_i}. \quad (1.23)$$

Let (V, h^V) be a Hermitian (or Euclidean) vector space. We endow $\text{Sym}^k V$ with a Hermitian metric induced by the induced metric on $V^{\otimes k}$ and the inclusion $\text{Sym}^k V \rightarrow V^{\otimes k}$, defined as

$$v_1 \odot \dots \odot v_k \mapsto \frac{1}{k!} \sum v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)}, \quad (1.24)$$

where the sum runs over all permutations σ on k indices. Clearly, if v_1, \dots, v_l form an orthonormal basis of V , then $\sqrt{\frac{k!}{\alpha!}} \cdot v^{\odot \alpha}$, $\alpha \in \mathbb{N}^k$, $|\alpha| = k$, forms an orthonormal basis of $\text{Sym}^k V$. The inclusion (1.24) gives also a natural isomorphism $\text{Sym}^k V^* \rightarrow (\text{Sym}^k V)^*$. For a fixed basis v_1, \dots, v_l of V and the dual basis u_1, \dots, u_l of V^* , we have the following relation

$$u^{\odot \beta}(v^{\odot \alpha}) = \begin{cases} 0, & \text{if } \alpha \neq \beta, \\ \frac{\alpha!}{k!}, & \text{otherwise.} \end{cases} \quad (1.25)$$

For a (V, h^V) be a Hermitian vector space with a complex structure J , we denote by $V^{1,0}$, $V^{0,1}$ the holomorphic and antiholomorphic components of $V \otimes \mathbb{C}$. We endow $V^{1,0}$ and $V^{0,1}$ with the natural metrics, verifying $\|v \pm \sqrt{-1}Jv\| = \sqrt{2}\|v\|$. In this way, when $V = \mathbb{C}^l$ is endowed with the standard Hermitian product and with the usual linear (complex) coordinates z_i , we have $\|dz_i\| = \sqrt{2}$ and $\|\frac{\partial}{\partial z_i}\| = \frac{1}{\sqrt{2}}$.

We use notations X, Y for complex manifolds and M, H for real manifolds. The complex (resp. real) dimensions of X, Y (resp. M, H) are denoted here by n, m (resp. $2n, 2m$). An operator ι always means an embedding $\iota : Y \rightarrow X$ (resp. $\iota : H \rightarrow M$). We denote by Res_Y (resp. Res_H) the restriction operator from X to Y (resp. M to H).

For a Riemannian manifold (M, g^{TM}) , we denote the Levi-Civita connection by ∇^{TM} , by R^{TM} the curvature of it, and by $dv_{g^{TM}}$ the Riemannian volume form. For a closed subset $W \subset M$,

$r \geq 0$, let $B_W^M(r)$ be the ball of radius r around W . For a Hermitian vector bundle (E, h^E) , note $B_r(E) := \{Z \in E : |Z|_{h^E} < r\}$.

For a fixed volume form dv_M on M , we denote by $L^2(dv_M, h^E)$ the space of L^2 -sections of E with respect to dv_M and h^E . When $dv_M = dv_{g^{TM}}$, we also use the notation $L^2(g^{TM}, h^E)$. When there is no confusion about the data, we also use the simplified notation $L^2(M, E)$ or $L^2(M)$.

For $n \in \mathbb{N}^*$, we denote by $dv_{\mathbb{C}^n}$ the standard volume form on \mathbb{C}^n . We view \mathbb{C}^m (resp. \mathbb{R}^{2m}) embedded in \mathbb{C}^n (resp. \mathbb{R}^{2n}) by the first m (resp. $2m$) coordinates. For $Z \in \mathbb{R}^k$, we denote by $Z_l, l = 1, \dots, k$, the coordinates of Z . If $Z \in \mathbb{R}^{2n}$, we denote by $z_i, i = 1, \dots, n$, the induced complex coordinates $z_i = Z_{2i-1} + \sqrt{-1}Z_{2i}$. We frequently use the decomposition $Z = (Z_Y, Z_N)$, where $Z_Y = (Z_1, \dots, Z_{2m})$ and $Z_N = (Z_{2m+1}, \dots, Z_{2n})$. For a fixed frame (e_1, \dots, e_{2n}) in $T_x X$, $x \in X$, (resp. $y \in Y$) we implicitly identify Z (resp. Z_Y, Z_N) to an element in $T_x X$ (resp. $T_y Y, N_y$) by

$$Z = \sum_{i=1}^{2n} Z_i e_i, \quad Z_Y = \sum_{i=1}^{2m} Z_i e_i, \quad Z_N = \sum_{i=2m+1}^{2n} Z_i e_i. \quad (1.26)$$

If the frame e_i satisfies the condition

$$J e_{2i-1} = e_{2i}, \quad (1.27)$$

we denote $\frac{\partial}{\partial z_i} := \frac{1}{2}(e_{2i-1} - \sqrt{-1}e_{2i})$, $\frac{\partial}{\partial \bar{z}_i} := \frac{1}{2}(e_{2i-1} + \sqrt{-1}e_{2i})$, and identify z, \bar{z} to vectors in $T_x X \otimes_{\mathbb{R}} \mathbb{C}$ as follows

$$z = \sum_{i=1}^n z_i \cdot \frac{\partial}{\partial z_i}, \quad \bar{z} = \sum_{i=1}^n \bar{z}_i \cdot \frac{\partial}{\partial \bar{z}_i}. \quad (1.28)$$

Clearly, in this identification, $Z = z + \bar{z}$. We define $z_Y, \bar{z}_Y \in T_y Y \otimes_{\mathbb{R}} \mathbb{C}$, $z_N, \bar{z}_N \in N_y \otimes_{\mathbb{R}} \mathbb{C}$ in a similar way.

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2 Second fundamental form and bounded geometry

The main goal of this section is to recall the basic facts about the geometry of manifolds of bounded geometry and the second fundamental form. More precisely, in Section 2.1, we recall the definition and various properties satisfied by the second fundamental form. We also recall the definition of manifolds (resp. pairs of manifolds, vector bundles) of bounded geometry. In Section 2.2, we recall the results about the convergence of exponential integrals on manifolds of bounded geometry.

2.1 The second fundamental form and bounded geometry assumption

Here we recall some basic facts about the second fundamental form and its relation with bounded geometry assumptions. Let H be an embedded submanifold of a Riemannian manifold (M, g^{TM}) , $g^{TH} := g^{TM}|_H$. We identify the normal bundle $N^{M|H}$ of H in M to an orthogonal complement of TH in TM as

$$TM|_H \rightarrow TH \oplus N^{M|H}. \quad (2.1)$$

We denote by $g^{N^{M|H}}$ the metric on $N^{M|H}$ induced by g^{TM} . We denote by $P_N^{M|H} : TM|_H \rightarrow N^{M|H}$, $P_H^{M|H} : TM|_H \rightarrow TH$, the projections induced by (2.1). Clearly, $\nabla^{N^{M|H}} := P_N^{M|H} \nabla^{TM}|_H$

defines a connection on $N^{M|H}$. Recall that the definition of the *second fundamental form* $A \in \mathcal{C}^\infty(H, T^*H \otimes \text{End}(TM|_H))$ is given by

$$A := \nabla^{TM}|_H - \nabla^{TH} \oplus \nabla^{N^{M|H}}. \quad (2.2)$$

Recall that the *mean curvature* $\nu^{M|H} \in \mathcal{C}^\infty(H, N^{M|H})$ of ι is defined as follows

$$\nu^{M|H} := \frac{1}{2m} \sum_{i=1}^{2m} A(e_i)e_i, \quad (2.3)$$

where the sum runs over an orthonormal basis of (TH, g^{TH}) .

Proposition 2.1 (cf. [17, Proposition 2.3]). The second fundamental form satisfies the following properties.

1. It takes values in skew-symmetric endomorphisms of $TM|_H$, interchanging TH and $N^{M|H}$.
2. For any $U, V \in TH$, we have $A(U)V = A(V)U$.

Assume, moreover, that (M, g^{TM}) is Kähler and H is a complex submanifold. Then the following holds.

3. A commutes with the action of the complex structure.
4. For any $U \in TH, V \in TM, U = u + \bar{u}, V = v + \bar{v}, u, v \in T^{1,0}M$, we have

$$\begin{aligned} A(U)v &= A(\bar{u})v, & A(U)\bar{v} &= A(u)\bar{v}, & \text{if } V \in N^{M|H}, \\ A(U)v &= A(u)v, & A(U)\bar{v} &= A(\bar{u})\bar{v}, & \text{if } V \in TH. \end{aligned} \quad (2.4)$$

5. We have $\nu^{M|H} = 0$.

Let us now recall the definitions of manifolds (resp. pairs of manifolds, vector bundles) of bounded geometry. For more detailed overview of this part, refer to [15], [37], [18], cf. also [16]. Now, for a Hermitian vector bundle (E, h^E) with a fixed connection ∇^E over M , we denote

$$\mathcal{C}_b^\infty(M, E) := \left\{ f \in \mathcal{C}^\infty(M, E) : \text{for any } k \in \mathbb{N}, \text{ there is } C > 0, \text{ so that } |\nabla^k f| \leq C \right\}, \quad (2.5)$$

where ∇ is the connection induced by ∇^E and the Levi-Civita connection on TM , and $|\cdot|$ is the norm induced by the metrics g^{TM}, h^E . When M is complex and E is holomorphic, we implicitly take ∇^E to be the associated Chern connection.

Definition 2.2. We say that a Riemannian manifold (M, g^{TM}) is of bounded geometry if the following two conditions are satisfied.

- (i) The injectivity radius of (M, g^{TM}) is bounded below by a positive constant r_M .
- (ii) For the Riemann curvature tensor R^{TM} of M , we have $R^{TM} \in \mathcal{C}_b^\infty(M, \Lambda^2 T^*M \otimes \text{End}(TM))$.

Remark 2.3. By Hopf-Rinow theorem, the condition (i) implies that (M, g^{TM}) is complete.

Definition 2.4. We say that the triple (M, H, g^{TM}) of a Riemannian manifold (M, g^{TM}) and a submanifold H is of bounded geometry if the following conditions are fulfilled.

- (i) The manifold (M, g^{TM}) is of bounded geometry.
- (ii) The injectivity radius of (H, g^{TH}) is bounded below by a positive constant r_H .
- (iii) There is a collar around H (a tubular neighborhood of fixed radius), i.e. there is $r_\perp > 0$ such that for any $x, y \in H$, the normal geodesic balls $B_{r_\perp}^\perp(x), B_{r_\perp}^\perp(y)$, obtained by the application of the exponential mapping to vectors, orthogonal to H , of norm bounded by r_\perp , are disjoint.
- (iv) The second fundamental form, A , satisfies $A \in \mathcal{C}_b^\infty(H, T^*M|_H \otimes \text{End}(TM|_H))$.

We will now introduce a local coordinate system in M , which is particularly well-adapted to study of triples of bounded geometry. We fix a point $y_0 \in H$ and an orthonormal frame (e_1, \dots, e_{2m}) (resp. $(e_{2m+1}, \dots, e_{2n})$) in $(T_{y_0}H, g_{y_0}^{TH})$ (resp. in $(N_{y_0}^{M|H}, g_{y_0}^{N^{M|H}})$). For $Z = (Z_H, Z_N)$, $Z_H \in \mathbb{R}^{2m}$, $Z_N \in \mathbb{R}^{2(n-m)}$, $Z_H = (Z_1, \dots, Z_{2m})$, $Z_N = (Z_{2m+1}, \dots, Z_{2n})$, $|Z_H| \leq r_H$, $|Z_N| \leq r_\perp$, we define a coordinate system $\psi_{y_0}^{M|H} : B_0^{\mathbb{R}^{2m}}(r_H) \times B_0^{\mathbb{R}^{2(n-m)}}(r_\perp) \rightarrow M$ by

$$\psi_{y_0}^{M|H}(Z_H, Z_N) := \exp_{\exp_{y_0}^H(Z_H)}^M(Z_N(Z_H)), \quad (2.6)$$

where $Z_N(Z_H)$ is the parallel transport of $Z_N \in N_{y_0}^{M|H}$ along $\exp_{y_0}^H(tZ_H)$, $t = [0, 1]$, with respect to the connection $\nabla^{N^{M|H}}$ on $N^{M|H}$. The coordinates $\psi_{y_0}^{M|H}$ are called the *Fermi coordinates* at y_0 .

In the special case when $H = M$ and $x_0 := y_0$, those coordinates correspond to the exponential coordinates $\phi_{x_0}^M : \mathbb{R}^{2n} \rightarrow M$, $x_0 \in M$, defined as follows

$$\phi_{x_0}^M(Z) := \exp_{x_0}^M(Z). \quad (2.7)$$

The importance of Fermi coordinates stems from the fact, cf. [37, Lemma 3.9], [18, Theorem 4.9], that for triples of bounded geometry the metric tensor has uniformly bounded derivatives in Fermi coordinates. From this we see, in particular, for a triple of bounded geometry (M, H, g^{TM}) , the Riemannian manifold (H, g^{TH}) has bounded geometry. Along with the assumption (1.5), this result implies that in the notations of (1.9), we have $\kappa_N \in \mathcal{C}_b^\infty(U)$.

Definition 2.5. Let (E, ∇^E, h^E) be a Hermitian vector bundle with a fixed Hermitian connection over a manifold (M, g^{TM}) of bounded geometry. We say that (E, ∇^E, h^E) is of bounded geometry if $R^E \in \mathcal{C}_b^\infty(M, \Lambda^2 T^*M \otimes \text{End}(E))$.

If (E, h^E) is a Hermitian vector bundle over a *complex manifold*, we say that it is of bounded geometry if (E, ∇^E, h^E) is of bounded geometry for the Chern connection ∇^E on (E, h^E) .

Let us now define several trivializations of vector bundles which are particularly well-adapted to the bounded geometry assumption. Let us fix $y_0 \in H$ and an orthonormal frame $f_1, \dots, f_r \in E_{x_0}$. Define the local orthonormal frame $\tilde{f}_1, \dots, \tilde{f}_r$ of E around y_0 by the parallel transport of f_1, \dots, f_r with respect to ∇^E , done first along the path $\psi(tZ_H, 0)$, $t \in [0, 1]$, and then along the path $\psi(Z_H, tZ_N)$, $t \in [0, 1]$, $Z_H \in \mathbb{R}^{2m}$, $Z_N \in \mathbb{R}^{2(n-m)}$, $|Z_H| < r_H$, $|Z_N| < r_\perp$. When $H = M$, we denote this frame by $\tilde{f}'_1, \dots, \tilde{f}'_r$.

Notation. For $g \in \mathcal{C}^\infty(M, F)$, by an abuse of notation, we write $g(\phi_{y_0}^M(Z)) \in \mathbb{R}^r$, $Z \in \mathbb{R}^{2n}$, $|Z| \leq R$, for coordinates of g in the frame $(\tilde{f}'_1, \dots, \tilde{f}'_r)$. We identify $g(\phi_{y_0}^M(Z))$ with an element in F_{y_0} using the frame (f_1, \dots, f_r) . Similarly, we denote by $g(\psi_{y_0}(Z)) \in \mathbb{R}^r$ the coordinates in the frame $(\tilde{f}_1, \dots, \tilde{f}_r)$ and identify them with an element from F_{y_0} . Similar notations are used for sections of F^* , $F \otimes L^p$, $(F \otimes L^p)^*$, $F \boxtimes F^*$, etc.

As it was explained in [15], [18], cf. [16, Section 2.1], if (E, ∇^E, h^E) has bounded geometry over (M, H, g^{TM}) and $g \in \mathcal{C}_b^\infty(M, E)$, then for any $y_0 \in H$ (resp. $x_0 \in M$), the function $Z \mapsto g(\psi_{y_0}^{M|H}(Z))$ (resp. $Z \mapsto g(\phi_{x_0}^M(Z))$) is in $\mathcal{C}_b^\infty(B_0^{\mathbb{R}^{2n}}(r), F_{y_0})$ (resp. $\mathcal{C}_b^\infty(B_0^{\mathbb{R}^{2n}}(r), F_{x_0})$) for some constant $r > 0$, independent of y_0 (resp. x_0).

2.2 Exponential bounds over manifolds of bounded geometry

The main goal of this section is to recall some results about the convergence of exponential integrals for triples of bounded geometry from [16], [17]. More precisely, fix a triple (M, H, g^{TM}) of bounded geometry. Let us recall the exponential bound from [16, Corollary 3.3], established using Bishop-Gromov inequality.

Proposition 2.6. There are $c, C' > 0$, which depend only on n, m, r_M, r_N, r_\perp and sup-norm on R^{TM}, R^{TH}, A , such that for any $y_0 \in H, l \geq c$, we have

$$\int_H \exp(-l \text{dist}_M(y_0, y)) dv_{g^{TH}}(y) < \frac{C'}{l^{2m}}. \quad (2.8)$$

Now, in addition to the triple (M, H, g^{TM}) of bounded geometry, we consider a Riemannian manifold (K, g^{TK}) with an embedding $\iota_1 : M \rightarrow K$, such that $\iota_1^* g^{TK} = g^{TM}$. We assume that the triple (K, M, g^{TK}) is of bounded geometry. Let (E, h^E) be a Hermitian vector bundle over M and $D : L^2(H, \iota_1^* E) \rightarrow L^2(M, E)$ be a fixed linear operator. Assume that there is $c > 0$ as in Proposition 2.6 and $C > 0$, such that for some $l \geq c$ and any $y \in H, x \in M$, the Schwartz kernel of D , evaluated with respect to $dv_{g^{TH}}$, satisfies the bound

$$|D(x, y)| \leq Cl^{2m} \exp(-l \text{dist}_K(x, y)). \quad (2.9)$$

By essentially relying on Heintze-Karcher estimate [19, Corollary 3.3.1] and Young's inequality for integral operators, we obtained in [17, Proposition 2.12] the following bound.

Proposition 2.7. There is $C' > 0$, which depends on the same data as C' from Proposition 2.6 and the analogous data on (K, M, g^{TK}) , such that

$$\|D\| \leq \frac{C'C}{l^{n-m}}. \quad (2.10)$$

We will now recall a related result about the bound on the composition of operators with Schwartz kernels having exponential bounds. More precisely, we fix $q \in \mathbb{N}, q \geq 2$, and operators $\mathcal{G}_t, \mathcal{A}_t^1, \dots, \mathcal{A}_t^q, t \in [0, 1]$, acting on the sections of the trivial vector bundle $\mathbb{C}^{r_0} \times \mathbb{C}^n$ over \mathbb{C}^n by the convolutions with smooth kernels $\mathcal{G}_t(Z, Z'), \mathcal{A}_t^1(Z, Z'), \dots, \mathcal{A}_t^q(Z, Z') \in \text{End}(\mathbb{C}^{r_0})$ with respect to the volume form $dv_{\mathbb{C}^n}$ on \mathbb{C}^n . We assume that there are $c_0, q_1 > 0$, such that for any $l \in \mathbb{N}$, there are $C > 0, Q_{h,1} \geq 0, h = 1, \dots, q$, such that for any $t \in [0, 1], Z, Z' \in \mathbb{R}^{2n}, \alpha, \alpha' \in \mathbb{N}^{2n}, |\alpha| + |\alpha'| \leq l$, we have

$$\begin{aligned} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \mathcal{A}_t^h(Z, Z') \right| &\leq C \left(1 + |Z| + |Z'|\right)^{Q_{h,1}+q_1 l} \exp\left(-c_0(|Z_Y - Z'_Y| + |Z_N| + |Z'_N|)\right), \\ \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \mathcal{G}_t(Z, Z') \right| &\leq C \left(1 + |Z| + |Z'|\right)^{Q_{1,1}+q_1 l} \exp\left(-c_0|Z - Z'|\right). \end{aligned} \quad (2.11)$$

Lemma 2.8 ([16, Lemma 3.5]). The operators $\mathcal{D}_t := \mathcal{A}_t^1 \circ \cdots \circ \mathcal{A}_t^q$, $\mathcal{D}'_t := \mathcal{G}_t \circ \mathcal{A}_t^2 \circ \cdots \circ \mathcal{A}_t^q$ are well-defined and have smooth Schwartz kernels $\mathcal{D}_t(Z, Z')$, $\mathcal{D}'_t(Z, Z')$ with respect to $dv_{\mathbb{C}^n}$. Moreover, for any $l \in \mathbb{N}$, there is $C > 0$, such that for any $t \in [0, 1]$, $Z, Z' \in \mathbb{R}^{2n}$, $\alpha, \alpha' \in \mathbb{N}^{2n}$, $|\alpha| + |\alpha'| \leq l$, we have

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \mathcal{R}_t(Z, Z') \right| \leq C \left(1 + |Z| + |Z'| \right)^{Q_{1,1} + \cdots + Q_{q,1} + q_1 l} \cdot \exp \left(- \frac{c_0}{8} (|Z_Y - Z'_Y| + |Z_N| + |Z'_N|) \right), \quad (2.12)$$

where \mathcal{R}_t designates either \mathcal{D}_t or \mathcal{D}'_t .

