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Some completeness results in derivational modal logic

Author version*

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Abstract

Alongside the traditional Kripke semantics, modal logic also enjoys a topological interpretation, which is becoming increasingly influential. In this paper, we present various developments related to the topological derivational semantics, based on the Cantor derivative operator. We provide several characterizations of the validity of the axioms of bounded depth. We also elucidate the topological interpretation of the axioms of directedness and connectedness – which come in different forms, all of which we examine. We then prove results of soundness and completeness for all of these logics, using a range of old and new techniques.

1 Introduction

While modal logic is traditionally associated with Saul Kripke’s relational semantics, it also enjoys a topological interpretation, which actually predates Kripke’s work. This approach can indeed be traced back to the work of McKinsey and Tarski [MT44], who proposed to interpret \diamond as the topological closure operator – hence introducing the *c-semantics* – and proved the celebrated result that **S4** is the logic of any separable metric dense-in-itself space. This was subsequently strengthened by Rasiowa and Sikorski [RS63] who eliminated the separability condition – for a good survey of these results we recommend [vBB07]. Since open sets can naturally be interpreted as *pieces of observation* [Vic96], this approach has recently gained momentum in fields such as formal epistemology [BBÖS19, Özg17] and learning theory [dBY10].

A less known close kin of the *c-semantics* is the *derivational semantics*, or *d-semantics*. It is obtained by interpreting \diamond not as the closure, but as the *derived set* or *derivative* operator which is attributed to Georg Cantor. This variant was also introduced by McKinsey and Tarski, and further investigated by Esakia and others – see e.g., [Esa81, Esa01]. First, it must be noted that it is more expressive than the *c-semantics*, in the sense that any modally expressible property with respect to the *c-semantics*, is also modally expressible with respect to the *d-semantics*. This approach thus enables a more refined classification of spaces. Further, while the logic of the *c-semantics* is **S4**, the logic of the *d-semantics* is **wK4**, as proved by Esakia [Esa01]. Since **wK4** is weaker than **S4**, it has more extensions, and thus more logics can be studied in the derivational setting.

In spite of these compelling features, the *d-semantics* has received much less attention than the *c-semantics*, and our knowledge of it is largely incomplete: the interpretation of many

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standard logics is missing, and so are proofs of their completeness. One example is the axiom bd_n (for any $n \geq 0$) which characterizes the weakly transitive Kripke frames that contain no proper path of length greater than n – in other words, those with *depth bounded by n* . With the topological semantics, bd_n also yields a notion of depth for spaces, and this kind of parameter is of great interest when it comes to classification: spaces with finite depth are generally easier to deal with, and their logics tend to have good properties (e.g., the finite model property). The topological interpretation of the concept of depth, however, is not obvious. This question was solved by Bezhanishvili et al. [BBLBvM17] for the c -semantics: they introduced for any space X a number called the *modal Krull dimension* of X , which is smaller than n exactly when bd_n is valid on X . However, the same problem for the d -semantics remained open prior to our work.

Another gap is the interpretation of the axioms sdir and scon . In the Kripke semantics, they define respectively the property of (*strong*) *directedness* and (*strong*) *connectedness*. In the landscape of the c -semantics, they have also been familiar faces for a while, with their well-known connection to extremally disconnected spaces [vBB07, BBLBvM15]. However, their interpretation in the d -semantics is still missing. Further, several axioms coincide with sdir or scon over reflexive and transitive frames, but turn out to be distinct when we step outside of $\mathbf{S4}$ and only work with $\mathbf{wK4}$. This generates some confusion, and preliminary work will be needed in the first place to clarify the situation. We will eventually identify five logics of interest: $\mathbf{wK4.2}$ and $\mathbf{wK4.3}$ which correspond directly to $\mathbf{S4.2}$ and $\mathbf{S4.3}$, plus $\mathbf{wK4d}$, $\mathbf{wK4sd}$ and $\mathbf{wK4sc}$. These last three logics will be particularly instructive. We will show that $\mathbf{wK4d}$ and $\mathbf{wK4sd}$ define classes of extremally disconnected spaces with some forbidden patterns, whose general shape is quite subtle. Also fertile will be the study of $\mathbf{wK4sc}$, from which we can derive the simpler axiom $\mathbf{aT} := \Box(p \rightarrow \Diamond p)$, leading to the logic $\mathbf{aS4} := \mathbf{wK4} + \mathbf{aT}$. While its Kripke semantics is rather unimpressive, its topological semantics will yield rich results, and will lead us to introduce the very intriguing *accumulative spaces*.

Along this road we will naturally be interested in the topological completeness of these logics. This raises the particular challenge of turning Kripke frames into appropriate topological spaces. Here by “appropriate” we mean in a way that preserves the truth of formulas with respect to the relevant semantics. This operation is crucial because it allows one to immediately transfer results of completeness from the Kripke semantics to the topological semantics. While this is straightforward in the c -semantics, the case of the d -semantics presents many difficulties related to reflexive points, that we will explain in detail. Our starting point will be the technique of *unfolding* a Kripke frame presented in [BBFD21]. While this strategy works well for $\mathbf{wK4d}$, $\mathbf{wK4sd}$ and $\mathbf{wK4.2}$, the topology of the unfolded frame turns out to be too coarse as soon as extensions of $\mathbf{aS4}$ are concerned. To remedy this, we will introduce the operation of *refined unfolding* which generates more open sets and successfully yields the topological completeness of $\mathbf{aS4}$ and $\mathbf{wK4sc}$. The remaining logics will be addressed via a more straightforward procedure called *dereflexivation*. The results of the present document are based on the author’s master thesis [Gou21].

The paper is structured as followed: in Section 2, we present the mathematical background and notations. In Section 3 we show that the c -semantics and d -semantics of bd_n coincide, and provide a number of helpful characterizations. In Section 4 we study the semantics of $\mathbf{wK4.2}$, $\mathbf{wK4d}$ and $\mathbf{wK4sd}$. In Section 5 we study the semantics of $\mathbf{aS4}$, $\mathbf{wK4.3}$ and $\mathbf{wK4sc}$. In Section 6 we address the completeness of all of these logics, and we end with some concluding remarks in Section 7.

2 Background

We denote by \mathbb{N} the set of all natural integers, including 0. Given $n, m \in \mathbb{N}$, the notation $[n, m]$ will denote the set $\{k \in \mathbb{N} \mid n \leq k \leq m\}$. Following [BBFD21], we present the semantics of modal logic in terms of derivative spaces, a modern framework which unifies the Kripke semantics, the

closure semantics and the derivational semantics in a concise and elegant manner.

2.1 Modal logic

Definition 2.1. We fix a countable set **Prop** of atomic propositions. The *modal language* \mathcal{L} is generated by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi.$$

The abbreviations \perp , \top , $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$ and $\Box\varphi$ are defined as usual. Let $\varphi, \psi_1, \dots, \psi_n \in \mathcal{L}$ and $p_1, \dots, p_n \in \mathbf{Prop}$, and let us write $\bar{\psi} := (\psi_1, \dots, \psi_n)$ and $\bar{p} := (p_1, \dots, p_n)$. We then denote by $\varphi[\bar{\psi}/\bar{p}]$ the formula φ where each ψ_i is uniformly substituted for p_i .

Definition 2.2. A *derivative space* is a pair $\mathcal{X} = (X, d)$, where X is a set of *points* and $d : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is an operator on subsets of X which satisfies the following properties, for all $A, B \subseteq X$:

- $d\emptyset = \emptyset$,
- $d(A \cup B) = dA \cup dB$,
- $ddA \subseteq A \cup dA$.

A *derivative model* based on \mathcal{X} is a tuple of the form $\mathfrak{M} = (X, d, \nu)$ with $\nu : \mathbf{Prop} \rightarrow \mathcal{P}(X)$ a *valuation*. Given $x \in X$ we then call (\mathfrak{M}, x) a *pointed derivative model*.

Definition 2.3. Given a derivative model $\mathfrak{M} = (X, d, \nu)$, we define by induction on a formula $\varphi \in \mathcal{L}$ the *extension* $\llbracket \varphi \rrbracket_{\mathfrak{M}}$ of φ in \mathfrak{M} by:

- $\llbracket p \rrbracket_{\mathfrak{M}} := \nu(p)$,
- $\llbracket \neg\varphi \rrbracket_{\mathfrak{M}} := X \setminus \llbracket \varphi \rrbracket_{\mathfrak{M}}$,
- $\llbracket \varphi \wedge \psi \rrbracket_{\mathfrak{M}} := \llbracket \varphi \rrbracket_{\mathfrak{M}} \cap \llbracket \psi \rrbracket_{\mathfrak{M}}$,
- $\llbracket \Diamond\varphi \rrbracket_{\mathfrak{M}} := d(\llbracket \varphi \rrbracket_{\mathfrak{M}})$.

We then write $\mathfrak{M}, x \models \varphi$ whenever $x \in \llbracket \varphi \rrbracket_{\mathfrak{M}}$. If \mathcal{X} is a space and $\mathfrak{M}, x \models \varphi$ for some model \mathfrak{M} based on \mathcal{X} , we say that φ is *satisfiable* on \mathcal{X} . If $\llbracket \varphi \rrbracket_{\mathfrak{M}} = X$, we write $\mathfrak{M} \models \varphi$. If $\mathfrak{M} \models \varphi$ for all models \mathfrak{M} based on \mathcal{X} we write $\mathcal{X} \models \varphi$ and we say that φ is *valid* on \mathcal{X} .

If $\mathcal{X} \models \varphi$ for all derivative spaces \mathcal{X} , we write $\models \varphi$. If $\models \varphi \leftrightarrow \psi$ we write $\varphi \equiv \psi$ and say that φ and ψ are *equivalent*. Given a class \mathcal{C} of derivative spaces, we write $\mathcal{C} \models \varphi$ whenever $\mathcal{X} \models \varphi$ for all $\mathcal{X} \in \mathcal{C}$. If Γ is a set of formulas we write $\mathfrak{M}, x \models \Gamma$ whenever $\mathfrak{M}, x \models \varphi$ for all $\varphi \in \Gamma$, and all of the other notations are adapted similarly.

In modal logic it is customary to study morphisms that preserve validity. In the context of derivative spaces, these are known as *d-morphisms*.

Definition 2.4. Let $\mathcal{X}_1 = (X_1, d_1)$ and $\mathcal{X}_2 = (X_2, d_2)$ be two derivative spaces. A map $f : X_1 \rightarrow X_2$ is called a *d-morphism* from \mathcal{X}_1 to \mathcal{X}_2 if it satisfies $f^{-1}[d_2 B] = d_1(f^{-1}[B])$ for all $B \subseteq X_2$. If in addition f is surjective, we will say that \mathcal{X}_2 is a *d-morphic image* of \mathcal{X}_1 .

Proposition 2.5 ([BBFD21, Cor. V.5]). *Suppose that \mathcal{X}_2 is a d-morphic image of \mathcal{X}_1 . If φ is a modal formula and $\mathcal{X}_1 \models \varphi$, then $\mathcal{X}_2 \models \varphi$.*

We will also be interested in an alternative semantics, induced by a specific operation on derivative models.

Definition 2.6. Let $\mathcal{X} = (X, d)$ be a derivative space. We define the operator $d^+ : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ by $d^+ A := A \cup dA$ for all $A \subseteq X$. We then write $\mathcal{X}^+ := (X, d^+)$. Clearly \mathcal{X}^+ is also a derivative space. If $\mathfrak{M} = (X, d, \nu)$ is a derivative model we write $\mathfrak{M}^+ := (X, d^+, \nu)$. Given a pointed derivative model (\mathfrak{M}, x) and a formula φ we then write $\mathfrak{M}, x \models^+ \varphi$ whenever $\mathfrak{M}^+, x \models \varphi$ holds.

The relationship between \models and \models^+ is reflected on the level of formulas by a syntactic translation.

Definition 2.7. Given a formula φ we write $\diamond^+\varphi := \varphi \wedge \diamond\varphi$ and $\square^+\varphi := \varphi \vee \square\varphi$. The formula φ^+ is then defined inductively by:

- $p^+ := p$,
- $(\neg\varphi)^+ := \neg\varphi^+$,
- $(\varphi \wedge \psi)^+ := \varphi^+ \wedge \psi^+$,
- $(\diamond\varphi)^+ := \diamond^+\varphi^+$.

Note that we also have $(\square\varphi)^+ \equiv \square^+\varphi^+$.

Proposition 2.8. Let φ be a modal formula. For all pointed derivative models (\mathfrak{M}, x) we have $\mathfrak{M}, x \models^+ \varphi$ if and only if $\mathfrak{M}, x \models \varphi^+$.

Proof. It suffices to prove that $\llbracket \varphi \rrbracket_{\mathfrak{M}^+} = \llbracket \varphi^+ \rrbracket_{\mathfrak{M}}$ by induction on φ . □

The following result will also be useful in the sequel.

Proposition 2.9. Let \mathcal{X}_1 and \mathcal{X}_2 be two derivative spaces. Every d -morphism from \mathcal{X}_1 to \mathcal{X}_2 is also a d -morphism from \mathcal{X}_1^+ to \mathcal{X}_2^+ .

Proof. We write $\mathcal{X}_1 = (X_1, d_1)$ and $\mathcal{X}_2 = (X_2, d_2)$. Let f be a d -morphism from \mathcal{X}_1 to \mathcal{X}_2 , and let $B \subseteq X_2$. We then have $f^{-1}[d_2^+ B] = f^{-1}[B \cup d_2 B] = f^{-1}[B] \cup f^{-1}[d_2 B] = f^{-1}[B] \cup d_1(f^{-1}[B]) = d_1^+(f^{-1}[B])$, and this proves the claim. □

2.2 Kripke semantics

The Kripke semantics consists in interpreting formulas in *Kripke frames*. A Kripke frame is a set of possible worlds along with an accessibility relation, indicating which worlds can be “seen” from a given world.

Definition 2.10. A *Kripke frame* is a pair $\mathfrak{F} = (W, R)$ with W a set of *worlds* and $R \subseteq W^2$ a binary relation on W .

Here are additional useful properties on frames:

Definition 2.11. Let $\mathfrak{F} = (W, R)$ be a Kripke frame.

- A world $w \in W$ is said to be *reflexive* if wRw , and *irreflexive* if *not* wRw . The frame \mathfrak{F} is reflexive (resp. irreflexive) if every $w \in W$ is reflexive (resp. irreflexive).
- \mathfrak{F} is *rooted* in $r \in W$ if for all $w \in W$, there exist $w_1, \dots, w_n \in W$ such that $w_1 = r$, $w_n = w$ and $w_i R w_{i+1}$ for all $i \in [1, n-1]$.
- \mathfrak{F} is *serial* if for all $w \in W$ there exists $u \in W$ such that wRu .
- \mathfrak{F} is *transitive* if wRu and uRv implies wRv .
- \mathfrak{F} is *weakly transitive* if wRu and uRv implies wRu or $w = u$.
- \mathfrak{F} is *directed* if wRu and wRv and $u \neq v$ implies the existence of $t \in W$ such that uRt and vRt .
- \mathfrak{F} is *strongly directed* if wRu and wRv implies the existence of $t \in W$ such that uRt and vRt .
- \mathfrak{F} is *connected* if wRu and wRv and $u \neq v$ implies uRv or vRu .
- \mathfrak{F} is *strongly connected* if wRu and wRv implies uRv or vRu .

- \mathfrak{F} is *converse well-founded* if there exists no sequence $(w_n)_{n \in \mathbb{N}} \in W^{\mathbb{N}}$ such that $w_n R w_{n+1}$ for all $n \in \mathbb{N}$.

In this paper we will extensively use a family of Kripke frames called *chains*.

Definition 2.12. Let $n \in \mathbb{N}$. A *n-chain* is a Kripke frame of the form $\mathfrak{F} = (W, R)$ with $W = \{w_i \mid 0 \leq i \leq n-1\}$ and

$$\{(w_i, w_j) \mid 0 \leq i < j \leq n-1\} \subseteq R \subseteq \{(w_i, w_j) \mid 0 \leq i \leq j \leq n-1\}$$

If $R = \{(w_i, w_j) \mid 0 \leq i < j \leq n-1\}$, we write $\mathfrak{F} = \underline{n}$. If $R = \{(w_i, w_j) \mid 0 \leq i \leq j \leq n-1\}$, we write $\mathfrak{F} = \underline{n}^+$.

Visually, a *n-chain* looks as in Figure 1. Each world may be either reflexive or irreflexive, so for all n there are 2^n different *n-chains* (up to isomorphism). We can see that \underline{n} denotes the irreflexive *n-chain*, and \underline{n}^+ the reflexive *n-chain*.

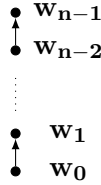


Figure 1: A *n-chain*

Finally, we observe that every weakly transitive Kripke frame $\mathfrak{F} = (W, R)$ induces a derivative space (W, d) with d defined by

$$dA := R^{-1}A := \{w \mid wRu \text{ and } u \in A\}.$$

Slightly abusing terminology, we will identify \mathfrak{F} and (W, d) , since one can be constructed from the other – that is to say, the application mapping a Kripke frame to its induced space is injective. Then (pointed) derivative models based on weakly transitive frames will be called (pointed) *Kripke models*. Accordingly, the semantics for derivative models gives rise to the *Kripke semantics* when restricted to Kripke frames. We also notice that the derivative space (W, d^+) is induced by the Kripke frame (W, R^+) , where $R^+ := R \cup \{(w, w) \mid w \in W\}$ is the *reflexive closure* of R .

2.3 Topological semantics

Here we introduce the closure and derivational semantics for modal logic. We first recall some basics of general topology.

Definition 2.13. Let X be a set of *points*. A *topology* on X is a set $\tau \subseteq \mathcal{P}(X)$ containing \emptyset and X , closed under arbitrary unions, and closed under finite intersections. The pair (X, τ) is then called a *topological space*. The elements of τ are called the *open* sets of X . The complement of an open set is called a *closed* set. If $x \in U \in \tau$ then U is called an *open neighbourhood* of x . If $A \subseteq X$, the *closure* $\text{Cl}(A)$ of A is the smallest closed set containing A , while the *interior* $\text{Int}(A)$ of A is the greatest open set contained in A .

Slightly abusing terminology, we will often keep τ implicit and let X refer to the space (X, τ) .

Definition 2.14. Let X be a topological space, $A \subseteq X$ and $x \in X$. The point x is said to be a *limit point* of A if for all open neighbourhoods U of x , we have $U \cap A \setminus \{x\} \neq \emptyset$. We denote by

dA the set of all limit points of A and call it the *derived set* of A . The dual of d is defined by $\widehat{d}A := X \setminus d(X \setminus A)$. In case of ambiguity, the notations $d_X A$ and $\widehat{d}_X A$ may be used to indicate the space wherein these two operators are evaluated.

When working with a space, a topological *base* is often very useful as it simplifies reasoning. A base is essentially a collection of open sets from which the whole topology can be generated by applying arbitrary unions.

Definition 2.15. Let X be a set. A *base* is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ such that $\bigcup \mathcal{B} = X$ and for all $U, V \in \mathcal{B}$ and $x \in U \cap V$, there exists $W \in \mathcal{B}$ such that $x \in W \subseteq U \cap V$. Then $\tau := \{\bigcup B \mid B \subseteq \mathcal{B}\}$ is a topology, called the topology *generated* by \mathcal{B} .

Given a topology τ generated by \mathcal{B} , it is well known that many statements about τ can be reduced to conditions involving the elements of \mathcal{B} only, e.g., we have $x \in \text{Int}(A)$ iff there exists $U \in \mathcal{B}$ such that $x \in U \subseteq A$.

Topologies also specialize to subsets, and induce *subspace topologies*.

Definition 2.16. Let (X, τ) be a topological space and $Y \subseteq X$. The *subspace topology* on Y is the topology $\tau_Y := \{U \cap Y \mid U \in \tau\}$. The space (Y, τ_Y) is then called a *subspace* of (X, τ) .

In this paper, any subset Y of X will implicitly be regarded as the subspace (Y, τ_Y) . Observe that the operators Cl_Y and d_Y associated to Y can be expressed in terms of Cl_X and d_X via the identities $\text{Cl}_Y(A) = \text{Cl}_X(A) \cap Y$ and $d_Y(A) = d_X(A) \cap Y$.

