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# NULL-CONTROLLABILITY OF THE GENERALIZED BAOUENDI-GRUSHIN HEAT LIKE EQUATIONS

PHILIPPE JAMING AND YUNLEI WANG

ABSTRACT. In this article, we prove null-controllability results for the heat equation associated to fractional Baouendi-Grushin operators

$$\partial_t u + (-\Delta_x - V(x)\Delta_y)^s u = 1_\Omega h$$

where  $V$  is a potential that satisfies some power growth conditions and the set  $\Omega$  is thick in some sense. This extends previously known results for potentials  $V(x) = |x|^{2k}$ .

To do so, we study Zhu-Zhuge's spectral inequality for Schrödinger operators with power growth potentials, and give a precised quantitative form of it.

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## 1. INTRODUCTION

The aim of this paper is to prove null-controllability and observability from equidistributed subsets of  $\mathbb{R}^d$  for heat equations associated to Baouendi-Grushin type operators

$$\mathcal{L}_V(x, y) = -\Delta_x - V(x)\Delta_y, \quad x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}^m \text{ or } \mathbb{T}^m$$

where the potential  $V$  has a power growth  $c_1|x|^{\beta_1} \leq V(x) \leq c_2|x|^{\beta_2}$  and an upper bound on the gradient (for precise assumptions on  $V$ , see Assumptions [A1-A2](#) below). Let us now make this more precise.

**1.1. Null-controllability, observability and spectral inequalities.** First recall that null-controllability is defined as follows:

**Definition 1.1** (Null-controllability). Let  $P$  be a closed operator on  $L^2(\mathbb{R}^d)$  which is the infinitesimal generator of a strongly continuous semigroup  $(e^{-tP})_{t \geq 0}$  on  $L^2(\mathbb{R}^d)$ ,  $T > 0$  and  $\Omega$  be a measurable

subset of  $\mathbb{R}^d$ . The equation

$$(1.1) \quad \begin{cases} (\partial_t + P)u(t, x) = h(t, x)1_\Omega(x), & x \in \mathbb{R}^d, t > 0, \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^d), \end{cases}$$

is said to be *null-controllable from the set  $\Omega$  in time  $T > 0$*  if, for any initial datum  $u_0 \in L^2(\mathbb{R}^d)$ , there exists  $h \in L^2((0, T) \times \mathbb{R}^d)$ , supported in  $(0, T) \times \Omega$ , such that the mild (or semigroup) solution of (1.1) satisfies  $u|_{t=T} = 0$ .

By the Hilbert Uniqueness Method, see [Co07, Theorem 2.44] or [Li88, TW09], the null controllability of the equation (1.1) is equivalent to the (final state) observability of the adjoint system

$$(1.2) \quad \begin{cases} (\partial_t + P^*)v(t, x) = 0, & x \in \mathbb{R}^d, t > 0, \\ v|_{t=0} = v_0 \in L^2(\mathbb{R}^d), \end{cases}$$

where  $P^*$  stands for the  $L^2(\mathbb{R}^d)$ -adjoint of  $P$ . This notion of observability is defined as follows:

**Definition 1.2** (Observability). Let  $T > 0$  and  $\Omega$  be a measurable subset of  $\mathbb{R}^d$ . Equation (1.2) is said to be *observable from the set  $\Omega$  in time  $T > 0$*  if there exists a positive constant  $C_T > 0$  such that, for any initial datum  $v_0 \in L^2(\mathbb{R}^d)$ , the mild (or semigroup) solution of (1.2) satisfies

$$\int_{\mathbb{R}^d} |v(T, x)|^2 dx \leq C_T \int_0^T \left( \int_{\Omega} |v(t, x)|^2 dx \right) dt.$$

Null-controllability/observability problems of heat equations associated to elliptic operators on domains is a an old and vast subject (see [Co07, Li88, TW09] and references therein). More recently, following the seminal work by K. Beauchard, P. Cannarsa and R. Guglielmi [CBG13], research about null-controllability properties of heat equations associated to degenerate hypo-elliptic operators of the Baouendi-Grushin type have attracted a lot of attention (see e.g. [FrMu16, Koe17, DK20, DKR22, BMM15, BDE20, ABM21, Lis22]).

In another direction, there has been a lot of recent activity on null-controllability/observability problems for operators  $P$  defined on the whole space  $\mathbb{R}^d$  and sets  $\omega$  satisfying various thickness conditions. The first result in this direction concerns the heat equation on  $\mathbb{R}^d$  ( $P = \Delta$ ), and observability from so-called thick sets. To define this notion, let us introduce the following notation: we denote by  $\mathcal{B}_r(z)$  the Euclidean ball with radius  $r$  centered on  $z$  and  $\mathcal{B}_r = \mathcal{B}_r(0)$ . We write  $\mathcal{Q}_r = [-r/2, r/2]^d$  and  $\mathcal{Q}_r(z) = z + \mathcal{Q}_r$  denotes the cube with side length  $r$  centered on  $z$ . The Lebesgue measure of a set  $E \subset \mathbb{R}^d$  will be denoted by  $|E|$ .

**Definition 1.3.** Let  $\gamma \in (0, 1)$  and  $\ell > 0$ . A set  $\omega \subset \mathbb{R}^d$  is said to be  $\gamma$ -thick (at scale  $\ell > 0$ ) if, for every  $z \in \mathbb{R}^d$ ,

$$|\omega \cap (z + \mathcal{Q}_\ell)| \geq \gamma |\mathcal{Q}_\ell|.$$

We will say that  $\omega$  is thick, if it is  $\gamma$ -thick at scale  $\ell$  for some  $\gamma \in (0, 1)$  and  $\ell > 0$ .

It is easy to check that the notion of thickness does not depend on the norm used to define it, in particular cubes may be replaced by balls. In other words,  $\omega$  is thick if and only if there is a  $\gamma' \in (0, 1)$  and an  $\ell' > 0$  such that, for every  $z \in \mathbb{R}^d$ ,  $|\omega \cap (z + \mathcal{B}_{\ell'})| \geq \gamma' |\mathcal{B}_{\ell'}|$ .

It was then proven in [EV18] and independently in [WWZZ19] that the heat equation on  $\mathbb{R}^d$  is controllable/observable from a set  $\omega$  if and only if  $\omega$  is thick:

**Theorem 1.4** (Egidi & Veselić [EV18], Wang, Wang, Zhang & Zhang [WWZZ19]). *Let  $\omega \subset \mathbb{R}^d$  and  $T > 0$ . Then the following are equivalent:*

- (i)  $\Omega$  is thick;
- (ii) *Observability: there exists a constant  $C > 0$  such that, for every  $f \in L^2(\mathbb{R}^d)$ ,*

$$\|e^{-T\Delta} f\|_{L^2(\mathbb{R}^d)}^2 \leq C \int_0^T \|e^{-t\Delta} f\|_{L^2(\omega)}^2 dt.;$$

- (iii) *Controllability: For every  $u_0 \in L^2(\mathbb{R}^d)$  there exists  $h \in L^2((0, T) \times \Omega)$  such that the solution of the heat equation  $\partial_t u - \Delta_x u = h(t, x)1_\Omega(x)$  with initial condition  $u(0, x) = u_0(x)$  satisfies  $u(T, x) = 0$ .*

Soon after, the case of heat equations associated to some Schrödinger operators

$$P = H_V := -\Delta_x + V(x), \quad x \in \mathbb{R}^d,$$

started to be investigated. For  $V = |x|^2$ ,  $P$  is the harmonic oscillator and observability inequalities from different kinds of  $\omega$  were established, see [BPS18, BJPS21, ES21, MPS22, DSV23]. For  $V = |x|^{2k}$ , null controllability and observability inequalities were established for  $P = H_V$  in [AS23, Alp20, MPS22, Mi09, Ma22]. Very recently, in [AS23], P. Alphonse and A. Seelman extended some of those results to Baouendi-Grushin operators  $P = \mathcal{L}_{|x|^{2k}}$ . Our aim here is to pursue this line of research by extending the class of potentials  $V$ .

All those results require various restrictions on  $\omega$ . Apart from thick sets that we already defined, we now need to introduce two further classes. The first one was introduced in control theory in [RMV13] and consists of a particular class of thick sets and are those one would naturally consider if one wants to construct such sets:

**Definition 1.5.** Let  $\gamma \in (0, 1)$  and  $0 \leq \sigma < 1$ .

- (i) A set  $\omega \subset \mathbb{R}^d$  is said to be  $\gamma$ -*equidistributed* or simply *equidistributed* if there exists a set of points  $\{z_k\}_{k \in \mathbb{Z}^d} \subset \mathbb{R}^d$  such that, for every  $k$

$$\omega \cap (k + \mathcal{Q}_1) \supset \mathcal{B}_\gamma(z_k).$$

Since [RMV13], a larger class has appeared:

- (ii)  $\omega$  is  $(\gamma, \sigma)$ -*distributed* if there exists a set of points  $\{z_k\}_{k \in \mathbb{Z}^d} \subset \mathbb{R}^d$  such that, for every  $k \in \mathbb{Z}^d$ ,

$$z_k \in k + \mathcal{Q}_1 \quad \text{and} \quad \omega \cap (k + \mathcal{Q}_1) \supset \mathcal{B}_{\gamma+|k|^\sigma}(z_k).$$

Note that a  $(\gamma, 0)$ -distributed set is  $\sqrt{\gamma}$ -equidistributed.

Next, thanks to the famous Lebeau-Robbiano method introduced in [LR95] (see also [TT11, BPS18, NTTV20, GST20]), the mentioned results are based on some form of Spectral Inequality. To be more precise, recall that a spectral inequality for a nonnegative selfadjoint operator  $P$  in  $L^2(\mathbb{R}^d)$  takes the form

$$(1.3) \quad \|\phi\|_{L^2(\mathbb{R}^d)}^2 \leq \kappa_0 e^{\kappa_1 \lambda^\zeta} \|\phi\|_{L^2(\omega)}^2, \quad \forall \phi \in \mathcal{E}_P(\lambda), \lambda \geq 0,$$

where  $\omega$  is a measurable subset of  $\mathbb{R}^d$ ,  $\mathcal{E}_P(\lambda)$  are the spectral sets associated to  $P$ , and  $\kappa_0, \kappa_1, \zeta$  are constants. Such an inequality is a quantitative version of a unique continuation property (i.e.,  $f = 0$  on  $\omega$  implies  $f = 0$  on  $\mathbb{R}^d$ ).

In the case of  $P = -\Delta$  being the Laplacian on  $\mathbb{R}^d$ , this inequality is a reformulation of Kovrijkine's sharp version of the Logvinenko-Sereda Uncertainty Principle [Kov01] and is valid if (and only if)  $\omega$  is  $\gamma$ -thick for some  $\gamma > 0$ . The strategy of proof of Kovrijkine is very powerful and the authors of several of the results mentioned so far have managed to implement this strategy in different settings.

Alternative strategies are based on Carleman estimates and allow for potentials  $V$  that are less regular at the price of having slightly more regular sets  $\omega$ . In this direction, Dicke, Seelmann and Veselić [DSV22] recently considered the Schrödinger operator with power growth potentials and a set  $\omega$  that is  $(\gamma, \sigma)$ -distributed. Precisely speaking, they establish the spectral inequality (1.3) when the potential  $V \in W_{\text{loc}}^{1, \infty}(\mathbb{R}^d)$  has suitable power growth.

Shortly after, Zhu and Zhuge [ZZ23] improved the exponent of  $\lambda$  in (1.3) for a slightly larger class of potentials (thus positively answering to the questions asked in [DSV22]), namely those satisfying the following assumption:

**Assumption A.**  $V \in L_{\text{loc}}^1(\mathbb{R}^d)$  is real valued and there are constants  $c_1, c_2 > 0$  and  $0 < \beta_1 \leq \beta_2$  such that

- (i) for every  $x \in \mathbb{R}^d$ ,

$$(1.4) \quad c_1(|x| - 1)_+^{\beta_1} \leq V(x).$$

where  $(a)_+ := \max\{a, 0\}$ ;

(ii) we can write  $V = V_1 + V_2$  with

$$|V_1(x)| + |\nabla V_1(x)| + |V_2(x)|^{\frac{4}{3}} \leq c_2(|x| + 1)^{\beta_2}$$

for every  $x \in \mathbb{R}^d$ .

The main result of Zhu and Zhuge [ZZ23] is then that, when  $P = H_V := -\Delta + V$  with  $V$  satisfying assumption **A** and  $\omega$  a  $(\gamma, \sigma)$ -distributed set,

$$(1.5) \quad \|\phi\|_{L^2(\mathbb{R}^d)}^2 \leq \kappa_0 \left(\frac{1}{\gamma}\right)^{\kappa_1 \lambda^\zeta} \|\phi\|_{L^2(\omega)}^2, \quad \forall \phi \in \mathcal{E}_P(\lambda), \lambda \geq 0,$$

where  $\zeta = \frac{2\sigma + \beta_2}{2\beta_1}$  (thus improving the exponent  $\zeta = \frac{3\sigma + 2\beta_2}{3\beta_1}$  in [DSV22]). Note for future use that [ZZ23, DSV22] do not provide estimates of the constants  $\kappa_0, \kappa_1$  in (1.3) in terms of  $c_1, c_2$  that we will need here. We will thus revisit their proofs in Theorem 3.1 below in order to obtain those estimates.

Also, in view of the results by L. Miller [Mi09] and J. Martin [Ma22], one should expect to be able to improve those results when  $\beta_1 > 2$  by taking much smaller sets  $\omega$  (in the case of the Baouendi-Grushin operator,  $\omega$  can be any open set). We do not pursue in this direction here.

**1.2. Generalized Baouendi-Grushin operator.** Let us now move to the focus of this article. We will here consider observability/controlability properties of evolution equations

$$\partial_t u + Pu = 0$$

where  $P$  is an operator of Baouendi-Grushin type associated to a real-valued potential  $V$ ,

$$P = \begin{cases} \mathcal{L}_V := -\Delta_x - V(x)\Delta_y & x \in \mathbb{R}^n, y \in \mathbb{R}^m \\ \mathcal{L}_{V,p} := -\Delta_x - V(x)\Delta_y & x \in \mathbb{R}^n, y \in \mathbb{T}^m \end{cases}$$

where  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$  (by abuse of notation, we will write  $\mathcal{L}_V$  for both operators).

The potential  $V$  will satisfy some assumptions which are a bit stronger than Assumption **A** that covers both the Grushin operator

$$\Delta_G = \mathcal{L}_{|x|^2} := -\Delta_x - |x|^2 \Delta_y$$

and the Baouendi-Grushin operator

$$\Delta_k = \mathcal{L}_{|x|^{2k}} := -\Delta_x - |x|^{2k} \Delta_y, \quad k \text{ a positive integer.}$$

The first assumption we will consider is satisfied in the above case  $V(x) = |x|^{2k}$ ,  $k \in \mathbb{N}^*$  and more generally when  $V(x) = |x|^\beta$ ,  $\beta > 1$  and is defined as follows:

**Assumption A1.**  $V \in L_{\text{loc}}^\infty(\mathbb{R}^n)$  and there exist positive  $c_1, c_2, \beta_1, \beta_2 > 0$  such that, for every  $x \in \mathbb{R}^n$ ,

$$c_1|x|^{\beta_1} \leq V(x) \text{ and } |V(x)| + |\nabla V(x)| \leq c_2|x|^{\beta_2}.$$

Next we notice that  $V(x) = |x|^\beta$  does not satisfy this assumption when  $0 < \beta \leq 1$  and we will replace Assumption **A1** with the following weaker one:

**Assumption A2.**  $V \in L_{\text{loc}}^\infty(\mathbb{R}^n)$  and there exist positive  $c_1, c_2, \beta_1, \beta_2 > 0$  such that the following two conditions are satisfied:

(i) for every  $x \in \mathbb{R}^n$ ,

$$(1.6) \quad c_1|x|^{\beta_1} \leq V(x).$$

(ii) We can write  $V = V_1 + V_2$  such that for every  $x \in \mathbb{R}^n$ ,

$$|V_1(x)| + |\nabla V_1(x)| + |V_2(x)|^{\frac{4}{3}} \leq c_2(|x| + 1)^{\beta_2}.$$

**Remark 1.6.** Notice that, as  $c_1 > 0$ ,  $V = 0$  is excluded.

Further, if  $V$  satisfies Assumption **A1**, then setting  $V_1 = V$  and  $V_2 = 0$  we obtain that  $V$  also satisfies Assumption **A2**. As expected, our results will be stronger under Assumption **A1** than under Assumption **A2**.

On the other hand, let us consider the standard case  $V = |x|^\beta$  with  $0 < \beta \leq 1$ . Take a smooth cut-off function  $\eta$  such that  $\eta = 1$  in  $\mathcal{B}_1$  and  $\eta = 0$  in  $\mathbb{R}^n \setminus \mathcal{B}_2$  and write  $V_1 = |x|^\beta(1 - \eta(x))$  and  $V_2(x) = |x|^\beta \eta(x)$ ,

then we see that  $V$  satisfies Assumption **A2**. This is the main reason for introduction of Assumption **A** in [ZZ23].

Also it should be noted that in [ZZ23] the original Assumption **A** is different from Assumption **A2** only in (1.4), in which the lower bound is  $c_1(|x| - 1)_+^{\beta_1}$  instead of  $c_1|x|^{\beta_1}$  in (1.6). This will play a key role in order to obtain a lower bound of the first eigenvalue of the Schrödinger operator  $-\Delta + V(x)$  with suitable dependence on the parameter  $c_1$  (see Subsection 2.1 for details).

