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# ANALOGICAL PROPORTIONS II

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**ABSTRACT.** Analogical reasoning is the ability to detect parallels between two seemingly distant objects or situations, a fundamental human capacity used for example in commonsense reasoning, learning, and creativity which is believed by many researchers to be at the core of human and artificial general intelligence. Analogical proportions are expressions of the form “ $a$  is to  $b$  what  $c$  is to  $d$ ” at the core of analogical reasoning. The author has recently introduced an abstract algebraic framework of analogical proportions within the general setting of universal algebra. It is the purpose of this paper to further develop the mathematical theory of analogical proportions within that framework as motivated by the fact that it has already been successfully applied to logic program synthesis in artificial intelligence.

## 1. INTRODUCTION

**Analogical reasoning** is the ability to detect parallels between two seemingly distant objects or situations, a fundamental human capacity used for example in commonsense reasoning, learning, and creativity which is believed by many researchers to be at the core of human and artificial general intelligence (see e.g. Gentner, Holyoak, & Kokinov, 2001; Gentner, 2012; Gust, Krumnack, Kühnberger, & Schwering, 2008; Hofstadter & Sander, 2013; Krieger, 2003; Pólya, 1954). Notable models of analogical reasoning are Gentner’s (1983) prominent *structure-mapping theory* (SMT) and its implementation within the *structure-mapping engine* (Falkenhainer, Forbus, & Gentner, 1989) and Hofstadter and Mitchell’s (1995) *copycat* algorithm. A formal model using second-order logic similar to SMT is *heuristic-driven theory projection* (Gust, Kühnberger, & Schmid, 2006). Winston (1980) is a classic paper demonstrating the use of analogy in learning. A formalization of analogical reasoning in law (Koszowski, 2019) using Gentzen’s sequent calculus is provided by Baaz (2005). For a short and somewhat outdated introduction to analogical reasoning we refer the reader to Prade and Richard (2014), and for a historic account of models of analogical reasoning we refer the reader to Hall (1989).

**Analogical proportions** are expressions of the form “ $a$  is to  $b$  what  $c$  is to  $d$ ” — written  $a : b :: c : d$  — at the core of analogical reasoning. Formal models of analogical proportions started to appear only recently, most notably Lepage’s (2001, 2003) axiomatic approach in the linguistic setting, Miclet and Prade’s (2009) logical approach in the 2-element boolean setting (cf. Prade & Richard, 2013, 2018), and Stroppa and Yvon’s (2006) and Gust et al.’s (2006) algebraic approaches. Barbot, Miclet, and Prade (2019) use analogical proportions to formalize analogies between concepts. See (Prade & Richard, 2021) for a short summary of applications of analogical proportions to AI (and see Correa, Prade, & Richard, 2012).

The author has recently introduced an abstract algebraic framework of analogical proportions in the general setting of universal algebra (Antić, 2022). It is **justification-based** in nature and thus part of the emerging field of Explainable AI (see e.g. Héder, 2023). The **purpose of this paper** is

to contribute to the **mathematical foundations** of artificial general intelligence by further developing the mathematical theory of that framework. This is **motivated** by the fact that it has already been successfully applied to **logic program synthesis** for artificial intelligence in Antić (2023d), and by the fact that it is capable of capturing two different prominent modellings of a **boolean proportion** given by Klein (1982) and Miclet and Prade (2009) in a single framework (Antić, 2023c). For an analysis of the framework in monounary algebras see Antić (2023b).

Specifically, we first observe in §3 that sets of justifications are principal filters, which motivates a peculiar change of notation. We then introduce some terminology which makes the Uniqueness Lemma and Functional Proportion Theorem in Antić (2022) — which are essential cornerstones of the framework — easier to apprehend.

We then prove in §4 a **Homomorphism Theorem** as a generalization of the First Isomorphism Theorem in Antić (2022) showing that *arrow* proportions — which are expressions of the form  $a \rightarrow b : \cdot c \rightarrow d$  (read as “ $a$  transforms into  $b$  as  $c$  transforms into  $d$ ”, see §2) — are compatible with homomorphisms.

In §5, we initiate the study of **fragments** of the framework where the form of justifications is syntactically restricted. Particularly, in §5.1–5.2, we show that we can capture **difference and geometric proportions** already in the simplest *monolinear* fragment consisting only of justifications with at most one occurrence of a single variable. This implies that Stroppa and Yvon’s (2006) notions of arithmetical proportions coincides with monolinear arithmetical proportions in our framework and is thus too restrictive to be considered *the* notion of arithmetical proportions. Moreover, in §5.3 we study monolinear word proportions.

Analogical proportion between words have found applications to computational linguistics and natural language processing and have been studied in that context by a number of authors (see e.g. Hofstadter & Mitchell, 1995; Lepage, 2001, 2003; Lim, Prade, & Richard, 2021; Stroppa & Yvon, 2006). In §6, we therefore study **word proportions** where in Theorem 42 and Example 43 we show that our framework interpreted in the word domain strictly generalizes the often used notion of a word proportion in Stroppa and Yvon (2006). This is particularly interesting as our framework has not been geared towards the word domain.

Trees are often used in computational linguistics and natural language processing for the representation of the internal structure of complex words and sentences and in first-order predicates used in the representation of the semantics of linguistic entities (see e.g. §2.2 in Stroppa & Yvon, 2006). In §7, we therefore study **tree proportions** and show that classical syntactic anti-unification (Reynolds, 1970; Plotkin, 1970; Huet, 1976) can be used to compute tree proportions via a simple syntactic check (Theorem 50).

Motivated by the previous observation, in §8 we show that algebraic **anti-unification** — which the author has recently introduced within the setting of universal algebra in Antić (2023a) — is related to analogical proportions by giving an illustrative Example 53. We shall point out that other authors have noted the connection between analogical proportions and anti-unification in other settings before: Krumnack et al. (2007) is a paper dealing with restricted higher-order anti-unification and analogy making, and Weller and Schmid (2007) use anti-unification for computing analogical proportions using regular tree grammars.

Finally, in §9 we show that analogical proportions can be computed in **finite algebras** using **tree automata** where we provide algorithms for the most important computational tasks.

## 2. PRELIMINARIES

This section recalls the abstract algebraic framework of analogical proportions in Antić (2022) where we assume the reader to be fluent in basic universal algebra as it is presented for example in Burris and Sankappanavar (2000, §II).

We denote the *empty word* by  $\varepsilon$ . As usual, we denote the set of all words over an alphabet  $A$  by  $A^*$  and define  $A^+ := A^* \cup \{\varepsilon\}$ . The *reverse* of a word  $\mathbf{a} = a_1 \dots a_n$ ,  $n \geq 1$ , is given by  $\mathbf{a}^r := a_n \dots a_1$ .

A *language*  $L$  of algebras is a set of *function symbols*<sup>1</sup> together with a *rank function*  $r : L \rightarrow \mathbb{N}$ , and a denumerable set  $X$  of *variables* distinct from  $L$ . Terms are formed as usual from variables and function symbols.

An *L-algebra*  $\mathfrak{A}$  consists of a non-empty set  $A$ , the *universe* of  $\mathfrak{A}$ , and for each function symbol  $f \in L$ , a function  $f^{\mathfrak{A}} : A^{rf} \rightarrow A$ , the *functions* of  $\mathfrak{A}$  (the *distinguished elements* of  $\mathfrak{A}$  are the 0-ary functions). Every term  $s$  induces a function  $s^{\mathfrak{A}}$  on  $\mathfrak{A}$  in the usual way. We call a term  $s$  *injective* in  $\mathfrak{A}$  iff  $s^{\mathfrak{A}}$  is an injective function. We denote the variables occurring in  $s$  by  $Xs$ .

Given a language  $L$  and set of variables  $X$ , the *term algebra*  $\mathfrak{T}_{LX}$  over  $L$  and  $X$  has as universe the set  $T_{LX}$  of all terms over  $L$  and  $X$  and each function symbol  $f \in L$  is interpreted by itself, that is,

$$f^{\mathfrak{T}_{LX}}(p_1, \dots, p_{rf}) := f(p_1, \dots, p_{rf}).$$

In the sequel, we do not distinguish between a function symbol  $f$  and its induced function  $f^{\mathfrak{T}_{LX}}$  and we call both *term functions*.

We say that a term  $t$  is a *generalization* of  $s$  — in symbols,  $s \lesssim t$  — iff there is a substitution  $\sigma$  such that  $s = t\sigma$ . The generalization ordering is reflexive and transitive and thus a pre-order. It is well-known that any two terms  $p$  and  $q$  have a *least general generalization* (Plotkin, 1970; Reynolds, 1970) in  $\mathfrak{T}_{LX}$  — in symbols,  $p \uparrow q$  — computed by Huet's (1976) algorithm as follows. Given an injective mapping  $\chi : T_{LX} \times T_{LX} \rightarrow X$  and two  $L$ -terms  $p, q \in T_{LX}$ , if

$$p = f(p_1, \dots, p_{rf}) \quad \text{and} \quad q = f(q_1, \dots, q_{rf}),$$

for some  $f \in L$  and  $p_1, \dots, p_{rf}, q_1, \dots, q_{rf} \in T_{LX}$ , then define

$$p \uparrow_{\chi} q := f(p_1, \dots, p_{rf}) \uparrow_{\chi} f(q_1, \dots, q_{rf}) := f(p_1 \uparrow_{\chi} q_1, \dots, p_{rf} \uparrow_{\chi} q_{rf});$$

otherwise define

$$p \uparrow_{\chi} q := p\chi q.$$

Notice that since each constant symbol  $a$  induces an 0-ary term function  $a()$ , we have

$$a \uparrow_{\chi} a = a() \uparrow_{\chi} a() = a.$$

**Convention 1.** We will always write  $s \rightarrow t$  instead of  $(s, t)$ , for any pair of  $L$ -terms  $s$  and  $t$  such that every variable in  $t$  occurs in  $s$ , that is,  $Xt \subseteq Xs$ . We call such expressions *L-rewrite rules* or *L-justifications* where we often omit the reference to  $L$ . We denote the set of all  $L$ -justifications with variables among  $X$  by  $J_{LX}$ . We make the convention that  $\rightarrow$  binds weaker than every other algebraic operation.

**Definition 2.** We define the *analogical proportion relation* as follows:

- (1) Define the *set of justifications* of an *arrow*  $a \rightarrow b$  in  $\mathfrak{A}$  by<sup>2</sup>

$$J_{s\mathfrak{A}}(a \rightarrow b) := \left\{ s \rightarrow t \in J_{LX} \mid a \rightarrow b = s^{\mathfrak{A}}\mathbf{o} \rightarrow t^{\mathfrak{A}}\mathbf{o}, \text{ for some } \mathbf{o} \in A^{|\mathbf{x}|} \right\},$$

<sup>1</sup>We omit constant symbols as we identify constants with 0-ary functions.

<sup>2</sup>For a sequence of objects  $\mathbf{o} = o_1 \dots o_n$  define  $|\mathbf{o}| := n$ .

extended to an **arrow proportion**  $a \rightarrow b : \cdot c \rightarrow d^3$  in  $\mathfrak{A}\mathfrak{B}$  by

$$Jus_{\mathfrak{A}\mathfrak{B}}(a \rightarrow b : \cdot c \rightarrow d) := Jus_{\mathfrak{A}}(a \rightarrow b) \cap Jus_{\mathfrak{B}}(c \rightarrow d).$$

- (2) A justification is **trivial** in  $\mathfrak{A}\mathfrak{B}$  iff it justifies every arrow proportion in  $\mathfrak{A}\mathfrak{B}$  and we denote the set of all such trivial justifications by  $\emptyset_{\mathfrak{A}\mathfrak{B}}$  simply written  $\emptyset$  in cases where the underlying algebras are understood from the context. Moreover, we say that  $J$  is a **trivial set of justifications** in  $\mathfrak{A}\mathfrak{B}$  iff every justification in  $J$  is trivial.
- (3) Now we say that  $a \rightarrow b : \cdot c \rightarrow d$  **holds** in  $\mathfrak{A}\mathfrak{B}$  — in symbols,

$$a \rightarrow b : \cdot_{\mathfrak{A}\mathfrak{B}} c \rightarrow d$$

iff

- (a) either  $Jus_{\mathfrak{A}}(a \rightarrow b) \cup Jus_{\mathfrak{B}}(c \rightarrow d) = \emptyset_{\mathfrak{A}\mathfrak{B}}$  consists only of trivial justifications, in which case there is neither a non-trivial relation from  $a$  to  $b$  in  $\mathfrak{A}$  nor from  $c$  to  $d$  in  $\mathfrak{B}$ ; or
- (b)  $Jus_{\mathfrak{A}\mathfrak{B}}(a \rightarrow b : \cdot c \rightarrow d)$  is maximal with respect to subset inclusion among the sets  $Jus_{\mathfrak{A}\mathfrak{B}}(a \rightarrow b : \cdot c \rightarrow d')$ ,  $d' \in B$ , containing at least one non-trivial justification, that is, for any element  $d' \in B$ ,<sup>4</sup>

$$\emptyset_{\mathfrak{A}\mathfrak{B}} \subsetneq Jus_{\mathfrak{A}\mathfrak{B}}(a \rightarrow b : \cdot c \rightarrow d) \subseteq Jus_{\mathfrak{A}\mathfrak{B}}(a \rightarrow b : \cdot c \rightarrow d')$$

implies

$$\emptyset_{\mathfrak{A}\mathfrak{B}} \subsetneq Jus_{\mathfrak{A}\mathfrak{B}}(a \rightarrow b : \cdot c \rightarrow d') \subseteq Jus_{\mathfrak{A}\mathfrak{B}}(a \rightarrow b : \cdot c \rightarrow d).$$

We abbreviate the above definition by simply saying that  $Jus_{\mathfrak{A}\mathfrak{B}}(a \rightarrow b : \cdot c \rightarrow d)$  is ***d*-maximal**.

