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# The Extended Finite Element and Level Set Methods for Non-Planar 3D Crack Growth

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**Abstract:** A methodology for treating non-planar three-dimensional cracks with geometries that are independent of the mesh is described. The method is based on the extended finite element method, in which the crack discontinuity is introduced as a Heaviside step function via a partition of unity. In addition, branch functions are introduced for all elements containing the crack front. The crack geometry is described by two signed distance functions (level sets), which in turn can be defined by nodal values. Consequently, no explicit representation of the crack is needed. A Hamilton-Jacobi equation is used to update the level sets as the crack grows. Numerical experiments show the robustness of the method in treating cracks with significant changes in topology. The method is readily extendable to inelastic fracture problems.

**Keywords:** Fracture, cracks, discontinuous approximation, finite elements, level sets.

## 1. Introduction

Three dimensional fracture analysis of engineering problem by standard finite element methods is still quite difficult because of the need to construct a mesh which conforms to both the crack surfaces and the surfaces of the component. If the crack surface is not aligned with the element boundaries, the displacement discontinuity and the traction conditions on the crack surface cannot be treated by standard finite element methods. Furthermore, for standard elements, the mesh must be designed so that it is substantially more refined around the crack than in the remainder of the model. The difficulties are further amplified when considering the growth of cracks, because then the model must be remeshed in

the vicinity of the crack. In addition to this, it must be borne in mind that initial cracks in many locations of the component must be considered for a complete engineering analysis.

This paper summarizes a method presented in Moës et al. [17] and Gravouil et al. [11], in which a three dimensional version of the extended finite element method is developed and applied. The extended finite element method alleviates much of the burden associated with mesh generation for objects with cracks by not requiring the finite elements to conform to the crack surface. Moreover, it provides a convenient way for incorporating near-tip asymptotic fields, so that good accuracy can be obtained for elastic fracture with relatively coarse meshes around the crack.

The essential idea in X-FEM is to use a displacement field approximation that can model an arbitrary discontinuity and the near-tip asymptotic crack fields. As a consequence it is often not necessary to modify the mesh to consider a specific crack; at most, moderate refinement must be introduced around the crack to achieve engineering accuracy in elastic fracture mechanics.

The methodology was first presented in Belytschko and Black [2], Moës et al. [16] and Dolbow et al. [8]. It was shown that discontinuous functions can be used to enrich finite element approximations via the partition of unity concept introduced by Melenk and Babuška [15]. The resulting approximation can treat cracks that are arbitrarily aligned in the finite element mesh with great accuracy. The concept was generalized in Daux et al. [6] and in Belytschko et al. [3], which described the application of the concept to arbitrary discontinuities. Sukumar et al. [22] illustrated the potential of combining the extended finite element method with level sets by solving several problems involving inclusions and holes. In Stolarska et al. [21], the extended finite element methodology was combined with a level set method to provide a general method for growing cracks. All of the preceding papers dealt with two dimensional problems.

The first application of the extended finite element method to three-dimensional cracks was Sukumar et al. [24], who solved several planar crack mode I problems and showed that the method compared well with analytical and benchmark solutions. Subsequently Sukumar et al. [23] coupled the method with the fast marching method to solve several planar crack growth problems in three dimensions.

In this paper, the methodology is extended and modified so that it can handle arbitrary cracks in three dimensions. A key development that facilitates treatment of cracks in three dimensions is the description of crack geometry in terms of two signed distance functions. The displacement field is also described in terms of these signed distance functions. This enables us to construct a near-tip asymptotic field with a discontinuity that conforms to the crack even when it is curved or kinked near a tip. Furthermore, it eliminates the need for a surface

model of the crack. As a consequence, no explicit representation of the crack is needed and the crack is entirely described by nodal data. Although the method will here be described for elastic fracture, it is not limited to linear problems and can easily be extended to nonlinear problems.