Our main result gives conditions for null-controllability of the heat equation associated to  $(-\Delta_x - V(x)\Delta_y)^s$  from sets of the form  $\omega \times \mathbb{R}^m$ ,  $\omega$  being  $\gamma$ -equidistributed.

**Theorem 1.7.** *Let  $\gamma \in (0, 1/2)$ ,  $\sigma \geq 0$ ,  $s, T > 0$ ,  $c_1, c_2, \beta_1, \beta_2 > 0$ . Take  $V \in L_{\text{loc}}^\infty(\mathbb{R}^n)$  satisfying Assumption **A2** with parameters  $c_1, c_2, \beta_1, \beta_2$ . Let  $s_* = \frac{\beta_1 + 2}{3}$ .*

*Let  $\omega \subset \mathbb{R}^n$  be a  $\gamma$ -equidistributed set. Then the fractional Baouendi-Grushin heat equation associated to  $V$*

$$(E_{BG,s}) \quad \begin{cases} \partial_t u(t, x, y) + (-\Delta_x - V(x)\Delta_y)^s u(t, x, y) = h(t, x, y)1_{\omega \times \mathbb{R}^m}(x, y) \\ u(0, x, y) = u_0(x, y) \quad \text{for } (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \end{cases} \quad \text{for } t > 0, (x, y) \in \mathbb{R}^n \times \mathbb{R}^m,$$

*with initial condition  $u_0 \in L^2(\mathbb{R}^n \times \mathbb{R}^m)$  is exactly null-controllable from  $\omega \times \mathbb{R}^m$  in any time  $T > 0$  if  $\beta_2 = \beta_1$  and  $s > s_*$ .*

*If  $V$  further satisfies Assumption **A1**, then the same holds with the critical power  $s_* = \frac{\beta_1 + 2}{4}$  instead of  $\frac{\beta_1 + 2}{3}$ .*

Our second result is that, in the presence of a second potential, the set  $\omega$  can be sparser. More precisely, we have the following:

**Proposition 1.8.** *Let  $\gamma \in (0, 1/2)$ ,  $\sigma \geq 0$ ,  $T > 0$ ,  $c_1, c_2, \beta_1, \beta_2 > 0$  and  $s > \frac{\beta_2 + 2\sigma}{2\beta_1}$ . Take  $V, \tilde{V} \in L_{\text{loc}}^\infty(\mathbb{R}^n)$  satisfying Assumption **A2** with parameters  $c_1, c_2, \beta_1, \beta_2$ . Let  $s_* = \frac{\beta_1 + 2}{3}$ . Let  $\omega \subset \mathbb{R}^n$  be a  $(\gamma, \sigma)$ -distributed set. Then the fractional Baouendi-Grushin-Schrödinger heat equation associated to  $V$  and  $\tilde{V}$*

$$(E_{BGS,s}) \quad \begin{cases} \partial_t u(t, x, y) + (-\Delta_x - V(x)\Delta_y + \tilde{V}(x))^s u(t, x, y) = h(t, x, y)1_{\omega \times \mathbb{R}^m}(x, y) \\ u(0, x, y) = u_0(x, y) \quad \text{for } (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \end{cases} \quad \text{for } t > 0, (x, y) \in \mathbb{R}^n \times \mathbb{R}^m,$$

*with initial condition  $u(0, \cdot, \cdot) = u_0 \in L^2(\mathbb{R}^n \times \mathbb{R}^m)$  is exactly null-controllable in any time  $T > 0$  if  $\beta_2 = \beta_1$  and  $s > s_*$ .*

*If  $V$  further satisfies Assumption **A1**, then the same holds with the critical power  $s_* = \frac{\beta_1 + 2}{4}$  instead of  $\frac{\beta_1 + 2}{3}$ .*

Finally, we also obtain a result on  $\mathbb{R}^n \times \mathbb{T}^m$ , that is, when the equation is periodic in the  $y$  variable. Here the sets  $\omega$  need again to be  $\gamma$ -equidistributed, but there is a relaxation on the potential as the upper bound can now be of a different order than the lower bound ( $\beta_2 > \beta_1$ ):

**Theorem 1.9.** *Let  $\gamma \in (0, 1/2)$ ,  $\sigma \geq 0$ ,  $s, T > 0$ ,  $c_1, c_2, \beta_1, \beta_2 > 0$ . Take  $V \in L_{\text{loc}}^\infty(\mathbb{R}^n)$  satisfying Assumption **A2** with parameters  $c_1, c_2, \beta_1, \beta_2$ . Let  $s_* = \frac{\beta_1 + 2}{3}$ .*

*Let  $\omega \subset \mathbb{R}^n$  be a  $\gamma$ -equidistributed set. Then the semi-periodic fractional Baouendi-Grushin heat equation associated to  $V$*

$$(E_{pBG,s}) \quad \begin{cases} \partial_t u(t, x, y) + (-\Delta_x - V(x)\Delta_y)^s u(t, x, y) = h(t, x, y)1_{\omega \times \mathbb{R}^m}(x, y) \\ u(0, x, y) = u_0(x, y) \quad \text{for } (x, y) \in \mathbb{R}^n \times \mathbb{T}^m \end{cases} \quad \text{for } t > 0, (x, y) \in \mathbb{R}^n \times \mathbb{T}^m,$$

with initial condition  $u(0, \cdot, \cdot) = u_0 \in L^2(\mathbb{R}^n \times \mathbb{T}^m)$  is exactly null-controllable in any time  $T > 0$  if  $s > s_*$ .

If  $V$  further satisfies Assumption A1, then the same holds with the critical power  $s_* = \frac{\beta_1 + 2}{4}$  instead of  $\frac{\beta_1 + 2}{3}$ .

**Remark 1.10.** For the sake of this last theorem, the main difference between the Baouendi-Grushin operator on  $\mathbb{R}^n \times \mathbb{R}^m$  and  $\mathbb{R}^n \times \mathbb{T}^m$  is that  $L^2(\mathbb{T}^m)$  has an orthonormal basis of eigenvectors of the Laplace operator on  $\mathbb{T}^m$ . The same result would be true e.g. if  $\mathbb{T}^m$  is replaced by any compact Riemannian manifold  $S$  and  $\Delta_y$  the Laplace-Beltrami operator on  $S$ .

Our proof does not extend to controllability from  $(\gamma, \sigma)$ -distributed sets mainly because 0 is an eigenvalue. This difficulty no longer arises when adding a second potential  $\tilde{V}$  satisfying the same assumption as  $V$  and considering

$$\begin{aligned} \partial_t u(t, x, y) + (-\Delta_x - V(x)\Delta_y + \tilde{V}(x))^s u(t, x, y) &= h(t, x, y)1_{\omega \times \mathbb{T}^m}(x, y) \\ t > 0, (x, y) &\in \mathbb{R}^n \times \mathbb{T}^m, \end{aligned}$$

instead of  $(E_{pBG,s})$ .

The results have been recasted in the synthetic Table 1, Section 5 where we will show the corresponding observability inequalities.

**Remark 1.11.** In the standard case of  $V(x) = |x|^\beta$ , the null-controllability from sets of the form  $\tilde{\omega} = \omega \times \mathbb{R}^m$  resp.  $\tilde{\omega} = \omega \times \mathbb{T}^m$  of the evolution equation

$$(E_{\beta,s}) \quad \begin{aligned} \partial_t u(t, x, y) + (-\Delta_x - |x|^\beta \Delta_y)^s u(t, x, y) &= h(t, x, y)1_{\omega \times \mathbb{T}^m}(x, y) \\ t > 0, x \in \mathbb{R}^n, y \in \mathbb{R}^n \text{ resp. } \mathbb{T}^m, \end{aligned}$$

with initial condition  $u(0, \cdot, \cdot) = u_0 \in L^2(\mathbb{R}^n \times \mathbb{R}^m)$  (resp.  $L^2(\mathbb{R}^n \times \mathbb{T}^m)$ ) is summarized in the following picture:

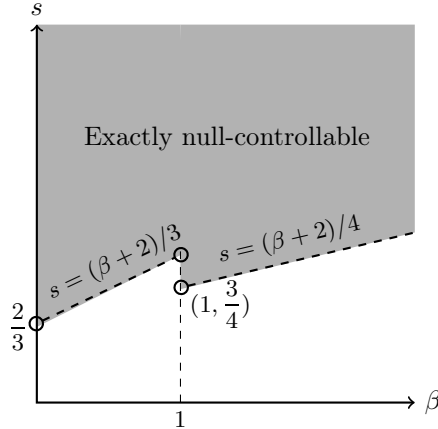


FIGURE 1. The standard case

Our results thus partially recover those of Alphonse and Seelmann [AS23]. There are two main losses compared to their results: one is that our control sets are a bit less general, and another is that we are not able to obtain the null-controllability of equations for the critical case  $s = s_*$ . The main gain is that our potentials are more general.

Further, when  $\beta = 2k$  and  $s = (\beta + 2)/4$ , Alphonse and Seelmann [AS23, Theorem 2.17] show that the evolution equation is never exactly null-controllable from any control support  $\omega \subset \mathbb{R}^n \times \mathbb{T}^m$  satisfying  $\bar{\omega} \cap \{x = 0\} = \emptyset$ . This implies  $s = (\beta + 2)/4$  is the critical value for  $\beta = 2k \geq 2$ . Based on our results, it is reasonable to conjecture that it is the critical value for all  $\beta \geq 1$  under Assumption A1 and  $\beta = \beta_1 = \beta_2$ .

For the case  $0 < \beta < 1$ , it seems that the critical value may be a little worse. Indeed, we only obtain exactly null-controllability results up to  $s = \frac{\beta + 2}{3} > \frac{\beta + 2}{4}$ . One of the possible reasons may be the wilder behavior of  $V = |x|^\beta$  around 0.

Our results presented above are in line with articles devoted to the study of null-controllability of  $(E_{\beta,s})$  in bounded domains, *see* [CBG13, ABM21, BDE20, BMM15, DKR22, DK20, Koe17]. All results in these articles give a fact that the null-controllability of  $(E_{\beta,s})$  is governed by minimal time in the critical case  $s = (\beta_1 + 2)/4$ . Note that we have a larger critical point  $s = (\beta_1 + 2)/3$  in Theorem 1.7 under Assumption A2. It is worthy to mention that in [Lis22, Theorem 1.2] the author considers the case of dimension  $n = m = 1, \beta = 2$  and  $s = 1$ , and then proved the equation  $(E_{\beta,s})$  is never exactly null-controllable from any control support of the form  $\mathbb{R} \times \omega$  whenever  $\bar{\omega} \neq \mathbb{R}$ .

**1.3. Strategy of proof and outline of the work.** Our strategy of proof is near to the one developed in Alphonse and Seelmann's work in [AS23] and based on an earlier idea of Beauchard, Cannarsa and Guglielmi [CBG13]. To avoid technicalities, we concentrate on the non-fractional case ( $s = 1$ ).

We observe that the operator  $-\Delta_x + V(x)\Delta_y$  is self-adjoint when  $V$  is real valued so that null-controllability is equivalent to observability. The first step consists in taking a partial Fourier transform in the  $y$ -variable of  $u$ :

$$u^\lambda(t, x) = \int_{\mathbb{R}^m} u(t, x, y) e^{-i(y, \lambda)} dy$$

so that we are lead to observability of the heat equation associated to Schrödinger operator

$$\partial_t u^\lambda - \Delta_x u^\lambda - \lambda^2 V(x) u^\lambda = 0.$$

We can then use results on the observability of this equation, in particular those of Zhu-Zhugue [ZZ23]. Note that the potential  $V_\lambda = \lambda^2 V(x)$  now depends on the frequency parameter  $\lambda$ . The main difficulty is precisely to deal with this parameter. This has been done by Alphonse and Seelmann when  $V$  is a power function and is more difficult here and requires two key differences in the proof:

- (1) We use a different kind of spectral inequality, which we call Zhu-Zhugue's inequality given in [ZZ23, Theorem 1]. However, the original form of this theorem cannot be used directly, since it does not provide explicit dependence of the cost constant on the parameters in Assumptions A1-A2. Our first task is thus to convert Zhu-Zhugue's proof into a more quantitative one which leads us to an explicit form of the cost constant in the spectral inequality in terms of the parameters. This is vital for our proofs in order to invert the partial Fourier transform in the  $y$ -variable. This allows to take into account the frequency parameter  $\lambda$ .
- (2) The lowest eigenvalue of the Schrödinger operator  $-\Delta + V$  is easily obtained in the case  $V = c|x|^{2k}, k \in \mathbb{N}$  by the rescaling approach. It does not work for our general potentials under Assumption A1 or A2. To overcome this difficulty, we do not calculate the exact value of the lowest eigenvalue, instead we just calculate a lower bound which satisfies our needs. This allows us to deal with more general potentials.

In Theorem 1.7 the set  $\omega$  is  $(\gamma, 0)$ -distributed rather than a sparser  $(\gamma, \sigma)$ -distributed set. This comes from the fact that the spectral multiplier associated to  $-\Delta_y$ , namely  $|\lambda|^2$  degenerates at  $\lambda = 0$ . This singularity is erased when adding a second potential as in Proposition 1.8 and explains why sparser sets are allowed there.

The remaining of the paper is organised as follows:

Section 2 is devoted to the spectral theory of Schrödinger operators  $H_{rV} = -\Delta + rV$  where  $V \in L^\infty_{\text{loc}}(\mathbb{R}^n)$  with a polynomial growth  $V(x) \geq c|x|^\beta, c, \beta > 0$  and a scaling parameter  $r > 0$ . In the Section 2.1, we provide a lower bound of the lowest eigenvalue of  $H_{rV}$  in terms of the scaling parameter while in the Section 2.2 we then provide a more quantitative version of the localization property of eigenfunctions of  $H_V$  first established in [DSV22, Theorem 1.4] and [ZZ23, Proposition 3].

Section 3 is then devoted to the precised form of Zhu-Zhugue's Spectral Inequality (1.5) in which we clarify the dependence of the constant  $\kappa_1$  appearing there in terms of the parameters  $c_1, c_2$  (and  $\beta_1, \beta_2$ ). This allows us in Section 3.2 to obtain precised version of (1.5) in which  $\kappa_1$  is estimated in terms of the scaling parameter  $r$  of the potential *i.e.* when applying (1.5) to  $H_{rV}$  instead of  $H_V$ . In Section 3.3, we conclude this section by computing an observability constant of  $H_{rV}$  from  $(\gamma, \sigma)$ -distributed sets and then bound this constant when  $r$  varies over  $]0, 1[$  or  $[1, +\infty)$ .



Section 4 is devoted to well-posedness of the considered equations and also introduces some notation for the last section.

Finally, we prove Theorem 1.7, Proposition 1.8 and their  $y$ -periodic counterparts in Section 5 by establishing the corresponding observability inequalities. In the simplest cases, we also provide an explicit form of the observability constants.

Throughout the paper, we will write  $A(x) \lesssim B(x)$  to say that there is a constant  $C$  that does not depend on the parameter  $x$  such that  $A \leq CB$ . At times, it will be more convenient to make the constants appear explicitly and in this case, they may change from line to line and still be called with the same letter.

## 2. EIGENFUNCTIONS OF THE SCHRÖDINGER OPERATOR

Let  $V$  be a real valued, non-negative function on  $\mathbb{R}^n$  with  $V \in L_{\text{loc}}^\infty(\mathbb{R}^n)$  and

$$\lim_{|x| \rightarrow +\infty} V(x) = +\infty$$

but for some integer  $m$ ,

$$\lim_{|x| \rightarrow +\infty} x^m V(x) = 0$$

and consider the associated Schrödinger operator

$$H_V f(x) = -\Delta f(x) + V(x)f(x).$$

This operator is well defined on  $\mathcal{S}(\mathbb{R}^n)$  and extends to an (unbounded) self-adjoint operator on  $L^2(\mathbb{R}^n)$ . Further it is positive since, for every  $f \in \mathcal{S}(\mathbb{R}^n)$ ,

$$\langle H_V f, f \rangle = \int_{\mathbb{R}^n} |\nabla f(x)|^2 + V(x)|f(x)|^2 dx \geq 0$$

The following is well-known (see e.g. [BS91]):

**Theorem 2.1.** *Let  $c, \beta > 0$  and  $V \in L_{\text{loc}}^\infty(\mathbb{R}^n)$  be such that  $V(x) \geq c|x|^\beta$ . Then there exists a sequence  $(\lambda_k)_{k \in \mathbb{N}}$ , with  $0 \leq \lambda_0 \leq \lambda_1 \leq \dots$  and  $|\lambda_k| \rightarrow +\infty$  and an orthonormal basis  $(\phi_k)_{k \in \mathbb{N}}$  of  $L^2(\mathbb{R}^n)$  consisting of eigenvectors of  $H_V$ ,  $\phi_k \in H^1(\mathbb{R}^n)$  and  $H_V \phi_k = \lambda_k \phi_k$ . Moreover, there exists  $a, C > 0$  such that, for every  $x$ ,*

$$|\phi_k(x)| \leq C \exp(-a|x|^{1+\frac{\beta}{2}}).$$

**Notation 2.2.** We will write  $\mathcal{E}_V(\lambda) = \mathcal{E}_{H_V}(\lambda)$  for the spectral set associated to  $H_V$ , that is

$$\mathcal{E}_V(\lambda) = \text{span}\{\phi_k : k \text{ s.t. } \lambda_k \leq \lambda\}.$$

In the remaining of this section, we assume that the potential  $V \in L_{\text{loc}}^\infty(\mathbb{R}^n)$  satisfies a global estimate

$$V(x) \geq c|x|^\beta$$

which is common to Assumptions A1 and A2. We first compute a lower bound of the lowest eigenvalue  $\lambda_0(V)$ . We then give a detailed decay estimate of linear combinations of eigenfunctions for the Schrödinger operator in terms of the parameters  $c, \beta$ .

