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RECURRENCE OF THE PLANE ELEPHANT RANDOM WALK

Nicolas CURIEN*, Lucile LAULIN†

Abstract

We give a short proof of the recurrence of the two-dimensional elephant random walk in the diffusive regime. This was recently established by Qin [5], but our proof mainly uses very rough comparison with the standard plane random walk. We hope that the method can be useful for other applications.

1 Introduction

The elephant random walk on \mathbb{Z}^d has been introduced in dimension 1 by Schütz and Trimper [6] and is a well-studied discrete process with reinforcement, see [3] for background and references. Its definition (see (2.1)) depends on a memory parameter¹ $\alpha \in (-\frac{1}{2d-1}, 1)$ and it exhibits a phase transition going from a diffusive when $\alpha < \alpha_c = \frac{1}{2}$ to a superdiffusive behavior when $\alpha > \alpha_c$. We focus here on the two-dimensional case and establish recurrence of the process in the diffusive regime.

Theorem 1.1. *In the diffusive regime $\alpha < \alpha_c = \frac{1}{2}$, the plane elephant random walk is recurrent.*

This has been recently proved by Qin [5] but our approach is different and much shorter, however it gives less quantitative information and does not directly apply in the critical regime $\alpha = \alpha_c$. We use a comparison to the simple random walk which could apply in dimension 1 as well since the simple random walk is also recurrent in that case.

It is worth pointing out that [5] established that the elephant random walk is always transient when $d \geq 3$, similar to the simple random walk.

Notation. We write \mathbf{e}_i the four directions of \mathbb{Z}^2 for $1 \leq i \leq 4$. We shall write $(X_k : k \geq 0)$ for the canonical underlying process starting from $\mathbf{0} := (0, 0) \in \mathbb{Z}^2$, we denote its steps by $\Delta X_k = X_{k+1} - X_k \in \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and we introduce for $1 \leq i \leq 4$ the centered counting direction processes $D_k^{[X]}(\mathbf{e}_i)$ defined by

$$D_k^{[X]}(\mathbf{e}_i) = \sum_{j=0}^{k-1} \mathbf{1}\{X_{j+1} - X_j = \mathbf{e}_i\} - \frac{k}{4}, \quad \text{in particular notice that} \quad \sum_{i=1}^4 D_n^{[X]}(\mathbf{e}_i) = 0. \quad (1.1)$$

For any stopping time θ , we denote by $X^{(\theta)}$ the shifted process $X_k^{(\theta)} = X_{\theta+k} - X_\theta$ for $k \geq 0$. Finally \mathcal{F}_n is the canonical filtration generated by the first n steps of the walk and we use $X_{[0,n]}$ as a shorthand for $(X_k : 0 \leq k \leq n)$.

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¹The usual definition uses a memory parameter $p \in [0, 1]$ which is the probability to reproduce a (uniform) former step of the walk, or to move in one of the 3 remaining directions with the same probability $(1-p)/3$ so that $\alpha = (4p-1)/3$, see [3, Eq. (1.4)].

2 Comparison between elephant and simple random walk

Under the law $\mathbb{P}_{\#}$ the underlying process (X) evolves as the standard simple random walk on \mathbb{Z}^2 , whereas under $\mathbb{P}_{\#}$, it evolves as the α -elephant random walk i.e. satisfying for $n \geq 0$

$$\mathbb{P}_{\#}(\Delta X_n = \mathbf{e}_i \mid \mathcal{F}_n) = \frac{1}{4} + \alpha \frac{D_n^{[X]}(\mathbf{e}_i)}{n}, \quad (2.1)$$

(where we interpret $0/0 = 0$ for $n = 0$). In particular, under $\mathbb{P}_{\#}$, the process $(D_k^{[X]}(\mathbf{e}_i) : 1 \leq i \leq 4, k \geq 0)$ is Markov and evolves as an urn process with four colors, which was crucially used in [1] to establish the phase transition diffusive/superdiffusive. The local evolution of the elephant random walk (for large times) resembles that of the simple random walk and this is quantified in the following proposition:

