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From Thin Concurrent Games to Generalized Species of Structures

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Abstract—Two families of denotational models have emerged from the semantic analysis of linear logic: *dynamic* models, typically presented as game semantics, and *static* models, typically based on a category of relations. In this paper we introduce a formal bridge between two-dimensional dynamic and static models: we connect the bicategory of *thin concurrent games and strategies*, based on event structures, to the bicategory of *generalized species of structures*, based on distributors.

In the first part of the paper, we construct an oplax functor from (the linear bicategory of) thin concurrent games to distributors. This explains how to view a strategy as a distributor, and highlights two fundamental differences: the composition mechanism, and the representation of resource symmetries.

In the second part of the paper, we adapt established methods from game semantics (visible strategies, payoff structure) to enforce a tighter connection between the two models. We obtain a cartesian closed pseudofunctor, which we exploit to shed new light on recent results in the bicategorical theory of the λ -calculus.

I. INTRODUCTION

The discovery of linear logic has had a deep influence on programming language semantics. The linear analysis of resources provides a refined perspective that leads, for instance, to important notions of program approximation [1] and differentiation [2]. Denotational models for higher-order programming languages can be constructed from this resource-aware perspective, exploiting the fact that every model of linear logic is also a model of the simply-typed λ -calculus.

In this paper, we clarify the relationship between two denotational models that arise in this way:

- *Thin concurrent games*, a framework for game semantics introduced by Castellán, Clairambault, and Winskel [3], in which programs are modelled as concurrent strategies.
- *Generalized species of structures*, a combinatorial model developed by Fiore, Gambino, Hyland, and Winskel [4], in which programs are interpreted as categorical distributors (or profunctors) over groupoids.

We carry out this comparison in a two-dimensional setting also including morphisms between strategies and morphisms between distributors. In the language of bicategory theory, our first key contribution is an *oplax* functor of bicategories

$$\begin{array}{ccc} \text{games, strategies} & \longrightarrow & \text{groupoids, distributors and} \\ \text{and maps} & & \text{natural transformations} \end{array} \quad (1)$$

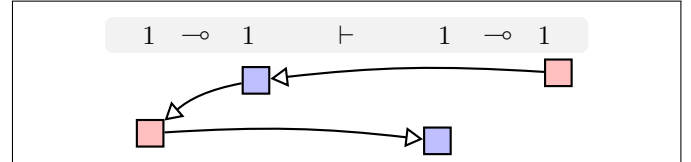
which shows, in particular, that the symmetries of a strategy can be explained using groupoid actions.

A. Static and dynamic models

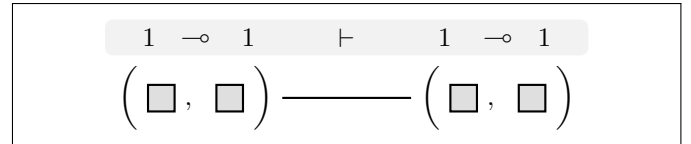
This work fits in a long line of research on the relationship between *static* and *dynamic* denotational models.

In a static model, programs are represented by their input/output behaviour, or by collecting representations of completed executions. The simplest example is given by the category of sets and relations: this is the relational model of linear logic (§II-A or [5]). In a dynamic model, programs are represented by their interactive behaviour with respect to every possible execution environment. This includes game semantics ([6], [7]), which has proved incredibly proficient at modelling various computational features or effects ([8], [9], [10]).

To illustrate the difference, the identity at type $1 \multimap 1$ in game semantics is a strategy that represents an exchange of information between the **program** and its **environment**:



In contrast, in the relational model, $1 \multimap 1$ is a one-element set containing a single input/output pair. The identity relation over it can be seen as a collapsed version of the strategy above:



This suggests a simple equation

$$\text{game semantics} = \text{relational model} + \text{time},$$

so that going from game semantics to the relational model is a simple matter of forgetting the temporal order of execution.

But this naive intuition hides a fundamental difference between the composition mechanisms in static and dynamic models: strategies may *deadlock*, while relations cannot. More precisely, in game semantics, two strategies can synchronize by performing the same actions *in the same order*, whereas in the collapsed version, only the actions matter and not the order. Thus, as was quickly established [11], one cannot simply

forget time in a functorial way, and composition is usually only preserved in an oplax manner, as in (1).

Static collapses of game semantics require an adequate notion of *position* for a game. This is difficult to define in traditional game semantics, but very natural with concurrent games, because we can look at configurations of the underlying event structure (§III-A, see also [12], [13]).

The subtle relationship between static and dynamic models was refined by many authors over two decades ([13], [14], [15], [16]), to identify settings in which functoriality can be restored. Leveraging this, we show (in §V) how the oplax functor (1) can be strictified to a *pseudofunctor*, that preserves composition up to isomorphism.

B. Proof-relevant models and symmetries

Our aim here is to take the static-dynamic relationship to a new level that takes into account the symmetries of resource usage. Symmetry plays an important role in game semantics (§IV, or [7], [12], [17], [18]), but so far connections only exist with static models whose symmetries are implicit or quotiented, like in the relational model. We argue that generalized species, which represent combinatorial structures in terms of their symmetries, provide a convenient target for a static collapse of thin concurrent games.