Definition 2.17. Let X be a topological space. A point $x \in X$ is said to be *isolated* if $\{x\}$ is open. The space X is called *dense-in-itself* if it contains no isolated point, and *discrete* if all of its points are isolated. The space X is called *scattered* if every non-empty subspace of X contains an isolated point. We say that X is *extremally disconnected* (or ED for short) if $\text{Cl}(U)$ is open for all open sets U . It is called *hereditarily extremally disconnected* (or HED for short) if every subspace of X is extremally disconnected. The space X is called T_1 if for all $x, y \in X$ such that $x \neq y$, there exists an open set U such that $x \in U$ and $y \notin U$. Finally, X is said to be T_D if for all $A \subseteq X$ we have $ddA \subseteq dA$.

Given a topological space X , it is easily observed that the pair (X, d) is a derivative space. Conversely, the topology τ can be recovered from d since we have the equivalence

$$A \text{ is closed} \iff dA \subseteq A$$

for all $A \subseteq X$. For this reason we choose, again, to identify (X, τ) and (X, d) . Then (pointed) derivative models based on topological spaces will be called (pointed) *topological models*. The semantics for derivative models then gives rise to the *derivational semantics* for topological models, or *d-semantics* for short. We also notice that d^+ coincides with the closure operator Cl . Thus, the semantics introduced in Definition 2.6 (carried by the relation \models^+) gives rise to the *closure semantics* (or *c-semantics*) for topological models. Proposition 2.8 then shows that the d-semantics is at least as expressive as the c-semantics, as any property expressed by the formula φ in the c-semantics is also expressed by the formula φ^+ in the d-semantics. In fact, the d-semantics is strictly *more* expressive than the c-semantics, as proved by Kuratowski in his seminal 1922 paper [Kur22, Sect. 5].

Given two topological spaces X and Y , an *interior map* from X to Y is a d-morphism from X^+ to Y^+ – that is, a map $f : X \rightarrow Y$ satisfying $f^{-1}[\text{Cl}(B)] = \text{Cl}(f^{-1}[B])$ for all $B \subseteq Y$. Alternatively, interior maps can be described as the maps $f : X \rightarrow Y$ such that $f[U]$ is open for all open sets $U \subseteq X$, and $f^{-1}[V]$ is open for all open sets $V \subseteq Y$ [RS63, Sect. III.3]. If in addition f is surjective, we will say that Y is an *interior image* of X . By Proposition 2.9, we see that every d-morphism from X to Y is also an interior map from X to Y .

Kripke frames are closely connected to topological spaces, via the following construction.

Definition 2.18. Let $\mathfrak{F} := (W, R)$ be a weakly transitive Kripke frame. A set $U \subseteq W$ is called an *upset* if $w \in U$ and wRu implies $u \in U$. The collection τ_R of all upsets is then a topology, and (W, τ_R) is called the topological space *induced* by \mathfrak{F} . If $\mathfrak{M} = (W, R, \nu)$ is a Kripke model based on \mathfrak{F} , then (W, τ_R, ν) is the topological model *induced* by \mathfrak{M} .

In this document we will largely abuse terminology and not distinguish a weakly transitive Kripke frame (resp. model) from the topological space (resp. model) induced by it. This convention is partly motivated by the following property, which states that the Kripke semantics and the d-semantics agree over irreflexive and weakly transitive Kripke frames.

Proposition 2.19. *Let φ be a modal formula. For all irreflexive and weakly transitive models $\mathfrak{M} = (W, R, \nu)$ and all worlds $w \in W$, we have $\mathfrak{M}, w \models \varphi$ if and only if $(W, \tau_R, \nu), w \models \varphi$.*

2.4 Axiomatization

We denote by \mathbf{K} the smallest normal modal logic (see [BdRV01, Sect. 4]). We will also be interested in the axioms and logics introduced below.

$$\begin{array}{ll}
4 := \Diamond\Diamond p \rightarrow \Diamond p & \mathbf{K4} := \mathbf{K} + 4 \\
w4 := \Diamond\Diamond p \rightarrow \Diamond^+ p & \mathbf{wK4} := \mathbf{K} + w4 \\
\mathbf{T} := p \rightarrow \Diamond p & \mathbf{S4} := \mathbf{K} + \mathbf{T} + 4 \\
dir := \Diamond(p \wedge \Box q) \rightarrow \Box(p \vee \Diamond q) & \\
sdir := \Diamond\Box p \rightarrow \Box\Diamond p & \mathbf{S4.2} := \mathbf{S4} + sdir \\
con := \Box(p \rightarrow \Diamond^+ q) \vee \Box(q \rightarrow \Diamond^+ p) & \\
scon := \Box(p \rightarrow \Diamond q) \vee \Box(q \rightarrow \Diamond p) & \mathbf{S4.3} := \mathbf{S4} + scon \\
gl := \Box(\Box p \rightarrow p) \rightarrow \Box p & \mathbf{GL} := \mathbf{K} + gl \\
bd_n := \begin{cases} \perp & \text{if } n = 0 \\ \Diamond(\Box p_n \wedge \neg bd_{n-1}) \rightarrow p_n & \text{if } n > 0 \end{cases} &
\end{array}$$

Definition 2.20. Let \mathcal{C}_0 be a class of derivative spaces. Given $\mathcal{C} \subseteq \mathcal{C}_0$, we say that \mathbf{L} *defines* the class \mathcal{C} within \mathcal{C}_0 if we have $\mathcal{C} = \{\mathcal{X} \in \mathcal{C}_0 \mid \mathcal{X} \models \mathbf{L}\}$. We say that \mathbf{L} is *sound and complete* for a class \mathcal{C} of derivative spaces if for all formulas φ , we have $\mathbf{L} \vdash \varphi$ iff $\mathcal{C} \models \varphi$.

Instead of $\mathcal{X} \models \mathbf{L}$ we will sometimes say that \mathcal{X} is an “ \mathbf{L} space” – for instance, weakly transitive Kripke frames will often be called “ $\mathbf{wK4}$ frames”. In this paper we are essentially interested in the following classes:

$$\begin{aligned}
\mathcal{C}_{\text{Kripke}} &:= \{\mathfrak{F} \mid \mathfrak{F} \text{ is a weakly transitive Kripke frame}\}, \\
\mathcal{C}_{\text{topo}}^+ &:= \{\mathcal{X}^+ \mid \mathcal{X} \text{ is a topological space}\}, \\
\mathcal{C}_{\text{topo}} &:= \{\mathcal{X} \mid \mathcal{X} \text{ is a topological space}\}.
\end{aligned}$$

So when \mathbf{L} defines a class \mathcal{C} within $\mathcal{C}_{\text{Kripke}}$, $\mathcal{C}_{\text{topo}}^+$ or $\mathcal{C}_{\text{topo}}$, we will simply omit the “within” part as it is generally obvious from the context, and say that “ \mathbf{L} defines \mathcal{C} ”. We call \mathbf{L} *Kripke complete* if it is sound and complete for some subclass of $\mathcal{C}_{\text{Kripke}}$, *topologically complete in the c-semantics* if it is sound and complete for some subclass of $\mathcal{C}_{\text{topo}}^+$, and *topologically complete in the d-semantics* if it is sound and complete for some subclass of $\mathcal{C}_{\text{topo}}$. If there is no ambiguity, the shorter statement “ \mathbf{L} is topologically complete” will often be sufficient. A particular case of Kripke completeness happens when \mathbf{L} is complete for a class of *finite* Kripke frames – in this case \mathbf{L} is said to have the *finite model property* [BdRV01, Sect. 2.3 & 3.4].

Theorem 2.21 ([BBLBvM15, BdRV01, CZ97, vBB07]). *Every logic \mathbf{L} in the first column of Table 1 defines the corresponding class of Kripke frames or topological spaces – within $\mathcal{C}_{\text{topo}}$ for the upper half, and $\mathcal{C}_{\text{topo}}^+$ for the lower half. All of these logics are also Kripke complete.*

Logic	Class of frames	Class of spaces	Semantics
wK4	weakly transitive	all	derivational
wK4 + $\diamond\top$	serial	dense-in-itself	
K4	transitive	T_D	
GL	transitive and converse well-founded	scattered	
S4	reflexive and transitive	all	closure
S4.2	refl., trans. and strongly directed	ED	
S4.3	refl., trans. and strongly connected	HED	

Table 1: Known definability results

2.5 The infinite path modality

The language \mathcal{L}^* is defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \diamond\varphi \mid \diamond^*\varphi.$$

Given a **wK4** model $\mathfrak{M} = (W, R, \nu)$ and $w \in W$, we write $\mathfrak{M}, w \models \diamond^*\varphi$ whenever there exists a sequence $(w_n)_{n \in \mathbb{N}}$ such that $w_0 = w$ and for all $n \in \mathbb{N}$, $w_n R w_{n+1}$ and $\mathfrak{M}, w_n \models \varphi$. This defines the Kripke semantics of \diamond^* , which we dub the *infinite path modality*. Here we briefly introduce it because it will be instrumental to prove some results of Section 3. We denote by **wK4**^{*} the logic **wK4** over \mathcal{L}^* extended with the *fixpoint axiom* $\diamond^*p \rightarrow p \wedge \diamond\diamond^*p$ and the *induction rule*:

$$\text{from } \varphi \rightarrow \diamond(\varphi \wedge \psi) \text{ infer } \varphi \rightarrow \diamond^*\psi.$$

The Kripke completeness of **wK4**^{*} can be obtained as an immediate consequence of the work of Baltag et al. on the topological μ -calculus [BBFD21]. Indeed, the modality \diamond^* can be expressed in the μ -calculus by $\diamond^*\varphi := \nu p. \varphi \wedge \diamond p$. In fact, \mathcal{L}^* is a *natural sublanguage* of the μ -calculus, and **wK4**^{*} is merely the specialization of the logic $\mu\mathbf{wK4}$ to \mathcal{L}^* , whence it follows that **wK4**^{*} is sound and complete for the class of weakly transitive Kripke frames [BBFD21, Sect. III].

3 Topological bounded depth

In this section we investigate modal depth in the d-semantics. In the c-semantics, it was shown [BBLBvM17] that the validity of \mathbf{bd}_n is captured by a parameter called *modal Krull dimension*. Roughly summarized, the modal Krull dimension of a space X is the size of a maximal stack of nested non-empty nowhere dense subspaces of X . In Section 3.1, we show that the modal Krull dimension also captures the validity of \mathbf{bd}_n in the d-semantics. This entails that the validity of \mathbf{bd}_n is the same regardless of whether it is interpreted in the c-semantics or in the d-semantics.

Though deeply meaningful from an algebraic perspective which is out of the scope of this document, the modal Krull dimension has barely anything to do with depth. A more natural characterization is provided by another result of [BBLBvM17], which states that the modal Krull dimension of X is the greatest integer n such that the reflexive $n + 1$ -chain is contained in X (see Theorem 3.6). Such a formulation is already much closer to the initial graph-theoretic notion of depth. What “contain” means in this context will be made precise soon, but we can already mention that it can be defined in two ways: either via the satisfiability of a Jankov-Fine formula, or by the existence of a surjective interior map. It should be noted, however, that both

definitions are typically designed for the c-semantics: Jankov-Fine formulas are evaluated in the c-semantics, and interior maps are the natural morphisms for the c-semantics. An analogue result, but relevant to the d-semantics, would be welcome, and this is investigated in Section 3.2. Finally, we briefly study in Section 3.3 how taking the derivative of a space affects its modal dimension.

First, let us define a notion of depth for Kripke frames. Definition 3.1 and Proposition 3.2 below were initially stated in [CZ97, Sect. 3.5] in the context of transitive frames, but they are easily adapted to the weakly transitive case. For convenience we provide a detailed proof of Proposition 3.2.

Definition 3.1. Let $\mathfrak{F} = (W, R)$ be a weakly transitive frame. A *proper path* in \mathfrak{F} is a sequence (w_1, \dots, w_n) where $w_1, \dots, w_n \in W$ are such that $w_i R w_{i+1}$ and *not* $w_{i+1} R w_i$ for all $i \in [1, n-1]$. The integer n is then called the *length* of (w_1, \dots, w_n) . The *depth* $\text{depth}(\mathfrak{F})$ of \mathfrak{F} is the smallest $n \in \mathbb{N}$ such that there exists no proper path of length $n+1$ in \mathfrak{F} , provided that such a n exists. Otherwise, we have $\text{depth}(\mathfrak{F}) := \infty$.

Proposition 3.2. *Let \mathfrak{F} be a weakly transitive frame and $n \in \mathbb{N}$. We have $\mathfrak{F} \models \text{bd}_n$ if and only if $\text{depth}(\mathfrak{F}) \leq n$.*

Proof. Let us write $\mathfrak{F} = (W, R)$. We proceed by induction on n . For $n = 0$, observe that any $w \in W$ trivially corresponds to a proper path (w) of length 1. Thus, we have $\text{depth}(\mathfrak{F}) > 0$ iff W is non-empty, iff $\mathfrak{F} \not\models \perp$.

Now suppose that the results holds for $n-1$ with $n > 0$. By contraposition, suppose that $\text{depth}(\mathfrak{F}) > n$. Then there exists a proper path (w_1, \dots, w_{n+1}) of length $n+1$ in \mathfrak{F} . Let $\mathfrak{F}' = (W', R')$ be the subframe of \mathfrak{F} generated by w_2 . Then (w_2, \dots, w_{n+1}) is a proper path of length n in \mathfrak{F}' , and so by the induction hypothesis, we have $\mathfrak{F}' \not\models \text{bd}_{n-1}$. Thus, there exists a valuation ν' on \mathfrak{F}' and $w \in W'$ such that $\mathfrak{F}', \nu', w \models \neg \text{bd}_{n-1}$. We also assume that $\nu'(p_n) = W' \setminus \{w_1\}$. Now define a valuation ν on \mathfrak{F} by $\nu(p_i) := \nu'(p_i)$ for all $i \in [1, n-1]$, and $\nu(p_n) := W \setminus \{w_1\}$. Then (\mathfrak{F}', ν') is a generated submodel of (\mathfrak{F}, ν) , whence $\mathfrak{F}, \nu, w \models \neg \text{bd}_{n-1}$. Further, suppose that $w R w_1$. Then since $w_2 R^+ w$ we obtain $w_2 R^+ w_1$ by weak transitivity. Since $w_1 R w_2$ we end up with $w_2 R w_1$ in all cases, a contradiction. So w does not see w_1 , and this proves that $\mathfrak{F}, \nu, w \models \Box p_n$. From $w_1 R w_2$ and $w_2 R^+ w$, we also have $w_1 R^+ w$. If $w_1 = w$ then $w_2 R^+ w_1$, which we have seen to be impossible. Thus $w_1 R w$, and it follows that $\mathfrak{F}, \nu, w_1 \models \Diamond(\Box p_n \wedge \neg \text{bd}_{n-1}) \wedge \neg p_n$. Therefore $\mathfrak{F} \not\models \text{bd}_n$.

Conversely, suppose that $\mathfrak{F} \not\models \text{bd}_n$. Then there exists a valuation ν on \mathfrak{F} and $w \in W$ such that $\mathfrak{F}, \nu, w \models \Diamond(\Box p_n \wedge \neg \text{bd}_{n-1}) \wedge \neg p_n$. This gives the existence of $u \in W$ such that $w R u$ and $\mathfrak{F}, \nu, u \models \Box p_n \wedge \neg \text{bd}_{n-1}$. As a result we have $\mathfrak{F}' \not\models \text{bd}_{n-1}$, where \mathfrak{F}' is the subframe of \mathfrak{F} generated by u . By the induction hypothesis, there exists a proper path (w_1, \dots, w_n) in \mathfrak{F}' . Since $w R u$ and $u R^+ w_1$, we have $w R^+ w_1$ by weak transitivity. If $w = w_1$, then $u R^+ w$, and since $w R u$ we have $u R w$ in all cases, contradicting the fact that $\mathfrak{F}, \nu, u \models \Box p_n$ and $\mathfrak{F}, \nu, w \not\models p_n$. Thus $w R w_1$. In addition, if $w_1 R w$, then by weak transitivity again we obtain $u R^+ w$, a contradiction. This proves that (w, w_1, \dots, w_n) is a proper path in \mathfrak{F} of length $n+1$, and we are done. \square

Definition 3.3 ([BBLBvM17, Sect. 3]). Let X be a topological space. A subspace $Y \subseteq X$ is said to be *nowhere dense* in X if $\text{Int}(\text{Cl}(Y)) = \emptyset$. The *modal Krull dimension* $\text{mdim}(X)$ of X is then defined as follows:

- $\text{mdim}(X) \leq -1$ if $X = \emptyset$,
- $\text{mdim}(X) \leq n+1$ if for all $Y \subseteq X$ nowhere dense in X , we have $\text{mdim}(Y) \leq n$,
- $\text{mdim}(X) := \inf \{n \in \mathbb{N} \cup \{-1\} \mid \text{mdim}(X) \leq n\}$ with the convention that $\inf \emptyset = \infty$.

Theorem 3.4 ([BBLBvM17, Th. 3.6]). *Let X be a topological space. For all $n \in \mathbb{N}$ we have $X \models^+ \text{bd}_n$ if and only if $\text{mdim}(X) \leq n-1$.*

We now address alternative characterizations of modal Krull dimension. Those involve Jankov-Fine formulas, which encode a “pattern” given by a finite Kripke frame.

Definition 3.5 ([BdRV01, Sect. 3.4]). Let $\mathfrak{F} = (W, R)$ be a finite rooted **S4** frame. We write $W = \{w_i \mid 0 \leq i \leq n\}$ where w_0 is a root. The *Jankov-Fine formula* $\chi_{\mathfrak{F}}$ of \mathfrak{F} is the conjunction of the following formulas:

1. q_0 ,
2. $\Box \bigvee_{i=0}^n q_i$,
3. $\Box \neg(q_i \wedge q_j)$ for all $i, j \in [0, n]$ such that $i \neq j$,
4. $\Box(q_i \rightarrow \Diamond q_j)$ for all $i, j \in [0, n]$ such that $w_i R w_j$,
5. $\Box(q_i \rightarrow \neg \Diamond q_j)$ for all $i, j \in [0, n]$ such that *not* $w_i R w_j$.

Theorem 3.6 ([BBLBvM17, Th. 3.6]). *Let X be a topological space and $n \in \mathbb{N}$. The following conditions are equivalent:*

1. $\text{mdim}(X) \leq n - 1$,
2. $X \models^+ \neg \chi_{n+1}^+$,
3. $\underline{n+1}^+$ is not an interior image of X ,
4. $\underline{n+1}^+$ is not an interior image of an open subspace of X .

3.1 Bounded depth and modal Krull dimension

Here we show that given $n \in \mathbb{N}$ and a space X , we have $X \models \text{bd}_n$ if and only if $\text{mdim}(X) \leq n - 1$. To achieve this goal, we proceed by induction on n . This will bring us to a point where we have a nowhere dense subspace Y of X , and try to prove that $Y \not\models \text{bd}_n$ if and only if $X \not\models \text{bd}_{n+1}$. From left to right, this is done by *shifting up* a valuation: basically, p_1 is relabelled as p_2 , p_2 is relabelled as p_3 , etc. From right to left, this is simply achieved by the reverse process of *shifting down* a valuation: p_2 is relabelled as p_1 , p_3 is relabelled as p_2 , etc. Below we define these operations formally.

Definition 3.7. If $n \in \mathbb{N}$, we write $P_n := \{p_i \mid 1 \leq i \leq n\}$. Let X be a topological space.

- Let ν be a valuation on X with domain P_{n+1} and $Y \subseteq X$. The *Y-downshift* of ν is the valuation ν_Y^∇ over Y with domain P_n defined by $\nu_Y^\nabla(p_i) := \nu(p_{i+1}) \cap Y$ for all $i \in [1, n]$.
- Let $Y \subseteq X$ and ν be a valuation on Y with domain P_n . The *Y-upshift* of ν is the valuation $\nu_Y^\hat{}$ with domain P_{n+1} defined by $\nu_Y^\hat{(p_1)} := X \setminus Y$ and $\nu_Y^\hat{(p_{i+1})} := \nu(p_i) \cup (X \setminus Y)$ for all $i \in [1, n]$.

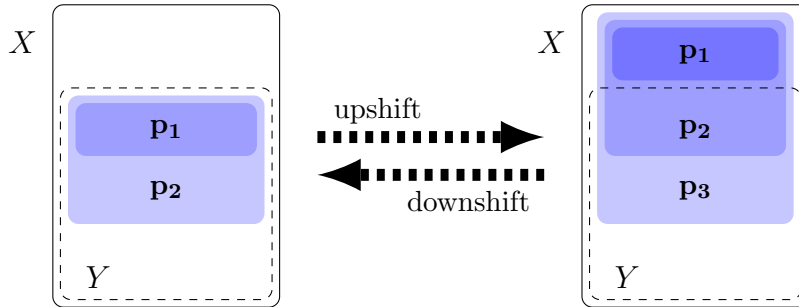


Figure 2: Depiction of valuation shifting

The effect of shifting up and down is depicted in Figure 2. In addition, in the implication from right to left, the subspace Y is not given, so it is our task to select a nowhere dense subspace Y of X with the desired property. We are going to show that the extension of the formula $\sigma := \diamond^+(\neg p_1 \wedge \diamond \square p_1)$ is qualified for being such Y . First, we prove that it is nowhere dense.