Proposition 2.1 (Markov contiguity). *For any $\varepsilon > 0$ and any $A > 0$, there exist $c_{\varepsilon, A} > 0$ and a sequence of events E_n satisfying $\liminf_{n \rightarrow \infty} \mathbb{P}_{\#}(X_{[0, n]} \in E_n) \geq 1 - \varepsilon$ such that for any measurable function f ,*

$$\mathbb{E}_{\#} \left[f(X_{[0, n]}^{(n)}) \mathbf{1}_{X_{[0, n]}^{(n)} \in E_n} \mid \mathcal{F}_n \text{ and } \frac{|D_n^{[X]}(\mathbf{e}_i)|}{\sqrt{n}} \leq A \text{ for all } 1 \leq i \leq 4 \right] \geq c_{\varepsilon, A} \cdot \mathbb{E}_{\#} \left[f(X_{[0, n]}) \mathbf{1}_{X_{[0, n]} \in E_n} \right].$$

Proof. In the event considered in the conditioning, we have $|D_n^{[X]}(\mathbf{e}_i)| \leq A\sqrt{n}$ for all $1 \leq i \leq 4$. By (2.1), the Radon–Nikodym derivative of $(X_{n+k} - X_n : 0 \leq k \leq n)$ under $\mathbb{P}_{\#}$ with respect to $\mathbb{P}_{\#}$ is given by

$$\text{RND}_n := \prod_{k=n}^{2n-1} \left(1 + 4\alpha \frac{D_k^{[X]}(\Delta X_k)}{k} \right) = \prod_{k=0}^{n-1} \left(1 + 4\alpha \frac{D_n^{[X]}(\Delta X_k^{(n)}) + D_k^{[X^{(n)}]}(\Delta X_k^{(n)})}{n+k} \right). \quad (2.2)$$

By Donsker's invariance principle, we can find a constant A_ε such that the event

$$G_n = \left\{ \max_i \sup_{0 \leq k \leq n} |D_k^{[X^{(n)}]}(\mathbf{e}_i)| \leq A_\varepsilon \sqrt{n} \right\}$$

has probability at least $1 - \varepsilon$ under $\mathbb{P}_{\#}$. On this event (and conditionally on the event of the statement of the proposition), the counting directions processes $D_{[n, 2n]}^{[X]}(\mathbf{e}_i)$ are in absolute value bounded by $(A + A_\varepsilon)\sqrt{n}$. In particular, using $\log(1 + \alpha x) \geq \alpha x - (\alpha x)^2$ for small $|x|$, we deduce that on this event, for n large enough, the Radon-Nikodym derivative in (2.2) is lower bounded by

$$\text{RND}_n \mathbf{1}_{G_n} \geq \exp \left(4\alpha M_n - (4\alpha)^2 (A + A_\varepsilon)^2 \right) \mathbf{1}_{G_n} \quad \text{where} \quad M_j = \sum_{k=0}^{j-1} \frac{D_{n+k}^{[X]}(\Delta X_{n+k})}{n+k}.$$

Using (1.1), it is trivial to check that $(M_j : 0 \leq j \leq n)$ is a (\mathcal{F}_{n+}) -martingale with quadratic variation

$$\mathbb{E}[M_{j+1}^2 - M_j^2 \mid \mathcal{F}_{n+j}] = \mathbb{E}[(M_{j+1} - M_j)^2 \mid \mathcal{F}_{n+j}] = \frac{1}{4} \frac{\sum_{i=1}^4 \left(D_{n+j}^{[X]}(\mathbf{e}_i) \right)^2}{(n+j)^2} \underset{\text{on } G_n}{\leq} \frac{(A + A_\varepsilon)^2}{n}.$$

Consider then the (\mathcal{F}_{n+}) -stopping time

$$\vartheta = \inf \left\{ j \geq 0 : \left\{ \max_i \sup_{0 \leq k \leq j} |D_k^{[X^{(n)}]}(\mathbf{e}_i)| \leq A_\varepsilon \sqrt{n} \right\} \right\}.$$

By the penultimate display, the stopped martingale $(M_{k \wedge \vartheta} : 0 \leq j \leq n)$ has quadratic variation bounded above by $n \times \frac{(A+A_\varepsilon)^2}{n}$ and it follows that $\mathbb{E}[M_n^2 \mathbf{1}_{G_n}] \leq \mathbb{E}[M_{n \wedge \vartheta}^2] \leq (A + A_\varepsilon)^2$.