We cannot list references to all of the competing methods, but we list some recent papers in the following. The remeshing approach appears to be the most advanced for problems of an industrial type; recent accounts are given by Carter et al. [5] and Neto et al. [18]. Methods which rely on boundary element formulations combined with finite elements are given in Dhondt [7], Gerstle et al. [10] and Nishioka and Atluri [19]. Duarte et al. [9] used the partition of unity concept with the visibility criterion to develop methods for dynamic three dimensional crack growth. Three dimensional dynamic crack growth by the element free Galerkin method has been reported by Krysl and Belytschko [13]. The crack surface was represented by a set of triangular elements in 3D, which would be very awkward in a finite element method. The use of finite elements with embedded discontinuities also makes it possible to grow cracks in 2D without remeshing, see Bolzon and Corigliano [4] and Jirasek and Zimmermann [12] for recent works on the topic.

The methodology for treating arbitrary three dimensional cracks and their evolution by X-FEM are described in this and a companion paper. This paper focuses on the description of cracks in three dimensions in terms of level sets, the computation of the elastic solution and the stress intensity factors (SIFs), whereas the companion paper deals with the update of the level sets needed to model crack growth.

The outline of the paper is as follows. In section 2, the methods for defining the crack geometry and the displacement fields are described. The level sets update scheme is given in Section 3. Section 4 reports the numerical experiment of a cracked beam under bending.

## 2. Crack and displacement field description

We consider a body  $\Omega$  with an outer surface  $\Gamma$  and interior crack surfaces  $\Gamma_{\text{cr}}$ . The crack can be treated as a single surface or as two surfaces:  $\Gamma_{\text{cr}}^+$  and  $\Gamma_{\text{cr}}^-$ . In the latter case, the initial crack surfaces are considered coincident and the outward normals to the surface of the crack are denoted by  $\mathbf{n}^+$  and  $\mathbf{n}^-$ , respectively.

Stolarska et al. [21] described a crack geometry in 2D by two signed distance functions. We also use two signed distance functions to describe a crack in 3D as shown in Fig. 1. Note that the definition of the two level set functions is only needed in a neighborhood of the crack. The signed distance function  $\phi(\mathbf{x})$  defines the surface of the crack. It is given by

$$\phi(\mathbf{x}) = \min_{\bar{\mathbf{x}} \in \Gamma_{\text{cr}}^{\text{ext}}} \|\mathbf{x} - \bar{\mathbf{x}}\| \text{sign}(\mathbf{n}^+ \cdot (\bar{\mathbf{x}} - \mathbf{x})) \quad (1)$$

where  $\mathbf{x} = [x, y, z]$  and  $\text{sign}(\cdot)$  is the sign function  $\text{sign}(x) = +1$  if  $x > 0$  and  $-1$  if  $x < 0$ . We also need a smooth extension of the crack surface  $\Gamma_{\text{cr}}$ , denoted by  $\Gamma_{\text{cr}}^{\text{ext}}$ , which includes the entire crack surface, i.e.  $\Gamma_{\text{cr}} \subset \Gamma_{\text{cr}}^{\text{ext}}$ . The distance function (1) gives the shortest distance of any point  $\mathbf{x}$  to the extension of the crack surface  $\Gamma_{\text{cr}}^{\text{ext}}$ . This corresponds to the orthogonal projection of  $\mathbf{x}$  on  $\Gamma_{\text{cr}}^{\text{ext}}$ .

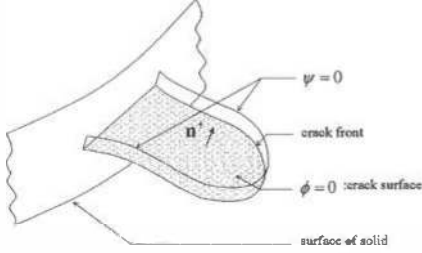


Figure 1. The two iso-zero level sets defining the crack location.

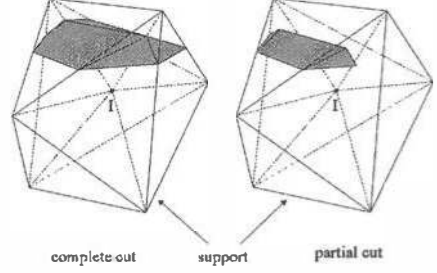


Figure 2. A node  $I$  whose support is completely and partially cut by the crack; the support is the volume of the elements connected to node  $I$  as shown.

