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ANALOGICAL PROPORTIONS II

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ABSTRACT. The purpose of this paper is to further develop the theory of analogical proportions within a recently introduced algebro-logical framework. Specifically, we show that analogical proportions are compatible with homomorphisms and related to semantic anti-unification; we initiate the study of fragments consisting of justifications of a restricted form; and we show how analogical proportions can be computed in finite algebras using tree automata.

1. INTRODUCTION

Analogical proportions are expressions of the form

“ a is to b what c is to d ”

written

$$a : b :: c : d$$

at the core of analogical reasoning, which itself is **at the core of artificial general intelligence** (e.g. Boden, 1998; Gentner, 1983; Gust et al., 2008; Hesse, 1966; Hofstadter, 2001; Hofstadter & Sander, 2013; Krieger, 2003; Pólya, 1954; Winston, 1980). They have numerous applications in AI such as

- computational linguistics (e.g. Lepage, 1998, 2001, 2003),
- image processing (e.g. Lepage, 2014),
- recommender systems (e.g. Hug et al., 2019),
- program synthesis (Antić, 2023e),

just to name a few (cf. Prade & Richard, 2021).

Formal models of analogical proportions started to appear only recently, most notably

- Lepage’s (2001) **axiomatic** approach in the linguistic setting,
- Miclet and Prade’s (2009) **logical** approach (cf. Prade & Richard, 2013; Prade et al., 2018),
- Stroppa and Yvon’s (2006) and Gust et al.’s (2006) **algebraic** approaches.

See Prade and Richard (2014) for a (somewhat outdated) short introduction to analogical reasoning, and Hall (1989) for a historic overview of approaches to analogical reasoning.

In a series of papers (some of which are under review), Antić (2022, 2023b, 2023d, 2023f) has recently introduced an **abstract algebro-logical framework of analogical proportions** in the general setting of **universal algebra**. It is a promising novel model of analogical proportions with appealing mathematical properties (cf. Antić, 2023c, 2023i, 2023h).

The **purpose of this paper** is to further develop the algebraic theory of the framework in Antić (2022):

In §2, we briefly recall the framework in Antić (2022), where we assume that the reader is fluent in basic universal algebra as it is presented for example in Burris and Sankappanavar (2000, §II) and Baader and Nipkow (1998, §3).

In §3, we observe that sets of justifications are principal filters, which motivates a change of notation. We then introduce some terminology which make the Uniqueness Lemma and Functional Proportion Theorem in Antić (2022) easier to state.

In §4, we prove a Homomorphism Theorem as an analogue to the First Isomorphism Theorem in Antić (2022) showing that arrow proportions — which are expressions of the form $a \rightarrow b : \cdot c \rightarrow d$ expressing that “ a transforms into b as c transforms into d ” (see §2) — are compatible with homomorphisms in the sense that in case H satisfies a mild condition, we have $a \rightarrow b : \cdot Ha \rightarrow Hb$.

In §5, we show that semantic anti-unification — as recently introduced in Antić (2023g) — is related to analogical proportions by giving an illustrative Example 9. Other authors have noted the connection between analogical proportions and anti-unification in other frameworks before: Krumnack et al. (2007) is a paper dealing with restricted higher-order anti-unification and analogy making, and Weller and Schmid (2007) use anti-unification for computing analogical proportions using regular tree grammars.

In §6, we initiate the study of fragments of the framework where the form of justifications is syntactically restricted. Particularly, in §6.1–6.2, we show that we can capture difference and geometric proportions already in the simplest monolinear fragment consisting only of justifications with at most one occurrence of a single variable. Moreover, in §6.3, we study monolinear word proportions.

Finally, in §7, we show that analogical proportions can be computed in *finite* algebras using tree automata where we provide algorithms for the most important computational tasks.

2. PRELIMINARIES

This section recalls the abstract algebraic framework of analogical proportions in Antić (2022). We assume the reader to be fluent in basic universal algebra as it is presented for example in Burris and Sankappanavar (2000, §II) and Baader and Nipkow (1998, §3).

A *language* L of algebras is a set of *function symbols*¹ together with a *rank function* $r : L \rightarrow \mathbb{N}$, and a denumerable set $Z = \{z, z_1, z_2, \dots\}$ of *Z-variables* distinct from L . Terms are formed as usual from variables and function symbols.

An *L-algebra* \mathfrak{A} consists of a non-empty set A , the *universe* of \mathfrak{A} , and for each function symbol $f \in L$, a function $f^{\mathfrak{A}} : A^{r(f)} \rightarrow A$, the *functions* of \mathfrak{A} (the *distinguished elements* of \mathfrak{A} are the 0-ary functions). Every term s induces a function $s^{\mathfrak{A}}$ on \mathfrak{A} in the usual way. We call a term t *injective* in \mathfrak{A} iff $t^{\mathfrak{A}}$ is an injective function.

We will always write $s(\mathbf{z}) \rightarrow t(\mathbf{z})$ or $s \rightarrow t$ instead of (s, t) , for any pair of L -terms s and t containing variables among \mathbf{z} such that every variable in t occurs in s , that is, $Z(t) \subseteq Z(s)$. We call such expressions *L-rewrite rules* or *L-justifications* where we often omit the reference to L . We denote the set of all L -justifications with variables among Z by $J_L(Z)$. We make the convention that \rightarrow binds weaker than every other algebraic operation.

We define the *analogical proportion entailment relation* in two steps:

- (1) Define the *set of justifications* of an arrow $a \rightarrow b$ in \mathfrak{A} by²

$$Jus_{\mathfrak{A}}(a \rightarrow b) := \left\{ s \rightarrow t \in J_L(Z) \mid a \rightarrow b = s^{\mathfrak{A}}(\mathbf{o}) \rightarrow t^{\mathfrak{A}}(\mathbf{o}), \text{ for some } \mathbf{o} \in A^{|\mathbf{z}|} \right\},$$

¹We omit constant symbols as we identify constants with 0-ary functions.

²For a sequence of objects $\mathbf{o} = o_1 \dots o_n$ define $|\mathbf{o}| := n$.

extended to an *arrow proportion* $a \rightarrow b : \cdot c \rightarrow d^3$ in $(\mathfrak{A}, \mathfrak{B})$ by

$$Jus_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot c \rightarrow d) := Jus_{\mathfrak{A}}(a \rightarrow b) \cap Jus_{\mathfrak{B}}(c \rightarrow d).$$

A justification is *trivial* in $(\mathfrak{A}, \mathfrak{B})$ iff it justifies every arrow proportion in $(\mathfrak{A}, \mathfrak{B})$, and we say that J is a *trivial set of justifications* in $(\mathfrak{A}, \mathfrak{B})$ iff every justification in J is trivial.

Now we say that $a \rightarrow b : \cdot c \rightarrow d$ *holds* in $(\mathfrak{A}, \mathfrak{B})$ — in symbols,

$$(\mathfrak{A}, \mathfrak{B}) \models a \rightarrow b : \cdot c \rightarrow d$$

iff

- (a) either $Jus_{\mathfrak{A}}(a \rightarrow b) \cup Jus_{\mathfrak{B}}(c \rightarrow d)$ consists only of trivial justifications, in which case there is neither a non-trivial relation from a to b in \mathfrak{A} nor from c to d in \mathfrak{B} ; or
- (b) $Jus_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot c \rightarrow d)$ is maximal with respect to subset inclusion among the sets $Jus_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot c \rightarrow d')$, $d' \in B$, containing at least one non-trivial justification, that is, for any element $d' \in \mathfrak{B}$,⁴

$$\emptyset \subsetneq Jus_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot c \rightarrow d) \subseteq Jus_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot c \rightarrow d')$$

implies

$$\emptyset \subsetneq Jus_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot c \rightarrow d') \subseteq Jus_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot c \rightarrow d).$$

We abbreviate the above definition by simply saying that $Jus_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot c \rightarrow d)$ is *d-maximal*.

- (2) Finally, the analogical proportion entailment relation is most succinctly defined by

$$\begin{aligned} a : b :: c : d \quad & :\Leftrightarrow \quad a \rightarrow b : \cdot c \rightarrow d \quad \text{and} \quad b \rightarrow a : \cdot d \rightarrow c \\ & \quad \quad \quad c \rightarrow d : \cdot a \rightarrow b \quad \text{and} \quad d \rightarrow c : \cdot b \rightarrow a. \end{aligned}$$

This means that in order to prove $(\mathfrak{A}, \mathfrak{B}) \models a : b :: c : d$, we need to check the first two relations in the first line with respect to \models in $(\mathfrak{A}, \mathfrak{B})$, and the last two relations in the same line in $(\mathfrak{B}, \mathfrak{A})$.

We will always write \mathfrak{A} instead of $(\mathfrak{A}, \mathfrak{A})$.

Example 1 (Antić (2022), Example 11). First consider the algebra $\mathfrak{A}_1 := (\{a, b, c, d\})$, consisting of four distinct elements with no functions and no constants:

$$b \quad d$$

$$a \quad c$$

Since $Jus_{\mathfrak{A}_1}(a', b') \cup Jus_{\mathfrak{A}_1}(c', d')$ contains only trivial justifications for *any distinct* elements $a', b', c', d' \in A'$, we have, for example:

$$\mathfrak{A}_1 \models a : b :: c : d \quad \text{and} \quad \mathfrak{A}_1 \models a : c :: b : d.$$

On the other hand, since

$$Jus_{\mathfrak{A}_1}(a, a) \cup Jus_{\mathfrak{A}_1}(a, d) = \{z \rightarrow z\} \neq \emptyset$$

and

$$\emptyset = Jus_{\mathfrak{A}_1}(a \rightarrow a : \cdot a \rightarrow d) \subsetneq Jus_{\mathfrak{A}_1}(a \rightarrow a : \cdot a \rightarrow a) = \{z \rightarrow z\},$$

³Read as “ a transforms into b as c transforms into d ”.