In particular, thanks to Markov inequality, for any $\varepsilon > 0$, the event $H_n = \{|M_n| \mathbf{1}_{G_n} < \frac{(A+A_\varepsilon)}{\sqrt{\varepsilon}}\}$ is of probability at least $1 - \varepsilon$. Gathering up the pieces, on the event $E_n = G_n \cap H_n$ which is of $\mathbb{P}_\#$ measure at least $1 - 2\varepsilon$, the Radon-Nikodym derivative of the elephant w.r.t. the simple random walk is at least $e^{-4\alpha \frac{(A+A_\varepsilon)}{\sqrt{\varepsilon}} - (4\alpha)^2 (A+A_\varepsilon)^2} =: c_{\varepsilon, A}$. \square

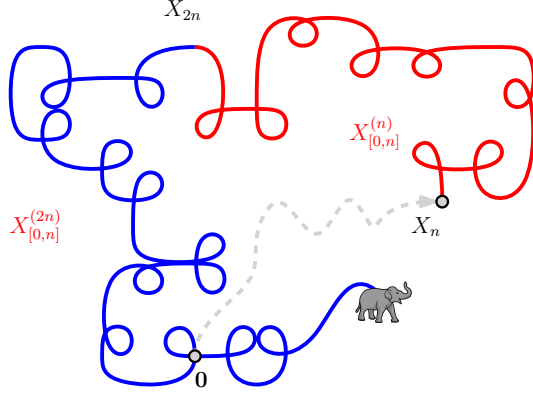


Figure 2.1: Illustration of the proof of Proposition 2.2. Conditionally on \mathcal{F}_n and on the fact that the counting directions processes are controlled at time n , the blue and red parts are independent on events of large probability. This is sufficient to imply a lower bound on the probability of return to $\mathbf{0}$.

It is classical that in the plane, the simple random walk started from $x \in \mathbb{Z}^2$ with $\|x\| \approx \sqrt{n}$ has a probability of order $\log^{-1} n$ to visit $(0, 0)$ within n steps. Our weak bound (Proposition 2.1) is sufficient to imply the same kind of estimate for the elephant random walk:

Proposition 2.2. *For any $A > 0$ there exists $c_A > 0$ such that*

$$\mathbb{E}_{\text{elephant}} \left[\exists \frac{5}{2}n \leq k \leq 3n : X_k = 0 \mid \mathcal{F}_n \text{ and } \frac{|D_n^{[X]}(\mathbf{e}_i)|}{\sqrt{n}} \leq A \text{ for all } 1 \leq i \leq 4 \right] \geq \frac{c_A}{\log n}.$$

Proof. Let us denote $\mathbf{x}_n = X_n$ which is fixed conditionally on \mathcal{F}_n . Using Proposition 2.1 twice, for any positive functions f and g and any $A, A' > 0$ and any $\varepsilon > 0$, we can find two sequences of events E_n and E'_n and constants $c_{\varepsilon, A}$ and $c_{\varepsilon, A'}$ such that

$$\begin{aligned} & \mathbb{E}_{\text{elephant}} \left[f(X_{[0,n]}^{(2n)}) g(X_{[0,n]}^{(n)}) \mid \mathcal{F}_n \right] \\ & \geq \mathbb{E}_{\text{elephant}} \left[f(X_{[0,n]}^{(2n)}) g(X_{[0,n]}^{(n)}) \mathbf{1}_{\frac{\|D_{2n}^{[X]}(\mathbf{e}_i)\|}{\sqrt{n}} \leq A', \forall 1 \leq i \leq 4} \mid \mathcal{F}_n \right] \\ & \geq c_{\varepsilon, A'} \cdot \mathbb{E}_\# \left[f(X_{[0,n]}) \mathbf{1}_{X_{[0,n]} \in E'_n} \right] \cdot \mathbb{E}_{\text{elephant}} \left[\mathbf{1}_{\frac{\|D_{2n}^{[X]}(\mathbf{e}_i)\|}{\sqrt{n}} \leq A', \forall 1 \leq i \leq 4} g(X_{[0,n]}^{(n)}) \mid \mathcal{F}_n \right] \\ & \geq c_{\varepsilon, A'} \cdot \mathbb{E}_\# \left[f(X_{[0,n]}) \mathbf{1}_{X_{[0,n]} \in E'_n} \right] \cdot c_{\varepsilon, A} \cdot \mathbb{E}_\# \left[g(X_{[0,n]}) \mathbf{1}_{X_{[0,n]} \in E_n} \mathbf{1}_{\frac{\|D_n^{[X]}(\mathbf{e}_i)\|}{\sqrt{n}} \leq A' - A, \forall 1 \leq i \leq 4} \right] \mathbf{1}_{\frac{\|D_n^{[X]}(\mathbf{e}_i)\|}{\sqrt{n}} \leq A, \forall 1 \leq i \leq 4}. \end{aligned}$$