Lemma 3.8. *Given a topological model \mathfrak{M} , the subspace $\llbracket \sigma \rrbracket_{\mathfrak{M}}$ is nowhere dense in X .*

Proof. The statement that we need to prove is $\text{Int}(\text{Cl}(\llbracket \sigma \rrbracket_{\mathfrak{M}})) = \emptyset$, or equivalently $\llbracket \square^+ \diamond^+ \diamond^+ (\neg p_1 \wedge \diamond \square p_1) \rrbracket_{\mathfrak{M}} = \emptyset$. By soundness of **wK4**, it suffices to show that **wK4** $\vdash \neg \square^+ \diamond^+ \diamond^+ (\neg p_1 \wedge \diamond \square p_1)$, which can be reduced to

$$\mathbf{wK4} \vdash \neg \square^+ \diamond^+ (\neg p_1 \wedge \diamond \square p_1).$$

So using Kripke completeness, we consider a **wK4** Kripke model $\mathfrak{M}' = (W, R, \nu)$ and $w \in W$ and suppose toward a contradiction that $\mathfrak{M}', w \models \square^+ \diamond^+ (\neg p_1 \wedge \diamond \square p_1)$. Then there exists $u \in W$ such that wR^+u and $\mathfrak{M}', u \models \neg p_1 \wedge \diamond \square p_1$. It follows that there exists $v \in W$ such that uRv and $\mathfrak{M}', v \models \square p_1$. By weak transitivity we have wR^+v and therefore $\mathfrak{M}', w \models \diamond^+ (\neg p_1 \wedge \diamond \square p_1)$, that is, there exists $t \in W$ such that vR^+t and $\mathfrak{M}', t \models \neg p_1 \wedge \diamond \square p_1$. If $t \neq v$ then vRt and since $\mathfrak{M}', v \models \square p_1$, it follows that $\mathfrak{M}', t \models p_1$, a contradiction. Therefore $t = v$ and we obtain $\mathfrak{M}', v \models \neg p_1 \wedge \diamond \square p_1$. The whole reasoning about u can then be applied again to v , giving us the existence of $v' \in W$ such that vRv' and $\mathfrak{M}', v' \models \neg p_1 \wedge \diamond \square p_1$. This contradicts $\mathfrak{M}', v \models \square p_1$, and we are done. \square

Let $n \in \mathbb{N}$. We introduce the formula $\theta_n := \diamond(\square p_{n+1} \wedge \neg \mathbf{bd}_n)$ which is the antecedent of \mathbf{bd}_{n+1} , that is, we have $\mathbf{bd}_{n+1} = \theta_n \rightarrow p_{n+1}$. Equivalently we have $\neg \mathbf{bd}_{n+1} \equiv \theta_n \wedge \neg p_{n+1}$, so for a topological model (X, ν) , the following result will help transferring the falsity of \mathbf{bd}_n from X to $Y := \llbracket \sigma \rrbracket_{(X, \nu)}$.

Lemma 3.9. *For all $n \geq 1$, we have $\mathbf{wK4} \vdash \theta_n \rightarrow \sigma$.*

Proof. We proceed by induction on n . For $n = 1$ this is immediate since $\theta_1 \equiv \diamond(\square p_2 \wedge \diamond \square p_1 \wedge \neg p_1)$. Suppose that this holds for n . Let $\mathfrak{M} = (W, R, \nu)$ be a **wK4** Kripke model and $w \in W$, and suppose that $\mathfrak{M}, w \models \theta_{n+1}$. We have

$$\theta_{n+1} \equiv \diamond(\square p_{n+2} \wedge \theta_n \wedge \neg p_{n+1})$$

so there exists $u \in W$ such that wRu and $\mathfrak{M}, u \models \square p_{n+2} \wedge \theta_n \wedge \neg p_{n+1}$. In particular $\mathfrak{M}, u \models \theta_n$ so there exists $v \in W$ such that uRv and $\mathfrak{M}, v \models \square p_{n+1} \wedge \neg \mathbf{bd}_n$. By weak transitivity we have wR^+v . If $w = v$ then vRu and from $\mathfrak{M}, v \models \square p_{n+1}$ and $\mathfrak{M}, u \models \neg p_{n+1}$ we derive a contradiction. Therefore wRv , and it follows that $\mathfrak{M}, w \models \diamond(\square p_{n+1} \wedge \neg \mathbf{bd}_n)$, that is, $\mathfrak{M}, w \models \theta_n$; by the induction hypothesis, we then obtain $\mathfrak{M}, w \models \sigma$. By Kripke completeness this proves **wK4** $\vdash \theta_{n+1} \rightarrow \sigma$. \square

We are now ready to prove that the operations of shifting up and down have the desired properties.

Lemma 3.10. *Let (X, ν) be a topological model, Y a subspace of X such that $\llbracket \sigma \rrbracket_{(X, \nu)} \subseteq Y$ and $n \in \mathbb{N}$. If $X, \nu, x \models \neg \mathbf{bd}_{n+1}$ then $x \in Y$ and $Y, \nu_Y^\nabla, x \models \neg \mathbf{bd}_n$.*

Proof. By induction on n . For $n = 0$, suppose that $X, \nu, x \models \neg \mathbf{bd}_1$, i.e., $X, \nu, x \models \theta_0 \wedge \neg p_1$. Recall that $\sigma = \diamond^+(\neg p_1 \wedge \diamond \square p_1)$; therefore $x \in \llbracket \sigma \rrbracket_{(X, \nu)}$, so $x \in Y$ and the rest is immediate since $\neg \mathbf{bd}_0 \equiv \top$.

Suppose that it holds for n and assume that $X, \nu, x \models \neg \mathbf{bd}_{n+2}$, that is, $X, \nu, x \models \theta_{n+1} \wedge \neg p_{n+2}$. Since $n+1 \geq 1$ we can apply Lemma 3.9 to obtain $x \in \llbracket \sigma \rrbracket_{(X, \nu)}$ and thus $x \in Y$. Now consider an open neighbourhood U of x in Y , of the form $U = U' \cap Y$ with U' open. From $X, \nu, x \models \theta_{n+1}$ and

$x \in U'$ we obtain the existence of some $y \in U'$ such that $y \neq x$ and $X, \nu, y \models \Box p_{n+2} \wedge \neg \mathbf{bd}_{n+1}$. Then by the induction hypothesis we obtain $y \in Y$ and $Y, \nu_Y^\nabla, y \models \neg \mathbf{bd}_n$. There also exists some open neighbourhood V of y such that $V \setminus \{y\} \subseteq \nu(p_{n+2})$, and thus $Y \cap V \setminus \{y\} \subseteq \nu_Y^\nabla(p_{n+1})$. All in all we have $y \in Y \cap U'$ with $y \neq x$ and $Y, \nu_Y^\nabla, y \models \Box p_{n+1} \wedge \neg \mathbf{bd}_n$. Therefore $Y, \nu_Y^\nabla, x \models \theta_n$. We also have $x \notin \nu(p_{n+2}) = \nu_Y^\nabla(p_{n+1})$ so finally $Y, \nu_Y^\nabla, x \models \theta_n \wedge \neg p_{n+1}$ as desired. \square

Lemma 3.11. *Let X be a topological space, Y nowhere dense in X , ν a valuation on Y , $x \in Y$ and $n \in \mathbb{N}$. If $Y, \nu, x \models \neg \mathbf{bd}_n$ then $X, \nu_Y^\Delta, x \models \neg \mathbf{bd}_{n+1}$.*

Proof. By induction on n . For $n = 0$, suppose that $x \notin \widehat{\mathbf{d}}\mathbf{d}(\nu_Y^\Delta(p_1))$. Since $\nu_Y^\Delta(p_1) = X \setminus Y$, this means that $x \in \widehat{\mathbf{d}}\mathbf{d}Y$. Then there exists an open neighbourhood U of x such that $U \setminus \{x\} \subseteq \mathbf{d}Y$. We note that x is not isolated, otherwise $\{x\} \subseteq Y \subseteq \mathbf{Cl}(Y)$ with $\{x\}$ open, contradicting the fact that Y is nowhere dense. We then prove that $x \in \mathbf{Cl}(Y)$. Indeed, let V be an open neighbourhood of x . Since x is not isolated there exists $y \in U \cap V$ such that $y \neq x$. Thus $y \in \mathbf{d}Y$, and it follows that $V \cap Y \neq \emptyset$ as desired. Therefore $x \in \mathbf{Cl}(Y)$ and since $U \setminus \{x\} \subseteq \mathbf{d}Y \subseteq \mathbf{Cl}(Y)$ it follows that $U \subseteq \mathbf{Cl}(Y)$ with U non-empty, a contradiction since Y is nowhere dense. Therefore $x \in X \setminus \widehat{\mathbf{d}}\mathbf{d}Y = \widehat{\mathbf{d}}\mathbf{d}(X \setminus Y)$, and so $X, \nu_Y^\Delta, x \models \Diamond \Box p_1 \wedge \neg p_1$.

Now suppose that it holds for n , and assume that $Y, \nu, x \models \neg \mathbf{bd}_{n+1}$, that is, $Y, \nu, x \models \theta_n \wedge \neg p_{n+1}$. It is then immediate that $X, \nu_Y^\Delta, x \models \neg p_{n+2}$, so we have to show $X, \nu_Y^\Delta, x \models \theta_{n+1}$. Let U be an open neighbourhood of x . By assumption there exists $y \in Y \cap U \setminus \{x\}$ such that $Y, \nu, y \models \Box p_{n+1} \wedge \neg \mathbf{bd}_n$. Thus there exists an open set W such that $y \in Y \cap W$ and $Y \cap W \setminus \{y\} \subseteq \nu(p_{n+1})$. Now let $z \in W \setminus \{y\}$; if $z \in X \setminus Y$ we have $z \in \nu_Y^\Delta(p_{n+2})$, and otherwise $z \in W \cap Y \subseteq \nu(p_{n+1}) \subseteq \nu_Y^\Delta(p_{n+2})$. Hence $W \setminus \{y\} \subseteq \nu_Y^\Delta(p_{n+2})$. From $Y, \nu, y \models \neg \mathbf{bd}_n$ we also get $X, \nu_Y^\Delta, y \models \neg \mathbf{bd}_{n+1}$ by the induction hypothesis. All in all we have $y \in U \setminus \{x\}$ and $X, \nu_Y^\Delta, y \models \Box p_{n+2} \wedge \neg \mathbf{bd}_{n+1}$. Therefore $X, \nu_Y^\Delta, x \models \theta_{n+1}$, and this concludes the proof. \square

Theorem 3.12. *Let X be a topological space and $n \in \mathbb{N}$. We then have $\mathbf{mdim}(X) \leq n - 1$ if and only if $X \models \mathbf{bd}_n$.*

Proof. By induction on n . For $n = 0$ this is immediate. Suppose that this holds for n , and assume $\mathbf{mdim}(X) > n$. Then there exists some Y nowhere dense in X such that $\mathbf{mdim}(Y) > n - 1$. By the induction hypothesis we have $Y \not\models \mathbf{bd}_n$, so there exists a valuation ν over Y and $x \in Y$ such that $Y, \nu, x \models \neg \mathbf{bd}_n$. Then by Lemma 3.11 it follows that $X, \nu_Y^\Delta, x \models \neg \mathbf{bd}_{n+1}$, and thus $X \not\models \mathbf{bd}_{n+1}$.

Conversely, suppose that $X \not\models \mathbf{bd}_{n+1}$. Then there exists a valuation ν and $x \in X$ such that $X, \nu, x \models \neg \mathbf{bd}_{n+1}$. We define $Y := \llbracket \sigma \rrbracket_{\mathfrak{M}}$ and then by Lemma 3.10 we know that $x \in Y$ and $Y, \nu_Y^\nabla, x \models \neg \mathbf{bd}_n$. Thus $Y \not\models \mathbf{bd}_n$ and by the induction hypothesis we obtain $\mathbf{mdim}(Y) > n - 1$. By Lemma 3.8, Y is nowhere dense in X and therefore $\mathbf{mdim}(X) > n$. \square

3.2 Bounded depth and chains

In this section we aim to prove a variant of Theorem 3.6 that is meaningful for the d-semantics. To be more precise, we aim to find an analogue of item 2 involving validity in the d-semantics instead of the c-semantics, and an analogue of items 3 and 4 involving d-morphisms instead of interior maps. To meet the first goal, we introduce a variant of Jankov-Fine formulas. They are adapted from the *subframe formulas* presented in [CZ97, Sect. 9.4], and slightly reworked to be made more readable.

Definition 3.13. Let $\mathfrak{F} = (W, R)$ be a finite rooted Kripke frame. We write $W := \{w_i \mid 0 \leq i \leq n\}$ where w_0 is a root. The *subframe formula* $\alpha_{\mathfrak{F}}$ associated to \mathfrak{F} is the conjunction of the following formulas:

1. q_0 ,

2. $\Box^+ \neg(q_i \wedge q_j)$ for all $i, j \in [0, n]$ such that $i < j$,
3. $\Box^+(q_i \rightarrow \Diamond q_j)$ for all $i, j \in [0, n]$ such that $w_i R w_j$,
4. $\Box^+(q_i \rightarrow \neg \Diamond q_j)$ for all $i, j \in [0, n]$ such that *not* $w_i R w_j$.

In words, $\alpha_{\mathfrak{F}}$ is just like $\chi_{\mathfrak{F}}$ apart from two points. First, the terms of the conjunction are under the scope of \Box^+ instead of \Box , but this is merely an adjustment to account for the fact that we are no longer restricted to reflexive frames. Second, the term $\Box^+ \bigvee_{i=0}^n q_i$ is not present. This means that satisfiability of $\alpha_{\mathfrak{F}}$ on a space X encodes the presence of the “pattern” given by \mathfrak{F} within *some* subspace of X , instead of only X itself. This is made precise by the following result.

Proposition 3.14. *Let X be a topological space. Then $\alpha_{\mathfrak{F}}$ is satisfiable on X if and only if \mathfrak{F} is a d -morphic image of some subspace of X .*

Proof. Suppose that there exists a set $Y \subseteq X$ and a surjective d -morphism $f : Y \rightarrow W$. We define a valuation ν by setting $\nu(q_i) := f^{-1}(w_i)$ for all $i \in [0, n]$. Since f is surjective, there exists $x \in \nu(q_0)$. We prove that $X, \nu, x \models \alpha_{\mathfrak{F}}$.

1. $X, \nu, x \models q_0$ by construction.
2. If $0 \leq i < j \leq n$ then $X, \nu, x \models \Box^+ \neg(q_i \wedge q_j)$ since any point in X has at most one image by f .
3. Suppose that $w_i R w_j$ and let $y \in Y$ such that $f(y) = w_i$. Then $y \in f^{-1}[R^{-1}\{w_j\}]$, so $y \in d_Y f^{-1}(w_j)$, that is, $y \in d f^{-1}(w_j) \cap Y$. Hence $X, \nu, y \models \Diamond q_j$. Since X is open it follows that $X, \nu, x \models \Box^+(q_i \rightarrow \Diamond q_j)$.
4. Suppose that *not* $w_i R w_j$ and let $y \in Y$ such that $f(y) = w_i$. Then $y \notin f^{-1}[R^{-1}\{w_j\}]$, so $y \notin d_Y f^{-1}(w_j)$, that is, $y \notin d f^{-1}(w_j) \cap Y$. Hence $X, \nu, y \models \neg \Diamond q_j$. Since X is open it follows that $X, \nu, x \models \Box^+(q_i \rightarrow \neg \Diamond q_j)$.

Conversely, suppose that there exists a valuation ν and a point $x \in X$ such that $X, \nu, x \models \alpha_{\mathfrak{F}}$. Since \Box^+ commutes with \wedge , we have $X, \nu, x \models \Box^+ \theta$ where

$$\theta := \left(\bigwedge_{0 \leq i < j \leq n} \neg(q_i \wedge q_j) \right) \wedge \left(\bigwedge_{\substack{0 \leq i, j \leq n \\ w_i R w_j}} q_i \rightarrow \Diamond q_j \right) \wedge \left(\bigwedge_{\substack{0 \leq i, j \leq n \\ \text{not } w_i R w_j}} q_i \rightarrow \neg \Diamond q_j \right).$$

Thus, there exists an open neighbourhood U of x such that $U \subseteq \llbracket \theta \rrbracket_{X, \nu}$. Then, given $y \in U$, if there exists some $i \in [0, n]$ such that $X, \nu, y \models q_i$, it is unique and we set $f(y) := w_i$. We define $Y := \text{Dom}(f)$ and prove that $f : Y \rightarrow W$ meets the requirements. First, we show that f is surjective. Let $i \in [0, n]$. Since \mathfrak{F} is rooted in w_0 there exist $w_{j_1}, \dots, w_{j_m} \in W$ such that $j_1 = 0$, $j_m = i$ and $w_{j_k} R w_{j_{k+1}}$ for all $k \in [0, m-1]$. Thus, for all $k \in [0, m-1]$ we have $U \subseteq \llbracket q_{j_k} \rightarrow \Diamond q_{j_{k+1}} \rrbracket_{X, \nu}$, that is, $U \cap \llbracket q_{j_k} \rrbracket_{X, \nu} \subseteq d \llbracket q_{j_{k+1}} \rrbracket_{X, \nu}$. Since $x \in U \cap \llbracket q_0 \rrbracket_{X, \nu}$, a straightforward induction yields $U \cap \llbracket q_{j_k} \rrbracket_{X, \nu} \neq \emptyset$ for all $k \in [0, m]$. Thus, there exists $y \in U$ such that $X, \nu, y \models q_i$, and consequently $f(y) = w_i$.

We then show that f is a d -morphism. If $j \in [0, n]$, we show that $f^{-1}[R^{-1}\{w_j\}] = d_Y f^{-1}(w_j)$. For consider $y \in f^{-1}[R^{-1}\{w_j\}]$. Writing $w_i := f(y)$, this means that $w_i R w_j$. It follows that $X, \nu, y \models q_i \rightarrow \Diamond q_j$ and thus $y \in d(\nu(q_j))$, so finally $y \in d(\nu(q_j)) \cap Y = d_Y(\nu(q_j))$. Conversely, suppose that $y \notin f^{-1}[R^{-1}\{w_j\}]$. Then *not* $w_i R w_j$, and thus $X, \nu, y \models q_i \rightarrow \neg \Diamond q_j$. Hence $y \notin d(\nu(q_j))$, and therefore $y \notin d_Y(\nu(q_j))$. Since W is finite and R^{-1} and d commute with \cup , it follows that $f^{-1}[R^{-1}A] = d_Y f^{-1}[A]$ for all $A \subseteq W$. This concludes the proof. \square

Now, our goal is to connect subframe formulas to modal dimension by showing that given a space X , we have $X \models \text{bd}_n$ iff $X \models \neg \alpha_{\mathfrak{F}}$ for all $n+1$ -chains \mathfrak{F} . We can see that we are now

quantifying over all chains instead of only the reflexive ones – which makes sense since we have moved to the weakly transitive setting. Here the challenging implication is

$$X \models \bigwedge \{ \neg \alpha_{\mathfrak{F}} \mid \mathfrak{F} \text{ is a } n+1\text{-chain} \} \implies X \models \text{bd}_n$$

and for convenience we will instead prove

$$X \models \bigwedge \{ \neg \alpha_{\mathfrak{F}} \mid \mathfrak{F} \text{ is a } n+1\text{-chain} \} \implies X \models \text{bd}_n^+$$

(we will see that this is sufficient). It will however be easier to work with Kripke frames, which is possible if we move to the syntactic level, and then use Kripke completeness. However, for a reason that will soon be clear, attempting to show

$$\mathbf{wK4} + \{ \neg \alpha_{\mathfrak{F}} \mid \mathfrak{F} \text{ is a } n+1\text{-chain} \} \vdash \text{bd}_n^+$$

will not succeed. At some point in the proof we will need the infinite path modality (see Section 2.5), and this is why we need to work within the stronger logic $\mathbf{wK4}^*$. Thus, the claim we are going to prove is

$$\mathbf{wK4}^* + \{ \neg \alpha_{\mathfrak{F}} \mid \mathfrak{F} \text{ is a } n+1\text{-chain} \} \vdash \text{bd}_n^+. \quad (1)$$

Unfortunately, while $\mathbf{wK4}^*$ is Kripke complete, we do not know whether this is the case for $\mathbf{wK4}^* + \{ \neg \alpha_{\mathfrak{F}} \mid \mathfrak{F} \text{ is a } n+1\text{-chain} \}$. Here the solution comes from the observation that only finitely many instances of the $\neg \alpha_{\mathfrak{F}}$'s are needed to derive bd_n^+ . Indeed we know that if bd_n^+ is refuted in a frame $\mathfrak{F}_0 = (W, R)$, then we have a proper path (w_0, \dots, w_n) of length $n+1$ in (W, R^+) . From this proper path we can construct a $n+1$ -chain \mathfrak{F}^1 , from which we can then derive an instantiation of the q_i 's satisfying $\alpha_{\mathfrak{F}}$ on \mathfrak{F}_0 . This instantiation is a tuple of formulas essentially describing the structure of \mathfrak{F} (i.e., which points are reflexive) along with some information retrieved from the falsity of bd_n — for this reason it is called the *refutation tuple* associated to \mathfrak{F} . Since there are finitely many $n+1$ -chains and each chain has only one refutation tuple, we end up as intended with finitely many instances of subframe formulas. Therefore (1) can be seen as an implication in $\mathbf{wK4}^*$.