⁴We ignore trivial justifications and write “ $\emptyset \subsetneq \dots$ ” instead of “ $\{\text{trivial justifications}\} \subsetneq \dots$ ” et cetera.

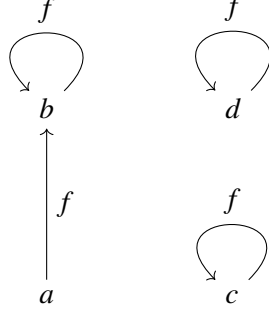
we have

$$\mathfrak{A}_1 \not\models a \rightarrow a : \cdot a \rightarrow d,$$

which implies

$$\mathfrak{A}_1 \not\models a : a :: a : d.$$

Now consider the slightly different algebra $\mathfrak{A}_2 := (\{a, b, c, d\}, f)$, where f is the unary function defined by



We expect $a : b :: c : d$ to fail in \mathfrak{A}_2 as it has no non-trivial justification. In fact,

$$Jus_{\mathfrak{A}_2}(a, b) \cup Jus_{\mathfrak{A}_2}(c, d) = \{z \rightarrow f^\ell(z) \mid \ell \geq 1\} \neq \emptyset$$

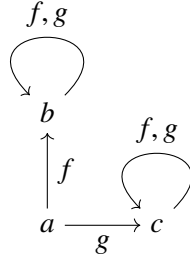
and

$$Jus_{\mathfrak{A}_2}(a \rightarrow b : \cdot c \rightarrow d) = \emptyset$$

show

$$\mathfrak{A}_2 \not\models a : b :: c : d.$$

In the algebra \mathfrak{A}_3 given by



we have

$$\mathfrak{A}_3 \not\models a : b :: a : c.$$

The intuitive reason is that $a : b :: a : b$ is a more plausible proportion than $a : b :: a : c$, which is reflected in the computation

$$\emptyset = Jus_{\mathfrak{A}_3}(a \rightarrow b : \cdot a \rightarrow c) \subsetneq Jus_{\mathfrak{A}_3}(a \rightarrow b : \cdot a \rightarrow b) = \{z \rightarrow f(z), \dots\}.$$

Computing all justifications of an arrow proportion is difficult in general, which fortunately can be omitted in many cases. We call a set J of justifications a *characteristic set of justifications* of $a \rightarrow b : \cdot c \rightarrow d$ in $(\mathfrak{A}, \mathfrak{B})$ iff J is a sufficient set of justifications in the sense that

- (1) $J \subseteq Jus_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot c \rightarrow d)$, and
- (2) $J \subseteq Jus_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot c \rightarrow d')$ implies $d' = d$, for each $d' \in B$.

In case $J = \{s \rightarrow t\}$ is a singleton set satisfying both conditions, we call $s \rightarrow t$ a *characteristic justification* of $a \rightarrow b : \cdot c \rightarrow d$ in $(\mathfrak{A}, \mathfrak{B})$.

Lepage (2003) (cf. Miclet et al., 2008, pp. 796-797) introduces the following axioms in the linguistic context as a guideline for formal models of analogical proportions (over a single universe), adapted here to our framework formulated above:⁵

- (1) $(\mathfrak{A}, \mathfrak{B}) \models a : b :: c : d \Leftrightarrow (\mathfrak{B}, \mathfrak{A}) \models c : d :: a : b$ (p-symmetry),
- (2) $\mathfrak{A} \models a : b :: c : d \Leftrightarrow \mathfrak{A} \models a : c :: b : d$ (central permutation),
- (3) $\mathfrak{A} \models a : a :: c : d \Rightarrow d = c$ (strong inner p-reflexivity),
- (4) $\mathfrak{A} \models a : b :: a : d \Rightarrow d = b$ (strong p-reflexivity).

Antić (2022, §4.3) argues why Lepage's original list of axioms is inadequate in the general setting justified by simple counterexamples and he replaces the above list by the following list of axioms (which have been considered by other authors as well):

- (5) $(\mathfrak{A}, \mathfrak{B}) \models a : b :: c : d \Leftrightarrow (\mathfrak{B}, \mathfrak{A}) \models c : d :: a : b$ (p-symmetry),
- (6) $(\mathfrak{A}, \mathfrak{B}) \models a : b :: c : d \Leftrightarrow (\mathfrak{A}, \mathfrak{B}) \models b : a :: d : c$ (inner p-symmetry),
- (7) $(\mathfrak{A}, \mathfrak{B}) \models a : a :: c : c$ (inner p-reflexivity),
- (8) $\mathfrak{A} \models a : b :: a : b$ (p-reflexivity),
- (9) $\mathfrak{A} \models a : a :: a : d \Leftrightarrow d = a$ (p-determinism).

Furthermore, the following properties are considered, for L -algebras $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ and elements $a, b \in A$, $c, d \in B$, $e, f \in C$:

$$\frac{(\mathfrak{A}, \mathfrak{B}) \models a : b :: c : d \quad (\mathfrak{B}, \mathfrak{C}) \models c : d :: e : f}{(\mathfrak{A}, \mathfrak{C}) \models a : b :: e : f} \text{ (p-transitivity),}$$

and, for elements $a, b, e \in A$ and $c, d, f \in B$, the property

$$\frac{(\mathfrak{A}, \mathfrak{B}) \models a : b :: c : d \quad (\mathfrak{A}, \mathfrak{B}) \models b : e :: d : f}{(\mathfrak{A}, \mathfrak{B}) \models a : e :: c : f} \text{ (inner p-transitivity),}$$

and, for elements $a \in A$, $b \in A \cap B$, $c \in B$ and \mathfrak{C} , and $d \in \mathfrak{C}$, the property

$$\frac{(\mathfrak{A}, \mathfrak{B}) \models a : b :: b : c \quad (\mathfrak{B}, \mathfrak{C}) \models b : c :: c : d}{(\mathfrak{A}, \mathfrak{C}) \models a : b :: c : d} \text{ (central p-transitivity).}$$

Notice that central transitivity follows from transitivity.

3. JUSTIFICATIONS

In this section, we observe that sets of justifications are principal filters, which will motivate a change of notation by replacing Jus by \uparrow thus expressing syntactically the close connection to generalizations more adequately (this has already been observed in Antić (2023h) in the context of tree proportions).

Recall that a *filter* F on a pre-ordered set (P, \leq) is a subset of P satisfying:

- (1) F is non-empty.
- (2) F is downward directed, that is, for every $a, b \in F$, there is some $c \in F$ such that $c \leq a, b$.
- (3) F is an upper set or upward closed, that is, for every $a \in F$ and $b \in P$, if $a \leq b$ then $b \in F$.

⁵Lepage (2003) formulates his axioms to hold in a single domain without any reference to an underlying structure \mathfrak{A} .

The smallest filter containing an element a is a *principal filter* and a is a *principal element* — it is given by

$$\uparrow_{(P, \leq)} a := \{b \in P \mid a \leq b\}.$$

We extend the generalization pre-ordering from terms to justification via

$$s \rightarrow t \lesssim s' \rightarrow t' \iff s \lesssim s' \quad \text{and} \quad t \lesssim t'.$$

Fact 2. *The set of all generalizations of a term forms a principal filter with respect to the generalization pre-ordering generated by that term. Moreover, the set of all justifications of an arrow forms a principal filter with respect to the generalization pre-ordering generated by that justification.*

Notation 3. Fact 2 motivates the following notation which we will use in the rest of the paper:

$$\uparrow_{\mathfrak{A}}(a \rightarrow b) := Jus_{\mathfrak{A}}(a \rightarrow b),$$

extended to an arrow proportion by

$$\uparrow_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot c \rightarrow d) := Jus_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot c \rightarrow d).$$

We shall now reformulate some key results in Antić (2022) using a different — hopefully more intuitive — terminology. For this, we first define, for a term $s \in T_L(Z)$ and element $a \in A$, the set

$$\langle s, a \rangle_{\mathfrak{A}} := \{\mathbf{o} \in A^{r(s)} \mid a = s^{\mathfrak{A}}(\mathbf{o})\},$$

consisting of all solutions to the polynomial equation $a = s(\mathbf{z})$ in \mathfrak{A} . We can now depict every justification $s \rightarrow t$ of $a \rightarrow b : \cdot c \rightarrow d$ as follows (see Antić, 2022, Convention 15):

$$\begin{array}{ccccc} & & & t & \\ & & & \swarrow & \searrow \\ & & \langle t, b \rangle & & \langle t, d \rangle \\ a & \rightarrow & b & \vdots & c & \rightarrow & d. \\ & & \swarrow & & \searrow & & \\ & & \langle s, a \rangle & & \langle s, c \rangle & & \\ & & s & & & & \end{array}$$

Moreover, we have

$$(10) \quad s \rightarrow t \in \uparrow(a \rightarrow b : \cdot c \rightarrow d) \iff \langle s, a \rangle \cap \langle t, b \rangle \neq \emptyset \quad \text{und} \quad \langle s, c \rangle \cap \langle t, d \rangle \neq \emptyset.$$

Define

$$\mathbb{1}_{\mathfrak{A}}(s) := \{a \in A \mid |\langle s, a \rangle_{\mathfrak{A}}| = 1\}.$$

We can now reformulate the rather opaque Uniqueness Lemma and Functional Proportion Theorem in Antić (2022) more cleanly using the above notions:

Lemma 4 (Uniqueness Lemma). *We have the following implications:*

$$\frac{s \rightarrow t \in \uparrow_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot c \rightarrow d) \quad c \in \mathbb{1}_{\mathfrak{B}}(s)}{(\mathfrak{A}, \mathfrak{B}) \models a \rightarrow b : \cdot c \rightarrow d}$$

and

$$\frac{s \rightarrow t \in \uparrow_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot c \rightarrow d) \quad a \in \mathbb{1}_{\mathfrak{A}}(s) \quad b \in \mathbb{1}_{\mathfrak{A}}(t) \quad c \in \mathbb{1}_{\mathfrak{B}}(s) \quad d \in \mathbb{1}_{\mathfrak{B}}(t)}{(\mathfrak{A}, \mathfrak{B}) \models a : b :: c : d}.$$

Theorem 5 (Functional Proportion Theorem). *For any L -term $t(z)$, we have the following implication:*

$$\frac{a \in \mathbb{1}_{\mathfrak{A}}(t) \quad c \in \mathbb{1}_{\mathfrak{B}}(t)}{(\mathfrak{A}, \mathfrak{B}) \models a : t^{\mathfrak{A}}(a) :: c : t^{\mathfrak{B}}(c)}.$$

In this case, we call $t^{\mathfrak{B}}(c)$ a functional solution of $a : b :: c : x$ in $(\mathfrak{A}, \mathfrak{B})$ characteristically justified by $z \rightarrow t(z)$. This holds in particular if t is injective in \mathfrak{A} and \mathfrak{B} .