Since the crack surface  $\Gamma_{\text{cr}}$  is a bounded surface with a crack front, it is also necessary to define the crack front. This is accomplished by using a second signed distance function,  $\psi(\mathbf{x})$ , which is approximately orthogonal to  $\phi(\mathbf{x})$  so that  $\nabla\phi \cdot \nabla\psi \sim 0$ . The intersection of the surfaces  $\psi(\mathbf{x}) = 0$  and  $\phi(\mathbf{x}) = 0$  gives the crack front; we define the sign of  $\psi$  so that  $\phi(\mathbf{x}) = 0, \psi(\mathbf{x}) < 0$  gives the crack surface  $\Gamma_{\text{cr}}$ .

The signed distance functions in this paper are approximated by the same shape functions as the displacement field. Therefore, in the computations the signed distance functions are given by

$$\phi = \sum_I N_I(\mathbf{x})\phi_I \quad (2)$$

$$\psi = \sum_I N_I(\mathbf{x})\psi_I \quad (3)$$

where  $N_I$  are the finite element shape functions and  $\phi_I$  and  $\psi_I$  are the nodal values of the distance function. This enables the crack shape to be described entirely in terms of nodal values.

The displacement field  $\mathbf{u}(\mathbf{x})$  for the body is decomposed into the continuous and discontinuous parts by

$$\mathbf{u} = \mathbf{u}_{\text{cont}} + \mathbf{u}_{\text{dis}} \quad (4)$$

where  $\mathbf{u}_{\text{cont}}$  is continuous in  $\Omega$ , whereas  $\mathbf{u}_{\text{dis}}$  may have several surfaces of discontinuity in  $\Omega$ . The locations of the discontinuities in  $\mathbf{u}_{\text{dis}}$  are assumed to coincide with  $\Gamma_{\text{cr}}$ .

A standard finite element approximation is used for  $\mathbf{u}_{\text{cont}}$ , i.e.

$$\mathbf{u}_{\text{cont}} = \sum_{I \in \mathcal{N}} N_I(\mathbf{x}) \mathbf{u}_I \quad (5)$$

where  $\mathcal{N}$  is the set of all nodes in the mesh,  $N_I$  are the classical  $C^0$  shape functions and  $\mathbf{u}_I$  are displacement nodal degrees of freedom.

For the purpose of constructing the discontinuous field, the nodes are subdivided into three sets :

- $I \in \mathcal{N}_{\text{cut}}$  : the set of nodes whose support (union of the elements connected to the node) are completely cut into two, i.e. bisected by the crack surface  $\Gamma_{\text{cr}}$ . An example of such a node is shown Figure 2.
- $I \in \mathcal{N}_{\text{branch}}$  : the set of nodes whose support are partially cut by the crack surface  $\Gamma_{\text{cr}}$ , see Figure 2 for an example of such a node.
- $I \in \mathcal{N} - \mathcal{N}_{\text{cut}} - \mathcal{N}_{\text{branch}}$  : the remaining nodes.

The discontinuous displacement fields are given as follows

$$\mathbf{u}_{\text{dis}} = \sum_{I \in \mathcal{N}_{\text{cut}}} N_I(\mathbf{x}) H(\phi(\mathbf{x})) \mathbf{a}_I + \sum_{I \in \mathcal{N}_{\text{branch}}} \sum_{\alpha} N_I(\mathbf{x}) B_{\alpha}(\phi(\mathbf{x}), \psi(\mathbf{x})) \mathbf{a}_{I\alpha}$$

In the above,  $H(\cdot)$  is the Heaviside step function,  $B_{\alpha}(\cdot, \cdot)$  are branch functions, and  $\mathbf{a}_I$  and  $\mathbf{a}_{I\alpha}$  are additional degrees of freedom for the displacement field. The branch functions are constructed in terms of the level sets functions

$$[B_{\alpha}] = [\sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2} \sin \theta, \sqrt{r} \cos \frac{\theta}{2} \sin \theta] \quad (6)$$

where

$$r = \sqrt{\phi^2 + \psi^2}, \quad \theta = \tan^{-1}\left(\frac{\phi}{\psi}\right) \quad (7)$$

Note that the branch functions have been expressed in terms of the level set functions. By expressing the branch functions  $B_{\alpha}$  in terms of the level set functions, it is guaranteed that the discontinuity always corresponds to  $\phi = 0$  and  $\psi < 0$ , i.e. to the surface of the crack. The resulting field will not contain the exact basis of the asymptotic near field, but it is more important to construct the discontinuity in the correct place than to match the exact near-tip asymptotic field.