- (4) Finally, the analogical proportion entailment relation is most succinctly defined by

$$a : b ::_{\mathfrak{A}\mathfrak{B}} c : d \quad :\Leftrightarrow \quad a \rightarrow b : \cdot_{\mathfrak{A}\mathfrak{B}} c \rightarrow d \quad \text{and} \quad b \rightarrow a : \cdot_{\mathfrak{A}\mathfrak{B}} d \rightarrow c \\ c \rightarrow d : \cdot_{\mathfrak{B}\mathfrak{A}} a \rightarrow b \quad \text{and} \quad d \rightarrow c : \cdot_{\mathfrak{B}\mathfrak{A}} b \rightarrow a.$$

We will always write  $\mathfrak{A}$  instead of  $\mathfrak{A}\mathfrak{A}$ . Moreover, we define

$$\mathfrak{S}_{\mathfrak{A}\mathfrak{B}}(a : b :: c : v) := \{d \in B \mid a : b ::_{\mathfrak{A}\mathfrak{B}} c : d\}.$$

**Example 3** (Antić (2022), Example 11). First consider the algebra  $\mathfrak{A}_1 := (\{a, b, c, d\})$ , consisting of four distinct elements with no functions and no constants:

$$\begin{array}{cc} b & d \\ & \\ a & c \end{array}$$

Since  $Jus_{\mathfrak{A}_1}(a' \rightarrow b') \cup Jus_{\mathfrak{A}_1}(c' \rightarrow d')$  contains only trivial justifications for *any distinct* elements  $a', b', c', d' \in A'$ , we have, for example:

$$a : b ::_{\mathfrak{A}_1} c : d \quad \text{and} \quad a : c ::_{\mathfrak{A}_1} b : d.$$

On the other hand, since

$$Jus_{\mathfrak{A}_1}(a \rightarrow a) \cup Jus_{\mathfrak{A}_1}(a \rightarrow d) = \{x \rightarrow x\} \neq \emptyset$$

and

$$\emptyset = Jus_{\mathfrak{A}_1}(a \rightarrow a : \cdot a \rightarrow d) \subsetneq Jus_{\mathfrak{A}_1}(a \rightarrow a : \cdot a \rightarrow a) = \{x \rightarrow x\},$$

<sup>3</sup>Read as “ $a$  transforms into  $b$  as  $c$  transforms into  $d$ ”.

<sup>4</sup>We ignore trivial justifications and write “ $\emptyset \subsetneq \dots$ ” instead of “ $\{\text{trivial justifications}\} \subsetneq \dots$ ” et cetera.

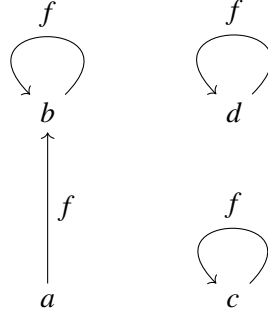
we have

$$a \rightarrow a ::_{\mathfrak{A}_1} a \rightarrow d,$$

which implies

$$a : a ::_{\mathfrak{A}_1} a : d.$$

Now consider the slightly different algebra  $\mathfrak{A}_2 := (\{a, b, c, d\}, f)$ , where  $f$  is the unary function defined by



We expect the proportion  $a : b :: c : d$  to fail in  $\mathfrak{A}_2$  as it has no non-trivial justification. In fact,

$$Jus_{\mathfrak{A}_2}(a \rightarrow b) \cup Jus_{\mathfrak{A}_2}(c \rightarrow d) = \{x \rightarrow f^n x \mid n \geq 1\} \neq \emptyset$$

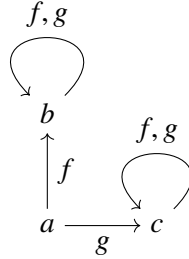
and

$$Jus_{\mathfrak{A}_2}(a \rightarrow b :: c \rightarrow d) = \emptyset$$

show

$$a : b ::_{\mathfrak{A}_2} c : d.$$

In the algebra  $\mathfrak{A}_3$  given by



we have

$$a : b \not\vdash_{\mathfrak{A}_3} a : c.$$

The intuitive reason is that  $a : b :: a : b$  is a more plausible proportion than  $a : b :: a : c$ , which is reflected in the computation

$$\emptyset = Jus_{\mathfrak{A}_3}(a \rightarrow b :: a \rightarrow c) \subsetneq Jus_{\mathfrak{A}_3}(a \rightarrow b :: a \rightarrow b) = \{x \rightarrow fx, \dots\}.$$

Computing all justifications of an arrow proportion is difficult in general, which fortunately can be omitted in many cases:

**Definition 4.** We call a set  $J$  of justifications a *characteristic set of justifications* of  $a \rightarrow b :: c \rightarrow d$  in  $\mathfrak{A}\mathfrak{B}$  iff  $J$  is a sufficient set of justifications in the sense that

- (1)  $J \subseteq Jus_{\mathfrak{A}\mathfrak{B}}(a \rightarrow b : \cdot c \rightarrow d)$ , and  
 (2)  $J \subseteq Jus_{\mathfrak{A}\mathfrak{B}}(a \rightarrow b : \cdot c \rightarrow d')$  implies  $d' = d$ , for each  $d' \in B$ .

In case  $J = \{s \rightarrow t\}$  is a singleton set satisfying both conditions, we call  $s \rightarrow t$  a **characteristic justification** of  $a \rightarrow b : \cdot c \rightarrow d$  in  $\mathfrak{A}\mathfrak{B}$ .

In the tradition of the ancient Greeks, Lepage (2003) introduced (in the linguistic context) a set of proportional properties as a guideline for formal models of analogical proportions. His proposed list has since been extended by a number of authors and can now be summarized as follows:

$$\begin{aligned}
 a : b ::_{\mathfrak{A}} a : b & \quad (\text{p-reflexivity}), \\
 a : b ::_{\mathfrak{A}\mathfrak{B}} c : d & \Leftrightarrow c : d ::_{(\mathfrak{B}, \mathfrak{A})} a : b \quad (\text{p-symmetry}), \\
 a : b ::_{\mathfrak{A}\mathfrak{B}} c : d & \Leftrightarrow b : a ::_{\mathfrak{A}\mathfrak{B}} d : c \quad (\text{inner p-symmetry}), \\
 a : a ::_{\mathfrak{A}} a : d & \Leftrightarrow d = a \quad (\text{p-determinism}), \\
 a : a ::_{\mathfrak{A}\mathfrak{B}} c : c & \quad (\text{inner p-reflexivity}), \\
 a : b ::_{\mathfrak{A}} c : d & \Leftrightarrow a : c ::_{\mathfrak{A}} b : d \quad (\text{central permutation}), \\
 a : a ::_{\mathfrak{A}} c : d & \Rightarrow d = c \quad (\text{strong inner p-reflexivity}), \\
 a : b ::_{\mathfrak{A}} a : d & \Rightarrow d = b \quad (\text{strong p-reflexivity}).
 \end{aligned}$$

Moreover, the following property is considered, for  $a, b \in A \cap B$ :

$$a : b ::_{\mathfrak{A}\mathfrak{B}} b : a \quad (\text{p-commutativity}).$$

Furthermore, the following properties are considered, for  $L$ -algebras  $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$  and elements  $a, b \in A$ ,  $c, d \in B$ ,  $e, f \in C$ :

$$\frac{a : b ::_{\mathfrak{A}\mathfrak{B}} c : d \quad c : d ::_{\mathfrak{B}\mathfrak{C}} e : f}{a : b ::_{\mathfrak{A}\mathfrak{C}} e : f} \quad (\text{p-transitivity}),$$

and, for elements  $a, b, e \in A$  and  $c, d, f \in B$ , the property

$$\frac{a : b ::_{\mathfrak{A}\mathfrak{B}} c : d \quad b : e ::_{\mathfrak{A}\mathfrak{B}} d : f}{a : e ::_{\mathfrak{A}\mathfrak{B}} c : f} \quad (\text{inner p-transitivity}),$$

and, for elements  $a \in A$ ,  $b \in A \cap B$ ,  $c \in B \cap C$ , and  $d \in C$ , the property

$$\frac{a : b ::_{\mathfrak{A}\mathfrak{B}} b : c \quad b : c ::_{\mathfrak{B}\mathfrak{C}} c : d}{a : b ::_{\mathfrak{A}\mathfrak{C}} c : d} \quad (\text{central p-transitivity}).$$

Notice that central p-transitivity follows from p-transitivity.

Finally, the following schema is considered, where  $\mathfrak{A}'$  and  $\mathfrak{B}'$  are  $L'$ -algebras, for some language  $L \subseteq L'$ :

$$\frac{a : b ::_{\mathfrak{A}\mathfrak{B}} c : d \quad \mathfrak{A} = \mathfrak{A}' \upharpoonright L \quad \mathfrak{B} = \mathfrak{B}' \upharpoonright L}{a : b ::_{\mathfrak{A}'\mathfrak{B}'} c : d} \quad (\text{p-monotonicity}).$$

### 3. JUSTIFICATIONS

In this section, we observe that sets of justifications are principal filters, which will motivate a change of notation by replacing  $Jus$  by  $\uparrow$  thus expressing syntactically the close connection to generalizations more adequately.

Recall that a **filter**  $F$  on a pre-ordered set  $(P, \leq)$  is a subset of  $P$  satisfying:

- (1)  $F$  is non-empty.
- (2)  $F$  is downward directed, that is, for every  $a, b \in F$ , there is some  $c \in F$  such that  $c \leq a, b$ .
- (3)  $F$  is an upper set or upward closed, that is, for every  $a \in F$  and  $b \in P$ , if  $a \leq b$  then  $b \in F$ .

The smallest filter containing an element  $a$  is a **principal filter** and  $a$  is a **principal element** — it is given by

$$\uparrow_{(P, \leq)} a := \{b \in P \mid a \leq b\}.$$

We extend the generalization pre-ordering from terms to justification via

$$s \rightarrow t \lesssim s' \rightarrow t' \iff s \lesssim s' \text{ and } t \lesssim t'.$$

**Fact 5.** *The set of all generalizations of a term forms a principal filter with respect to the generalization pre-ordering generated by that term. Moreover, the set of all justifications of an arrow forms a principal filter with respect to the generalization pre-ordering generated by that justification.*

**Notation 6.** Fact 5 motivates the following notation which we will use in the rest of the paper:

$$\uparrow_{\mathfrak{A}} (a \rightarrow b) := Jus_{\mathfrak{A}}(a \rightarrow b),$$

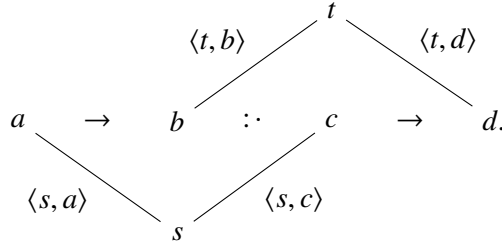
extended to an arrow proportion by

$$\uparrow_{\mathfrak{A}\mathfrak{B}} (a \rightarrow b : \cdot c \rightarrow d) := Jus_{\mathfrak{A}\mathfrak{B}}(a \rightarrow b : \cdot c \rightarrow d).$$

We shall now reformulate some key results in Antić (2022) using a different — hopefully more intuitive — terminology. For this, we first define, for a term  $s \in T_{LX}$  and element  $a \in A$ , the set

$$\langle s, a \rangle_{\mathfrak{A}} := \{\mathbf{o} \in A^{rs} \mid a = s^{\mathfrak{A}} \mathbf{o}\},$$

consisting of all solutions to the polynomial equation  $a = s(\mathbf{x})$  in  $\mathfrak{A}$ . We can now depict every justification  $s \rightarrow t$  of  $a \rightarrow b : \cdot c \rightarrow d$  as follows (see Antić, 2022, Convention 15):



Moreover, we have

$$(1) \quad s \rightarrow t \in \uparrow (a \rightarrow b : \cdot c \rightarrow d) \iff \langle s, a \rangle \cap \langle t, b \rangle \neq \emptyset \text{ und } \langle s, c \rangle \cap \langle t, d \rangle \neq \emptyset.$$

Define

$$\mathbb{1}_{\mathfrak{A}}(s) := \{a \in A \mid |\langle s, a \rangle_{\mathfrak{A}}| = 1\}.$$

We can now reformulate the rather opaque Uniqueness Lemma and Functional Proportion Theorem in Antić (2022) more cleanly using the above notions:

**Lemma 7** (Uniqueness Lemma). *We have the following implications:*

$$\frac{s \rightarrow t \in \uparrow_{\mathfrak{A}\mathfrak{B}} (a \rightarrow b : \cdot c \rightarrow d) \quad c \in \mathbb{1}_{\mathfrak{B}}(s)}{a \rightarrow b : \cdot_{\mathfrak{A}\mathfrak{B}} c \rightarrow d}$$

and

$$\frac{s \rightarrow t \in \uparrow_{\mathfrak{A}\mathfrak{B}} (a \rightarrow b : \cdot c \rightarrow d) \quad a \in \mathbb{1}_{\mathfrak{A}}(s) \quad b \in \mathbb{1}_{\mathfrak{A}}(t) \quad c \in \mathbb{1}_{\mathfrak{B}}(s) \quad d \in \mathbb{1}_{\mathfrak{B}}(t)}{a : b : \cdot_{\mathfrak{A}\mathfrak{B}} c : d}.$$

**Theorem 8** (Functional Proportion Theorem). *For any L-term  $tx$ , we have the following implication:*



$$\frac{a \in \mathbb{1}_{\mathfrak{A}}(t) \quad c \in \mathbb{1}_{\mathfrak{B}}(t)}{a : t^{\mathfrak{A}}(a) ::_{\mathfrak{A}\mathfrak{B}} c : t^{\mathfrak{B}}(c)}.$$

In this case, we call  $t^{\mathfrak{B}}(c)$  a **functional solution** of  $a : b :: c : v$  in  $\mathfrak{A}\mathfrak{B}$  characteristically justified by  $x \rightarrow tx$ . This holds in particular if  $t$  is injective in  $\mathfrak{A}$  and  $\mathfrak{B}$ .