4. HOMOMORPHISM THEOREM

In Antić (2022), the First Isomorphism Theorems is shown saying that for any isomorphism $H : \mathfrak{A} \rightarrow \mathfrak{B}$,

$$(\mathfrak{A}, \mathfrak{B}) \models a : b :: Ha : Hb, \quad \text{for all } a, b \in A,$$

and it was argued, by giving a counterexample, that homomorphisms are in general not compatible with analogical proportions in the same way.

In this section, we shall recover a part of the result by showing that homomorphisms are compatible with *arrow* proportions. We first show an auxiliary lemma (analogous to the Isomorphism Lemma in Antić (2022)):

Lemma 6 (Homomorphism Lemma). *For any homomorphism $H : \mathfrak{A} \rightarrow \mathfrak{B}$ and $a, b \in A$,*

$$(11) \quad \uparrow_{\mathfrak{A}}(a \rightarrow b) \subseteq \uparrow_{\mathfrak{B}}(Ha \rightarrow Hb).$$

Proof. We have

$$\begin{aligned} s \rightarrow t \in \uparrow_{\mathfrak{A}}(a \rightarrow b) &\Leftrightarrow a \rightarrow b = s^{\mathfrak{A}}(\mathbf{o}) \rightarrow t^{\mathfrak{A}}(\mathbf{o}), \quad \text{for some } \mathbf{o} \in A^{r(s)} \\ &\Rightarrow Ha \rightarrow Hb = H(s^{\mathfrak{A}}(\mathbf{o})) \rightarrow H(t^{\mathfrak{A}}(\mathbf{o})) = s^{\mathfrak{B}}(H\mathbf{o}) \rightarrow t^{\mathfrak{B}}(H\mathbf{o}) \\ &\Rightarrow s \rightarrow t \in \uparrow_{\mathfrak{B}}(Ha \rightarrow Hb). \end{aligned}$$

□

Theorem 7 (Homomorphism Theorem). *For any homomorphism $H : \mathfrak{A} \rightarrow \mathfrak{B}$ and elements $a, b \in A$, we have the following implication:*

$$\frac{\uparrow_{\mathfrak{A}}(a \rightarrow b) = \emptyset \quad \Rightarrow \quad \uparrow_{\mathfrak{B}}(Ha \rightarrow Hb) = \emptyset}{(\mathfrak{A}, \mathfrak{B}) \models a \rightarrow b : \cdot Ha \rightarrow Hb}.$$

Proof. By the Homomorphism Lemma 6, we have

$$\uparrow_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot Ha \rightarrow Hb) = \uparrow_{\mathfrak{A}}(a \rightarrow b) \cap \uparrow_{\mathfrak{B}}(Ha \rightarrow Hb) = \uparrow_{\mathfrak{A}}(a \rightarrow b),$$

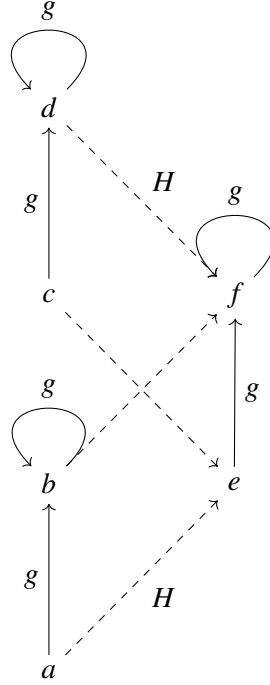
which shows the *Hb*-maximality of $\uparrow_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot Ha \rightarrow Hb)$.

It remains to show that we cannot have

$$\uparrow_{\mathfrak{A}}(a \rightarrow b) \cup \uparrow_{\mathfrak{B}}(Ha \rightarrow Hb) \neq \emptyset \quad \text{whereas} \quad \uparrow_{(\mathfrak{A}, \mathfrak{B})}(a \rightarrow b : \cdot Ha \rightarrow Hb) = \emptyset.$$

This is a direct consequence of (11) and the assumption that $\uparrow_{\mathfrak{A}}(a \rightarrow b) = \emptyset$ implies $\uparrow_{\mathfrak{B}}(Ha \rightarrow Hb) = \emptyset$. □

Example 8. Let us now analyze the counterexample in Antić (2022, Example 39). Let $\mathfrak{A} := (\{a, b, c, d\}, g)$ and $\mathfrak{B} := (\{e, f\}, g)$ and $H : \mathfrak{A} \rightarrow \mathfrak{B}$ be given by



In Antić (2022, Example 39) it is shown that

$$a \rightarrow d \not\vdash Ha \rightarrow Hd.$$

This is not a contradiction to the Homomorphism Theorem 7, since

$$\uparrow (a \rightarrow d) = \emptyset \quad \Rightarrow \quad \uparrow (Ha \rightarrow Hd) = \emptyset,$$

shows that we *cannot* apply the theorem. What we do have is

$$a \rightarrow b \vdash Ha \rightarrow Hb \quad \text{and} \quad c \rightarrow d \vdash Hc \rightarrow Hd.$$

In fact, we even have

$$a : b :: Ha : Hb \quad \text{and} \quad c : d :: Hc : Hd.$$

5. ANTI-UNIFICATION

Antić (2023g) has recently introduced *semantic* anti-unification as a generalization of classical syntactic anti-unification, which we shall now repeat here briefly.

Define the set of *minimally general generalizations* (or *mggs*) of two elements $a, b \in A$ in $(\mathfrak{A}, \mathfrak{B})$ by

$$a \sqcap_{(\mathfrak{A}, \mathfrak{B})} b := \min_{\sqsubseteq_{(\mathfrak{A}, \mathfrak{B})}} (a \uparrow_{(\mathfrak{A}, \mathfrak{B})} b),$$

where

$$a \uparrow_{(\mathfrak{A}, \mathfrak{B})} b := (\uparrow_{\mathfrak{A}} a) \cap (\uparrow_{\mathfrak{B}} b)$$

and

$$\uparrow_{\mathfrak{A}} a := \{s \in T_L(Z) \mid a = s^{\mathfrak{A}}(\mathbf{o}), \text{ for some } \mathbf{o} \in A^{r(s)}\}.$$

In case $a \sqcap_{(\mathfrak{A}, \mathfrak{B})} b = \{s\}$ contains a single generalization, we call s the *least general generalization* of a and b in $(\mathfrak{A}, \mathfrak{B})$.

We now wish to *initiate* the study of connections between semantic anti-unification and analogical proportions with an illustrative example:

Example 9. In Antić (2022, Example 66), the author has computed the solutions of $20 : 4 :: 30 : x$ in the multiplicative algebra $\mathfrak{M} := (\mathbb{N}_2, \cdot, \mathbb{N}_2)$, where $\mathbb{N}_2 := \{2, 3, \dots\}$, given by

$$\text{Sol}_{\mathfrak{M}}(20 : 4 :: 30 : x) = \{6, 9\}.$$

The two solutions are characteristically justified respectively by

$$10z \rightarrow 2z \quad \text{and} \quad 10z \rightarrow z^2.$$

We see that in both cases, the generalization $10z$ occurs on the left-hand side of the respective justifications — this seems not to be an accident as we have

$$20 \sqcap 30 = \{10z\}$$

as we are now going to show; in other words, $10z$ is the least general generalization of 20 and 30 in \mathfrak{M} .

Recall from Antić (2022, Example 66) that we have (we omit the subscript \mathfrak{M})

$$\begin{aligned} \uparrow 4 &= \{4, 2z, z_1z_2, z^2, z\}, \\ \uparrow 20 &= \left\{ \begin{array}{ccc} 20 & 10z & 5z^2 \\ & 4z & 5z_1z_2 \\ 2z_1z_2 & & z_1z_2z_3 \\ 2z & 5z & z_1z_2 \\ z_1^2z_2 & z & \end{array} \right\}, \\ \uparrow 30 &= \left\{ \begin{array}{ccc} 30 & 15z & \\ 10z & 6z & 5z_1z_2 \\ 2z_1z_2 & 3z_1z_3 & z_1z_2z_3 \\ 2z & 5z & z_1z_2 \\ & 3z & z \end{array} \right\}. \end{aligned}$$

Hence, we have

$$20 \uparrow 30 = \{10z, 2z_1z_2, 5z_1z_2, z_1z_2z_3, 2z, 5z, z_1z_2, z\}.$$

We now want to find the \sqsubseteq -minimal generalizations in $20 \sqcap 30$. The generalization z is, of course, not minimal. We have

$$\downarrow 2z_1z_2 = \{8, 12, 18, \dots\} \subset \{4, 6, 8, 10, 12, \dots\} = \downarrow 2z$$

which shows

$$2z \sqsubset 2z_1z_2.$$

Hence, we can exclude $2z_1z_2$. Analogously,

$$5z \sqsubset 5z_1z_2.$$

shows that we can exclude $5z_1z_2$. Moreover, we clearly have

$$10z \sqsubset 2z \quad \text{und} \quad 10z \sqsubset 5z,$$

which means that we can exclude $2z$ and $5z$. We are thus left with the generalizations

$$10z \quad \text{and} \quad z_1z_2 \quad \text{and} \quad z_1z_2z_3.$$

We clearly have

$$10z \sqsubset z_1 z_2 \quad \text{und} \quad 10z \sqsubset z_1 z_2 z_3,$$

which means that we are left with

$$20 \sqcap 30 = \min_{\sqsubseteq} (20 \uparrow 30) = \{10z\}.$$

Of course, a single example is not enough to establish a strong connection between two concepts — this brief section is to be understood only as an *inspiration* for a deeper investigation of the relation between semantic anti-unification (Antić, 2023g) and analogical proportions.⁶