This also explains why we work with the infinite path modality: if \mathfrak{F} is a $n+1$ -chain containing reflexive points, then $\alpha_{\mathfrak{F}}$ contains subformulas of the form $\Box^+(q_i \rightarrow \Diamond q_i)$, which imposes a self-referential condition on the formula that is substituted for q_i . When the infinite path modality is available, constructing such a formula is very easy.

Definition 3.15. Let $n \in \mathbb{N}$ and \mathfrak{F} be a $n+1$ -chain, whose worlds are denoted w_0, \dots, w_n . Define by downward recursion on $k \in [0, n]$ the formulas ψ_k and φ_k as follows:

$$\psi_k := \begin{cases} \bigwedge_{1 \leq i \leq n} \Box^+ p_i & \text{if } k = n \\ \Diamond \varphi_{k+1} \wedge \neg p_{n-k} \wedge \bigwedge_{n-k+1 \leq i \leq n} \Box^+ p_i & , \end{cases}$$

$$\varphi_k := \begin{cases} \Diamond^* \psi_k & \text{if } w_k \text{ is reflexive} \\ \psi_k \wedge \neg \Diamond \psi_k & . \end{cases}$$

The *refutation tuple* associated to \mathfrak{F} is then $t_{\mathfrak{F}} := (\varphi_0, \dots, \varphi_n)$.

Lemma 3.16. Let us write $\bar{q} := (q_0, \dots, q_n)$. We then have

$$\mathbf{wK4}^* \vdash \bigwedge \{ \Box^+ \neg \alpha_{\mathfrak{F}}[t_{\mathfrak{F}}/\bar{q}] \mid \mathfrak{F} \text{ is a } n+1\text{-chain} \} \rightarrow \text{bd}_n^+.$$

¹Warning: this $n+1$ -chain is not necessarily the one given by $\{w_0, \dots, w_n\}$!

Proof. By completeness (see Section 2.5), it suffices to consider a weakly transitive Kripke model $\mathfrak{M} = (W, R, \nu)$ and $w \in W$ such that $\mathfrak{M}, w \models \Box^+ \neg \alpha_{\mathfrak{F}}[t_{\mathfrak{F}}/\bar{q}]$ for all $n+1$ -chains \mathfrak{F} , and to prove that $\mathfrak{M}, w \models \text{bd}_n^+$. For suppose not. Then there exist w_0, \dots, w_n with $w_0 = w$ such that $\mathfrak{M}, w_0 \models \neg p_n$ and $\mathfrak{M}, w_n \models \Box^+ p_1$ and for all $k \in [1, n-1]$, $w_k R^+ w_{k+1}$ and $\mathfrak{M}, w_k \models \neg p_{n-k} \wedge \Box^+ p_{n-k+1}$. We introduce the $n+1$ -chain \mathfrak{F} with worlds u_0, \dots, u_n and such that for all $i \in [0, n]$, the world u_i is reflexive iff $\mathfrak{M}, w_i \models \Diamond^* \psi_i$. Then, let ψ_0, \dots, ψ_n and $\varphi_0, \dots, \varphi_n$ be the formulas defined in Definition 3.15 relatively to \mathfrak{F} .

We now construct by downward recursion a sequence of worlds (w'_0, \dots, w'_n) satisfying $w_k R^+ w'_k$ and $\mathfrak{M}, w'_k \models \varphi_k$ for all $k \in [0, n]$ (see Figure 3). First, since $\mathfrak{M}, w_{n-i+1} \models \Box^+ p_i$ for all $i \in [1, n]$ and \mathfrak{M} is weakly transitive, we have $\mathfrak{M}, w_n \models \psi_n$. If $\mathfrak{M}, w_n \models \Diamond^* \psi_n$ then u_n is reflexive, hence $\varphi_n = \Diamond^* \psi_n$, so we set $w'_n := w_n$ and we are done. Otherwise, there exists a path $v_1 R \dots R v_m$ with $v_1 = w_n$ and such that $\mathfrak{M}, v_i \models \psi_n$ for all $i \in [1, m-1]$ and $\mathfrak{M}, v_m \not\models \Diamond \psi_n$. In this case we also have $\varphi_n = \psi_n \wedge \neg \Diamond \psi_n$. By weak transitivity we obtain $w_0 R^+ v_m$, so we set $w'_0 := v_m$ and we are done.

Next, let $k \in [0, n-1]$. We have $w_{k+1} R^+ w'_{k+1}$ and $\mathfrak{M}, w'_{k+1} \models \varphi_{k+1}$ by the induction hypothesis. Then from $w_k R^+ w_{k+1}$ we obtain $w_k R^+ w'_{k+1}$, but $w_k = w'_{k+1}$ is impossible since $\mathfrak{M}, w_k \models \neg p_{n-k}$, while $\mathfrak{M}, w_{k+1} \models \Box^+ p_{n-k}$ yields $\mathfrak{M}, w'_{k+1} \models p_{n-k}$. Hence $w_k R w'_{k+1}$, and since $\mathfrak{M}, w'_{k+1} \models \varphi_{k+1}$, we obtain $\mathfrak{M}, w_k \models \Diamond \varphi_{k+1}$. We also have $\mathfrak{M}, w_k \models \neg p_{n-k}$ and $\mathfrak{M}, w_k \models \Box^+ p_i$ for all $i \in [n+1-k, n+1]$ by the same argument as above. Therefore $\mathfrak{M}, w_k \models \psi_k$ and the construction of w'_k is analogous to the case $k=0$.

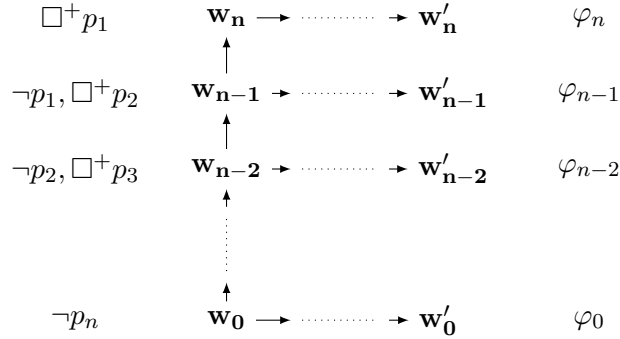


Figure 3: The construction of the desired chain

We now show that $\mathfrak{M}, w'_0 \models \alpha_{\mathfrak{F}}[t_{\mathfrak{F}}/\bar{q}]$.

1. We know that $\mathfrak{M}, w'_0 \models \varphi_0$.
2. If $0 \leq i < j \leq n$ then $n-j+1 \leq n-i$, and so $\models \varphi_j \rightarrow \Box^+ p_{n-i}$ whereas $\models \varphi_i \rightarrow \neg p_{n-i}$. Hence $\mathfrak{M}, w'_0 \models \Box^+ \neg(\varphi_i \wedge \varphi_j)$.
3. Suppose that $0 \leq i \leq j \leq n$. In case $i = j$, we also suppose that u_i is reflexive in \mathfrak{F} . Then $\varphi_i = \Diamond^* \psi_i$. It follows that $\mathfrak{M}, w'_0 \models \Box^+(\varphi_i \rightarrow \Diamond \varphi_i)$, as an immediate consequence of the fixpoint axiom.
In case $i < j$ we show that $\mathfrak{M}, w'_0 \models \Box^+(\varphi_i \rightarrow \Diamond \varphi_j)$. For suppose that $w'_0 R^+ v$ and $\mathfrak{M}, v \models \varphi_i$. Since for all $k \in [i+1, j]$ we have $\models \varphi_k \rightarrow \Diamond \varphi_{k+1}$ we obtain the existence of a path $v_1 R \dots R v_m$ with $v_1 = v$ and such that $\mathfrak{M}, v_m \models \varphi_j$. By weak transitivity it follows that $v R^+ v_m$ and by 2 we cannot have $v = v_m$, so $v R v_m$, and this proves that $\mathfrak{M}, v \models \Diamond \varphi_j$.
4. Suppose that $0 \leq j \leq i \leq n$. In case $i = j$, we also suppose that u_i is irreflexive in \mathfrak{F} . Then $\varphi_i = \psi_i \wedge \neg \Diamond \psi_i$. It is then clear that $\mathfrak{M}, w'_0 \models \Box^+(\varphi_i \rightarrow \neg \Diamond \varphi_i)$.
In case $j < i$ we show that $\mathfrak{M}, w'_0 \models \Box^+(\varphi_i \rightarrow \neg \Diamond \varphi_j)$. For suppose that $w'_0 R^+ v$ and

$\mathfrak{M}, v \models \varphi_i$, and that vRu with $\mathfrak{M}, u \models \varphi_j$. Then since $j < i$ we have $n - i + 1 \leq n - j$ and so $\mathfrak{M}, v \models \Box^+ p_{n-j}$, whereas $\mathfrak{M}, u \models \neg p_{n-j}$, a contradiction. This proves the claim.

Since $wR^+w'_0$, it follows that $\mathfrak{M}, w \not\models \Box^+ \neg \alpha_{\mathfrak{F}}[t_{\mathfrak{F}}/\bar{q}]$, a contradiction. This concludes the proof. \square

Lemma 3.17. *We have $\mathbf{wK4}^* + \{\neg \alpha_{\mathfrak{F}} \mid \mathfrak{F} \text{ is a } n+1\text{-chain}\} \vdash \mathbf{bd}_n^+$.*

Proof. Let \mathfrak{F}' be a $n+1$ -chain. By the rules of necessitation and substitution we have

$$\mathbf{wK4}^* + \{\neg \alpha_{\mathfrak{F}} \mid \mathfrak{F} \text{ is a } n\text{-chain}\} \vdash \Box^+ \neg \alpha_{\mathfrak{F}}[t_{\mathfrak{F}'}/\bar{q}]$$

and we conclude by Lemma 3.16. \square

Theorem 3.18. *Let X be a topological space and $n \in \mathbb{N}$. The following conditions are equivalent:*

1. $\mathbf{mdim}(X) \leq n - 1$,
2. $X \models \neg \alpha_{\mathfrak{F}}$ for all $n+1$ -chains \mathfrak{F} ,
3. No $n+1$ -chain is a d -morphic image of a subspace of X .

Proof. From 3 to 2, suppose that $X \not\models \neg \alpha_{\mathfrak{F}}$ for some $n+1$ -chain \mathfrak{F} , that is, $\alpha_{\mathfrak{F}}$ is satisfiable on X . Then by Proposition 3.14, there exists a subspace Y of X and a surjective d -morphism from Y to some \mathfrak{F} .

From 1 to 3, suppose that there exists a surjective d -morphism f from an open set $Y \subseteq X$ to some $n+1$ -chain \mathfrak{F} . In particular, f is an interior map, and it can be seen as an interior map from Y to the reflexive $n+1$ -chain (since all $n+1$ -chains induce the same topological space). Then from Theorem 3.6, we obtain $\mathbf{mdim}(X) > n - 1$.

From 2 to 1, suppose that $X \models \neg \alpha_{\mathfrak{F}}$ for all $n+1$ -chains \mathfrak{F} . In other words,

$$X \models \mathbf{wK4}^* + \{\neg \alpha_{\mathfrak{F}} \mid \mathfrak{F} \text{ is a } n+1\text{-chain}\}$$

and from Lemma 3.17 it follows that $X \models \mathbf{bd}_n^+$. Then by Proposition 2.8 we obtain $X \models^+ \mathbf{bd}_n$, and therefore $\mathbf{mdim}(X) \leq n - 1$ by Theorem 3.6. \square

Remark 3.19. Items 3 and 4 of Theorem 3.6 may suggest that, similarly:

- $\mathbf{mdim}(X) \leq n - 1$ iff no $n+1$ -chain is a d -morphic image of X ,
- $\mathbf{mdim}(X) \leq n - 1$ iff no $n+1$ -chain is a d -morphic image of some open subspace of X .

This differs from $1 \iff 3$ above on the quantification over subspaces of X , that is, we only consider X itself in the first case, and its *open* subspaces in the second case. Yet this is not the case in general: if we take $X := \{0, 1, 2, 3\}$ and $\tau := \{\emptyset, \{1, 2\}, \{3\}, X\}$ we find that $\mathbf{mdim}(X) > 0$ – yet no 2-chain is a d -morphic image of X , nor any open subspace of X .

3.3 Bounded depth and derivative

We conclude this section with a last, modest result: we show that taking the derivative of a space decreases its modal dimension by at most 1. This proposition can be split into two parts: on the one hand, given a space X we have $\mathbf{mdim}(X) \geq \mathbf{mdim}(dX) + 1$; on the other hand, we have $\mathbf{mdim}(dX) \leq \mathbf{mdim}(X)$. Note that this last claim stems from the stronger property that $\mathbf{mdim}(Y) \leq \mathbf{mdim}(X)$ whenever Y is a subspace of X . Though this result was already proved in [BBLBvM17] by means of algebraic arguments, it is worth observing that we can recover it with the techniques displayed in Section 3.1.

Definition 3.20. Let $n \in \mathbb{N}$, X a space, $Y \subseteq X$ and ν a valuation on Y with domain P_n . The *Y-stuffing* ν_Y° of ν is the valuation on X with domain P_n defined by $\nu_Y^\circ(p_k) := \nu(p_k) \cup (X \setminus Y)$ for all $k \in [1, n]$.

Lemma 3.21. Let (X, ν) be a topological model, $x \in Y \subseteq X$ and $n \in \mathbb{N}$. If $Y, \nu, x \models \neg \mathbf{bd}_n$ then $X, \nu_Y^\circ, x \models \neg \mathbf{bd}_n$.

Proof. By induction on n . For $n = 0$ this is trivial since $\neg \mathbf{bd}_0 \equiv \top$. Suppose that it holds for n , and assume that $Y, \nu, x \models \neg \mathbf{bd}_{n+1}$, i.e., $Y, \nu, x \models \theta_n \wedge \neg p_{n+1}$. Let U be an open neighbourhood of x . By assumption there exists $y \in U \cap Y \setminus \{x\}$ such that $Y, \nu, y \models \Box p_{n+1} \wedge \neg \mathbf{bd}_n$. Thus there exists an open neighbourhood V of y such that $V \cap Y \setminus \{y\} \subseteq \nu(p_{n+1})$. If $z \in V \setminus \{y\}$ we have either $z \in Y$, in which case $z \in \nu(p_{n+1}) \subseteq \nu_Y^\circ(p_{n+1})$, or $z \notin Y$ which yields $z \in \nu_Y^\circ(p_{n+1})$. This proves $X, \nu_Y^\circ, y \models \Box p_{n+1}$, and we also have $X, \nu_Y^\circ, y \models \neg \mathbf{bd}_n$ by the induction hypothesis. Since $y \in U \setminus \{x\}$ we obtain $X, \nu_Y^\circ, x \models \theta_n$ and we are done. \square

Proposition 3.22 ([BBLBvM17, Lemma 3.3]). Let X be a topological space and $Y \subseteq X$ a subspace of X . Then $\mathbf{mdim}(Y) \leq \mathbf{mdim}(X)$.

Proof. If $\mathbf{mdim}(X) = \infty$ this is obvious, so assume that $n := \mathbf{mdim}(X)$ is finite. Suppose that $\mathbf{mdim}(Y) > n$. Then by Theorem 3.12 we have $Y \not\models \mathbf{bd}_{n+1}$, that is, there exists a valuation ν and $x \in dX$ such that $Y, \nu, x \models \neg \mathbf{bd}_{n+1}$. Then by Lemma 3.21 we obtain $X, \nu_Y^\circ, x \models \neg \mathbf{bd}_{n+1}$ and thus $X \not\models \mathbf{bd}_{n+1}$. By Theorem 3.12 again we get $\mathbf{mdim}(X) > n$, a contradiction. Therefore $\mathbf{mdim}(Y) \leq n$. \square

Lemma 3.23. Let X be a topological space. If $\mathbf{mdim}(dX) \leq n$ then $\mathbf{mdim}(X) \leq n + 1$.

Proof. By contraposition, suppose that $\mathbf{mdim}(X) > n + 1$. Then $X \not\models \mathbf{bd}_{n+2}$ by Theorem 3.12, so there exists a valuation ν and $x \in X$ such that $X, \nu, x \models \neg \mathbf{bd}_{n+2}$. It is clear that $\llbracket \sigma \rrbracket_{(X, \nu)} \subseteq dX$ so we can apply Lemma 3.10 to $Y := dX$ and obtain $x \in dX$ and $dX, \nu_Y^\circ, x \models \neg \mathbf{bd}_{n+1}$. Hence $dX \not\models \neg \mathbf{bd}_{n+1}$ and it follows that $\mathbf{mdim}(dX) > n$. \square

Theorem 3.24. We have $\mathbf{mdim}(X) - 1 \leq \mathbf{mdim}(dX) \leq \mathbf{mdim}(X)$ for all topological spaces X , with the convention that $\infty - 1 = \infty$.

Proof. If $\mathbf{mdim}(X)$ is finite then this is an immediate consequence of Proposition 3.22 and Lemma 3.23. If $\mathbf{mdim}(X) = \infty$, we show that $\mathbf{mdim}(dX) = \infty$ too. If not, we have $\mathbf{mdim}(dX) \leq n$ for some n , so $\mathbf{mdim}(X) \leq n + 1$ by Lemma 3.23, a contradiction. This concludes the proof. \square

4 Topological directedness

In this section we study the logics of directness. In the c-semantics, the only logic of reference for directness is **S4.2**, which is usually defined as **S4.2** := **S4** + **sdir**. Recall that **S4.2** defines the class of strongly directed **S4** frames, as well as the class of extremally disconnected spaces. However, as soon as we dive below **S4**, things become more complex, and **sdir** is not the only axiom that matters. Indeed, in [CZ97, Sect. 3.5] we also find the logic **K4.2** := **K4** + **dir** which defines the class of transitive directed frames. We will thus address the following three logics:

$$\begin{aligned} \mathbf{wK4d} &:= \mathbf{wK4} + \mathbf{dir}, \\ \mathbf{wK4sd} &:= \mathbf{wK4} + \mathbf{sdir}, \\ \mathbf{wK4.2} &:= \mathbf{wK4} + \mathbf{sdir}^+. \end{aligned}$$

Obviously, Proposition 2.8 immediately provides the semantics of **wK4.2**, so no surprise will come from this logic. It deserves some attention nonetheless, because it is unknown whether it is topologically complete. First we recall some results of Kripke completeness.