**2.1. Lower bound of the first eigenvalue.** We start with the estimate of the first eigenvalue.

**Proposition 2.3.** *Let  $V \in L_{\text{loc}}^1(\mathbb{R}^n)$  and assume that there are  $c, \beta > 0$  such that  $V \geq c|x|^\beta$ . Let  $\lambda_0(V)$  be the lowest eigenvalue of the operator  $H_V = -\Delta_x + V(x)$ . Then we have*

$$(2.1) \quad \lambda_0(V) \geq \mu_* := c^{\frac{2}{\beta+2}} \lambda_*$$

where  $\lambda_*$  is a positive constant and depends only  $\beta$  and  $n$ .

The proof is based on the following result by Barnes, Brascamp and Lieb:

**Theorem 2.4** (Barnes, Brascamp and Lieb, [BBL76]). *For all  $a > 0$ , define*

$$I_V(a) = \int_{\mathbb{R}^n} e^{-aV(x)} dx$$

and assume that, for every  $a > 0$ ,  $I_V(a) < +\infty$ . Then we have

$$\lambda_0(V) \geq \sup_{t>0} t \left[ n + \frac{n}{2} \ln \frac{\pi}{t} - \ln I_V \left( \frac{1}{t} \right) \right].$$

Now we finish the proof of Proposition 2.3.

*Proof of Proposition 2.3.* In the case of  $V(x) = c|x|^\beta$ , a change into polar coordinates and then a change of variable  $s = acr^\beta$  shows that

$$I_{c|x|^\beta}(a) = \int_{\mathbb{R}^n} e^{-ac|x|^\beta} dx = \sigma_n \int_0^{+\infty} e^{-acr^\beta} r^{n-1} dr = \sigma_n \int_0^{+\infty} e^{-s} \left( \frac{s}{ac} \right)^{\frac{n}{\beta}} \frac{ds}{\beta s} = \frac{\sigma_n}{\beta(ac)^{n/\beta}} \Gamma \left( \frac{n}{\beta} \right)$$

where  $\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$  is the surface measure of the unit ball in  $\mathbb{R}^n$ , and  $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$  is the Gamma function. It follows that

$$\lambda_0(c|x|^\beta) \geq \sup_{t>0} t \left[ n - \ln \frac{2}{\beta} \frac{\Gamma \left( \frac{n}{\beta} \right)}{\Gamma \left( \frac{n}{2} \right)} - n \left( \frac{1}{\beta} + \frac{1}{2} \right) \ln t + \frac{n}{\beta} \ln c \right].$$

The maximum is attained when

$$n - \ln \frac{2\pi^{\frac{n}{2}}}{\beta} \frac{\Gamma \left( \frac{n}{\beta} \right)}{\Gamma \left( \frac{n}{2} \right)} - n \left( \frac{1}{\beta} + \frac{1}{2} \right) \ln t + \frac{n}{\beta} \ln c = n \left( \frac{1}{\beta} + \frac{1}{2} \right)$$

so that

$$\lambda_0(c|x|^\beta) \geq n \frac{\beta+2}{2\beta} \exp \left( \frac{\beta-2}{\beta+2} \right) \left( \frac{\beta}{2\pi^{\frac{n}{2}}} \frac{\Gamma \left( \frac{n}{\beta} \right)}{\Gamma \left( \frac{n}{2} \right)} \right)^{\frac{2\beta}{n(\beta+2)}} c^{\frac{2}{\beta+2}} := \lambda_* c^{\frac{2}{\beta+2}}.$$

Note that, by definition,  $\lambda_*$  depends only on  $\beta$  and  $n$ .

Finally, if  $V(x) \geq c|x|^\beta$ , it is obvious that  $I_V(a) \leq I_{c|x|^\beta}(a)$ . Hence we obtain

$$\lambda_0(V) \geq \lambda_0(c|x|^\beta) \geq \lambda_* c^{\frac{2}{\beta+2}}$$

as claimed.  $\square$

From this, we immediately obtain the following:

**Corollary 2.5.** *Let  $V \in L^1_{\text{loc}}(\mathbb{R}^n)$  and assume that there are  $c, \beta > 0$  such that  $V(x) \geq c|x|^\beta$  for every  $x \in \mathbb{R}^n$ . Let  $r > 0$   $H_{rV} = -\Delta_x + rV(x)$ . Then, for every  $f \in L^2(\mathbb{R}^n)$  and every  $s, t > 0$ ,*

$$\left\| e^{-tH_{rV}^s} f \right\|_{L^2(\mathbb{R}^n)} \leq e^{-tr \frac{2s}{\beta+2} \mu_*} \|f\|_{L^2(\mathbb{R}^n)}.$$

where  $\mu_*$  is defined in (2.1) and is thus a positive constant that depends only  $c, \beta$  and  $n$ .

**2.2. Localization property of eigenfunctions.** In this section we still assume that  $V(x) \geq c|x|^\beta$  and prove the following localization (or decay) property of eigenfunctions which is adapted from [DSV22, Theorem 1.4] and [ZZ23, Proposition 3] but with more precise quantitative estimates.

**Proposition 2.6.** *Assume that  $V \in L^\infty_{\text{loc}}(\mathbb{R}^n)$  is such that  $V(x) \geq c|x|^\beta$  and let  $H_V = -\Delta + V$ . Let  $\lambda > 0$  and  $\phi \in \mathcal{E}_V(\lambda)$  then*

$$\|\phi\|_{L^2(\mathbb{R}^n)} \leq 2\|\phi\|_{L^2(\mathcal{B}_\rho(0))} \quad \text{and} \quad \|\phi\|_{H^1(\mathbb{R}^n)} \leq 2\|\phi\|_{H^1(\mathcal{B}_\rho(0))}$$

with

$$(2.2) \quad \rho = \hat{C} \left( \left( \frac{n}{\beta} + \frac{n+2}{2} \right) \log_+ \frac{\lambda+1}{c} + \frac{n+2}{2} \log_+ c + \left( \frac{\lambda+2}{c} \right)^{1/\beta} + 1 \right).$$

and  $\hat{C}$  depending only on  $n$ .

Before the proof of Proposition 2.6, we give several lemmas.

**Lemma 2.7.** *Let  $V \in L_{\text{loc}}^\infty(\mathbb{R}^n)$  be such that  $V(x) \geq c|x|^\beta$ . Let  $\phi$  be an eigenvector of  $H_V = -\Delta + V$  with eigenvalue  $\lambda_k$  and let  $R_k = \max\{(\lambda_k + 2)/c, 1\}$ . Then we have*

$$(2.3) \quad \|e^{|\cdot|/2} \phi_k\|_{L^2(\mathbb{R}^n)}^2 \leq 7e^{R_k^{1/\beta} + 1} \|\phi_k\|_{L^2(\mathbb{R}^n)}^2$$

and

$$(2.4) \quad \|e^{|\cdot|/2} \nabla \phi_k\|_{L^2(\mathbb{R}^n)}^2 \leq C e^{R_k^{1/\beta}} \|\phi_k\|_{L^2(\mathbb{R}^n)}^2$$

where  $C > 0$  is a constant that depends on  $n$  only.

The first part is [DSV22, Proposition 2.3]. For the second part, we first prove a local Caccioppoli inequality:

**Lemma 2.8.** *Under the notation of Lemma 2.7: for every  $\rho > 0$  there exists a positive constant  $C(\rho)$  depending on  $\rho$  only such that, for every  $z \in \mathbb{R}^n$ ,*

$$\|\nabla \phi_k\|_{L^2(\mathcal{B}_\rho(z))}^2 \leq C(\rho) (1 + \lambda_k) \|\phi_k\|_{L^2(\mathcal{B}_{2\rho}(z))}^2.$$

*Proof of Lemma 2.8.* Choose a cutoff function  $\eta \in C_c^\infty(\mathcal{B}_{2\rho}(z))$  and  $\eta = 1$  in  $\mathcal{B}_\rho(z)$  and  $|\nabla \eta| < \frac{2}{\rho}$ . Let  $\psi_k = \eta^2 \phi_k$ , then

$$\int_{\mathcal{B}_{2\rho}(z)} \nabla \phi_k(x) \cdot \nabla \psi_k(x) \, dx = - \int_{\mathcal{B}_{2\rho}(z)} \psi_k(x) \Delta \phi_k(x) \, dx = - \int_{\mathcal{B}_{2\rho}(z)} \psi_k(x) (V(x) - \lambda_k) \phi_k(x) \, dx.$$

Further

$$(2.5) \quad \begin{aligned} \int_{\mathcal{B}_{2\rho}(z)} |\eta(x) \nabla \phi_k(x)|^2 \, dx &= \int_{\mathcal{B}_{2\rho}(z)} \eta(x)^2 \nabla \phi_k(x) \cdot \nabla \phi_k(x) \, dx \\ &= - \int_{\mathcal{B}_{2\rho}(z)} 2\eta(x) (\nabla \eta(x) \cdot \nabla \phi_k(x)) \phi_k(x) \, dx - \int_{\mathcal{B}_{2\rho}(z)} \eta(x)^2 (V(x) - \lambda_k) \phi_k(x)^2 \, dx \\ &\leq \int_{\mathcal{B}_{2\rho}(z)} \frac{4}{\rho} |\eta(x) \nabla \phi_k(x)| |\phi_k(x)| \, dx - \int_{\mathcal{B}_{2\rho}(z)} V(x) |\eta(x) \phi_k(x)|^2 \, dx + \lambda_k \|\phi_k\|_{L^2(\mathcal{B}_{2\rho}(z))}^2 \\ &\leq \int_{\mathcal{B}_{2\rho}(z)} \frac{4}{\rho} |\eta(x) \nabla \phi_k(x)| |\phi_k(x)| \, dx + \lambda_k \|\phi_k\|_{L^2(\mathcal{B}_{2\rho}(z))}^2 \end{aligned}$$

since  $V \geq 0$ . Note that

$$\begin{aligned} \int_{\mathcal{B}_{2\rho}(z)} \frac{4}{\rho} |\eta(x) \nabla \phi_k(x)| |\phi_k(x)| \, dx &= \int_{\mathcal{B}_{2\rho}(z)} 2 \left| \frac{1}{\sqrt{2}} \eta(x) \nabla \phi_k(x) \right| \left| \frac{2\sqrt{2}}{\rho} \phi_k(x) \right| \, dx \\ &\leq \frac{1}{2} \int_{\mathcal{B}_{2\rho}(z)} |\eta(x) \nabla \phi_k(x)|^2 \, dx + \frac{8}{\rho^2} \int_{\mathcal{B}_{2\rho}(z)} |\phi_k(x)|^2 \, dx \end{aligned}$$

so that (2.5) implies

$$\begin{aligned} \int_{\mathcal{B}_{2\rho}(z)} |\eta(x) \nabla \phi_k(x)|^2 \, dx &\leq \frac{1}{2} \int_{\mathcal{B}_{2\rho}(z)} |\eta(x) \nabla \phi_k(x)|^2 \, dx + \frac{8}{\rho^2} \|\phi_k\|_{L^2(\mathcal{B}_{2\rho}(z))}^2 + \lambda_k \|\phi_k\|_{L^2(\mathcal{B}_{2\rho}(z))}^2 \\ &\leq \left(1 + \frac{8}{\rho^2}\right) (1 + \lambda_k) \|\phi_k\|_{L^2(\mathcal{B}_{2\rho}(z))}^2. \end{aligned}$$

As  $\eta = 1$  on  $\mathcal{B}_\rho(z)$  we conclude that

$$\|\nabla \phi_k\|_{L^2(\mathcal{B}_\rho(z))}^2 \leq C(\rho) (1 + \lambda_k) \|\phi_k\|_{L^2(\mathcal{B}_{2\rho}(z))}^2$$

with  $C(\rho) = 1 + \frac{8}{\rho^2}$ . □

We can now complete the proof of Lemma 2.7.

*Proof of (2.4) in Lemma 2.7.* First we fix  $z \in \mathbb{R}^n$  and apply Lemma 2.7 with  $\rho = 1$  to obtain

$$\|\nabla\phi_k\|_{L^2(\mathcal{B}_1(z))}^2 \leq A(1 + \lambda_k)\|\phi_k\|_{L^2(\mathcal{B}_2(z))}^2$$

where  $A = C(1)$  is a numerical constant ( $A = 9$  with the previous proof). Further, as for  $x \in \mathcal{B}_2(z)$ ,

$$e^{-1}e^{|z|/2} \leq e^{|x|/2} \leq ee^{|z|/2}$$

we get

$$\begin{aligned} \|e^{|\cdot|/2}\nabla\phi_k\|_{L^2(\mathcal{B}_1(z))}^2 &\leq e^2e^{|z|}\|\nabla\phi_k\|_{L^2(\mathcal{B}_1(z))}^2 \leq e^2Ae^{|z|}(1 + \lambda_k)\|\phi_k\|_{L^2(\mathcal{B}_2(z))}^2 \\ &\leq e^4A(1 + \lambda_k)\|e^{|\cdot|/2}\phi_k\|_{L^2(\mathcal{B}_2(z))}^2. \end{aligned}$$

We then cover  $\mathbb{R}^n$  with a family of balls  $\mathbb{R}^n = \bigcup_{i \in \mathbb{N}} \mathcal{B}_1(z_i)$  such that the balls  $\{\mathcal{B}_2(z_i)\}$  have a finite covering number *i.e.* such that  $N := \max_{z \in \mathbb{R}^n} \#\{i : z \in \mathcal{B}_2(z_i)\} < +\infty$ . Then

$$\begin{aligned} \|e^{|\cdot|/2}\nabla\phi_k\|_{L^2(\mathbb{R}^n)}^2 &\leq \sum_{i \in \mathbb{N}} \|e^{|\cdot|/2}\nabla\phi_k\|_{L^2(\mathcal{B}_1(z_i))}^2 \leq e^4A(1 + \lambda_k) \sum_{i \in \mathbb{N}} \|e^{|\cdot|/2}\phi_k\|_{L^2(\mathcal{B}_2(z_i))}^2 \\ &\leq e^4AN(1 + \lambda_k)\|e^{|\cdot|/2}\phi_k\|_{L^2(\mathbb{R}^n)}^2 \\ &\leq e^4AN(1 + \lambda_k)7e^{R_k^{1/\beta}+1}\|\phi_k\|_{L^2(\mathbb{R}^n)}^2 \end{aligned}$$

with (2.3). As  $N$  depends on  $n$  only, we obtain (2.4) with  $C = e^5AN$ .  $\square$

Define

$$N(\lambda) := \#\{\lambda_k | \lambda_k \leq \lambda\}.$$

Note that, from

$$N(\lambda) \leq \sum_{k=1}^{N(\lambda)} (\lambda + 1 - \lambda_k)$$

and the lower bound  $V(x) \geq c|x|^\beta$ , the right hand side can be estimated explicitly by means of the classic Lieb-Thirring bound from [LT01, Theorem 1]. More precisely, for  $\lambda > 0$  we have

$$\begin{aligned} \sum_{k=1}^{N(\lambda)} (\lambda + 1 - \lambda_k) &\lesssim_n \int_{\mathbb{R}^n} \max\{\lambda + 1 - V(x), 0\}^{n/2+1} dx \\ &\leq \int_{\mathcal{B}_0(((\lambda+1)/c)^{1/\beta})} (\lambda + 1)^{n/2+1} dx \\ &\leq \kappa_n \frac{(\lambda + 1)^{\frac{n}{\beta} + \frac{n+2}{2}}}{c^{\frac{n}{\beta}}} \end{aligned}$$

with  $\kappa_n$  depending on  $n$  only. It follows that

$$(2.6) \quad N(\lambda) \leq \kappa_n c^{\frac{n+2}{2}} \left( \frac{\lambda + 1}{c} \right)^{\frac{n}{\beta} + \frac{n+2}{2}}$$

We are now in position to prove the main result of this section.