The two models we consider are “proof-relevant” [19], in the sense that the interpretation of a program provides, for each possible execution, a set of proofs or *witnesses* that this execution can be realized. This high degree of intensionality is useful for modelling languages with non-deterministic features [20], [21]. In a proof-relevant model, symmetries arise naturally in the linear duplication of witnesses. In §II-B we discuss the limitations of a proof-relevant model without symmetries.

Proof-relevant models are naturally organized into *bicategories*: programs are interpreted as structured objects (e.g. strategies or distributors) which themselves support a notion of morphism. By constructing functors of bicategories we clarify the relationship at the two-dimensional level.

C. Bicategorical models of the λ -calculus

To motivate this further, we note that the two-dimensional and proof-relevant aspects are significant on the syntactic side. The interpretation of λ -terms in generalized species has a presentation in terms of an *intersection type system* [22], that takes into account the symmetries and can be exploited to characterize computational properties and equational theories of the λ -calculus [19]. More generally, the structural 2-cells in a cartesian closed bicategory have a syntactic interpretation as $\beta\eta$ -rewriting steps in the simply-typed λ -calculus ([23], [24]).

In §V we connect to this line of work by constructing a cartesian closed pseudofunctor, which preserves the semantics of λ -terms in both typed and untyped settings.

D. Outline of the paper and key contributions

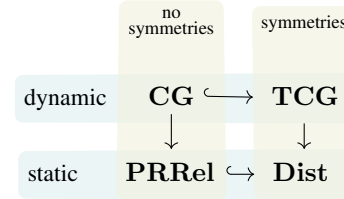
Review of static models: In §II we recall static semantics, including a bicategory **PRRel** of proof-relevant relations, and a bicategory **Dist** of distributors. One can view **PRRel**

as the sub-model of **Dist** with no symmetries, so there is an embedding **PRRel** \hookrightarrow **Dist**. The bicategory **Esp** of generalized species is defined in terms of **Dist**.

Collapsing concurrent games to static models: In §III we introduce the bicategory **CG** of “plain” concurrent games without symmetries, and we show that a collapse operation gives an oplax functor **CG** \rightarrow **PRRel** (Theorem 2).

Then in §IV we add symmetry: we define the bicategory **TCG** of thin concurrent games, with **CG** \hookrightarrow **TCG** as the sub-model with no symmetries. We show that every strategy has a distributor of *positive witnesses* (Proposition 4), and that this extends to an oplax functor **TCG** \rightarrow **Dist** (Theorem 3).

Thus we have the following situation:



Cartesian closed structure and the λ -calculus: In §V we introduce a refined version of **TCG** called **Vis**, using methods from game semantics: payoff ([13], [25]) and visible strategies [26]. Roughly speaking, this is to ensure that the composition of strategies yields no deadlocks and this behaves like that of distributors. Thus we obtain a pseudofunctor

$$\mathbf{Vis} \longrightarrow \mathbf{Dist} \quad (\text{Theorem 4})$$

and by also refining our categorical structure with a *relative pseudo-comonad*, we derive a cartesian closed pseudofunctor **Vis**_! \rightarrow **Esp** (Theorem 7). We apply this result to the semantics of untyped λ -calculus: we build a reflexive object in **Vis** and show that, under our pseudofunctor, this is sent to an extensional, *categorified graph model* D^* in **Esp** [19].

II. A TOUR OF STATIC SEMANTICS

In this section we present three static models: the basic relational model **Rel** (§II-A), a proof-relevant version of it which we call **PRRel** (§II-B), and the model **Dist** of groupoids and distributors (§II-C).

A. The relational model of linear logic

We start with the *relational model*, which gives a denotational interpretation in the category **Rel** of sets and relations. A type A is interpreted as a *set* $\llbracket A \rrbracket$, and a program $\vdash M : A$ is interpreted as a subset $\llbracket M \rrbracket \subseteq \llbracket A \rrbracket$. The set $\llbracket A \rrbracket$ is often called the *web* of A , and we think of its elements as representations of completed program executions. The subset $\llbracket M \rrbracket$ contains the executions that the program M can realize.

Example 1. *The ground type for booleans is interpreted as $\llbracket \mathbb{B} \rrbracket = \{\mathbf{tt}, \mathbf{ff}\}$, and the constant $\vdash \mathbf{tt} : \mathbb{B}$ as $\llbracket \mathbf{tt} \rrbracket = \{\mathbf{tt}\}$.*

The interpretation of a program M is computed compositionally, following the methodology of denotational semantics, using the categorical structure we describe below.

1) *Basic categorical structure*: The category \mathbf{Rel} is defined to have sets as objects, and as morphisms the *relations* from A to B , i.e. the subsets $R \subseteq A \times B$, with the usual notions of identity and composition for relations. Its monoidal product is the cartesian product of sets. If $R_i \in \mathbf{Rel}[A_i, B_i]$ for $i = 1, 2$, then the relation $R_1 \times R_2 \in \mathbf{Rel}[A_1 \times A_2, B_1 \times B_2]$ is defined to contain the pairs $((a_1, a_2), (b_1, b_2))$ with $(a_i, b_i) \in R_i$. The unit I is a fixed singleton set, say $\{*\}$. This monoidal structure is closed, with linear arrow $A \multimap B = A \times B$.