Up to increasing A' we may suppose that the event $H_n = E_n \cap E'_n \cap \left\{ \frac{\|D_n^{[X]}(\mathbf{e}_i)\|}{\sqrt{n}} \leq A' - A, \forall 1 \leq i \leq 4 \right\}$ has probability at least $1 - 3\varepsilon$ and particularizing the inequality above, we deduce that for some constant $\tilde{c}_{\varepsilon, A} > 0$ the probability in the proposition is lower bounded by

$$\tilde{c}_{\varepsilon, A} \cdot \mathbb{P}_{\#} \left(\exists \frac{3}{2}n \leq k \leq 2n : X_k = -\mathbf{x}_n \text{ and } \begin{array}{l} X_{[0, n]}^{(0)} \in H_n \\ X_{[0, n]}^{(n)} \in H_n \end{array} \right),$$

so that we can apply the following lemma to conclude.

Lemma 2.3. *For any $A > 0$, there exists $\varepsilon > 0$ and $\delta_A > 0$ so that if $\mathbf{x}_n \in \mathbb{Z}^2$ is such that $\|\mathbf{x}_n\| \leq A\sqrt{n}$ and if E_n is a sequence of events such that $\mathbb{P}_{\#}(X_{[0, n]} \in E_n) \geq 1 - \varepsilon$ then we have*

$$\mathbb{P}_{\#} \left(\exists \frac{3}{2}n \leq k \leq 2n : X_k = -\mathbf{x}_n \text{ and } \begin{array}{l} X_{[0, n]}^{(0)} \in E_n \\ X_{[0, n]}^{(n)} \in E_n \end{array} \right) \geq \frac{\delta_A}{\log n}.$$

Proof. We use a second-moment method on the random variable

$$\mathcal{N}_{\mathbf{x}_n}^{E_n} := \# \left\{ \frac{3}{2}n \leq k \leq 2n : X_k = -\mathbf{x}_n \right\} \mathbf{1}_{X_{[0, n]}^{(0)} \in E_n} \mathbf{1}_{X_{[0, n]}^{(n)} \in E_n}.$$

We denote by $p_k^{E_n}(y) = \mathbb{E}_{\#}[\mathbf{1}_{X_k=y} \mathbf{1}_{X_{[0, n]} \in E_n}]$ and $p_k(y) = \mathbb{P}_{\#}(X_k = y)$ for the heat kernels. By the standard local limit theorem (or just Stirling approximation on the binomial coefficients) there exists $C > 0$ such that $p_k(y) \leq \frac{C}{k}$ for all $k \geq 1$ and $y \in \mathbb{Z}^2$. First, by lifting the restrictions on E_n we have

$$\begin{aligned} \mathbb{E}_{\#} \left[\left(\mathcal{N}_{\mathbf{x}_n}^{E_n} \right)^2 \right] &\leq \mathbb{E}_{\#} \left[\left(\sum_{k=3/2n}^{2n} \mathbf{1}_{X_k = -\mathbf{x}_n} \right)^2 \right] \leq 2 \sum_{\frac{3}{2}n \leq k \leq k' \leq 2n} p_k(-\mathbf{x}_n) p_{k'-k}(\mathbf{0}) \\ &\leq 2 \sum_{\frac{3}{2}n \leq k \leq k' \leq 2n} \frac{C}{n} \frac{C}{k' - k} \leq C' \log(n), \end{aligned}$$

for some $C' > 0$ (independent of n). To evaluate the first moment, introduce the (truncated) Green functions $g^{E_n}(y) = \sum_{k=n/2}^n p_k^{E_n}(y)$ and similarly $g(y) = \sum_{k=n/2}^n p_k(y)$. In particular, since $\mathbb{P}_{\#}(E_n) \geq 1 - \varepsilon$ we have $\|p - p^{E_n}\|_1 := \sum_y p(y) - p^{E_n}(y) \leq \varepsilon$ and similarly and $\|g - g^{E_n}\|_1 = \sum_y g(y) - g^{E_n}(y) \leq \varepsilon n$. Recalling that $\frac{C}{n} \geq p_n^{E_n}(y) \geq p_n(y)$ and $2C \geq g^{E_n}(y) \geq g(y)$, we have