*Proof of Proposition 2.6.* Let us write

$$(2.7) \quad \phi = \sum_{k \leq N(\lambda)} \psi_k \quad \text{where } \psi_k = c_k \phi_k, \quad c_k \in \mathbb{C}.$$

For every  $\rho > 0$ , we have

$$(2.8) \quad \begin{aligned} \|\phi\|_{H^1(\mathbb{R}^n \setminus \mathcal{B}_\rho(0))}^2 &= \|\phi\|_{L^2(\mathbb{R}^n \setminus \mathcal{B}_\rho(0))}^2 + \|\nabla\phi\|_{L^2(\mathbb{R}^n \setminus \mathcal{B}_\rho(0))}^2 \\ &\leq e^{-\rho} \left( \|e^{|\cdot|/2}\phi\|_{L^2(\mathbb{R}^n)}^2 + \|e^{|\cdot|/2}\nabla\phi\|_{L^2(\mathbb{R}^n)}^2 \right). \end{aligned}$$

Moreover, using the expansion (2.7) and Cauchy-Schwartz, we obtain

$$(2.9) \quad \|e^{|\cdot|/2}\phi\|_{L^2(\mathbb{R}^n)}^2 \leq \left( \sum_{k=1}^{N(\lambda)} \|e^{|\cdot|/2}\psi_k\|_{L^2(\mathbb{R}^n)} \right)^2 \leq N(\lambda) \sum_{k=1}^{N(\lambda)} \|e^{|\cdot|/2}\psi_k\|_{L^2(\mathbb{R}^n)}^2$$

as well as

$$(2.10) \quad \|e^{|\cdot|/2}\nabla\phi\|_{L^2(\mathbb{R}^n)}^2 \leq N(\lambda) \sum_{k=1}^{N(\lambda)} \|e^{|\cdot|/2}\nabla\psi_k\|_{L^2(\mathbb{R}^n)}^2.$$

Taking (2.9) and (2.10) into (2.8), we obtain

$$(2.11) \quad \|\phi\|_{H^1(\mathbb{R}^n \setminus \mathcal{B}_\rho(0))}^2 \leq e^{-\rho} N(\lambda) \left( \sum_{k=1}^{N(\lambda)} \|e^{|\cdot|/2}\psi_k\|_{L^2(\mathbb{R}^n)}^2 + \sum_{k=1}^{N(\lambda)} \|e^{|\cdot|/2}\nabla\psi_k\|_{L^2(\mathbb{R}^n)}^2 \right).$$

Taking (2.3), (2.4) and (2.6) into (2.11) we obtain

$$\|\phi\|_{H^1(\mathbb{R}^n \setminus \mathcal{B}_\rho(0))}^2 \leq C e^{-\rho} c^{\frac{n+2}{2}} \left( \frac{\lambda+1}{c} \right)^{\frac{n}{\beta} + \frac{n+2}{2}} \sum_{k=1}^{N(\lambda)} e^{R_k^{1/\beta}} \|\psi_k\|_{L^2(\mathbb{R}^n)}^2$$

where  $R_k = \max\{(\lambda_k + 2)/c, 1\}$  and  $C$  is a constant depending only on  $n$ . As the  $\psi_k$ 's are orthogonal and  $R_k \leq R := \max\{(\lambda + 2)/c, 1\}$ , we get

$$\|\phi\|_{H^1(\mathbb{R}^n \setminus \mathcal{B}_\rho(0))}^2 \leq C e^{-\rho} \left( \frac{\lambda+1}{c} \right)^{\frac{n}{\beta} + \frac{n+2}{2}} c^{\frac{n+2}{2}} e^{R^{1/\beta}} \|\phi\|_{L^2(\mathbb{R}^n)}^2$$

We now chose  $\hat{C}$  large enough so that

$$\rho = \hat{C} \left( \left( \frac{n}{\beta} + \frac{n+2}{2} \right) \log_+ \frac{\lambda+1}{c} + \frac{n+2}{2} \log_+ c + \left( \frac{\lambda+2}{c} \right)^{1/\beta} + 1 \right)$$

satisfies

$$\rho \geq \log 2 + \log C + \left( \frac{n}{\beta} + \frac{n+2}{2} \right) \log \frac{\lambda+1}{c} + \frac{n+2}{2} \log_+ c + R^{1/\beta} + 1.$$

Note that  $\hat{C}$  depends on  $C$  only so that it depends only on  $n$ . With this choice,

$$\begin{aligned} \|\phi\|_{L^2(\mathbb{R}^n \setminus \mathcal{B}_\rho(0))}^2 + \frac{1}{2} \|\nabla\phi\|_{L^2(\mathbb{R}^n \setminus \mathcal{B}_\rho(0))}^2 &\leq \|\phi\|_{H^1(\mathbb{R}^n \setminus \mathcal{B}_\rho(0))}^2 \\ &\leq \frac{1}{2} \|\phi\|_{L^2(\mathbb{R}^n)}^2 = \frac{1}{2} \|\phi\|_{L^2(\mathbb{R}^n \setminus \mathcal{B}_\rho(0))}^2 + \frac{1}{2} \|\phi\|_{L^2(\mathcal{B}_\rho(0))}^2. \end{aligned}$$

This yields

$$\|\phi\|_{L^2(\mathbb{R}^n \setminus \mathcal{B}_\rho(0))}^2 \leq \|\phi\|_{H^1(\mathbb{R}^n \setminus \mathcal{B}_\rho(0))}^2 \leq \|\phi\|_{L^2(\mathcal{B}_\rho(0))}^2$$

and finally  $\|\phi\|_{L^2(\mathbb{R}^n)}^2 \leq 2\|\phi\|_{L^2(\mathcal{B}_\rho(0))}^2$  as well as  $\|\phi\|_{H^1(\mathbb{R}^n)}^2 \leq 2\|\phi\|_{H^1(\mathcal{B}_\rho(0))}^2$ .  $\square$

### 3. SPECTRAL INEQUALITY FOR THE SCHRÖDINGER OPERATOR

**3.1. Precised form of Zhu-Zhuge's spectral inequality.** The aim of this section is to prove the following precised form of [ZZ23, Theorem 1] :

**Theorem 3.1** (Zhu-Zhuge, precised form). *Let  $V \in L_{\text{loc}}^\infty(\mathbb{R}^n)$  be such that Assumption A2 is satisfied. Let  $\sigma \geq 0$ ,  $\gamma \in (0, 1/2)$  and  $\omega \subset \mathbb{R}^n$  be a  $(\gamma, \sigma)$ -distributed set. Then there exists a constant  $C$  depending only on  $n$  such that for every  $\lambda \geq \lambda_0(V)$  and every  $\phi \in \mathcal{E}_V(\lambda)$ ,*

$$\|\phi\|_{L^2(\mathbb{R}^n)} \leq \left( \frac{1}{\gamma} \right)^{C\mathcal{J}_V(\lambda)} \|\phi\|_{L^2(\omega)},$$

where

$$\begin{aligned} \mathcal{J}_V(\lambda) &:= \mathcal{J}(c_1, c_2, \lambda) = \mathbf{J}(c_1, \lambda)^{2\sigma/\beta_2} \widehat{\mathbf{J}}(c_1, c_2, \lambda), \\ (3.1) \quad \text{with } \mathbf{J}(c_1, \lambda) &= \left( 1 + \left( \frac{\lambda + 2}{c_1} \right)^{1/\beta_1} + \frac{n+2}{2} \log_+ c_1 \right)^{\frac{\beta_2}{2}} \quad \text{and } \widehat{\mathbf{J}}(c_1, c_2, \lambda) = \lambda^{\frac{1}{2}} + c_2^{\frac{1}{2}} \mathbf{J}(c_1, \lambda). \end{aligned}$$

The main difference with Zhu & Zhuge's result is that the exponent  $\mathcal{J}$  depends explicitly on the parameters  $c_1$  and  $c_2$  in Assumption A2. Recall that those are given by  $V = V_1 + V_2 \in L_{\text{loc}}^\infty(\mathbb{R}^n)$  and

$$V(x) \geq c_1 |x|^{\beta_1} \quad , \quad |V_1(x)| + |\nabla V_1(x)| + |V_2(x)|^{4/3} \leq c_2 (|x| + 1)^{\beta_2}.$$

To obtain this result we follow step by step the proof in [ZZ23]. This starts with two kinds of three-ball inequalities that are already given in a quantitative form sufficient for our needs. Then, we follow the strategy in [DSV22, ZZ23] to prove the spectral inequality.

Let us start with some notation:

**Notation 3.2.** Given  $L > 0$ , recall that  $\mathcal{Q}_L := \left[ -\frac{L}{2}, \frac{L}{2} \right]^n$ ,  $\mathcal{B}_r(x) \subset \mathbb{R}^n$  is the ball with radius  $r$  and center  $x$ . We denote by  $\mathbb{B}_r(x)$  the ball in  $\mathbb{R}^{n+1}$  centered at  $x$  and radius  $r$ .

Let  $\delta \in (0, \frac{1}{2})$ ,  $b = (0, \dots, 0, -b_{n+1}) \in \mathbb{R}^{n+1}$  and  $b_{n+1} = \frac{\delta}{100}$ . Define

$$W_1 = \left\{ y \in \mathbb{R}_+^{n+1} : |y - b| \leq \frac{1}{4} \delta \right\} \quad , \quad W_2 = \left\{ y \in \mathbb{R}_+^{n+1} : |y - b| \leq \frac{2}{3} \delta \right\},$$

so that  $W_1 \subset W_2 \subset \mathbb{B}_\delta(b)$ . Write  $\Lambda_L := \mathcal{Q}_L \cap \mathbb{Z}^n$  and define

$$W_j(z_i) := (z_i, 0) + W_j, \quad j = 1, 2.$$

as well as

$$P_j(L) = \bigcup_{i \in \Lambda_L} W_j(z_i), \quad j = 1, 2 \quad \text{and} \quad D_\delta(L) = \bigcup_{i \in \Lambda_L} \mathcal{B}_\delta(z_i).$$

Define  $R = 9\sqrt{n}$  and

$$X_1 = \mathcal{Q}_L \times [-1, 1] \quad \text{and} \quad \tilde{X}_R = \mathcal{Q}_{L+R} \times [-R, R].$$

The first three-ball inequality in  $\mathbb{R}^{n+1}$  we need is the following:

**Lemma 3.3** ([ZZ23, Lemma 1]). *Let  $\delta \in (0, \frac{1}{2})$ . There exist  $0 < \alpha < 1$  and  $C > 0$ , depending only on  $n$  such that, if  $v$  is the solution of*

$$\begin{cases} -(\Delta_x + \partial_{x_{n+1}}^2)v(x, x_{n+1}) + V(x)v(x, x_{n+1}) = 0 & x \in \mathbb{R}^n, x_{n+1} \in \mathbb{R} \\ v(x, 0) = 0 & x \in \mathbb{R}^n \end{cases}$$

then

$$(3.2) \quad \|v\|_{H^1(P_1(L))} \leq \delta^{-\alpha} \exp(C(1 + \mathcal{G}(V_1, V_2, 9\sqrt{n}L))) \|v\|_{H^1(P_2(L))}^\alpha \left\| \frac{\partial v}{\partial y_{n+1}} \right\|_{L^2(D_\delta(L))}^{1-\alpha},$$

where

$$(3.3) \quad \mathcal{G}(V_1, V_2, L) = \|V_1\|_{W^{1,\infty}(\mathcal{Q}_L)}^{\frac{1}{2}} + \|V_2\|_{L^\infty(\mathcal{Q}_L)}^{\frac{2}{3}}.$$

**Remark 3.4.** Under Assumption A2,  $\mathcal{G}(V_1, V_2, L) \leq c_2^{1/2}(1+L)^{\beta_2/2}$ .

The second inequality is:

**Lemma 3.5** ([ZZ23, Lemma 2]). *Let  $\delta \in (0, \frac{1}{2})$ . There exist  $C > 0$  depending only on  $n$ ,  $0 < \alpha < 1$  depending on  $\delta$  and  $n$  such that, if  $v$  is the solution of*

$$\begin{cases} -(\Delta_x + \partial_{x_{n+1}}^2)v(x, x_{n+1}) + V(x)v(x, x_{n+1}) = 0 & x \in \mathbb{R}^n, x_{n+1} \in \mathbb{R} \\ v(x, -y) = -v(x, y) & x \in \mathbb{R}^n, y \in \mathbb{R} \end{cases}$$

then

$$(3.4) \quad \|v\|_{H^1(X_1)} \leq \delta^{-2\alpha_1} \exp\left(C(1 + \mathcal{G}(V_1, V_2, 9\sqrt{n}L))\right) \|v\|_{H^1(\tilde{X}_R)}^{1-\alpha_1} \|v\|_{H^1(P_1(L))}^{\alpha_1},$$

where  $\mathcal{G}(V_1, V_2, L)$  is given by (3.3). Further,  $\alpha_1$  can be given in the form

$$(3.5) \quad 0 < \alpha_1 = \frac{\epsilon_1}{|\log \delta| + \epsilon_2} < 1$$

with positive constants  $\epsilon_1$  and  $\epsilon_2$  depending only on  $n$ .

Now let  $\phi \in \mathcal{E}_\lambda(H)$  and define

$$\Phi(x, x_{n+1}) = \sum_{0 < \lambda_k \leq \lambda} \alpha_k \phi_k(x) \frac{\sinh(\sqrt{\lambda_k} x_{n+1})}{\sqrt{\lambda_k}}.$$

Then  $\Phi(x, x_{n+1})$  satisfies the equation

$$(3.6) \quad H\Phi := -\Delta\Phi + V(x)\Phi = 0, \quad (x, x_{n+1}) \in \mathbb{R}^{n+1}.$$

We need to mention that  $\Delta = \Delta_x + \partial_{n+1}^2$  with  $\Delta_x = \sum_{j=1}^n \partial_j^2$  in (3.6). It is easy to check that

$$\partial_{n+1}\Phi(x, 0) = \phi(x) \text{ and } \Phi(x, 0) = 0.$$

The following estimate for  $\Phi$  is standard and can be found in [ZZ23, Lemma 3].

**Lemma 3.6.** *Let  $\phi \in \mathcal{E}_V(\lambda)$  and  $\Phi$  be given in (3.6). For any  $\rho > 0$ , we have*

$$2\rho \|\phi\|_{L^2(\mathbb{R}^n)}^2 \leq \|\Phi\|_{H^1(\mathbb{R}^n \times (-\rho, \rho))}^2 \leq 2\rho \left(1 + \frac{\rho^2}{3}(1 + \lambda)e^{2\rho\sqrt{\lambda}}\right) \|\phi\|_{L^2(\mathbb{R}^n)}^2.$$

To use Proposition 2.6, we need to extend it from  $\phi$  to  $\Phi$ . Indeed, we have the following corollary:

**Corollary 3.7.** *Given the same condition as in Proposition 2.6, we have*

$$(3.7) \quad \|\Phi\|_{H^1(\mathbb{R}^n \times (-1, 1))}^2 \leq 2\|\Phi\|_{H^1(\mathcal{B}_r \times (-1, 1))}^2.$$

*Proof.* Since  $\Phi(\cdot, x_{n+1}) \in \mathcal{E}_\lambda(H)$ , by Proposition 2.6 we obtain

$$(3.8) \quad \|\Phi(\cdot, x_{n+1})\|_{H^1(\mathbb{R}^n)}^2 \leq 2\|\Phi(\cdot, x_{n+1})\|_{H^1(\mathcal{B}_r(0))}^2.$$

Since  $\partial_{n+1}\Phi(\cdot, x_{n+1}) \in \mathcal{E}_\lambda(H)$ , we obtain

$$(3.9) \quad \|\partial_{n+1}\Phi(\cdot, x_{n+1})\|_{L^2(\mathbb{R}^n)}^2 \leq \|\partial_{n+1}\Phi(\cdot, x_{n+1})\|_{H^1(\mathbb{R}^n)}^2 \leq 2\|\partial_{n+1}\Phi(\cdot, x_{n+1})\|_{L^2(\mathcal{B}_r(0))}^2.$$

Then we have

$$\begin{aligned} \|\Phi\|_{H^1(\mathbb{R}^n \times (-1, 1))}^2 &= \int_{-1}^1 \|\Phi(\cdot, x_{n+1})\|_{L^2(\mathbb{R}^n)}^2 dx_{n+1} + \int_{-1}^1 \sum_{j=1}^n \|\partial_j \Phi(\cdot, x_{n+1})\|_{L^2(\mathbb{R}^n)}^2 dx_{n+1} \\ &\quad + \int_{-1}^1 \|\partial_{n+1}\Phi(\cdot, x_{n+1})\|_{L^2(\mathbb{R}^n)}^2 dx_{n+1} \\ &\leq \int_{-1}^1 \|\Phi(\cdot, x_{n+1})\|_{H^1(\mathbb{R}^n)}^2 dx_{n+1} + \int_{-1}^1 2\|\partial_{n+1}\Phi(\cdot, x_{n+1})\|_{L^2(\mathcal{B}_r(0))}^2 dx_{n+1} \end{aligned}$$

with (3.9). Using (3.8) we then obtain

$$\begin{aligned} \|\Phi\|_{H^1(\mathbb{R}^n \times (-1, 1))}^2 &\leq \int_{-1}^1 2\|\Phi(\cdot, x_{n+1})\|_{H^1(\mathcal{B}_r(0))}^2 dx_{n+1} + \int_{-1}^1 2\|\partial_{n+1}\Phi(\cdot, x_{n+1})\|_{L^2(\mathcal{B}_r(0))}^2 dx_{n+1} \\ &= 2\|\Phi\|_{H^1(\mathcal{B}_r(0))}^2 \end{aligned}$$

as claimed.  $\square$

We can now prove Theorem 3.1:

*Proof of Theorem 3.1.* Let  $L = 2[r] + 1$ , where  $r$  is given in Proposition 2.6 and  $[a]$  means the largest integer smaller than  $a + 1$ . Then we have  $\mathcal{B}_r(0) \subset \mathcal{Q}_L$ . Moreover, we can decompose  $\mathcal{Q}_L$  as

$$\mathcal{Q}_L = \bigcup_{k \in \Lambda_L} \left( k + \left[ -\frac{1}{2}, \frac{1}{2} \right]^n \right).$$

For each  $k \in \Lambda_L$ , we have  $|k| \leq \sqrt{n} \lceil r \rceil$ . As  $\gamma \in (0, \frac{1}{2})$ , we get

$$\delta := \gamma^{1+(\sqrt{n}\lceil r \rceil)^\sigma} \leq \gamma^{1+|k|^\sigma}, \quad \forall k \in \Lambda_L \cap \mathbb{Z}^n.$$

Now we show an interpolation inequality. Note that  $\Phi$  is odd in  $x_{n+1}$ , so taking  $v = \Phi$ , we combine (3.2) in Lemma 3.3 and (3.4) in Lemma 3.5 with  $\delta$  and  $L$  defined above to get

$$\begin{aligned} \|\Phi\|_{H^1(X_1)} &\leq \delta^{-2\alpha_1} \exp\left(C(1 + \mathcal{G}(V_1, V_2, 9\sqrt{n}L))\right) \|\Phi\|_{H^1(\tilde{X}_R)}^{1-\alpha_1} \|\Phi\|_{H^1(P_1(L))}^{\alpha_1} \\ &\leq \delta^{-2\alpha_1 - \alpha\alpha_1} \exp\left(C(1 + \mathcal{G}(V_1, V_2, 9\sqrt{n}L))\right) \|\Phi\|_{H^1(P_2(L))}^{\alpha\alpha_1} \left\| \frac{\partial\Phi}{\partial y_{n+1}} \right\|_{L^2(D_\delta(L))}^{\alpha_1(1-\alpha)} \|\Phi\|_{H^1(\tilde{X}_R)}^{1-\alpha_1} \\ &\leq \delta^{-3\alpha_1} \exp\left(C(1 + \mathcal{G}(V_1, V_2, 9\sqrt{n}L))\right) \|\phi\|_{L^2(D_\delta(L))}^{\hat{\alpha}} \|\Phi\|_{H^1(\tilde{X}_R)}^{1-\hat{\alpha}}, \end{aligned}$$

where  $\hat{\alpha} = \alpha_1(1 - \alpha)$  and we have used the facts  $P_2(L) \subset \tilde{X}_R$  and  $\frac{\partial\Phi}{\partial y_{n+1}}(\cdot, 0) = \phi$ . Here and below, the symbol  $C$  may represent different positive constants depending on  $n$ .