3 Kernel calculus and asymptotic criteria of Toeplitz operators

The main goal of this section is to study the basic properties of Schwartz kernels of Toeplitz operators. More precisely, in Section 3.1, we consider the model situation, for which an explicit formula for the Schwartz kernels of Bergman projectors, the extension and restriction operators can be given. We then study the composition rules for the operators with related kernels. Then, in Section 3.2, we recall a definition and an asymptotic characterization of Toeplitz operators with weak exponential decay.

3.1 Model operators and the Fock-Bargmann space

In this section, we consider the model situation, for which an explicit formula for the Schwartz kernels of Bergman projectors, the extension and restriction operators can be given. We then use these explicit formulas to give a description for compositions of operators, the Schwartz kernels of which can be expressed using the above kernels. This section is motivated in many ways by the works of Ma-Marinescu [24], [22] and Dai-Liu-Ma [10], cf. also [16, §3.2].

Endow \mathbb{C}^n with the standard Riemannian metric and consider a trivialized holomorphic line bundle L_0 on \mathbb{C}^n . We endow L_0 with the Hermitian metric h^{L_0} , given by

$$\|1\|_{h^{L_0}}(Z) = \exp \left(- \frac{\pi}{2} |Z|^2 \right), \quad (3.1)$$

where Z is the natural real coordinate on \mathbb{C}^n , and 1 is the trivializing section of L_0 . An easy verification shows that (3.1) implies that (1.2) holds in our setting. From [22, Theorem 4.1.20], we know that the functions

$$\left(\frac{\pi^{|\beta|}}{\beta!} \right)^{\frac{1}{2}} z^\beta \exp \left(- \frac{\pi}{2} \sum_{i=1}^n |z_i|^2 \right), \quad (3.2)$$

viewed as sections of L_0 using the orthonormal trivialization of L_0 by $1 \exp(\frac{\pi}{2}|Z|^2)$, form an orthonormal basis of $H_{(2)}^0(\mathbb{C}^n, L_0)$, endowed with the induced L^2 -metric. From this, we see that $H_{(2)}^0(\mathbb{C}^n, L_0)$, endowed with this L^2 -metric, coincides with the Fock-Bargmann space, cf. [42]. From (3.2), cf. [22, (4.1.84)], the Bergman kernel \mathcal{P}_n of \mathbb{C}^n , which is an orthogonal projection from the space of L^2 -sections of L_0 , $L^2(\mathbb{C}^n, L_0)$, to $H_{(2)}^0(\mathbb{C}^n, L_0)$, is given by

$$\mathcal{P}_n(Z, Z') = \exp \left(- \frac{\pi}{2} \sum_{i=1}^n (|z_i|^2 + |z'_i|^2 - 2z_i \bar{z}'_i) \right), \quad \text{for } Z, Z' \in \mathbb{C}^n. \quad (3.3)$$

From (3.2), we see easily, cf. [16, (3.28), (3.29)], that the Schwartz kernel of the *orthogonal Bergman kernel of order* $k \in \mathbb{N}$, $\mathcal{P}_{n,m}^{\perp,k}$, corresponding to the projection onto the subspace of $H_{(2)}^0(\mathbb{C}^n, L_0 \otimes \mathcal{J}_{\mathbb{C}^m}^k)$, orthogonal to $H_{(2)}^0(\mathbb{C}^n, L_0 \otimes \mathcal{J}_{\mathbb{C}^m}^{k+1})$, is given by

$$\begin{aligned} \mathcal{P}_{n,m}^{\perp,k}(Z, Z') = \exp\left(-\frac{\pi}{2} \sum_{i=1}^m (|z_i|^2 + |z'_i|^2 - 2z_i \bar{z}'_i) - \frac{\pi}{2} \sum_{i=m+1}^n (|z_i|^2 + |z'_i|^2)\right) \\ \cdot \pi^k \cdot \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=k}} \frac{z_N^\beta \cdot (\bar{z}'_N)^\beta}{\beta!}. \end{aligned} \quad (3.4)$$

Remark that if we view z_N and \bar{z}'_N as $n - m$ vectors in a natural way, we also have

$$\sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=k}} \frac{z_N^\beta \cdot (\bar{z}'_N)^\beta}{\beta!} = \frac{(z_N \cdot \bar{z}'_N)^k}{k!}. \quad (3.5)$$

Let us now denote $V := \mathbb{C}^{n-m}$ and consider the holomorphic and antiholomorphic parts $V^{1,0}$, $V^{0,1}$ of the complexification $V \otimes_{\mathbb{R}} \mathbb{C}$. We denote by ∇ the standard covariant derivative on V . Remark now that $\nabla^{|\beta|} z_N^\beta \in ((V^{1,0})^*)^{\otimes |\beta|}$ satisfies the following identity

$$\nabla^{|\beta|} z_N^\beta \left(\frac{\partial}{\partial z_{\sigma(1)}} \otimes \cdots \otimes \frac{\partial}{\partial z_{\sigma(|\beta|)}} \right) = \frac{\partial}{\partial z_{\sigma(1)}} \cdots \frac{\partial}{\partial z_{\sigma(|\beta|)}} z_N^\beta = \begin{cases} \beta!, & \text{if } \#\sigma^{-1}(i) = \beta_i, \\ 0, & \text{otherwise,} \end{cases} \quad (3.6)$$

where $\sigma : [1, N] \rightarrow [m+1, n]$ is any map. From this, we conclude that $\nabla^{|\beta|} z_N^\beta$ lies in the image of the natural map $\text{Sym}^{|\beta|}((V^{1,0})^*) \rightarrow ((V^{1,0})^*)^{\otimes |\beta|}$, and under this map, from (1.25) we have

$$\nabla^{|\beta|} z_N^\beta = |\beta|! \cdot dz_N^{\otimes \beta}. \quad (3.7)$$

From (1.25) and (3.2), the L^2 -extension operator $\mathcal{E}_{n,m}^k : H_{(2)}^0(\mathbb{C}^m, \text{Sym}^k(V^{1,0})^* \otimes L_0) \rightarrow H_{(2)}^0(\mathbb{C}^n, L_0 \otimes \mathcal{J}_{\mathbb{C}^m}^k)$, mapping to the sections with minimal L^2 -norm as in (1.6), is given in the orthonormal trivialization $1 \exp(\frac{\pi}{2}|Z|^2)$ of L_0 by the multiplication by $\exp(-\frac{\pi}{2} \sum_{i=m+1}^n |z_i|^2)$. From this and (3.3), the Schwartz kernel $\mathcal{E}_{n,m}^k(Z, Z'_Y)$ of $\mathcal{E}_{n,m}^k$ is given by

$$\begin{aligned} \mathcal{E}_{n,m}^k(Z, Z'_Y) = \exp\left(-\frac{\pi}{2} \sum_{i=1}^m (|z_i|^2 + |z'_i|^2 - 2z_i \bar{z}'_i) - \frac{\pi}{2} \sum_{i=m+1}^n |z_i|^2\right) \\ \cdot \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=k}} \frac{1}{\beta!} \cdot z_N^\beta \cdot \left(\frac{\partial}{\partial z_N}\right)^{\otimes \beta}, \end{aligned} \quad (3.8)$$

where we implicitly identified $(\text{Sym}^k(V^{1,0})^*)^*$ and $\text{Sym}^k(V^{1,0})$ as in (1.25). Remark that under our identification, the following identities hold

$$\begin{aligned} \text{Id}_{\text{Sym}^k(V^{1,0})^*} &= \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=k}} \frac{k!}{\beta!} \cdot (dz_N)^{\otimes \beta} \cdot \left(\frac{\partial}{\partial z_N}\right)^{\otimes \beta}, \\ \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=k}} \frac{1}{\beta!} \cdot z_N^\beta \cdot \left(\frac{\partial}{\partial z_N}\right)^{\otimes \beta} &= \frac{(z_N)^{\otimes k}}{k!}, \end{aligned} \quad (3.9)$$

where for the second identity we used the notations (1.28) for z_N . From (3.3), (3.4), (3.8) and (3.9), we can verify that the following natural relation holds

$$\text{Res}_{\mathbb{C}^m} \circ \nabla^k \mathcal{E}_{n,m}^k = \mathcal{P}_m \cdot \text{Id}_{\text{Sym}^k(V^{1,0})^*}, \quad (3.10)$$

where we identified $H_{(2)}^0(\mathbb{C}^m, \text{Sym}^k(V^{1,0})^* \otimes L_0)$ with $H_{(2)}^0(\mathbb{C}^m, L_0) \otimes \text{Sym}^k(V^{1,0})^*$.

For $k \in \mathbb{N}$, recall that the logarithmic Bergman kernel, $\mathcal{P}_{n,m}^{\mathbb{C}^n, k\mathbb{C}^m}$, corresponding to the pair $(\mathbb{C}^n, k \cdot \mathbb{C}^k)$, is the orthogonal projection from $L^2(\mathbb{C}^n, L_0)$ onto $H_{(2)}^0(\mathbb{C}^n, L_0 \otimes \mathcal{J}_{\mathbb{C}^m}^k)$. It clearly satisfies the following identity

$$\mathcal{P}_{n,m}^{\mathbb{C}^n, k\mathbb{C}^m} := \mathcal{P}_n - \sum_{l=0}^{k-1} \mathcal{P}_{n,m}^{\perp, l}. \quad (3.11)$$

From (3.2), (3.3) and (3.4), we can write

$$\begin{aligned} \mathcal{P}_{n,m}^{\mathbb{C}^n, k\mathbb{C}^m}(Z, Z') &= \exp\left(-\frac{\pi}{2} \sum_{i=1}^m (|z_i|^2 + |z'_i|^2 - 2z_i \bar{z}'_i) - \frac{\pi}{2} \sum_{i=m+1}^n (|z_i|^2 + |z'_i|^2)\right) \\ &\quad \cdot \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta| \geq k}} \pi^{|\beta|} \cdot \frac{z_N^\beta \cdot (\bar{z}'_N)^\beta}{\beta!}. \end{aligned} \quad (3.12)$$

Hence, by (3.7), the Schwartz kernel of the operator $\mathcal{R}_{n,m}^k := \text{Res}_{\mathbb{C}^m} \circ \nabla^k \mathcal{P}_{n,m}^{\mathbb{C}^n, k\mathbb{C}^m}$ is given by

$$\begin{aligned} \mathcal{R}_{n,m}^k(Z_Y, Z') &= \exp\left(-\frac{\pi}{2} \sum_{i=1}^m (|z_i|^2 + |z'_i|^2 - 2z_i \bar{z}'_i) - \frac{\pi}{2} \sum_{i=m+1}^n |z'_i|^2\right) \\ &\quad \cdot \pi^k \cdot \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=k}} \frac{k!}{\beta!} \cdot (dz_N)^{\odot \beta} \cdot (\bar{z}'_N)^\beta. \end{aligned} \quad (3.13)$$

Let us consider $f \in H_{(2)}^0(\mathbb{C}^n, L_0 \otimes \mathcal{J}_{\mathbb{C}^m}^k)$ (resp. $g \in H_{(2)}^0(\mathbb{C}^m, \text{Sym}^k(V^{1,0})^* \otimes L_0)$), given in the holomorphic trivialization of L_0 by $f = z_N^\beta$ (resp. $g = dz_N^{\odot \beta'}$), where $\beta, \beta' \in \mathbb{N}^{2(n-m)}$, $|\beta|, |\beta'| = k$. By (3.7), we have

$$\mathcal{R}_{n,m}^k f = k! \cdot dz_N^{\odot \beta}, \quad \mathcal{E}_{n,m}^k g = \frac{1}{k!} \cdot z_N^{\beta'}. \quad (3.14)$$

From the discussion after (1.24) and (3.2), we see that

$$\langle \mathcal{R}_{n,m}^k f, g \rangle_{L^2(Y)} = 2^k \cdot \beta! \cdot \delta_{\beta\beta'} = (2\pi)^k \cdot k! \cdot \langle f, \mathcal{E}_{n,m}^k g \rangle_{L^2(X)}, \quad (3.15)$$

where $\delta_{\beta\beta'}$ is the Kronecker delta. This implies the following identity

$$(\mathcal{R}_{n,m}^k)^* = (2\pi)^k \cdot k! \cdot \mathcal{E}_{n,m}^k, \quad (3.16)$$

which is compatible with (3.8) and (3.13) in view of the identity

$$((dz_N)^{\odot \beta})^* = 2^{|\beta|} \cdot \left(\frac{\partial}{\partial z_N}\right)^{\odot \beta}. \quad (3.17)$$

Similarly, we deduce the following equality

$$\mathcal{P}_{n,m}^{\perp,k} = \mathcal{E}_{n,m}^k \circ \mathcal{R}_{n,m}^k. \quad (3.18)$$

Later we will see that the analogues of (3.16) and (3.18) hold in the approximate sense for any pair (X, Y) instead of the model case $(\mathbb{C}^n, \mathbb{C}^m)$.

Now, a lot of calculations in this article will have something to do with compositions of operators having Schwartz kernels, given by the product of polynomials with the above kernels. For that reason, the following lemma will be of utmost importance in what follows.

Lemma 3.1. For any polynomials $A_1(Z, Z'), A_2(Z, Z'), Z, Z' \in \mathbb{R}^{2n}$, there is a polynomial $A_3 := \mathcal{K}_{n,m}[A_1, A_2]$, the coefficients of which are polynomials of the coefficients of A_1, A_2 , such that

$$(A_1 \cdot \mathcal{P}_{n,m}^{\perp,0}) \circ (A_2 \cdot \mathcal{P}_{n,m}^{\perp,0}) = A_3 \cdot \mathcal{P}_{n,m}^{\perp,0}. \quad (3.19)$$

Moreover, $\deg A_3 \leq \deg A_1 + \deg A_2$. Also, if both polynomials A_1, A_2 are even or odd (resp. one is even, another is odd), then the polynomial A_3 is even (resp. odd).

Also, for any polynomials $A(Z, Z'_Y), B(Z_Y, Z'_Y), Z \in \mathbb{R}^{2n}, Z_Y, Z'_Y \in \mathbb{R}^{2m}$, there is a polynomial $A'_3 := \mathcal{K}_{n,m}^{EP}[A, B]$ with the same properties as A_3 , such that

$$(A \cdot \mathcal{E}_{n,m}^0) \circ (B \cdot \mathcal{P}_m) = A'_3 \cdot \mathcal{E}_{n,m}. \quad (3.20)$$

Finally, for any polynomials $A(Z, Z'_Y), C(Z_Y, Z'), Z, Z' \in \mathbb{R}^{2n}, Z_Y, Z'_Y \in \mathbb{R}^{2m}$, there is a polynomial $A''' := \mathcal{K}_{n,m}^{ER}[A, C]$ with the same properties as A_3 , such that

$$(A \cdot \mathcal{E}_{n,m}^0) \circ (C \cdot \mathcal{R}_{n,m}^0) = A''' \cdot \mathcal{P}_{n,m}^{\perp,0}. \quad (3.21)$$

Remark 3.2. The statement (3.19) for $n = m$ is due to Ma-Marinescu [22, Lemma 7.1.1, (7.1.6)].

Proof. Statements (3.19) and (3.21) were proved in [17, Lemma 3.1]. Let us recall the relation between $\mathcal{K}_{n,m}^{EP}$ and $\mathcal{K}_{n,m}$, established in [17, (3.20)].

We represent $A(Z, Z'_Y) := \sum Z_N^\alpha \cdot A^\alpha(Z_Y, Z'_Y)$. Then, from (3.3) and (3.4), the following equation holds

$$(A^\alpha \cdot \mathcal{E}_{n,m}^0) \circ (B \cdot \mathcal{P}_m) = \exp\left(-\frac{\pi}{2}|Z_N|^2\right) \cdot (A^\alpha \cdot \mathcal{P}_m) \circ (B \cdot \mathcal{P}_m). \quad (3.22)$$

By this and (3.19), we clearly have (3.20) for

$$\mathcal{K}_{n,m}^{EP}[A, D] = \sum_{\alpha} Z_N^\alpha \cdot \mathcal{K}_{m,m}[A^\alpha, D]. \quad (3.23)$$

To establish (3.21), we decompose polynomials $A(Z, Z'_Y), C(Z_Y, Z')$ as follows

$$A(Z, Z'_Y) = \sum_{\alpha} Z_N^\alpha \cdot A^\alpha(Z_Y, Z'_Y), \quad C(Z_Y, Z') = \sum_{\alpha'} C^{\alpha'}(Z_Y, Z'_Y) Z'_N{}^{\alpha'}, \quad (3.24)$$

where $\alpha, \alpha' \in \mathbb{N}^{2(n-m)}$ verify $|\alpha| \leq \deg A, |\alpha'| \leq \deg C$.