Moreover \mathbf{Rel} has finite cartesian products: the product of sets A and B is given by the disjoint union $A + B = \{1\} \times A \uplus \{2\} \times B$, and the empty set is a terminal object.

2) *The exponential modality*: The exponential modality of \mathbf{Rel} is based on *finite multisets*. If A is a set, we write $\mathcal{M}(A)$ for the set of finite multisets on A . To denote specific multisets we use a list-like notation, as in e.g. $[0, 1, 1] \in \mathcal{M}(\mathbb{N})$.

Given a set A , its **bang** $!A$ is the set $\mathcal{M}(A)$. This extends to a comonad $!$ on \mathbf{Rel} , satisfying the required conditions for a model of intuitionistic linear logic: the *Seely isomorphisms*

$$\mathcal{M}(A + B) \cong \mathcal{M}(A) \times \mathcal{M}(B) \quad \mathcal{M}(\emptyset) \cong I$$

make $!$ a symmetric monoidal functor from $(\mathbf{Rel}, +, \emptyset)$ to $(\mathbf{Rel}, \times, I)$, satisfying a coherence axiom [27, §7.3]. From this, we obtain that the Kleisli category $\mathbf{Rel}_!$ is cartesian closed and thus a model of the simply-typed λ -calculus.

3) *Conditionals and non-determinism*: The category $\mathbf{Rel}_!$ also supports further primitives in a call-by-name setting.

Example 2. Considering the term $\vdash M : \mathbb{B} \rightarrow \mathbb{B}$ of PCF

$$\vdash \lambda x^{\mathbb{B}}. \text{if } x \text{ then } x \\ \text{else if } x \text{ then } \mathbf{ff} \text{ else } \mathbf{tt} : \mathbb{B} \rightarrow \mathbb{B},$$

then $\llbracket M \rrbracket = \{([\mathbf{tt}, \mathbf{tt}], \mathbf{tt}), ([\mathbf{tt}, \mathbf{ff}], \mathbf{ff}), ([\mathbf{ff}, \mathbf{ff}], \mathbf{tt})\}$, representing the possible executions given a multiset of values for x .

As a further example, \mathbf{Rel} supports the interpretation of non-deterministic computation: consider $\vdash \text{choice} : \mathbb{B}$ a non-deterministic primitive that may evaluate to either \mathbf{tt} or \mathbf{ff} . Then we can set $\llbracket \text{choice} \rrbracket = \{\mathbf{tt}, \mathbf{ff}\}$ so that we have

$$\llbracket \text{if choice then } \mathbf{tt} \text{ else } \mathbf{tt} \rrbracket = \{\mathbf{tt}\}. \quad (2)$$

The relational model is a cornerstone of static semantics, and the foundation of many recent developments in denotational semantics [28], [29], [30]. In this paper we are concerned with its *proof-relevant* extensions – roughly speaking, one motivation is to keep separate different execution paths that lead to the same value, as with value \mathbf{tt} in (2).

B. Proof-relevant relations

To showcase this, we consider a notion of proof-relevant relation between sets (e.g. [31], [32]). The idea is to record not only the executions that a program may achieve, but also the distinct ways in which each execution is realized. We replace relations $\llbracket M \rrbracket \subseteq A \times B$ with *proof-relevant* relations

$$\llbracket M \rrbracket : A \times B \rightarrow \mathbf{Set},$$

so that each point of the web has an associated set of *witnesses*. In this model, $\llbracket \text{if choice then } \mathbf{tt} \text{ else } \mathbf{tt} \rrbracket(\mathbf{tt})$ from (2) should be a set $\{*_1, *_2\}$ containing two witnesses, because there are two possible paths to the value \mathbf{tt} .

Formally, the model is organized as a categorical structure with sets as objects, functors $\alpha : A \times B \rightarrow \mathbf{Set}$ (with $A \times B$ viewed as a discrete category) as morphisms, composed with

$$(\beta \circ \alpha)(a, c) = \sum_{b \in B} \alpha(a, b) \times \beta(b, c) \quad (3)$$

and with identity morphisms given by $\text{id}_A(a, a') = \{*\}$ if $a = a'$ and empty otherwise¹. An important observation is that this does not form a category. Categorical laws are only *isomorphisms*, with for instance $(\text{id}_B \circ \alpha)(a, b) = \alpha(a, b) \times \{*\} \cong \alpha(a, b)$. We obtain a *bicategory*: a two-dimensional structure incorporating *2-cells* – morphisms between morphisms – with categorical laws holding only up to coherent invertible 2-cells.

We call this bicategory \mathbf{PRRel} . This model shares (in a bicategorical sense) much of the structure of \mathbf{Rel} , and may be used to interpret e.g. the linear λ -calculus. We use \mathbf{PRRel} as a static collapse of a basic dynamic model in §III-C.

Limitations of a proof-relevant model without symmetry: Unfortunately, the finite multiset functor on \mathbf{Rel} does not seem to extend to \mathbf{PRRel} . Intuitively, the objective of keeping track of individual execution witnesses is in tension with the quotient involved in constructing finite multisets, which blurs out the identity of individual resource accesses.² Proof-relevant models that do support an exponential modality do so by replacing finite multisets with a categorification, such as finite lists related by explicit permutations – *symmetries*.

C. Distributors and generalized species of structures

Distributors are symmetry-aware proof-relevant relations – here we consider distributors on *groupoids*, i.e. small categories in which every morphism is invertible.