6. THE (k, ℓ) -FRAGMENTS

Since computing the set of *all* justifications is rather difficult in general, it is reasonable to study fragments of the framework. For this, we introduce in this section the (k, ℓ) -fragments. Formally, let $Z_k := \{z_1, \dots, z_k\}$, for some $k, \ell \in \mathbb{N} \cup \{\infty\}$ so that $Z_\infty = Z$. Define

$$\uparrow_{\mathfrak{A}}^{(k, \ell)} a := (\uparrow_{\mathfrak{A}} a) \cap \{s(z_1, \dots, z_k) \in T_L(Z_k) \mid \text{each of the } k \text{ variables in } Z_k \text{ occurs at most } \ell \text{ times in } s\}.$$

We write k instead of (k, ∞) so that

$$\uparrow_{\mathfrak{A}}^k a = (\uparrow_{\mathfrak{A}} a) \cap T_L(Z_k).$$

We extend the above notions from elements to justifications by

$$\uparrow_{\mathfrak{A}}^{(k, \ell)} (a \rightarrow b) := \left\{ s \rightarrow t \in \uparrow_{\mathfrak{A}} (a \rightarrow b) \mid s \in \uparrow_{\mathfrak{A}}^{(k, \ell)} a \text{ and } t \in \uparrow_{\mathfrak{A}}^{(k, \ell)} b \right\},$$

extended to arrow proportions by

$$\uparrow_{(\mathfrak{A}, \mathfrak{B})}^{(k, \ell)} (a \rightarrow b : \cdot c \rightarrow d) := \uparrow_{\mathfrak{A}}^{(k, \ell)} (a \rightarrow b) \cap \uparrow_{\mathfrak{B}}^{(k, \ell)} (c \rightarrow d).$$

The entailment relation $\models_{(k, \ell)}$ is defined in the same way as \models with \uparrow replaced by $\uparrow^{(k, \ell)}$. In case the underlying algebras are clear from the context, we will often write

$$a : b ::_{(k, \ell)} c : d$$

to denote the analogical proportion relation in the (k, ℓ) -fragment.

A fragment which we will study in the next subsections in the arithmetical and word domain is the so-called monilinear fragment $(1, 1)$ consisting only of justifications with at most one occurrence of a single variable on each side, which we will denote by m .

6.1. Difference proportions. This section studies additive monilinear number proportions. We begin by noting that the set of monilinear justifications of $a \rightarrow b$ in $(\mathbb{Z}, +, \mathbb{Z})$ is given by

$$\begin{aligned} \uparrow^m (a \rightarrow b) = & \{k + z \rightarrow \ell + z \mid a \rightarrow b = k + o \rightarrow \ell + o, \text{ for some } o \in \mathbb{Z}\} \\ & \cup \{k + z \rightarrow b \mid a \rightarrow b = k + o \rightarrow b, \text{ for some } o \in \mathbb{Z}\} \cup \{a \rightarrow b\}. \end{aligned}$$

Remark 10. In $(\mathbb{Z}, +)$ containing no constants, the only monilinear rewrite rule is $z \rightarrow z$ which justifies only inner reflexive proportions of the form $a : a ::_m c : c$. This explains why we instead consider the algebra $(\mathbb{Z}, +, \mathbb{Z})$ in which every integer is a distinguished element.

Interestingly, it turns out that monilinear additive number proportions are characterized by difference proportions.

⁶Similar connections have been observed in other frameworks of analogical proportions by Krumnack et al. (2007) and Weller and Schmid (2007).

Theorem 11 (Difference Proportion Theorem).

$$\begin{aligned} (\mathbb{Z}, +, \mathbb{Z}) \models_m a : b :: c : d &\Leftrightarrow a = k + o, \quad b = \ell + o, \quad c = k + u, \quad d = \ell + u, \quad k, \ell, o, u \in \mathbb{N} \\ &\Leftrightarrow a - b = c - d \quad (\text{difference proportion}). \end{aligned}$$

Proof. We first show

$$(12) \quad a = k + o, \quad b = \ell + o, \quad c = k + u, \quad d = \ell + u \Leftrightarrow a - b = c - d,$$

for some integers $k, \ell, o, u \in \mathbb{Z}$. The direction from left to right holds trivially. For the other direction, we proceed as follows. We can always write $a = k + o$ and $b = \ell + o$, for some $k, \ell, o \in \mathbb{Z}$. We then have $a - b = k - \ell$. Analogously, we can always write $c = k + u$ and $d = \ell' + u$, for some $\ell', u \in \mathbb{Z}$. We then have $c - d = k - \ell'$. By assumption, we have $a - b = c - d$ which implies $k - \ell = k - \ell'$ and therefore $\ell = \ell'$ and finally $d = \ell + u$.

We now proceed to show the first equivalence in the statement of the theorem.

(\Rightarrow) By assumption, we have $(\mathbb{Z}, +, \mathbb{Z}) \models_m a \rightarrow b : \cdot c \rightarrow d$ which holds iff either

$$\uparrow^m (a \rightarrow b) \cup \uparrow^m (c \rightarrow d) = \emptyset$$

or $\uparrow^m (a \rightarrow b : \cdot c \rightarrow d)$ is non-empty and subset maximal with respect to d . In the first case, notice that neither $\uparrow^m (a \rightarrow b)$ nor $\uparrow^m (c \rightarrow d)$ can be empty as we always have

$$a \rightarrow b \in \uparrow^m (a \rightarrow b) \quad \text{and} \quad c \rightarrow d \in \uparrow^m (c \rightarrow d).$$

In the second case, by assumption we must have some monilinear justification $s(z) \rightarrow t(z)$ of $a \rightarrow b : \cdot c \rightarrow d$ in $(\mathbb{Z}, +, \mathbb{Z})$. We distinguish the following cases:

- (1) If $s(z) \rightarrow t(z)$ equals $a \rightarrow b$ or $c \rightarrow d$, we must have $a = c$ and $b = d$.
- (2) Else if $s(z) \rightarrow t(z)$ equals $k + z \rightarrow \ell + z$, we must have $a = k + o, b = \ell + o, c = k + u,$ and $d = \ell + u$, for some integers $o, u \in \mathbb{Z}$, which is equivalent to $a - b = c - d$ by (12).
- (3) Else if $s(z) \rightarrow t(z)$ equals $k + z \rightarrow b$, we must have $b = d$. Then, by assumption, we must also have

$$a \rightarrow b : \cdot_m c \rightarrow b \quad \text{and} \quad b \rightarrow a : \cdot_m b \rightarrow c.$$

So, either we have

$$\uparrow^m (b \rightarrow a) \cup \uparrow^m (b \rightarrow c) = \emptyset$$

or $\uparrow^m (b \rightarrow a : \cdot b \rightarrow c)$ is non-empty and subset maximal with respect to c . Again, the sets $\uparrow^m (b \rightarrow a)$ and $\uparrow^m (b \rightarrow c)$ cannot be empty as they certainly contain $b \rightarrow a$ and $b \rightarrow c$, respectively. Hence, $\uparrow^m (b \rightarrow a : \cdot b \rightarrow c)$ must contain at least one non-trivial monilinear justification $s'(z) \rightarrow t'(z)$. We distinguish the following cases:

- (a) If $s'(z) \rightarrow t'(z)$ equals $b \rightarrow a$ or $b \rightarrow c$, we must have $c = a$.
- (b) Else if $s'(z) \rightarrow t'(z)$ equals $k' + z \rightarrow \ell' + z$, we must have $b = k' + o, a = \ell' + o, b = k' + u,$ and $c = \ell' + u$, for some $o, u \in \mathbb{Z}$, which implies $b = k' + o = k' + u$ and therefore $o = u$ and hence $a = \ell' + o = \ell' + u = c$.
- (c) Finally, if $s'(z) \rightarrow t'(z)$ equals $k' + z \rightarrow a$, we must have $a = c$.

(\Leftarrow) Every justification of the form $k + z \rightarrow \ell + z$ is a characteristic justification by the Uniqueness Lemma 4 since $k + z$ and $\ell + z$ are injective in $(\mathbb{Z}, +, \mathbb{Z})$. Since $a - b = c - d$ holds by assumption, $z \rightarrow z + b - a$ is a characteristic justification of $a : b :: c : d$ in $(\mathbb{Z}, +, \mathbb{Z})$. \square

Interestingly, additive monilinear number proportions are equivalent to number proportions in the domain of natural numbers (\mathbb{N}, S) with the successor function $S(a) := a + 1$.

Corollary 12. $(\mathbb{Z}, +, \mathbb{Z}) \models_m a : b :: c : d \Leftrightarrow (\mathbb{N}, S) \models a : b :: c : d$.