Now, an easy verification, based on (3.3), (3.4) and (3.13), shows that

$$\begin{aligned} \left((A^\alpha \cdot \mathcal{E}_{n,m}^0) \circ (C^{\alpha'} \cdot \mathcal{R}_{n,m}^0) \right) (Z, Z') &= \exp \left(-\frac{\pi}{2} (|Z_N|^2 + |Z'_N|^2) \right) \\ &\cdot \left((A^\alpha \cdot \mathcal{P}_m) \circ (C^{\alpha'} \cdot \mathcal{P}_m) \right) (Z_Y, Z'_Y). \end{aligned} \quad (3.25)$$

From (3.24) and (3.25), we see that (3.21) holds for

$$\mathcal{K}_{n,m}^{ER}[A, C] = \sum_{\alpha, \alpha'} Z_N^\alpha \cdot Z_N^{\alpha'} \cdot \mathcal{K}_{m,m}[A^\alpha, C^{\alpha'}]. \quad (3.26)$$

This clearly finishes the proof, as the statements about the degree, the coefficients and the parity now follow from (3.26). \square

From the above, we see that to compute the polynomials from Lemma 3.1, it suffices to give an algorithm for the calculation of $\mathcal{K}_{n,m}$. Below, we explain how to do this. Directly from the definitions, we see that $\mathcal{K}_{n,m}[1 \cdot P(Z'), A] = \mathcal{K}_{n,m}[1, P(Z) \cdot A]$ for any polynomial A . Also, we trivially have $\mathcal{K}_{n,m}[P(Z) \cdot A(Z, Z'), A'(Z, Z')] = P(Z) \mathcal{K}_{n,m}[A(Z, Z'), A'(Z, Z')]$ for any polynomials P, A, A' . Hence, it is enough to give an algorithm for the calculation of $\mathcal{K}_{n,m}$ where the first argument is given by 1. For this, remark that for any $i = 1, \dots, n$, $a, b \in \mathbb{N}$, we have

$$\mathcal{K}_{n,m}[1, P_i(Z) z_i^a \bar{z}_i^b] = \mathcal{K}_{n,m}[1, P_i(Z)] \cdot \mathcal{K}_{n,m}[1, z_i^a \bar{z}_i^b], \quad (3.27)$$

where the polynomial $P_i(Z)$ doesn't depend on z_i and \bar{z}_i . Hence, to understand $\mathcal{K}_{n,m}$, it suffices to know how to calculate it for polynomials $z_i^a \bar{z}_i^b$. For $i \leq m$, by [17, (3.28)], we have

$$\mathcal{K}_{n,m}[1, z_i^a \bar{z}_i^b] = \sum_{\substack{l+k=b \\ a \geq k}} \frac{1}{\pi^k} \frac{a! b!}{(a-k)! l! k!} z_i^{a-k} \bar{z}_i^l. \quad (3.28)$$

For $m+1 \leq i \leq n$, by [17, (3.30)], we have

$$\mathcal{K}_{n,m}[1, z_i^a \bar{z}_i^b] = \delta_{ab} \frac{a!}{\pi^a}. \quad (3.29)$$

3.2 Toeplitz operators with weak exponential decay and their properties

The main goal of this section is to recall the definition of Toeplitz operators with weak exponential decay and to recall the asymptotic characterization of them in terms of their Schwartz kernels.

More precisely, for a section $f \in \mathcal{C}_b^\infty(X, \text{End}(F))$, we associate a sequence of linear operators $T_{f,p}^X \in \text{End}(L^2(X, L^p \otimes F))$, $p \in \mathbb{N}$, called *Berezin-Toeplitz operator*, by

$$T_{f,p}^X(g) := B_p^X(f \cdot B_p^X g). \quad (3.30)$$

We fix some Riemannian manifold (Z, g^{TZ}) and an embedding $\iota' : X \rightarrow Z$, such that $(\iota')^* g^{TZ} = g^{TX}$, and such that the triple (Z, X, g^{TZ}) is of bounded geometry.

Definition 3.3. A sequence of linear operators $T_p^X \in \text{End}(L^2(X, L^p \otimes F))$, $p \in \mathbb{N}$, verifying $B_p^X \circ T_p^X \circ B_p^X = T_p^X$, is called a *Toeplitz operator with weak exponential decay with respect to Z* if there is a sequence $f_i \in \mathcal{C}_b^\infty(X, \text{End}(F))$ and $c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $r, l \in \mathbb{N}$, there is

$C > 0$, such that for any $p \geq p_1$, the Schwartz kernel, evaluated with respect to dv_X , satisfies for any $x_1, x_2 \in X$, the following estimate

$$\left| T_p^X(x_1, x_2) - \sum_{j=0}^r p^{-j} T_{f_j, p}^X(x_1, x_2) \right|_{\mathcal{C}^l} \leq Cp^{n-r+\frac{l}{2}} \cdot \exp(-c\sqrt{p} \cdot \text{dist}_Z(x_1, x_2)), \quad (3.31)$$

where the pointwise \mathcal{C}^l -norm at a point $(x_1, x_2) \in X \times X$ is interpreted as in (1.21). The sections f_i will later be denoted by $[T_p^X]_i$.

Remark 3.4. a) From Proposition 2.7, we see that (3.31) implies that for any $k \in \mathbb{N}$, there is $C > 0$, such that for any $p \geq p_1$, we have $\|T_p^X - \sum_{j=0}^r p^{-j} T_{f_j, p}^X\| \leq Cp^{-r}$. In particular, for compact X , the sequence of operators T_p^X , $p \in \mathbb{N}$, forms a Toeplitz operator in the sense of Ma-Marinescu [22, §7].

b) In [17, Corollary 3.13], we showed that the sections f_i , $i \in \mathbb{N}$, verifying (3.31), are uniquely defined. Hence, the notation $[\cdot]_i$, $i \in \mathbb{N}$, from Definition 3.3 is well-defined.

For the theorem below, we use the notational convention introduced after Definition 2.5. Let us fix some further notation. Recall that geodesic coordinates were defined in (2.7). We fix $x_0 \in X$. Define the function $\kappa_{\phi, x_0}^X : B_0^{\mathbb{R}^{2n}}(r_X) \rightarrow \mathbb{R}$, by

$$((\phi_{x_0}^X)^* dv_X)(Z) = \kappa_{\phi, x_0}^X dZ_1 \wedge \cdots \wedge dZ_{2n}. \quad (3.32)$$

Theorem 3.5. *A family of operators $T_p^X \in \text{End}(L^2(X, L^p \otimes F))$, $p \in \mathbb{N}$, forms a Toeplitz operator with weak exponential decay with respect to Z if and only if the following conditions hold*

1. For any $p \in \mathbb{N}$, $T_p^X = B_p^X \circ T_p^X \circ B_p^X$.
2. There is $p_1 \in \mathbb{N}$, such that for any $l \in \mathbb{N}$, there is $C > 0$, such that for any $p \geq p_1$, the Schwartz kernel $T_p^X(x_1, x_2)$; $x_1, x_2 \in Y$, of T_p^X , evaluated with respect to dv_X , satisfies

$$\left| T_p^X(x_1, x_2) \right|_{\mathcal{C}^l} \leq Cp^{n+\frac{l}{2}} \cdot \exp(-c\sqrt{p} \cdot \text{dist}_Z(x_1, x_2)). \quad (3.33)$$

3. For any $x_0 \in X$, $r \in \mathbb{N}$, there are $I_r^X(Z, Z') \in \text{End}(F_{x_0})$ polynomials in $Z, Z' \in \mathbb{R}^{2n}$ of the same parity as r , such that the coefficients of I_r^X lie in $\mathcal{C}_b^\infty(X, \text{End}(F))$, and for $F_r := I_r^X \cdot \mathcal{P}_n$, the following holds. There are $\epsilon, c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $r, l, l' \in \mathbb{N}$, there are $C, Q > 0$, such that for any $x_0 \in X$, $p \geq p_1$, $Z, Z' \in \mathbb{R}^{2n}$, $|Z|, |Z'| \leq \epsilon$, $\alpha, \alpha' \in \mathbb{N}^{2n}$, $|\alpha| + |\alpha'| \leq l$, the following bound holds

$$\begin{aligned} & \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(\frac{1}{p^n} T_p^X(\phi_{x_0}^X(Z), \phi_{x_0}^X(Z')) \right. \right. \\ & \quad \left. \left. - \sum_{j=0}^r p^{-\frac{j}{2}} F_j(\sqrt{p}Z, \sqrt{p}Z') \kappa_{\phi, x_0}^X(Z)^{-\frac{1}{2}} \kappa_{\phi, x_0}^X(Z')^{-\frac{1}{2}} \right) \right|_{\mathcal{C}^{l'}} \\ & \leq Cp^{-\frac{r+1-l}{2}} \left(1 + \sqrt{p}|Z| + \sqrt{p}|Z'| \right)^Q \exp(-c\sqrt{p}|Z - Z'|), \quad (3.34) \end{aligned}$$

where the $\mathcal{C}^{l'}$ -norm is taken with respect to x_0 .

Moreover, for any sequence of operators T_p^X , verifying the above assumptions, the polynomial I_0^X is constant and it is related to the expansion from Definition 3.3 by the identity $I_0^X(0, 0) = [T_p^X]_0$.

Proof. For Toeplitz operators in the sense of Ma-Marinescu [22, §7], cf. Remark 3.4.a), the analogous result was established by Ma-Marinescu in [23, Theorem 4.9]. The proof for Toeplitz operators with weak exponential decay was done in [17, Theorem 3.18]. \square

Proposition 3.6. Assume that a family of linear operators $T_p^X \in \text{End}(L^2(X, L^p \otimes F))$, $p \in \mathbb{N}$ satisfies all the three assumptions of Theorem 3.5, with the only exception that the parity of I_r^X equals to the parity of $r + 1$ instead of r . Then $I_0^X = 0$, $I_1^X(Z, Z')$ is a constant polynomial, $\sqrt{p}T_p^X$ forms a Toeplitz operator with weak exponential decay with respect to Z , and we have $[\sqrt{p}T_p^X]_0 = I_1^X(0, 0)$.

Proof. In the proof of Ma-Marinescu of [23, Theorem 4.9], authors established that regardless of the parity assumption on I_r^X , the polynomial $I_0^X(Z, Z')$ is always constant. This was proved in the compact setting, but the proof remains verbatim for manifolds of bounded geometry, cf. [17, proof of Theorem 3.19]. Now, since the parity of I_0^X is odd by our assumption, the above result implies $I_0^X = 0$. Hence all the assumptions of Theorem 3.5 are satisfied for the sequence of operators $\sqrt{p}T_p^X$, $p \in \mathbb{N}^*$, which implies Proposition 3.6. \square

Now, for technical reasons we will need to consider sequences of operators $T_p^X : L^2(X, L^p \otimes F_1) \rightarrow L^2(X, L^p \otimes F_2)$, $p \in \mathbb{N}$, where (F_i, h_i^F) , $i = 1, 2$, are Hermitian vector bundles of bounded geometry over X . For such sequences of operators, we have a notion of (F_1, F_2) -Toeplitz operators with weak exponential decay, analogous to Definition 3.3. The only difference between this definition and the one for $(F_1, h_1^F) = (F_2, h_2^F)$ is that the Berezin-Toeplitz operators are now associated to $f \in \mathcal{C}_b^\infty(X, \text{Hom}(F_1, F_2))$ as follows $T_{f,p}^X(g) := B_{2,p}^X(f \cdot B_{1,p}^X g)$, where $B_{i,p}^X$, $i = 1, 2$, are the Bergman kernels associated to $L^p \otimes F_i$. We similarly use the notation $[T_p^X]_i$, $i \in \mathbb{N}$, to designate elements of $\mathcal{C}_b^\infty(X, \text{Hom}(F_1, F_2))$, corresponding to the asymptotic expansion of T_p^X .

Proposition 3.7. A family of linear operators $T_p^X : L^2(X, L^p \otimes F_1) \rightarrow L^2(X, L^p \otimes F_2)$, $p \in \mathbb{N}$, forms a (F_1, F_2) -Toeplitz operator with weak exponential decay with respect to Z if and only if the following conditions hold

1. For any $p \in \mathbb{N}$, $T_p^X = B_{2,p}^X \circ T_p^X \circ B_{1,p}^X$.
2. Exponential bound analogous to (3.33) holds.
3. For any $x_0 \in X$, $r \in \mathbb{N}$, there are polynomials $I_r^X(Z, Z') \in \text{Hom}(F_{1,x_0}, F_{2,x_0})$, satisfying the same assumptions as in (3.34), and for which an analogue of the expansion (3.34) holds.

Moreover, for any sequence of operators T_p^X , verifying the above assumptions, the polynomial $I_0^X(Z, Z')$ is constant and we have $I_0^X(0, 0) = [T_p^X]_0$.

Proof. Consider a sequence of operators $G_p^X \in \text{End}(L^2(X, L^p \otimes (F_1 \oplus F_2)))$, $p \in \mathbb{N}$, which in a matrix form associated to the decomposition $L^2(X, L^p \otimes (F_1 \oplus F_2)) = L^2(X, L^p \otimes F_1) \oplus L^2(X, L^p \otimes F_2)$ corresponds to

$$G_p^X = \begin{pmatrix} 0 & T_p^X \\ 0 & 0 \end{pmatrix}. \quad (3.35)$$

An easy verification shows that the assumptions of Theorem 3.5 are satisfied for G_p^X if and only if the corresponding assumptions from Proposition 3.7 are satisfied for T_p^X . Proposition 3.7 now follows from this and Theorem 3.5. \square

Remark 3.8. From the above proof and Proposition 3.6, we see that a proposition, analogous to Proposition 3.6 holds for (F_1, F_2) -Toeplitz operator with weak exponential decay.

Later on, we will need a precise bound on the degrees of the polynomials from (3.34). For this, we fix any $f \in \mathcal{C}_b^\infty(X, \text{End}(F))$, and denote by $I_{f,r}^X$ the polynomials associated to $T_{f,p}^X$ as in (3.34).

Proposition 3.9. For any $f \in \mathcal{C}_b^\infty(X, \text{End}(F))$, we have $\deg I_{f,r}^X \leq 3r$.

This result is definitely well-known to experts, but we were not able to find a precise reference for it anywhere in the literature. Due to this and to the fact that the preliminary statement, used in the proof of Proposition 3.9, will later be used in this article several times, we will explain the proof below. To state the result in the generality we will need later on, we fix some further notation. Recall that geodesic coordinates were defined in (2.7). We assume as in Introduction that the triple (X, Y, g^{TX}) has bounded geometry. Similarly to (3.32), for $y_0 \in Y$, we define the function $\kappa_{\phi, y_0}^Y : B_0^{\mathbb{R}^{2m}}(r_Y) \rightarrow \mathbb{R}$, by

$$((\phi_{y_0}^Y)^* dv_Y)(Z_Y) = \kappa_{\phi, y_0}^Y dZ_1 \wedge \cdots \wedge dZ_{2m}. \quad (3.36)$$

Recall that Fermi coordinates were defined in (2.6). Define the function $\kappa_{\psi, y_0}^{X|Y} : B_0^{\mathbb{R}^{2m}}(r_Y) \times B_0^{\mathbb{R}^{2(n-m)}}(r_\perp) \rightarrow \mathbb{R}$ by

$$((\psi_{y_0}^{X|Y})^* dv_X)(Z) = \kappa_{\psi, y_0}^{X|Y} dZ_1 \wedge \cdots \wedge dZ_{2n}. \quad (3.37)$$

Recall that the function κ_N was defined in (1.9). Clearly, for $Z = (Z_Y, Z_N) \in \mathbb{R}^{2n}$, $Z_Y \in \mathbb{R}^{2m}$, we have the following relation between different κ -functions

$$\kappa_{\psi, y_0}^{X|Y}(Z) = \kappa_N(\psi_{y_0}(Z)) \cdot \kappa_{\phi, y_0}^Y(Z_Y). \quad (3.38)$$

Also, under assumptions (1.10), we have $\kappa_{\psi, y_0}^{X|Y}(0) = \kappa_{\phi, y_0}^Y(0) = 1$.

Theorem 3.10. For any $r \in \mathbb{N}$, $y_0 \in Y$, there are $J_r^{X|Y}(Z, Z') \in \text{End}(F_{y_0})$ polynomials in $Z, Z' \in \mathbb{R}^{2n}$, with the same parity as r and $\deg J_r^{X|Y} \leq 3r$, whose coefficients are polynomials in R^{TX} , A , R^F , $(dv_X/dv_{g^{TX}})^{\pm \frac{1}{2n}}$, $(dv_Y/dv_{g^{TY}})^{\pm \frac{1}{2n}}$, and their derivatives of order $\leq 2r$, all evaluated at y_0 , such that for the functions $F_r^{X|Y} := J_r^{X|Y} \cdot \mathcal{P}_n$ over $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$, the following holds. There are $\epsilon, c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $r, l, l' \in \mathbb{N}$, there exists $C > 0$, such that for any $y_0 \in Y$, $p \geq p_1$, $Z, Z' \in \mathbb{R}^{2n}$, $|Z|, |Z'| \leq \epsilon$, $\alpha, \alpha' \in \mathbb{N}^{2n}$, $|\alpha| + |\alpha'| \leq l$, $Q_{r,l,l'}^1 := 3(n+r+l'+2) + l$:

$$\begin{aligned} & \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(\frac{1}{p^n} B_p^X(\psi_{y_0}(Z), \psi_{y_0}(Z')) \right. \right. \\ & \quad \left. \left. - \sum_{j=0}^r p^{-\frac{j}{2}} F_j^{X|Y}(\sqrt{p}Z, \sqrt{p}Z') \kappa_\psi^{X|Y}(Z)^{-\frac{1}{2}} \kappa_\psi^{X|Y}(Z')^{-\frac{1}{2}} \right) \right|_{\mathcal{C}^{l'}} \\ & \leq Cp^{-\frac{r+1-l}{2}} \left(1 + \sqrt{p}|Z| + \sqrt{p}|Z'| \right)^{Q_{r,l,l'}^1} \exp\left(-c\sqrt{p}|Z-Z'|\right), \end{aligned} \quad (3.39)$$

where the $\mathcal{C}^{l'}$ -norm is taken with respect to y_0 . Also, the following identity holds

$$J_0^{X|Y}(Z, Z') = \text{Id}_{F_{y_0}}. \quad (3.40)$$

Moreover, under the assumption (1.10), we have

$$J_1^{X|Y}(Z, Z') = \text{Id}_{F_{y_0}} \cdot \pi \left(g(z_N, A(\bar{z}_Y - \bar{z}'_Y)(\bar{z}_Y - \bar{z}'_Y)) \right. \\ \left. + g(\bar{z}'_N, A(z_Y - z'_Y)(z_Y - z'_Y)) \right). \quad (3.41)$$

Proof. For $X = Y$, the result is due to Dai-Liu-Ma [10] and the calculation of $J_1^{X|X}$ is due to Ma-Marinescu [22, Remark 4.1.26]. Related results were previously obtained by Tian [38], Bouche [5], Zelditch [40], Catlin [8], Lu [21], Wang [39]. The proof of the general case is done in [16, Theorem 5.5] by relying on the result of [10] and some local calculations. \square

Proof of Proposition 3.9. In [22, Lemma 7.2.4], Ma-Marinescu proved that for compact manifolds, for any $r \in \mathbb{N}$, the following identity holds

$$I_{f,r}^X := \sum_{\alpha \in \mathbb{N}^{2m}} \sum_{a+b+|\alpha|=r} \mathcal{K}_{n,n} \left[J_a^{X|X}, \frac{\partial^\alpha f(\phi_{x_0}^X(Z))}{\partial Z^\alpha}(0) \cdot \frac{Z^\alpha}{\alpha!} \cdot J_b^{X|X} \right], \quad (3.42)$$

where $\mathcal{K}_{n,n}$ was defined in Lemma 3.1. As it was explained in [17, (3.51)], since by the result of Ma-Marinescu [25], cf. Theorem 4.7, the Bergman kernel decays exponentially away from the diagonal, the same proof holds for manifolds of bounded geometry. The result now follows from (3.28), the bound on the degrees of $J_r^{X|X}$ from Theorem 3.10 and (3.42). \square

The final result about Toeplitz operators we will need is given below.