$$\begin{aligned} \mathbb{E}_{\#}[\mathcal{N}_{\mathbf{x}_n}^{E_n}] &= \sum_{y \in \mathbb{Z}^2} p_n^{E_n}(y) g^{E_n}(-y - \mathbf{x}_n) = \sum_{y \in \mathbb{Z}^2} \left(\begin{array}{l} p_n^{E_n}(y) g^{E_n}(-y - \mathbf{x}_n) - p_n^{E_n}(y) g(-y - \mathbf{x}_n) \\ -p_n(y) g(-y - \mathbf{x}_n) + p_n^{E_n}(y) g(-y - \mathbf{x}_n) \\ +p_n(y) g(-y - \mathbf{x}_n) \end{array} \right) \\ &\geq \sum_y p_n(y) g(-y - \mathbf{x}_n) - \|p_n^{E_n}\|_{\infty} \|g - g^{E_n}\|_1 - \|g\|_{\infty} \|p_n - p_n^{E_n}\|_1 \\ &\geq \sum_y p_n(y) g(-y - \mathbf{x}_n) - 3C^2 \varepsilon. \end{aligned}$$

However, since $\|\mathbf{x}_n\| \leq A\sqrt{n}$, the local limit theorem implies that $\sum_y p_n(y) g(-y - \mathbf{x}_n) > c_A$ for some $c_A > 0$ independently of n and so one can choose $\varepsilon > 0$ small enough so that if $\mathbb{P}_{\#}(E_n) \geq 1 - \varepsilon$ then we have $\mathbb{E}_{\#}[\mathcal{N}_{\mathbf{x}_n}^{E_n}] > c_A/2$. We conclude by the second moment method that

$$\mathbb{P}_{\#}(\mathcal{N}_{\mathbf{x}_n}^{E_n} > 0) \geq \mathbb{E}_{\#}[\mathcal{N}_{\mathbf{x}_n}^{E_n}] / \mathbb{E}_{\#} \left[\left(\mathcal{N}_{\mathbf{x}_n}^{E_n} \right)^2 \right] \geq \frac{c_A}{2C' \log n}.$$

□

3 Proof of Theorem 1.1

Proof of Theorem 1.1. Let us denote $\mathcal{P}_{3^j} = \mathbb{P}_{\text{elephant}}(\exists 3^j \leq k \leq 3^{j+1}, X_k = \mathbf{0} \mid \mathcal{F}_{3^j})$. When $\alpha < \alpha_c$, i.e. the diffusive regime, Bertenghi [1, Theorem 4.2] showed that under $\mathbb{P}_{\text{elephant}}$ we have

$$\left(\frac{D_n^X(\mathbf{e}_i)}{\sqrt{n}} \right)_{1 \leq i \leq 4} \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{X}_i)_{1 \leq i \leq 4},$$

for some random variable \mathcal{X} (whose distribution is irrelevant for our purposes). Together with our Proposition 2.2, this shows that in the diffusive regime, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for large j 's we have

$$\mathbb{P}_{\text{elephant}}(j \cdot \mathcal{P}_{3^j} > \delta) \geq 1 - \varepsilon. \quad (3.1)$$

Notice that the variables \mathcal{P}_{3^j} are not independent, but Jeulin's lemma [4, Proposition 3.2] gives

$$\sum_{k=1}^{\infty} \mathcal{P}_{3^k} = \infty, \quad \mathbb{P}_{\text{elephant}} - a.s. \quad (3.2)$$

To be honest we rather use the proof that the lemma itself, and since the argument is short let us reproduce it here: Suppose by contradiction that there exists $\varepsilon, M > 0$ so that the event $A = \{\sum_{k=1}^{\infty} \mathcal{P}_{3^k} < M\}$ has probability at least $\varepsilon > 0$. Using (3.1) we take $\delta > 0$ so that $\mathbb{P}_{\text{elephant}}(j \cdot \mathcal{P}_{3^j} > \delta) \geq 1 - \frac{\varepsilon}{2}$ and write

$$M \geq \mathbb{E} \left[\mathbf{1}_A \sum_{j \geq 1} \mathcal{P}_{3^j} \right] \geq \sum_{j \geq 1} \frac{\delta}{j} \cdot \underbrace{\mathbb{P}_{\text{elephant}} \left(A \cap \left\{ \mathcal{P}_{3^j} > \frac{\delta}{j} \right\} \right)}_{\geq \varepsilon - (1 - (1 - \frac{\varepsilon}{2})) = \varepsilon/2} = \infty,$$

which is a contradiction. Given (3.2), the conditional Borel-Cantelli lemma ([2, Theorem 4.3.4]) then implies that the events $\{\exists 3^j \leq k \leq 3^{j+1} : X_k = \mathbf{0}\}$ happen for infinitely many j 's with probability one, implying recurrence of the process.

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