Proof. A direct consequence of the Difference Proportion Theorem in Antić (2023a). \square

Theorem 13. *All the proportional axioms hold in $(\mathbb{Z}, +, \mathbb{Z})$ with respect to \models_m except for p-commutativity.*

Proof. We have the following proofs:

- The proofs for p-symmetry, inner p-symmetry, inner p-reflexivity, p-reflexivity, and p-determinism are analogous to the original proofs in the proof of Theorem 28 in Antić (2022).
- p-Commutativity fails since $a - b \neq b - a$ whenever $a \neq b$.
- Central permutation follows from the fact that $a - b = c - d$ iff $a - c = b - d$.
- Strong inner p-reflexivity follows from the fact that $a - a = c - d$ implies $d = c$.
- Strong p-reflexivity follows from the fact that $a - b = a - d$ implies $d = b$.
- p-Determinism follows from the fact that $a - a = a - d$ iff $d = a$.
- p-Transitivity follows from

$$a - b = c - d \quad \text{and} \quad c - d = e - f \quad \Rightarrow \quad a - b = e - f.$$

- Inner p-transitivity follows from

$$\frac{\frac{a - b = c - d \quad b - e = d - f}{a - b + b - e = c - d + d - f}}{a - e = c - f}.$$

- Central p-transitivity is a direct consequence of p-transitivity. Explicitly, we have

$$a - b = b - c \quad \text{and} \quad b - c = c - d \quad \Rightarrow \quad a - b = c - d.$$

\square

Theorem 14. $\frac{a : b ::_m c : d \quad a' : b' ::_m c' : d'}{a + a' : b + b' ::_m c + c' : d + d'}.$

Proof.

$$\frac{\frac{a : b ::_m c : d}{a - b = c - d} \quad 11 \quad \frac{a' : b' ::_m c' : d'}{a' - b' = c' - d'} \quad 11}{(a + a') - (b + b') = a - b + a' - b' = c - d + c' - d' = (c + c') - (d + d')}}{a + a' : b + b' ::_m c + c' : d + d'}.$$

\square

Theorem 14 shows that we can decompose number proportions; for example,

$$4 : 5 :: 0 : 1 = (2 : 3 :: 0 : 1) + (2 : 2 :: 0 : 0).$$

The following notion of an number proportion is an instance of the more general definition due to Stroppa and Yvon (2006, Proposition 2) given for abelian semigroups adapted to the additive setting of this paper (cf. Antić, 2022, §5.3):

Definition 15. For any integers $a, b, c, d \in \mathbb{Z}$, define

$$(\mathbb{Z}, +, \mathbb{Z}) \models_{SY} a : b :: c : d \quad :\Leftrightarrow \quad a = k + o, \quad b = \ell + o, \quad c = k + u, \quad d = \ell + u$$

for some $k, \ell, o, u \in \mathbb{Z}$.

By the Difference Proportion Theorem 11, we have

$$(13) \quad (\mathbb{Z}, +, \mathbb{Z}) \models_{SY} a : b :: c : d \quad \Leftrightarrow \quad (\mathbb{Z}, +, \mathbb{Z}) \models_m a : b :: c : d.$$

This shows that Stroppa and Yvon's (2006) notion of an additive number proportion coincides with the restrictive monolinear fragment of our framework and should therefore *not* be used as a general definition of an additive number proportion — this is demonstrated in the next example:

Example 16. The number proportion

$$(\mathbb{Z}, +, \mathbb{Z}) \models 0 : 0 :: 1 : 2,$$

is characteristically justified by $z \rightarrow z + z$, which is non-monolinear since z occurs more than once on the right-hand side; on the other hand, this simple proportion is not captured within Stroppa and Yvon's (2006) framework:

$$(14) \quad (\mathbb{Z}, +, \mathbb{Z}) \not\models_{SY} 0 : 0 :: 1 : 2.$$

6.2. Geometric proportions. This section studies multiplicative monolinear number proportions in $(\mathbb{Q}, \cdot, \mathbb{Q})$ where \mathbb{Q} denotes the rational numbers. We begin by noting that the set of justifications of $a \rightarrow b$ in $(\mathbb{Q}, \cdot, \mathbb{Q})$ is given by

$$\begin{aligned} \uparrow^m (a \rightarrow b) = & \{kz \rightarrow \ell z \mid a \rightarrow b = ko \rightarrow \ell o, \text{ for some } k, \ell, o \in \mathbb{Q}\} \\ & \cup \{kz \rightarrow b \mid a \rightarrow b = ko \rightarrow b, \text{ for some } k, o \in \mathbb{Q}\} \cup \{a \rightarrow b\}. \end{aligned}$$

This leads to the following characterization of the monolinear entailment relation with respect to multiplication:

Theorem 17 (Geometric Proportion Theorem). *For any $a, b, c, d \in \mathbb{Q}$,*

$$\begin{aligned} (\mathbb{Q}, \cdot, \mathbb{Q}) \models_m a : b :: c : d & \Leftrightarrow a = ko, \quad b = \ell o, \quad c = ku, \quad d = \ell u, \quad k, \ell, o, u \in \mathbb{Q} \\ & \Leftrightarrow \frac{a}{b} = \frac{c}{d} \quad (\text{geometric proportion}). \end{aligned}$$

The first equivalence holds in $(\mathbb{N}, \cdot, \mathbb{N})$ as well.⁷

Proof. We first show the second equivalence

$$a = ko, \quad b = \ell o, \quad c = ku, \quad d = \ell u \quad \Leftrightarrow \quad \frac{a}{b} = \frac{c}{d},$$

for some $k, \ell, o, u \in \mathbb{Q}$. The direction from left to right holds trivially. For the other direction, notice that $\frac{a}{b} = \frac{c}{d}$ implies

$$a = \left(\frac{c}{d}\right)b, \quad b = 1b, \quad c = \left(\frac{c}{d}\right)d, \quad d = 1d.$$

The rest of the proof is similar to the proof of Theorem 11. □

The Geometric Proportion Theorem 17 shows that monolinear multiplicative number proportions can be geometrically interpreted as analogical proportions between rectangles. Moreover, the simple characterization of the monolinear relation in Theorem 17 allows us to analyze the proportional axioms within the monolinear setting:

Theorem 18. *All the proportional axioms hold in $(\mathbb{Q}, \cdot, \mathbb{Q})$ with respect to \models_m .*

Proof. We have the following proofs:

- The proofs for p-symmetry, inner p-symmetry, inner p-reflexivity, p-reflexivity, and p-determinism are analogous to the original proofs in the proof of Theorem 28 in Antić (2022).
- p-Commutativity follows from Theorem 17 together with

$$ko : \ell o ::_m \ell o : ko, \quad \text{for all } k, \ell, o \in \mathbb{Q}.$$

- Central permutation follows from Theorem 17 together with

$$ko : \ell o ::_m ku : \ell u \quad \Leftrightarrow \quad ok : uk ::_m ol : ul.$$

⁷This will be essential in §6.2.1 when we study primes.

- Strong inner p-reflexivity follows from Theorem 17 together with

$$ko : ko ::_m ku : d \Leftrightarrow d = ku.$$

- Strong p-reflexivity follows from Theorem 17 together with

$$ko : lo ::_m ko : d \Leftrightarrow d = lo.$$

- p-Determinism follows from $\frac{a}{a} = \frac{a}{d}$ iff $d = a$.
- p-Transitivity follows from Theorem 17 together with

$$\frac{a}{b} = \frac{c}{d} \quad \text{and} \quad \frac{c}{d} = \frac{e}{f} \quad \Rightarrow \quad \frac{a}{b} = \frac{e}{f}.$$

- Inner p-transitivity follows from the following derivation:

$$\frac{\frac{a : b ::_m c : d}{\frac{\frac{a}{b} = \frac{c}{d}}{a = \frac{bc}{d}}} \text{Theorem 17} \quad \frac{\frac{b : e ::_m d : f}{\frac{\frac{b}{e} = \frac{d}{f}}{e = \frac{bf}{d}}} \text{Theorem 17}}{\frac{\frac{a}{e} = \frac{\frac{bc}{d}}{\frac{bf}{d}} = \frac{bcd}{bfd} = \frac{c}{f}}}{a : e ::_m c : f.}$$

- Central p-transitivity is an immediate consequence of transitivity. □

Theorem 19. $\frac{a : b ::_m c : d \quad a' : b' ::_m c' : d'}{aa' : bb' ::_m cc' : dd'}.$

Proof. We have the following derivation:

$$\frac{\frac{a : b ::_m c : d}{ko : lo ::_m ku : lu} \text{Theorem 17} \quad \frac{a' : b' ::_m c' : d'}{k'o' : \ell'o' ::_m k'u' : \ell'u'} \text{Theorem 17}}{\frac{(ko)(k'o') : (\ell'o)(\ell'o') ::_m (ku)(k'u') : (\ell'u)(\ell'u')}{(kk')(oo') : (\ell\ell')(oo') ::_m (kk')(uu') : (\ell\ell')(uu')}}}{aa' : bb' ::_m cc' : dd'}.$$

□

6.2.1. *Primes.* We shall now prove some properties of the monolinear entailment relation with respect to primes. In this subsection, the underlying algebra is $(\mathbb{N}, \cdot, \mathbb{N})$ where \mathbb{N} denotes the natural numbers.

Proposition 20. *Let p, q, p', q' be primes. We have*

$$p : q ::_m p' : q' \Leftrightarrow (p = q \text{ and } p' = q') \text{ or } (p = p' \text{ and } q = q').$$

Proof. By Theorem 17, we have

$$p : q ::_m p' : q' \Leftrightarrow p = ko, \quad q = lo, \quad c = ko', \quad d = \ell'o', \quad \text{for some } k, \ell, o, o' \in \mathbb{N}.$$

We distinguish two cases. First, if $k = 1$ and $o = p$, then $q = \ell p$ which implies $\ell = 1$ and therefore $q' = o'$ and $p' = o'$. Second, if $o = p$ and $o = 1$, then $q = \ell$ and $p' = po'$ which implies $o' = 1$ and therefore $q' = \ell = q$. □

Proposition 21. *Let p, q be primes, and let $c, d \in \mathbb{N}$. We have*

$$p : q ::_m c : d \Leftrightarrow (p = q \text{ and } c = d) \text{ or} \\ (p \neq q \text{ and } c = pu \text{ and } d = qu, \text{ for some } u \in \mathbb{N}).$$

Proof. By Theorem 17, we have

$$p : q ::_m c : d \Leftrightarrow p = ko, \quad q = \ell o, \quad c = ku, \quad d = \ell u, \quad \text{for some } k, \ell, o, u \in \mathbb{N}.$$

We distinguish two cases. First, if $k = 1$ and $o = p$, then $q = \ell p$ and thus $\ell = 1$ and $q = p$ and $c = d = u$. Second, if $k = p$ and $o = 1$, then $q = \ell$, $c = pu$, and $d = qu$, for some $u \in \mathbb{N}$. \square

6.3. Monolinear word proportions. In the rest of this section, A denotes a finite non-empty alphabet and \cdot denotes concatenation of words. We denote the *empty word* by ε . As usual, we denote the set of all words over A by A^* and define $A^* := A^* \cup \{\varepsilon\}$.