Only the first of the functions in Eq. (6) is discontinuous across  $\phi = 0$ . The others were added to improve the accuracy in elastic fracture problems. The

above functions span the near-tip asymptotic solution for an elastic crack in two dimensions. In this study and previous studies, see Sukumar et al. [24], we have also found this basis to be quite accurate for three-dimensional cracks, although we have only considered smooth crack fronts.

This technique of adding asymptotic solutions through the partition of unity in finite elements can be considered an asymptotic matching technique. The displacement fields in the other elements provide the far-field, whereas the elements with the branch functions (6) provide the near field. The finite element procedure then matches these fields so that equilibrium is approximately satisfied.

Table 1. Scheme for level set update.

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1 – extend  $V_\psi$  to the domain

$$\frac{\partial V_\psi}{\partial \tau} + \text{sign}(\psi) \frac{\nabla \psi}{\|\nabla \psi\|} \cdot \nabla V_\psi = 0 \quad \frac{\partial V_\psi}{\partial \tau} + \text{sign}(\phi) \frac{\nabla \phi}{\|\nabla \phi\|} \cdot \nabla V_\psi = 0 \quad (8)$$

2 – extend  $V_\phi$  to the domain

$$\frac{\partial V_\phi}{\partial \tau} + \text{sign}(\phi) \frac{\nabla \phi}{\|\nabla \phi\|} \cdot \nabla V_\phi = 0 \quad \frac{\partial V_\phi}{\partial \tau} + \text{sign}(\psi) \frac{\nabla \psi}{\|\nabla \psi\|} \cdot \nabla V_\phi = 0 \quad (9)$$

3 – adjustment to prevent modification of previous crack surface

$$\bar{V}_\phi = H(\psi) \frac{V_\phi \psi}{V_\psi \Delta t} \quad (10)$$

4 – update and reinitialize the  $\phi$  level set

$$\frac{\partial \phi}{\partial t} + \bar{V}_\phi \|\nabla \phi\| = 0 \quad \frac{\partial \phi}{\partial \tau} + \text{sign}(\phi) (\|\nabla \phi\| - 1) = 0 \quad (11)$$

5 – update the  $\psi$  level set

$$\frac{\partial \psi}{\partial t} + \bar{V}_\psi \|\nabla \psi\| = 0 \quad (12)$$

6 – orthogonalize and reinitialize the  $\psi$  level set

$$\frac{\partial \psi}{\partial t} + \text{sign}(\phi) \frac{\nabla \phi}{\|\nabla \phi\|} \cdot \nabla V_\psi = 0 \quad \frac{\partial \psi}{\partial \tau} + \text{sign}(\psi) (\|\nabla \psi\| - 1) = 0 \quad (13)$$


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### 3. Level sets update

The initial level sets representing the crack location are given as data. For many cracks, level set functions which are not signed distance functions can easily be constructed. Then, they may be transformed into true signed distance function through a process called initialization, see Gravouil et al. [11]. The crack velocity on the crack front is given by

$$\mathbf{V} = V_\psi \mathbf{n}_\psi + V_\phi \mathbf{n}_\phi, \quad \mathbf{n}_\psi = \frac{\nabla \psi}{\|\nabla \psi\|}, \quad \mathbf{n}_\phi = \frac{\nabla \phi}{\|\nabla \phi\|} \quad (14)$$

where  $\mathbf{n}_\psi$  and  $\mathbf{n}_\phi$  are the unit normal vectors to the crack front and the crack surface, respectively. The computation of the velocity on the crack front is detailed in the next section. Table 1 summarizes the scheme for the level sets update. In steps 1 and 2, the crack front velocity is extended to all the nodes around the crack front. This extension procedure is classical in the level set method (Sethian [20]). The  $\tau$  indicates a dummy time-like variable whereas the  $t$  symbol in the other steps of Table 1 indicates the true time. Equations (8) and (9) are solved in the “ $\tau$ ” time until  $V_\psi$  and  $V_\phi$  are stationary, i.e. their derivatives with respect to  $\tau$  is zero. The steps in Table 1 are explained in more details in Gravouil et al. [11].