Proposition 4.1. *The logic $\mathbf{wK4d}$ defines the class of directed $\mathbf{wK4}$ frames. The logic $\mathbf{wK4sd}$ defines the class of strongly directed $\mathbf{wK4}$ frames. The logic $\mathbf{wK4.2}$ defines the class of $\mathbf{wK4}$ frames \mathfrak{F} such that \mathfrak{F}^+ is strongly directed, and also the class of ED spaces. In addition, these three logics are Kripke complete.*

Proof. For $\mathbf{wK4d}$ and $\mathbf{wK4sd}$ see [CZ97, Sect. 3.5]. For $\mathbf{wK4.2}$ this is a direct consequence of Theorem 2.21 and Proposition 2.8. Finally, Kripke completeness follows from the fact that $\mathbf{w4}$, \mathbf{dir} , \mathbf{sdir} and \mathbf{sdir}^+ are Sahlqvist formulas [BdRV01, Sect. 3.6 & 5.6]. \square

Corollary 4.2. *The logic $\mathbf{wK4sd}$ is an extension of $\mathbf{wK4d}$, and the logic $\mathbf{wK4d}$ is an extension of $\mathbf{wK4.2}$.*

Proof. This stems from Kripke completeness. Indeed, it is clear that if \mathfrak{F} is strongly directed then \mathfrak{F} is directed, and that if \mathfrak{F} is directed then \mathfrak{F}^+ is strongly directed. \square

Since $\mathbf{wK4d}$ and $\mathbf{wK4sd}$ are very similar, it only makes sense to study the two together. So, throughout this section, we will continuously make progress toward understanding both logics. Because their semantics is quite convoluted, we find it appropriate to first give a brief description of the underlying thought process. Let us discuss $\mathbf{wK4sd}$, which will be the simplest of the two. First, as a consequence of Corollary 4.2, we see that every $\mathbf{wK4sd}$ space is ED. Yet the converse is not true in general, as witnessed by the ED spaces X_1 and X_2 depicted in Figure 4, which falsify $\mathbf{wK4sd}$.



Figure 4: Four spaces

However, we also realized that “most” ED spaces are $\mathbf{wK4sd}$ spaces. Upon closer examination, the reason why $\mathbf{wK4sd}$ is invalid on X_1 and X_2 turned out to be the presence of the subspaces X'_1 and X'_2 . Spaces that do not contain these patterns will be called *strongly dense-in-itself* and, as we will prove see in Proposition 4.14, strongly dense-in-itself ED spaces are well-behaved enough to validate $\mathbf{wK4sd}$. On the other hand, we see that X'_1 and X'_2 are not always problematic, since they are $\mathbf{wK4sd}$ spaces themselves. So we will prove that, in the general case, one can break any $\mathbf{wK4sd}$ space into two open subspaces: a strongly dense-in-itself ED space, and a space called *almost discrete* which gathers all the subspaces of the form X'_1 and X'_2 . This decomposition will then give us a complete characterization of $\mathbf{wK4sd}$ spaces. Below we define these properties formally.

Definition 4.3. Let X be a topological space. An open set U is said to be *atomic* if it is non-empty and there exists no open set V such that $\emptyset \subset V \subset U$.² Then X is called:

- *almost discrete* if every point has an atomic open neighbourhood U such that $|U| \leq 2$,

²Here $A \subset B$ means that $A \subseteq B$ and $A \neq B$.

- *strongly dense-in-itself* if for every non-empty open set U we have $|U| \geq 3$.

Remark 4.4. To understand how this definition relates to discrete and dense-in-itself spaces, observe that X is discrete iff every point has an (atomic) open neighbourhood U such that $|U| \leq 1$, and X dense-in-itself iff for every non-empty open set U we have $|U| \geq 2$. The link is then clear, and the names are consistent since any discrete space is almost discrete, and any strongly dense-in-itself space is dense-in-itself. See Figure 5 for a visual depiction.

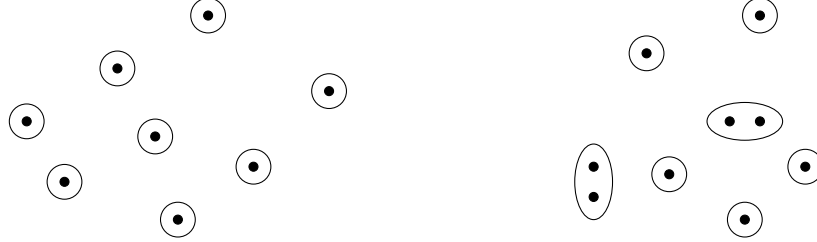


Figure 5: A discrete space (left) and an almost discrete space (right)

The case of **wK4d** is more complex. In addition to its almost discrete and strongly dense-in-itself components, every **wK4d** space will also contain a subspace with *hyperdense discreteness*. An example of such space is the space X_1 of Figure 4 (on which **wK4d** is indeed valid), but their general shape is much more intricate.

Definition 4.5. Let X be a topological space. We say that X has *hyperdense discreteness* if there exists an open discrete subspace Y of X such that for all $x \in X$ and $A \subseteq Y$, either $A \cup \{x\}$ is open or $(Y \setminus A) \cup \{x\}$ is open.

Example 4.6. Let Y and Z be two disjoint sets. We consider an *ultrafilter* \mathfrak{U} on Y , that is, a set $\mathfrak{U} \subseteq \mathcal{P}(Y)$ such that:

- $\emptyset \notin \mathfrak{U}$,
- $A \in \mathfrak{U}$ and $A \subseteq B \subseteq Y$ implies $B \in \mathfrak{U}$,
- $A, B \subseteq \mathfrak{U}$ implies $A \cap B \in \mathfrak{U}$,
- for all $A \subseteq Y$, we have $A \in \mathfrak{U}$ or $Y \setminus A \in \mathfrak{U}$.

Note that there always exist an ultrafilter on Y [DP02, Chapter 10]. We then endow the set $X := Y \cup Z$ with the topology $\tau := \mathcal{P}(Y) \cup \{A \cup B \mid A \in \mathfrak{U} \text{ and } B \subseteq Z\}$. We let the reader check that τ is indeed a topology. By construction, Y is open and discrete. Let $x \in X$ and $A \subseteq Y$. We have either $A \in \mathfrak{U}$ or $Y \setminus A \in \mathfrak{U}$, and thus either $A \cup \{x\}$ is open or $(Y \setminus A) \cup \{x\}$ is open. Therefore X has hyperdense discreteness.

The following observation, though elementary, will be helpful in the sequel.

Lemma 4.7. *We have $\mathbf{wK4} \vdash (\Box\Box\perp \wedge \Diamond\top) \rightarrow \Diamond\Box\perp$.*

Proof. This can be proved by Kripke completeness. Indeed, assume that \mathfrak{M} is weakly transitive and that $\mathfrak{M}, w \vDash \Box\Box\perp \wedge \Diamond\top$. Then there exists u such that wRu . Since $\mathfrak{M}, w \vDash \Box\Box\perp$ we also have $\mathfrak{M}, u \vDash \Box\perp$. Therefore $\mathfrak{M}, w \vDash \Diamond\Box\perp$. \square

Proposition 4.8. *The logic $\mathbf{wK4} + (\Diamond p \rightarrow \Box p) + \Box\Box\perp$ defines the class of spaces that have hyperdense discreteness.*

Proof. Suppose that X has hyperdense discreteness. Then there exists Y as in Definition 4.5. Since Y is open and discrete we have $Y \subseteq \widehat{d}\emptyset$. Let $x \in X$. Applying the assumption to $A := Y$, we obtain that $\{x\}$ is open or $\{x\} \cup Y$ is open. In both cases it follows that $x \in \widehat{d}Y$, whence $x \in \widehat{d}\widehat{d}\emptyset$. Therefore $X \models \square\square\perp$. Now let $x \in X$ and $A \subseteq X$ and assume that $x \in dA$. By assumption, either $A \cup \{x\}$ is open or $(Y \setminus A) \cup \{x\}$ is open. However the second case contradicts $x \in dA$, as $((Y \setminus A) \cup \{x\}) \cap A \subseteq \{x\}$. Thus $A \cup \{x\}$ is open, and we have $(A \cup \{x\}) \setminus \{x\} = A \setminus \{x\} \subseteq A$. This proves that $X \models \diamond p \rightarrow \square p$.

Now suppose that $X \models \diamond p \rightarrow \square p$ and $X \models \square\square\perp$. Let Y be the set of all isolated points of X , which by construction is open and discrete. Let $x \in X$ and $A \subseteq Y$. If $x \in Y$ then $A \cup \{x\}$ is immediately open. Otherwise $x \in dX$. By Lemma 4.7 we have $\mathbf{wK4} + \square\square\perp \vdash \diamond\top \rightarrow \diamond\square\perp$, so $X \models \diamond\top \rightarrow \diamond\square\perp$, and it follows that $x \in d\widehat{d}\emptyset$, that is, $x \in dY$. Let $B := Y \setminus A$. Note that A and B are open, as $A, B \subseteq Y$ and Y is open and discrete. We then have $dY = d(A \cup B) = dA \cup dB \subseteq \widehat{d}A \cup \widehat{d}B$ by assumption. If $x \in \widehat{d}A$, there exists an open neighbourhood U of x such that $U \setminus \{x\} \subseteq A$, and thus $A \cup \{x\} = A \cup U$ is open. Likewise, if $x \in \widehat{d}B$ then $B \cup \{x\} = (Y \setminus A) \cup \{x\}$ is open. This proves that X has hyperdense discreteness. \square

Remark 4.9. As a consequence of Lemma 4.7, we have $\mathbf{wK4} + \square\square\perp \vdash \neg\diamond\top \vee \diamond\square\perp$, whence $\mathbf{wK4} + \square\square\perp \vdash \diamond^+\square\perp$ since $\neg\diamond\top \equiv \square\perp$. Thus, whenever $X \models \square\square\perp$, we have $X = \text{Cl}(\widehat{d}\emptyset)$, and so the set of isolated points of X is dense in X (and of course discrete). This explains the name ‘‘hyperdense discreteness’’.³

Lemma 4.10.

1. If X is almost discrete then $X \models \mathbf{wK4sd}$.
2. If X has hyperdense discreteness and $X \models \mathbf{wK4sd}$ then X is discrete.

Proof. 1. Suppose that X is almost discrete. Let $x \in X$ and $A \subseteq X$ with $x \in d\widehat{d}A$. Let U be an atomic open neighbourhood of x such that $|U| \leq 2$. If $U = \{x\}$ then $x \in \widehat{d}dA$ is immediate. Otherwise we have $U = \{x, y\}$ with $x \neq y$. Then $y \in \widehat{d}A$ by assumption, and since U is atomic this entails $U \setminus \{y\} \subseteq A$, that is, $x \in A$. Hence $y \in dA$ – again because U is atomic – and this proves $x \in \widehat{d}dA$.

2. First, after substituting \perp for p in sdir , we get $\mathbf{wK4sd} \vdash \diamond\square\perp \rightarrow \square\diamond\perp$, which boils down to $\mathbf{wK4sd} \vdash \diamond\square\perp \rightarrow \square\perp$. By Lemma 4.7, we also have $\mathbf{wK4sd} \vdash (\square\square\perp \wedge \diamond\top) \rightarrow \diamond\square\perp$. By combining the two, we obtain $\mathbf{wK4sd} \vdash (\square\square\perp \wedge \diamond\top) \rightarrow \square\perp$, and thus $\mathbf{wK4sd} \vdash \square\square\perp \rightarrow \square\perp$ after simplification. Then, if $X \models \mathbf{wK4sd}$ and X has hyperdense discreteness, we have $X \models \square\square\perp$ by Proposition 4.8. We then obtain $X \models \square\perp$, and this proves the claim. \square

Moving toward our goal, the hardest part will be to show that dir and sdir are valid on all strongly dense-in-itself ED spaces. To achieve this result, a few preliminaries will be needed. We begin by introducing the notion of *strong* extremal disconnectedness.

Definition 4.11. A topological space X is said to be *strongly extremally disconnected* (or SED for short) if whenever $U \subseteq X$ is open, dU is open too.

Proposition 4.12.

1. Any SED space is also ED.
2. Any dense-in-itself ED space is SED.

Proof. 1. Let X be SED. If U is open in X then so is $U \cup dU = \text{Cl}(U)$.

³Here the prefix *hyper-* is meant to make clear that the property so defined is *stronger* than the mere existence of a dense open discrete subspace.

2. Let X be dense-in-itself and ED. If U is open in X , we show that $\text{Cl}(U) = dU$. For consider $x \in \text{Cl}(U)$ and V an open neighbourhood of x . Then $U \cap V$ is also an open neighbourhood of x , and since X is dense-in-itself there must exist $y \in U \cap V$ such that $y \neq x$, so that $x \in dU$. This proves the claim, and since $\text{Cl}(U)$ is open by assumption, dU is open too. \square

As foreshadowed by the name, being SED is thus stronger than being ED. Note that in general, an ED space is not SED, as witnessed by the space induced by the reflexive 2-chain. Without surprise, the class of SED spaces is defined by a modal formula which is very similar to sdir .

Theorem 4.13. *The logic $\mathbf{wK4} + (\diamond\Box^+p \rightarrow \Box\diamond p)$ defines the class of SED spaces.*

Proof. Let X be a topological space and suppose that $X \models \diamond\Box^+p \rightarrow \Box\diamond p$. Consider U open in X . By assumption we have $d(\text{Int}(U)) \subseteq \widehat{d}dU$, that is, $dU \subseteq \widehat{d}dU$. Then $dU \subseteq dU \cap \widehat{d}dU = \text{Int}(dU)$, which means that dU is open. Conversely, suppose that X is SED. If $A \subseteq X$, we know that $d(\text{Int}(A))$ is open. Thus $d(\text{Int}(A)) = \text{Int}(d(\text{Int}(A))) \subseteq \widehat{d}d(\text{Int}(A)) \subseteq \widehat{d}dA$ and this proves that $X \models \diamond\Box^+p \rightarrow \Box\diamond p$. \square

Proposition 4.14. *Let X be a strongly dense-in-itself space. Then $X \models \mathbf{wK4sd}$ if and only if X is ED.*

Proof. Combining Proposition 4.12 and Theorem 4.13, it suffices to prove that $X \models \diamond\Box p \rightarrow \Box\diamond p$ iff $X \models \diamond\Box^+p \rightarrow \Box\diamond p$. From left to right, suppose $X \models \diamond\Box p \rightarrow \Box\diamond p$, and let $A \subseteq X$. Then by assumption we obtain $d(\text{Int}(A)) \subseteq \widehat{d}dA \subseteq \widehat{d}dA$.

From right to left, suppose that $X \models \diamond\Box^+p \rightarrow \Box\diamond p$, and let $A \subseteq X$ and $x \in \widehat{d}dA$. Then for all open neighbourhoods U of x there exists $y_U \in U \setminus \{x\}$ such that $y_U \in \widehat{d}A$. We then set $A' := A \cup \{y_U \mid U \text{ open and } x \in U\}$, so that every y_U belongs to $\text{Int}(A')$. Then by construction we have $x \in d(\text{Int}(A'))$. By assumption it follows that $x \in \widehat{d}d(A')$, that is, there exists an open neighbourhood V of x such that $V \setminus \{x\} \subseteq d(A')$. Then it suffices to prove that $V \setminus \{x\} \subseteq dA$. For consider $y \in V \setminus \{x\}$ and W an open neighbourhood of y . Then there exists $z \in W \setminus \{y\}$ such that $z \in A'$. If $z \in A$ we are done, otherwise $z = y_U$ for some open neighbourhood U of x . Since $y_U \in \widehat{d}A$ there exists an open neighbourhood T of y_U such that $T \setminus \{y_U\} \subseteq A$. Then $y_U \in W \cap T$ and $W \cap T$ is open, and since X is strongly dense-in-itself, there exists $t, t' \in W \cap T \setminus \{y_U\}$ such that $t \neq t'$. Then we have $t, t' \in W \cap A$, and one of t and t' has to be different from y . Therefore $y \in dA$, and this concludes the proof. \square

We are now ready to prove the announced decomposition theorems. We first settle the case of $\mathbf{wK4d}$, before adapting the result to $\mathbf{wK4sd}$.

Theorem 4.15. *Given a topological space X , we have $X \models \mathbf{wK4d}$ iff there exist three pairwise disjoint open subspaces Y, Z, Z' of X such that $X = Y \cup Z \cup Z'$, Y is strongly dense-in-itself and ED, Z is almost discrete, and Z' has hyperdense discreteness.*

Proof. From left to right, suppose that $X \models \mathbf{wK4d}$. We define

$$Z := \bigcup \{U \mid U \text{ is open and atomic and } |U| = 2\}$$

as well as $Z_0 := \{z \in X \mid z \text{ is isolated}\}$, $Z' := \text{Cl}(Z_0)$ and $Y := X \setminus (Z \cup Z')$.

- Obviously $Y \cap Z = Y \cap Z' = \emptyset$. Suppose that there exists $z \in Z \cap Z'$. Let U be an atomic open neighbourhood of z such that $|U| = 2$. Then from $z \in \text{Cl}(Z_0)$ it follows that U contains an isolated point, contradicting the fact that U is atomic. Therefore $Z \cap Z' = \emptyset$.

- Z is clearly open and almost discrete.
- Observe that Z' is the extension of $\diamond^+\square\perp$ in X (under any valuation). Thus, to prove that Z' has the required properties, we first prove that $\mathbf{wK4d} \vdash \diamond^+\square\perp \rightarrow (\square\diamond^+\square\perp \wedge \square\square\perp \wedge (\diamond p \rightarrow \square p))$. To this end we apply Proposition 4.1. So let $\mathfrak{M} = (W, R, \nu)$ be a directed $\mathbf{wK4}$ model and $w \in W$, and suppose that $\mathfrak{M}, w \models \diamond^+\square\perp$. If w has no successor we are done. Otherwise w has, by assumption, a successor u which has no successor. Suppose toward a contradiction that w has an other successor v , different from u . Then since \mathfrak{M} is directed there exists $t \in W$ such that uRt and vRt , contradicting the assumption on u . Thus u is the sole successor of w and we are done.

From $\mathbf{wK4d} \vdash \diamond^+\square\perp \rightarrow \square\diamond^+\square\perp$ it follows that $Z' \subseteq \widehat{d}Z'$, that is, Z' is open. From $X \models \mathbf{wK4d}$ we then obtain $Z' \models \mathbf{wK4d}$. Therefore $Z' \models \square\square\perp \wedge (\diamond p \rightarrow \square p)$. From Proposition 4.8 we conclude that Z' has hyperdense discreteness.

- By construction, Z' is closed. We show that Z is closed too. First, for all atomic open sets V such that $|V| = 2$, we write $V = \{z_1^V, z_2^V\}$, that is, we specify a first and a second element. We then define

$$\begin{aligned} Z_1 &:= \{z_1^V \mid V \text{ open and atomic and } |V| = 2\}, \\ Z_2 &:= \{z_2^V \mid V \text{ open and atomic and } |V| = 2\}. \end{aligned}$$

Obviously, we have $Z = Z_1 \cup Z_2$. We claim that $d(Z_1) \subseteq Z_2$. First, we clearly have $Z_1 \subseteq \widehat{d}Z_2$. We then apply the assumption that $X \models \mathbf{wK4d}$ to Z_1 and Z_2 and we obtain $d(Z_1) = d(Z_1 \cap \widehat{d}(Z_2)) \subseteq \widehat{d}(Z_1 \cup d(Z_2))$. We can also see that:

- $Z_1 \cap Z_2 = \emptyset$, because if $z_1^V = z_2^W$ for some V and W , then $\{z_1^V\} = V \cap W \subseteq V$ contradicts the fact that V is atomic and of size 2;
- $d(Z_2) \cap Z_2 = \emptyset$, because if $z_2^V \in d(Z_2)$ for some V , then $z_1^V \in Z_2$, in contradiction with the previous fact.

By combining these various facts we arrive at $d(Z_1) \subseteq \widehat{d}(X \setminus Z_2)$. Now let $x \in d(Z_1)$. Then there exists an open neighbourhood U of x such that $U \setminus \{x\} \subseteq X \setminus Z_2$. Then since $x \in d(Z_1)$ there exists $z_1^V \in Z_1 \cap U \setminus \{x\}$. Then $V \subseteq U$, for otherwise $V \cap U = \{z_1^V\}$, contradicting the fact that V is atomic. Thus $z_2^V \in U$, but since $U \setminus \{x\} \subseteq X \setminus Z_2$ the only possibility is $z_2^V = x$. Therefore $x \in Z_2$.

Likewise we can prove that $d(Z_2) \subseteq Z_1$, so all in all $dZ = dZ_1 \cup dZ_2 \subseteq Z_2 \cup Z_1 = Z$. This proves that Z is closed. Since Z and Z' are closed, we deduce that Y is open.

- We prove that Y is strongly dense-in-itself. First, we write $Z'' := Z \cup Z'$ and claim that for all open sets U such that $|U| \leq 2$, we have $U \subseteq Z''$. If U is atomic, this is immediate. Otherwise U is of the form $U = \{x, y\}$ with $x \neq y$ and y isolated. If x is isolated then $U \subseteq Z_0 \subseteq Z'$. Otherwise, for all open neighbourhoods V of x we have $U \cap V \neq \{x\}$ and thus $y \in U \cap V$. This proves that $x \in \text{Cl}(Z_0)$, and thus $U \subseteq Z'$.