1) *The bicategory of groupoids and distributors*: If A and B are groupoids, a **distributor** from A to B (also known as a *profunctor* or *bimodule*) is a functor

$$\alpha : A^{\text{op}} \times B \rightarrow \mathbf{Set}.$$

Thus, for every $a \in A$ and $b \in B$ we have a set $\alpha(a, b)$ of witnesses, but unlike in \mathbf{PRRel} we also have symmetries, in the form of an action by morphisms in A and B . If $x \in \alpha(a, b)$ and $g \in B(b, b')$, we write $g \cdot x$ for the functorial action $\alpha(\text{id}, g)(x) \in \alpha(a, b')$. Similarly, if $f \in A(a', a)$, we write $x \cdot f \in \alpha(a', b)$ for $\alpha(f, \text{id})$. The actions must commute, so we can write $g \cdot x \cdot f$ for $(g \cdot x) \cdot f = g \cdot (x \cdot f) \in \alpha(a', b')$.

We define a bicategory \mathbf{Dist} with groupoids as objects, distributors as morphisms, and natural transformations as 2-cells ([33], [34]). The **identity distributor** on A is

$$\text{id}_A = A[-, -] : A^{\text{op}} \times A \rightarrow \mathbf{Set},$$

¹A more standard presentation of this model is via the bicategory of *spans* of sets, with sets as objects and spans $A \leftarrow S \rightarrow B$ as morphisms.

²In technical terms, the functor $\mathcal{M}(-)$ on \mathbf{Set} is not cartesian – does not preserve pullbacks – and so does not preserve the composition of spans.

the hom-set functor. The **composition** of two distributors $\alpha : A^{\text{op}} \times B \rightarrow \mathbf{Set}$ and $\beta : B^{\text{op}} \times C \rightarrow \mathbf{Set}$ is obtained as a categorified version of (3), defined in terms of a coend:

$$(\beta \bullet \alpha)(a, c) = \int^{b \in B} \alpha(a, b) \times \beta(b, c).$$

Concretely, $(\beta \bullet \alpha)(a, c)$ consists in pairs (x, y) , where $x \in \alpha(a, b)$ and $y \in \beta(b, c)$ for some $b \in B$, quotiented by $(g \cdot x, y) \sim (x, y \cdot g)$ for $x \in \alpha(a, b)$, $g \in B(b, b')$ and $y \in \beta(b', c)$.

The bicategory **Dist** has a symmetric monoidal structure given by the cartesian product $A \times B$ of groupoids, extended pointwise to distributors. There is a closed structure given by $A \multimap B = A^{\text{op}} \times B$. Finally, **Dist** has cartesian products given by the disjoint union $A + B$ of groupoids.

2) *The exponential modality*: In this model with explicit symmetries, the exponential modality is not given by finite multisets, but instead by finite lists with explicit permutations.

Definition 1. For a groupoid A , there is a groupoid $\mathbf{Sym}(A)$ with as objects the finite lists $(a_1 \dots a_n)$ of objects of A , and as morphisms $(a_1 \dots a_n) \rightarrow (a'_1 \dots a'_m)$ the pairs $(\pi, (f_i)_{1 \leq i \leq n})$, where $\pi : \{1, \dots, n\} \cong \{1, \dots, m\}$ is a bijection and $f_i \in A(a_i, a'_{\pi(i)})$ for each $i = 1, \dots, n$.

More abstractly, $\mathbf{Sym}(A)$ is the *free symmetric (strict) monoidal category* over A . This extends to a pseudo-comonad on **Dist**, where *pseudo* means that the comonad laws only hold up to coherent invertible 2-cells [4], [35].

3) *Generalized species of structures*: The Kleisli bicategory $\mathbf{Dist}_{\mathbf{Sym}}$ is denoted **Esp**, and the morphisms in **Esp** are called *generalized species of structures* [4]. Concretely, **Esp** has the same objects as **Dist**; morphisms are generalized species³, defined as distributors from $\mathbf{Sym}(A)$ to B ; and 2-cells are natural transformations. Equipped with

$$\mathbf{Sym}(A + B) \simeq \mathbf{Sym}(A) \times \mathbf{Sym}(B) \quad \mathbf{Sym}(\emptyset) \simeq I$$

the Seely *equivalences*, **Dist** is a bicategorical model of linear logic. In particular, the bicategory **Esp** is cartesian closed.

Any functor $F : A \rightarrow B$ determines a pair of (adjoint) species $\hat{F} \in \mathbf{Esp}[A, B]$ and $\check{F} \in \mathbf{Esp}[B, A]$, defined as $\hat{F}((a_1, \dots, a_n), b) = \mathbf{Sym}(B)((F(a_1), \dots, F(a_n)), (b))$ and $\check{F}((b_1, \dots, b_n), a) = \mathbf{Sym}(B)((b_1, \dots, b_n), (F(a)))$.

4) *Relationship with PRRel*: Distributors conservatively extend the proof-relevant relations of §II-B: if we regard sets as discrete groupoids, we get an embedding $\mathbf{PRRel} \hookrightarrow \mathbf{Dist}$ that preserves the symmetric monoidal closed structure and the cartesian structure. Explicit symmetries appear essential in defining an exponential modality in a proof-relevant model: even when A is a discrete groupoid, $\mathbf{Sym}(A)$ is not discrete.