Lemma 3.11. For non-zero $f \in \mathcal{C}_b^\infty(X, \text{End}(F))$, as $p \rightarrow \infty$, the following asymptotics holds

$$\|T_{f,p}^X\| \sim \sup_{x \in X} \|f(x)\|. \quad (3.43)$$

Proof. For compact manifolds X , this result is due to Bordemann-Meinrenken-Schlichenmaier [4, Theorem 4.1] (for (F, h^F) trivial) and Ma-Marinescu [24, Theorem 3.19, (3.91)] (for any (F, h^F)). Essentially the same proof works in our more general situation. The details are given in the end of the proof of [16, Theorem 1.1]. \square

4 Multiplicative defect and optimal extensions

The main goal of this section is to make a comparison between the dual of the restriction operator and the extension operator. For this, we prove the existence of a sequence of operators, which we call the multiplicative defect. It will play a central role in most of the proofs of this article, as it relates the Bergman kernel, studied extensively in the past, and the extension operator, which is the main object of this paper. We conserve the notations from Introduction.

More precisely, for $k \in \mathbb{N}$, we denote by $B_p^{\perp,k}$ the orthogonal projection from $L^2(X, L^p \otimes F)$ to the orthogonal complement of $H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^{k+1})$ in $H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k)$. We call the operator $B_p^{\perp,k}$ the *orthogonal Bergman kernel of order k* . Clearly, the logarithmic Bergman kernel of order k , introduced before Theorem 1.5, relates to orthogonal Bergman kernels as

$$B_p^{X,kY} = B_p^X - \sum_{l=0}^{k-1} B_p^{\perp,l}. \quad (4.1)$$

The main result of this section is as follows.

Theorem 4.1. *For any $k \in \mathbb{N}$, there is $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, there is a unique operator $A_{k,p} \in \text{End}(H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F)))$, verifying*

$$(\text{Res}_Y \circ \nabla^k B_p^{X,kY})^* = E_{k,p} \circ A_{k,p}. \quad (4.2)$$

Remark 4.2. The sequence of operators $A_{k,p}$, will be later called ‘‘multiplicative defect’’. For $k = 0$, Theorem 4.1 was proved in [17, Theorem 4.3] in its relation with the problem of transitivity of optimal holomorphic extensions.

To establish this result, we rely on several statements of independent interest. The proofs of these extend our methods from [16], [17]. The first result we need is the following semi-classical analogue of the trace theorem from the theory of Sobolev spaces, to be proved in Section 4.1.

Theorem 4.3. *For any $k \in \mathbb{N}$, $p \in \mathbb{N}$, and $f \in H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k)$ the section $(\nabla^k f)|_Y \in \mathcal{C}^\infty(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$ is holomorphic. Also, for any $k \in \mathbb{N}$, there are $p_1 \in \mathbb{N}^*$, $C > 0$, such that for any $f \in H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k)$, $p \geq p_1$, in the notations of (1.6), we have $\text{Res}_{k,p}(f) \in H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$, and the following bound is satisfied*

$$\|\text{Res}_{k,p}(f)\|_{L^2(Y)} \leq Cp^{\frac{n-m+k}{2}} \|f\|_{L^2(X)} \quad (4.3)$$

The second result is an asymptotic version of Ohsawa-Takegoshi extension theorem for holomorphic jets, which we prove in Section 4.3 by relying on the results from Section 4.2.

Theorem 4.4. *For any $k \in \mathbb{N}$, there are $C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$ and $g \in H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$, there is $f \in H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k)$, such that $\text{Res}_{k,p}(f) = g$ and the following bound holds*

$$\|f\|_{L^2(X)} \leq \frac{C}{p^{\frac{n-m+k}{2}}} \|g\|_{L^2(Y)}. \quad (4.4)$$

Remark 4.5. For $k = 0$, Theorems 4.3 and 4.4 were proved in [16, §4.1 and Theorem 4.4].

Proof of Theorem 4.1. Clearly, it suffices to prove that for any $k \in \mathbb{N}$, there is $p_1 \in \mathbb{N}$, such that the kernels and the images of the operators $(\text{Res}_Y \circ \nabla^k B_p^{X,kY})^*$ and $E_{k,p}$ coincide for $p \geq p_1$, where we view both operators acting on the space $L^2(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$.

For this, we first of all note

$$\ker(\text{Res}_Y \circ \nabla^k B_p^{X,kY})^* = (\text{Im}(\text{Res}_Y \circ \nabla^k B_p^{X,kY}))^\perp. \quad (4.5)$$

Now, in Theorem 4.3, we proved that there is $p_1 \in \mathbb{N}$, such that for any $p \geq p_1$, $\text{Res}_Y \circ \nabla^k B_p^{X,kY}$ has its image inside of $H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$. In Theorem 4.4, we proved that there is $p_1 \in \mathbb{N}$, such that for any $p \geq p_1$, the image of $\text{Res}_Y \circ \nabla^k B_p^{X,kY}$ coincides exactly with $H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$. On another hand, the extension operator is defined as the optimal holomorphic extension of the Bergman projection of a section from $L^2(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$. Hence, the kernel of the extension operator coincides with the kernel of the Bergman projection, which is the orthogonal complement of $H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$. From this, and (4.5), we see that the kernels of $(\text{Res}_Y \circ \nabla^k B_p^{X,kY})^*$ and $E_{k,p}$ coincide.

Let us now establish that the images of these operators coincide as well. Indeed, we have

$$\operatorname{Im}(\operatorname{Res}_Y \circ \nabla^k B_p^{X,kY})^* = (\ker(\operatorname{Res}_Y \circ \nabla^k B_p^{X,kY}))^\perp. \quad (4.6)$$

Recall that $B_p^{X,kY}$ vanishes on $H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k)^\perp$. Hence, $\operatorname{Res}_Y \circ \nabla^k B_p^{X,kY}$ vanishes on the vector space spanned by the kernel of $B_p^{X,kY}$ and $H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^{k+1})$. In particular, $(\ker(\operatorname{Res}_Y \circ \nabla^k B_p^{X,kY}))^\perp$ is the orthogonal complement of $H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^{k+1})$ in $H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k)$. But by Theorem 4.4, this space coincides with the image of $E_{k,p}$.

Hence, the kernels and the images of the operators $(\operatorname{Res}_Y \circ \nabla^k B_p^{X,kY})^*$ and $E_{k,p}$ coincide for $p \geq p_1$, which implies that there is a unique sequence of operators $A_{k,p}$ as in (4.2). \square

Remark 4.6. As we see from the proof, we never used the precise calculations of the constants on the right-hand side of Theorems 4.3 and 4.4. We nevertheless decided to include the precise estimates there due to the fact that our proof becomes no easier if one wishes to get simply the existence.

4.1 Bounds on jets of holomorphic sections

The main goal of this section is to prove Theorem 4.3, i.e. to give an estimate of the L^2 -norm of a jet of a holomorphic section in terms of the L^2 -norm of the section. The following result is crucial for this proof and later arguments.

Theorem 4.7 (Ma-Marinescu [25, Theorem 1]). *There are $c > 0$, $p_1 \in \mathbb{N}$, such that for any $r \in \mathbb{N}$, there is $C > 0$, such that for any $p \geq p_1$, $x_1, x_2 \in X$, we have*

$$\left| B_p^X(x_1, x_2) \right|_{\mathcal{C}^r} \leq Cp^{n+\frac{r}{2}} \cdot \exp(-c\sqrt{p} \cdot \operatorname{dist}(x_1, x_2)), \quad (4.7)$$

where the pointwise \mathcal{C}^r -norm at a point $(x_1, x_2) \in X \times X$ is defined as in (1.21).

Proof of Theorem 4.3. The main idea of our proof is to study the Schwartz kernel of the operator sending a holomorphic section to its k -jet and to use the exponential bound on the Bergman kernel.

First of all, let us verify that for any f as above, $(\nabla^k f)|_Y$ is a holomorphic section of the vector bundle $\operatorname{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F)$. As this is a local statement, let us fix a point $y_0 \in Y$ with holomorphic coordinates $t_1, \dots, t_m, z_1, \dots, z_{n-m}$, around y_0 in X in such way that z_1, \dots, z_{n-m} vanish along Y . From Weierstrass division theorem, cf. [13, Theorem II.2.3], we see that in a local holomorphic trivialization of L and F around y_0 , we may represent f in the following way

$$f = \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=k}} z^\beta \cdot g_\beta(t, z), \quad (4.8)$$

where g_β are some local holomorphic functions. Using this representation and (3.7), we see that in the induced local frame over Y , the following identity holds

$$(\nabla^k f)|_Y = k! \cdot \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=k}} (dz)^{\odot \beta} \cdot g_\beta(t, 0). \quad (4.9)$$

Since g_β are holomorphic, we see that $(\nabla^k f)|_Y$ is a holomorphic section.

Let us now verify that the restriction satisfies the stated L^2 -bound. Consider the operator $T_p : L^2(X, L^p \otimes F) \rightarrow \mathcal{C}^\infty(Y, \text{Sym}^k(\iota^* T X^{1,0})^* \otimes \iota^*(L^p \otimes F))$, defined as follows

$$T_p = \text{Res}_Y \circ ((\nabla^{1,0})^k B_p^X), \quad (4.10)$$

where $\nabla^{1,0}$ represents above the holomorphic component of the covariant derivative, induced by the Chern connection on L , F and $T X$. Let us now calculate the norm of T_p .

For this, we remark that the Schwartz kernel $T_p(y, x)$, $y \in Y$, $x \in X$, evaluated with respect to the volume form dv_X , of T_p is given by

$$T_p(y, x) = (\nabla^{1,0})^k B_p^X(y, x). \quad (4.11)$$

Consider now the operator $U_p := T_p \circ T_p^*$. Directly from (4.11) and the fact that $B_p^X \circ B_p^X = B_p^X$, we conclude that the Schwartz kernel $U_p(y, y')$, $y, y' \in Y$, of U_p , evaluated with respect to the volume form dv_Y , satisfies

$$U_p(y, y') = ((\nabla^{1,0})^k B_p^X (\nabla^{1,0;*})^k)(y, y'), \quad (4.12)$$

where $\nabla^{1,0;*}$ is the induced connection on the dual of $L^p \otimes F$ and $T^{1,0} X$. Directly from Proposition 2.7, Theorem 4.7 and (4.12), we deduce that there are $C > 0$ and $p_1 \in \mathbb{N}$, such that for $p \geq p_1$, we have $\|U_p\| \leq Cp^{n-m+k}$. We deduce the bound (4.3) by this, the trivial remark $\|U_p\| = \|T_p\|^2$ and the fact that for any $f \in H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k)$, we have $T_p(f) = \text{Res}_{k,p}(f)$. \square

4.2 Taylor expansion of the holomorphic differential near a submanifold

The main goal of this section is to recall the calculation of the first two terms of the Taylor expansion of $\bar{\partial}^{L^p \otimes F}$ -operator, considered in a shrinking neighborhood of Y of size $\frac{1}{\sqrt{p}}$, as $p \rightarrow \infty$. This section is taken almost entirely from [16, §4.1].

We consider a triple (X, Y, g^{TX}) of bounded geometry. By means of the exponential map as in (1.9), we identify a neighborhood of the zero section $B_{r_\perp}(N)$ in the normal bundle N , to a neighborhood $U := B_Y^X(r_\perp)$ of Y in X .

Recall that the projection $\pi_0 : U \rightarrow Y$ and the identifications of L, F to $\pi_0^*(L|_Y), \pi_0^*(F|_Y)$ in $B_Y^X(r_\perp)$ were defined before (1.13). We similarly identify $T X$ to $\pi_0^*(T X|_Y)$ over $B_Y^X(r_\perp)$ using the parallel transport with respect to the Levi-Civita connection ∇^{TX} . Remark that since g^{TX} is Kähler by (1.2), the decomposition $T X \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0} X \oplus T^{0,1} X$ is preserved by ∇^{TX} , cf. [22, Theorem 1.2.8]. In other words, the identification of $T X$ with $\pi_0^*(T X|_Y)$ induces the identifications

$$\tau : \pi_0^*(T^{1,0} X|_Y) \rightarrow T^{1,0} X|_U, \quad \tau : \pi_0^*(T^{0,1} X|_Y) \rightarrow T^{0,1} X|_U. \quad (4.13)$$

We define the 1-form Γ^F with values in $\text{End}(\pi_0^*(F|_Y))$

$$\Gamma^F = \nabla^F - \pi_0^*(\nabla^F|_Y), \quad (4.14)$$

where we implicitly used the above isomorphism. Similarly, we define Γ^L . Recall also that the connection ∇^N on N was introduced before (2.2). The connection ∇^N induces the splitting

$$T N = N \oplus T^H N \quad (4.15)$$

of the tangent space of the total space of N . Here $T^H N$ is the horizontal part of N with respect to the connection on N . If $U \in TY$, we denote by $U^H \in T^H N$ the horizontal lift of U in $T^H N$.

For $\epsilon > 0$, we denote by $\mathbb{E}(\epsilon)$ (resp. \mathbb{E}) the set of smooth sections of $\pi_0^*(L^p|_Y \otimes F|_Y)$ on $B_\epsilon(N)$ (resp. on the total space of N). We also denote by $\mathbb{E}^{(0,1)}(\epsilon)$ (resp. $\mathbb{E}^{(0,1)}$) the set of smooth sections of $\pi_0^*(T^{*(0,1)}X|_Y) \otimes \pi_0^*(L^p|_Y \otimes F|_Y)$ on $B_\epsilon(N)$ (resp. on the total space of N).

Clearly, the above isomorphisms allow us to see $\bar{\partial}^{L^p \otimes F}$ as an operator

$$\bar{\partial}^{L^p \otimes F} : \mathbb{E}(r_\perp) \rightarrow \mathbb{E}^{(0,1)}(r_\perp). \quad (4.16)$$

We fix a point $y_0 \in Y$ and an orthonormal frame (e_1, \dots, e_{2m}) (resp. $(e_{2m+1}, \dots, e_{2n})$) in $(T_{y_0}Y, g^{TY})$ (resp. in $(N_{y_0}, g_{y_0}^N)$) such that (1.27) is satisfied. Using the exponential coordinates on Y and the parallel transport of $(e_{2m+1}, \dots, e_{2n})$ along the geodesics on Y , as in Fermi coordinates ψ_{y_0} in (2.6), we introduce complex coordinates z_1, \dots, z_m on Y and linear ‘‘vertical’’ coordinates z_{m+1}, \dots, z_n on N . Using those coordinates, we define the operators

$$\bar{\partial}_H^{L^p \otimes F}, \mathcal{L}_N^{L^p \otimes F} : \mathbb{E} \rightarrow \mathbb{E}^{(0,1)}, \quad (4.17)$$

by prescribing their action at a point (y_0, Z_N) , $Z_N \in \mathbb{R}^{2(n-m)}$, as follows

$$\bar{\partial}_H^{L^p \otimes F} = \sum_{i=1}^m d\bar{z}_i|_{y_0} \cdot \left(\frac{\partial}{\partial \bar{z}_i} \Big|_{y_0} \right)^H, \quad \mathcal{L}_N^{L^p \otimes F} = \sum_{i=m+1}^n d\bar{z}_i|_{y_0} \cdot \left(\frac{\partial}{\partial \bar{z}_i} + \frac{\pi z_i}{2} \right). \quad (4.18)$$

The first differentiation in (4.18) is well-defined because $\pi_{0*}(\frac{\partial}{\partial \bar{z}_i}|_{y_0})^H = \frac{\partial}{\partial \bar{z}_i}|_{y_0}$ is of type $(0, 1)$, and the second derivation is well-defined because the vector bundles are trivialized along fibers of π_0 .

Below a variable $t \in \mathbb{R}$ is related to $p \in \mathbb{N}$ by

$$t = \frac{1}{\sqrt{p}}. \quad (4.19)$$

For any $\epsilon > 0$ define the rescaling operator $F_t : \mathbb{E}(\epsilon) \rightarrow \mathbb{E}(\frac{\epsilon}{t})$ for $f \in \mathbb{E}(\epsilon)$ as follows

$$(F_t f)(y, Z_N) := f(y, tZ_N), \quad (y, Z_N) \in B_{\frac{\epsilon}{t}}(N). \quad (4.20)$$

The operator $F_t : \mathbb{E}^{(0,1)}(\epsilon) \rightarrow \mathbb{E}^{(0,1)}(\frac{\epsilon}{t})$ is defined in an analogous way.

Theorem 4.8 ([16, Theorem 4.3]). *As $p \rightarrow \infty$, we have*

$$F_t \circ \bar{\partial}^{L^p \otimes F} \circ F_t^{-1} = \frac{1}{t} \mathcal{L}_N^{L^p \otimes F} + \bar{\partial}_H^{L^p \otimes F} + O(t|Z_N|^2 \nabla_N + t|Z_N| |\nabla_H + t|Z_N|), \quad (4.21)$$

where $O(t|Z_N|^2 \nabla_N + t|Z_N| |\nabla_H + t|Z_N|)$ is an operator of the form $\sum_{i=1}^{2m} a_i(t, y, Z_N) \cdot dx_i|_{y_0} \cdot (\frac{\partial}{\partial x_i}|_{y_0})^H + \sum_{j=2m+1}^{2n} b_j(t, y, Z_N) \cdot dx_j|_{y_0} \cdot \frac{\partial}{\partial x_j} + c(t, y, Z_N)$, such that there is a constant $C > 0$, for which $|a_i(t, y, Z_N)| \leq Ct|Z_N|^2$, $|b_j(t, y, Z_N)| \leq Ct|Z_N|$, $|c(t, y, Z_N)| \leq Ct|Z_N|$ holds for any $y \in Y$, $|Z_N| < r_\perp$, $i = 1, \dots, m$, and $j = m+1, \dots, n$.

Remark 4.9. Bismut-Lebeau in [2, Theorem 8.18] established an analogue of Theorem 4.8, which corresponds to trivial (L, h^L) in our setting. In this case the operator $\mathcal{L}_N^{L^p \otimes F}$ from (4.18) doesn't have an additional $\frac{\pi z_i}{2}$ term. Another closely related Taylor expansion is due to Dai-Liu-Ma [10, Theorem 4.6], and it corresponds to Y equal to a point in our setting.

4.3 Asymptotic Ohsawa-Takegoshi extension theorem for holomorphic jets

The main goal of this section is to establish the asymptotic analogue of Ohsawa-Takegoshi extension theorem for holomorphic jets. In other words, we show that Theorem 4.4 holds.

The main idea of the proof of Theorem 4.4, which follows rather closely our proof from [16, Theorem 4.4], corresponding to the case $k = 0$, is to pass through the general framework of the proof of Ohsawa-Takegoshi extension theorem. We choose a smooth extension of g over X , and then obtain the holomorphic extension by modifying the smooth one using a solution of $\bar{\partial}$ -equation with singular weight, which forces the solution to annihilate along Y .

The novelty here as well as in [16] is that instead of choosing an arbitrary smooth extension, we choose a specific one, given by the operator (1.13). This allows us to significantly simplify the original proof of Ohsawa-Takegoshi extension theorem (in our asymptotic setting). In particular, instead of considering a double sequence of singular weights (depending on p and on additional parameter ϵ), which would dampen the fact that we do not know much of an arbitrary smooth extension, it would suffice to consider a single sequence of weights (only depending on p).

Recall that the function ρ was defined in (1.11). Let us now consider the functions $\delta_Y : X \setminus Y \rightarrow \mathbb{R}$, $\alpha_Y : X \rightarrow \mathbb{R}$, defined as

$$\begin{aligned}\delta_Y(x) &:= \log(\text{dist}_X(x, Y)) \cdot \rho\left(\frac{\text{dist}_X(x, Y)}{r_\perp}\right), \\ \alpha_Y(x) &:= \text{dist}_X(x, Y)^2 \cdot \rho\left(\frac{\text{dist}_X(x, Y)}{r_\perp}\right) + \left(1 - \rho\left(\frac{\text{dist}_X(x, Y)}{r_\perp}\right)\right).\end{aligned}\tag{4.22}$$

As $\text{dist}_X(x, Y)^2$ is smooth on the r_\perp -geodesic tubular neighborhood of Y , δ_Y and α_Y are smooth. We assume in what follows for simplicity $r_\perp \leq 1$. Then $\delta_Y \leq 0$.