#### 4. HOMOMORPHISM THEOREM

Analogical proportions are compatible with structure-preserving mappings in the following way: The First Isomorphism Theorem in Antić (2022) says that for any isomorphism  $H : \mathfrak{A} \rightarrow \mathfrak{B}$ ,

$$a : b ::_{\mathfrak{A}\mathfrak{B}} Ha : Hb, \quad \text{for all } a, b \in A.$$

A simple counterexample shows that homomorphisms are in general *not* compatible with analogical proportions in the same way (see Example 11). In this section, we shall recover a part of the result by showing that homomorphisms are compatible with *arrow* proportions. We first show an auxiliary lemma (analogous to the Isomorphism Lemma in Antić (2022)):

**Lemma 9** (Homomorphism Lemma). *For any homomorphism  $H : \mathfrak{A} \rightarrow \mathfrak{B}$  and  $a, b \in A$ ,*

$$(2) \quad \uparrow_{\mathfrak{A}}(a \rightarrow b) \subseteq \uparrow_{\mathfrak{B}}(Ha \rightarrow Hb).$$

*In case  $H$  is an isomorphism, we have*

$$(3) \quad \uparrow_{\mathfrak{A}}(a \rightarrow b) = \uparrow_{\mathfrak{B}}(Ha \rightarrow Hb).$$

*Proof.* We have

$$\begin{aligned} s \rightarrow t \in \uparrow_{\mathfrak{A}}(a \rightarrow b) &\Leftrightarrow a \rightarrow b = s^{\mathfrak{A}}\mathbf{o} \rightarrow t^{\mathfrak{A}}\mathbf{o}, \quad \text{for some } \mathbf{o} \in A^{rs} \\ &\Rightarrow Ha \rightarrow Hb = Hs^{\mathfrak{A}}\mathbf{o} \rightarrow Ht^{\mathfrak{A}}\mathbf{o} = s^{\mathfrak{B}}H\mathbf{o} \rightarrow t^{\mathfrak{B}}H\mathbf{o} \\ &\Rightarrow s \rightarrow t \in \uparrow_{\mathfrak{B}}(Ha \rightarrow Hb). \end{aligned}$$

The second part has already been shown in the proof of Antić (2022, Isomorphism Lemma).  $\square$

**Theorem 10** (Homomorphism Theorem). *For any homomorphism  $H : \mathfrak{A} \rightarrow \mathfrak{B}$  and elements  $a, b \in A$ , we have the following implication:*

$$\frac{\uparrow_{\mathfrak{A}}(a \rightarrow b) = \emptyset_{\mathfrak{A}} \quad \Rightarrow \quad \uparrow_{\mathfrak{B}}(Ha \rightarrow Hb) = \emptyset_{\mathfrak{B}}}{a \rightarrow b : \cdot_{\mathfrak{A}\mathfrak{B}} Ha \rightarrow Hb}.$$

*Proof.* By the Homomorphism Lemma 9, we have

$$\uparrow_{\mathfrak{A}\mathfrak{B}}(a \rightarrow b : \cdot Ha \rightarrow Hb) = \uparrow_{\mathfrak{A}}(a \rightarrow b) \cap \uparrow_{\mathfrak{B}}(Ha \rightarrow Hb) = \uparrow_{\mathfrak{A}}(a \rightarrow b),$$

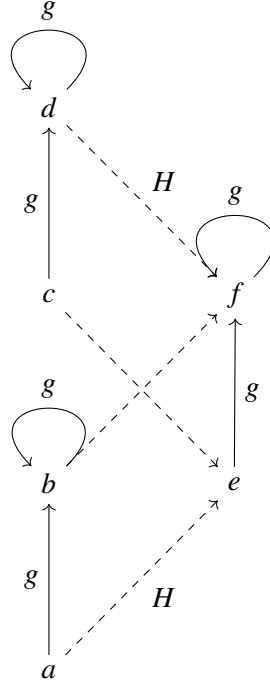
which shows the  $Hb$ -maximality of  $\uparrow_{\mathfrak{A}\mathfrak{B}}(a \rightarrow b : \cdot Ha \rightarrow Hb)$ .

It remains to show that we cannot have

$$\uparrow_{\mathfrak{A}}(a \rightarrow b) \cup \uparrow_{\mathfrak{B}}(Ha \rightarrow Hb) \neq \emptyset_{\mathfrak{A}\mathfrak{B}} \quad \text{whereas} \quad \uparrow_{\mathfrak{A}\mathfrak{B}}(a \rightarrow b : \cdot Ha \rightarrow Hb) = \emptyset_{\mathfrak{A}\mathfrak{B}}.$$

This is a direct consequence of (2) and the assumption that  $\uparrow_{\mathfrak{A}}(a \rightarrow b) = \emptyset_{\mathfrak{A}}$  implies  $\uparrow_{\mathfrak{B}}(Ha \rightarrow Hb) = \emptyset_{\mathfrak{B}}$ .  $\square$

**Example 11.** Let us now analyze the counterexample in Antić (2022, Example 39). Let  $\mathfrak{A} := \{a, b, c, d\}$ ,  $\mathfrak{B} := \{e, f, g\}$ , and  $H : \mathfrak{A} \rightarrow \mathfrak{B}$  be given by



In Antić (2022, Example 39) it is shown that

$$a \rightarrow d \not\vdash Ha \rightarrow Hd.$$

This is not a contradiction to the Homomorphism Theorem 10, since

$$\uparrow (a \rightarrow d) = \emptyset \quad \Leftrightarrow \quad \uparrow (Ha \rightarrow Hd) = \emptyset,$$

shows that we *cannot* apply the theorem. What we do have is

$$a \rightarrow b \vdash Ha \rightarrow Hb \quad \text{and} \quad c \rightarrow d \vdash Hc \rightarrow Hd.$$

In fact, we even have

$$a : b :: Ha : Hb \quad \text{and} \quad c : d :: Hc : Hd.$$

Antić (2023e) introduces the notion of a **proportional homomorphism**  $H : \mathfrak{A} \rightarrow \mathfrak{B}$  satisfying

$$a : b ::_{\mathfrak{A}} c : d \quad \Leftrightarrow \quad Ha : Hb ::_{\mathfrak{B}} Hc : Hd, \quad \text{for all } a, b, c, d \in A.$$

**Theorem 12** (Isomorphism Theorem). *Every isomorphism is a proportional homomorphism.*

*Proof.* A direct consequence of the Homomorphism Lemma 9. □

**Remark 13.** The Isomorphism Theorem shows that isomorphisms are **analogy-preserving functions** in the sense of Couceiro and Lehtonen (2023, Definition 6) which are a variant of the **analogical jump** in Davies and Russell (1987).

## 5. THE $(k, \ell)$ -FRAGMENTS

Since computing the set of *all* justifications is rather difficult in general, it is reasonable to study fragments of the framework. For this, we introduce in this section the  $(k, \ell)$ -fragments:

**Definition 14.** Let  $X_k := \{x_1, \dots, x_k\}$ , for some  $k \in \mathbb{N} \cup \{\infty\}$ , so that  $X_\infty = X$ . Let  $\ell \in \mathbb{N} \cup \{\infty\}$ . Define

$$\uparrow_{\mathfrak{A}}^{(k,\ell)} a := (\uparrow_{\mathfrak{A}} a) \cap \{s(x_1, \dots, x_k) \in T_{LX_k} \mid \text{each of the } k \text{ variables in } X_k \text{ occurs at most } \ell \text{ times in } s\}.$$

We write  $k$  instead of  $(k, \infty)$  so that

$$\uparrow_{\mathfrak{A}}^k a = (\uparrow_{\mathfrak{A}} a) \cap T_{LX_k}.$$

We extend the above notions from elements to justifications by

$$\uparrow_{\mathfrak{A}}^{(k,\ell)} (a \rightarrow b) := \left\{ s \rightarrow t \in \uparrow_{\mathfrak{A}} (a \rightarrow b) \mid s \in \uparrow_{\mathfrak{A}}^{(k,\ell)} a \text{ and } t \in \uparrow_{\mathfrak{A}}^{(k,\ell)} b \right\},$$

extended to arrow proportions by

$$\uparrow_{\mathfrak{A}\mathfrak{B}}^{(k,\ell)} (a \rightarrow b : \cdot c \rightarrow d) := \uparrow_{\mathfrak{A}}^{(k,\ell)} (a \rightarrow b) \cap \uparrow_{\mathfrak{B}}^{(k,\ell)} (c \rightarrow d).$$

The **analogical proportion**  $(k, \ell)$ -**relation**  $::_{(k,\ell)}$  is defined in the same way as  $::$  with  $\uparrow$  replaced by  $\uparrow^{(k,\ell)}$ . In case the underlying algebras are clear from the context, we will often write  $a : b ::_{(k,\ell)} c : d$  to denote the analogical proportion relation in the  $(k, \ell)$ -fragment. The **monolinear fragment**  $(1, 1)$  consists only of justifications with at most one occurrence of a single variable on each side and is denoted by  $m$ .

**Remark 15.** The unrestricted framework occurs as the special “fragment”  $(\infty, \infty)$  where every justification may have arbitrary many variables occurring arbitrary often.

**5.1. Monolinear additive arithmetical proportions.** In this subsection, we work in  $(\mathbb{Z}, +, \mathbb{Z})$  (see Remark 16). We begin by noting that the set of monolinear justifications of  $a \rightarrow b$  is given by

$$\begin{aligned} \uparrow^m (a \rightarrow b) &= \{k + x \rightarrow \ell + x \mid a \rightarrow b = k + o \rightarrow \ell + o, \text{ for some } o \in \mathbb{Z}\} \\ &\cup \{k + x \rightarrow b \mid a \rightarrow b = k + o \rightarrow b, \text{ for some } o \in \mathbb{Z}\} \cup \{a \rightarrow b\}. \end{aligned}$$

**Remark 16.** In  $(\mathbb{Z}, +)$  containing no constants, the only monolinear rewrite rule is  $x \rightarrow x$  which justifies only inner reflexive proportions of the form  $a : a ::_m c : c$ . This explains why we instead consider the algebra  $(\mathbb{Z}, +, \mathbb{Z})$  in which *every* integer is a distinguished element.

Interestingly, it turns out that monolinear additive number proportions are characterized by difference proportions:

**Theorem 17** (Difference Proportion Theorem).

$$\begin{aligned} a : b ::_m c : d &\Leftrightarrow a = k + o, \quad b = \ell + o, \quad c = k + u, \quad d = \ell + u, \quad k, \ell, o, u \in \mathbb{N} \\ &\Leftrightarrow a - b = c - d \quad (\text{difference proportion}). \end{aligned}$$

*Proof.* We first show

$$(4) \quad a = k + o, \quad b = \ell + o, \quad c = k + u, \quad d = \ell + u \Leftrightarrow a - b = c - d,$$

for some integers  $k, \ell, o, u \in \mathbb{Z}$ . The direction from left to right holds trivially. For the other direction, we proceed as follows. We can always write  $a = k + o$  and  $b = \ell + o$ , for some  $k, \ell, o \in \mathbb{Z}$ . We then have  $a - b = k - \ell$ . Analogously, we can always write  $c = k + u$  and  $d = \ell' + u$ , for some  $\ell', u \in \mathbb{Z}$ . We then have  $c - d = k - \ell'$ . By assumption, we have  $a - b = c - d$  which implies  $k - \ell = k - \ell'$  and therefore  $\ell = \ell'$  and finally  $d = \ell + u$ .

We now proceed to show the first equivalence in the statement of the theorem:

( $\Rightarrow$ ) By assumption, we have  $a \rightarrow b : \cdot_m c \rightarrow d$  which holds iff either<sup>5</sup>

$$\uparrow^m (a \rightarrow b) \cup \uparrow^m (c \rightarrow d) = \emptyset$$

<sup>5</sup>Recall from Definition 2 that  $\emptyset$  here really means the set of all trivial monolinear justifications  $\emptyset_{(\mathbb{Z}, +, \mathbb{Z})}$ .

or  $\uparrow^m (a \rightarrow b : \cdot c \rightarrow d)$  contains at least one non-trivial monilinear justification and is subset maximal with respect to  $d$ . In the first case, notice that neither  $\uparrow^m (a \rightarrow b)$  nor  $\uparrow^m (c \rightarrow d)$  can consist only of trivial justifications as we always have

$$a \rightarrow b \in \uparrow^m (a \rightarrow b) \quad \text{and} \quad c \rightarrow d \in \uparrow^m (c \rightarrow d).$$

In the second case, by assumption we must have some monilinear justification  $sx \rightarrow tx$  of  $a \rightarrow b : \cdot c \rightarrow d$  in  $(\mathbb{Z}, +, \mathbb{Z})$ . We distinguish the following cases:

- (1) If  $sx \rightarrow tx$  equals  $a \rightarrow b$  or  $c \rightarrow d$ , we must have  $a = c$  and  $b = d$ .
- (2) Else if  $sx \rightarrow tx$  equals  $k + x \rightarrow \ell + x$ , we must have  $a = k + o$ ,  $b = \ell + o$ ,  $c = k + u$ , and  $d = \ell + u$ , for some integers  $o, u \in \mathbb{Z}$ , which is equivalent to  $a - b = c - d$  by (4).
- (3) Else if  $sx \rightarrow tx$  equals  $k + x \rightarrow b$ , we must have  $b = d$ . Then, by assumption, we must also have

$$a \rightarrow b : \cdot_m c \rightarrow b \quad \text{and} \quad b \rightarrow a : \cdot_m b \rightarrow c.$$

So, either we have

$$\uparrow^m (b \rightarrow a) \cup \uparrow^m (b \rightarrow c) = \emptyset$$

or  $\uparrow^m (b \rightarrow a : \cdot b \rightarrow c)$  is non-empty and subset maximal with respect to  $c$ . Again, the sets  $\uparrow^m (b \rightarrow a)$  and  $\uparrow^m (b \rightarrow c)$  cannot be empty as they certainly contain  $b \rightarrow a$  and  $b \rightarrow c$ , respectively. Hence,  $\uparrow^m (b \rightarrow a : \cdot b \rightarrow c)$  must contain at least one non-trivial monilinear justification  $s'x \rightarrow t'x$ . We distinguish the following cases:

- (a) If  $s'x \rightarrow t'x$  equals  $b \rightarrow a$  or  $b \rightarrow c$ , we must have  $c = a$ .
- (b) Else if  $s'x \rightarrow t'x$  equals  $k' + x \rightarrow \ell' + x$ , we must have  $b = k' + o$ ,  $a = \ell' + o$ ,  $b = k' + u$ , and  $c = \ell' + u$ , for some  $o, u \in \mathbb{Z}$ , which implies  $b = k' + o = k' + u$  and therefore  $o = u$  and hence  $a = \ell' + o = \ell' + u = c$ .
- (c) Finally, if  $s'x \rightarrow t'x$  equals  $k' + x \rightarrow a$ , we must have  $a = c$ .