Now let U be an open set such that $|U \cap Y| \leq 2$. Then since Y is open, $U \cap Y$ is open too, and it follows that $U \cap Y \subseteq Z'' = X \setminus Y$, whence $U \cap Y = \emptyset$. This proves that Y is strongly dense-in-itself.

- Finally, since Y is open in X and $X \models \diamond\square p \rightarrow \square\diamond p$, we have $Y \models \diamond\square p \rightarrow \square\diamond p$ as well. From Proposition 4.14 it then follows that Y is ED.

From right to left, suppose that there exist three pairwise disjoint open subspaces Y, Z, Z' of X such that $X = Y \cup Z \cup Z'$, Y is strongly dense-in-itself and ED, Z is almost discrete, and Z' has hyperdense discreteness. Since Y, Z and Z' are open, and $\mathbf{wK4sd}$ is an extension of $\mathbf{wK4d}$, it is enough to prove that $Y \models \mathbf{wK4sd}$, $Z \models \mathbf{wK4sd}$ and $Z' \models \mathbf{wK4d}$. For Y this immediately stems from Proposition 4.14. For Z , this stems from Lemma 4.10. For Z' , this is a consequence of Proposition 4.8, which yields $Z' \models \diamond p \rightarrow \square p$. Substituting $p \wedge \square q$ for p , we obtain $Z' \models \diamond(p \wedge \square q) \rightarrow \square(p \wedge \square q)$, and thus $Z' \models \diamond(p \wedge \square q) \rightarrow \square(p \vee \diamond q)$. \square

Theorem 4.16. *Given a topological space X , we have $X \models \mathbf{wK4sd}$ iff there exist two disjoint open subspaces Y, Z of X such that $X = Y \cup Z$, Y is strongly dense-in-itself and ED, and Z is almost discrete.*

Proof. Suppose that $X \models \mathbf{wK4sd}$. Then in particular $X \models \mathbf{wK4d}$, so by Theorem 4.15 there exist three pairwise disjoint open subspaces Y, Z, Z' of X such that $X = Y \cup Z \cup Z'$, Y is strongly dense-in-itself and ED, Z is almost discrete, and Z' has hyperdense discreteness. Since Z' is open in X we have $Z' \models \mathbf{wK4sd}$, and thus Z' is discrete by Lemma 4.10. Therefore $Z'' := Z \cup Z'$ is open and almost discrete and we are done.

Conversely, suppose that $X = Y \cup Z$, with Y, Z disjoint and open, Y strongly dense-in-itself and ED, and Z almost discrete. Since Y and Z are open, it is enough to prove that $Y \models \mathbf{wK4sd}$ and $Z \models \mathbf{wK4sd}$. The former stems from Proposition 4.14, and the latter from Lemma 4.10. \square

For future purposes (related to the topological completeness of $\mathbf{wK4d}$ and $\mathbf{wK4sd}$), we also prove similar decomposition results for Kripke frames. Below, the frames \bullet , \uparrow and \downarrow are understood to be irreflexive.

Proposition 4.17. *Let $\mathfrak{F} = (W, R)$ be a $\mathbf{wK4}$ frame. Then $\mathfrak{F} \models \mathbf{wK4d}$ if and only if \mathfrak{F} is the disjoint union of three Kripke frames $\mathfrak{F}_0, \mathfrak{F}_1, \mathfrak{F}_2$ such that:*

1. \mathfrak{F}_0 is directed,
2. every world in \mathfrak{F}_0 has a reflexive successor, or has at least two distinct successors,
3. \mathfrak{F}_1 is a disjoint union of frames of the form \bullet or \uparrow ,
4. \mathfrak{F}_2 is a disjoint union of frames of the form \downarrow .

Proof. Suppose that there exists such of decomposition of \mathfrak{F} into $\mathfrak{F}_0, \mathfrak{F}_1$ and \mathfrak{F}_2 . It is then clear that $\mathfrak{F}_0, \mathfrak{F}_1$ and \mathfrak{F}_2 are directed. Thus \mathfrak{F} is directed, and it follows that $\mathfrak{F} \models \mathbf{dir}$.

Conversely, suppose that $\mathfrak{F} \models \mathbf{dir}$. We denote by W_0 the set of worlds in W that have a reflexive successor, or at least two distinct successors. Then let \mathfrak{F}_0 be the subframe of \mathfrak{F} generated by W_0 . We then let \mathfrak{F}_1 be the disjoint union of all generated subframes of \mathfrak{F} of the form \bullet or \uparrow , and \mathfrak{F}_2 the disjoint union of all generated subframes of \mathfrak{F} of the form \downarrow . We denote by W_1 and W_2 the sets of worlds of \mathfrak{F}_1 and \mathfrak{F}_2 , respectively.

We prove that \mathfrak{F}_0 is a generated subframe of \mathfrak{F} . So suppose that $w \in W_0$ and wRu . If u is reflexive then $u \in W_0$. Otherwise, w has a successor v different from u . Then since $\mathfrak{F} \models \mathbf{dir}$, the frame \mathfrak{F} is directed and there exists $t \in W$ such that uRt and vRt . If $t = u$ then u is reflexive, so suppose that $u \neq t$. By weak transitivity we have either $w = t$ or wRt . If $w = t$ then uRw and so u has either a reflexive successor (in case $w = v$) or two distinct successors (in case $w \neq v$, by weak reflexivity). If instead wRt then by directedness there exists $z \in W$ such that uRz and tRz . If $t = z$ then t is a reflexive successor of u , and if $t \neq z$ then t and z are two distinct successors of u . In all cases we see that $u \in W_0$, as desired.

By construction, the frames \mathfrak{F}_1 and \mathfrak{F}_2 are disjoint. Further, \mathfrak{F}_0 cannot contain any generated subframe of the form \bullet , \uparrow or \downarrow , as the first pattern contains a point with no successor, and the last two contain a point with a single irreflexive successor. Therefore, the frames $\mathfrak{F}_0, \mathfrak{F}_1$ and \mathfrak{F}_2 are pairwise disjoint. We then prove that $W = W_0 \cup W_1 \cup W_2$. Let $w \in W$. If w has no successor then $w \in W_1$. If w has at least two distinct successors then $w \in W_0$. So assume that w has exactly one successor u . If u has no successor then $w \in W_2$. If u is reflexive then $w \in W_0$. Otherwise, u has a successor $v \neq u$. If $v \neq w$ then wRv by weak transitivity, so w has at least two successors, a contradiction. Thus $v = w$, and we obtain $w \in W_1$. Further, \mathfrak{F}_0 is directed since it is a generated subframe of \mathfrak{F} , which is directed. Finally, conditions 2 to 4 are satisfied by definition, and we are done. \square

Proposition 4.18. *Let $\mathfrak{F} = (W, R)$ be a **wK4** frame. Then $\mathfrak{F} \models \mathbf{wK4sd}$ if and only if \mathfrak{F} is the disjoint union of two Kripke frames \mathfrak{F}_0 and \mathfrak{F}_1 such that:*

1. \mathfrak{F}_0 is directed,
2. every world in \mathfrak{F}_0 has a reflexive successor, or has at least two distinct successors,
3. \mathfrak{F}_1 is a disjoint union of frames of the form \bullet or \downarrow .

Proof. Suppose that there exists such of decomposition of \mathfrak{F} into \mathfrak{F}_0 and \mathfrak{F}_1 . Here it suffices to prove that \mathfrak{F}_0 and \mathfrak{F}_1 are strongly directed. For \mathfrak{F}_1 this is clear. For \mathfrak{F}_0 , let w, u, v be three worlds of \mathfrak{F}_0 such that wRu and wRv . Since \mathfrak{F}_0 is directed, the only non-trivial case is when $u = v$. Since u lies in \mathfrak{F}_0 , the world u has at least one successor, which is then a common successor of u and v . This proves the claim.

Conversely, suppose that $\mathfrak{F} \models \mathbf{sdir}$. Then in particular $\mathfrak{F} \models \mathbf{dir}$, so let $\mathfrak{F}_0, \mathfrak{F}_1$ and \mathfrak{F}_2 be the frames obtained by applying Proposition 4.17. The pattern \downarrow is not a **wK4sd** frame, since it refutes the formula $\diamond\Box\perp \rightarrow \Box\diamond\perp$, which is an instantiation of **sdir**. Thus \mathfrak{F}_2 is actually empty, and we are done. □

5 Topological (modal) connectedness

In this section we study the logics of connectedness.⁴ In the c-semantics, the logic of reference is **S4.3**, which defines the class of strongly connected **S4** frames, as well as the class of hereditarily extremally disconnected spaces. When moving to the d-semantics, we find ourselves in a situation similar to that of the beginning of Section 4. Indeed, in [CZ97, Sect. 3.8] we find the definition **S4.3** := **S4** + **scon** but it is also noticed that this logic is equal to **S4** + **con**. Though **scon** and **con** coincide over **S4**, this is not the case in general, so it is worth investigating the semantics of both **wK4** + **scon** and **wK4** + **con**. The axiom **con**, however, turns out to coincide with **scon**⁺.

Proposition 5.1. *We have $\mathbf{wK4} + \mathbf{con} = \mathbf{wK4} + \mathbf{scon}^+$. In addition, this logic defines the class of weakly transitive connected Kripke frames \mathfrak{F} , and is Kripke complete. In the d-semantics, this logic defines the class of HED spaces.*

Proof. First, given a **wK4** frame \mathfrak{F} , we know that $\mathfrak{F} \models \mathbf{scon}$ iff \mathfrak{F} is strongly connected [CZ97, Sect. 3.5], so by Proposition 2.8 it follows that $\mathfrak{F} \models \mathbf{scon}^+$ iff \mathfrak{F}^+ is strongly connected. Then it is clear that \mathfrak{F}^+ is strongly connected iff \mathfrak{F} is connected. We also know that \mathfrak{F} is connected iff $\mathfrak{F} \models \mathbf{con}$ [CZ97, Sect. 3.5]. All in all we obtain $\mathfrak{F} \models \mathbf{con}$ iff $\mathfrak{F} \models \mathbf{scon}^+$. Then, we observe that **con** can be expressed as $\diamond(p \wedge \Box^+q) \rightarrow \Box^+(q \vee \diamond p)$, and **scon**⁺ as $\diamond^+(p \wedge \Box^+q) \rightarrow \Box^+(q \vee \diamond^+p)$, both being Sahlqvist formulas. Therefore **wK4** + **con** and **wK4** + **scon**⁺ are Kripke complete, and since they define the same classes of frames, it follows that they are equal. Finally, if X is a topological space, we know from Proposition 2.8 that $X \models \mathbf{scon}^+$ iff $X \models^+ \mathbf{scon}$, and from Theorem 2.21 it follows that $X \models \mathbf{scon}^+$ iff X is HED. □

This leads us to introduce the following names:

$$\mathbf{wK4sc} := \mathbf{wK4} + \mathbf{scon},$$

$$\mathbf{wK4.3} := \mathbf{wK4} + \mathbf{scon}^+.$$

⁴What we call “the logics of connectedness” are, in reality, the logics of *connected frames* (and their variants). They have nothing to do with *connected topological spaces*, namely the spaces that cannot be represented as the union of two disjoint non-empty open subsets. This paper does not deal with this kind of space, and thus presents no risk of ambiguity. But in other contexts, one may favor the terms *modal connectedness* and *spatial connectedness* to distinguish the two concepts. Note that the latter is not definable in the basic modal language [vBB07, Prop. 1.49], but can be defined with the universal modality [She99].

Again, the semantics of **wK4.3** can immediately be deduced from Proposition 2.8, but the question of completeness remains non-trivial. Further, if we substitute p for q in the axiom **scon**, we obtain $\mathbf{aT} := \Box(p \rightarrow \Diamond p)$, whose name stands for “almost T” (recall that $\mathbf{T} = p \rightarrow \Diamond p$). Whereas this formula is a mere tautology in the c-semantics, it contains a surprising amount of information in the d-semantics, and this justifies making a detour to examine this axiom. At the level of Kripke frames, its semantics is quite simple.

Definition 5.2. A Kripke frame (W, R) is called *almost reflexive* if for all $w, u \in W$, wRu implies uRu .

Proposition 5.3. *The logic $\mathbf{wK4} + \mathbf{aT}$ defines the class of transitive and almost reflexive Kripke frames, and is Kripke complete.*

Proof. If \mathfrak{F} is transitive and almost reflexive then it is clear that $\mathfrak{F} \models \mathbf{wK4} + \mathbf{aT}$. Conversely, suppose that $\mathfrak{F} \models \mathbf{wK4} + \mathbf{aT}$. If wRu , we define a valuation ν by $\nu(p) := \{u\}$; then from $\mathfrak{F}, \nu, w \models \Box(p \rightarrow \Diamond p)$ it follows that $\mathfrak{F}, \nu, u \models p \rightarrow \Diamond p$ and thus uRu . Since $\mathfrak{F} \models \mathbf{wK4}$ we also know that \mathfrak{F} is weakly transitive, and we show that it is in fact transitive. Indeed, suppose that wRu and uRv . If $w \neq v$ we obtain wRv and we are done. If $w = v$, then since uRv we know that vRv , and it follows that wRv too. Finally, since **w4** and **aT** are Sahlqvist formulas, the logic $\mathbf{wK4} + \mathbf{aT}$ is Kripke complete. \square

Note that since $\mathbf{wK4} + \mathbf{aT}$ defines a class of transitive frames, it is, by completeness, equal to $\mathbf{K4} + \mathbf{aT}$. Just as $\mathbf{K4} + \mathbf{T}$ is called **S4**, we choose the name $\mathbf{aS4} := \mathbf{K4} + \mathbf{aT}$. On the topological side, it is connected to a class of spaces that we call *accumulative*.

Definition 5.4. A space X is said to be *accumulative* if for all $A \subseteq X$ such that $dA \neq \emptyset$, there exists an open set U such that $\emptyset \neq A \cap U \subseteq dA$.

As the concept of accumulative space might look somewhat unintuitive, we immediately provide an example.

Example 5.5. We call $A \subseteq \mathbb{N}$ *cofinite* if $\mathbb{N} \setminus A$ is finite. We then endow \mathbb{N} with the topology $\tau := \{A \mid A \text{ is cofinite}\} \cup \{\emptyset\}$. It is easy to see that for all $A \subseteq \mathbb{N}$, we have $dA \neq \emptyset$ iff $dA = \mathbb{N}$ iff dA is infinite. Thus $dA \neq \emptyset$ implies $\emptyset \neq A \cap \mathbb{N} \subseteq dA$, and it follows that (\mathbb{N}, τ) is accumulative.

This example is not fully satisfying though: the property that $dA = \emptyset$ or $dA = \mathbb{N}$ for all $A \subseteq \mathbb{N}$ is absurdly strong and not representative of what accumulative spaces look like in general. A more refined range of examples would be welcome, and this can be obtained by generalizing example 5.5 to any pre-order.

Definition 5.6. Let (X, \preceq) be a pre-order. For all $x \in X$ we write $\uparrow x := \{y \in X \mid x \preceq y\}$. The *cofinite topology* on X is the topology generated by the base $\mathcal{B}_{\preceq} := \{\uparrow x \setminus A \mid x \in X \text{ and } A \text{ finite}\}$.

That \mathcal{B}_{\preceq} is a base is easily verified: for all $x \in X$ we have $x \in \uparrow x$, and for all $x, y \in X$, A and B finite and $z \in (\uparrow x \setminus A) \cap (\uparrow y \setminus B)$ we have $z \in \uparrow z \setminus (A \cup B) \subseteq (\uparrow x \setminus A) \cap (\uparrow y \setminus B)$. The name “cofinite topology” is motivated by the fact that when A is finite, the set $\uparrow x \setminus A$ can be seen as “cofinite in $\uparrow x$ ”. We are then going to prove that X is accumulative whenever X has “almost finite depth” and “almost finite width” in a sense that we define below.

Definition 5.7. Let (X, \preceq) be a pre-order. We denote by ω the first infinite ordinal.

- A set $A \subseteq X$ is called an *antichain* if for all $x, y \in A$ such that $x \neq y$, we have $x \not\preceq y$. We say that (X, \preceq) has *locally finite width* if there is no antichain $A \subseteq X$ and no $x \in X$ such that $\uparrow x \cap A$ is infinite. We say that (X, \preceq) is *total* if it contains no antichain of size 2.
- Given an ordinal α , a *descending α -sequence* is a sequence $(x_\xi)_{\xi < \alpha} \in X^\alpha$ such that whenever $\xi < \eta < \alpha$ we have $x_\eta \preceq x_\xi$ and $x_\xi \not\preceq x_\eta$. We call (X, \preceq) *almost well-founded* if it contains no descending $\omega + 1$ -sequence.

- Given $A \subseteq X$, an element $x \in A$ is said to be *minimal* in A if there is no $y \in A$ such that $y \preceq x$ and $x \not\preceq y$.
- Given $A \subseteq X$, we denote by A^* the set of all $x \in A$ such that $\uparrow x \cap A$ is finite.

The definition of locally finite width is similar to the property of finite width, which says that no antichain in (X, \preceq) is infinite; what it states instead is that no antichain in (X, \preceq) is infinite *above a point*. Likewise, almost well-founded pre-orders are reminiscent of well-founded pre-orders, i.e., those that contain no descending ω -sequence. We will prove that the conjunction of these two conditions is necessary and sufficient for X to be accumulative (with respect to the cofinite topology). First, we need a technical result.

Lemma 5.8. *Suppose that (X, \preceq) is almost well-founded. Let $x, y \in X$ and $A \subseteq X$, and assume that $y \in \uparrow x \cap A$. Then there is some element z minimal in $\uparrow x \cap A$ such that $z \preceq y$.*

Proof. For suppose not. We define the sequence $(x_n)_{n < \omega} \in (\uparrow x \cap A)^\omega$ by induction, with $x_n \preceq y$ for all $n < \omega$. First, we set $x_0 := y$. Then, if x_n is defined, it is by assumption not minimal in $\uparrow x \cap A$. Thus, there exists $x_{n+1} \in \uparrow x \cap A$ such that $x_{n+1} \preceq x_n$ and $x_n \not\preceq x_{n+1}$, and we have $x_{n+1} \preceq y$ too. Setting $x_\omega := x$, we know that $(x_n)_{n \leq \omega}$ cannot be a descending $\omega + 1$ -sequence, so there exists $n < \omega$ such that $x_n \preceq x_\omega$. But then $x_n \preceq x_\omega \preceq x_{n+1}$, a contradiction. This proves the claim. \square

Lemma 5.9. *Suppose that (X, \preceq) is almost well-founded and has locally finite width. Then for all $x \in X$ and $A \subseteq X$, the set $\uparrow x \cap A^*$ is finite.*

Proof. Toward a contradiction, we assume that $\uparrow x \cap A^*$ is infinite. We construct a sequence $(x_n)_{n < \omega} \in (\uparrow x \cap A^*)^\omega$ by induction. First, we select some $x_0 \in \uparrow x \cap A^*$. Next, suppose that x_0, \dots, x_n are defined. Then for all $k \in [0, n]$, we have $x_k \in A^*$, so we know that $\uparrow x_k \cap A$ is finite. Thus, since $\uparrow x \cap A^*$ is infinite, there exists $y_n \in \uparrow x \cap A^* \setminus \bigcup_{k=0}^n \uparrow x_k$. By Lemma 5.8, there exists $x_{n+1} \preceq y_n$ minimal in $\uparrow x \cap A^*$. We then claim that $\{x_n : n < \omega\}$ is an infinite antichain. Indeed, let $n < m < \omega$. If $x_n \preceq x_m$, then we also have $x_n \preceq y_{m-1}$, contradicting $y_{m-1} \notin \uparrow x_n$. Thus $x_n \not\preceq x_m$, and by minimality of x_n it follows that $x_m \not\preceq x_n$ as well. Since $\{x_n : n < \omega\} \subseteq \uparrow x$, this contradicts the fact that (X, \preceq) has locally finite width. \square

We can then prove the announced equivalence.