Now, recall that a function $f : X \rightarrow [-\infty, +\infty[$ on a complex Hermitian manifold (X, ω) is called quasi-plurisubharmonic if it is upper-semicontinuous, and there is a constant $C \in \mathbb{R}$, such that the following inequality holds in the distributional sense

$$\sqrt{-1}\partial\bar{\partial}f \geq -C\omega.\tag{4.23}$$

We denote by $\text{PSH}(X, C\omega)$ the set of quasi-plurisubharmonic functions f , verifying (4.23).

Theorem 4.10 ([16, Theorem 2.31]). *There is $C > 0$, such that $\delta_Y, \alpha_Y, -\alpha_Y \in \text{PSH}(X, C\omega)$.*

Another result we will use concerns the L^2 -bounds of derivatives of holomorphic sections.

Proposition 4.11 ([16, Proposition 4.5]). *For any $k \in \mathbb{N}$, there are $C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$ and $f \in H_{(2)}^0(X, L^p \otimes F)$, we have*

$$\|\nabla^k f\|_{L^2(X)} \leq Cp^{\frac{k}{2}} \|f\|_{L^2(X)},\tag{4.24}$$

where ∇ is the covariant derivative with respect to the induced Chern and Levi-Civita connections.

Proof of Theorem 4.4. Recall that the operator $E_{k,p}^0$ was defined in (1.13). We would like to verify that for any $g \in H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$, the form $\alpha := \bar{\partial}^{L^p \otimes F}(E_{k,p}^0 g)$ vanishes at least up to order $k + 1$ over Y .

Indeed, let us work in a neighborhood $V := B_Y^X(\frac{r_\perp}{4})$ of Y in X . Recall that $t \in \mathbb{R}_+$ and F_t were defined in (4.20). Then in the notations of (1.13), on V , we have

$$E_{k,p}^0 g = \frac{F_t^{-1} \tilde{g}}{\sqrt{p^k}}, \quad \tilde{g}(y, Z_N) = z_N^{\otimes k} \cdot g(y) \cdot \exp\left(-\frac{\pi}{2}|Z_N|^2\right). \quad (4.25)$$

Recall that $\bar{\partial}_H^{L^p \otimes F}$, $\mathcal{L}_N^{L^p \otimes F}$ were defined in (4.18). A trivial calculation shows that on V , we have

$$\mathcal{L}_N^{L^p \otimes F} \tilde{g} = 0. \quad (4.26)$$

Also, since ∇^N preserves g^N , in the notations of (4.18), similarly to [2, (8.97)], we have

$$\left(\left(\frac{\partial}{\partial \bar{z}_i}\Big|_{y_0}\right)^H \tilde{g}\right)(y_0, Z_N) = \left(\frac{\partial}{\partial \bar{z}_i}(z_N^{\otimes k} \cdot g)\right)(y_0) \exp\left(-\frac{\pi}{2}|Z_N|^2\right). \quad (4.27)$$

As a consequence of (4.27) and the fact that g is holomorphic, we obtain

$$\bar{\partial}_H^{L^p \otimes F} \tilde{g} = 0. \quad (4.28)$$

From (4.25), (4.26), (4.28) and the fact that all the residue terms in Theorem 4.8 contain $|Z_N|$, we deduce that α vanishes at least up to order $k+1$ along Y .

Now, using the L^2 -estimates, let us construct a holomorphic perturbation of $E_{k,p}^0 g$, satisfying the assumptions of Theorem 4.4. Recall that $\delta_Y : X \setminus Y \rightarrow \mathbb{R}$, $\alpha_Y : X \rightarrow \mathbb{R}$, were defined in (4.22). For $\epsilon > 0$, let us now define the weight $\delta_p : X \setminus Y \rightarrow \mathbb{R}$ as follows

$$\delta_p := 2(n - m + k)\delta_Y - \epsilon p \alpha_Y. \quad (4.29)$$

By taking ϵ small, from Theorem 4.10, we see that there exists $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, over X , the following inequality holds in the distributional sense

$$p\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \delta_p > \frac{p}{2} \omega. \quad (4.30)$$

Let us fix ϵ small enough, so that it verifies the above inequality and for any $|Z_N| < r_\perp$, we have

$$\frac{\pi}{2}|Z_N|^2 - \epsilon \alpha_Y(y, Z_N) \geq \frac{\pi}{4}|Z_N|^2. \quad (4.31)$$

We will now prove that there are $C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, $g \in H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$ and $\alpha := \bar{\partial}^{L^p \otimes F}(E_{k,p}^0 g)$, we have

$$\int_{X \setminus Y} |\alpha|^2 e^{-\delta_p} dv_X \leq C \|g\|_{L^2(Y)}^2. \quad (4.32)$$

Remark first that α has support in $B_Y^X(\frac{r_\perp}{2})$, and so we can work in (y, Z_N) , $y \in Y$, $Z_N \in N_y$ coordinates. Over $B_Y^X(\frac{r_\perp}{2}) \setminus B_Y^X(\frac{r_\perp}{4})$, we then have $-p\frac{\pi}{2}|Z_N|^2 - \delta_p \leq -p\frac{\pi}{4}|Z_N|^2 \leq -p\frac{\pi}{64}r_\perp^2$. By this, the coordinate-wise description of bounded geometry from Section 2.1, we see that there are $c, C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, we have

$$\int_{X \setminus B_Y^X(\frac{r_\perp}{4})} |\alpha|^2 e^{-\delta_p} dv_X \leq C \exp(-cp) \left(\|g\|_{L^2(Y)}^2 + \|\nabla g\|_{L^2(Y)}^2 \right). \quad (4.33)$$

Now, as α has support in $B_Y^X(\frac{r_\perp}{2})$, it is enough to work in (y, Z_N) , $y \in Y$, $Z_N \in N_y$ coordinates. To estimate the integral over $B_Y^X(\frac{r_\perp}{4})$, we use (4.25) and make the change of variables by F_t to get

$$\begin{aligned} & \int_{B_Y^X(\frac{r_\perp}{4})} |\alpha|^2 e^{-\delta_p} dv_Y \wedge dv_N \\ &= \int_{B_{\frac{r_\perp}{4t}}(N)} \left| (F_t \circ \bar{\partial}^{L^p \otimes F} \circ F_t^{-1}) \tilde{g} \right|^2 (y, Z_N) \frac{e^{\epsilon p \alpha_Y(y, t Z_N)}}{|Z_N|^{2(n-m+k)}} dv_Y \wedge dv_N. \end{aligned} \quad (4.34)$$

We see that to estimate the right-hand side of (4.34), we may apply Theorem 4.8. From (4.26), (4.28), we see that the first two terms of the asymptotic expansion of $(F_t \circ \bar{\partial}^{L^p \otimes F} \circ F_t^{-1}) \tilde{g}$ vanish. In particular, we conclude that $(F_t \circ \bar{\partial}^{L^p \otimes F} \circ F_t^{-1}) \tilde{g}$ vanishes up to order $k+1$ along Y . From this and the trivial fact that there is $C > 0$, such that for $j = 1, 2$, we have

$$\int_{\mathbb{R}^{2(n-m)}} \frac{|Z_N|^{2k+j} \exp(-\frac{\pi}{4}|Z_N|^2) dv_{\mathbb{R}^{2(n-m)}}(Z_N)}{|Z_N|^{2(n-m+k)}} < C, \quad (4.35)$$

we conclude that there are $C > 0$, $p_1 \in \mathbb{N}$, such that for any $p \geq p_1$, $g \in H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$ and $\alpha := \bar{\partial}^{L^p \otimes F}(E_{k,p}^0 g)$, we have

$$\int_{B_Y^X(\frac{r_\perp}{4})} |\alpha|^2 e^{-\delta_p} dv_X \leq \frac{C}{p} \left(\|g\|_{L^2(Y)}^2 + \|\nabla g\|_{L^2(Y)}^2 \right). \quad (4.36)$$

From Proposition 4.11, (4.33) and (4.36), we deduce (4.32).

By [11, Theorem 1.5], $X \setminus Y$ is a complete Kähler manifold. Hence, by (4.30), we may resolve the $\bar{\partial}$ -equation on $X \setminus Y$, see [12, Proposition 13.4]. From this, the trivial fact that $\bar{\partial}^{L^p \otimes F} \alpha = 0$ and (4.32), we see that there are $C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, $g \in H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$, there is a section $f_0 \in \mathcal{C}^\infty(X \setminus Y, L^p \otimes F)$, such that

$$\bar{\partial}^{L^p \otimes F} f_0 = \alpha, \quad \int_{X \setminus Y} |f_0|^2 e^{-\delta_p} dv_X \leq \frac{C}{p} \|g\|_{L^2(Y)}^2. \quad (4.37)$$

Let us prove that $f := E_{k,p}^0 g - f_0$ verifies the assumptions of Theorem 4.4.

From (4.37), we see that over $X \setminus Y$, $\bar{\partial}^{L^p \otimes F} f = 0$. Also, by (4.37), we easily get that $f \in L^2(X, L^p \otimes F)$. By the standard regularity result, [11, Lemme 6.9], f extends smoothly and the equation $\bar{\partial}^{L^p \otimes F} f = 0$ holds on X . In particular, f_0 extends smoothly as well. However, since $\exp(-2(n-m+k)\delta_Y)$ is not integrable, the L^2 -bound (4.37) implies that f_0 has to vanish at least up to order $k+1$ along Y . Hence, we conclude that $(\nabla^k f)|_Y = g$. It is only left to verify that f satisfies the needed L^2 -bound (4.4).

From the Gaussian integral calculation, using the coordinate-wise description of bounded geometry condition from Section 2.1 and (1.5), cf. [16, (4.38)], we see that there are $c, C > 0$, such that we have

$$\frac{c}{p^{\frac{n-m+k}{2}}} \|g\|_{L^2(Y)} \leq \|E_{k,p}^0 g\|_{L^2(X)} \leq \frac{C}{p^{\frac{n-m+k}{2}}} \|g\|_{L^2(Y)}. \quad (4.38)$$

Let us now prove the following bound

$$\int_X |f_0|^2 e^{-\delta_p} dv_X \geq Cp^{n-m+k} \int_X |f_0|^2 dv_X. \quad (4.39)$$

This would finish the proof, as from the L^2 -bound in (4.37), (4.38) and (4.39), we would then deduce the L^2 -bound (4.4).

First of all, since $\alpha_Y \geq \min\{\frac{1}{2}(\frac{r_\perp}{4})^2, \frac{1}{2}\}$ on $X \setminus B_Y^X(\frac{r_\perp}{4})$, there are $c, C > 0$, such that

$$\int_{X \setminus B_Y^X(\frac{r_\perp}{4})} |f_0|^2 e^{-\delta_p} dv_X \geq C \exp(\epsilon cp) \int_{X \setminus B_Y^X(\frac{r_\perp}{4})} |f_0|^2 dv_X \quad (4.40)$$

It is now only left to give the lower bound for the integrand on the left-hand side of (4.39), where the integration is done over $B_Y^X(\frac{r_\perp}{4})$. But remark that from (4.22) and (4.29), over $B_Y^X(\frac{r_\perp}{4})$, there is $C > 0$, such that for any $p \in \mathbb{N}^*$, we have $e^{-\delta_p} \geq Cp^{n-m+k}$. To see this, for any $i \in \mathbb{N}$, we have $\delta_Y(x) \leq \log(\frac{i+1}{\sqrt{p}})$ and $\alpha_Y(x) \geq \frac{i^2}{p}$ for any $x \in B_Y^X(\frac{i+1}{\sqrt{p}}) \setminus B_Y^X(\frac{i}{\sqrt{p}})$, $i \leq \frac{r_\perp \sqrt{p}}{4}$. So we obtain that over $B_Y^X(\frac{i+1}{\sqrt{p}}) \setminus B_Y^X(\frac{i}{\sqrt{p}})$, we have $\delta_p \leq -(n-m+k) \log p + 2(n-m+k) \log(i+1) - \epsilon i^2$. From this, and the fact that logarithm grows slower at infinity than the square function, we deduce

$$\int_{B_Y^X(\frac{r_\perp}{4})} |f_0|^2 e^{-\delta_p} dv_X \geq Cp^{n-m+k} \int_{B_Y^X(\frac{r_\perp}{4})} |f_0|^2 dv_X. \quad (4.41)$$

From (4.40) and (4.41), we obtain (4.39). \square

Remark 4.12. Our proof shows that there is $E_{k,p}^1 : H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F)) \rightarrow H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k)$, verifying $(\nabla^k E_{k,p}^1 g)|_Y = g$ for $g \in H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$, and such that (1.14) holds for $E_{k,p}^1$ instead of $E_{k,p}^0$.

5 Asymptotics of the extension operator for holomorphic jets

The main goal of this section is to study the asymptotic expansion of the extension operator for holomorphic jets. For this, in Section 5.1, we prove the exponential bounds for the Schwartz kernels of the extension operator and orthogonal Bergman kernel of order k , and study their asymptotics. We also deduce from these statements Theorems 1.1 and 1.5. Then, in Section 5.2, we establish the announced exponential bounds and asymptotics, and deduce Theorem 1.3 from our methods. Finally, in Section 5.3, we apply the results from Section 5.1 to the case when the submanifold corresponds to a fixed point and deduce the asymptotics of higher order peak sections.

5.1 Schwartz kernels of the extension and orthogonal Bergman projectors

The main goal of this section is to study the Schwartz kernels of the extension operator and orthogonal Bergman kernels of order $k \in \mathbb{N}$. In particular, we prove the exponential estimates for the Schwartz kernels of these operators, and show that these Schwartz kernels admit a full asymptotic expansion, as powers of the line bundle tend to infinity.

We use notations from Section 1 and assume that the triple (X, Y, g^{TX}) is of bounded geometry. Our first main result goes as follows.

Theorem 5.1. *For any $k \in \mathbb{N}$, there are $c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $r \in \mathbb{N}$, there is $C > 0$, such that for any $p \geq p_1$, $x_1, x_2 \in X$, $y \in Y$, the following estimates hold*

$$\begin{aligned} a) & |E_{k,p}(x_1, y)|_{\mathcal{C}^r} \leq Cp^{m+\frac{r-k}{2}} \exp(-c\sqrt{p} \cdot \text{dist}(x_1, y)), \\ b) & |B_{k,p}^\perp(x_1, x_2)|_{\mathcal{C}^r} \leq Cp^{n+\frac{r}{2}} \exp(-c\sqrt{p} \cdot (\text{dist}(x_1, x_2) + \text{dist}(x_1, Y) + \text{dist}(x_2, Y))). \end{aligned} \quad (5.1)$$

Remark 5.2. For $k = 0$, the corresponding result was proved in [16, Theorems 1.5, 1.8]. Our proof here is different even in the case $k = 0$.

Theorem 5.1 implies that to understand fully the asymptotics of the Schwartz kernel of the extension operator and orthogonal Bergman kernels of order k , it suffices to do so in a neighborhood of a fixed point $(y_0, y_0) \in Y \times Y$ in $X \times Y$ and $X \times X$. Our next result shows that after a reparametrization by a homothety with factor \sqrt{p} in Fermi coordinates around (y_0, y_0) , the kernel of these operators admit a complete asymptotic expansion in integer powers of \sqrt{p} , as $p \rightarrow \infty$.

Theorem 5.3. *For any $k, r \in \mathbb{N}$, $y_0 \in Y$, there are polynomials $J_{k,r}^E(Z, Z'_Y) \in \text{Sym}^k(N_{y_0}^{1,0}) \otimes \text{End}(F_{y_0})$ in $Z \in \mathbb{R}^{2n}$, $Z'_Y \in \mathbb{R}^{2m}$, of parity $k + r$, $\deg J_{k,r}^E \leq k^2 + k^2r + 3r$, whose coefficients have the same properties as the coefficients of the polynomials from Theorem 3.10, and which vanish at least up to order k along $\mathbb{R}^{2m} \times \mathbb{R}^{2m} \subset \mathbb{R}^{2n} \times \mathbb{R}^{2m}$, such that for $F_{k,r}^E := J_{k,r}^E \cdot \mathcal{E}_{n,m}^0$, the following holds.*

For any $k \in \mathbb{N}$, there are $\epsilon, c > 0$, $p_1 \in \mathbb{N}^$, such that for any $r, l, l' \in \mathbb{N}$, there are $C, Q > 0$, such that for any $y_0 \in Y$, $p \geq p_1$, $Z = (Z_Y, Z_N)$, $Z_Y, Z'_Y \in \mathbb{R}^{2m}$, $Z_N \in \mathbb{R}^{2(n-m)}$, $|Z|, |Z'_Y| \leq \epsilon$, $\alpha \in \mathbb{N}^{2n}$, $\alpha' \in \mathbb{N}^{2m}$, $|\alpha| + |\alpha'| \leq l$, we have*

$$\begin{aligned} & \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(\frac{1}{p^{m-\frac{k}{2}}} \text{E}_{k,p}(\psi_{y_0}^{X|Y}(Z), \phi_{y_0}^Y(Z'_Y)) \right. \right. \\ & \quad \left. \left. - \sum_{j=0}^r p^{-\frac{j}{2}} F_{k,j}^E(\sqrt{p}Z, \sqrt{p}Z'_Y) \kappa_\psi^{X|Y}(Z)^{-\frac{1}{2}} \kappa_\phi^Y(Z'_Y)^{-\frac{1}{2}} \right) \right|_{\mathcal{C}^{l'}} \\ & \leq Cp^{-\frac{r+1-l}{2}} \cdot \left(1 + \sqrt{p}|Z| + \sqrt{p}|Z'_Y| \right)^Q \exp\left(-c\sqrt{p}(|Z_Y - Z'_Y| + |Z_N|)\right), \end{aligned} \quad (5.2)$$

where the $\mathcal{C}^{l'}$ -norm is taken with respect to y_0 . Also, the following identity holds

$$J_{k,0}^E(Z, Z'_Y) = \text{Id}_{F_{y_0}} \cdot \kappa_N^{\frac{1}{2}}(y_0) \cdot \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=k}} \frac{1}{\beta!} \cdot z_N^\beta \cdot \left(\frac{\partial}{\partial z'_N} \right)^{\odot \beta}. \quad (5.3)$$

Moreover, under the assumption (1.10), we have

$$J_{k,1}^E(Z, Z'_Y) = \text{Id}_{F_{y_0}} \cdot g(z_N, A(\bar{z}_Y - \bar{z}'_Y)(\bar{z}_Y - \bar{z}'_Y)) \cdot \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=k}} \frac{1}{\beta!} \cdot z_N^\beta \cdot \left(\frac{\partial}{\partial z'_N} \right)^{\odot \beta}, \quad (5.4)$$

where A is the second fundamental form of ι , introduced in (2.2).