Recall  $\alpha_1$  in (3.5), we have  $\alpha_1 \approx \hat{\alpha} \approx \frac{1}{|\log \delta|}$  for any  $\delta \in (0, \frac{1}{2})$ . Hence  $\delta^{-3\alpha_1} \leq C$  and then

$$(3.10) \quad \|\Phi\|_{H^1(X_1)} \leq \exp\left(C(1 + \mathcal{G}(V_1, V_2, 9\sqrt{n}L))\right) \|\phi\|_{L^2(\omega \cap \mathcal{Q}_L)}^{\hat{\alpha}} \|\Phi\|_{H^1(\tilde{X}_R)}^{1-\hat{\alpha}},$$

where we have also used the fact  $D_\delta(L) \subset \omega \cap \mathcal{Q}_L$ .

Substituting  $L = 2 \lceil r \rceil + 1$  and (2.2) into  $\mathcal{G}(V_1, V_2, L)$  and by Assumption A2, we have

$$\begin{aligned} \mathcal{G}(V_1, V_2, 9\sqrt{n}L) &\lesssim c_2^{\frac{1}{2}} (2 \lceil r \rceil + 2)^{\frac{\beta_2}{2}} \lesssim c_2^{\frac{1}{2}} \left( \frac{n+4}{2\beta_1} \log_+ \frac{\lambda+1}{c_1} + \left( \frac{\lambda+2}{c_1} \right)^{1/\beta_1} + \frac{n+2}{2} \log_+ c_1 + 1 \right)^{\frac{\beta_2}{2}} \\ &\lesssim c_2^{\frac{1}{2}} \left( \left( \frac{\lambda+2}{c_1} \right)^{1/\beta_1} + \frac{n+2}{2} \log_+ c_1 + 1 \right)^{\frac{\beta_2}{2}} := c_2^{\frac{1}{2}} \mathbf{J}(c_1, \lambda) \end{aligned}$$

where  $\mathbf{J}(c_1, \lambda)$  was defined in (3.1). We can then write (3.10) as

$$(3.11) \quad \|\Phi\|_{H^1(X_1)} \leq \exp\left(C c_2^{\frac{1}{2}} \mathbf{J}(c_1, \lambda)\right) \|\phi\|_{L^2(\omega \cap \mathcal{Q}_L)}^{\hat{\alpha}} \|\Phi\|_{H^1(\tilde{X}_R)}^{1-\hat{\alpha}}.$$

We now bound  $\|\Phi\|_{H^1(\mathbb{R}^n \times (-\rho, \rho))}^2$  from above and below by respectively taking  $\rho = R$  and  $\rho = 1$  in Lemma 3.6. This gives

$$\frac{\|\Phi\|_{H^1(\mathbb{R}^n \times (-R, R))}^2}{\|\Phi\|_{H^1(\mathbb{R}^n \times (-1, 1))}^2} \leq R \left( 1 + \frac{R^2}{3} (1 + \lambda) \right) \exp\left(2R\sqrt{\lambda}\right) \leq \exp\left(C_2\sqrt{\lambda}\right).$$

With the aid of (3.7) and  $\mathcal{B}_r(0) \subset \mathcal{Q}_L$ , we get

$$(3.12) \quad \begin{aligned} \|\Phi\|_{H^1(\mathbb{R}^n \times (-R, R))} &\leq \exp\left(\frac{1}{2} C_2 \sqrt{\lambda}\right) \|\Phi\|_{H^1(\mathbb{R}^n \times (-1, 1))} \\ &\leq \sqrt{2} \exp\left(\frac{1}{2} C_2 \sqrt{\lambda}\right) \|\Phi\|_{H^1(\mathcal{Q}_L \times (-1, 1))}. \end{aligned}$$

Recall that  $X_1 = \mathcal{Q}_L \times (-1, 1)$ , substituting (3.12) into (3.11) we obtain

$$\|\Phi\|_{H^1(\mathbb{R}^n \times (-R, R))} \leq \exp\left(C_3 \hat{\mathbf{J}}(c_1, c_2, \lambda)\right) \|\phi\|_{L^2(\omega \cap \mathcal{Q}_L)}^{\hat{\alpha}} \|\Phi\|_{H^1(\tilde{X}_R)}^{1-\hat{\alpha}}$$

where  $\hat{\mathbf{J}}(c_1, c_2, \lambda)$  was defined in (3.1). Since  $\tilde{X}_R \subset \mathbb{R}^n \times (-R, R)$ , it follows that

$$\|\Phi\|_{H^1(\mathbb{R}^n \times (-R, R))} \leq \exp\left(\hat{\alpha}^{-1} C_3 \hat{\mathbf{J}}(c_1, c_2, \lambda)\right) \|\phi\|_{L^2(\omega \cap \mathcal{Q}_L)}.$$

Recall that

$$\hat{\alpha}^{-1} \approx \alpha_1^{-1} \approx |\log \delta| \approx |\log \gamma| \mathbf{J}^{\frac{2\sigma}{\beta_2}}$$



we obtain

$$\|\Phi\|_{H^1(\mathbb{R}^n \times (-R, R))} \leq \left(\frac{1}{\gamma}\right)^{C\mathcal{J}^{\frac{2\sigma}{\beta_2}}\hat{\mathcal{J}}} \|\phi\|_{L^2(\omega \cap \mathcal{Q}_L)}.$$

Finally, using the lower bound in Lemma 3.6 with  $\rho = R$ , we obtain

$$\|\phi\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{1}{2R}\right)^{\frac{1}{2}} \|\Phi\|_{H^1(\mathbb{R}^n \times (-R, R))} \leq \left(\frac{1}{\gamma}\right)^{C\mathcal{J}^{\frac{2\sigma}{\beta_2}}\hat{\mathcal{J}}} \|\phi\|_{L^2(\omega)}$$

where  $C$  is a positive constant depending only on  $n$ .  $\square$

**3.2. Scaling the potential.** The aim of this section is to consider the ‘‘scaled’’ Schrödinger operator

$$(3.13) \quad H_{rV} = -\Delta_x + rV(x), \quad r > 0$$

and to evaluate the influence of  $r$  on the exponent  $\mathcal{J}$  in Theorem 3.1. This will be slightly different according to when  $V$  satisfies Assumption A1 or A2.

First, we assume that  $V$  in (3.13) satisfies Assumption A1. Then the potential  $rV$  satisfies Assumption A1 but replacing  $c_1, c_2$  with  $rc_1, rc_2$  respectively, while  $\beta_1, \beta_2$  are unchanged. We thus need to estimate

$$\mathcal{J}^1(r, \lambda) := \mathcal{J}_{rV} = \mathcal{J}(rc_1, rc_2, \lambda)$$

for  $\lambda \geq \lambda_0(rV)$ .

On the other hand, if we assume that  $V$  in (3.13) satisfies Assumption A1. Then the potential  $rV$  satisfies Assumption A1 by replacing  $c_1, c_2$  with  $rc_1, r^{4/3}c_2$  respectively, and the same  $\beta_1, \beta_2$ . We thus need to estimate

$$\mathcal{J}^2(r, \lambda) := \mathcal{J}(rc_1, r^{4/3}c_2, \lambda)$$

for  $\lambda \geq \lambda_0(rV)$ .

Further, according to Proposition 2.3,  $\lambda_0(V) \geq c_1^{\frac{2}{\beta_1+2}} \lambda_*$  where  $\lambda_*$  depends on  $\beta_1$  and  $n$  only. Thus  $\lambda_0(rV) \geq r^{\frac{2}{\beta_1+2}} c_1^{\frac{2}{\beta_1+2}} \lambda_* := r^{\frac{2}{\beta_1+2}} \mu_*$ . Recall that  $\mu_*$  depends only on  $c_1, \beta_1, n$ .

The estimates of  $\mathcal{J}^1, \mathcal{J}^2$  we need are given in the following proposition:

**Proposition 3.8.** *Fix  $\sigma \geq 0, \gamma \in (0, 1/2)$  and let  $\omega \subset \mathbb{R}^n$  be a  $(\gamma, \sigma)$ -distributed set. Fix  $c_1, c_2, \beta_1, \beta_2 > 0$  and let  $V \in L_{\text{loc}}^\infty(\mathbb{R}^n)$  be such that one of Assumptions A1-A2 is satisfied with parameters  $c_1, c_2, \beta_1, \beta_2$ . Let  $\zeta = \frac{\beta_2 + 2\sigma}{2\beta_1}$  and  $\varepsilon > 0$ .*

*Then there exists a constant  $C_{V, \omega} \geq 1$  depending on  $n, c_1, c_2, \beta_1, \beta_2, \sigma$  and  $\varepsilon$  such that, for every  $r > 0$  and for every  $\phi \in \mathcal{E}_{rV}(\lambda)$*

$$(3.14) \quad \|\phi\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{1}{\gamma}\right)^{C_{V, \omega} \mathcal{J}_{rV}(\lambda)} \|\phi\|_{L^2(\omega)},$$

with

$$(3.15) \quad \mathcal{J}_{rV}(\lambda) = \begin{cases} r^{a_-} + r^{b_-} \lambda^\zeta & \text{if } 0 < r < 1 \\ r^{a_+ + \varepsilon} + r^{b_+ + \varepsilon} \lambda^\zeta & \text{if } r \geq 1 \end{cases}$$

where  $a_-, b_-, a_+, b_+$  are given as follows:

(i) if  $V$  satisfies Assumption A1, then

$$a_- = \frac{1}{2} - \zeta \leq 0, \quad b_- = \frac{1}{2} - \zeta \leq 0,$$

$$a_+ = \frac{1}{2}, \quad b_+ = \begin{cases} \frac{1 - \sigma - 2\zeta}{\beta_1 + 2} \leq 0 & \text{if } (\beta_1 - \beta_2)\sigma = 0 \\ \max\left(\frac{1}{2} - \frac{\sigma}{\beta_1 + 2}, \frac{1}{\beta_1 + 2}\right) - \frac{2\zeta}{\beta_1 + 2} & \text{if } (\beta_1 - \beta_2)\sigma \neq 0 \end{cases}.$$

(ii) If  $V$  satisfies Assumption A2 with  $\beta_* := 3\beta_2 - 4\beta_1 - 2 \leq 0$ , then

$$a_- = \frac{2}{3} - \zeta, \quad b_- = \begin{cases} -\left(\frac{\sigma}{\beta_1} + \frac{\beta_2 - \beta_1}{(\beta_1 + 2)\beta_1}\right) \leq 0 & \text{if } \beta_* \sigma = 0 \\ \min\left(\frac{2}{3} - \frac{\beta_2}{2\beta_1} - \frac{\sigma}{\beta_1 + 2}, \frac{1}{\beta_1 + 2} - \frac{\sigma}{\beta_1}\right) - \frac{2\zeta}{\beta_1 + 2} < 0 & \text{if } \beta_* \sigma \neq 0 \end{cases},$$

$$a_+ = \frac{2}{3}, \quad b_+ = -\frac{\sigma}{\beta_1 + 2} + \frac{2}{3} - \frac{\beta_2}{2\beta_1} - \frac{2\zeta}{\beta_1 + 2}.$$

(iii) If  $V$  satisfies Assumption A2 with  $\beta_* > 0$ , then

$$a_- = \frac{2}{3} - \zeta, \quad b_- = -\left(\frac{\beta_*}{2(\beta_1 + 2)} + \frac{\sigma}{\beta_1 + 2} + \frac{2\zeta - 1}{\beta_1 + 2}\right) < 0,$$

$$a_+ = \frac{2}{3}, \quad b_+ = \begin{cases} -\frac{\beta_2 - \beta_1}{(\beta_1 + 2)\beta_1} \leq 0 & \text{if } \sigma = 0 \\ \max\left(\frac{2}{3} - \frac{\sigma}{\beta_1 + 2}, \frac{1}{\beta_1 + 2}\right) - \frac{2\zeta}{\beta_1 + 2} & \text{if } \sigma \neq 0 \end{cases}.$$

**Remark 3.9.** The separation of cases at  $r = 1$  in (3.15) is arbitrary and can be replaced with

$$\mathcal{J}_{rV}(\lambda) = \begin{cases} r^{a_-} + r^{b_-} \lambda^\zeta & \text{if } 0 < r < \alpha \\ r^{a_+ + \varepsilon} + r^{b_+ + \varepsilon} \lambda^\zeta & \text{if } r \geq \alpha \end{cases}$$

for any  $\alpha > 0$ . This only influences the constant  $C_{V,\omega}$  in (3.14) but not the exponents  $a_\pm, b_\pm$  and  $\zeta$ .

**Remark 3.10.** Note that if  $V, \tilde{V}$  both satisfy either Assumption A1 or Assumption A2 with same exponents  $\beta_1, \beta_2$  and (without loss of generality) same coefficients  $c_1, c_2$ , then  $rV + \tilde{V}$  satisfy the same assumption, still with exponents  $\beta_1, \beta_2$ , but with parameters  $(r+1)c_1, (r+1)c_2$  in case of Assumption A1 and  $(r+1)c_1, (r^{4/3} + 1)c_2$ . As  $(r^{4/3} + 1) \lesssim (1+r)^{4/3}$ , we get that  $\mathcal{J}_{rV + \tilde{V}}$  satisfies the same bounds but with  $r$  replaced by  $r+1 \geq 1$ .

*Proof.* Recall that  $\mathcal{J}(c_1, c_2, \lambda)$  is given in (3.1). As  $\lambda_0(rV) \geq r^{\frac{2}{\beta_1+2}} \mu_*$  we may write  $\lambda = r^{\frac{2}{\beta_1+2}} \mu$  with  $\mu \gtrsim 1$ . We thus want to estimate  $\mathcal{J}^1(r, \lambda) = \mathcal{J}(rc_1, rc_2, r^{\frac{2}{\beta_1+2}} \mu)$ .

As  $\mu \gtrsim 1$ , We will use that, for  $a, b, c, d \in \mathbb{R}, \alpha, \beta \geq 0$

$$\begin{aligned} (r^a + r^b \mu^\alpha)(r^c + r^d \mu^\beta) &= r^{a+c} + r^{a+d} \mu^\beta + r^{b+c} \mu^\alpha + r^{b+d} \mu^{\alpha+\beta} \\ &\lesssim \begin{cases} r^{a+c} + r^{\min(a+d, b+c, b+d)} \mu^{\alpha+\beta} & \text{when } 0 < r < 1 \\ r^{a+c} + r^{\max(a+d, b+c, b+d)} \mu^{\alpha+\beta} & \text{when } r \geq 1 \end{cases} \end{aligned}$$

when  $\delta := (b-a)\beta - (d-c)\alpha \neq 0$ . Here we use that  $\mu \gtrsim 1$  to absorb  $\mu^\alpha, \mu^\beta$  into  $\mu^{\alpha+\beta}$  and then keep only the smallest power of  $r$  when  $r < 1$  and the largest one otherwise.

When  $\delta = (b-a)\beta - (d-c)\alpha = 0$ , one can improve this as follows: observe that for  $t \geq 0$ ,  $t^\alpha, t^\beta \leq \max(1, t^{\alpha+\beta})$  so that  $(1+t^\alpha)(1+t^\beta) \leq 3(1+t^{\alpha+\beta})$ . Thus, with  $t = r^{\frac{b-a}{\alpha}} \mu = r^{\frac{d-c}{\beta}} \mu$  we get

$$\begin{aligned} (r^a + r^b \mu^\alpha)(r^c + r^d \mu^\beta) &= r^{a+c} \left(1 + \left(r^{\frac{b-a}{\alpha}} \mu\right)^\alpha\right) \left(1 + \left(r^{\frac{d-c}{\beta}} \mu\right)^\beta\right) \\ &\lesssim r^{a+c} \left(1 + \left(r^{\frac{b-a}{\alpha}} \mu\right)^{\alpha+\beta}\right) \\ &\lesssim r^{a+c} + r^{b+d} \mu^{\alpha+\beta}. \end{aligned}$$

Now

$$\mathbf{J}(rc_1, r^{\frac{2}{\beta_1+2}} \mu) \lesssim \left(\frac{r^{\frac{2}{\beta_1+2}} \mu + 1}{rc_1}\right)^{\frac{\beta_2}{2\beta_1}} + (\log_+ rc_1)^{\frac{\beta_2}{2}} + 1 \lesssim 1 + (\log_+ r)^{\frac{\beta_2}{2}} + r^{-\frac{\beta_2}{2\beta_1}} + r^{-\frac{\beta_2}{2(\beta_1+2)}} \mu^{\frac{\beta_2}{2\beta_1}}$$

with the implied constants depending on  $\beta_1, \beta_2$  and  $c_1$ , as well as

$$(3.19) \quad \begin{aligned} \mathbf{J}(rc_1, r^{\frac{2}{\beta_1+2}}\mu)^{2\sigma/\beta_2} &\lesssim \left( \frac{r^{\frac{2}{\beta_1+2}}\mu + 1}{rc_1} \right)^{\frac{\sigma}{\beta_1}} + (\log_+ r)^\sigma + 1 \lesssim 1 + (\log_+ r)^\sigma + r^{-\frac{\sigma}{\beta_1}} + r^{-\frac{\sigma}{\beta_1+2}}\mu^{\frac{\sigma}{\beta_1}} \\ &\lesssim \begin{cases} r^{-\frac{\sigma}{\beta_1}} (1 + (r^{\frac{2}{\beta_1+2}}\mu)^{\frac{\sigma}{\beta_1}}) & \text{if } 0 < r < 1 \\ \left(1 + r^{-\frac{\sigma}{\beta_1+2}}\mu^{\frac{\sigma}{\beta_1}}\right) \log^\sigma(r+1) & \text{if } r \geq 1 \end{cases} \end{aligned}$$

with the implied constants depending on  $\sigma, \beta_1, \beta_2$  and  $c_1$ .