( $\Leftrightarrow$ ) Every justification of the form  $k + x \rightarrow \ell + x$  is a characteristic justification by the Uniqueness Lemma 7 since  $k + x$  and  $\ell + x$  are injective in  $(\mathbb{Z}, +, \mathbb{Z})$ . Since  $a - b = c - d$  holds by assumption,  $x \rightarrow x + b - a$  is a characteristic justification of  $a : b :: c : d$  in  $(\mathbb{Z}, +, \mathbb{Z})$ .  $\square$

Interestingly, additive monilinear number proportions are equivalent to number proportions in the domain of natural numbers  $(\mathbb{N}, S)$  with the **successor function**  $Sx := x + 1$ .

**Corollary 18.**  $a : b ::_m c : d \Leftrightarrow a : b ::_{(\mathbb{N}, S)} c : d$ .

*Proof.* A direct consequence of the Difference Proportion Theorem in Antić (2023b).  $\square$

**Theorem 19.** *All the properties listed in §2 hold in  $(\mathbb{Z}, +, \mathbb{Z})$  in the monilinear fragment except for  $p$ -commutativity.*

*Proof.* We have the following proofs:

- The proofs for  $p$ -symmetry, inner  $p$ -symmetry, inner  $p$ -reflexivity,  $p$ -reflexivity, and  $p$ -determinism are analogous to the original proofs in the proof of Theorem 28 in Antić (2022).
- $p$ -Commutativity fails since  $a - b \neq b - a$  whenever  $a \neq b$ .
- Central permutation follows from the fact that  $a - b = c - d$  iff  $a - c = b - d$ .
- Strong inner  $p$ -reflexivity follows from the fact that  $a - a = c - d$  implies  $d = c$ .
- Strong  $p$ -reflexivity follows from the fact that  $a - b = a - d$  implies  $d = b$ .
- $p$ -Determinism follows from the fact that  $a - a = a - d$  iff  $d = a$ .

- p-Transitivity follows from

$$a - b = c - d \quad \text{and} \quad c - d = e - f \quad \Rightarrow \quad a - b = e - f.$$

- Inner p-transitivity follows from

$$\frac{a - b = c - d \quad b - e = d - f}{\frac{a - b + b - e = c - d + d - f}{a - e = c - f.}}$$

- Central p-transitivity is a direct consequence of p-transitivity. Explicitly, we have

$$a - b = b - c \quad \text{and} \quad b - c = c - d \quad \Rightarrow \quad a - b = c - d.$$

□

**Fact 20.**  $\frac{a : b ::_m c : d \quad e : f ::_m g : h}{a + e : b + f ::_m c + g : d + h}.$

*Proof.*

$$\frac{\frac{a : b ::_m c : d}{a - b = c - d} \quad 17 \quad \frac{e : f ::_m g : h}{e - f = g - h} \quad 17}{(a + e) - (b + f) = a - b + e - f = c - d + g - h = (c + g) - (d + h)} \\ a + e : b + f ::_m c + g : d + h.$$

□

Fact 20 shows that we can decompose number proportions; for example,

$$4 : 5 :: 0 : 1 = (2 : 3 :: 0 : 1) + (2 : 2 :: 0 : 0).$$

The following notion of a number proportion is an instance of the more general definition due to Stroppa and Yvon (2006, Proposition 2) given for abelian semigroups, adapted to the additive setting of this section:

**Definition 21.** For any integers  $a, b, c, d \in \mathbb{Z}$ , define (recall that the underlying algebra is  $(\mathbb{Z}, +, \mathbb{Z})$ )

$$a : b ::_{SY} c : d \quad :\Leftrightarrow \quad a = k + o, \quad b = \ell + o, \quad c = k + u, \quad d = \ell + u, \quad \text{for some } k, \ell, o, u \in \mathbb{Z}.$$

We have the following equivalence which, surprisingly, shows that the notion of an additive number proportion in Stroppa and Yvon (2006) coincides with the rather restricted monilinear fragment of our framework and is therefore too weak to be used as a *general* definition of an additive number proportion (and see Example 23):

**Theorem 22.**  $a : b ::_{SY} c : d \quad \Leftrightarrow \quad a : b ::_m c : d.$

*Proof.* A direct consequence of the Difference Proportion Theorem 17. □

**Example 23.** The natural number proportion

$$2 : 4 :: 3 : 6$$

is characteristically justified within our framework by  $x \rightarrow x + x$ , which is non-monilinear since  $x$  occurs more than once on the right-hand side; on the other hand, this simple proportion is not captured within Stroppa and Yvon's (2006) framework, that is,

$$(5) \quad 2 : 4 \not::_{SY} 3 : 6.$$

**5.2. Monolinear multiplicative arithmetical proportions.** In this subsection, we work in  $(\mathbb{Q}, \cdot, \mathbb{Q})$ , where we study multiplicative monolinear number proportions. We begin by noting that the set of justifications of  $a \rightarrow b$  is given by

$$\begin{aligned} \uparrow^m (a \rightarrow b) = & \{kx \rightarrow \ell x \mid a \rightarrow b = ko \rightarrow \ell o, \text{ for some } k, \ell, o \in \mathbb{Q}\} \\ & \cup \{kx \rightarrow b \mid a \rightarrow b = ko \rightarrow b, \text{ for some } k, o \in \mathbb{Q}\} \cup \{a \rightarrow b\}. \end{aligned}$$

This leads to the following characterization of the monolinear entailment relation with respect to multiplication:

**Theorem 24** (Geometric Proportion Theorem). *For any  $a, b, c, d \in \mathbb{Q}$ ,*

$$\begin{aligned} a : b ::_m c : d & \Leftrightarrow a = ko, \quad b = \ell o, \quad c = ku, \quad d = \ell u, \quad k, \ell, o, u \in \mathbb{Q} \\ & \Leftrightarrow \frac{a}{b} = \frac{c}{d} \quad (\text{geometric proportion}). \end{aligned}$$

*The first equivalence holds in  $(\mathbb{N}, \cdot, \mathbb{N})$  as well.<sup>6</sup>*

*Proof.* We first show the second equivalence

$$a = ko, \quad b = \ell o, \quad c = ku, \quad d = \ell u \quad \Leftrightarrow \quad \frac{a}{b} = \frac{c}{d},$$

for some  $k, \ell, o, u \in \mathbb{Q}$ . The direction from left to right holds trivially. For the other direction, notice that  $\frac{a}{b} = \frac{c}{d}$  implies

$$a = \left(\frac{c}{d}\right)b, \quad b = 1b, \quad c = \left(\frac{c}{d}\right)d, \quad d = 1d.$$

The rest of the proof is similar to the proof of Theorem 17. □

The Geometric Proportion Theorem 24 shows that monolinear multiplicative number proportions can be geometrically interpreted as analogical proportions between rectangles. Moreover, the simple characterization of the monolinear relation in Theorem 24 allows us to analyze the proportional properties within the monolinear setting:

**Theorem 25.** *All the properties listed in §2 hold in  $(\mathbb{Q}, \cdot, \mathbb{Q})$  in the monolinear fragment.*

*Proof.* We have the following proofs:

- The proofs for p-symmetry, inner p-symmetry, inner p-reflexivity, p-reflexivity, and p-determinism are analogous to the original proofs in the proof of Theorem 28 in Antić (2022).
- p-Commutativity follows from Theorem 24 together with

$$ko : \ell o ::_m \ell o : ko, \quad \text{for all } k, \ell, o \in \mathbb{Q}.$$

- Central permutation follows from Theorem 24 together with

$$ko : \ell o ::_m ku : \ell u \quad \Leftrightarrow \quad ok : uk ::_m ol : ul.$$

- Strong inner p-reflexivity follows from Theorem 24 together with

$$ko : ko ::_m ku : d \quad \Leftrightarrow \quad d = ku.$$

- Strong p-reflexivity follows from Theorem 24 together with

$$ko : \ell o ::_m ko : d \quad \Leftrightarrow \quad d = \ell o.$$

- p-Determinism follows from  $\frac{a}{a} = \frac{a}{a}$  iff  $d = a$ .

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<sup>6</sup>This will be essential in §5.2.1 when we study primes.

- p-Transitivity follows from Theorem 24 together with

$$\frac{a}{b} = \frac{c}{d} \quad \text{and} \quad \frac{c}{d} = \frac{e}{f} \quad \Rightarrow \quad \frac{a}{b} = \frac{e}{f}.$$

- Inner p-transitivity follows from the following derivation:

$$\frac{\frac{a : b ::_m c : d}{\frac{\frac{a}{b} = \frac{c}{d}}{a = \frac{bc}{d}}} \text{ Theorem 24} \quad \frac{\frac{b : e ::_m d : f}{\frac{\frac{b}{e} = \frac{d}{f}}{e = \frac{bf}{d}}} \text{ Theorem 24}}{\frac{\frac{a}{e} = \frac{\frac{bc}{d}}{\frac{bf}{d}} = \frac{bcd}{bfd} = \frac{c}{f}}}{a : e ::_m c : f.}$$

- Central p-transitivity is an immediate consequence of transitivity. □

**Theorem 26.**  $\frac{a : b ::_m c : d \quad a' : b' ::_m c' : d'}{aa' : bb' ::_m cc' : dd'}.$

*Proof.*

$$\frac{\frac{a : b ::_m c : d}{ko : \ell o ::_m ku : \ell u} \text{ 24} \quad \frac{a' : b' ::_m c' : d'}{k'o' : \ell'o' ::_m k'u' : \ell'u'} \text{ 24}}{\frac{(ko)(k'o') : (\ell o)(\ell'o') ::_m (ku)(k'u') : (\ell u)(\ell'u')}{(kk')(oo') : (\ell\ell')(oo') ::_m (kk')(uu') : (\ell\ell')(uu')}}}{aa' : bb' ::_m cc' : dd'}.$$

□

5.2.1. *Primes.* We shall now prove some properties of the monolinear entailment relation with respect to primes. In this subsection, the underlying algebra is  $(\mathbb{N}, \cdot, \mathbb{N})$  where  $\mathbb{N}$  denotes the natural numbers.

**Proposition 27.** *Let  $p, q, p', q'$  be primes. We have*

$$p : q ::_m p' : q' \Leftrightarrow (p = q \text{ and } p' = q') \text{ or } (p = p' \text{ and } q = q').$$

*Proof.* By Theorem 24, we have

$$p : q ::_m p' : q' \Leftrightarrow p = ko, \quad q = \ell o, \quad c = ko', \quad d = \ell o', \quad \text{for some } k, \ell, o, o' \in \mathbb{N}.$$

We distinguish two cases. First, if  $k = 1$  and  $o = p$ , then  $q = \ell p$  which implies  $\ell = 1$  and therefore  $q' = o'$  and  $p' = o'$ . Second, if  $o = p$  and  $o = 1$ , then  $q = \ell$  and  $p' = po'$  which implies  $o' = 1$  and therefore  $q' = \ell = q$ . □

**Proposition 28.** *Let  $p, q$  be primes, and let  $c, d \in \mathbb{N}$ . We have*

$$p : q ::_m c : d \Leftrightarrow (p = q \text{ and } c = d) \text{ or} \\ (p \neq q \text{ and } c = pu \text{ and } d = qu, \text{ for some } u \in \mathbb{N}).$$

*Proof.* By Theorem 24, we have

$$p : q ::_m c : d \Leftrightarrow p = ko, \quad q = \ell o, \quad c = ku, \quad d = \ell u, \quad \text{for some } k, \ell, o, u \in \mathbb{N}.$$

We distinguish two cases. First, if  $k = 1$  and  $o = p$ , then  $q = \ell p$  and thus  $\ell = 1$  and  $q = p$  and  $c = d = u$ . Second, if  $k = p$  and  $o = 1$ , then  $q = \ell$ ,  $c = pu$ , and  $d = qu$ , for some  $u \in \mathbb{N}$ . □

**5.3. Monolinear word proportions.** In this subsection, we work in the word domain  $(A^*, \cdot, A^*)$ , where the set of monolinear justifications of  $\mathbf{a} \rightarrow \mathbf{b}$  is given by

$$\begin{aligned} \uparrow^m(\mathbf{a} \rightarrow \mathbf{b}) = & \{ \mathbf{a}_1 x \mathbf{a}_3 \rightarrow \mathbf{b}_1 x \mathbf{b}_3 \mid \mathbf{a} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_3 \in A^* \} \\ & \cup \{ \mathbf{a}_1 x \mathbf{a}_3 \rightarrow \mathbf{b} \mid \mathbf{a} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \rightarrow \mathbf{b}; \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in A^* \} \cup \{ \mathbf{a} \rightarrow \mathbf{b} \}. \end{aligned}$$