In the monolinear word domain, the set of monolinear justifications of $\mathbf{a} \rightarrow \mathbf{b}$ in (A^*, \cdot, A^*) is given by

$$\begin{aligned} \uparrow^m(\mathbf{a} \rightarrow \mathbf{b}) = & \{ \mathbf{a}_1 z \mathbf{a}_3 \rightarrow \mathbf{b}_1 z \mathbf{b}_3 \mid \mathbf{a} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3; \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_3 \in A^* \} \\ & \cup \{ \mathbf{a}_1 z \mathbf{a}_3 \rightarrow \mathbf{b} \mid \mathbf{a} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \rightarrow \mathbf{b}; \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in A^* \} \cup \{ \mathbf{a} \rightarrow \mathbf{b} \}. \end{aligned}$$

This implies

$$\begin{aligned} \uparrow^m(\mathbf{a} \rightarrow \mathbf{b} : \cdot \mathbf{c} \rightarrow \mathbf{d}) = & \left\{ \mathbf{a}_1 z \mathbf{a}_3 \rightarrow \mathbf{b}_1 z \mathbf{b}_3 \mid \begin{array}{l} \mathbf{a} \rightarrow \mathbf{b} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{a}_2 \mathbf{b}_3 \\ \mathbf{c} \rightarrow \mathbf{d} = \mathbf{a}_1 \mathbf{c}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{c}_2 \mathbf{b}_3 \\ \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_3, \mathbf{c}_2 \in A^* \end{array} \right\} \\ & \cup \left\{ \mathbf{a}_1 z \mathbf{a}_3 \rightarrow \mathbf{b} \mid \begin{array}{l} \mathbf{b} = \mathbf{d} \\ \mathbf{a} \rightarrow \mathbf{b} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \rightarrow \mathbf{b} \\ \mathbf{c} \rightarrow \mathbf{d} = \mathbf{a}_1 \mathbf{c}_2 \mathbf{a}_3 \rightarrow \mathbf{b} \\ \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{c}_2 \in A^* \end{array} \right\} \cup \{ \mathbf{a} \rightarrow \mathbf{b} \mid \mathbf{a} = \mathbf{c}, \mathbf{b} = \mathbf{d} \}. \end{aligned}$$

This leads to the following characterization of the monolinear entailment relation:

Theorem 22.

$$\mathbf{a} : \mathbf{b} ::_m \mathbf{c} : \mathbf{d} \Leftrightarrow \mathbf{a} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3, \quad \mathbf{b} = \mathbf{b}_1 \mathbf{a}_2 \mathbf{b}_3, \quad \mathbf{c} = \mathbf{a}_1 \mathbf{b}_2 \mathbf{a}_3, \quad \mathbf{d} = \mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3, \\ \text{for some } \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3.$$

Proof. (\Rightarrow) By assumption, we have $(A^*, \cdot, A^*) \models_m \mathbf{a} \rightarrow \mathbf{b} : \cdot \mathbf{c} \rightarrow \mathbf{d}$ which holds iff either

$$\uparrow^m(\mathbf{a} \rightarrow \mathbf{b}) \cup \uparrow^m(\mathbf{c} \rightarrow \mathbf{d}) = \emptyset,$$

or $\uparrow^m(\mathbf{a} \rightarrow \mathbf{b} : \cdot \mathbf{c} \rightarrow \mathbf{d})$ is non-empty and subset maximal with respect to d . In the first case, notice that neither $\uparrow^m(\mathbf{a} \rightarrow \mathbf{b})$ nor $\uparrow^m(\mathbf{c} \rightarrow \mathbf{d})$ can be empty since we always have

$$\mathbf{a} \rightarrow \mathbf{b} \in \uparrow^m(\mathbf{a} \rightarrow \mathbf{b}) \quad \text{and} \quad \mathbf{c} \rightarrow \mathbf{d} \in \uparrow^m(\mathbf{c} \rightarrow \mathbf{d}).$$

In the second case, by assumption we must have some monolinear justification $s(z) \rightarrow t(z)$ of $\mathbf{a} \rightarrow \mathbf{b} : \cdot \mathbf{c} \rightarrow \mathbf{d}$ in (A^*, \cdot, A^*) . We distinguish the following cases:

(1) If $s(z) \rightarrow t(z)$ equals $\mathbf{a} \rightarrow \mathbf{b}$ or $\mathbf{c} \rightarrow \mathbf{d}$, we must have

$$\mathbf{a} = \mathbf{c} \quad \text{and} \quad \mathbf{b} = \mathbf{d}.$$

(2) Else if $s(z) \rightarrow t(z)$ equals $\mathbf{a}_1 z \mathbf{a}_3 \rightarrow \mathbf{b}_1 z \mathbf{b}_3$, we must have

$$\mathbf{a} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \quad \text{and} \quad \mathbf{b} = \mathbf{b}_1 \mathbf{a}_2 \mathbf{b}_3 \quad \text{and} \quad \mathbf{c} = \mathbf{a}_1 \mathbf{b}_2 \mathbf{a}_3 \quad \text{and} \quad \mathbf{d} = \mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3$$

for some $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in A^*$.

- (3) Else if $s(z) \rightarrow t(z)$ equals $\mathbf{a}_1 z \mathbf{a}_3 \rightarrow \mathbf{b}$, we must have $\mathbf{a} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3$, $\mathbf{c} = \mathbf{a}_1 \mathbf{b}_2 \mathbf{a}_3$, and $\mathbf{d} = \mathbf{b}$, for some $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_2 \in A^*$. Then, by assumption, we must also have $\mathbf{a} : \mathbf{b} ::_m \mathbf{c} : \mathbf{b}$ and, by inner symmetry, $\mathbf{b} : \mathbf{a} ::_m \mathbf{b} : \mathbf{c}$ and therefore $\mathbf{b} \rightarrow \mathbf{a} : \cdot_m \mathbf{b} \rightarrow \mathbf{c}$. So, either we have $\uparrow^m(\mathbf{b} \rightarrow \mathbf{a}) \cup \uparrow^m(\mathbf{b} \rightarrow \mathbf{c}) = \emptyset$ or $\uparrow^m(\mathbf{b} \rightarrow \mathbf{a} : \cdot \mathbf{b} \rightarrow \mathbf{c})$ is non-empty and subset maximal with respect to \mathbf{c} . Again, the sets $\uparrow^m(\mathbf{b} \rightarrow \mathbf{a})$ and $\uparrow^m(\mathbf{b} \rightarrow \mathbf{c})$ cannot be empty as they contain $\mathbf{b} \rightarrow \mathbf{a}$ and $\mathbf{b} \rightarrow \mathbf{c}$, respectively. Hence, $\uparrow^m(\mathbf{b} \rightarrow \mathbf{a} : \cdot \mathbf{b} \rightarrow \mathbf{c})$ contains at least one non-trivial monilinear justification $s'(z) \rightarrow t'(z)$. We distinguish the following cases:
- (a) If $s'(z) \rightarrow t'(z)$ equals $\mathbf{b} \rightarrow \mathbf{a}$ or $\mathbf{b} \rightarrow \mathbf{c}$, we must have $\mathbf{a} = \mathbf{c}$.
- (b) Else if $s'(z) \rightarrow t'(z)$ equals $\mathbf{b}'_1 z \mathbf{b}'_3 \rightarrow \mathbf{a}'_1 z \mathbf{a}'_3$, for some $\mathbf{b}'_1, \mathbf{b}'_3, \mathbf{a}'_1, \mathbf{a}'_3 \in A^*$, we must have $\mathbf{b} = \mathbf{b}'_1 \mathbf{b}'_2 \mathbf{b}'_3 = \mathbf{b}'_1 \mathbf{c}'_2 \mathbf{b}'_3$ and $\mathbf{a} = \mathbf{a}'_1 \mathbf{b}'_2 \mathbf{a}'_3$ and $\mathbf{c} = \mathbf{a}'_1 \mathbf{c}'_2 \mathbf{a}'_3$, for some $\mathbf{b}'_2, \mathbf{c}'_2 \in A^*$. The identity $\mathbf{b}'_1 \mathbf{b}'_2 \mathbf{b}'_3 = \mathbf{b}'_1 \mathbf{c}'_2 \mathbf{b}'_3$ implies $\mathbf{b}'_2 = \mathbf{c}'_2$ and again $\mathbf{a} = \mathbf{c}$.
- (c) Finally, if $s'(z) \rightarrow t'(z)$ equals $\mathbf{b}'_1 z \mathbf{b}'_3 \rightarrow \mathbf{a}$, we must also have $\mathbf{a} = \mathbf{c}$.
- (\Leftarrow) The monilinear justification $\mathbf{a}_1 z \mathbf{a}_3 \rightarrow \mathbf{b}_1 z \mathbf{b}_3$ is a characteristic justification of

$$\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{a}_2 \mathbf{b}_3 : \cdot \mathbf{a}_1 \mathbf{c}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{c}_2 \mathbf{b}_3 \quad \text{and} \quad \mathbf{a}_1 \mathbf{c}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{c}_2 \mathbf{b}_3 : \cdot \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{a}_2 \mathbf{b}_3$$

in (A^*, \cdot, A^*) by the Uniqueness Lemma 4 since $\mathbf{a}_1 z \mathbf{a}_3$ and $\mathbf{b}_1 z \mathbf{b}_3$ both induce injective word functions. Analogously, $\mathbf{b}_1 z \mathbf{b}_3 \rightarrow \mathbf{a}_1 z \mathbf{a}_3$ is a characteristic justification of

$$\mathbf{a}_1 \mathbf{c}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{c}_2 \mathbf{b}_3 : \cdot \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{a}_2 \mathbf{b}_3 \quad \text{and} \quad \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{a}_2 \mathbf{b}_3 : \cdot \mathbf{a}_1 \mathbf{c}_2 \mathbf{a}_3 \rightarrow \mathbf{b}_1 \mathbf{c}_2 \mathbf{b}_3.$$

Hence, we have shown the theorem. \square

Corollary 23. $\mathbf{a} : \mathbf{eaf} ::_m \mathbf{c} : \mathbf{ecf}$.