On the other hand

$$\begin{aligned} \widehat{\mathbf{J}}(rc_1, rc_2, r^{\frac{2}{\beta_1+2}}\mu) &= r^{\frac{1}{\beta_1+2}}\mu^{1/2} + r^{1/2}c_2^{1/2}\mathbf{J}(rc_1, r^{\frac{2}{\beta_1+2}}\mu) \\ &\lesssim r^{\frac{1}{\beta_1+2}}\mu^{1/2} + r^{1/2} + r^{1/2}(\log_+ r)^{\frac{\beta_2}{2}} + r^{-\frac{\beta_2-\beta_1}{2\beta_1}} + r^{\frac{1}{2}-\frac{\beta_2}{2(\beta_1+2)}}\mu^{\frac{\beta_2}{2\beta_1}} \\ &\lesssim r^{1/2} + r^{1/2}(\log_+ r)^{\frac{\beta_2}{2}} + r^{-\frac{\beta_2-\beta_1}{2\beta_1}} + \left(r^{\frac{1}{\beta_1+2}} + r^{\frac{1}{2}-\frac{\beta_2}{2(\beta_1+2)}}\right)\mu^{\frac{\beta_2}{2\beta_1}} \end{aligned}$$

where, in the last line, we used that  $\mu \gtrsim 1$  and  $\frac{\beta_2}{2\beta_1} \geq \frac{1}{2}$  so that  $\mu^{\frac{1}{2}} \lesssim \mu^{\frac{\beta_2}{2\beta_1}}$ . The implied constant here depends on  $\sigma, \beta_1, \beta_2$  and  $c_1, c_2$ . Now note that  $\beta_2 \geq \beta_1$  implies

$$\frac{1}{\beta_1+2} \geq \frac{2-(\beta_2-\beta_1)}{2(\beta_1+2)} = \frac{1}{2} - \frac{\beta_2}{2(\beta_1+2)}.$$

It follows that

$$(3.20) \quad \widehat{\mathbf{J}}(rc_1, rc_2, r^{\frac{2}{\beta_1+2}}\mu) \lesssim \begin{cases} r^{-\frac{\beta_2-\beta_1}{2\beta_1}} (1 + (r^{\frac{2}{\beta_1+2}}\mu)^{\frac{\beta_2}{2\beta_1}}) & \text{if } 0 < r < 1 \\ \left(r^{\frac{1}{2}} + r^{\frac{1}{\beta_1+2}}\mu^{\frac{\beta_2}{2\beta_1}} + r^{\frac{1}{2}}\right) \log^{\frac{\beta_2}{2}}(r+1) & \text{if } r \geq 1 \end{cases}.$$

For  $0 < r < 1$ , we multiply (3.19) with (3.20) by using (3.17) and obtain

$$(3.21) \quad \mathcal{J}_{rV}(r, \lambda) \lesssim r^{-\frac{\sigma}{\beta_1} - \frac{\beta_2-\beta_1}{2\beta_1}} (1 + (r^{\frac{2}{\beta_1+2}}\mu)^{\frac{\beta_2+2\sigma}{2\beta_1}}), \quad 0 < r < 1.$$

For  $r \geq 1$ , we take  $a = 0, b = -\frac{\sigma}{\beta_1+2}, c = \frac{1}{2}, d = \frac{1}{\beta_1+2}, \alpha = \frac{\sigma}{\beta_1}, \beta = \frac{\beta_2}{2\beta_1}$  and then obtain

$$\delta = (b-a)\beta - (d-c)\alpha = \frac{(\beta_1 - \beta_2)\sigma}{2(\beta_1+2)\beta_1}.$$

Then  $\delta = 0$  if and only if  $\beta_1 = \beta_2$  or  $\sigma = 0$ . Multiplying (3.19) and (3.20) and using (3.18) when  $\delta = 0$  (resp. (3.16) when  $\delta \neq 0$ ) we obtain

$$(3.22) \quad \mathcal{J}_{rV}(r, \lambda) \lesssim \begin{cases} \left(r^{\frac{1}{2}} + r^{-\frac{\sigma}{\beta_1+2} + \frac{1}{\beta_1+2}}\mu^{\frac{\beta_2+2\sigma}{2\beta_1}}\right) \log^{\sigma+\frac{\beta_2}{2}}(r+1) & \text{if } (\beta_1 - \beta_2)\sigma = 0 \\ \left(r^{\frac{1}{2}} + r^{\tilde{b}_+} \mu^{\frac{2\sigma+\beta_2}{2\beta_1}}\right) \log^{\sigma+\frac{\beta_2}{2}}(r+1) & \text{if } (\beta_1 - \beta_2)\sigma \neq 0 \end{cases}, \quad r \geq 1$$

where

$$\tilde{b}_+ := \max\left(\frac{1}{2} - \frac{\sigma}{\beta_1+2}, \frac{1}{\beta_1+2}\right) \geq \frac{1}{\beta_1+2} - \frac{\sigma}{\beta_1+2}.$$

Replacing  $\mu$  in (3.21) and (3.22) by  $r^{-\frac{2}{\beta_1+2}}\lambda$ , and absorbing the log term in the power with the help of  $\varepsilon$  we obtain the claimed inequality.

To estimate  $\mathcal{J}_{rV}$  under Assumption A2, the main difference is that we need to estimate

$$\begin{aligned} \widehat{\mathbf{J}}(rc_1, r^{4/3}c_2, r^{\frac{2}{\beta_1+2}}\mu) &= r^{\frac{1}{\beta_1+2}}\mu^{1/2} + r^{2/3}c_2^{1/2}\mathbf{J}(rc_1, r^{\frac{2}{\beta_1+2}}\mu) \\ &\lesssim r^{\frac{2}{3}} + r^{\frac{2}{3}-\frac{\beta_2}{2\beta_1}} + r^{\frac{1}{\beta_1+2}}\mu^{1/2} + r^{\frac{2}{3}-\frac{\beta_2}{2(\beta_1+2)}}\mu^{\frac{\beta_2}{2\beta_1}} + r^{\frac{2}{3}}(\log_+ r)^{\frac{\beta_2}{2}} \\ &\lesssim r^{\frac{2}{3}} + r^{\frac{2}{3}-\frac{\beta_2}{2\beta_1}} + \left(r^{\frac{1}{\beta_1+2}} + r^{\frac{2}{3}-\frac{\beta_2}{2(\beta_1+2)}}\right)\mu^{\frac{\beta_2}{2\beta_1}} + r^{\frac{2}{3}}(\log_+ r)^{\frac{\beta_2}{2}}. \end{aligned}$$

There are now two cases to be distinguished since

$$\frac{1}{\beta_1+2} \leq \frac{2}{3} - \frac{\beta_2}{2(\beta_1+2)}$$

if and only if  $3\beta_2 \leq 4\beta_1 + 2$  or equivalently  $\frac{1}{2} \leq \frac{\beta_2}{2\beta_1} \leq \frac{2}{3} + \frac{1}{3\beta_1}$ .

So assume first that  $3\beta_2 \leq 4\beta_1 + 2$ , then

$$(3.23) \quad \widehat{\mathbf{J}}(rc_1, r^{4/3}c_2, r^{\frac{2}{\beta_1+2}}\mu) \lesssim \begin{cases} r^{\frac{2}{3}-\frac{\beta_2}{2\beta_1}} + r^{\frac{1}{\beta_1+2}}\mu^{\frac{\beta_2}{2\beta_1}} & \text{if } 0 < r < 1 \\ \left(r^{\frac{2}{3}} + r^{\frac{2}{3}-\frac{\beta_2}{2(\beta_1+2)}}\mu^{\frac{\beta_2}{2\beta_1}}\right) \log^{\frac{\beta_2}{2}}(r+1) & \text{if } r \geq 1 \end{cases}.$$

For  $0 < r < 1$ , we take  $a = -\frac{\sigma}{\beta_1}$ ,  $b = -\frac{\sigma}{\beta_1+2}$ ,  $c = \frac{2}{3} - \frac{\beta_2}{2\beta_1}$ ,  $d = \frac{1}{\beta_1+2}$ ,  $\alpha = \frac{\sigma}{\beta_1}$  and  $\beta = \frac{\beta_2}{2\beta_1}$  so that

$$\delta := (b-a)\beta - (d-c)\alpha = \frac{(2+4\beta_1-3\beta_2)\sigma}{6(\beta_1+2)\beta_1}.$$

Hence  $\delta \neq 0$  if and only if  $3\beta_2 < 4\beta_1 + 2$  and  $\sigma \neq 0$ . Using (3.16) and (3.18) when  $\delta \neq 0$  and  $\delta = 0$  respectively to multiply (3.19) and (3.23), we obtain

$$(3.24) \quad \mathcal{J}_{rV}(r, \lambda) \lesssim \begin{cases} r^{\frac{2}{3}-\frac{\beta_2+2\sigma}{2\beta_1}} + r^{\frac{1-\sigma}{\beta_1+2}}\mu^{\frac{\beta_2+2\sigma}{2\beta_1}} & \text{if } (2+4\beta_1-3\beta_2)\sigma = 0 \\ r^{\frac{2}{3}-\frac{\beta_2+2\sigma}{2\beta_1}} + r^{\tilde{b}_-}\mu^{\frac{\beta_2+2\sigma}{2\beta_1}} & \text{if } (2+4\beta_1-3\beta_2)\sigma \neq 0 \end{cases}, \quad 0 < r < 1$$

where

$$\tilde{b}_- := \min\left(\frac{2}{3} - \frac{\beta_2}{2\beta_1} - \frac{\sigma}{\beta_1+2}, \frac{1}{\beta_1+2} - \frac{\sigma}{\beta_1}\right) \leq \frac{1}{\beta_1+2} - \frac{\sigma}{\beta_1+2}.$$

For  $r \geq 1$ , we take  $a = 0$ ,  $b = -\frac{\sigma}{\beta_1+2}$ ,  $c = \frac{2}{3}$ ,  $d = \frac{2}{3} - \frac{\beta_2}{2\beta_1}$ ,  $\alpha = \frac{\sigma}{\beta_1}$ ,  $\beta = \frac{\beta_2}{2\beta_1}$  and then obtain

$$\delta := (b-a)\beta - (d-c)\alpha = 0.$$

Using (3.18) to multiply (3.19) and (3.23), we obtain

$$(3.25) \quad \mathcal{J}_{rV}(r, \lambda) \lesssim r^{\frac{2}{3}} + r^{-\frac{\sigma}{\beta_1+2} + \frac{2}{3} - \frac{\beta_2}{2\beta_1}}\mu^{\frac{\beta_2+2\sigma}{2\beta_1}}, \quad r \geq 1.$$

Replacing  $\mu$  in (3.24) and (3.25) by  $r^{-\frac{2}{\beta_1+2}}\lambda$ , we obtain the claimed inequality.

On the other hand, if  $3\beta_2 > 4\beta_1 + 2$ , then

$$(3.26) \quad \widehat{\mathbf{J}}(rc_1, r^{4/3}c_2, r^{\frac{2}{\beta_1+2}}\mu) \lesssim \begin{cases} r^{\frac{2}{3}-\frac{\beta_2}{2\beta_1}} + r^{\frac{2}{3}-\frac{\beta_2}{2(\beta_1+2)}}\mu^{\frac{\beta_2}{2\beta_1}} & \text{if } 0 < r < 1 \\ \left(r^{\frac{2}{3}} + r^{\frac{1}{\beta_1+2}}\mu^{\frac{\beta_2}{2\beta_1}}\right) \log^{\frac{\beta_2}{2}}(r+1) & \text{if } r \geq 1 \end{cases}.$$

For  $0 < r < 1$ , we take  $a = -\frac{\sigma}{\beta_1}$ ,  $b = -\frac{\sigma}{\beta_1+2}$ ,  $c = \frac{2}{3} - \frac{\beta_2}{2\beta_1}$ ,  $d = \frac{2}{3} - \frac{\beta_2}{2(\beta_1+2)}$ ,  $\alpha = \frac{\sigma}{\beta_1}$ ,  $\beta = \frac{\beta_2}{2\beta_1}$  and then obtain

$$\delta := (b-a)\beta - (d-c)\alpha = 0.$$

Using (3.18) to multiply (3.19) and (3.23), we obtain

$$(3.27) \quad \mathcal{J}_{rV}(r, \lambda) \lesssim r^{\frac{2}{3}-\frac{\beta_2+2\sigma}{2\beta_1}} + r^{-\frac{\sigma}{\beta_1+2} + \frac{2}{3} - \frac{\beta_2}{2(\beta_1+2)}}\mu^{\frac{\beta_2+2\sigma}{2\beta_1}}, \quad 0 < r < 1.$$

For  $r \geq 1$ , we take  $a = 0$ ,  $b = -\frac{\sigma}{\beta_1+2}$ ,  $c = \frac{2}{3}$ ,  $d = \frac{1}{\beta_1+2}$ ,  $\alpha = \frac{\sigma}{\beta_1}$ ,  $\beta = \frac{\beta_2}{2\beta_1}$  and then obtain

$$\delta := (b-a)\beta - (d-c)\alpha = -\frac{(3\beta_2-4\beta_1-2)\sigma}{6(\beta_1+2)\beta_1} \leq 0.$$

Hence  $\delta \neq 0$  if and only if  $\sigma \neq 0$ . Using (3.16) and (3.18) when  $\delta \neq 0$  and  $\delta = 0$  respectively to multiply (3.19) and (3.26), we obtain

$$(3.28) \quad \mathcal{J}_{rV}(r, \lambda) \lesssim \begin{cases} \left(r^{\frac{2}{3}} + r^{\frac{1}{\beta_1+2}}\mu^{\frac{\beta_2}{2\beta_1}}\right) \log^{\frac{\beta_2}{2}}(r+1) & \text{if } \sigma = 0 \\ \left(r^{\frac{2}{3}} + r^{\tilde{b}_+}\mu^{\frac{\beta_2+2\sigma}{2\beta_1}}\right) \log^{\sigma+\frac{\beta_2}{2}}(r+1) & \text{if } \sigma \neq 0 \end{cases}, \quad r \geq 1$$

where

$$\tilde{b}_+ = \max\left(-\frac{\sigma}{\beta_1+2} + \frac{2}{3}, \frac{1}{\beta_1+2}\right) \geq \frac{1}{\beta_1+2} - \frac{\sigma}{\beta_1+2}$$

Absorbing the log term in the power with the help of  $\varepsilon$ , we again conclude by replacing  $\mu$  in (3.27) and (3.28) by  $r^{-\frac{2}{\beta_1+2}}\lambda$ .  $\square$

**3.3. From the spectral inequality to observability of scaled Schrödinger operators.** We can now prove observability inequalities for “scaled” Schrödinger operators  $H_{rV} = -\Delta_x + rV(x)$  on  $\mathbb{R}^n$  in which we give an explicit dependence on  $r$ . To do so, we will use the following result from [NTTV20] that allows to go from a spectral inequality to an observability inequality. (see [BPS18, Theorem 2.1] for a similar result in which constants are less explicit).