This implies

$$\begin{aligned} \uparrow^m(\mathbf{a} \rightarrow \mathbf{b} : \cdot \mathbf{c} \rightarrow \mathbf{d}) = & \left\{ \mathbf{a}_1 x \mathbf{a}_3 \rightarrow \mathbf{b}_1 x \mathbf{b}_3 \mid \begin{array}{l} \mathbf{a} \rightarrow \mathbf{b} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{a}_2 \mathbf{b}_3 \\ \mathbf{c} \rightarrow \mathbf{d} = \mathbf{a}_1 \mathbf{c}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{c}_2 \mathbf{b}_3 \\ \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_3, \mathbf{c}_2 \in A^* \end{array} \right\} \\ & \cup \left\{ \mathbf{a}_1 x \mathbf{a}_3 \rightarrow \mathbf{b} \mid \begin{array}{l} \mathbf{b} = \mathbf{d} \\ \mathbf{a} \rightarrow \mathbf{b} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \rightarrow \mathbf{b} \\ \mathbf{c} \rightarrow \mathbf{d} = \mathbf{a}_1 \mathbf{c}_2 \mathbf{a}_3 \rightarrow \mathbf{b} \\ \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{c}_2 \in A^* \end{array} \right\} \cup \{ \mathbf{a} \rightarrow \mathbf{b} \mid \mathbf{a} = \mathbf{c}, \mathbf{b} = \mathbf{d} \}. \end{aligned}$$

This leads to the following characterization of the monolinear entailment relation:

**Theorem 29.**  $\mathbf{a} : \mathbf{b} ::_m \mathbf{c} : \mathbf{d} \iff \mathbf{a} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3, \mathbf{b} = \mathbf{b}_1 \mathbf{a}_2 \mathbf{b}_3, \mathbf{c} = \mathbf{a}_1 \mathbf{b}_2 \mathbf{a}_3, \mathbf{d} = \mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3.$

*Proof.*  $(\implies)$  By assumption, we have  $\mathbf{a} \rightarrow \mathbf{b} : \cdot_m \mathbf{c} \rightarrow \mathbf{d}$  which holds iff either<sup>7</sup>

$$\uparrow^m(\mathbf{a} \rightarrow \mathbf{b}) \cup \uparrow^m(\mathbf{c} \rightarrow \mathbf{d}) = \emptyset,$$

or  $\uparrow^m(\mathbf{a} \rightarrow \mathbf{b} : \cdot \mathbf{c} \rightarrow \mathbf{d})$  is non-empty and subset maximal with respect to  $\mathbf{d}$ . In the first case, notice that neither  $\uparrow^m(\mathbf{a} \rightarrow \mathbf{b})$  nor  $\uparrow^m(\mathbf{c} \rightarrow \mathbf{d})$  can be empty since we always have

$$\mathbf{a} \rightarrow \mathbf{b} \in \uparrow^m(\mathbf{a} \rightarrow \mathbf{b}) \quad \text{and} \quad \mathbf{c} \rightarrow \mathbf{d} \in \uparrow^m(\mathbf{c} \rightarrow \mathbf{d}).$$

In the second case, by assumption we must have some monolinear justification  $sx \rightarrow tx$  of  $\mathbf{a} \rightarrow \mathbf{b} : \cdot \mathbf{c} \rightarrow \mathbf{d}$  in  $(A^*, \cdot, A^*)$ . We distinguish the following cases:

(1) If  $sx \rightarrow tx$  equals  $\mathbf{a} \rightarrow \mathbf{b}$  or  $\mathbf{c} \rightarrow \mathbf{d}$ , we must have

$$\mathbf{a} = \mathbf{c} \quad \text{and} \quad \mathbf{b} = \mathbf{d}.$$

(2) Else if  $sx \rightarrow tx$  equals  $\mathbf{a}_1 x \mathbf{a}_3 \rightarrow \mathbf{b}_1 x \mathbf{b}_3$ , we must have

$$\mathbf{a} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \quad \text{and} \quad \mathbf{b} = \mathbf{b}_1 \mathbf{a}_2 \mathbf{b}_3 \quad \text{and} \quad \mathbf{c} = \mathbf{a}_1 \mathbf{b}_2 \mathbf{a}_3 \quad \text{and} \quad \mathbf{d} = \mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3$$

for some  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in A^*$ .

(3) Else if  $sx \rightarrow tx$  equals  $\mathbf{a}_1 x \mathbf{a}_3 \rightarrow \mathbf{b}$ , we must have  $\mathbf{a} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3$ ,  $\mathbf{c} = \mathbf{a}_1 \mathbf{b}_2 \mathbf{a}_3$ , and  $\mathbf{d} = \mathbf{b}$ , for some  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_2 \in A^*$ . Then, by assumption, we must also have  $\mathbf{a} : \mathbf{b} ::_m \mathbf{c} : \mathbf{b}$  and, by inner p-symmetry,  $\mathbf{b} : \mathbf{a} ::_m \mathbf{b} : \mathbf{c}$  and therefore  $\mathbf{b} \rightarrow \mathbf{a} : \cdot_m \mathbf{b} \rightarrow \mathbf{c}$ . So, either we have  $\uparrow^m(\mathbf{b} \rightarrow \mathbf{a}) \cup \uparrow^m(\mathbf{b} \rightarrow \mathbf{c}) = \emptyset$  or  $\uparrow^m(\mathbf{b} \rightarrow \mathbf{a} : \cdot \mathbf{b} \rightarrow \mathbf{c})$  is non-empty and subset maximal with respect to  $\mathbf{c}$ . Again, the sets  $\uparrow^m(\mathbf{b} \rightarrow \mathbf{a})$  and  $\uparrow^m(\mathbf{b} \rightarrow \mathbf{c})$  cannot be empty as they contain  $\mathbf{b} \rightarrow \mathbf{a}$  and  $\mathbf{b} \rightarrow \mathbf{c}$ , respectively. Hence,  $\uparrow^m(\mathbf{b} \rightarrow \mathbf{a} : \cdot \mathbf{b} \rightarrow \mathbf{c})$  contains at least one non-trivial monolinear justification  $s'x \rightarrow t'x$ . We distinguish the following cases:

(a) If  $s'x \rightarrow t'x$  equals  $\mathbf{b} \rightarrow \mathbf{a}$  or  $\mathbf{b} \rightarrow \mathbf{c}$ , we must have  $\mathbf{a} = \mathbf{c}$ .

(b) Else if  $s'x \rightarrow t'x$  equals  $\mathbf{b}'_1 x \mathbf{b}'_3 \rightarrow \mathbf{a}'_1 x \mathbf{a}'_3$ , for some  $\mathbf{b}'_1, \mathbf{b}'_3, \mathbf{a}'_1, \mathbf{a}'_3 \in A^*$ , we must have  $\mathbf{b} = \mathbf{b}'_1 \mathbf{b}'_2 \mathbf{b}'_3 = \mathbf{b}'_1 \mathbf{c}'_2 \mathbf{b}'_3$  and  $\mathbf{a} = \mathbf{a}'_1 \mathbf{b}'_2 \mathbf{a}'_3$  and  $\mathbf{c} = \mathbf{a}'_1 \mathbf{c}'_2 \mathbf{a}'_3$ , for some  $\mathbf{b}'_2, \mathbf{c}'_2 \in A^*$ . The identity  $\mathbf{b}'_1 \mathbf{b}'_2 \mathbf{b}'_3 = \mathbf{b}'_1 \mathbf{c}'_2 \mathbf{b}'_3$  implies  $\mathbf{b}'_2 = \mathbf{c}'_2$  and again  $\mathbf{a} = \mathbf{c}$ .

(c) Finally, if  $s'x \rightarrow t'x$  equals  $\mathbf{b}'_1 x \mathbf{b}'_3 \rightarrow \mathbf{a}$ , we must also have  $\mathbf{a} = \mathbf{c}$ .

<sup>7</sup>Recall from Definition 2 that  $\emptyset$  here really means the set of all trivial monolinear justifications  $\emptyset_{(A^*, \cdot, A^*)}$ .



( $\Leftarrow$ ) The monolinear justification  $\mathbf{a}_1 \mathbf{x} \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{x} \mathbf{b}_3$  is a characteristic justification of

$$\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{a}_2 \mathbf{b}_3 : \cdot \mathbf{a}_1 \mathbf{c}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{c}_2 \mathbf{b}_3 \quad \text{and} \quad \mathbf{a}_1 \mathbf{c}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{c}_2 \mathbf{b}_3 : \cdot \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{a}_2 \mathbf{b}_3$$

in  $(A^*, \cdot, A^*)$  by the Uniqueness Lemma 7 since  $\mathbf{a}_1 \mathbf{x} \mathbf{a}_3$  and  $\mathbf{b}_1 \mathbf{x} \mathbf{b}_3$  both induce injective word functions. Analogously,  $\mathbf{b}_1 \mathbf{x} \mathbf{b}_3 \rightarrow \mathbf{a}_1 \mathbf{x} \mathbf{a}_3$  is a characteristic justification of

$$\mathbf{a}_1 \mathbf{c}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{c}_2 \mathbf{b}_3 : \cdot \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{a}_2 \mathbf{b}_3 \quad \text{and} \quad \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{a}_2 \mathbf{b}_3 : \cdot \mathbf{a}_1 \mathbf{c}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{c}_2 \mathbf{b}_3.$$

Hence, we have shown the theorem.  $\square$

**Corollary 30.**  $\mathbf{a} : \mathbf{e} \mathbf{a} \mathbf{f} ::_m \mathbf{c} : \mathbf{e} \mathbf{c} \mathbf{f}$ .

**Corollary 31.**  $\mathbf{a} \mathbf{b} = \mathbf{c} \mathbf{d} \Leftrightarrow \mathbf{a} : \mathbf{b} ::_m \mathbf{c} : \mathbf{d}$ .

*Proof.* For example, by Theorem 29 we have  $\mathbf{a} : \mathbf{b} \not\prec_m \varepsilon : \mathbf{a} \mathbf{b}$ .  $\square$

**Corollary 32.** *There are words such that  $\mathbf{a} : \mathbf{a}^r \not\prec_m \mathbf{c} : \mathbf{c}^r$ .*

*Proof.* For example, by Theorem 29 we have  $\mathbf{a} \mathbf{b} : \mathbf{b} \mathbf{a} \not\prec_m \mathbf{b} \mathbf{a} : \mathbf{a} \mathbf{b}$ .  $\square$

The simple characterization of the monolinear proportion relation in Theorem 29 allows us to analyze the proportional properties within the monolinear word setting:

**Theorem 33.** *The monolinear word proportion relation satisfies*

- *symmetry,*
- *inner p-symmetry,*
- *p-reflexivity,*
- *p-determinism,*
- *strong inner p-reflexivity,*
- *strong p-reflexivity,*
- *transitivity,*
- *central transitivity,*

*and, in general, it dissatisfies*

- *central permutation,*
- *commutativity,*
- *inner transitivity.*

*Proof.* We have the following proofs:

- The proofs for p-symmetry, inner p-symmetry, inner p-reflexivity, p-reflexivity, and p-determinism are analogous to the original proofs in the proof of Theorem 28 in Antić (2022).
- p-Determinism is by Theorem 29 equivalent to

$$\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 : \mathbf{b}_1 \mathbf{a}_2 \mathbf{b}_3 ::_m \mathbf{a}_1 \mathbf{b}_2 \mathbf{a}_3 : \mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3 \quad \Leftrightarrow \quad \mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3 = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3$$

where

$$\mathbf{a} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 = \mathbf{b}_1 \mathbf{a}_2 \mathbf{b}_3 = \mathbf{a}_1 \mathbf{b}_2 \mathbf{a}_3 \quad \text{and} \quad \mathbf{d} = \mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3.$$

This follows from

$$\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 = \mathbf{a}_1 \mathbf{b}_2 \mathbf{a}_3 \quad \Leftrightarrow \quad \mathbf{b}_2 = \mathbf{a}_2 \quad \Leftrightarrow \quad \mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3 = \mathbf{a} \quad \Leftrightarrow \quad \mathbf{d} = \mathbf{a}.$$

- Central permutation fails<sup>8</sup>, for example, given the alphabet  $A := \{a_1, a_2, a_3, b_1, b_3, c_2\}$  since as a consequence of Theorem 29, we have

$$a_1a_2a_3 : b_1a_2b_3 ::_m a_1c_2a_3 : b_1c_2b_3$$

whereas

$$a_1a_2a_3 : a_1c_2a_3 ::_m b_1a_2b_3 : b_1c_2b_3.$$

- Strong inner p-reflexivity and strong p-reflexivity are immediate consequences of Theorem 29.
- p-Commutativity fails, for example, in  $A := \{a, b\}$  since as a consequence of Theorem 29, we have

$$a : b ::_m b : a.$$

- p-Transitivity is an immediate consequence of Theorem 29.
- Inner p-transitivity fails, for example, in  $A := \{a_1, a_2, a_3, b_1, b_3, c_2, d_2, e_1, e_3\}$  since as a consequence of Theorem 29, we have

$$a_1a_2a_3 : b_1a_2b_3 ::_m a_1c_2a_3 : b_1c_2a_3$$

and

$$b_1a_2b_3 : e_1a_2e_3 ::_m b_1d_2b_3 : e_1d_2e_3$$

whereas

$$a_1a_2a_3 : e_1a_2e_3 ::_m a_1c_2a_3 : e_1d_2e_3.$$

- Finally, central p-transitivity is an immediate consequence of transitivity already shown above.

□

**Remark 34.** The fact that central permutation fails gives a negative answer to Problem 30 in Antić (2022) in the monolinear setting.

**Remark 35.** Notice that we cannot prove an analogue of Theorems 20 and 26 in the word domain since by Theorem 29, we in general have

$$\mathbf{a} : \mathbf{b} ::_m \mathbf{c} : \mathbf{d} \quad \text{and} \quad \mathbf{a}' : \mathbf{b}' ::_m \mathbf{c}' : \mathbf{d}' \quad \Rightarrow \quad \mathbf{aa}' : \mathbf{bb}' ::_m \mathbf{cc}' : \mathbf{dd}'.$$

## 6. WORD PROPORTIONS

In §5.3, we studied word proportions in the monolinear fragment where justifications contain at most one occurrence of a single variable. This section studies word proportions in the full framework. Specifically, we show in Theorem 42 and Example 43 that our framework strictly subsumes Stroppa and Yvon's (2006) notion of a word proportion.