III. CONCURRENT GAMES AND STATIC COLLAPSE

We now construct a dynamic model based on concurrent games and strategies, without symmetries. We show that it has a static collapse in the model **PRRel** introduced in §II-B.

³If $A = B = 1$, then this corresponds to a *species* in the classical combinatorial sense [36]. Note that this can be further generalized to arbitrary small categories A and B [4], but we do not need this generality.

A. Rudiments of concurrent games

Game semantics presents computation in terms of a two-player game: *Player* plays for the program under scrutiny, while *Opponent* plays for the execution environment. So a program is interpreted as a *strategy* for Player, and this strategy is constrained by a notion of *game*, specified by the type. The framework of *concurrent games* ([37], [38], [39]) is not merely a game semantics for concurrency, but a deep reworking of the basic mechanisms of game semantics using causal “truly concurrent” structures from concurrency theory [40].

1) *Event structures*: Concurrent games and strategies are based on event structures. An event structure represents the behaviour of a system as a set of possible computational events equipped with dependency and incompatibility constraints.

Definition 2. An *event structure (es)* is $E = (|E|, \leq_E, \#_E)$, where $|E|$ is a (countable) set of *events*, \leq_E is a partial order called *causal dependency* and $\#_E$ is an irreflexive symmetric binary relation on $|E|$ called *conflict*, satisfying:

- (1) $\forall e \in |E|, [e]_E = \{e' \in |E| \mid e' \leq_E e\}$ is finite,
- (2) $\forall e_1 \#_E e_2, \forall e_2 \leq_E e'_2, e_1 \#_E e'_2$.

Operationally, an event can occur if *all* its dependencies are met, and *no* conflicting events have occurred. A finite set $x \subseteq_f |E|$ down-closed for \leq_E and comprising no conflicting pair is called a **configuration** – we write $\mathcal{C}(E)$ for the set of configurations on E , naturally ordered by inclusion. If $x \in \mathcal{C}(E)$ and $e \in |E|$ is such that $e \notin x$ but $x \cup \{e\} \in \mathcal{C}(E)$, we say that e is **enabled** by x and write $x \vdash_E e$. For $e_1, e_2 \in |E|$ we write $e_1 \rightarrow_E e_2$ for the **immediate causal dependency**, i.e. $e_1 <_E e_2$ with no event strictly in between.

There is an accompanying notion of *map*: a **map of event structures** from E to F is a function $f : |E| \rightarrow |F|$ such that: (1) for all $x \in \mathcal{C}(E)$, the direct image $fx \in \mathcal{C}(F)$; and (2) for all $x \in \mathcal{C}(E)$ and $e, e' \in x$, if $fe = fe'$ then $e = e'$. There is a category **ES** of event structures and maps.

2) *Games and strategies*: Throughout this paper, we will gradually refine our notion of game. For now, a **plain game** is simply an event structure A together with a **polarity** function $\text{pol}_A : |A| \rightarrow \{-, +\}$ which specifies, for each event $a \in A$, whether it is **positive** (i.e. due to Player / the program) or **negative** (i.e. due to Opponent / the environment). Events are often called **moves**, and annotated with their polarity.

A strategy is an event structure with a projection map to A :

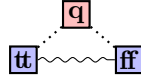
Definition 3. Consider A a plain game. A **strategy** on A , written $\sigma : A$, is an event structure σ together with a map $\partial_\sigma : \sigma \rightarrow A$ called the **display map**, satisfying:

- (1) for all $x \in \mathcal{C}(\sigma)$ and $\partial_\sigma x \vdash_A a^-$, there is a unique $x \vdash_\sigma s$ such that $\partial_\sigma s = a$.
- (2) for all $s_1 \rightarrow_\sigma s_2$, if $\text{pol}_A(\partial_\sigma(s_1)) = +$ or $\text{pol}_A(\partial_\sigma(s_2)) = -$, then $\partial_\sigma(s_1) \rightarrow_A \partial_\sigma(s_2)$.

Informally, the two conditions (called *receptivity* and *courtesy*) ensure that the strategy does not constrain the behaviour of Opponent any more than the game does. They are essential

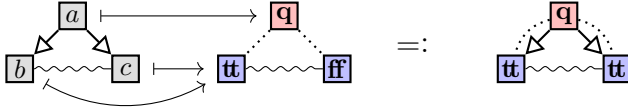
for the bicategorical structure we describe below [41], but they do not play a major role in this paper.

As a simple example, the usual game \mathbb{B} for booleans is



drawn from top to bottom (Player moves are blue, and Opponent moves are red): Opponent initiates computation with the first move q , to which Player can react with either tt or ff . The wiggly line indicates that tt and ff are in conflict.

Strategies are proof-relevant, in the sense that moves of the game can have multiple witnesses in the strategy. For example, on the left below, b and c are mapped to the same move tt :



Note that we denote immediate causality by \rightarrow in strategies, while we use dotted lines for games. This lets us represent the strategy in a single diagram, as on the right above.

3) *Morphisms between strategies*: For σ and τ two strategies on A , a **morphism** from σ to τ , written $f : \sigma \Rightarrow \tau$, is a map of event structures $f : \sigma \rightarrow \tau$ preserving the dependency relation \leq (we say it is **rigid**) and s.t. $\partial_\tau \circ f = \partial_\sigma$.