**Theorem 3.11** ([NTTV20, Theorem 2.8]). *Let  $P$  be a non-negative selfadjoint operator on  $L^2(\mathbb{R}^n)$ ,  $s > 0$  and  $\omega \subset \mathbb{R}^n$  be measurable. Suppose that there are  $\alpha_0 \geq 1$ ,  $\alpha_1 \geq 0$  and  $0 < \zeta < s$  such that, for all  $\lambda \geq 0$  and  $\phi \in \mathcal{E}_\lambda(P)$ ,*

$$(3.29) \quad \|\phi\|_{L^2(\mathbb{R}^n)}^2 \leq \alpha_0 e^{\alpha_1 \lambda^\zeta} \|\phi\|_{L^2(\omega)}^2.$$

*Then there exist positive constants  $\kappa_1, \kappa_2, \kappa_3 > 0$  depending only on  $n, \zeta$  and  $s$ , such that for all  $T > 0$  and  $g \in L^2(\mathbb{R}^n)$  we have the observability estimate*

$$\|e^{-tP^s} g\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{C_{\text{obs}}}{T} \int_0^T \|e^{-tP^s} g\|_{L^2(\omega)}^2 dt,$$

where the positive constant  $C_{\text{obs}} > 0$  is given by

$$C_{\text{obs}} = \kappa_1 \alpha_0^{\kappa_2} \exp\left(\kappa_3 \alpha_1^{\frac{s}{s-\zeta}} T^{-\frac{\zeta}{s-\zeta}}\right).$$

In [NTTV20], the theorem is stated with  $s = 1$ , but this form follows directly from the transformation formula for spectral measures (see [Sch12, Proposition 4.24]), for all  $s > 0$  and  $\lambda \geq 0$  we have

$$\mathcal{E}_\lambda(P^s) = \mathcal{E}_{\lambda^{\frac{1}{s}}}(P).$$

We can now prove the following:

**Proposition 3.12.** *Let  $r > 0$  and  $V \in L^1_{\text{loc}}(\mathbb{R}^n)$  satisfying one of Assumptions A1-A2, with corresponding constants  $c_1, c_2, \beta_1, \beta_2$ . Let  $H_{rV} = -\Delta + rV$  be the Schrödinger operator associated to  $rV$ . Let  $0 < \gamma < 1$  and  $\sigma \geq 0$  and  $\omega \subset \mathbb{R}^n$  be a  $(\gamma, \sigma)$ -distributed set. Let  $a_\pm, b_\pm$  and  $\varepsilon$  be defined in Proposition 3.8. Let  $\zeta = \frac{\beta_2+2\sigma}{2\beta_1}$  and  $s > \zeta$ .*

*For  $T > 0$  and  $g \in L^2(\mathbb{R}^n)$  we have*

$$\|e^{-TH_{rV}^s} g\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{C_{\text{obs}}(T, s, rV, \omega)}{T} \int_0^T \|e^{-tH_r^s} g\|_{L^2(\omega)}^2 dt$$

where the positive constant  $C_{\text{obs}}$  is given by

$$(3.30) \quad \begin{cases} C_0 \exp \left[ C_1 \log \left( \frac{1}{\gamma} \right) r^{a_-} + C_2 T^{-\frac{\zeta}{s-\zeta}} \log^{\frac{s}{s-\zeta}} \left( \frac{1}{\gamma} \right) r^{\frac{s}{s-\zeta} b_-} - C_3 T r^{\frac{2s}{\beta_1+2}} \right] & \text{for } 0 < r < 1 \\ C_0 \exp \left[ C_1 \log \left( \frac{1}{\gamma} \right) r^{a_+ + \varepsilon} + C_2 T^{-\frac{\zeta}{s-\zeta}} \log^{\frac{s}{s-\zeta}} \left( \frac{1}{\gamma} \right) r^{\frac{s}{s-\zeta} (b_+ + \varepsilon)} - C_3 T r^{\frac{2s}{\beta_1+2}} \right] & \text{for } r \geq 1 \end{cases}$$

with positive constants  $C_0, C_1, C_2, C_3$  depending on  $n, \sigma, \varepsilon$  and on  $c_1, c_2, \beta_1, \beta_2$ .

*Proof.* For  $P = H_{rV}$ , Inequality (3.29) was established in Proposition 3.8. Recall that  $a_-, b_-$  is for  $0 < r < 1$  and  $a_+, b_+$  is for  $r \geq 1$ . For  $0 < r < 1$ , we can apply Theorem 3.11 with  $\alpha_0 = C \exp \left[ C_{V, \omega} \log \left( \frac{1}{\gamma} \right) r^{a_-} \right] \geq 1$  (we can assume that  $C \geq 1$ ),  $\alpha_1 = C \log \left( \frac{1}{\gamma} \right) r^{b_-}$ . we thus get, the following:

For  $T > 0$  and  $g \in L^2(\mathbb{R}^n)$  we have

$$\|e^{-\frac{T}{2} H_r^s} g\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{\tilde{C}_{\text{obs}}(T/2, s, rV, \omega)}{T} \int_0^{T/2} \|e^{-tH_r^s} g\|_{L^2(\omega)}^2 dt$$

where the positive constant  $\tilde{C}_{\text{obs}}(T/2, s, rV, \omega)$  is given by

$$(3.31) \quad \tilde{C}_{\text{obs}}(T/2, s, rV, \omega) = C_0 \exp \left[ C_1 \log \left( \frac{1}{\gamma} \right) r^{a_-} + C_2 T^{-\frac{\zeta}{s-\zeta}} \log^{\frac{s}{s-\zeta}} \left( \frac{1}{\gamma} \right) r^{\frac{s}{s-\zeta} b_-} \right]$$

with constants  $C_0, C_1, C_2 > 0$  depending on  $n, \sigma$  and on  $c_1, c_2, \beta_1, \beta_2$  of Assumptions **A1-A2** only.

Next, using Corollary **2.5**, we get

$$(3.32) \quad \|e^{-TH_r^s} g\|_{L^2(\mathbb{R}^n)}^2 = \|e^{-\frac{T}{2}H_r^s} (e^{-\frac{T}{2}H_r^s} g)\|_{L^2(\mathbb{R}^n)}^2 \leq e^{-\frac{\mu_*}{2} r^{\frac{2s}{\beta_1+2}} T} \|e^{-\frac{T}{2}H_r^s} g\|_{L^2(\mathbb{R}^n)}^2$$

with  $\mu_*$  depending only on  $c_1, \beta_1$  and  $n$ . Combining (3.31) and (3.32) we obtain (3.30) with  $C_3 = \mu_*/2$  and  $C_{\text{obs}}(T, s, rV, \omega) = \tilde{C}_{\text{obs}}(T/2, s, rV, \omega) \exp(-C_3 T r^{\frac{2s}{\beta_1+2}})$ .

For  $r \geq 1$ , the proof is the same by replacing  $a_-$  and  $b_-$  with  $a_+ + \varepsilon$  and  $b_+ + \varepsilon$ , and the constants will also depend on  $\varepsilon$ .  $\square$

Simple calculus shows the following:

**Lemma 3.13.** *With the notation and conditions of Proposition 3.12*

(i) *for every  $T > 0$ ,  $\sup_{0 < r < 1} C_{\text{obs}}(T, s, rV, \omega) < +\infty$  if and only if  $a_-, b_- \geq 0$  in which case we obtain*

$$\sup_{0 < r < 1} C_{\text{obs}}(T, s, rV, \omega) \leq A_0(T, s, \gamma) := C_0 \left(\frac{1}{\gamma}\right)^{C_2} \exp\left[C_2 T^{-\frac{\zeta}{s-\zeta}} \log \frac{s}{s-\zeta} \left(\frac{1}{\gamma}\right)\right].$$

(ii) *If  $\frac{2s}{\beta_1+2} > \nu := \max\left(a_+, \frac{s}{s-\zeta} b_+\right)$  then  $\sup_{r \geq 1} C_{\text{obs}}(T, s, rV, \omega) < +\infty$ . Further,*

$$\sup_{r \geq 1} C_{\text{obs}}(T, s, rV, \omega) \leq A_1(T, s, \gamma) := \begin{cases} C_0 \exp C_4 T^{-(\delta + \frac{\zeta}{s-\zeta}(1+\delta))} \log \frac{s}{s-\zeta} (1+\delta) \left(\frac{1}{\gamma}\right) & \text{for } T \leq 1 \\ C_0 \exp C_5 T^{-\delta} \log \frac{s}{s-\zeta} (1+\delta) \left(\frac{1}{\gamma}\right) & \text{otherwise} \end{cases}$$

$$\text{where } \delta = \left(\frac{s}{\beta_1+2} - \frac{\nu}{2}\right)^{-1} \text{ and } C_4, C_5 \text{ do not depend on } T, \gamma.$$

*Proof.* The first bound is trivial, for the second one, we use that

$$\sup_{r \geq 1} C_{\text{obs}}(T, s, rV, \omega) \leq C_0 \sup_{r \geq 1} \exp\left[C_1 \min(1, T)^{-\frac{\zeta}{s-\zeta}} \log \frac{s}{s-\zeta} \left(\frac{1}{\gamma}\right) r^{\nu + \frac{s}{s-\zeta} \varepsilon} - C_3 T r^{\frac{2s}{\beta_1+2}}\right]$$

where  $\varepsilon$  is chosen to be

$$0 < \varepsilon = \frac{1}{2} \left(\frac{2s}{\beta_1+2} - \nu\right) \frac{s-\zeta}{s} < \left(\frac{2s}{\beta_1+2} - \nu\right) \frac{s-\zeta}{s}.$$

and then that  $\sup_{r \geq 1} \exp(Ar^u - Br^v) = \exp \frac{v-u}{u} A \left(\frac{Au}{Bv}\right)^{1/(v-u)}$  when  $0 < u \leq v$ .  $\square$

**Remark 3.14.** From Remark 3.10 we get that, if  $V, \tilde{V}$  both satisfy Assumption **A1** or Assumption **A2** with same parameters  $c_1, c_2, \beta_1$  and  $\beta_2$ , then

$$\sup_{r \geq 0} C_{\text{obs}}(T, s, rV + \tilde{V}, \omega) \lesssim \sup_{r \geq 1} C_{\text{obs}}(T, s, rV, \omega).$$

**Lemma 3.15.** *With the notation and conditions of Proposition 3.12 and assuming that  $s > \zeta$ , the condition  $\frac{2s}{\beta_1+2} > \nu := \max\left(a_+, \frac{s}{s-\zeta} b_+\right)$  (resp.  $\frac{2s}{\beta_1+2} = \nu$ ) simplifies to the following:*

- (i) *if  $V$  satisfies Assumption **A1**,  $s > s_{A1} := \frac{\beta_1+2}{4}$ ;*
- (ii) *if  $V$  satisfies Assumption **A2**,  $s > s_{A2} := \frac{\beta_1+2}{3}$ .*

*Proof.* (i) Let us first assume that  $V$  satisfies Assumption **A1**. Remember that

$$a_+ = \frac{1}{2}, \quad b_+ = \begin{cases} \frac{1-\sigma-2\zeta}{\beta_1+2} \leq 0 & \text{if } (\beta_1 - \beta_2)\sigma = 0 \\ \max\left(\frac{1}{2} - \frac{\sigma}{\beta_1+2}, \frac{1}{\beta_1+2}\right) - \frac{2\zeta}{\beta_1+2} & \text{if } (\beta_1 - \beta_2)\sigma \neq 0 \end{cases}.$$

In particular,  $\frac{2s}{\beta_1+2} > a_+$  is satisfied if and only if  $s > s_{A_1}$ . We will thus assume that  $s > s_{A_1}$  and we only need to prove that  $\frac{2s}{\beta_1+2} > \frac{s}{s-\zeta}b_+$  as well. As  $s > \zeta$ , this is equivalent to

$$(3.33) \quad s > \frac{\beta_1+2}{2}b_+ + \zeta.$$

Using again that  $s > \zeta$ , this is obviously satisfied if  $b_+ \leq 0$ , hence we only need to consider the case  $b_+ > 0$ , *i.e.*,

$$b_+ = \max\left(\frac{1}{2} - \frac{\sigma}{\beta_1+2}, \frac{1}{\beta_1+2}\right) - \frac{2\zeta}{\beta_1+2} > 0.$$

As  $\zeta \geq \frac{1}{2}$ ,  $b_+$  can only be positive when

$$b_+ = \frac{1}{2} - \frac{\sigma}{\beta_1+2} - \frac{2\zeta}{\beta_1+2}.$$

Substitute this into (3.33) we obtain the equivalent form

$$s > \frac{\beta_1+2}{4} - \frac{\sigma}{2} = s_{A_1} - \frac{\sigma}{2}$$

which is clearly satisfied under our assumption  $s > s_{A_1}$ .

(ii) Let us now assume that  $V$  satisfies Assumption A2 with  $3\beta_2 \leq 4\beta_1+2$  and  $s > s_{A_2}$ . Remember that

$$a_+ = \frac{2}{3}, \quad b_+ = \frac{2}{3} - \left(\frac{\sigma}{\beta_1+2} + \frac{\beta_2}{2\beta_1} + \frac{2\zeta}{\beta_1+2}\right)$$

Again,  $\frac{2s}{\beta_1+2} > a_+$  is equivalent to  $s > s_{A_2}$ . We thus assume that  $s > s_{A_2}$  and we only need to prove (3.33) which, in this case, is equivalent to

$$s > s_{A_2} - \frac{\beta_1+2}{2} \left(\frac{\sigma}{\beta_1+2} + \frac{\beta_2}{2\beta_1} + \frac{2\zeta}{\beta_1+2}\right)$$

and is thus clearly satisfied when  $s > s_{A_2}$ .

Assume that  $V$  satisfies Assumption A2 with  $3\beta_2 > 4\beta_1+2$ . Remember that

$$a_+ = \frac{2}{3}, \quad b_+ = \begin{cases} \frac{1}{\beta_1+2} - \frac{\beta_2}{(\beta_1+2)\beta_1} \leq 0 & \text{if } \sigma = 0 \\ \max\left(\frac{2}{3} - \frac{\sigma}{\beta_1+2}, \frac{1}{\beta_1+2}\right) - \frac{2\zeta}{\beta_1+2} & \text{if } \sigma \neq 0 \end{cases}.$$

As for the previous case, we have to impose  $s > s_{A_2}$  and only need to show that  $\frac{2s}{\beta_1+2} > \frac{s}{s-\zeta}b_+$ . This is satisfied when  $b_+ \leq 0$  so that we only need to consider the case  $\sigma > 0$  and

$$b_+ = \frac{2}{3} - \frac{2\zeta + \sigma}{\beta_1+2}$$

since  $2\zeta \geq 1$ . Substitute this into (3.33) we want  $s > s_{A_2} - (4\zeta + 2\sigma)$  which follows from the assumption  $s > s_{A_2}$ .  $\square$

Let us now determine under which conditions

$$\sup_{r>0} C_{\text{obs}}(T, s, rV, \omega) < +\infty.$$

By Lemma 3.13 (i) it is necessary to ensure

$$a_- \geq 0 \quad \text{and} \quad b_- \geq 0.$$

From Proposition 3.8, this only happens in the following cases:

- (i) If  $V$  satisfies Assumption A1,  $a_-, b_- \geq 0$  is equivalent to  $\zeta := \frac{\beta_2+2\sigma}{2\beta_1} = \frac{1}{2}$  which, as  $\beta_2 \geq \beta_1$  and  $\sigma \geq 0$ , reduces to  $\beta_2 = \beta_1$  and  $\sigma = 0$ .

- (ii) If  $V$  satisfies Assumption **A2** and  $3\beta_2 - 4\beta_1 - 2 \leq 0$ , then  $b_- < 0$  unless  $\sigma(3\beta_2 - 4\beta_1 - 2) = 0$  and in this case  $b_- = -\left(\frac{\sigma}{\beta_1 + 2} + \frac{\beta_2 - \beta_1}{\beta_1(\beta_1 + 2)}\right) < 0$  unless  $\beta_2 = \beta_1$  and  $\sigma = 0$ . Note that again  $\zeta = \frac{1}{2}$  in this case.

Note also that if  $\beta_1 = \beta_2$  then  $3\beta_2 - 4\beta_1 - 2 = -(\beta_1 + 2) < 0$  so that, if  $V$  satisfies Assumption **A2**,  $b_- \geq 0$  if and only if  $\beta_2 = \beta_1$  and  $\sigma = 0$ .

Now it remains to make sure that

$$\sup_{r \geq 1} C_{\text{obs}}(T, s, rV, \omega) < +\infty.$$

By Lemma 3.15 it is sufficient to assume that  $s > s_{A1}$  (resp.  $s > s_{A2}$ ) under Assumption **A1** (resp. Assumption **A2**). Hence we obtain the following corollary from Lemma 3.13 and Lemma 3.15.

**Corollary 3.16.** *With the notation and conditions of Proposition 3.12, then*

$$\sup_{r > 0} C_{\text{obs}}(T, s, rV, \omega) < +\infty$$

if

- (i)  $V$  satisfies Assumption **A1** with  $\beta_1 = \beta_2, \sigma = 0$  and  $s > s_{A1}$ ;
- (ii)  $V$  satisfies Assumption **A2** with  $\beta_1 = \beta_2, \sigma = 0$  and  $s > s_{A2}$ .

Further if  $V$  satisfies Assumption **A1** (resp.  $V$  satisfies Assumption **A2**) and the above conditions is satisfied, then set  $s_A = s_{A1}$  (resp.  $s_A = s_{A2}$ ),  $\delta = \frac{2(\beta_1 + 2)}{s - s_A}$ , then

- (a) if  $s > s_A$  and  $T \leq 1$ ,

$$(3.34) \quad B_-(T, s, rV, \omega) := \sup_{r > 0} C_{\text{obs}}(s, rV, \omega) \lesssim \left(\frac{1}{\gamma}\right)^{C_1} \exp\left[C_2 T^{-\frac{1}{2s-1}} \log^{-\frac{2s}{2s-1}}\left(\frac{1}{\gamma}\right)\right] \\ + \exp\left[C_3 T^{-\delta - \frac{1+\delta}{2s-1}} \log^{-\frac{2s}{2s-1}(1+\delta)}\left(\frac{1}{\gamma}\right)\right];$$

- (b) if  $s > s_A$  and  $T \geq 1$ ,

$$(3.35) \quad B_+(T, s, rV, \omega) := \sup_{r > 0} C_{\text{obs}}(s, rV, \omega) \lesssim \left(\frac{1}{\gamma}\right)^{C_1} \exp\left[C_2 T^{-\frac{1}{2s-1}} \log^{-\frac{2s}{2s-1}}\left(\frac{1}{\gamma}\right)\right] \\ + \exp\left[C_3 T^{-\delta} \log^{-\frac{2s}{2s-1}(1+\delta)}\left(\frac{1}{\gamma}\right)\right];$$

#### 4. ESSENTIAL SELF ADJOINTNESS OF BAOUENDI-GRUSHIN OPERATORS

The results and the proofs in this section are essentially the same as for [DM21, Proposition 3.1].