In this section, we shall work in the word algebra  $(A^+, \cdot, A^+)$ , where  $\cdot$  denotes concatenation (always omitted) and  $A$  is a finite non-empty alphabet. We will often omit the reference to the word algebra for readability.

**Notation 36.** In the other sections, we have used boldface letters to denote sequences of elements. In this section, we shall use boldface letters to denote words (i.e. sequences of letters) and we use the vector symbol to denote sequences of words.

<sup>8</sup>See Remark 34

**Theorem 37.** *We have*

$$\mathbf{a}_1 \mathbf{o}_1 \mathbf{a}_2 \dots \mathbf{a}_n \mathbf{o}_n \mathbf{a}_{n+1} : \mathbf{b}_1 \mathbf{o}_1 \mathbf{b}_2 \dots \mathbf{b}_n \mathbf{o}_n \mathbf{b}_{n+1} :: \mathbf{a}_1 \mathbf{u}_1 \mathbf{a}_2 \dots \mathbf{a}_n \mathbf{u}_n \mathbf{a}_{n+1} : \mathbf{b}_1 \mathbf{u}_1 \mathbf{b}_2 \dots \mathbf{b}_n \mathbf{u}_n \mathbf{b}_{n+1},$$

for all  $\mathbf{a}_1, \mathbf{a}_{n+1}, \mathbf{b}_1, \mathbf{b}_{n+1} \in A^*$ ,  $\mathbf{a}_2, \dots, \mathbf{a}_n, \mathbf{b}_2, \dots, \mathbf{b}_n \in A^+$ , and  $\mathbf{o}_i, \mathbf{u}_i \in A^*$ ,  $n \geq 1$ .

*Proof.* Since  $\mathbf{a}_1 x_1 \mathbf{a}_2 \dots \mathbf{a}_n x_n \mathbf{a}_{n+1}$  and  $\mathbf{b}_1 x_1 \mathbf{b}_2 \dots \mathbf{b}_n x_n \mathbf{b}_{n+1}$  are injective in  $A^+$  whenever  $\mathbf{a}_2, \dots, \mathbf{a}_n$  and  $\mathbf{b}_2, \dots, \mathbf{b}_n$  are non-empty words (notice that two variables side by side yield non-injective word terms), the justification  $\mathbf{a}_1 x_1 \mathbf{a}_2 \dots \mathbf{a}_n x_n \mathbf{a}_{n+1} \rightarrow \mathbf{b}_1 x_1 \mathbf{b}_2 \dots \mathbf{b}_n x_n \mathbf{b}_{n+1}$  is a characteristic justification by the Uniqueness Lemma 7.  $\square$

**Corollary 38.**  $\mathbf{a} : \mathbf{eaf} :: \mathbf{c} : \mathbf{ecf}$  and  $\mathbf{a} : \mathbf{b} :: \mathbf{eaf} : \mathbf{ebf}$ .

**Proposition 39.** *Given non-empty words  $\mathbf{a}, \mathbf{b} \in A^+$  of equal length, we have*

$$(6) \quad \mathbf{a} : \mathbf{a}^r :: \mathbf{b} : \mathbf{b}^r.$$

Consequently,

$$\mathbf{a} : \mathbf{a}^r :: \mathbf{a}^r : \mathbf{a}.$$

*Proof.* Since  $|\mathbf{a}| = |\mathbf{b}|$  holds by assumption (here  $|\cdot|$  denotes the length of  $\mathbf{a}$ ), the rule

$$x_1 \dots x_{|\mathbf{a}|} \rightarrow x_{|\mathbf{a}|} \dots x_1$$

is a characteristic justification of  $\mathbf{a} : \mathbf{a}^r :: \mathbf{b} : \mathbf{b}^r$  in  $A^+$  by the Uniqueness Lemma 7. The second assertion is a consequence of the first together with the identity  $\mathbf{a}^{rr} = \mathbf{a}$ .  $\square$

In the proof of Proposition 39, two assumptions are essential: first, we exclude the empty word since otherwise the Uniqueness Lemma 7 is no longer applicable; and second, we assume that  $\mathbf{a}$  and  $\mathbf{b}$  have the same length for the same reason. We can avoid both assumptions by simply adding the *injective* reverse operation to our algebra since by the Functional Proportion Theorem 8, we then have in  $(A^*, \cdot, \cdot^r, A^*)$  (notice that we now have included the empty word) the proportions

$$\mathbf{a} : \mathbf{a}^r :: \mathbf{b} : \mathbf{b}^r, \quad \text{for all } \mathbf{a}, \mathbf{b} \in A^*.$$

This is a common pattern and we encourage users of our framework to add (preferable injective) algebraic operations to the word algebra as needed for intended applications.

The following notion of word proportion is an instance of Stroppa and Yvon (2006, Definition 2), originally given in the more general context of semigroups:

**Definition 40** (Stroppa and Yvon (2006)). Given  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in A^+$ , define

$$\mathbf{a} : \mathbf{b} ::_{SY} \mathbf{c} : \mathbf{d} \quad :\Leftrightarrow \quad \mathbf{a} = a_1 \dots a_n, \quad \mathbf{b} = b_1 \dots b_n, \quad \mathbf{c} = c_1 \dots c_n, \quad \mathbf{d} = d_1 \dots d_n,$$

for some symbols  $a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n, d_1, \dots, d_n \in A$  and  $n \geq 0$  such that

$$(a_i = b_i \quad \text{and} \quad c_i = d_i) \quad \text{or} \quad (a_i = c_i \quad \text{and} \quad b_i = d_i) \quad \text{holds for all } 1 \leq i \leq n.$$

**Example 41.**  $a : a ::_{SY} bb : bb$  and  $abc : abd ::_{SY} bbc : bbd$ .

We have the following implication:

**Theorem 42.**  $\mathbf{a} : \mathbf{b} ::_{SY} \mathbf{c} : \mathbf{d} \quad \Rightarrow \quad \mathbf{a} : \mathbf{b} :: \mathbf{c} : \mathbf{d}$ .

*Proof.* Let  $\mathbf{a} = a_1 \dots a_n$ ,  $\mathbf{b} = b_1 \dots b_n$ ,  $\mathbf{c} = c_1 \dots c_n$ , and  $\mathbf{d} = d_1 \dots d_n$ ,  $n \geq 1$ , be decompositions of  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$  into letters of the alphabet. Let  $I$  be the set of indices  $i$  such that  $a_i \neq c_i$  and  $b_i \neq d_i$ ,  $1 \leq i \leq n$ , and let  $m$  be the finite cardinality of  $I = \{i_1, \dots, i_m\}$ .

If  $m = 0$  then we have  $\mathbf{a} = \mathbf{c}$  and  $\mathbf{b} = \mathbf{d}$  which together with p-reflexivity of word proportions implies  $\mathbf{a} : \mathbf{b} :: \mathbf{c} : \mathbf{d}$ .

Otherwise, we define the terms  $s$  and  $t$  over  $A^*$  and  $\{x_{i_1}, \dots, x_{i_m}\}$  as follows. For every  $i \in [1, n]$ , if  $a_i = c_i$  and  $b_i = d_i$  define  $s_i := a_i$  and  $t_i := c_i$  and, otherwise, define  $s_i := x_i$  and  $t_i := x_i$ ; finally, define

$$s(x_{i_1}, \dots, x_{i_m}) := s_1 \dots s_n \quad \text{and} \quad t(x_{i_1}, \dots, x_{i_m}) := t_1 \dots t_n.$$

By construction, we have

$$\begin{aligned} s(x_{i_1}, \dots, x_{i_m}) &= a_1 \dots a_{i_1-1} x_{i_1} a_{i_1+1} \dots a_{i_m-1} x_{i_m} a_{i_m+1} \dots a_n, \\ t(x_{i_1}, \dots, x_{i_m}) &= c_1 \dots c_{i_1-1} x_{i_1} c_{i_1+1} \dots c_{i_m-1} x_{i_m} c_{i_m+1} \dots c_n, \end{aligned}$$

and

$$(7) \quad \mathbf{a} \rightarrow \mathbf{b} = s(a_{i_1}, \dots, a_{i_m}) \rightarrow t(a_{i_1}, \dots, a_{i_m}) \quad \text{and} \quad \mathbf{c} \rightarrow \mathbf{d} = s(c_{i_1}, \dots, c_{i_m}) \rightarrow t(c_{i_1}, \dots, c_{i_m}),$$

which shows that  $s \rightarrow t$  is a justification of  $\mathbf{a} \rightarrow \mathbf{b} : \mathbf{c} \rightarrow \mathbf{d}$  in  $A^*$ . Notice that if  $s$  does

It remains to show that  $s \rightarrow t$  is a characteristic justification. For this, we seek to apply the Uniqueness Lemma 7. Let  $\vec{\mathbf{e}} = (\mathbf{e}_1, \dots, \mathbf{e}_m) \in (A^*)^m$  be an arbitrary sequence of words satisfying  $\mathbf{a} = s(\vec{\mathbf{e}})$ . We need to show  $\mathbf{b} = t(\vec{\mathbf{e}})$ . By (7), we have

$$\mathbf{b} = c_1 \dots c_{i_1-1} a_{i_1} c_{i_1+1} \dots c_{i_m-1} a_{i_m} c_{i_m+1} \dots c_n.$$

From  $\mathbf{a} = s(\vec{\mathbf{e}})$ , we deduce

$$\mathbf{a} = a_1 \dots a_{i_1-1} \mathbf{e}_1 a_{i_1+1} \dots a_{i_m-1} \mathbf{e}_m a_{i_m+1} \dots a_n,$$

which entails

$$a_1 \dots a_{i_1-1} a_{i_1} a_{i_1+1} \dots a_{i_m-1} a_{i_m} a_{i_m+1} \dots a_n = a_1 \dots a_{i_1-1} \mathbf{e}_1 a_{i_1+1} \dots a_{i_m-1} \mathbf{e}_m a_{i_m+1} \dots a_n.$$

This implies

$$a_{i_1} \dots a_{i_m} = \mathbf{e}_1 \dots \mathbf{e}_m.$$

From this it is easy to see that

$$t(\vec{\mathbf{e}}) = c_1 \dots c_{i_1-1} \mathbf{e}_1 c_{i_1+1} \dots c_{i_m-1} \mathbf{e}_m c_{i_m+1} \dots c_n = \mathbf{b}.$$

The other cases of Item 4 are analogous. □

Notice that Stroppa and Yvon (2006) define word proportions only for words over the same alphabet, which means that we cannot expect the converse of Theorem 42 to be true — the following example shows that it may fail even in the case of a single alphabet:

**Example 43.** Let  $A := \{a, b, c\}$ . Since  $xy$  and  $xcy$  are injective in  $A^+$ , the rewrite rule

$$xby \rightarrow xcy$$

is a characteristic justification of

$$(8) \quad ab : ac :: bc : cc$$

in  $(A^+, \cdot, A^+)$  by the Functional Proportion Theorem 8. This solution formalizes the intuitive observation that  $ac$  is obtained from  $ab$  by replacing  $b$  by  $c$  — analogously,  $cc$  is obtained from  $bc$  by again replacing  $b$  by  $c$ . This solution cannot be obtained with respect to  $::_{SY}$  as the first letter of  $ab$  and  $ac$  is identical, whereas the first letter of  $bc$  and  $cc$  differs. (However, we should mention that (8) does hold with respect to  $::_{SY}$  in case the empty word is allowed).

## 7. TREE PROPORTIONS

In this section, we study analogical proportions between trees called tree proportions within the free term algebra  $\mathfrak{T}_{LX}$  (see §2).

The following simple observation highlights a peculiar property of the tree setting:

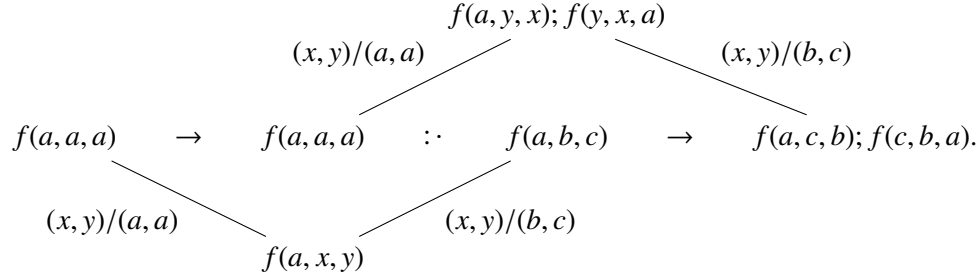
**Fact 44.** *Every term function is injective in the term algebra and thus every justification is a characteristic one.*

*Proof.* A direct consequence of the Uniqueness Lemma 7. □

**Example 45.** The following example shows that there may be terms  $p, q, r, u, u', u \neq u'$ , such that

$$p \rightarrow q : \cdot r \rightarrow u \quad \text{and} \quad p \rightarrow q : \cdot r \rightarrow u'$$

both hold in  $\mathfrak{T}_{LX}$ :<sup>9</sup>



The diagram is a compact representation of the fact that

$$f(a, x, y) \rightarrow f(a, y, x)$$

is a justification of

$$f(a, a, a) \rightarrow f(a, a, a) : \cdot f(a, b, c) \rightarrow f(a, c, b)$$

and

$$f(a, x, y) \rightarrow f(y, x, a)$$

is a justification of

$$f(a, a, a) \rightarrow f(a, a, a) : \cdot f(a, c, b) \rightarrow f(c, b, a),$$

and Fact 44 shows that they are in fact characteristic ones.