Theorem 5.10. *Let (X, \preceq) be a pre-order. Then X (with the cofinite topology) is accumulative if and only if (X, \preceq) is almost well-founded and has locally finite width.*

Proof. Suppose that (X, \preceq) is almost well-founded and has locally finite width. Let $A \subseteq X$ and suppose that $x \in \text{d}A$. By Lemma 5.9, the set $\uparrow x \cap A^*$ is finite. We introduce $U := \uparrow x \setminus (A^* \setminus \{x\})$, which is then an open neighbourhood of x . Then since $x \in \text{d}A$ we have $U \cap A \neq \emptyset$, and we also prove that $U \cap A \subseteq \text{d}A$. For consider $y \in U \cap A$. Then $y \notin A^* \setminus \{x\}$. If $y = x$ then $y \in \text{d}A$ immediately, otherwise we have $y \notin A^*$. Now consider an open neighbourhood of y of the form $\uparrow z \setminus B$ with B finite. In this case we have $z \preceq y$ and so $\uparrow y \subseteq \uparrow z$. Since $y \notin A^*$, the set $\uparrow y \cap A$ is infinite, and so there exists $t \in \uparrow y \cap A \setminus (B \cup \{y\})$. Therefore $t \in A \cap (\uparrow z \setminus B)$ and $t \neq y$. This proves that $y \in \text{d}A$, as desired.

Conversely, suppose that X is accumulative. To prove that (X, \preceq) is almost well-founded, suppose toward a contradiction that there exists a descending $\omega + 1$ -sequence $(x_n)_{n \leq \omega}$. We define $A := \{x_n \mid n < \omega\}$ and claim that $x_\omega \in \text{d}A$. Indeed, suppose that $x_\omega \in \uparrow x \setminus B$ with B finite. Since A is infinite there exists $n < \omega$ such that $x_n \notin B \cup \{x_\omega\}$, and from $x \preceq x_\omega \preceq x_n$ it follows that $x_n \in A \cap (\uparrow x \setminus B) \setminus \{x_\omega\}$. Therefore $\text{d}A \neq \emptyset$, so by assumption there exists an open set U such that $\emptyset \neq A \cap U \subseteq \text{d}A$. Thus there exists $x_n \in A \cap U$. Then $V := \uparrow x_n \setminus \{x_k \mid k < n\}$ is an open neighbourhood of x_n , but we have $V \cap A = \{x_n\}$, and this contradicts $x_n \in \text{d}A$.

To prove that (X, \preceq) has locally finite width, suppose toward a contradiction that there exists $x \in X$ and an antichain A such that $\uparrow x \cap A$ is infinite. Then if $x \in \uparrow y \setminus B$ with B finite, there exists $a \in (\uparrow x \cap A) \setminus (B \cup \{x\})$, and it follows that $a \in A \cap (\uparrow y \setminus B) \setminus \{x\}$. This proves $x \in dA$, and by assumption we obtain the existence of an open set U such that $\emptyset \neq U \cap A \subseteq dA$. Then there exists $a \in U \cap A$, but since A is an antichain we have $\uparrow a \cap A = \{a\}$, contradicting $a \in dA$. \square

We also prove two simple properties of accumulative spaces. Proposition 5.11 states that this condition is *hereditary*, i.e., preserved by taking subspaces, and that all accumulative spaces satisfy the separation axiom T_1 .

Proposition 5.11.

1. If X is accumulative then so is any subspace of X .
2. Any accumulative space is also T_1 .

Proof. 1. Suppose that X is accumulative and let $Y \subseteq X$. Let $A \subseteq Y$ and assume that $d_Y A \neq \emptyset$, i.e., $dA \cap Y \neq \emptyset$. Then $dA \neq \emptyset$, so by assumption there exists an open set U such that $\emptyset \neq U \cap A \subseteq dA$, and since $A \subseteq Y$ we have $(U \cap Y) \cap A = U \cap A \neq \emptyset$. In addition, if $x \in (U \cap Y) \cap A$ we obtain $x \in Y \cap dA = d_Y A$. This proves the claim.

2. Let X be accumulative. Suppose that X is not T_1 , i.e., there exists $x \in X$ such that $d\{x\} \neq \emptyset$. By assumption we obtain the existence of an open set U such that $\emptyset \neq \{x\} \cap U \subseteq d\{x\}$. It follows that $x \in d\{x\}$, a contradiction. \square

Now that we are more familiar with the landscape of accumulative spaces, we prove that they are defined by the logic **aS4**.

Theorem 5.12. *The logic **aS4** defines the class of accumulative spaces.*

Proof. Let X be a space and suppose that $X \models \mathbf{aT}$. Let $A \subseteq X$ such that $dA \neq \emptyset$. Then there exists $x \in dA$, and by assumption there is an open neighbourhood U of x such that $U \setminus \{x\} \subseteq (X \setminus A) \cup dA$. Since $x \in dA$ we have in fact $U \subseteq (X \setminus A) \cup dA$, and this entails $U \cap A \subseteq dA$. Since $x \in U$ and $x \in dA$ we also have $U \cap A \neq \emptyset$.

Conversely, suppose that X is accumulative. Let \mathfrak{M} be a model based on X and $x \in X$, and suppose $\mathfrak{M}, x \not\models \mathbf{aT}$. Then $\mathfrak{M}, x \models \diamond(p \wedge \Box \neg p)$, that is, $dA \neq \emptyset$ if we write $A := \llbracket p \wedge \Box \neg p \rrbracket_{\mathfrak{M}}$. Then by assumption there exists an open set U such that $\emptyset \neq A \cap U \subseteq dA$. So there exists $y \in U \cap A$, and we have $\mathfrak{M}, y \models \Box \neg p$, but from $y \in dA$ we also get $\mathfrak{M}, y \models \diamond p$, a contradiction. \square

When it comes to **wK4sc**, we have mentioned earlier that **aT** is instance of **scon**. Thus **wK4sc** contains **aS4** and in particular **K4**, so it is equal to **K4** + **scon**.

Theorem 5.13. *The logic **wK4sc** defines the class of HED accumulative spaces.*

Proof. Suppose that $X \models \mathbf{scon}$. Then we obviously have $X \models \mathbf{con}$ as well, and thus X is HED by Proposition 5.1. As mentioned earlier we also have $X \models \mathbf{aT}$, and thus X is accumulative by Theorem 5.12.

Conversely, suppose that X is HED and accumulative. If \mathfrak{M} is a model based on X and $x \in X$, we prove that $\mathfrak{M}, x \models \mathbf{scon}$. Since X is HED we know that $X \models \mathbf{scon}^+$, and by substituting $p \wedge \Box \neg q$ for p and $q \wedge \Box \neg p$ for q we obtain

$$\mathfrak{M}, x \models \Box^+(p \wedge \Box \neg q \rightarrow \diamond^+(q \wedge \Box \neg p)) \vee \Box^+(q \wedge \Box \neg p \rightarrow \diamond^+(p \wedge \Box \neg q)).$$

Then we assume that $\mathfrak{M}, x \models \Box^+(p \wedge \Box \neg q \rightarrow \diamond^+(q \wedge \Box \neg p))$ without loss of generality. This gives the existence of an open neighbourhood U of x such that $U \subseteq \llbracket p \wedge \Box \neg q \rightarrow \diamond^+(q \wedge \Box \neg p) \rrbracket_{\mathfrak{M}}$.

Suppose toward a contradiction that $\mathfrak{M}, x \not\models \Box(p \rightarrow \Diamond q)$. Then $\mathfrak{M}, x \models \Diamond(p \wedge \Box\neg q)$. If V is an open neighbourhood of x , so is $U \cap V$, and thus there exists $y \in U \cap V \setminus \{x\}$ such that $\mathfrak{M}, y \models p \wedge \Box\neg q$. Since $y \in U$ it follows that $\mathfrak{M}, y \models \Diamond^+(q \wedge \Box\neg p)$. We cannot have $\mathfrak{M}, y \models \Diamond(q \wedge \Box\neg p)$ because this contradicts $\mathfrak{M}, y \models \Box\neg q$, so we obtain $\mathfrak{M}, y \models q \wedge \Box\neg p$. In particular $\mathfrak{M}, y \models p \wedge \Box\neg p$ and this proves $\mathfrak{M}, x \models \Diamond(p \wedge \Box\neg p)$.

Writing $A := \llbracket p \wedge \Box\neg p \rrbracket_{\mathfrak{M}}$, this yields $dA \neq \emptyset$, so by assumption there exists an open set U such that $\emptyset \neq A \cap U \subseteq dA$. So there exists $y \in A \cap dA$, but then $y \in A$ yields $\mathfrak{M}, y \models \Box\neg p$ while $y \in dA$ yields $\mathfrak{M}, y \models \Diamond p$, a contradiction. Therefore $\mathfrak{M}, x \models \Box(p \rightarrow \Diamond q)$ and this concludes the proof. \square

In the c-semantics, **S4.2** is the logic of ED spaces, while **S4.3** is the logic of HED spaces. Thus **S4.3** is exactly “hereditary **S4.2**”, but we do not have this pattern in the d-semantics in the case of **wK4sd** and **wK4sc**. Indeed, the two-element space with the coarse topology is almost discrete and so all of its subspaces are **wK4sd** spaces, but it is not T_1 and thus not a **wK4sc** space. However we can show that the converse holds, that is, any **wK4sc** space is “hereditarily **wK4sd**”.

Lemma 5.14. *We have $\mathbf{wK4} + \mathbf{sdir}^+ + \mathbf{aT} \vdash \mathbf{sdir}$.*

Proof. As mentioned earlier, **w4**, **sdir**⁺ and **aT** are Sahlqvist formulas, so the logic **wK4** + **sdir**⁺ + **aT** is Kripke complete. Hence, it suffices to show that, given a **wK4** frame $\mathfrak{F} = (W, R)$, if \mathfrak{F} is almost reflexive and \mathfrak{F}^+ is strongly directed, then \mathfrak{F} is strongly directed. For suppose wRu and wRv . Then there exists $t \in W$ such that uR^+t and vR^+t . Since \mathfrak{F} is almost reflexive, the world t is reflexive, and thus uRt and vRt in all cases. \square

Proposition 5.15. *If $X \models \mathbf{wK4sc}$ then for any subspace Y of X we have $Y \models \mathbf{wK4sd}$.*

Proof. Suppose that $X \models \mathbf{wK4sc}$. By Theorem 5.13 we know that X is HED and accumulative. If Y is a subspace of X , it is then ED and also accumulative by proposition 5.11. It follows that $Y \models \mathbf{sdir}^+$ and $Y \models \mathbf{aT}$ by Theorem 5.12, whence $Y \models \mathbf{wK4sd}$ by Lemma 5.14. \square

6 Completeness results

We now address the completeness of the logics we have encountered in this paper. The challenge of proving the topological completeness of a logic **L** can be broken down into two sub-goals: a result of Kripke completeness for **L**, and a procedure to transform a Kripke frame into a topological space that satisfies at least the same formulas. What we expect from such a procedure is that it turns certain Kripke frames into certain spaces. For instance, in the case of **aS4**, we want to turn almost reflexive frames into accumulative spaces. Typically, mapping a frame to its induced space (see Definition 2.18) will not suffice. Indeed, if we consider the frame \mathfrak{F} consisting of a single reflexive point, and denote its induced space by X , we see that $\mathfrak{F} \not\models \Box\perp$ whereas $X \models \Box\perp$. This is because the information that the point is reflexive is lost in the process.⁵ This prompts the use of alternative techniques that better “conserve” the information of the initial frame. In Section 6.1, we employ the method of *unfolding* introduced in [BBFD21].⁶ However, most spaces obtained by classical unfolding fail to be T_1 , and in particular accumulative. In Section 6.2, we thus design our own variant of unfolding, tailored to the extensions of **aS4**. In Section 6.3, we apply the technique of dereflexivation, adapted from [BEG09] and more suited to **wK4** + **bd**_{*n*} and its extensions.

⁵As witnessed by the fact that the single *irreflexive* point generates the same space.

⁶It should be mentioned that the name “unfolding” is not from the original reference. It is our initiative to baptize the technique as such.

6.1 Completeness via unfolding

Here we employ unfolding to prove the topological completeness of **wK4d**, **wK4sd** and **wK4.2**. The construction essentially consists in replacing every reflexive point w of a frame by countably many copies of w , and to arrange them all into a dense-in-itself subspace, so that to mimic the reflexivity of w in the d-semantics.

Definition 6.1. Let $\mathfrak{F} = (W, R)$ be a **wK4** frame. We denote by W^r the set of reflexive worlds of \mathfrak{F} , and by W^i the set of irreflexive worlds of \mathfrak{F} . We then introduce the *unfolding* of \mathfrak{F} as the space $X_{\mathfrak{F}} := (W^r \times \omega) \cup (W^i \times \{\omega\})$ and the collection $\tau_{\mathfrak{F}} \subseteq \mathcal{P}(X_{\mathfrak{F}})$ of all sets U such that for all $(w, \alpha) \in U$:

1. there exists $n_{w,\alpha}^U < \omega$ such that for all $(u, \beta) \in X_{\mathfrak{F}}$, if wRu , uRw and $\beta \geq n_{w,\alpha}^U$ then $(u, \beta) \in U$,
2. if $(u, \beta) \in X_{\mathfrak{F}}$, wRu and *not* uRw then $(u, \beta) \in U$.

Proposition 6.2 ([BBFD21, Lemma VIII.2]). *The pair $(X_{\mathfrak{F}}, \tau_{\mathfrak{F}})$ is a topological space, and the map $\pi : X_{\mathfrak{F}} \rightarrow W$ defined by $\pi(w, \alpha) := w$ is a surjective d-morphism.*

In order to apply this technique to the targeted logics, we first need to prove that unfolding turns some specific frames into ED spaces.

Lemma 6.3. *We have $\text{Cl}(U) = \{(w, \alpha) \in X_{\mathfrak{F}} \mid \exists (u, \beta) \in U, wR^+u\}$ for all open sets U in $X_{\mathfrak{F}}$.*

Proof. Let $(w, \alpha) \in \text{Cl}(U)$. The set $V := \{(u, \beta) \in X_{\mathfrak{F}} \mid wR^+u\}$ is clearly an open neighbourhood of (w, α) , so $V \cap U \neq \emptyset$.

Conversely, suppose that $(w, \alpha) \in X_{\mathfrak{F}}$, $(u, \beta) \in U$ and wR^+u . Let V be an open neighbourhood of (w, α) . First assume that *not* wRu . Then $w = u$ and w is irreflexive, so we have $\alpha = \beta = \omega$ and thus $(w, \alpha) = (u, \beta) \in U \cap V$. Otherwise, we have wRu . If *not* uRw , then $(u, \beta) \in V$ since V is open. So assume that uRw . If u is irreflexive then $\beta = \omega \geq n_{w,\alpha}^V$, so $(u, \beta) \in V$. If u is reflexive, we define $n := \max \{n_{w,\alpha}^V, n_{u,\beta}^U\}$; then from uRu it follows that $(u, n) \in V$, and from wRu and uRw it follows that $(u, n) \in U$. In all cases we end up with $U \cap V \neq \emptyset$ and this proves that $(w, \alpha) \in \text{Cl}(U)$. \square

Lemma 6.4. *If \mathfrak{F}^+ is strongly directed, then $X_{\mathfrak{F}}$ is extremally disconnected.*

Proof. Suppose that \mathfrak{F}^+ is strongly directed, and let U be open in $X_{\mathfrak{F}}$. To show that $\text{Cl}(U)$ is open, consider $(w, \alpha) \in \text{Cl}(U)$. By Lemma 6.3 we know that wR^+u for some $(u, \beta) \in U$.

1. We claim that $n_{(w,\alpha)}^{\text{Cl}(U)} := 0$ satisfies the first condition. Indeed, if we suppose $(v, \gamma) \in X_{\mathfrak{F}}$, wRv and vRw , we obtain vR^+u and thus $(v, \gamma) \in \text{Cl}(U)$.
2. For the second condition, suppose $(v, \gamma) \in X_{\mathfrak{F}}$, wRv and *not* vRw . Then by assumption there exists $t \in W$ such that uR^+t and vR^+t . If tRu then vR^+u and thus $(v, \gamma) \in \text{Cl}(U)$. Otherwise we define $\delta := 0$ if t is reflexive and $\delta := \omega$ otherwise. We have $(u, \beta) \in U$, uRt and *not* tRu so $(t, \delta) \in U$ since U is open. Then from vR^+t we obtain $(v, \gamma) \in \text{Cl}(U)$. \square

Theorem 6.5. *The logics **wK4.2**, **wK4d** and **wK4sd** are topologically complete.*

Proof. Suppose that **wK4.2** $\not\models \neg\varphi$. By Proposition 4.1, there exists a **wK4** Kripke frame \mathfrak{F} on which φ is satisfiable, and such that \mathfrak{F}^+ is strongly directed. Since π is a surjective d-morphism from $X_{\mathfrak{F}}$ to \mathfrak{F} , we obtain from Proposition 2.5 that φ is satisfiable on $X_{\mathfrak{F}}$ as well. By Lemma 6.4, $X_{\mathfrak{F}}$ is ED, and we are done.

Suppose that **wK4d** $\not\models \neg\varphi$. By Proposition 4.1, **wK4d** is Kripke complete, so there exists a Kripke frame $\mathfrak{F} = (W, R)$ on which φ is satisfiable and such that $\mathfrak{F} \models \mathbf{wK4d}$. Let \mathfrak{F}_0 , \mathfrak{F}_1 and

\mathfrak{F}_2 be the Kripke frames introduced in Proposition 4.17, and generated by respectively W_0, W_1 and W_2 . We then consider the subspaces $Y := \pi^{-1}[W_0]$, $Z := \pi^{-1}[W_1]$ and $Z' := \pi^{-1}[W_2]$ of $X_{\mathfrak{F}}$. We claim that Y, Z, Z' satisfy the conditions of Theorem 4.15.

- Since W_0, W_1 and W_2 are pairwise disjoint, so are Y, Z and Z' . Since $W = W_0 \cup W_1 \cup W_2$, we have $X = Y \cup Z \cup Z'$.
- Since π is a d-morphism, it is also an interior map. Since $\mathfrak{F}_0, \mathfrak{F}_1$ and \mathfrak{F}_2 are generated subframes of \mathfrak{F} , the sets W_0, W_1 and W_2 are open in \mathfrak{F} , and therefore Y, Z and Z' are open in X .
- The frame \mathfrak{F}_1 is irreflexive, so $Z = W_1 \times \{\omega\}$. As a result, the restriction of π to Z is bijective, and thus a homeomorphism. Then, since \mathfrak{F}_1 is almost discrete, so is Z .
- Likewise, \mathfrak{F}_2 is irreflexive and has hyperdense discreteness, so Z' has hyperdense discreteness too.
- Since \mathfrak{F} is directed, it is clear that \mathfrak{F}^+ is strongly directed. Then by Lemma 6.4 it follows that $X_{\mathfrak{F}}$ is ED. Since Y is open in $X_{\mathfrak{F}}$, it is ED too.
- We show that Y is strongly dense-in-itself. Let $(w, \alpha) \in Y$ and let U be an open neighbourhood of (w, α) in Y . If w is reflexive, we set $n := 1 + \max \{n_{w,\alpha}^U, \alpha\}$, and we obtain $(w, n) \in U$ and $(w, n+1) \in U$ with $(w, \alpha), (w, n)$ and $(w, n+1)$ all different. Otherwise, since w is in W_0 it has two distinct successors u, v both different from w . Since $\omega \geq n_{w,\alpha}^U$, we have $(u, \omega) \in U$ – regardless of whether uRw or not – and likewise $(v, \omega) \in U$. Since $(w, \omega), (u, \omega)$ and (v, ω) are all different, we are done.

Therefore $X_{\mathfrak{F}} \models \mathbf{wK4d}$. The formula φ is satisfiable on \mathfrak{F} , so again it follows that φ is satisfiable on $X_{\mathfrak{F}}$. For $\mathbf{wK4sd}$ we follow exactly the same proof scheme, using Proposition 4.18 and Theorem 4.16 instead. \square

6.2 Completeness via refined unfolding

Unfolding is powerful, but it has some limits. When addressing $\mathbf{aS4}$ and $\mathbf{wK4sc}$, we are interested in accumulative spaces, which are known to satisfy T_1 separation. Unfortunately, spaces obtained by unfolding critically fail to be T_1 . Indeed, if $\mathfrak{F} = (W, R)$ is a Kripke frame and $(w, \alpha), (u, \beta) \in X_{\mathfrak{F}}$ are such that wRu and not uRw , then it is clear that any open neighbourhood of (w, α) is also an open neighbourhood of (u, β) . In words, the topology of $X_{\mathfrak{F}}$ is too coarse to guarantee enough separation, but this can be fixed by adding more open sets. We first observe that we can endow $X_{\mathfrak{F}}$ with the pre-order \preceq defined by

$$(w, \alpha) \preceq (u, \beta) \iff wR^+u$$

and that the topology of $X_{\mathfrak{F}}$ is generated by

$$\{\uparrow(w, \alpha) \setminus A \mid (w, \alpha) \in X_{\mathfrak{F}} \text{ and } A \text{ finite and for all } (u, \beta) \in A, \text{ not } uRw\}.$$

After dropping the condition “for all $(u, \beta) \in A$, not uRw ”, we obtain a finer topology, which appears to coincide with the cofinite topology associated to \preceq . This leads to the following definition.