Corollary 24. $\mathbf{ab} = \mathbf{cd} \Rightarrow \mathbf{a} : \mathbf{b} ::_m \mathbf{c} : \mathbf{d}$.

Proof. For example, by Theorem 22 we have $\mathbf{a} : \mathbf{b} \not\prec_m \varepsilon : \mathbf{ab}$. \square

Definition 25. We define the *reverse* of a word $\mathbf{a} = a_1 \dots a_n$, $n \geq 1$, as usual by $\mathbf{a}^R := a_n \dots a_1$.

Corollary 26. *There are words such that $\mathbf{a} : \mathbf{a}^R \not\prec_m \mathbf{c} : \mathbf{c}^R$.*

Proof. For example, by Theorem 22 we have $\mathbf{ab} : \mathbf{ba} \not\prec_m \mathbf{ba} : \mathbf{ab}$. \square

The simple characterization of the monilinear proportion relation in Theorem 22 allows us to analyze the proportional axioms within the monilinear word setting:

Theorem 27. *The monilinear word proportion relation satisfies*

- *symmetry,*
- *inner symmetry,*
- *reflexivity,*
- *determinism,*
- *strong inner reflexivity,*
- *strong reflexivity,*
- *transitivity,*
- *central transitivity,*

and, in general, it dissatisfies

- *central permutation,*
- *commutativity,*
- *inner transitivity.*

Proof. We have the following proofs:

- The proofs for p-symmetry, inner p-symmetry, inner p-reflexivity, p-reflexivity, and p-determinism are analogous to the original proofs in the proof of Theorem 28 in Antić (2022).
- p-Determinism is by Theorem 22 equivalent to

$$\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 : \mathbf{b}_1\mathbf{a}_2\mathbf{b}_3 ::_m \mathbf{a}_1\mathbf{b}_2\mathbf{a}_3 : \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3 \Leftrightarrow \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3 = \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$$

where

$$\mathbf{a} = \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 = \mathbf{b}_1\mathbf{a}_2\mathbf{b}_3 = \mathbf{a}_1\mathbf{b}_2\mathbf{a}_3 \quad \text{and} \quad \mathbf{d} = \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3.$$

This follows from

$$\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 = \mathbf{a}_1\mathbf{b}_2\mathbf{a}_3 \Leftrightarrow \mathbf{b}_2 = \mathbf{a}_2 \Leftrightarrow \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3 = \mathbf{a} \Leftrightarrow \mathbf{d} = \mathbf{a}.$$

- Central permutation fails⁸, for example, given the alphabet $A := \{a_1, a_2, a_3, b_1, b_3, c_2\}$ since as a consequence of Theorem 22, we have

$$a_1a_2a_3 : b_1a_2b_3 ::_m a_1c_2a_3 : b_1c_2b_3$$

whereas

$$a_1a_2a_3 : a_1c_2a_3 ::_m b_1a_2b_3 : b_1c_2b_3.$$

- Strong inner p-reflexivity and strong reflexivity are immediate consequences of Theorem 22.
- p-Commutativity fails, for example, in $A := \{a, b\}$ since as a consequence of Theorem 22, we have

$$a : b ::_m b : a.$$

- p-Transitivity is an immediate consequence of Theorem 22.
- Inner p-transitivity fails, for example, in $A := \{a_1, a_2, a_3, b_1, b_3, c_2, d_2, e_1, e_3\}$ since as a consequence of Theorem 22, we have

$$a_1a_2a_3 : b_1a_2b_3 ::_m a_1c_2a_3 : b_1c_2a_3$$

and

$$b_1a_2b_3 : e_1a_2e_3 ::_m b_1d_2b_3 : e_1d_2e_3$$

whereas

$$a_1a_2a_3 : e_1a_2e_3 ::_m a_1c_2a_3 : e_1d_2e_3.$$

- Finally, central p-transitivity is an immediate consequence of transitivity already shown above.

□

Remark 28. The fact that central permutation fails gives a negative answer to Problem 30 in Antić (2022) in the monilinear setting.

Remark 29. Notice that we cannot prove an analogue of Theorems 14 and 19 in the word domain since by Theorem 22, we in general have

$$\mathbf{a} : \mathbf{b} ::_m \mathbf{c} : \mathbf{d} \quad \text{and} \quad \mathbf{a}' : \mathbf{b}' ::_m \mathbf{c}' : \mathbf{d}' \quad \Rightarrow \quad \mathbf{aa}' : \mathbf{bb}' ::_m \mathbf{cc}' : \mathbf{dd}'.$$

⁸See Remark 28

7. FINITE ALGEBRAS

In this section, we provide an algorithm for the computation of the analogical proportion relation in finite algebras via tree automata and we therefore expect the reader to be familiar with the basics of tree automata as they are presented for example in Gécseg and Steinby (2015).

Recall that a (*frontier-to-root*) *tree automaton*

$$\mathcal{T}_{k,\alpha,A'}(\mathfrak{A}) := (\mathfrak{A}, L, Z_k, \alpha, A')$$

consists of

- a *finite* L -algebra \mathfrak{A} ,
- an *initial assignment* $\alpha : Z_k \rightarrow A$, and
- a set $A' \subseteq A$ of *final states*.

The *forest* recognized by $\mathcal{T}_{k,\alpha,A'}(\mathfrak{A})$ is given by

$$\|\mathcal{T}_{k,\alpha,A'}(\mathfrak{A})\| := \{s \in T_L(Z_k) \mid s^{\mathfrak{A}}(\alpha) \in A'\}.$$

We can thus rewrite the set of k -generalizations of a in \mathfrak{A} (see §5),

$$\uparrow_{\mathfrak{A}}^k a := (\uparrow_{\mathfrak{A}} a) \cap T_L(Z_k),$$

by

$$\uparrow_{\mathfrak{A}}^k a = \bigcup_{\alpha \in A^{Z_k}} \|\mathcal{T}_{k,\alpha,\{a\}}(\mathfrak{A})\|,$$

and the set of k -justifications of an arrow $a \rightarrow b$ in \mathfrak{A} in terms of tree automata as

$$\uparrow_{\mathfrak{A}}^k (a \rightarrow b) = \bigcup_{\alpha \in A^{Z_k}} [(\|\mathcal{T}_{k,\alpha,\{a\}}(\mathfrak{A})\| \rightarrow \|\mathcal{T}_{k,\alpha,\{b\}}(\mathfrak{A})\|) \cap \{s \rightarrow t \mid Z(t) \subseteq Z(s)\}],$$

where for two forests S and T ,

$$S \rightarrow T := \{s \rightarrow t \mid s \in S, t \in T\}.$$

Notice that the set A^{Z_k} of all initial assignments $Z_k \rightarrow A$ is *finite* by our assumption that Z_k is a finite set of variables. Since it is well-known that tree automata are closed under finite unions, there is a tree automaton $\mathcal{T}_{k,a \rightarrow b}(\mathfrak{A})$, for every arrow $a \rightarrow b$, such that

$$\uparrow_{\mathfrak{A}}^k (a \rightarrow b) = \|\mathcal{T}_{k,a \rightarrow b}(\mathfrak{A})\| \cap \{s \rightarrow t \mid Z(t) \subseteq Z(s)\}.$$

Now since tree automata are closed under intersection as well, there is a tree automaton $\mathcal{T}_{k,a \rightarrow b : c \rightarrow d}(\mathfrak{A}, \mathfrak{B})$, for every arrow proportion $a \rightarrow b : c \rightarrow d$, such that

$$\uparrow_{(\mathfrak{A}, \mathfrak{B})}^k (a \rightarrow b : c \rightarrow d) = \|\mathcal{T}_{k,a \rightarrow b : c \rightarrow d}(\mathfrak{A}, \mathfrak{B})\| \cap \{s \rightarrow t \mid Z(t) \subseteq Z(s)\}.$$

Since \mathfrak{B} is finite by assumption, checking the d -maximality of $\uparrow_{(\mathfrak{A}, \mathfrak{B})}^k (a \rightarrow b : c \rightarrow d)$ can thus be easily achieved with a search linear in the size of \mathfrak{B} ; checking the emptiness of $\uparrow_{\mathfrak{A}}^k (a \rightarrow b)$ and $\uparrow_{\mathfrak{B}}^k (c \rightarrow d)$ is well-known to be decidable as well (cf. Comon et al., 2008, p. 40); and checking $Z(t) \subseteq Z(s)$ is a simple syntactic comparison, which in total gives us an algorithm for deciding

$$(\mathfrak{A}, \mathfrak{B}) \models_k a \rightarrow b : c \rightarrow d$$

and thus for deciding

$$(\mathfrak{A}, \mathfrak{B}) \models_k a : b :: c : d.$$

7.1. Algorithms. This section lists algorithms for the most important problems on analogical proportions in finite algebras.