Given some  $L$ -term  $p$  and  $s \in \uparrow p$ , by Fact 44 there is always a *unique*  $\mathbf{o} \in T_{LX}^{rs}$  such that  $p = s\mathbf{o}$  which we will denote by  $o(s, p)$ , that is,

$$so(s, p) = p.$$

Moreover, define, for some  $\mathbf{o} \in T_{LX}^k, k \geq 1$ ,

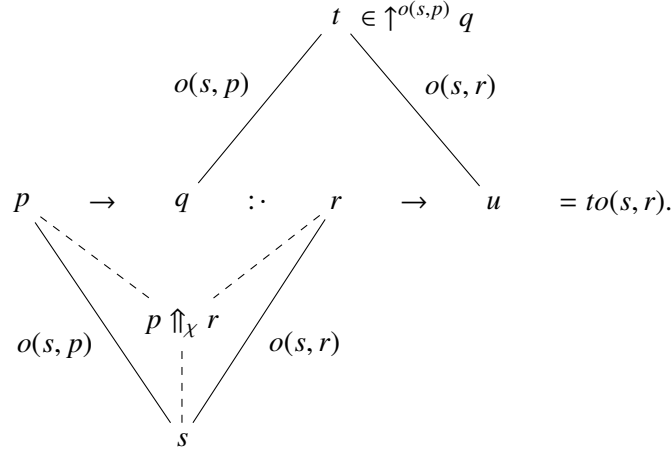
$$\uparrow^{\mathbf{o}} q := \{t \in T_{LX} \mid q = t\mathbf{o}\}.$$

We are now ready to prove a simple characterization of solutions to proportional term equations:

**Theorem 46.**  $\mathcal{S}(p \rightarrow q : \cdot r \rightarrow v) = \bigcup_{s \in \uparrow(p \uparrow_x r)} (\uparrow^{o(s, p)} q)(o(s, r))$ .

*Proof.* Every justification  $s \rightarrow t$  of  $p \rightarrow q : \cdot r \rightarrow u$  has the following form:

<sup>9</sup>Here we have joined two diagrams into one for brevity separated by semicolons.



Since every justification is a characteristic one by Fact 44, every solution  $u$  to  $p \rightarrow q :: r \rightarrow x$  has the form  $u = to(s, r)$ , for some  $s \in \uparrow(p \uparrow_\chi r)$  and  $t \in \uparrow^{o(s,p)} q$ .  $\square$

For a set of terms  $S$  and a term  $s$ , define

$$X_S s := Xs \setminus XS.$$

Moreover, we write  $s\langle p_i/q_i \mid i \in I \rangle$  for the term which we obtain from  $s$  by replacing one or more occurrences of the subterm  $p_i$  in  $s$  by  $q_i$ , for every  $i \in I$  (this is different from substitutions which replace *all* occurrences of *variables* at once). Notice that we have  $s\langle \rangle = s$ .

**Lemma 47.**  $u = r\langle p/q \mid p\chi q \in X_{\{p,q,r,u\}}(r \uparrow_\chi u) \rangle$ .

*Proof.* By structural induction on the shape of  $u$  and  $r$ :

- $u = a \in L_0$ :
  - $r = a$ :  $X_{\{p,q,r,u\}}(r \uparrow_\chi u) = \emptyset$  and  $a = a\langle \rangle$ .
  - $r \neq a$ :  $X_{\{p,q,r,u\}}(r \uparrow_\chi u) = \{r\chi a\}$  and  $a = r\langle r/a \rangle$ .
- $u = x \in X$ :
  - $r = x$ :  $X_{\{p,q,r,u\}}(r \uparrow_\chi u) = X_{\{p,q,r,u\}}(x \uparrow_\chi z) = X_{\{p,q,r,u\}}x = \emptyset$  and  $x = x\langle \rangle$ .
  - $r \neq x$ :  $X_{\{p,q,r,u\}}(r \uparrow_\chi u) = \{r\chi x\}$  and  $x = r\langle r/x \rangle$ .
- $u = f(u_1, \dots, u_{rf}) \in T_{LX}$ :
  - $r = f(r_1, \dots, r_{rf})$ : By induction hypothesis, we have

$$u_i = r_i\langle p/q \mid p\chi q \in X_{\{p,q,r,u\}}(u_i \uparrow_\chi r_i) \rangle.$$

This implies

$$\begin{aligned} & f(u_1, \dots, u_{rf}) \\ &= f(r_1\langle p/q \mid p\chi q \in X_{\{p,q,r,u\}}(u_1 \uparrow_\chi r_1) \rangle, \dots, r_{rf}\langle p/q \mid p\chi q \in X_{\{p,q,r,u\}}(u_{rf} \uparrow_\chi r_{rf}) \rangle) \\ &= f(r_1, \dots, r_{rf}) \left\langle p/q \mid p\chi q \in \bigcup_{i=1}^{rf} X_{\{p,q,r,u\}}(u_i \uparrow_\chi r_i) \right\rangle. \\ &–  $r = g(r_1, \dots, r_{rg})$ :  $X_{\{p,q,r,u\}}(r \uparrow_\chi u) = \{r\chi u\}$  and  $u = r\langle r/u \rangle$ . \end{aligned}$$

$\square$

**Lemma 48.**  $X_{\{p,q,r,u\}}(q \uparrow_\chi u) \subseteq X_{\{p,q,r,u\}}(p \uparrow_\chi r) \Leftrightarrow (p \uparrow_\chi r) \rightarrow (q \uparrow_\chi u) \in \uparrow(p \rightarrow q :: r \rightarrow u)$ .

*Proof.* The direction from right to left holds by definition of justifications. We prove the other direction by nested structural induction on the shape of  $p, q, r, u$ :

- $p = a \in L_0$ :
  - $r = a$ :  $X_{\{p,q,r,u\}}(p \uparrow_\chi r) = X_{\{p,q,r,u\}}(a \uparrow_\chi a) = \emptyset$  implies  $X_{\{p,q,r,u\}}(q \uparrow_\chi u) = \emptyset$  implies  $q = u$ . We then have
 
$$(p \uparrow_\chi r) \rightarrow (q \uparrow_\chi u) = a \rightarrow q \in \uparrow (a \rightarrow q : \cdot a \rightarrow q).$$
  - $r \neq a$ :  $X_{\{p,q,r,u\}}(p \uparrow_\chi r) = \{a\chi r\}$ :
    - \*  $X_{\{p,q,r,u\}}(q \uparrow_\chi u) = \emptyset$  implies  $q = u$ . Hence,
 
$$(p \uparrow_\chi r) \rightarrow (q \uparrow_\chi u) = a\chi r \rightarrow q \in \uparrow (a \rightarrow q : \cdot r \rightarrow q).$$
    - \*  $X_{\{p,q,r,u\}}(q \uparrow_\chi u) = \{a\chi r\}$  implies  $u = q\langle s/t \mid s\chi t \in X_{\{p,q,r,u\}}(q \uparrow_\chi u) \rangle = q\langle a/r \rangle$  (Lemma 47). Hence,
 
$$(p \uparrow_\chi r) \rightarrow (q \uparrow_\chi u) = a\chi r \rightarrow q\langle a/a\chi r \rangle \in \uparrow (a \rightarrow q : \cdot r \rightarrow q\langle a/r \rangle).$$
- $p = x \in X$ :  $X_{\{p,q,r,u\}}(p \uparrow_\chi r) = \{x\chi r\}$ :
  - $X_{\{p,q,r,u\}}(q \uparrow_\chi u) = \emptyset$  implies  $q = u$  and
 
$$(p \uparrow_\chi r) \rightarrow (q \uparrow_\chi u) = x\chi r \rightarrow q \in \uparrow (x \rightarrow q : \cdot r \rightarrow q).$$
  - $X_{\{p,q,r,u\}}(q \uparrow_\chi u) = \{x\chi r\}$  implies  $u = q\langle s/t \mid s\chi t \in X_{\{p,q,r,u\}}(q \uparrow_\chi u) \rangle = q\langle x/r \rangle$  (Lemma 47). Hence,
 
$$(p \uparrow_\chi r) \rightarrow (q \uparrow_\chi u) = x \rightarrow q\langle x/x\chi r \rangle \in \uparrow (x \rightarrow q : \cdot r \rightarrow q\langle x/r \rangle).$$
- $p = f(p_1, \dots, p_{rf}) \in T_{LX}$ :
  - $r = f(r_1, \dots, r_{rf})$ :
    - \*  $q = a \in L_0$ :
      - $u = a \in L_0$ :  $(p \uparrow_\chi r) \rightarrow (q \uparrow_\chi u) = p \uparrow_\chi r \rightarrow a \in \uparrow (p \rightarrow a : \cdot r \rightarrow a)$ .
      - $u \neq a$ :  $X_{\{p,q,r,u\}}(a \uparrow_\chi u) = \{a\chi u\} \subseteq X_{\{p,q,r,u\}}(p \uparrow_\chi r)$  implies  $a\chi u \in X_{\{p,q,r,u\}}(p \uparrow_\chi r)$ . By induction hypothesis, we have
 
$$p_i \uparrow_\chi r_i \rightarrow a\chi u \in \uparrow (p_i \rightarrow a : \cdot r_i \rightarrow u),$$

for all  $1 \leq i \leq rf$ . This holds iff

$$p_i = (p_i \uparrow_\chi r_i)(\mathbf{o}_i),$$

$$r_i = (p_i \uparrow_\chi r_i)(\mathbf{o}'_i),$$

for some  $\mathbf{o}_i = (o_{i,1}, \dots, o_{i,|X_{\dots}(p_i \uparrow_\chi r_i)|}) \in T_{LX}^{|X_{\dots}(p_i \uparrow_\chi r_i)|}$  such that  $o_{i,j} = a\chi u$ , for some  $1 \leq j \leq |X_{\dots}(p_i \uparrow_\chi r_i)|$ . We then have:

$$\begin{array}{c}
 \begin{array}{ccc}
 & a\chi u & \\
 \mathbf{x}/(\mathbf{o}_1, \dots, \mathbf{o}_{rf}) & \nearrow & \searrow \mathbf{x}/(\mathbf{o}'_1, \dots, \mathbf{o}'_{rf}) \\
 & & \\
 f(p_1, \dots, p_{rf}) \rightarrow a & \text{::} & f(r_1, \dots, r_{rf}) \rightarrow u. \\
 & & \\
 \mathbf{x}/(\mathbf{o}_1, \dots, \mathbf{o}_{rf}) & \searrow & \nearrow \mathbf{x}/(\mathbf{o}'_1, \dots, \mathbf{o}'_{rf}) \\
 & & \\
 f(p_1 \uparrow_\chi r_1, \dots, p_{rf} \uparrow_\chi r_{rf})(x \mid x \in X_{\{p,q,r,u\}}(p_i \uparrow_\chi r_i), 1 \leq i \leq rf)
 \end{array}
 \end{array}$$

This shows

$$(p \uparrow_{\chi} r) \rightarrow (q \uparrow_{\chi} u) = f(p_1 \uparrow_{\chi} r_1, \dots, p_{rf} \uparrow_{\chi} r_{rf}) \rightarrow a\chi u \in \uparrow (p \rightarrow a : \cdot r \rightarrow u).$$

\*  $q = x \in X$ :

$$\cdot u = x: p \uparrow_{\chi} r \rightarrow x \in \uparrow (p \rightarrow z : \cdot r \rightarrow z).$$

$$\cdot u \neq x: X_{\{p,q,r,u\}}(q \uparrow_{\chi} u) = \{x\chi u\} \subseteq X_{\{p,q,r,u\}}(p \uparrow_{\chi} r) \text{ implies } x\chi u \in X_{\{p,q,r,u\}}(p \uparrow_{\chi} r). \text{ Now proceed as in the case “}u \neq a\text{” above.}$$

\*  $q = g(q_1, \dots, q_{r(g)})$ :

$$\cdot u = q: \text{trivial.}$$

$$\cdot u \neq q: X_{\{p,q,r,u\}}(q \uparrow_{\chi} u) = \{q\chi u\} \subseteq X_{\{p,q,r,u\}}(p \uparrow_{\chi} r) \text{ implies } q\chi u \in X_{\{p,q,r,u\}}(p \uparrow_{\chi} r). \text{ Now proceed as in the case “}u \neq a\text{” above.}$$

\*  $r = g(r_1, \dots, r_{rr})$ :<sup>10</sup>  $p \uparrow_{\chi} r = p\chi r$  and  $X_{\{p,q,r,u\}}(q \uparrow_{\chi} u) \subseteq \{p\chi r\}$ . We distinguish two cases:

$$(1) X_{\{p,q,r,u\}} = \emptyset \text{ implies } q = u.$$

$$(2) X_{\{p,q,r,u\}} = \{p\chi r\} \text{ implies } u = q\langle p/r \rangle \text{ (Lemma 47). Hence,}$$

$$p\chi r \rightarrow q\langle p/p\chi r \rangle \in \uparrow (p \rightarrow q : \cdot r \rightarrow u).$$

□

**Lemma 49.**  $X_{\{p,q,r,u\}}(q \uparrow_{\chi} u) \subseteq X_{\{p,q,r,u\}}(p \uparrow_{\chi} r)$  implies  $p \rightarrow q : \cdot \mathfrak{z}_{LX} r \rightarrow u$ .

*Proof.* A direct consequence of Fact 44 and Lemma 48. □

We have thus arrived at the following sufficient condition for tree proportions:

**Theorem 50.**  $X_{\{p,q,r,u\}}(q \uparrow_{\chi} u) = X_{\{p,q,r,u\}}(p \uparrow_{\chi} r)$  implies  $p : q :: \mathfrak{z}_{LX} r : u$ .

*Proof.* A direct consequence of Lemma 49 and the definition of a tree proportion in terms of arrow proportions. □

**Corollary 51.** For every term function  $f : T_{LX} \rightarrow T_{LX}$ ,

$$p : q :: \mathfrak{z}_{LX} r : u \Leftrightarrow fp : fq :: \mathfrak{z}_{LX} fr : fu.$$

*Proof.* A direct consequence of Theorem 50 and the fact that

$$\begin{aligned} X_{\{p,q,r,u\}}(q \uparrow_{\chi} u) &= X_{\{p,q,r,u\}}(p \uparrow_{\chi} r) \\ &\Leftrightarrow X_{\{fp,fq,fr,fu\}}(fq \uparrow_{\chi} fu) = X_{\{fp,fq,fr,fu\}}(fp \uparrow_{\chi} fr). \end{aligned}$$

□

## 8. ANTI-UNIFICATION

In the previous section, we have seen that classical syntactic anti-unification can be used to yield tree proportions and motivated by that result, we initiate in this section the study of interactions between analogical proportions and anti-unification beyond the free term algebras. For this, we first need a notion of anti-unification generalized from term to arbitrary algebras which has been recently provided by the author (Antić, 2023a) and which we shall now briefly recall:

<sup>10</sup>Notice that  $r$  means here two things: (i) an  $L$ -term; and (ii) the rank function. Hence,  $rr$  stands for the rank of the term  $r$ .