4) *+covered configurations*: We now describe a useful technical tool: a strategy is completely characterized by a subset of its configurations, called **+covered**.

For a strategy σ on a game A , a configuration $x \in \mathcal{C}(\sigma)$ is **+covered** if all its maximal events are positive, so every Opponent move has at least one Player successor. We write $\mathcal{C}^+(\sigma)$ for the partial order of **+covered** configurations of σ .

Lemma 1. Consider a plain game A , and strategies $\sigma, \tau : A$.

If $f : \mathcal{C}^+(\sigma) \cong \mathcal{C}^+(\tau)$ is an order-isomorphism such that $\partial_\tau \circ f = \partial_\sigma$, then there is a unique isomorphism of strategies $\hat{f} : \sigma \cong \tau$ such that for all $x \in \mathcal{C}^+(\sigma)$, $\hat{f}(x) = f(x)$.

B. A bicategory of concurrent games and strategies

We now define our bicategory of concurrent games.

1) *Strategies between games*: If A is a plain game, its **dual** A^\perp has the same components as A except for the reversed polarity. In particular $\mathcal{C}(A) = \mathcal{C}(A^\perp)$. The **tensor** $A \otimes B$ of A and B is simply A and B side by side, with no interaction – its events are the tagged disjoint union $|A \otimes B| = |A| + |B| = \{1\} \times |A| \uplus \{2\} \times |B|$, and other components inherited. We write $x_A \otimes x_B$ for the configuration of $A \otimes B$ that has $x_A \in \mathcal{C}(A)$ on the left and $x_B \in \mathcal{C}(B)$ on the right, so

$$- \otimes - : \mathcal{C}(A) \times \mathcal{C}(B) \cong \mathcal{C}(A \otimes B).$$

Finally, the **hom** $A \vdash B$ is $A^\perp \otimes B$; as above its configurations are denoted $x_A \vdash x_B$ for $x_A \in \mathcal{C}(A)$ and $x_B \in \mathcal{C}(B)$.

Definition 4. A **strategy from A to B** is a strategy on the game $A \vdash B$. If $\sigma : A \vdash B$ and $x^\sigma \in \mathcal{C}(\sigma)$, by convention we write $\partial_\sigma(x^\sigma) = x_A^\sigma \vdash x_B^\sigma \in \mathcal{C}(A \vdash B)$.

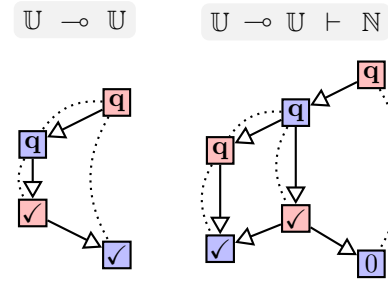


Fig. 1. An example of matching but causally incompatible configurations, in the composition of $\sigma : \mathbb{U} \multimap \mathbb{U}$ and $\tau : \mathbb{U} \multimap \mathbb{U} \vdash \mathbb{N}$. The underlying games are left undefined, but can be recovered by removing the arrows \multimap . The configurations are matching on $\mathbb{U} \multimap \mathbb{U}$, but the arrows \multimap impose incompatible orders (i.e. a cycle) between the two occurrences of \checkmark .

Our first example of a strategy between games is the **copycat** strategy \mathbf{c}_A , which is the identity morphism on A in our bicategory. Concretely, copycat on A has the same events as $A \vdash A$, but adds immediate causal links between copies of the same move across components. By Lemma 1, the following characterizes copycat up to isomorphism.

Proposition 1. If A is a game, there is an order-isomorphism

$$\mathbf{c}_{(-)} : \mathcal{C}(A) \cong \mathcal{C}^+(\mathbf{c}_A)$$

such that for all $x \in \mathcal{C}(A)$, $\partial_{\mathbf{c}_A}(\mathbf{c}_x) = x \vdash x$.

2) *Composition*: Consider $\sigma : A \vdash B$ and $\tau : B \vdash C$. We define their composition $\tau \odot \sigma : A \vdash C$. This is a dynamic model, and to successfully synchronize, σ and τ must agree to play the same events *in the same order*.

We say that configurations $x^\sigma \in \mathcal{C}(\sigma)$ and $x^\tau \in \mathcal{C}(\tau)$ are

- **matching**, if they reach the same configuration on B , i.e. $x_B^\sigma = x_B^\tau = x_B$; and
- **causally compatible** if, additionally, the synchronization is deadlock-free, in the sense that combining the causal dependencies in x^σ and x^τ gives an acyclic relation [41].

We illustrate a deadlock situation in Figure 1.

The **composition** of σ and τ is a strategy whose **+covered** configurations are causally compatible pairs of **+covered** configurations. Write $\mathbf{CC}(\sigma, \tau)$ for the set of causally compatible pairs $(x^\sigma, x^\tau) \in \mathcal{C}^+(\sigma) \times \mathcal{C}^+(\tau)$, ordered componentwise.