Remark 5.4. In particular, from (3.8), we have $F_{k,0}^E(Z, Z'_Y) = \mathcal{E}_{n,m}^k$ and under the assumption (1.10), we have

$$F_{k,1}^E(Z, Z'_Y) = \text{Id}_{F_{y_0}} \cdot g(z_N, A(\bar{z}_Y - \bar{z}'_Y)(\bar{z}_Y - \bar{z}'_Y)) \cdot \mathcal{E}_{n,m}^k. \quad (5.5)$$

Theorem 5.5. *For any $k, r \in \mathbb{N}$, $y_0 \in Y$, there are polynomials $J_{k,r}^\perp(Z, Z') \in \text{End}(F_{y_0})$, $Z, Z' \in \mathbb{R}^{2n}$, of parity k , $\deg J_{k,r}^\perp \leq 2k^2 + k^2r + 3r$, whose coefficients have the same properties as the coefficients of the polynomials from Theorem 3.10, and which vanish at least up to order k along*

$\mathbb{R}^{2m} \times \mathbb{R}^{2n} \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ and $\mathbb{R}^{2n} \times \mathbb{R}^{2m} \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, such that for $F_{k,r}^\perp := J_{k,r}^\perp \cdot \mathcal{P}_{n,m}^{\perp,0}$, the following holds.

For any $k \in \mathbb{N}$, there are $\epsilon, c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $r, l, l' \in \mathbb{N}$, there are $C, Q > 0$, such that for any $y_0 \in Y$, $p \geq p_1$, $Z = (Z_Y, Z_N)$, $Z' = (Z'_Y, Z'_N)$, $Z_Y, Z'_Y \in \mathbb{R}^{2m}$, $Z_N, Z'_N \in \mathbb{R}^{2(n-m)}$, $|Z|, |Z'| \leq \epsilon$, $\alpha, \alpha' \in \mathbb{N}^{2n}$, $|\alpha| + |\alpha'| \leq l$, we have

$$\begin{aligned} & \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(\frac{1}{p^n} B_{k,p}^\perp(\psi_{y_0}(Z), \psi_{y_0}(Z')) \right. \right. \\ & \quad \left. \left. - \sum_{j=0}^r p^{-\frac{j}{2}} F_{k,j}^\perp(\sqrt{p}Z, \sqrt{p}Z') \kappa_\psi^{X|Y}(Z)^{-\frac{1}{2}} \kappa_\psi^{X|Y}(Z')^{-\frac{1}{2}} \right) \right|_{\mathcal{C}^{l'}} \\ & \leq Cp^{-\frac{r+1-l}{2}} \cdot \left(1 + \sqrt{p}|Z| + \sqrt{p}|Z'| \right)^Q \exp \left(-c\sqrt{p}(|Z_Y - Z'_Y| + |Z_N| + |Z'_N|) \right), \end{aligned} \quad (5.6)$$

where the $\mathcal{C}^{l'}$ -norm is taken with respect to y_0 . Also, we have

$$J_{k,0}^\perp(Z, Z') = \pi^k \cdot \text{Id}_{F_{y_0}} \cdot \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=k}} \frac{z_N^\beta \cdot (\bar{z}'_N)^\beta}{\beta!}. \quad (5.7)$$

Moreover, under the assumption (1.10), we have

$$\begin{aligned} J_{k,1}^\perp(Z, Z') &= \pi^{k+1} \cdot \text{Id}_{F_{y_0}} \cdot \left(g(z_N, A(\bar{z}_Y - \bar{z}'_Y)(\bar{z}_Y - \bar{z}'_Y)) \right. \\ & \quad \left. + g(\bar{z}'_N, A(z_Y - z'_Y)(z_Y - z'_Y)) \right) \cdot \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=k}} \frac{z_N^\beta \cdot (\bar{z}'_N)^\beta}{\beta!}. \end{aligned} \quad (5.8)$$

Remark 5.6. a) In particular, from (3.4), we have $F_{k,0}^\perp(Z, Z') = \mathcal{P}_{n,m}^{\perp,k}$ and under the assumption (1.10), we have

$$\begin{aligned} F_{k,1}^\perp(Z, Z') &= \pi \cdot \text{Id}_{F_{y_0}} \cdot \left(g(z_N, A(\bar{z}_Y - \bar{z}'_Y)(\bar{z}_Y - \bar{z}'_Y)) \right. \\ & \quad \left. + g(\bar{z}'_N, A(z_Y - z'_Y)(z_Y - z'_Y)) \right) \cdot \mathcal{P}_{n,m}^{\perp,k}. \end{aligned} \quad (5.9)$$

b) Theorems 5.3 and 5.5 were established for $k = 0$ in [16, Theorems 1.6, 1.8]. Our proof here is different from the one from [16], even for $k = 0$.

c) Our methods allow to give a precise estimate for Q from Theorems 5.3 and 5.5, but as this is quite lengthy and cumbersome, we leave their derivation to the interested reader.

Proof of Theorem 1.1 assuming Theorems 5.1.a) and 5.3. The main idea of the proof is to compare the Schwartz kernels of $E_{k,p}$ and $E_{k,p}^0$. For this, we introduce $K_p := E_{k,p} - E_{k,p}^0$. From (1.13), remark that the Schwartz kernel, $E_{k,p}^0(x, y)$, $x = (y', Z_N) \in B_Y^X(r_\perp)$, $y \in Y$, of $E_{k,p}^0$, evaluated with respect to dv_Y , equals to

$$E_{k,p}^0(x, y) = \rho \left(\frac{|Z_N|}{r_\perp} \right) \cdot \exp \left(-p \frac{\pi}{2} |Z_N|^2 \right) \cdot Z_N^{\otimes k} \cdot B_{k,p}^Y(y', y). \quad (5.10)$$

From this, the fact that for any $k \in \mathbb{N}$, there is $C > 0$, such that for any $Z \in \mathbb{R}$, we have $\exp(-p|Z|^2)|Z|^k \leq Cp^{-\frac{k}{2}}$, and Theorem 4.7, we conclude that there are $c, C > 0, p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1, x \in X, y \in Y$, the following estimate holds

$$\left| E_{k,p}^0(x, y) \right| \leq Cp^{m-\frac{k}{2}} \exp(-c\sqrt{p}\text{dist}(x, y)). \quad (5.11)$$

This along with Theorem 5.1.a) implies that there are $c, C > 0, p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1, x \in X, y \in Y$, the following estimate holds

$$\left| K_p(x, y) \right| \leq Cp^{m-\frac{k}{2}} \exp(-c\sqrt{p}\text{dist}(x, y)). \quad (5.12)$$

Now, by studying the asymptotic expansion of the Schwartz kernel $K_p(x, y)$, we improve the estimate (5.12) by lowering the degree of p in the polynomial part of the right-hand side of (5.12).

We denote by $J_{k,r}^{Y|Y}$ the polynomials from Theorem 3.10 associated to $X = Y$ and $F = \text{Sym}^k(N^{1,0})^* \otimes \iota^* F$. From Theorem 3.10, the second equation from (3.9) and (5.10), we deduce that for polynomials $J_{k,r}^{E0}(Z, Z'_Y)$, defined for $r \in \mathbb{N}$ as follows

$$J_{k,r}^{E0}(Z, Z'_Y) := \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=k}} \frac{1}{\beta!} \cdot z_N^\beta \cdot \left(\frac{\partial}{\partial z_N} \right)^{\odot \beta} \cdot J_{k,r}^{Y|Y}(Z_Y, Z'_Y), \quad (5.13)$$

and the functions $F_{k,r}^{E0} := J_{k,r}^{E0} \cdot \mathcal{O}_{n,m}^0$ over $\mathbb{R}^{2n} \times \mathbb{R}^{2m}$, the following holds. There are $\epsilon, c > 0, p_1 \in \mathbb{N}^*$, such that for any $r \in \mathbb{N}$, there are $C, Q > 0$, such that for any $y_0 \in Y, p \geq p_1, Z = (Z_Y, Z_N), Z_Y, Z'_Y \in \mathbb{R}^{2m}, Z_N \in \mathbb{R}^{2(n-m)}, |Z|, |Z'_Y| \leq \epsilon$, we have

$$\left| \frac{1}{p^{m-\frac{k}{2}}} E_{k,p}^0(\psi_{y_0}(Z), \phi_{y_0}^Y(Z'_Y)) - \sum_{j=0}^r p^{-\frac{j}{2}} F_{j,K}^{E0}(\sqrt{p}Z, \sqrt{p}Z'_Y) \kappa_\phi^Y(Z_Y)^{-\frac{1}{2}} \kappa_\phi^Y(Z'_Y)^{-\frac{1}{2}} \right| \leq Cp^{-\frac{r+1}{2}} \left(1 + \sqrt{p}|Z| + \sqrt{p}|Z'_Y| \right)^Q \exp(-c\sqrt{p}(|Z_Y - Z'_Y| + |Z_N|)). \quad (5.14)$$

For $r \in \mathbb{N}$, let us expand $\kappa_N(\psi(Z))^{\frac{1}{2}}$ in a neighborhood of $Z = 0$ as follows

$$\kappa_N(\psi(Z))^{\frac{1}{2}} = \sum_{i=0}^r \kappa_{N,[i]}^{\frac{1}{2}}(Z) + O(|Z|^{r+1}), \quad (5.15)$$

where $\kappa_{N,[i]}^{\frac{1}{2}}(Z)$ are homogeneous polynomials of degree i .

From Theorem 5.3, (3.38) and (5.14), we deduce that for polynomials $J_{k,r}^{EK}(Z, Z'_Y)$, defined for $r \in \mathbb{N}$ as follows

$$J_{k,r}^{EK}(Z, Z'_Y) := J_{k,r}^E(Z, Z'_Y) - \sum_{a+b=r} \kappa_{N,[a]}^{\frac{1}{2}}(Z) \cdot J_{k,b}^{E0}, \quad (5.16)$$

and the functions $F_{k,r}^{EK} := J_{k,r}^{EK} \cdot \mathcal{O}_{n,m}^0$ over $\mathbb{R}^{2n} \times \mathbb{R}^{2m}$, the following holds. There are $\epsilon, c > 0, p_1 \in \mathbb{N}^*$, such that for any $r \in \mathbb{N}$, there are $C, Q > 0$, such that for any $y_0 \in Y, p \geq p_1, Z = (Z_Y, Z_N), Z_Y, Z'_Y \in \mathbb{R}^{2m}, Z_N \in \mathbb{R}^{2(n-m)}, |Z|, |Z'_Y| \leq \epsilon$, we have

$$\left| \frac{1}{p^{m-\frac{k}{2}}} K_p(\psi_{y_0}(Z), \phi_{y_0}^Y(Z'_Y)) - \sum_{j=0}^r p^{-\frac{j}{2}} F_{k,j}^{EK}(\sqrt{p}Z, \sqrt{p}Z'_Y) \kappa_{\psi}^{X|Y}(Z)^{-\frac{1}{2}} \kappa_{\phi}^Y(Z'_Y)^{-\frac{1}{2}} \right| \leq Cp^{-\frac{r+1}{2}} \left(1 + \sqrt{p}|Z| + \sqrt{p}|Z'_Y|\right)^Q \exp\left(-c\sqrt{p}(|Z_Y - Z'_Y| + |Z_N|)\right). \quad (5.17)$$

From (3.40), (5.3), (5.13) and (5.16), we deduce

$$J_{k,0}^{EK}(Z, Z'_Y) = 0. \quad (5.18)$$

In particular, from (5.17) and (5.18), we see that we can improve (5.12) as follows

$$\left| K_p(x, y) \right| \leq Cp^{m-\frac{1+k}{2}} \exp\left(-c\sqrt{p}\text{dist}(x, y)\right). \quad (5.19)$$

From Proposition 2.7 and (5.19), we deduce that there are $C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, we have $\|K_p\| \leq \frac{C}{p^{\frac{n-m+k+1}{2}}}$, which is exactly (1.14) by the definition of K_p .

Now, similarly to the derivation of (5.18), we see that under assumption (1.10), from (3.41), (5.16) and the fact from [16, (5.35)], stating

$$\frac{\partial}{\partial Z_N} \kappa_N = 0, \quad (5.20)$$

we deduce that we have

$$J_{k,1}^{EK}(Z, Z'_Y) = J_{k,1}^E(Z, Z'_Y). \quad (5.21)$$

Let us now prove (1.15). Recall that for any $\beta \in \mathbb{N}^{n-m}$, $|\beta| = k$, we have

$$\int_{\mathbb{R}^{2(n-m)}} |z_N^{2\beta}| \cdot \exp(-\pi|Z_N|^2) \cdot dZ_{2m+1} \wedge \cdots \wedge dZ_{2n} = \frac{\beta!}{\pi^k}. \quad (5.22)$$

Remark also from (1.24) and the description below (1.25) that

$$\|dz_N^{\odot\beta}\|^2 = 2^k \cdot \frac{\beta!}{k!}. \quad (5.23)$$

In particular, we have

$$\begin{aligned} \int_{\mathbb{R}^{2(n-m)}} \kappa_N(y, \sqrt{p}Z_N) \cdot \frac{|z_N^{2\beta}|}{k!^2} \cdot \exp(-p\pi|Z_N|^2) \rho\left(\frac{|Z_N|}{r_{\perp}}\right)^2 dZ_{2m+1} \wedge \cdots \wedge dZ_{2n} \\ = \frac{\kappa_N(y)}{p^{n-m+k}} \cdot \frac{1}{k! \cdot (2\pi)^k} \cdot \|dz_N^{\odot\beta}\|^2 + O\left(\frac{1}{p^{n-m+k+\frac{1}{2}}}\right). \end{aligned} \quad (5.24)$$

From (1.9) and (5.24), cf. [16, (4.38)] for the case $k = 0$, we see that for any $g \in L^2(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$, we have

$$\|E_{k,p}^0 g\|_{L^2(dv_X)}^2 = \frac{1}{p^{n-m+k}} \cdot \frac{1}{k! \cdot (2\pi)^k} \cdot \left\| \kappa_N(y)^{\frac{1}{2}} \cdot B_{k,p}^Y g \right\|_{L^2(Y)}^2 + O\left(\frac{\|g\|_{L^2(Y)}^2}{p^{n-m+k+\frac{1}{2}}}\right). \quad (5.25)$$

Consider the Toeplitz operator $T_{\kappa_N, p}^Y \in \text{End}(L^2(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F)))$, given by

$$T_{\kappa_N, p}^Y g := B_{k,p}^Y (\kappa_N \cdot B_{k,p}^Y g). \quad (5.26)$$

Then, we clearly have

$$\langle T_{\kappa_N, p}^Y g, g \rangle_{L^2(Y)} = \langle \kappa_N \cdot B_{k, p}^Y g, B_{k, p}^Y g \rangle_{L^2(Y)}. \quad (5.27)$$

Thus, by (5.25) and (5.27), we have

$$\|E_{k, p}^0\| = \frac{1}{p^{\frac{n-m+k}{2}}} \cdot \frac{1}{\sqrt{k! \cdot (2\pi)^k}} \cdot \|T_{\kappa_N, p}^Y\|^{\frac{1}{2}} + O\left(\frac{1}{p^{\frac{n-m+k+1}{2}}}\right). \quad (5.28)$$

From Lemma 3.11 and (5.28), we deduce (1.15).

Now it is only left to prove that if $A \neq 0$, then under additional assumption (1.10), one can not replace $p^{-\frac{n-m+k+1}{2}}$ by an asymptotically better estimate. For this, remark that as long as $A \neq 0$, by (5.5) and (5.21), the operator, acting on \mathbb{C}^n with the kernel $F_{k, 1}^{EK}(Z, Z'_Y)$, has non-zero norm. Then, by the calculations, similar to (5.25), we see that the operator, acting on \mathbb{C}^n with the kernel $F_{k, 1}^{EK}(\sqrt{p}Z, \sqrt{p}Z'_Y)$, has norm of order $p^{-\frac{n-m+k}{2}}$, as $p \rightarrow \infty$. We deduce from this that as long as $A \neq 0$, one can not replace $p^{-\frac{n-m+k+1}{2}}$ by an asymptotically better estimate. \square

Proof of Theorem 1.5 assuming Theorems 5.1.b) and 5.5. First of all, remark that by definition

$$B_p^X - B_p^{X, kY} = \sum_{l=0}^{k-1} B_{l, p}^\perp. \quad (5.29)$$

Clearly, the estimate (1.21) follows from Theorem 5.1.b) and (5.29). From Theorems 3.10, 5.5, (3.11), (3.12), and the use of (5.29), we deduce that for any $k \in \mathbb{N}$, there are $\epsilon, c > 0, p_1 \in \mathbb{N}^*$, such that for any $r, l' \in \mathbb{N}$, there are $C, Q > 0$, such that for any $y_0 \in Y, p \geq p_1, Z = (Z_Y, Z_N), Z' = (Z'_Y, Z'_N), Z_Y, Z'_Y \in \mathbb{R}^{2m}, Z_N, Z'_N \in \mathbb{R}^{2(n-m)}, |Z|, |Z'| \leq \epsilon, \alpha \in \mathbb{N}^{2n}, \alpha' \in \mathbb{N}^{2n}, |\alpha| + |\alpha'| \leq l'$, we have

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z_Y^{\alpha'}} \left(\frac{1}{p^n} B_p^{X, kY}(Z, Z') - \mathcal{P}_{n, m}^{\mathbb{C}^n, k\mathbb{C}^m}(\sqrt{p}Z, \sqrt{p}Z') \cdot \kappa_\psi^{X|Y}(y_0)^{-\frac{1}{2}} \cdot \kappa_\psi^{X|Y}(y_0)^{-\frac{1}{2}} \right) \right|_{\mathcal{C}^r} \leq Cp^{\frac{l'+r-1}{2}} \cdot (1 + \sqrt{p}|Z| + \sqrt{p}|Z'|)^Q \cdot \exp\left(-c\sqrt{p} \cdot (|Z_Y - Z'_Y| + |Z_N| + |Z'_N|)\right). \quad (5.30)$$

By using the fact that the term under the parentheses of the left-hand side of (5.30) vanishes up to order k along Y in each variable Z and Z' and taking Taylor expansion, we deduce from (5.30) that for any $k \in \mathbb{N}$, there are $\epsilon, c > 0, p_1 \in \mathbb{N}^*$, such that for any $r \in \mathbb{N}$, there are $C, Q > 0$, such that for any $y_0 \in Y, p \geq p_1, Z = (Z_Y, Z_N), Z' = (Z'_Y, Z'_N), Z_Y, Z'_Y \in \mathbb{R}^{2m}, Z_N, Z'_N \in \mathbb{R}^{2(n-m)}, |Z|, |Z'| \leq \epsilon$, we have

$$\left| \frac{1}{p^n} B_p^{X, kY}(Z, Z') - \mathcal{P}_{n, m}^{\mathbb{C}^n, k\mathbb{C}^m}(\sqrt{p}Z, \sqrt{p}Z') \cdot \kappa_\psi^{X|Y}(y_0)^{-\frac{1}{2}} \cdot \kappa_\psi^{X|Y}(y_0)^{-\frac{1}{2}} \right|_{\mathcal{C}^r} \leq Cp^{k+\frac{r-1}{2}} \cdot |Z_N|^k |Z'_N|^k \cdot (1 + \sqrt{p}|Z| + \sqrt{p}|Z'|)^Q \cdot \exp\left(-c\sqrt{p} \cdot (|Z_Y - Z'_Y| + |Z_N| + |Z'_N|)\right). \quad (5.31)$$

However, directly from (3.12), we obtain

$$\mathcal{P}_{n, m}^{\mathbb{C}^n, k\mathbb{C}^m}(Z, Z') = \mathcal{P}_{m, m}(Z_Y, Z'_Y) \cdot \exp\left(-\frac{\pi}{2}(|Z_N|^2 + |Z'_N|^2)\right) \cdot \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta| \geq k}} \pi^{|\beta|} \cdot \frac{z_N^\beta \cdot (\bar{z}'_N)^\beta}{\beta!}. \quad (5.32)$$

From Theorem 3.10, (3.38), (5.31) and (5.32), we deduce (1.22). \square

5.2 Multiplicative defect: from Bergman projections to extension operators

The main goal of this section is to prove Theorems 5.1, 5.3 and 5.5. As a consequence of this, we also establish Theorem 1.3. Our proof relies heavily on the existence of the multiplicative defect operator, $A_{k,p} \in \text{End}(H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F)))$, $p \geq p_1$, $k \in \mathbb{N}$, established in Theorem 4.1. More precisely, the following statement plays a crucial role in our approach.