Definition 6.6. Let $\mathfrak{F} = (W, R)$ be a $\mathbf{wK4}$ frame. We introduce the *refined unfolding* of \mathfrak{F} as the space $X_{\mathfrak{F}}^* := (W^r \times \omega) \cup (W^i \times \{\omega\})$, endowed with the cofinite topology associated to the pre-order $(X_{\mathfrak{F}}^*, \preceq)$.

We can already see that this defines a T_1 space, since whenever $(w, \alpha) \neq (u, \beta)$, the open set $\uparrow(w, \alpha) \setminus \{(u, \beta)\}$ separates (u, β) from (w, α) . On top of that, we will be able to reuse our results on cofinite topologies, and thus easily carry out the proof of completeness. The downside, however, is that the projection π is no longer a d-morphism in most cases. Though the proof still goes through for $\mathbf{aS4}$ and $\mathbf{wK4sc}$, this dramatically limits the scope of this method.

Proposition 6.7. *The map $\pi : X_{\mathfrak{F}}^* \rightarrow W$ defined by $\pi(w, \alpha) := w$ is a d-morphism if and only if \mathfrak{F} is almost reflexive.*

Proof. Suppose that π is a d-morphism, and let $w, u \in W$ be such that wRu . Suppose toward a contradiction that u is irreflexive. Then we introduce the open set U defined by

$$U := \begin{cases} \uparrow(w, 0) \setminus \{(u, \omega)\} & \text{if } w \text{ is reflexive,} \\ \uparrow(w, \omega) \setminus \{(u, \omega)\} & \text{otherwise.} \end{cases}$$

By assumption, π is an interior map, so $\pi[U]$ is open in \mathfrak{F} . Yet $w \in \pi[U]$ and $u \notin \pi[U]$, a contradiction. This proves that \mathfrak{F} is almost reflexive.

Conversely, suppose that \mathfrak{F} is almost reflexive. Let $A \subseteq W$. We prove that $d(\pi^{-1}[A]) = \pi^{-1}[R^{-1}A]$. So let $(w, \alpha) \in d(\pi^{-1}[A])$. Let $U := \uparrow(w, \alpha)$, which is open. By assumption there exists $(u, \beta) \in (U \cap \pi^{-1}[A]) \setminus \{(w, \alpha)\}$. By construction we have $(u, \beta) \preceq (w, X_{\mathfrak{F}}^*)$ and thus wR^+u . Toward a contradiction, assume that *not* wRu . Then $w = u$ and w is irreflexive, whence $\alpha = \beta = \omega$. It follows that $(u, \beta) = (w, \alpha)$, a contradiction. Therefore wRu . Since in addition $(u, \beta) \in \pi^{-1}[A]$, we obtain $u \in A$ and thus $w \in R^{-1}A$. Therefore $(w, \alpha) \in \pi^{-1}[R^{-1}A]$.

Now let $(w, \alpha) \in \pi^{-1}[R^{-1}A]$. Then wRu for some $u \in A$. Since \mathfrak{F} is almost reflexive, it follows that u is reflexive. Consider an open neighbourhood of (w, α) of the form $\uparrow(v, \beta) \setminus B$ with B finite. Then $(v, \beta) \preceq (w, \alpha)$, that is, vR^+w . By weak reflexivity we obtain vRu . Since u is reflexive and B is finite, we obtain the existence of some $n < \omega$ such that $n \neq \alpha$ and $(u, n) \notin B$. Then $(u, n) \in \pi^{-1}[A] \cap \uparrow(v, \beta) \setminus B$ and $(u, n) \neq (w, \alpha)$. This proves that $(w, \alpha) \in d(\pi^{-1}[A])$. \square

The completeness proof will also require the finite model property of **aS4** and **wK4sc**. This is provided by the results of [BGJ11], which asserts the finite model property for a class of extensions of **wK4** called *cofinal subframe logics*. Most logics in this paper can easily be proven to be cofinal subframe.

Definition 6.8 ([BGJ11]). Let $\mathfrak{F} = (W, R)$ be a Kripke frame. A subframe $\mathfrak{F}' = (W', R')$ of \mathfrak{F} is called a *cofinal subframe* of \mathfrak{F} if $w' \in W'$ and $w'Rw$ implies the existence of $u' \in W'$ such that wR^+u' .

Definition 6.9 ([BGJ11]). Let **L** be an extension of **K**. The logic **L** is called *cofinal subframe* if whenever $\mathfrak{F} \models \mathbf{L}$ and \mathfrak{F}' is a cofinal subframe of \mathfrak{F} , we have $\mathfrak{F}' \models \mathbf{L}$.

Theorem 6.10 ([BGJ11]). *Every extension of **wK4** which is a cofinal subframe logic has the finite model property.*

Theorem 6.11. *The logic **aS4** is topologically complete.*

Proof. Suppose that **aS4** $\not\models \neg\varphi$. We can check that **aS4** is a cofinal subframe logic, so by Theorem 6.10 there exists a finite transitive and almost reflexive Kripke frame $\mathfrak{F} = (W, R)$ on which φ is satisfiable. That \mathfrak{F} is finite immediately implies that $(X_{\mathfrak{F}}^*, \preceq)$ has locally finite width and is almost well-founded, and from Theorem 5.10 it follows that $X_{\mathfrak{F}}^*$ is accumulative. Since \mathfrak{F} is almost reflexive, we also know from Proposition 6.7 that π is a d-morphism from $X_{\mathfrak{F}}^*$ to \mathfrak{F} . Thus, since φ is satisfiable on \mathfrak{F} , it is also satisfiable on $X_{\mathfrak{F}}^*$ and we are done. \square

We now address the topological completeness of **wK4sd**. To this end we first establish a sufficient condition for unfolded spaces to be HED.

Lemma 6.12. *If (X, \preceq) is a total pre-order then the space X (with the cofinite topology) is HED.*

Proof. Let (X, \preceq) be a total pre-order and $Y \subseteq X$. Let $x \in X$ and let $A \subseteq X$ be finite. We show that

$$U := \text{Cl}_Y((\uparrow x \setminus A) \cap Y) = Y \cap \text{Cl}((\uparrow x \setminus A) \cap Y)$$

is open. Given $y \in U$, we show that the open neighbourhood $Y \cap \uparrow y \setminus (A \setminus \{y\})$ of y is included in U . Indeed, let $z \in Y \cap \uparrow y \setminus (A \setminus \{y\})$. Then in particular we have $y \preceq z$. If $z = y$ it is immediate that $z \in U$, so we can assume that $y \prec z$, and it follows that $z \notin A$ – for otherwise we would have $z \in A \setminus \{y\}$. Suppose toward a contradiction that $x \not\preceq z$. Since \preceq is total we then have $z \preceq x$. Suppose that $z \in Y \cap (\uparrow t \setminus B)$ with B finite; we can assume $B \subseteq \uparrow t$. Then suppose toward a contradiction that $Y \cap (\uparrow x \setminus A) \cap (\uparrow t \setminus B) = \emptyset$. Since $t \preceq z \preceq x$ this boils down to $Y \cap (\uparrow x \setminus A) \setminus B = \emptyset$, and thus $Y \cap (\uparrow x \setminus A) \subseteq B$. If $y \notin B$ then $Y \cap (\uparrow y \setminus B)$ is an open neighbourhood of y in Y which does not intersect $\uparrow x \setminus A$, and this contradicts $y \in U$. Therefore $y \in B$, and it follows that $t \preceq y$. Then $Y \cap \uparrow t \setminus (B \setminus \{y\})$ is an open neighbourhood of y in Y , so there exists

$$u \in Y \cap (\uparrow x \setminus A) \cap (\uparrow t \setminus (B \setminus \{y\}))$$

but since $Y \cap (\uparrow x \setminus A) \setminus B = \emptyset$ the only possibility is $u = y$. In particular $y \in \uparrow x$, and from $y \preceq z$ it follows that $x \preceq z$, a contradiction. Therefore $x \preceq z$. This yields $z \in Y \cap (\uparrow x \setminus A)$ and then $z \in U$. \square

Theorem 6.13. *The logic $\mathbf{wK4sc}$ is topologically complete.*

Proof. Suppose that $\mathbf{wK4sc} \not\models \neg\varphi$. We can check that $\mathbf{wK4sc}$ is a cofinal subframe logic, so by Theorem 6.10, φ is satisfiable on a finite transitive and strongly connected frame $\mathfrak{F} = (W, R)$. We can also assume that \mathfrak{F} is rooted, so for all $w, u \in W$ we have either wR^+u or uR^+w . As a result, the pre-order \preceq on $X_{\mathfrak{F}}^*$ is total, and because \mathfrak{F} is finite it is also almost well-founded and has locally finite width. Then from Theorem 5.10 and Lemma 6.12 it follows that $X_{\mathfrak{F}}^*$ is HED and accumulative. In addition, since $\mathfrak{F} \models \mathbf{wK4sc}$ we also have $\mathfrak{F} \models \mathbf{aS4}$, so \mathfrak{F} is almost reflexive, and from Proposition 6.7 we obtain that π is a d-morphism. Then since φ is satisfiable on \mathfrak{F} , it is also satisfiable on $X_{\mathfrak{F}}^*$, and we are done. \square

6.3 Completeness via dereflexivation

An other inconvenience of unfolding is that it sometimes distorts the original frame too much, and thus does not preserve the desired properties. For instance, the frame $\mathbf{u} \rightarrow \mathbf{w} \supset$ is a $\mathbf{wK4.3}$ frame, but its unfolding is not HED since it admits the subspace $Y := \{(w, \omega), (u, 0), (u, 1)\}$ with $(u, 0)$ and $(u, 1)$ isolated. We see that Y is not ED since $\{(u, 0)\}$ is open but $\text{Cl}\{(u, 0)\} = \{(u, 0), (w, \omega)\}$ is not. Unfolding is thus unsuitable to address $\mathbf{wK4.3}$, among other logics. In this case, it is more appropriate to make only minimal changes, and this motivates the operation of *dereflexivation*, adapted from [BEG09, Def. 8]. This consists in copying every reflexive point, according to the pattern depicted in Figure 6. The resulting Kripke frame then remains similar to the original, and is also irreflexive, enabling the use of Proposition 2.19 to perform completeness proofs. The exact construction, as well as the proof of Lemma 6.15 below, can be found in the proof of [BEG09, Th. 5].

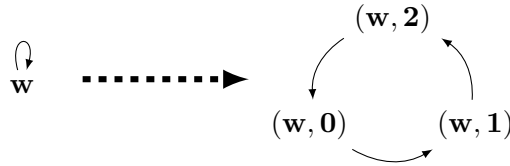


Figure 6: The action of dereflexivation

Definition 6.14. Let $\mathfrak{F} = (W, R)$ be a Kripke frame. We denote by W_r the set of reflexive worlds of W , and W_i the set of irreflexive worlds of W . Let $\overline{W} := (W_i \times \{0\}) \cup (W_r \times \{0, 1, 2\})$. Let $\pi : \overline{W} \rightarrow W$ be defined by $\pi(w, k) := w$. Then the *dereflexivation* of \mathfrak{F} is the Kripke frame $\overline{\mathfrak{F}} = (\overline{W}, \overline{R})$ with

$$\overline{R} := \{(x, y) \in \overline{W}^2 \mid \pi(x)R\pi(y) \text{ and } x \neq y\}.$$

Lemma 6.15. *If \mathfrak{F} is a **wK4** frame, then the map $\pi : \overline{\mathfrak{F}} \rightarrow \mathfrak{F}$ is a d -morphism.*

We then use dereflexivation to prove the topological completeness of **wK4.3**, as well as **wK4 + bd_n** and many of its extensions. As a bonus, we also prove the completeness of various extensions of **GL**. Since their frames are already irreflexive, they are easily dealt with.

Lemma 6.16. *Let $\mathfrak{F} = (W, R)$ be a **wK4** frame.*

1. *If $\text{depth}(\mathfrak{F}) \leq n$, then $\text{depth}(\overline{\mathfrak{F}}) \leq n$.*
2. *If \mathfrak{F}^+ is strongly directed, then so is $(\overline{\mathfrak{F}})^+$.*
3. *If \mathfrak{F} is directed, then so is $\overline{\mathfrak{F}}$.*
4. *If \mathfrak{F} is strongly directed, then so is $\overline{\mathfrak{F}}$.*
5. *If \mathfrak{F} is connected, then so is $\overline{\mathfrak{F}}$.*

Proof. 1. Assume that $\text{depth}(\mathfrak{F}) \leq n$. We prove that $\text{depth}(\overline{\mathfrak{F}}) \leq n$. For suppose there exists a proper path $((w_1, k_1), \dots, (w_{n+1}, k_{n+1}))$ in $\overline{\mathfrak{F}}$. It is then clear that (w_1, \dots, w_{n+1}) is a proper path in \mathfrak{F} , contradicting $\text{depth}(\mathfrak{F}) \leq n$. Therefore $\text{depth}(\overline{\mathfrak{F}}) \leq n$.



2. First, the equivalence $y\overline{R}^+z \iff \pi(y)R^+\pi(z)$ is easily verified. Now assume that \mathfrak{F}^+ is strongly directed. Let $x, y, z \in \overline{W}$ such that $x\overline{R}^+y$ and $x\overline{R}^+z$. Then $\pi(x)R^+\pi(y)$ and $\pi(x)R^+\pi(z)$, so there exists $w \in W$ such that $\pi(y)R^+w$ and $\pi(z)R^+w$. Thus, $y\overline{R}^+(w, 0)$ and $z\overline{R}^+(w, 0)$, and we are done.

3. Assume that \mathfrak{F} is directed. Let $x, y, z \in \overline{W}$ such that $x\overline{R}y$, $x\overline{R}z$ and $y \neq z$. Then $\pi(x)R\pi(y)$ and $\pi(x)R\pi(z)$. First consider the case $\pi(y) \neq \pi(z)$. Then by assumption there exists $w \in W$ such that $\pi(y)Rw$ and $\pi(z)Rw$. If $\pi(y) \neq w$ and $\pi(z) \neq w$ then $y\overline{R}(w, 0)$ and $z\overline{R}(w, 0)$. Otherwise w is reflexive and so there exists $k \in \{0, 1, 2\}$ such that $y \neq (w, k)$ and $z \neq (w, k)$, whence $y\overline{R}(w, k)$ and $z\overline{R}(w, k)$.

In case $\pi(y) = \pi(z)$, then since $y \neq z$ this means that y and z are of the form $y = (w, i)$ and $z = (w, j)$ with w reflexive. Then there exists $k \in \{0, 1, 2\} \setminus \{i, j\}$, and we have $y\overline{R}(w, k)$ and $z\overline{R}(w, k)$.

4. Assume that \mathfrak{F} is strongly directed. Let $x, y, z \in \overline{W}$ such that $x\overline{R}y$ and $x\overline{R}z$. Then $\pi(x)R\pi(y)$ and $\pi(x)R\pi(z)$, so there exists $w \in W$ such that $\pi(y)Rw$ and $\pi(z)Rw$. If $\pi(y) \neq w$ and $\pi(z) \neq w$ then $y\overline{R}(w, 0)$ and $z\overline{R}(w, 0)$. Otherwise w is reflexive and so there exists $k \in \{0, 1, 2\}$ such that $y \neq (w, k)$ and $z \neq (w, k)$, whence $y\overline{R}(w, k)$ and $z\overline{R}(w, k)$.

5. Assume that \mathfrak{F} is connected. Let $x, y, z \in \overline{W}$ such that $x\overline{R}y$ and $x\overline{R}z$ and $y \neq z$. Then $\pi(x)R\pi(y)$ and $\pi(x)R\pi(z)$. If $\pi(y) \neq \pi(z)$ then by assumption we have $\pi(y)R\pi(z)$ or $\pi(z)R\pi(y)$, whence it follows that $y\overline{R}z$ or $z\overline{R}y$. If $\pi(y) = \pi(z)$ then $y\overline{R}z$ is immediate. \square

Remark 6.17. The idea of creating three copies of each reflexive point, instead of two, may seem erratic. This is because the variant with two copies fails to preserve directedness, as witnessed by the frame  which becomes  after the procedure. For all of the other properties mentioned in Lemma 6.16, this variant is perfectly fine.

Theorem 6.18. *In the d -semantics, the logics presented in Table 2 are topologically complete.*

wK4.3	wK4 + bd_n	wK4 + gl = GL
	wK4.2 + bd_n	wK4.2 + gl
	wK4d + bd_n	wK4d + gl
	wK4sd + bd_n	wK4sd + gl
	wK4.3 + bd_n	wK4.3 + gl
		aS4 + gl = K + □⊥
		wK4sc + gl = K + □⊥

Table 2: A number of topologically complete logics

Proof. First, it is easy to check that all of these logics are cofinal subframe logics. By Theorem 6.10, they are thus Kripke complete. For the logics of the first and second columns, the result follows from Lemma 6.15, Lemma 6.16, Proposition 2.19 and the usual proof scheme. For the extensions of **GL**, we notice that a converse well-founded frame is necessarily irreflexive, so it suffices to apply Proposition 2.19. \square

The logics **aS4 + bd_n** and **wK4sc + bd_n** are absent from Table 2, because dereflexivation does not yield T_1 spaces, so for the reasons already explained it is not adapted to address extensions of **aS4**. We also notice that the logic **aS4 + gl** is so strong that it actually collapses to a trivial logic (and thus so does **wK4sc + gl**).

Proposition 6.19. *We have $\mathbf{aS4} + \mathbf{gl} = \mathbf{K} + \square\perp$.*

Proof. We have $\mathbf{aS4} \vdash \square(\square p \rightarrow p)$ and $\mathbf{gl} = \square(\square p \rightarrow p) \rightarrow \square p$, so $\mathbf{aS4} \vdash \square p$. Substituting \perp for p , we then obtain $\mathbf{aS4} \vdash \square\perp$. Conversely, the axioms **w4**, **aT** and **gl** are derivable in $\mathbf{K} + \square\perp$, as a straightforward consequence of Kripke completeness (seeing that $\square\perp$ is a Sahlqvist formula): every Kripke frame on which $\square\perp$ is valid, is also weakly transitive, almost reflexive, and converse well-founded. \square

Remark 6.20. Due to the simplicity of dereflexivation, one may wonder why it was not used to prove completeness of **wK4d**, **wK4sd** and **wK4.2**. This is because the operation of unfolding has other good properties that we did not mention, e.g., it turns transitive frames into T_D spaces [BBFD21], so it is best if it remains the standard method whenever this is possible.

7 Conclusion

We presented various advances in the derivational semantics for modal logic. In particular, we elucidated the semantics of the axioms of bounded depth, and proved a number of characterizations of depth that are useful for the d-semantics. These results show the relevance of topological depth for the derivational framework, and will hopefully lead to the apparition of this parameter in future classifications. We have also provided a comprehensive account of the many variants of the axioms **sdir** and **scon**. Interestingly, these are known to be related to the axioms of *bounded width* **bw_n** (with $n \in \mathbb{N}$) [CZ97, Sect. 3.5], so in some way they talk about the width of spaces, and thus elegantly accompany our work on bounded depth. More precisely, **scon**⁺ is merely equivalent to **bw₁**, so a natural line of research would be to generalize our results to **bw_n**. Our definability results are summarized in Table 3. Finally, our work on completeness led us to rediscover standard techniques such as the unfolding of a Kripke frame. Though powerful, it is also very technical: we have seen more than one way to do it, and doing

Logic	Defines:	Reference
wK4 + bd_n	spaces with modal Krull dimension $\leq n - 1$	Theorem 3.12
wK4.2	ED spaces	Proposition 4.1
wK4d	topological sums of (1) a strongly dense-in-itself ED space, (2) an almost discrete space, (3) a space with hyperdense discreteness	Theorem 4.15
wK4sd	topological sums of (1) a strongly dense-in-itself ED space, (2) an almost discrete space	Theorem 4.16
aS4	accumulative spaces	Theorem 5.12
wK4.3	HED spaces	Proposition 5.1
wK4sc	accumulative HED spaces	Theorem 5.13

Table 3: Summary of our definable classes

it right requires precision. This demonstrates the richness of the method, and may thus be a source of inspiration to future work.

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