Proposition 2 ([26]). There is a strategy $\tau \odot \sigma : A \vdash C$, unique up to isomorphism, with an order-isomorphism

$$- \odot - : \mathbf{CC}(\sigma, \tau) \cong \mathcal{C}^+(\tau \odot \sigma)$$

s.t. for all $x^\sigma \in \mathcal{C}^+(\sigma)$ and $x^\tau \in \mathcal{C}^+(\tau)$ causally compatible,

$$\partial_{\tau \odot \sigma}(x^\tau \odot x^\sigma) = x_A^\sigma \vdash x_C^\tau.$$

This description of composition emphasizes the conceptual difference between a static model, in which composition is based on matching pairs as in (3), and a dynamic model, based on causal compatibility and sensitive to deadlocks.

Theorem 1. *There is a bicategory \mathbf{CG} with: objects, plain games; morphisms from A to B , strategies on $A \vdash B$; and 2-cells, morphisms between strategies.*

C. A static collapse of concurrent games

We describe an oplax functor $\|- \| : \mathbf{CG} \rightarrow \mathbf{PRRel}$ – oplax because composition is not preserved: instead we have a coherent, non-invertible 2-cell $\|\tau \odot \sigma\| \rightarrow \|\tau\| \circ \|\sigma\|$ which embeds the causally compatible pairs into the matching pairs.

The image of a plain game A is the set $\mathcal{C}(A)$. To a strategy $\sigma : A \vdash B$, we associate the proof-relevant relation

$$\|\sigma\|(x_A, x_B) = \{x^\sigma \in \mathcal{C}^+(\sigma) \mid \partial_\sigma(x^\sigma) = x_A \vdash x_B\}$$

and to $f : \sigma \Rightarrow \sigma'$ we associate the function which to any $x^\sigma \in \|\sigma\|(x_A, x_B)$ associates $f(x^\sigma) \in \|\sigma'\|(x_A, x_B)$.

Proposition 1 induces an isomorphism of $\|\mathbf{c}_A\|$ with the identity proof-relevant relation. From Proposition 2 we obtain a function $\|\tau \odot \sigma\| \rightarrow \|\tau\| \circ \|\sigma\|$, not invertible in general because some matching pairs are not causally compatible.

Theorem 2. *The above data determines an oplax functor of bicategories $\|- \| : \mathbf{CG} \rightarrow \mathbf{PRRel}$.*

In summary, we can regard \mathbf{CG} as a *dynamic* version of \mathbf{PRRel} , where the witnesses in \mathbf{PRRel} are reached over time and composition is sensitive to deadlocks.

IV. ACCOMMODATING SYMMETRY

In this section, we look at a model of concurrent games enriched with symmetry, known as thin concurrent games [3]. We start by explaining the basics of event structures with symmetry and thin concurrent games (§IV-A). Then we explain how strategies in thin concurrent games can be viewed as distributors (§IV-B). We then define the bicategory \mathbf{TCG} (§IV-C) and construct an oplax functor $\mathbf{TCG} \rightarrow \mathbf{Dist}$ (§IV-D). Finally we discuss the exponential modality (§IV-E).

A. Symmetry and thin concurrent games

Recall that we went from \mathbf{PRRel} to \mathbf{Dist} by replacing sets with groupoids. We now go from \mathbf{CG} to \mathbf{TCG} by replacing the set of configurations $\mathcal{C}(A)$ with a groupoid of configurations $\mathcal{S}(A)$ whose morphisms are chosen bijections called *symmetries*, that behave well w.r.t. the causal order.

1) *Event structures with symmetry:* Our model is based on the following notion of event structure with symmetry [42]:

Definition 5. *An isomorphism family on es E is a groupoid $\mathcal{S}(E)$ having as objects all configurations, and as morphisms certain bijections between configurations, satisfying:*

- restriction: for all $\theta : x \simeq y \in \mathcal{S}(E)$ and $x \supseteq x' \in \mathcal{C}(E)$, there is $\theta \supseteq \theta' \in \mathcal{S}(E)$ such that $\theta' : x' \simeq y'$.
- extension: for all $\theta : x \simeq y \in \mathcal{S}(E)$, $x \subseteq x' \in \mathcal{C}(E)$, there is $\theta \subseteq \theta' \in \mathcal{S}(E)$ such that $\theta' : x' \simeq y'$.

We call $(E, \mathcal{S}(E))$ an **event structure with symmetry (ess)**.

We refer to morphisms in $\mathcal{S}(E)$ as **symmetries**, and write $\theta : x \cong_E y$ if $\theta : x \simeq y$ with $\theta \in \mathcal{S}(E)$. The **domain** $\text{dom}(\theta)$ of $\theta : x \cong_E y$ is x , and likewise its **codomain** $\text{cod}(\theta)$ is y .

A **map of ess** $E \rightarrow F$ is a map of event structures that preserves symmetry: for every $\theta : x \cong_E y$, the bijection

$$f\theta \stackrel{\text{def}}{=} fx \stackrel{f^{-1}}{\simeq} x \stackrel{\theta}{\simeq} y \stackrel{f}{\simeq} fy,$$

is in $\mathcal{S}(F)$. (Recall that f restricted to any y is bijective.) This makes $f : \mathcal{S}(E) \rightarrow \mathcal{S}(F)$ a functor of groupoids.