**Definition 52.** Define the set of *minimally general generalizations* (or *mggs*) of two elements  $a \in A$  and  $b \in B$  in  $\mathfrak{A}\mathfrak{B}$  by

$$a \uparrow_{\mathfrak{A}\mathfrak{B}} b := \min_{\sqsubseteq_{\mathfrak{A}\mathfrak{B}}} (a \uparrow_{\mathfrak{A}\mathfrak{B}} b),$$

where

$$a \uparrow_{\mathfrak{A}\mathfrak{B}} b := (\uparrow_{\mathfrak{A}} a) \cap (\uparrow_{\mathfrak{B}} b)$$

and

$$\uparrow_{\mathfrak{A}} a := \{s \in T_{LX} \mid a = s^{\mathfrak{A}} \mathbf{o}, \text{ for some } \mathbf{o} \in A^{rs}\}.$$

In case  $a \uparrow_{\mathfrak{A}\mathfrak{B}} b = \{s\}$  contains a single generalization, we call  $s$  the *least general generalization* of  $a$  and  $b$  in  $\mathfrak{A}\mathfrak{B}$ .

We now wish to *initiate* the study of connections between algebraic anti-unification and analogical proportions with an illustrative example:

**Example 53.** In Antić (2022, Example 66), the author has computed the solutions of  $20 : 4 :: 30 : v$  in the multiplicative algebra  $\mathfrak{M} := (\mathbb{N}_2, \cdot, \mathbb{N}_2)$ , where  $\mathbb{N}_2 := \{2, 3, \dots\}$ , as

$$\mathfrak{S}_{\mathfrak{M}}(20 : 4 :: 30 : v) = \{6, 9\}.$$

The two solutions are characteristically justified respectively by

$$10x \rightarrow 2x \quad \text{and} \quad 10x \rightarrow x^2.$$

We see that in both cases, the generalization  $10x$  occurs on the left-hand side of the respective justifications — this seems not to be an accident as we have

$$20 \uparrow 30 = \{10x\}$$

as we are now going to show; in other words,  $10x$  is the least general generalization of 20 and 30 in  $\mathfrak{M}$ .

Recall from Antić (2022, Example 66) that we have (we omit the subscript  $\mathfrak{M}$ )

$$\begin{aligned} \uparrow 4 &= \{4, 2x, xy, x^2, x\}, \\ \uparrow 20 &= \left\{ \begin{array}{ccc} 20 & 10x & 5x^2 \\ & 4x & 5xy \\ 2xy & & xyz \\ 2x & 5x & xy \\ x^2y & x & \end{array} \right\}, \\ \uparrow 30 &= \left\{ \begin{array}{ccc} 30 & 15x & \\ 10x & 6z & 5xy \\ 2xy & 3xy & xyz \\ 2x & 5x & xy \\ & 3x & x \end{array} \right\}. \end{aligned}$$

Hence, we have

$$20 \uparrow 30 = \{10x, 2xy, 5xy, xyz, 2x, 5x, xy, x\}.$$

We now want to find the  $\sqsubseteq$ -minimal generalizations in  $20 \uparrow 30$ . The generalization  $x$  is, of course, not minimal. We have

$$\downarrow 2xy = \{8, 12, 18, \dots\} \sqsubset \{4, 6, 8, 10, 12, \dots\} = \downarrow 2x$$

which shows

$$2x \sqsubset 2xy.$$

Hence, we can exclude  $2xy$ . Analogously,

$$5x \sqsubset 5xy.$$

shows that we can exclude  $5xy$ . Moreover, we clearly have

$$10x \sqsubset 2x \quad \text{und} \quad 10x \sqsubset 5x,$$

which means that we can exclude  $2x$  and  $5x$ . We are thus left with the generalizations

$$10x \quad \text{and} \quad xy \quad \text{and} \quad xyz.$$

We clearly have

$$10x \sqsubset xy \quad \text{und} \quad 10x \sqsubset xyz,$$

which means that we are left with

$$20 \uparrow\uparrow 30 = \min_{\sqsubseteq} (20 \uparrow 30) = \{10x\}.$$

Of course, a single example is not enough to establish a strong connection between two concepts — this brief section is to be understood only as an *inspiration* for a deeper investigation of the relation between algebraic anti-unification and analogical proportions.

## 9. FINITE ALGEBRAS

In this section, we provide an algorithm for the computation of the analogical proportion relation within the  $k$ -fragment (cf. §5) in finite algebras via tree automata.

Recall that a (**frontier-to-root**) **tree automaton** (see e.g. Gécseg & Steinby, 2015)

$$\mathcal{T}_{\mathfrak{A},k,\alpha,F} := (\mathfrak{A}, L, X_k, \alpha, F)$$

consists of

- a *finite*  $L$ -algebra  $\mathfrak{A}$ ,
- an **initial assignment**  $\alpha : X_k \rightarrow A$ , and
- a set  $F \subseteq A$  of **final states**.

The **regular tree language** recognized by  $\mathcal{T}_{\mathfrak{A},k,\alpha,F}$  is given by

$$\|\mathcal{T}_{\mathfrak{A},k,\alpha,F}\| := \{s \in T_{LX_k} \mid s^{\mathfrak{A}}\alpha \in F\}.$$

We can thus rewrite the set of  $k$ -generalizations of  $a$  in  $\mathfrak{A}$  (see §5),

$$\uparrow_{\mathfrak{A}}^k a := (\uparrow_{\mathfrak{A}} a) \cap T_{LX_k},$$

by

$$\uparrow_{\mathfrak{A}}^k a = \bigcup_{\alpha \in A^{X_k}} \|\mathcal{T}_{\mathfrak{A},k,\alpha,\{a\}}\|,$$

and the set of  $k$ -justifications of an arrow  $a \rightarrow b$  in  $\mathfrak{A}$  in terms of tree automata as

$$\uparrow_{\mathfrak{A}}^k (a \rightarrow b) = \bigcup_{\alpha \in A^{X_k}} [(\|\mathcal{T}_{\mathfrak{A},k,\alpha,\{a\}}\| \rightarrow \|\mathcal{T}_{\mathfrak{A},k,\alpha,\{b\}}\|) \cap \{s \rightarrow t \mid Xt \subseteq Xs\}],$$

where for two forests  $S$  and  $T$ ,

$$S \rightarrow T := \{s \rightarrow t \mid s \in S, t \in T\}.$$

Notice that the set  $A^{X_k}$  of all initial assignments  $X_k \rightarrow A$  is *finite* by our assumption that  $X_k$  is a finite set of variables. Since it is well-known that tree automata are closed under finite unions, there is a tree automaton

$$\mathcal{T}_{\mathfrak{A},k,a \rightarrow b},$$

for every arrow  $a \rightarrow b$ , such that

$$\uparrow_{\mathfrak{A}}^k(a \rightarrow b) = \|\mathcal{T}_{\mathfrak{A},k,a \rightarrow b}\| \cap \{s \rightarrow t \mid Xt \subseteq Xs\}.$$

Now since tree automata are closed under intersection as well, there is a tree automaton

$$\mathcal{T}_{\mathfrak{A}\mathfrak{B},k,a \rightarrow b : c \rightarrow d},$$

for every arrow proportion  $a \rightarrow b : c \rightarrow d$ , such that

$$\uparrow_{\mathfrak{A}\mathfrak{B}}^k(a \rightarrow b : c \rightarrow d) = \|\mathcal{T}_{\mathfrak{A}\mathfrak{B},k,a \rightarrow b : c \rightarrow d}\| \cap \{s \rightarrow t \mid Xt \subseteq Xs\}.$$

Since  $\mathfrak{B}$  is finite by assumption, checking the  $d$ -maximality of  $\uparrow_{\mathfrak{A}\mathfrak{B}}^k(a \rightarrow b : c \rightarrow d)$  can thus be easily achieved with a search linear in the size of  $\mathfrak{B}$ ; checking the emptiness of  $\uparrow_{\mathfrak{A}}^k(a \rightarrow b)$  and  $\uparrow_{\mathfrak{B}}^k(c \rightarrow d)$  is well-known to be decidable as well (cf. Comon et al., 2008, p. 40); and checking  $Xt \subseteq Xs$  is a simple syntactic comparison, which in total gives us an algorithm for deciding

$$a \rightarrow b : \cdot_{\mathfrak{A}\mathfrak{B},k} c \rightarrow d$$

and thus for deciding

$$a : b ::_{\mathfrak{A}\mathfrak{B},k} c : d.$$

**Pseudocode 54** (Decision algorithm). Given  $k \geq 1$ ,  $a, b \in A$ ,  $c, d \in B$ , and a pair of finite  $L$ -algebras  $\mathfrak{A}\mathfrak{B}$ , we design an algorithm for deciding whether

$$a : b \stackrel{?}{::}_{\mathfrak{A}\mathfrak{B},k} c : d.$$

We first provide an algorithm for deciding whether

$$(9) \quad a \rightarrow b : \cdot_{\mathfrak{A}\mathfrak{B},k} c \rightarrow d.$$

- (1) Construct the tree automata  $\mathcal{T}_{\mathfrak{A},k,a \rightarrow b}$  and  $\mathcal{T}_{\mathfrak{B},k,c \rightarrow d}$  as described above. If

$$(\|\mathcal{T}_{\mathfrak{A},k,a \rightarrow b}\| \cup \|\mathcal{T}_{\mathfrak{B},k,c \rightarrow d}\|) \cap \{s \rightarrow t \mid Xt \subseteq Xs\} = \emptyset$$

then stop with answer “yes”.

- (2) Otherwise, construct the tree automaton  $\mathcal{T}_{\mathfrak{A}\mathfrak{B},k,a \rightarrow b : c \rightarrow d}$  as described above and compute the forest

$$T := \|\mathcal{T}_{\mathfrak{A}\mathfrak{B},k,a \rightarrow b : c \rightarrow d}\|.$$

- (3) Compute the set of  $k$ -justifications

$$J := T \setminus \{s \rightarrow t \mid Xt \not\subseteq Xs\}$$

applying a simple syntactic check on each rule in  $T$  (notice that  $T$  may be infinite).

- (4) We now want to check whether  $J$  is  $d$ -maximal:

- (a) For each  $d' \neq d \in B$ :

- (i) Construct the set of  $k$ -justifications

$$J' := \|\mathcal{T}_{\mathfrak{A}\mathfrak{B},k,a \rightarrow b : c \rightarrow d'}\| \setminus \{s \rightarrow t \mid Xt \not\subseteq Xs\}$$

as above.

- (ii) If  $J \subseteq J'$  then stop with answer “no”.

- (b) Return the answer “yes”.
- (5) At this point, we have decided (9). Now repeat the above steps for the remaining arrow proportions

$$b \rightarrow a : \cdot d \rightarrow c \quad \text{and} \quad c \rightarrow d : \cdot a \rightarrow b \quad \text{and} \quad d \rightarrow c : \cdot b \rightarrow a$$

and return “yes” iff the answer is “yes” in each case.

**Pseudocode 55.** Given a pair of  $L$ -algebras  $\mathfrak{A}\mathfrak{B}$  and  $k \geq 1$ , computing the set

$$\{(a, b, c, d) \in A^2 \times B^2 \mid a : b ::_{\mathfrak{A}\mathfrak{B}, k} c : d\}$$

can be done using Pseudocode 54 to decide  $a : b ::_{\mathfrak{A}\mathfrak{B}, k}^? c : d$  on each of the finitely many tuples.

**Pseudocode 56** (Solving proportional equations). Given a proportional equation

$$a : b ::_k c : v,$$

finding some/all  $d \in B$  such that  $a : b ::_{\mathfrak{A}\mathfrak{B}, k} c : d$  can be achieved using Pseudocode 55.

## 10. DISCUSSION

The purpose of this paper was to expand the mathematical theory of analogical proportions within the abstract algebraic framework recently introduced by the author within the general setting of universal algebra.

We shall now discuss some lines of potential future research.

Section §5 introduced the  $(k, \ell)$ -fragments and in §5.1 – §5.3 we have studied the monilinear  $(1, 1)$ -fragment in the setting of numbers and words. The monilinear fragment for sets appears more challenging given that union and intersection are highly non-injective operations. The next step is to study the **linear fragment**  $(\infty, 1)$  — consisting of justifications with arbitrary many variables occurring at most once — for numbers, words, and sets. From a theoretical point, it is interesting to analyze relationships between different such fragments as it may be the case that we obtain a  $(k, \ell)$ -**hierarchy** of increasing expressive power.

In §6, we have derived some first partial results regarding word proportions where we showed in Theorem 42 that an important notion of word proportions is subsumed by our framework. However, a **full characterization of the word proportion relation** — analogous to the one provided in the monilinear fragment in Theorem 29 — remains a challenging and practically relevant open problem.

**Infinite trees** naturally arise in the study of programming languages (see e.g. Courcelle, 1983). It is thus interesting to generalize the concepts and results of §7 to tree proportions between infinite trees.

The algorithms of §9 regarding analogical proportions in finite algebras are restricted to the  $k$ -fragment where justifications may contain at most  $k$  different variables and is thus bounded. This is necessary for the techniques of the theory of tree automata to be applicable. It appears challenging to **generalize those algorithms for finite algebras to the full framework** where the number of variables in justifications is unbounded. Even more challenging is the task to **find algorithms for the computation of analogical proportions beyond finite algebras**, most importantly in finitely representable algebras such as automatic structures (Blumensath & Grädel, 2000, 2004).

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