Pseudocode 30 (Decision algorithm). Given $k \geq 1$, $a, b \in A$, $c, d \in B$, and $(\mathfrak{A}, \mathfrak{B})$, we design an algorithm for deciding whether

$$(\mathfrak{A}, \mathfrak{B}) \stackrel{?}{\models}_k a : b :: c : d.$$

We first provide an algorithm for deciding whether

$$(15) \quad (\mathfrak{A}, \mathfrak{B}) \stackrel{?}{\models}_k a \rightarrow b :: c \rightarrow d.$$

- (1) Construct the tree automata $\mathcal{T}_{k,a \rightarrow b}(\mathfrak{A})$ and $\mathcal{T}_{k,c \rightarrow d}(\mathfrak{B})$ as described above. If

$$(\|\mathcal{T}_{k,a \rightarrow b}(\mathfrak{A})\| \cup \|\mathcal{T}_{k,c \rightarrow d}(\mathfrak{B})\|) \cap \{s \rightarrow t \mid Z(t) \subseteq Z(s)\} = \emptyset$$

then stop with answer “yes”.

- (2) Otherwise, construct the tree automaton $\mathcal{T}_{k,a \rightarrow b : c \rightarrow d}(\mathfrak{A}, \mathfrak{B})$ as described above and compute the forest

$$T := \|\mathcal{T}_{k,a \rightarrow b : c \rightarrow d}(\mathfrak{A}, \mathfrak{B})\|.$$

- (3) Compute the set of k -justifications

$$J := T - \{s \rightarrow t \mid Z(t) \not\subseteq Z(s)\}$$

applying a simple syntactic check on each rule in T .

- (4) We now want to check whether J is d -maximal:

- (a) For each $d' \neq d \in B$:

- (i) Construct the set of k -justifications

$$J' := \|\mathcal{T}_{k,a \rightarrow b : c \rightarrow d'}(\mathfrak{A}, \mathfrak{B})\| - \{s \rightarrow t \mid Z(t) \not\subseteq Z(s)\}$$

as above.

- (ii) If $J \subsetneq J'$ then stop with answer “no”.

- (b) Return the answer “yes”.

- (5) At this point, we have decided (15). Now repeat the above steps for the remaining arrow proportions

$$b \rightarrow a :: d \rightarrow c \quad \text{and} \quad c \rightarrow d :: a \rightarrow b \quad \text{and} \quad d \rightarrow c :: b \rightarrow a$$

and return “yes” iff the answer is “yes” in each case.

Pseudocode 31. Given $(\mathfrak{A}, \mathfrak{B})$ and k , computing the set

$$\{a : b :: c : d \mid (\mathfrak{A}, \mathfrak{B}) \models_k a : b :: c : d\}$$

can be done using Pseudocode 30 to decide $(\mathfrak{A}, \mathfrak{B}) \stackrel{?}{\models}_k a : b :: c : d$ on each of the finitely many proportions $a : b :: c : d$.

Pseudocode 32 (Solving proportional equations). Given a proportional equation

$$a : b :: c : x,$$

finding some/all $d \in B$ such that $(\mathfrak{A}, \mathfrak{B}) \models_k a : b :: c : d$ can be achieved using Pseudocode 31.

REFERENCES

- Antić, C. (2022). Analogical proportions. *Annals of Mathematics and Artificial Intelligence*, 90(6), 595–644.
- Antić, C. (2023a). Analogical proportions in monounary algebras. *Annals of Mathematics and Artificial Intelligence*, accepted, <https://arxiv.org/pdf/2208.06829.pdf>.
- Antić, C. (2023b). Analogical proportions via abstract justifications. *Transactions on Computational Logic*, under review, <https://hal.archives-ouvertes.fr/hal--03879901/document>.
- Antić, C. (2023c). Boolean proportions. *Logical Methods in Computer Science*, under review, <https://arxiv.org/pdf/2109.00388.pdf>.
- Antić, C. (2023d). Logic-based analogical proportions. *Annals of Mathematics and Artificial Intelligence*, under review, https://www.researchgate.net/publication/369327579_Logic-based_analogical_proportions.
- Antić, C. (2023e). Logic program proportions. *Annals of Mathematics and Artificial Intelligence*, accepted. <https://arxiv.org/pdf/1809.09938.pdf>.
- Antić, C. (2023f). Proportional algebras. *Journal of Artificial Intelligence Research*, under review, <https://arxiv.org/pdf/2210.01751.pdf>.
- Antić, C. (2023g). Semantic anti-unification in general algebras. *Information Processing Letters*, under review, <https://hal.science/hal--04207922>.
- Antić, C. (2023h). Tree proportions. *Information Processing Letters*, under review, https://www.researchgate.net/publication/373339041_Tree_proportions.
- Antić, C. (2023i). Word proportions. *Information Processing Letters*, under review, https://www.researchgate.net/publication/369327505_Word_proportions.
- Baader, F., & Nipkow, T. (1998). *Term Rewriting and All That*. Cambridge University Press, Cambridge UK.
- Boden, M. A. (1998). Creativity and artificial intelligence. *Artificial Intelligence*, 103(1-2), 347–356.
- Burris, S., & Sankappanavar, H. (2000). *A Course in Universal Algebra*. <http://www.math.hawaii.edu/~ralph/Classes/619/univ-algebra.pdf>.
- Comon, H., Dauchet, M., Gilleron, R., Jacquemard, F., Lugiez, D., Tison, S., & Tommasi, M. (2008). *Tree Automata Techniques and Applications*. <https://hal.inria.fr/hal-03367725>.
- Gécseg, F., & Steinby, M. (2015). *Tree Automata* (2 edition). <https://arxiv.org/pdf/1509.06233.pdf>.
- Gentner, D. (1983). Structure-mapping: a theoretical framework for analogy. *Cognitive Science*, 7(2), 155–170.
- Gust, H., Krumnack, U., Kühnberger, K.-U., & Schwering, A. (2008). Analogical reasoning: a core of cognition. *Künstliche Intelligenz*, 22(1), 8–12.
- Gust, H., Kühnberger, K.-U., & Schmid, U. (2006). Metaphors and heuristic-driven theory projection (HDTP). *Theoretical Computer Science*, 354(1), 98–117.
- Hall, R. P. (1989). Computational approaches to analogical reasoning: a comparative analysis. *Artificial Intelligence*, 39(1), 39–120.
- Hesse, M. B. (1966). *Models and Analogies in Science*. University of Notre Dame Press.
- Hofstadter, D. (2001). Analogy as the core of cognition. In Gentner, D., Holyoak, K. J., & Kokinov, B. K. (Eds.), *The Analogical Mind: Perspectives from Cognitive Science*, pp. 499–538. MIT Press/Bradford Book, Cambridge MA.
- Hofstadter, D., & Sander, E. (2013). *Surfaces and Essences. Analogy as the Fuel and Fire of Thinking*. Basic Books, New York.

- Hug, N., Prade, H., Richard, G., & Serrurier, M. (2019). Analogical proportion-based methods for recommendation — first investigations. *Fuzzy Sets and Systems*, 366, 110–132.
- Krieger, M. H. (2003). *Doing Mathematics: Convention, Subject, Calculation, Analogy*. World Scientific, New Jersey.
- Krumnack, U., Schwering, A., Gust, H., & Kühnberger, K.-U. (2007). Restricted higher-order anti-unification for analogy making. In Orgun, M., & Thornton, J. (Eds.), *AI 2007*, LNAI 4830, pp. 273–282. Springer-Verlag, Berlin/Heidelberg.
- Lepage, Y. (1998). Solving analogies on words: an algorithm. In Boitet, C., & Whitelock, P. (Eds.), *COLING-ACL 1998*, pp. 728–735. Morgan Kaufmann Publishers.
- Lepage, Y. (2001). Analogy and formal languages. *Electronic Notes in Theoretical Computer Science*, 53, 180–191.
- Lepage, Y. (2003). *De L'Analogie. Rendant Compte de la Commutation en Linguistique*. Habilitation à diriger les recherches, Université Joseph Fourier, Grenoble.
- Lepage, Y. (2014). Analogies between binary images: application to Chinese characters. In Prade, H., & Richard, G. (Eds.), *Computational Approaches to Analogical Reasoning: Current Trends*, Studies in Computational Intelligence 548, pp. 25–57. Springer-Verlag.
- Miclet, L., Bayouduh, S., & Delhay, A. (2008). Analogical dissimilarity: definition, algorithms and two experiments in machine learning. *Journal of Artificial Intelligence Research*, 32, 793–824.
- Miclet, L., & Prade, H. (2009). Handling analogical proportions in classical logic and fuzzy logics settings. In Sossai, C., & Chemello, G. (Eds.), *ECSQARU 2009*, LNAI 5590, pp. 638–650. Springer-Verlag, Berlin/Heidelberg.
- Pólya, G. (1954). *Induction and Analogy in Mathematics*, Vol. 1 of *Mathematics and Plausible Reasoning*. Princeton University Press, Princeton, New Jersey.
- Prade, H., & Richard, G. (2013). From analogical proportion to logical proportions. *Logica Universalis*, 7, 441–505.
- Prade, H., & Richard, G. (2014). A short introduction to computational trends in analogical reasoning. In Prade, H., & Richard, G. (Eds.), *Approaches to Analogical Reasoning: Current Trends*, Studies in Computational Intelligence 548, pp. 1–22. Springer-Verlag, Berlin/Heidelberg.
- Prade, H., & Richard, G. (2021). Analogical proportions: why they are useful in AI. In Zhou, Z.-H. (Ed.), *IJCAI 2021*, pp. 4568–4576.
- Prade, H., Richard, G., Gabbay, D. M., & Guentner, F. (2018). Homogenous and heterogenous logical proportions: an introduction. In Gabbay, D. M., & Guentner, F. (Eds.), *Handbook of Philosophical Logic*. Springer Nature Switzerland AG.
- Stroppa, N., & Yvon, F. (2006). Formal models of analogical proportions. Technical Report D008, Telecom ParisTech - École Nationale Supérieure de Télécommunications, Télécom Paris.
- Weller, S., & Schmid, U. (2007). Solving proportional analogies using E-generalization. In Freksa, C., Kohlhase, M., & Schmill, K. (Eds.), *KI 2006*, LNAI 4314, pp. 64–75. Springer-Verlag.
- Winston, P. H. (1980). Learning and reasoning by analogy. *Communications of the ACM*, 23(12), 689–703.