The equations in Table 1 are Hamilton-Jacobi equations of the form

$$\frac{\partial f}{\partial t} + H(\nabla f, \mathbf{x}, t) = 0, \quad f(\mathbf{x}, 0) = f_0(\mathbf{x}) \quad (15)$$

where  $H$  is the Hamiltonian. In order to solve these equations they must be discretized in space and time. The level sets are approximated by finite elements:

$$\phi(\mathbf{x}, t) = \sum_I N_I(\mathbf{x}) \phi_I(t), \quad \psi(\mathbf{x}, t) = \sum_I N_I(\mathbf{x}) \psi_I(t) \quad (16)$$

The velocities are approximated by the same shape function:

$$V_\phi(\mathbf{x}, t) = \sum_I N_I(\mathbf{x}) V_{\phi I}(t), \quad V_\psi(\mathbf{x}, t) = \sum_I N_I(\mathbf{x}) V_{\psi I}(t) \quad (17)$$

The level set method has previously been applied to unstructured meshes by Barth and Sethian [1], and we used the same procedures.

### 4. Numerical experiment

A numerical experiment is performed for the growth of a crack in a beam under bending. The material properties are elastic and isotropic with Young’s modulus  $E = 280$  GPa and Poisson’s ratio  $\nu = 0.3$ . We consider fatigue crack growth



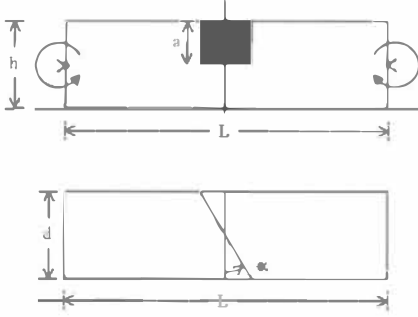


Figure 3. Side and top views of the beam with an initial crack.

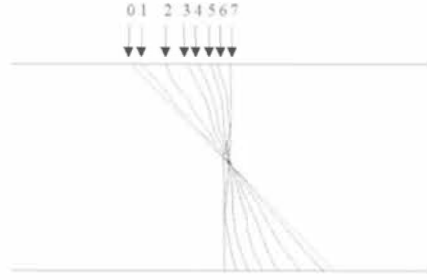


Figure 4. Evolution of the crack front in the beam seen from the top.

governed by the Paris law, which gives the rate of crack growth in mode I in terms of load cycles  $N$  by

$$\frac{da}{dN} = CG^m \quad (18)$$

where  $C$  is a constant to fit experimental results and  $G$  is the maximum energy release rate. We consider the cycles  $N$  a time-like variable so the expression for the crack front velocity is

$$\mathbf{V} = CG^m (\cos \theta_c \mathbf{n}_\psi + \sin \theta_c \mathbf{n}_\phi) \quad (19)$$

where  $\theta_c$ , the angle of the velocity to the plane tangent to the crack (or  $\mathbf{n}_\phi$ ), is obtained by

$$\theta_c = 2 \arctan \frac{1}{4} \left( \frac{K_I}{K_{II}} - \text{sign}(K_{II}) \sqrt{\left( \frac{K_I}{K_{II}} \right)^2 + 8} \right) \quad (20)$$

Thus, the crack growth direction depends on mode  $I$  and  $II$  stress intensity factors, whereas the crack speed on all three through the energy release rate  $G$ . In the numerical study, we chose  $m = 1$  and  $C = 1$ .

We consider the beam shown in Figure 3 subjected to a bending load (see Lazarus et al. [14] for a theoretical study of this problem). The beam dimensions are  $h = 0.02$  m,  $l = 0.1$  m and  $d = 0.01$  m. An initial edge crack along a plane at an angle  $\alpha$  to the plane of the cross-section is used to initiate crack growth with  $a = 0.01$  m and  $\alpha = 45^\circ$  (see Figure 3). The initial geometry is discretized with an unstructured mesh of 2553 nodes and 13393 tetrahedrons. It is emphasized that the mesh does not conform to the crack geometry, and that the same mesh is used throughout the simulation. The crack is driven by Paris fatigue law with the maximum circumferential stress hypothesis. for the direction of crack

propagation. From 15 to 25 points were used on the crack front to compute the velocity and the stress intensity factors. (The stress intensity factor computation is detailed in Moës et al. [17] ). In this study, the beam is completely cut into two after 17 time steps. Figure 4 shows the evolution of the crack front from a top view. We can observe that the crack grows asymptotically to a plane orthogonal to the axis of the beam.