For  $f \in L^1(\mathbb{R}^n \times \mathbb{R}^m)$ , we write its partial Fourier transform as

$$\mathcal{F}_2[f](x, \eta) = f^\eta(x) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} f(x, y) e^{-i\langle y, \eta \rangle} d\eta \quad \eta \in \mathbb{R}^m.$$

For  $f \in L^1(\mathbb{R}^n \times \mathbb{R}^m) \cap L^2(\mathbb{R}^n \times \mathbb{R}^m)$ , Parseval's relation writes  $\|\mathcal{F}_2[f]\|_{L^2(\mathbb{R}^{n+m})} = \|f\|_{L^2(\mathbb{R}^{n+m})}$ . We may thus extend  $\mathcal{F}$  into a unitary transform on  $L^2(\mathbb{R}^n \times \mathbb{R}^m)$  and in particular  $f^\eta$  is well defined for almost every  $\eta$  and, for  $u \in L^2(\mathbb{R}^n)$ ,  $v \in L^2(\mathbb{R}^m)$ ,

$$\int_{\mathbb{R}^{n+m}} f(x, y) \overline{u(x)v(y)} dx dy = \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \mathcal{F}_2[f](x, \eta) \overline{u(x)} dx \widehat{\overline{v}}(\eta) d\eta$$

where  $\widehat{u}$  (resp.  $\widehat{v}$ ) is the usual Fourier transforms of  $u$  in  $L^2(\mathbb{R}^n)$ , resp. of  $v$  in  $L^2(\mathbb{R}^m)$ .

We will use the same notation for  $f \in L^1(\mathbb{R}^n \times \mathbb{T}^m)$ , and its partial Fourier coefficient as

$$\mathcal{F}_2^p[f](x, k) = f^k(x) = (2\pi)^{-m/2} \int_{\mathbb{T}^m} f(x, y) e^{-i\langle y, k \rangle} d\eta \quad k \in \mathbb{Z}^m.$$



For  $f \in L^1(\mathbb{R}^n \times \mathbb{T}^m) \cap L^2(\mathbb{R}^n \times \mathbb{T}^m)$ , Parseval's relation writes  $\|\mathcal{F}_2[f]\|_{L^2(\mathbb{R}^n, \ell^2(\mathbb{Z}^m))} = \|f\|_{L^2(\mathbb{R}^n \times \mathbb{T}^m)}$ . We may thus extend  $\mathcal{F}$  into a unitary transform from  $L^2(\mathbb{R}^n \times \mathbb{T}^m)$  to  $L^2(\mathbb{R}^n, \ell^2(\mathbb{Z}^m))$ . In particular, for  $u \in L^2(\mathbb{R}^n)$ ,  $v \in L^2(\mathbb{T}^m)$ ,

$$\int_{\mathbb{R}^n \times \mathbb{T}^m} f(x, y) \overline{u(x)v(y)} \, dx \, dy = \sum_{k \in \mathbb{Z}^m} \int_{\mathbb{R}^n} \mathcal{F}_2[f](x, k) \overline{u(x)} \, dx \, \overline{c_k(v)}$$

where  $(c_k(v))_{k \in \mathbb{Z}^m}$  are the usual Fourier coefficients of  $v$ .

**Proposition 4.1.** *Let  $V \in L^1_{\text{loc}}(\mathbb{R}^n)$  and assume that  $V$  satisfies assumption [A2](#). Let  $\mathcal{L}_V = -\Delta_x - V(x)\Delta_y$ , then  $\mathcal{L}$  is essentially self-adjoint on  $L^2(\mathbb{R}^n \times \mathbb{R}^m)$  as well as on  $L^2(\mathbb{R}^n \times \mathbb{T}^m)$ .*

*Proof.* Let  $\mathcal{L}^*$  be the Hilbert adjoint of  $\mathcal{L}$  on  $L^2(\mathbb{R}^{n+m})$ . According to [[RS72](#), Corollary of Theorem VIII.3] it is enough to show that, for every  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ ,  $\mathcal{L}^* - \lambda$  is one-to-one. We thus want to show that, if  $f \in L^2(\mathbb{R}^{n+m})$  is such that, for every  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+m})$ ,

$$\int_{\mathbb{R}^{n+m}} f(x, y) \overline{(-\Delta_x - V(x)\Delta_y - \lambda)\varphi(x, y)} \, dx \, dy = 0$$

then  $f = 0$ .

Taking  $\varphi(x, y) = u(x)v(y)$  with  $u, v$  smooth and compactly supported, we assume that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(x, y) \overline{(-\Delta u(x) - \lambda u(x))} \, dx \, \overline{v(y)} \, dy - \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(x, y) V(x) u(x) \, dx \, \overline{\Delta v(y)} \, dy \\ &= \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \mathcal{F}_2[f](x, \eta) \overline{(-\Delta + |\eta|^2 V(x) - \lambda)u(x)} \, dx \, \overline{\widehat{v}(\eta)} \, d\eta. \end{aligned}$$

This implies that, for every  $u \in \mathcal{C}_c(\mathbb{R}^n)$ ,

$$\int_{\mathbb{R}^n} \mathcal{F}_2[f](x, \eta) \overline{(-\Delta + |\eta|^2 V(x) - \lambda)u(x)} \, dx = 0$$

for almost every  $\eta$ . Consider the Schrödinger operator  $H_\eta = -\Delta + |\eta|^2 V(x)$  which, under assumption [A2](#) is essentially self-adjoint so that, if  $g \in L^2(\mathbb{R}^n)$ , then

$$\int_{\mathbb{R}^n} g(x) \overline{(H_\eta - \lambda)u(x)} \, dx = 0$$

for every  $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$  implies that  $g = 0$  a.e. We conclude that  $\mathcal{F}_2[f](x, \eta) = 0$  for almost every  $x$  and almost every  $\eta$  so that  $f = 0$  a.e.

On  $L^2(\mathbb{R}^n \times \mathbb{T}^m)$ , we replace partial Fourier transform with partial Fourier coefficients.  $\square$

## 5. PROOF OF THE EXACT OBSERVABILITY INEQUALITIES

We are now ready to prove the main theorem. Let us recall what we want to prove in the synthetic table, Table [1](#).

	Assumption <a href="#">A2</a>	Assumption <a href="#">A1</a>
$s > (\beta_1 + 2)/3$	$s > (\beta + 2)/3$ , exactly null-controllable for any $T > 0$	exactly null-controllable
$s > (\beta_1 + 2)/4$	$s \leq (\beta_1 + 2)/3$	
$s \leq (\beta_1 + 2)/4$	not known under Assumption <a href="#">A2</a> see Remark <a href="#">1.11</a> for the standard case	not known under Assumption <a href="#">A1</a> see Remark <a href="#">1.11</a> for the standard case

TABLE 1. Exactly null-controllability results from equidistributed sets. For the case  $y \in \mathbb{R}^m$ , we need to set  $\beta_1 = \beta_2$ , while for the case  $y \in \mathbb{T}^m$  we do not need this condition.

As already mentioned, it is enough to prove the observability properties. Let us start with the non-periodic case. The observability equation corresponding to  $(E_{BG,s})$  is the following:

**Theorem 5.1.** *Let  $T > 0$ ,  $\beta_1 = \beta_2 > 0$ , and  $\gamma > 0$ . Let  $V \in L^1_{\text{loc}}(\mathbb{R}^n)$  that satisfies Assumption A1 (resp. Assumption A2) and let  $\mathcal{L}_V = -\Delta_x - V(x)\Delta_y$  be the corresponding Baouendi-Grushin operator on  $\mathbb{R}^n \times \mathbb{R}^m$ . Let  $\omega \subset \mathbb{R}^n$  be a  $\gamma$ -equidistributed set. If  $s > s_{A1}$  (resp.  $s > s_{A2}$ ), then there exists a constant  $C_{\text{obs}}^{\mathcal{L}}(T, s, V, \omega)$  such that the inequality*

$$\left\| e^{-T\mathcal{L}_V^s} f \right\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_{\text{obs}}^{\mathcal{L}}(T, s, V, \omega) \int_0^T \left\| e^{-t\mathcal{L}_V^s} f \right\|_{L^2(\omega \times \mathbb{R}^m)} dt$$

holds for every  $f \in L^2(\mathbb{R}^n \times \mathbb{R}^m)$ .

*Proof.* We have, using Fubini, Parseval (for  $\mathcal{F}_2$ ) and Proposition 3.12

$$\begin{aligned} \left\| e^{-T\mathcal{L}_V^s} u \right\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)}^2 &= \left\| \mathcal{F}_2[e^{-T\mathcal{L}_V^s} u] \right\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)}^2 = \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \left| e^{-TH^s_{|\eta|^2 V}} \mathcal{F}_2[u](x, \eta) \right|^2 dx d\eta \\ &\leq \int_{\mathbb{R}^m} \frac{C_{\text{obs}}(T, s, |\eta|^2 V, \omega)}{T} \int_0^T \int_{\omega} \left| e^{-tH^s_{|\eta|^2 V}} \mathcal{F}_2[u](x, \eta) \right|^2 dx d\eta dt \\ &\leq \frac{\sup_{r>0} C_{\text{obs}}(T, s, rV, \omega)}{T} \int_0^T \int_{\omega} \int_{\mathbb{R}^m} \left| e^{-tH^s_{|\eta|^2 V}} \mathcal{F}_2[u](x, \eta) \right|^2 dx d\eta dt \\ &= \frac{\sup_{r>0} C_{\text{obs}}(T, s, rV, \omega)}{T} \int_0^T \int_{\omega} \int_{\mathbb{R}^m} \left| e^{-t\mathcal{L}_V^s} u(x, y) \right|^2 dx dy dt \end{aligned}$$

provided

$$C_{\text{obs}}^{\mathcal{L}}(T, s, V, \omega) := \frac{1}{T} \sup_{r>0} C_{\text{obs}}(T, s, rV, \omega) < +\infty.$$

It remains to use Proposition 3.12 and Lemma 3.13 to conclude that the observability inequality holds in the listed case.  $\square$

**Remark 5.2.** The proof actually also provides an estimate of the observability constants, namely, when  $s > s_{A1}$  (resp.  $s > s_{A2}$ ) and  $C_{\text{obs}}^{\mathcal{L}}(T, s, V, \omega) \lesssim \frac{B_-(T, s, \omega, V)}{T}$  when  $T < 1$  and  $C(T, s, \omega, V) \lesssim \frac{B_+(T, s, \omega, V)}{T}$  when  $T \geq 1$  where  $B_-$  and  $B_+$  are defined in (3.34)-(3.35);

Next, we show that adding a zero-order term to  $\mathcal{L}_V$  allows to obtain observability corresponding to (EBGS<sub>s</sub>) from smaller sets:

**Proposition 5.3.** *Let  $T > 0$ ,  $\sigma \geq 0$ , and  $\gamma > 0$ . Let  $V, \tilde{V} \in L^1_{\text{loc}}(\mathbb{R}^n)$  that satisfy Assumption A2 (resp. Assumption A1) with same parameters  $\beta_1, \beta_2$  and let  $\mathcal{L}_{V, \tilde{V}} = -\Delta_x - V(x)\Delta_y + \tilde{V}$  be the corresponding Baouendi-Grushin-Schrödinger operator on  $\mathbb{R}^n \times \mathbb{R}^m$ . Let  $\omega \subset \mathbb{R}^n$  be a  $(\gamma, \sigma)$ -distributed set. Assume further that  $s, \beta_1, \beta_2, \sigma, T$  satisfy (ii) or (iii) with in Lemma 3.13.*

*Then there exists a constant  $C_{\text{obs}}^{\mathcal{L}}(T, s, V, \tilde{V}, \omega)$  such that for every  $f \in L^2(\mathbb{R}^n \times \mathbb{R}^m)$*

$$\left\| e^{-T\mathcal{L}_{V, \tilde{V}}^s} f \right\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_{\text{obs}}^{\mathcal{L}}(T, s, V, \tilde{V}, \omega) \int_0^T \left\| e^{-t\mathcal{L}_{V, \tilde{V}}^s} f \right\|_{L^2(\omega \times \mathbb{R}^m)} dt.$$

*Proof.* We have, using Fubini, Parseval (for  $\mathcal{F}_2$ ) and Proposition 3.12

$$\begin{aligned} \left\| e^{-T\mathcal{L}_{V, \tilde{V}, v_0}^s} u \right\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)}^2 &= \left\| \mathcal{F}_2[e^{-T\mathcal{L}_{V, \tilde{V}, v_0}^s} u] \right\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)}^2 = \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \left| e^{-TH^s_{|\eta|^2 V + v_0}} \mathcal{F}_2[u](x, \eta) \right|^2 dx d\eta \\ &\leq \int_{\mathbb{R}^m} \frac{C_{\text{obs}}(T, s, (1 + |\eta|^2)V, \omega)}{T} \int_0^T \int_{\omega} \left| e^{-tH^s_{|\eta|^2 V + v_0}} \mathcal{F}_2[u](x, \eta) \right|^2 dx d\eta dt. \end{aligned}$$

Here we use Remark 3.14 to absorb  $\tilde{V}$  into  $V$ . We thus get

$$\left\| e^{-T\mathcal{L}_{V, \tilde{V}, v_0}^s} u \right\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)}^2 \leq \frac{\sup_{r \geq 1} C_{\text{obs}}(T, s, rV, \omega)}{T} \int_0^T \int_{\omega} \int_{\mathbb{R}^m} \left| e^{-t\mathcal{L}_{V, v_0}^s} u(x, y) \right|^2 dx dy dt$$

provided

$$C_{\text{obs}}^{\mathcal{L}}(T, s, V, \tilde{V}, \omega) := \frac{1}{T} \sup_{r > 1} C_{\text{obs}}(T, s, rV, \omega) < +\infty.$$

It remains to use Lemmas 3.13 and 3.15 to conclude.  $\square$

We finally treat the semi-periodic case:

**Theorem 5.4.** *Let  $T > 0, \beta_2 \geq \beta_1 > 0$ , and  $\gamma > 0$ . Let  $V \in L^1_{\text{loc}}(\mathbb{R}^n)$  that satisfies Assumption A1 (resp. Assumption A2) and let  $\mathcal{L}_V = -\Delta_x - V(x)\Delta_y$  be the corresponding Baouendi-Grushin operator on  $\mathbb{R}^n \times \mathbb{T}^m$ . Let  $\omega \subset \mathbb{R}^n$  be a  $\gamma$ -equidistributed set. If  $s > s_{A1}$  (resp.  $s > s_{A2}$ ), then there exists a constant  $C_{\text{obs}}^{\mathcal{L}}(T, s, V, \omega)$  such that the inequality*

$$\left\| e^{-T\mathcal{L}_V^s} f \right\|_{L^2(\mathbb{R}^n \times \mathbb{T}^m)} \leq C_{\text{obs}}^{\mathcal{L}}(T, s, V, \omega) \int_0^T \left\| e^{-t\mathcal{L}_V^s} f \right\|_{L^2(\omega \times \mathbb{T}^m)} dt$$

holds for every  $f \in L^2(\mathbb{R}^n \times \mathbb{T}^m)$ .

*Proof.* We have, using Fubini and Parseval (for  $\mathcal{F}_2^p$ )

$$\|e^{-T\mathcal{L}_V^s} u\|_{L^2(\mathbb{R}^n \times \mathbb{T}^m)}^2 = \|\mathcal{F}_2^p[e^{-T\mathcal{L}_V^s} u]\|_{L^2(\mathbb{R}^n \times \mathbb{T}^m)}^2 = \sum_{k \in \mathbb{Z}^m} \int_{\mathbb{R}^n} \left| e^{-TH_{|k|^2}^s} \mathcal{F}_2^p[u](x, k) \right|^2 dx.$$

For  $k = 0$ , we use that a  $\gamma$ -equidistributed set is also  $\gamma$ -thick. Then, as  $H_{0V} = \Delta$  we can appeal to Theorem 1.4 to bound

$$\int_{\mathbb{R}^n} \left| e^{-TH_{0V}^s} \mathcal{F}_2^p[u](x, k) \right|^2 dx \leq C \int_0^T \int_{\omega} \left| e^{-tH_{0V}^s} \mathcal{F}_2^p[u](x, k) \right|^2 dx dt.$$

For  $k \neq 0$ , we have  $|k| \geq 1$ . Then we apply Lemma 3.13, Lemma 3.15 and Proposition 3.12 to obtain

$$\sum_{k \in \mathbb{Z}^m \setminus \{0\}} \int_{\mathbb{R}^n} \left| e^{-TH_{|k|^2}^s} \mathcal{F}_2^p[u](x, k) \right|^2 dx \leq C \sum_{k \in \mathbb{Z}^m \setminus \{0\}} \int_0^T \int_{\omega} \left| e^{-tH_{|k|^2}^s} \mathcal{F}_2^p[u](x, k) \right|^2 dx dt.$$

Grouping both estimates and applying Parseval's relation gives us the desired observability inequality.  $\square$

Note that the  $k = 0$  case in the above proof requires  $\omega$  to be  $\gamma$ -thick so that, for this proof to work for  $(\gamma, \sigma)$ -distributed sets, one needs  $\sigma = 0$ .

## 6. DATA AVAILABILITY

No data has been generated or analysed during this study.

## 7. FUNDING AND/OR CONFLICTS OF INTERESTS/COMPETING INTERESTS

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