Theorem 5.7. *For any $k \in \mathbb{N}$, the sequence of operators $\frac{1}{p^{n-m+k}} A_{k,p}$, $p \geq p_1$, viewed as a sequence of elements from $\text{End}(L^2(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F)))$ by precomposing with the Bergman projector $B_{k,p}^Y$, forms a Toeplitz operator with weak exponential decay with respect to X . Moreover, the first term of the asymptotic expansion of it can be calculated as follows*

$$\left[\frac{1}{p^{n-m+k}} A_{k,p} \right]_0 = (2\pi)^k \cdot k! \cdot \kappa_N^{-1}|_Y \cdot \text{Id}_{\text{Sym}^k N_{y_0}^{1,0}} \otimes \text{Id}_{F_{y_0}}. \quad (5.33)$$

Remark 5.8. For $k = 0$, Theorem 5.7 was established in [17, Theorem 4.3].

The core of our argument in the proofs of Theorems 5.1, 5.3 and 5.5 lies in the following inductive argument. We fix $k_0 \in \mathbb{N}$.

Lemma 5.9. Assume that for any $k < k_0$, Theorems 5.1.b) and 5.5 hold. Then for $k := k_0$,

- a) Theorem 5.7 holds,
- b) Theorems 5.1.a) and 5.3 hold,
- c) Theorems 5.1.b) and 5.5 hold.

The following statement will be useful in our proof of Lemma 5.9.

Lemma 5.10. Assume that for any $k < k_0$, Theorems 5.1.b) and 5.5 hold. Then for $k := k_0$, and any $r \in \mathbb{N}$, $y_0 \in Y$, there are polynomials $J_{k,r}^{R,0}(Z_Y, Z') \in \text{Sym}^k(N_{y_0}^{1,0})^* \otimes \text{End}(F_{y_0})$ in $Z' \in \mathbb{R}^{2n}$, $Z_Y \in \mathbb{R}^{2m}$, with the same properties as those from Theorem 5.3 (including the analogous vanishing, degree and parity requirements), such that for $F_{k,r}^{R,0} := J_{k,r}^{R,0} \cdot \mathcal{R}_{n,m}^0$, the following holds.

There are $\epsilon, c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $r, l, l' \in \mathbb{N}$, there are $C, Q > 0$, such that for any $y_0 \in Y$, $p \geq p_1$, $Z' = (Z'_Y, Z'_N)$, $Z_Y, Z'_Y \in \mathbb{R}^{2m}$, $Z'_N \in \mathbb{R}^{2(n-m)}$, $|Z_Y|, |Z'| \leq \epsilon$, $\alpha \in \mathbb{N}^{2m}$, $\alpha' \in \mathbb{N}^{2n}$, $|\alpha| + |\alpha'| \leq l$, the following bound holds

$$\begin{aligned} & \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z_Y^\alpha \partial Z'^{\alpha'}} \left(\frac{1}{p^{n+\frac{k}{2}}} \cdot (\text{Res}_Y \circ \nabla^k B_p^{X,kY})(\phi_{y_0}^Y(Z_Y), \psi_{y_0}(Z')) \right. \right. \\ & \quad \left. \left. - \sum_{j=0}^r p^{-\frac{j}{2}} F_{k,j}^{R,0}(\sqrt{p}Z_Y, \sqrt{p}Z') \kappa_\phi^Y(Z_Y)^{-\frac{1}{2}} \kappa_\psi^X(Z')^{-\frac{1}{2}} \right) \right|_{\mathcal{C}^{l'}} \\ & \leq Cp^{-\frac{r+1-l}{2}} \left(1 + \sqrt{p}|Z_Y| + \sqrt{p}|Z'| \right)^Q \exp \left(-c\sqrt{p}(|Z_Y - Z'_Y| + |Z'_N|) \right), \end{aligned} \quad (5.34)$$

where the $\mathcal{C}^{l'}$ -norm is taken with respect to y_0 . Also, the following identity holds

$$J_{k,0}^{R,0}(Z_Y, Z') = \pi^k \cdot \text{Id}_{F_{y_0}} \cdot \kappa_N^{-\frac{1}{2}}(y_0) \cdot \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=k}} \frac{k!}{\beta!} \cdot (dz_N)^{\odot \beta} \cdot (\bar{z}'_N)^\beta. \quad (5.35)$$

In particular, from (3.13), we have $F_{k,0}^{R,0}(Z_Y, Z') = \mathcal{R}_{n,m}^k$. Moreover, under the assumption (1.10)

$$J_{k,1}^{R,0}(Z_Y, Z') = \pi^{k+1} \cdot \text{Id}_{F_{y_0}} \cdot g(\bar{z}'_N, A(z_Y - z'_Y)(z_Y - z'_Y)) \cdot \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=k}} \frac{k!}{\beta!} \cdot (dz_N)^{\odot \beta} \cdot (\bar{z}'_N)^\beta. \quad (5.36)$$

Proof. From Section 3.1, remark the following trivial identities

$$\begin{aligned} \mathcal{P}_n(Z, Z') &= \mathcal{R}_{n,m}^0(Z_Y, Z') \cdot \frac{\mathcal{P}_{n-m}(Z_N, Z'_N)}{\mathcal{P}_{n-m,0}^{\perp,0}(0, Z'_N)}, \\ \mathcal{P}_{n,m}^{\perp,0}(Z, Z') &= \mathcal{R}_{n,m}^0(Z_Y, Z') \cdot \frac{\mathcal{P}_{n-m,0}^{\perp,0}(Z_N, Z'_N)}{\mathcal{P}_{n-m,0}^{\perp,0}(0, Z'_N)}. \end{aligned} \quad (5.37)$$

From the fact that in the notations introduced in the end of Section 2.1, we have $\text{Res}_Y(\tilde{f}_1^{X|Y}, \dots, \tilde{f}_r^{X|Y}) = \tilde{f}_1^{Y}, \dots, \tilde{f}_r^{Y}$, the fact that $\frac{\partial \psi}{\partial Z_j}|_{Z_N=0}$, $j = 2m+1, \dots, 2n$, are parallel along $B_0^{\mathbb{R}^{2m}}(r_Y) \subset \mathbb{R}^{2n}$, (3.38), (3.39), (5.6), (5.29) and (5.37), we see that the expansion (5.34) holds for polynomials $J_{k,r}^{R,0}(Z_Y, Z') \in \text{Sym}^k(N_{y_0}^{1,0})^* \otimes \text{End}(F_{y_0})$, defined as

$$J_{k,r}^{R,0}(Z_Y, Z') := \nabla_{Z_N}^k \left[\sum_{a+b=r} \left(J_a^{X|Y}(Z, Z') \cdot \frac{\mathcal{P}_{n-m}(Z_N, Z'_N)}{\mathcal{P}_{n-m,0}^{\perp,0}(0, Z'_N)} - \sum_{i=0}^{k-1} J_{i,a}^{\perp}(Z, Z') \cdot \frac{\mathcal{P}_{n-m,0}^{\perp,0}(Z_N, Z'_N)}{\mathcal{P}_{n-m,0}^{\perp,0}(0, Z'_N)} \right) \cdot \kappa_{N,[b]}^{-\frac{1}{2}}(Z) \right] \Big|_{Z_N=0}. \quad (5.38)$$

Remark, however, that since the first multiplicand after the summation sign in (5.38) corresponds to the asymptotic expansion of $B_p^{X,kY}$, it has to vanish up to order k along Y . In particular, we obtain

$$J_{k,r}^{R,0}(Z_Y, Z') := \sum_{a+b=r} \nabla_{Z_N}^k \left(J_a^{X|Y}(Z, Z') \cdot \mathcal{P}_{n-m}(Z_N, Z'_N) - \sum_{i=0}^{k-1} J_{i,a}^{\perp}(Z, Z') \cdot \mathcal{P}_{n-m,0}^{\perp,0}(Z_N, Z'_N) \right) \Big|_{Z_N=0} \cdot \frac{\kappa_{N,[b]}^{-\frac{1}{2}}(Z_Y)}{\mathcal{P}_{n-m,0}^{\perp,0}(0, Z'_N)}. \quad (5.39)$$

Now, from (3.4) and (5.7), we deduce that $J_{i,0}^{\perp}(Z, Z') \cdot \mathcal{P}_{n-m,0}^{\perp,0}(Z_N, Z'_N) = \text{Id}_{F_{y_0}} \cdot \mathcal{P}_{n-m,0}^{\perp,i}(Z_N, Z'_N)$. From this, (3.40) and (5.39), we conclude that

$$J_{k,0}^{R,0}(Z_Y, Z') = \nabla_{Z_N}^k \left(\mathcal{P}_{n-m}(Z_N, Z'_N) - \sum_{i=0}^{k-1} \mathcal{P}_{n-m,0}^{\perp,i}(Z_N, Z'_N) \right) \Big|_{Z_N=0} \cdot \frac{\kappa_N^{-\frac{1}{2}}(y_0) \cdot \text{Id}_{F_{y_0}}}{\mathcal{P}_{n-m,0}^{\perp,0}(0, Z'_N)}. \quad (5.40)$$

But from the calculations as in (3.12), we have

$$\mathcal{P}_{n-m}(Z_N, Z'_N) - \sum_{i=0}^{k-1} \mathcal{P}_{n-m,0}^{\perp,i}(Z_N, Z'_N) = \left(\sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta| \geq k}} \pi^{|\beta|} \cdot \frac{z_N^\beta \cdot (\bar{z}'_N)^\beta}{\beta!} \right) \cdot \mathcal{P}_{n-m,0}^{\perp,0}(Z_N, Z'_N). \quad (5.41)$$

From this, (3.7) and (5.40), we deduce (5.35). Moreover, under the assumption (1.10), from (5.20) and (5.39), we also obtain that

$$J_{k,1}^{R,0}(Z_Y, Z') = \nabla_{Z_N}^k \left(J_1^{X|Y}(Z, Z') \cdot \mathcal{P}_{n-m}(Z_N, Z'_N) - \sum_{i=0}^{k-1} J_{i,1}^\perp(Z, Z') \cdot \mathcal{P}_{n-m,0}^\perp(Z_N, Z'_N) \right) \Big|_{Z_N=0} \cdot \frac{\kappa_N^{-\frac{1}{2}}(y_0) \cdot \text{Id}_{F_{y_0}}}{\mathcal{P}_{n-m,0}^\perp(0, Z'_N)}. \quad (5.42)$$

From this, using (5.41) as in the proof of (5.35), but now using (3.41) and (5.8), we deduce (5.36).

The fact that the parity of $J_{k,r}^{R,0}$ coincides with the parity of $k+r$ follows from the analogous statements from Theorems 3.10, 5.5 and (5.38). From the bounded geometry assumption and the boundness properties of the coefficients of $J_a^{X|Y}$, $J_{i,a}^\perp$ from Theorems 3.10, 5.5, we see that the coefficients of $J_{k,r}^{R,0}$ are bounded with all their derivatives. From (5.38), we see that

$$\deg(J_{k,r}^{R,0}) \leq \max \left\{ \deg(J_a^{X|Y}) + k, \deg(J_{i,a}^\perp) + k \right\}, \quad (5.43)$$

where the maximum is taken over $a = 0, \dots, k$ and $i = 0, \dots, k-1$. From (5.43) and the corresponding bounds on the degrees of $J_a^{X|Y}$ and $J_{i,a}^\perp$ from Theorems 3.10 and 5.5, we deduce the needed bound on the degree of $J_{k,r}^{R,0}$.

It is only left to establish the vanishing of $J_{k,r}^{R,0}$ at least up to order k along $\mathbb{R}^{2m} \times \mathbb{R}^{2m} \subset \mathbb{R}^{2m} \times \mathbb{R}^{2n}$. For this, let us define the polynomials $J_{k,r}^{E,0}(Z, Z'_Y) \in \text{Sym}^k(N_{y_0}^{1,0}) \otimes \text{End}(F_{y_0})$ in $Z \in \mathbb{R}^{2n}$, $Z'_Y \in \mathbb{R}^{2m}$ as follows

$$J_{k,r}^{E,0}(Z, Z'_Y) := (J_{k,r}^{R,0}(Z'_Y, Z))^*. \quad (5.44)$$

Clearly, it is enough to establish the analogous vanishing property for the polynomials $J_{k,r}^{E,0}$. Assume that $r_0 \in \mathbb{N}$ is the first index, for which this vanishing property doesn't hold. We denote by $k' < k$ the order of vanishing of $J_{k,r_0}^{E,0}$ along $\mathbb{R}^{2m} \times \mathbb{R}^{2m} \subset \mathbb{R}^{2n} \times \mathbb{R}^{2m}$.

First of all, directly from (5.34), we obtain that for $F_{k,r}^{E,0} := J_{k,r}^{E,0} \cdot \mathcal{E}_{n,m}^0$, the following holds. There are $\epsilon, c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $r, l, l' \in \mathbb{N}$, there are $C, Q > 0$, such that for any $y_0 \in Y$, $p \geq p_1$, $Z = (Z_Y, Z_N)$, $Z_Y, Z'_Y \in \mathbb{R}^{2m}$, $Z_N \in \mathbb{R}^{2(n-m)}$, $|Z|, |Z'_Y| \leq \epsilon$, $\alpha \in \mathbb{N}^{2m}$, $\alpha' \in \mathbb{N}^{2n}$, $|\alpha| + |\alpha'| \leq l$, the following bound holds

$$\begin{aligned} & \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z_Y^\alpha \partial Z'^{\alpha'}} \left(\frac{1}{p^{n+\frac{k}{2}}} \cdot (\text{Res}_Y \circ \nabla^k B_p^{X,kY})^*(\psi_{y_0}(Z), \phi_{y_0}^Y(Z'_Y)) \right. \right. \\ & \quad \left. \left. - \sum_{j=0}^r p^{-\frac{j}{2}} F_{k,j}^{E,0}(\sqrt{p}Z, \sqrt{p}Z'_Y) \kappa_\psi^{X|Y}(Z)^{-\frac{1}{2}} \kappa_\phi^Y(Z'_Y)^{-\frac{1}{2}} \right) \right|_{\mathcal{C}^{l'}} \\ & \leq Cp^{-\frac{r+1-l}{2}} \left(1 + \sqrt{p}|Z| + \sqrt{p}|Z'_Y| \right)^Q \exp \left(-c\sqrt{p}(|Z_Y - Z'_Y| + |Z_N|) \right), \quad (5.45) \end{aligned}$$

where the $\mathcal{C}^{l'}$ -norm is taken with respect to y_0 .

Let us consider the operator $T_p^Y : L^2(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F)) \rightarrow L^2(Y, \text{Sym}^{k'}(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$, given by the following formula

$$T_p^Y := p^{m-n-\frac{k+k'}{2}} \cdot \text{Res}_Y \circ \nabla^{k'} (\text{Res}_Y \circ \nabla^k B_p^{X,kY})^*. \quad (5.46)$$

First of all, remark that since the image of $(\text{Res}_Y \circ \nabla^k B_p^{X,kY})^*$ is included in the space of holomorphic sections of L^k vanishing to order k (see the proof of Theorem 4.1), we deduce that $T_p^Y = 0$.

We will now establish that the sequence of operators T_p^Y satisfies the assumptions of Proposition 3.7 or Remark 3.8, depending on the parity of $k + k'$. Clearly, the first property from Proposition 3.7 follows from Theorem 4.3 and (5.46).

From Theorem 4.7 and Theorem 5.1.b), applied for $k \leq k_0 - 1$, we deduce that there are $c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $r \in \mathbb{N}$, there is $C > 0$, such that for any $p \geq p_1$, $x_1, x_2 \in X$, the following estimate holds

$$\left| B_p^{X,kY}(x_1, x_2) \right|_{\mathcal{E}^r} \leq Cp^{n+\frac{r}{2}} \exp(-c\sqrt{p} \cdot \text{dist}(x_1, x_2)). \quad (5.47)$$

From (5.47), we deduce that there are $c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $r \in \mathbb{N}$, there is $C > 0$, such that for any $p \geq p_1$, $y_1, y_2 \in Y$, the following estimate holds

$$\left| T_p^Y(y_1, y_2) \right|_{\mathcal{E}^r} \leq Cp^{m+\frac{r}{2}} \exp(-c\sqrt{p} \cdot \text{dist}_X(y_1, y_2)). \quad (5.48)$$

This implies the second property from Proposition 3.7 with respect to X .

We will now show that the third property is a direct consequence of Theorems 3.10 and 5.5 for $k \leq k_0 - 1$. Remark first that from (3.16) we have $(\mathcal{R}_{n,m}^0)^* = \mathcal{E}_{n,m}^0$, and thus, directly from Section 3.1, remark the following trivial identity

$$\mathcal{E}_{n,m}^0(Z, Z'_Y) = \mathcal{P}_{n-m}(Z_Y, Z'_Y) \cdot \mathcal{P}_{n-m,0}^{\perp,0}(Z_N, 0), \quad (5.49)$$

From this, (5.45) and the reasoning similar to (5.38), we see that the expansion (3.34) for T_p^Y as above holds for the polynomials $I_r^Y(Z_Y, Z'_Y)$, $Z_Y, Z'_Y \in \mathbb{R}^{2m}$, defined as follows

$$I_r^Y(Z_Y, Z'_Y) = \nabla_{Z_N}^k \left(\sum_{a+b=r} J_{k,a}^{E,0}(Z, Z'_Y) \cdot \mathcal{P}_{n-m,0}^{\perp,0}(Z_N, 0) \cdot \kappa_{N,[b]}^{-\frac{1}{2}}(Z)^{-\frac{1}{2}} \right) \Big|_{Z_N=0}. \quad (5.50)$$

From the parity statements on $J_{k,r}^{R,0}$ from above (implying the corresponding statements for $J_{k,r}^{E,0}$), we see that the parity of I_r^Y equals to the parity of $r + k + k'$. From (5.50), by our choice of r_0 , we also see that $I_r^Y(Z_Y, Z'_Y) = 0$ for $r < r_0$ and $I_{r_0}^Y(Z_Y, Z'_Y) = \nabla_{z_N}^{k'} J_{k,r_0}^{E,0}(Z, Z'_Y)|_{Z_N=0}$. Hence the sequence of operators $p^{\frac{r_0}{2}} \cdot T_p^Y$ still satisfies the assumptions of either Proposition 3.7 or Remark 3.8, and the first term of the associated asymptotic expansion (3.34) holds with nonzero term by our choice of k' and r_0 . This, however, goes in contradiction with the last part of Proposition 3.7 or Remark 3.8 and the fact that $T_p^Y = 0$. \square

Proof of Lemma 5.9.a). Let us verify that the validity of Theorems 5.1.b) and 5.5 for any $k < k_0$ imply that the sequence of operators $\frac{1}{p^{n-m+k}} A_{k,p}$, $p \geq p_1$, for $k := k_0$, satisfies all the properties of Theorem 3.5.