Figure 5 shows the crack after 12 time steps, and the vector velocity field on the crack front. The colors give the value of the signed distance function  $\psi$  which defines the signed distance from the crack front. An important remark concerns the topological properties of the method: with the level set approach, the crack does not need any specific treatment on the boundary, and the crack can completely cut the beam.

Figure 6 shows the evolution of a semi-circular crack. Again, we observe that the crack grows asymptotically to plane orthogonal to the axis of the beam. Here, the crack front first grows to the lateral free surfaces, then normal to the surface. No changes are needed in the algorithm to account for these topological changes. Other crack propagation examples may be found in Gravouil et al. [11].

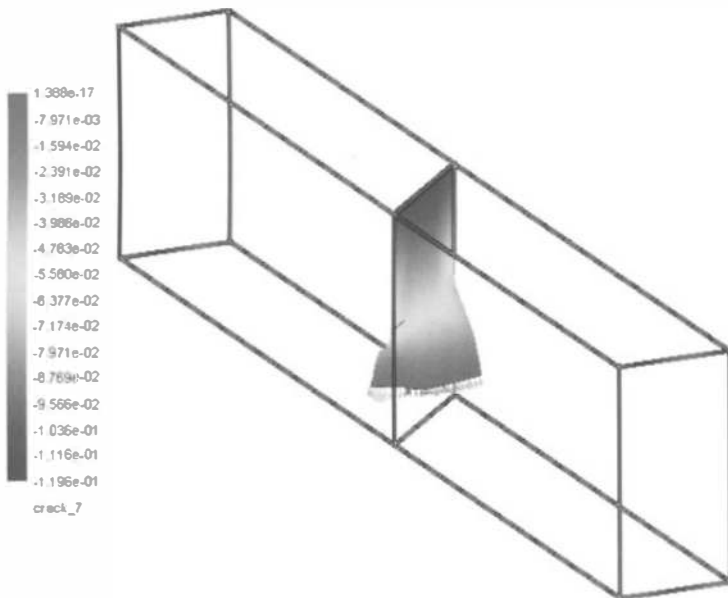


Figure 5. Crack location after 12 times steps.

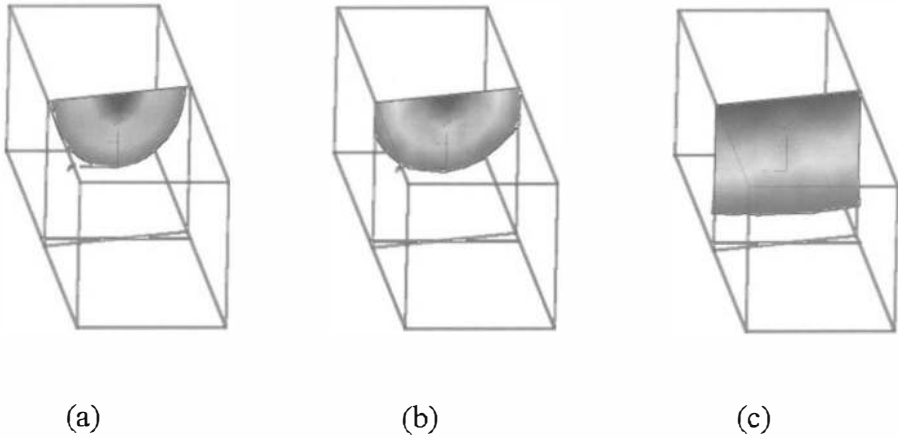


Figure 6. Evolution of a semi-circular crack in a beam.

## 5. Conclusions

A level set method for arbitrary non-planar cracks in three-dimensional bodies has been presented. The level set technique couples naturally to the extended finite element method, wherein the discontinuous and near-tip asymptotic fields are constructed through a partition of unity. The resulting combined method requires no explicit representation of the crack except in its visualization. Instead, the crack and its growth are described entirely in forms of nodal data. This simplifies the structure of the software and leads to great versatility in treating complex problems in crack growth.

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