Remark first that by Theorems 4.3, 4.4, there is $p_1 \in \mathbb{N}$, such that for any $p \geq p_1$, we have $\text{Res}_Y \circ \nabla^k E_{k,p} = B_{k,p}^Y$. From this and (4.2), we conclude that for any $p \geq p_1$, we have

$$A_{k,p} = \text{Res}_Y \circ \nabla^k (\text{Res}_Y \circ \nabla^k B_p^{X,kY})^*. \quad (5.51)$$

From the above identity, we see that $\frac{1}{p^{n-m+k}} A_{k,p}$ coincides with the operator T_p^Y , considered in (5.46) for $k' = k$. From the proof of Lemma 5.10, we see that the sequence of operators

$\frac{1}{p^{n-m+k}} A_{k,p}$, $p \geq p_1$, for $k = k_0$, forms a Toeplitz operator with weak exponential decay with respect to X . Let us estimate the first term of this asymptotic expansion.

Recall that the polynomials $J_{k,r}^{E,0}$ were defined in (5.44). From (3.17), (5.35) and (5.44), we obtain that

$$J_{k,0}^{E,0}(Z, Z'_Y) = (2\pi)^k \cdot k! \cdot \text{Id}_{F_{y_0}} \cdot \kappa_N^{-\frac{1}{2}}(y_0) \cdot \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta|=k}} \frac{1}{\beta!} \cdot z_N^\beta \cdot \left(\frac{\partial}{\partial z_N}\right)^{\odot \beta}. \quad (5.52)$$

From this, (3.7), (3.9) and (5.50), we deduce that

$$I_0^Y(Z_Y, Z'_Y) = (2\pi)^k \cdot k! \cdot \kappa_N^{-1}(y_0) \cdot \text{Id}_{\text{Sym}^k(N_{y_0}^{1,0})^*} \otimes \text{Id}_{F_{y_0}}. \quad (5.53)$$

From the last statement of Theorem 3.5, we deduce (5.33), which finishes our proof. \square

In the proof of Lemma 5.9.b), we will need to express the extension operator in terms of the orthogonal Bergman projector. For this, we need to invert the operators $A_{k,p}$. The following result gives a sufficient condition for inverting Toeplitz operators with weak exponential decay.

Lemma 5.11 ([17, Lemma 4.5]). Assume that a sequence of operators G_p , $p \in \mathbb{N}$, forms a Toeplitz operator with weak exponential decay with respect to a manifold Z in the notations from Definition 3.3. Assume that for $f := [G_p]_0$, and any $y \in Y$, the endomorphism $f(y)$ is invertible and $f^{-1} \in \mathcal{C}_b^\infty(Y, \text{End}(l^*F))$. Then there is $p_1 \in \mathbb{N}$, such that for any $p \geq p_1$, the operators G_p are invertible. Moreover, the sequence of operators G_p^{-1} , $p \geq p_1$, forms a Toeplitz operator with weak exponential decay with respect to the same manifold Z and we have $[(G_p)^{-1}]_0 = f^{-1}$.

Proof of Lemma 5.9.b). Remark that by Lemma 5.9.a) and Lemma 5.11, for any $k \in \mathbb{N}$, there is $p_1 \in \mathbb{N}$, such that the operator $A_{k,p}$ is invertible for any $p \geq p_1$. The main idea of our proof is to use the following formula

$$\frac{1}{p^{m-\frac{k}{2}}} E_{k,p} = \frac{1}{p^{n+\frac{k}{2}}} \cdot (\text{Res}_Y \circ \nabla^k B_p^{X,kY})^* \circ \left(\frac{1}{p^{n-m+k}} A_{k,p}\right)^{-1}, \quad (5.54)$$

which follows from (4.2). On the level of Schwartz kernels, the identity (5.54) means

$$\frac{1}{p^{m-\frac{k}{2}}} E_{k,p}(x, y) = \int_{y_1 \in Y} \frac{1}{p^{n+\frac{k}{2}}} \cdot (\text{Res}_Y \circ \nabla^k B_p^{X,kY})^*(x, y_1) \cdot \left(\frac{1}{p^{n-m+k}} A_{k,p}\right)^{-1}(y_1, y) dv_Y(y_1). \quad (5.55)$$

From Proposition 2.6, Theorem 5.7, Lemma 5.11, (3.33) and (5.47), we deduce Theorem 5.1.a).

Now, let $\epsilon > 0$ be the minimum of ϵ from Theorems 3.10 and 5.5 for $k \leq k_0 - 1$. We put $\epsilon_0 := \frac{\epsilon}{2}$. Let $y_0 \in Y$ and $y_1, y_2 \in B_{y_0}^Y(\epsilon_0)$. We decompose the integral in (5.55) into two parts: the first one over $B_{y_0}^Y(\epsilon)$, and the second one is over its complement, which we denote by Q . Clearly, for $y_3 \in Q$, we get

$$\text{dist}(y_1, y_3) + \text{dist}(y_3, y_2) \geq \epsilon. \quad (5.56)$$

Hence, from Proposition 2.6, Theorems 4.7, 5.1.b), 5.7 and (1.5), we see that the contribution from the integration over Q is smaller than $\exp(-c\sqrt{p}(1 + \text{dist}(y_1, y_2)))$ for some constant $c > 0$.

Consequently, only the integration over $y_1 \in B_{y_0}^Y(\epsilon)$ is non-negligible. We use the notations introduced before Theorem 3.10. We denote by $I_{A_{k,p},b}^{Y^{-1}}$ the polynomials associated to $(\frac{1}{p^{n-m+k}} A_{k,p})^{-1}$ as in Theorem 3.5 and by $J_{k,a}^{E,0}$ the polynomials introduced in (5.44). After the change of variables $Z \mapsto \sqrt{p}Z$, an estimate, similar to the one which bounded the integral over Q , Lemma 2.8, the second part of Lemma 3.1, applied for $n := m$, (3.34) and (5.45), we see that (5.2) holds for

$$J_{k,r}^E(Z, Z'_Y) := \sum_{a+b=r} \mathcal{K}_{n,m}^{EP} [J_{k,a}^{E,0}, I_{A_{k,p},b}^{Y^{-1}}]. \quad (5.57)$$

From the corresponding statements about the parity of $J_{k,r}^{E,0}, I_{A_{k,p},r}^Y$ from Theorem 3.5 and Lemma 5.10, and from the properties of $\mathcal{K}_{n,m}^{EP}$ from Lemma 3.1, we deduce that the parity of $J_{k,r}^E$ coincides with $k+r$. From Proposition 3.9 and (5.57), we deduce that

$$\deg(J_{k,r}^E) \leq \max \left\{ \deg(J_{k,a}^{E,0}) + \deg(I_{A_{k,p},b}^Y) \right\}, \quad (5.58)$$

where the maximum is taken over $a+b=r$. From Proposition 3.9, Lemma 5.10, (5.44), (5.58) and the bound on the degree of $J_{k,a}^{E,0}$ from Lemma 5.10, we deduce the bound on the degree of $J_{k,r}^E$ from Theorem 5.3.

Directly from (5.33), (5.52) and (5.57), we see that the formula (5.3) holds for $J_{k,0}^E$. Moreover, under assumption (1.10), by Theorem 5.7, for $i=0,1$, we have

$$I_{A_{k,p},i}^Y = \frac{1}{(2\pi)^k \cdot k!} \cdot J_{k,i}^{Y|Y}, \quad (5.59)$$

where $J_{k,i}^{Y|Y}$ are the polynomials from Theorem 3.10, associated to $X, Y := Y$ and $F := \text{Sym}^k(N^{1,0})^* \otimes \iota^*(F)$. From this, (3.40), (3.41), (5.57), we conclude that

$$J_{k,1}^E(Z, Z'_Y) = \frac{1}{(2\pi)^k \cdot k!} \cdot \mathcal{K}_{n,m}^{EP} [J_{k,1}^{E,0}, 1]. \quad (5.60)$$

We deduce (5.4) from this, (3.23), (5.36) and (5.44).

The proof of the fact that $J_{k,r}^E$ vanishes at least up to order k along $\mathbb{R}^{2m} \times \mathbb{R}^{2m} \subset \mathbb{R}^{2n} \times \mathbb{R}^{2m}$ proceeds along the same lines as the corresponding result from Lemma 5.10. In total, this finishes the proof of Theorem 5.3 for $k := k_0$. \square

Proof of Lemma 5.9.c). First of all, by Theorems 4.3, 4.4, there is $p_1 \in \mathbb{N}$, such that for any $p \geq p_1$, we have

$$\frac{1}{p^n} B_p^{\perp,k} = \frac{1}{p^{m-\frac{k}{2}}} E_{k,p} \circ \frac{1}{p^{n-m+\frac{k}{2}}} (\text{Res}_Y \circ \nabla^k B_p^{X,kY}). \quad (5.61)$$

On the level of Schwartz kernels, the identity (5.61) basically means that

$$\frac{1}{p^n} B_p^{\perp,k}(x_1, x_2) = \int_{y \in Y} \frac{1}{p^{m-\frac{k}{2}}} E_{k,p}(x_1, y) \cdot \frac{1}{p^{n-m+\frac{k}{2}}} (\text{Res}_Y \circ \nabla^k B_p^{X,kY})(y, x_2) dv_Y(y). \quad (5.62)$$

Remark first that Lemma 5.9.b) says that Theorem 5.1.a) holds for $k = k_0$. From this, Proposition 2.6 and Theorems 4.7, 5.1.b) for $k \leq k_0 - 1$, we see by (5.62) that Theorem 5.1.b) holds for $k = k_0$.

Remark now that Lemma 5.9.b) says that Theorem 5.3 holds for $k = k_0$. From Theorem 5.3, Lemma 5.10, (5.61) and the third part of Lemma 3.1, we see that the expansion (5.6) holds for $J_{k,r}^\perp$, defined as follows

$$J_{k,r}^\perp(Z, Z') := \sum_{a+b=r} \mathcal{K}_{n,m}^{ER} [J_{k,a}^E, J_{k,b}^{R,0}]. \quad (5.63)$$

From this (3.26), (3.28), (5.3) and (5.35) (resp. (5.4) and (5.36)), we deduce (5.7) (resp. (5.8)). The parity statement about the polynomials $J_{k,r}^\perp$ follows from Lemma 3.1 and the corresponding statements for $J_{k,a}^E$ and $J_{k,b}^{R,0}$, proved in Theorem 5.3 and Lemma 5.10. The vanishing up to order k of the polynomials $J_{k,r}^\perp$ follows from (3.26), (5.63) and the corresponding statements for $J_{k,a}^E$ and $J_{k,b}^{R,0}$, proved in Theorem 5.3 and Lemma 5.10. From Proposition 3.9 and (5.63), we deduce that

$$\deg(J_{k,r}^\perp) \leq \max \left\{ \deg(J_{k,a}^E) + \deg(J_{k,b}^{R,0}) \right\}, \quad (5.64)$$

where the maximum is taken over $a + b = r$. From Lemma 5.10, (5.64) and the bound on the degree of $J_{k,a}^{R,0}$ (resp. $J_{k,a}^E$) from Lemma 5.10 (resp. Theorem 5.3), we deduce the needed bound on the degree of $J_{k,r}^\perp$. \square

Proof of Theorems 5.1, 5.3, 5.5, 5.7. Follows directly from Lemma 5.9 by induction. \square

Proof of Theorem 1.3. First of all, let us establish that for any $k \in \mathbb{N}$, there are $p_1 \in \mathbb{N}$, $C > 0$, such that if the assumption (1.10) holds, then for any $p \geq p_1$, we have

$$\begin{aligned} \left| \|E_{k,p}\| - \frac{1}{p^{\frac{n-m+k}{2}}} \cdot \frac{1}{\sqrt{k! \cdot (2\pi)^k}} \right| &\leq \frac{C}{p^{\frac{n-m+k+2}{2}}}, \\ \left| \|\text{Res}_{k,p}\| - p^{\frac{n-m+k}{2}} \cdot \sqrt{k! \cdot (2\pi)^k} \right| &\leq Cp^{\frac{n-m+k-2}{2}}. \end{aligned} \quad (5.65)$$

Indeed, remark that from Proposition 2.7 and Theorem 5.7, for any $k \in \mathbb{N}$, there are $p_1 \in \mathbb{N}$, $C > 0$, such that for any $p \geq p_1$, we have

$$\left\| \frac{1}{p^{n-m+k}} A_{k,p} - k! \cdot (2\pi)^k \cdot B_{k,p}^Y \right\| \leq \frac{C}{p}. \quad (5.66)$$

In particular, we conclude that

$$\left| \|A_{k,p}\| - p^{n-m+k} \cdot k! \cdot (2\pi)^k \right| \leq Cp^{n-m+k-1}. \quad (5.67)$$

Using Lemma 5.11 and similar arguments, we see that

$$\left| \|A_{k,p}^{-1}\| - \frac{1}{p^{n-m+k}} \cdot \frac{1}{k! \cdot (2\pi)^k} \right| \leq \frac{C}{p^{n-m+k+1}}. \quad (5.68)$$

From Theorems 4.3 and 4.4, there is $p_1 \in \mathbb{N}$, such that for any $p \geq p_1$, we have

$$\text{Res}_{k,p} \circ E_{k,p} = B_{k,p}^Y. \quad (5.69)$$

From this and (4.2), there is $p_1 \in \mathbb{N}$, such that for any $p \geq p_1$, we have the following identities

$$(E_{k,p})^* \circ E_{k,p} = ((A_{k,p})^*)^{-1}, \quad \text{Res}_{k,p} \circ (\text{Res}_{k,p})^* = A_{k,p}. \quad (5.70)$$

Clearly, we have $\|(E_{k,p})^* \circ E_{k,p}\| = \|E_{k,p}\|^2$ and $\|\text{Res}_{k,p} \circ (\text{Res}_{k,p})^*\| = \|\text{Res}_{k,p}\|^2$. The bounds (5.65) now follow from this observation, (5.67), (5.68) and (5.70).

Now, from (5.65) and (5.69), we conclude that there are $p_1 \in \mathbb{N}$, $C > 0$, such that for any $k \in \mathbb{N}$, $p \geq p_1$, $g \in H_{(2)}^0(Y, \text{Sym}^k(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$, $g \neq 0$, we have

$$\left| \frac{\|E_{k,p}(g)\|_{L^2(X, L^p \otimes F)}}{\|g\|_{k, L^2(Y, L^p \otimes F)}} - \frac{1}{p^{\frac{n-m+k}{2}}} \cdot \frac{1}{\sqrt{k! \cdot (2\pi)^k}} \right| \leq \frac{C}{p^{\frac{n-m+k+2}{2}}}. \quad (5.71)$$

Now, let us fix $f \in H_{(2)}^0(X, L^p \otimes F)$, and denote by $[f]$ the element it represents in the quotient space $H_{(2)}^0(X, L^p \otimes F) / H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_Y^{k+1})$. Directly from the definition, we have

$$\|[f]\|_{L^2(X, L^p \otimes F)}^2 = \sum_{l=0}^k \|B_{l,p}^\perp f\|_{L^2(X, L^p \otimes F)}^2. \quad (5.72)$$

We let $\text{Jet}_{k,p}(f) = (g_0, \dots, g_k)$, where $g_i \in H_{(2)}^0(Y, \text{Sym}^i(N^{1,0})^* \otimes \iota^*(L^p \otimes F))$. From the definition of the map $\text{Jet}_{k,p}$ from (1.16) and the characterization of g_i from (1.17), we see that

$$g_i = \text{Res}_{i,p}(B_{i,p}^\perp f). \quad (5.73)$$

We conclude from (5.65) and (5.73) that

$$\|g_i\|_{k, L^2(Y, L^p \otimes F)} = \left(p^{\frac{n-m+k}{2}} \cdot \sqrt{k! \cdot (2\pi)^k} + O(p^{\frac{n-m+k-2}{2}}) \right) \cdot \|B_{i,p}^\perp f\|_{L^2(X, L^p \otimes F)}. \quad (5.74)$$

The result now follows from (5.71), (5.72), (5.74) and the definition of the scalar product $\langle \cdot, \cdot \rangle_{\text{Jet}_{k,p}}$ from (1.19). \square

5.3 Higher order peak sections as a special case of extension theorem

The main goal of this section is to illustrate Theorems 5.1, 5.3 on a simple example when the submanifold corresponds to a fixed point. We denote the extension operator in this case by $E_{k,p}^{\{x\}}$, i.e.

$$E_{k,p}^{\{x\}} : \text{Sym}^k T^{1,0} X_x^* \otimes (L^p \otimes F)_x \rightarrow H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_x^k). \quad (5.75)$$

Clearly, for any $v \in \text{Sym}^k(N_x^{1,0})^* \otimes (L^p \otimes F)_x$, the section $s_{k,p}^{x,v} := E_{k,p}^{\{x\}}(v)$ minimizes the L^2 -norm among all sections from $H_{(2)}^0(X, L^p \otimes F \otimes \mathcal{J}_x^k)$, having k -th jet equal to v .

The sections $s_{k,p}^{x,v}$ were defined in complex geometry by Tian [38]. For $k = 0$, they bear the name ‘‘peak sections’’. Due to this reason, for $k \geq 1$, we call $s_{k,p}^{x,v}$ the *higher order peak sections*.

Theorem 5.12. *For any $k \in \mathbb{N}$, there are $c > 0$, $p_1 \in \mathbb{N}$, such that for any $x, y \in X$, $p \in \mathbb{N}$, $p \geq p_1$, $v \in \text{Sym}^k(N_x^{1,0})^* \otimes (L^p \otimes F)_x$, we have*

$$|s_{k,p}^{x,v}(y)|_{\mathcal{C}^r} \leq C \cdot |v| \cdot p^{\frac{r-k}{2}} \cdot \exp(-c\sqrt{p}\text{dist}(x, y)). \quad (5.76)$$

Moreover, for any $k \in \mathbb{N}$, there are $\epsilon > 0$, $c, C > 0$, such that for any $x \in X$, for any $Z \in \mathbb{R}^{2n}$, $|Z| < \epsilon$, $p \in \mathbb{N}$, $p \geq p_1$, $v \in \text{Sym}^k(N_x^{1,0})^ \otimes (L^p \otimes F)_x$, we have*

$$\left| s_{k,p}^{x,v}(\phi_x^X(Z)) - (v \cdot Z^{\otimes k}) \cdot \exp\left(-\frac{\pi}{2}p|Z|^2\right) \right|_{\mathcal{C}^r} \leq C \cdot |v| \cdot p^{\frac{r}{2}-1} \cdot |Z|^k \cdot \exp(-c\sqrt{p}|Z|), \quad (5.77)$$

where the \mathcal{C}^r -norm is taken with respect to x, Z .

Remark 5.13. For $r \leq 2$, similar results were obtained by Tian [38, Lemma 1.2]. For $k = 0$, this statement follows directly from Dai-Liu-Ma [10].

Proof. The first statement follows directly by applying Theorem 5.1.a) for $Y := \{y_0\}$. Remark that for Y as above, the second fundamental form vanishes, so the second term of the asymptotic expansion from Theorem 5.3 vanishes according to (5.4). The second statement of Theorem 5.12 then follows from this remark and the application of Theorem 5.3 for $Y := \{y_0\}$ and $r := 1$. \square

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SIARHEI FINSKI, CNRS-CMLS, ÉCOLE POLYTECHNIQUE F-91128 PALAISEAU CEDEX, FRANCE.

E-mail : finski.siarhei@gmail.com.