We can define a 2-category \mathbf{ESS} of ess, maps of ess, and natural transformations between the induced functors. For $f, g : E \rightarrow F$ such a natural transformation is necessarily unique [42], and corresponds to the fact that for every $x \in \mathcal{C}(E)$ the bijection $\theta^x = \{(fs, gs) \mid s \in x\} : fx \simeq gx$ is in $\mathcal{S}(F)$. So this is really an equivalence, denoted $f \sim g$.

2) *Thin games:* We define games with symmetry. To match the polarized structure, a game is an ess with two sub-symmetries, one for each player (see e.g. [3], [18], [43]).

Definition 6. *A thin concurrent game (tcg) is a game A with isomorphism families $\mathcal{S}(A), \mathcal{S}_+(A), \mathcal{S}_-(A)$ s.t. $\mathcal{S}_+(A), \mathcal{S}_-(A) \subseteq \mathcal{S}(A)$, symmetries preserve polarity, and*

- (1) if $\theta \in \mathcal{S}_+(A) \cap \mathcal{S}_-(A)$, then $\theta = \text{id}_x$ for $x \in \mathcal{C}(A)$,
- (2) if $\theta \in \mathcal{S}_-(A)$, $\theta \subseteq^- \theta' \in \mathcal{S}(A)$, then $\theta' \in \mathcal{S}_-(A)$,
- (3) if $\theta \in \mathcal{S}_+(A)$, $\theta \subseteq^+ \theta' \in \mathcal{S}(A)$, then $\theta' \in \mathcal{S}_+(A)$,

where $\theta \subseteq^p \theta'$ is $\theta \subseteq \theta'$ with (pairs of) events of polarity p .

Elements of $\mathcal{S}_+(A)$ (resp. $\mathcal{S}_-(A)$) are called **positive** (resp. **negative**); they intuitively correspond to symmetries carried by positive (resp. negative) moves, and thus introduced by Player (resp. Opponent). We write $\theta : x \cong_A^- y$ (resp. $\theta : x \cong_A^+ y$) if $\theta \in \mathcal{S}_-(A)$ (resp. $\theta \in \mathcal{S}_+(A)$).

Each symmetry has a unique positive-negative factorization:

Lemma 2 ([3]). *For A a tcg and $\theta : x \cong_A z$, there are unique $y \in \mathcal{C}(A)$, $\theta_- : x \cong_A^- y$ and $\theta_+ : y \cong_A^+ z$ s.t. $\theta = \theta_+ \circ \theta_-$.*

Sketch. Existence is proved by induction on θ , using conditions (2) and (3); uniqueness follows from (1) with the fact that $\mathcal{S}_-(A)$ and $\mathcal{S}_+(A)$ are groupoids. \square

We extend with symmetry the basic constructions on games:

- The **dual** A^\perp has the same symmetries as A , but $\mathcal{S}_+(A^\perp) = \mathcal{S}_-(A)$ and $\mathcal{S}_-(A^\perp) = \mathcal{S}_+(A)$.
- The **tensor** $A_1 \otimes A_2$ has symmetries of the form $\theta_1 \otimes \theta_2 : x_1 \otimes x_2 \cong_{A_1 \otimes A_2} y_1 \otimes y_2$, where each $\theta_i : x_i \cong_{A_i} y_i$, and similarly for positive and negative symmetries.
- As before, the **hom** $A \vdash B$ is $A^\perp \otimes B$.

3) *Thin strategies:* We now define strategies on thin concurrent games, as an extension of strategies on plain games.

Definition 7. *Consider A a tcg. A strategy on A , written $\sigma : A$, is an ess σ equipped with a morphism of ess $\partial_\sigma : \sigma \rightarrow A$ forming a strategy in the sense of Definition 3, and such that:*

- (1) if $\theta \in \mathcal{S}(\sigma)$ and $\partial_\sigma \theta \vdash_A (a^-, b^-)$, there are unique $\theta \vdash_\sigma (s, t)$ s.t. $\partial_\sigma s = a$ and $\partial_\sigma t = b$.
- (2) if $\theta : x \cong_\sigma y$, $\partial_\sigma \theta \in \mathcal{S}_+(A)$, then $x = y$ and $\theta = \text{id}_x$.

As before, a **strategy from A to B** is a strategy on $\sigma : A \vdash B$.

The first condition forces σ to acknowledge Opponent symmetries in A ; the notation $\theta \vdash_A (a, b)$ means $(a, b) \notin \theta$ and $\theta \cup \{(a, b)\} \in \mathcal{S}(A)$. The second condition is **thinness**: it means that any non-identity symmetry in the strategy must originate from an Opponent symmetry.

4) *Comparison with the “saturated” approach*: The “thin” approach is only one possible way of adding symmetry to games. Other models (e.g. [17], [44], [45]) follow a different approach, where strategies satisfy a *saturation* condition.

We explain the difference in the language of concurrent games. Consider a strategy $\sigma : A$ on a tcg, in the sense of Definition 7 *without* conditions (1) and (2). The saturation condition [17] corresponds to a fibration property of the functor $\partial_\sigma : \mathcal{S}(\sigma) \rightarrow \mathcal{S}(A)$: for $x \in \mathcal{C}(\sigma)$ and $\psi : \partial_\sigma x \cong_A y$, there is a unique $\varphi : x \cong_\sigma z$ such that $\partial_\sigma \varphi = \psi$:

$$\begin{array}{ccc} \mathcal{S}(\sigma) & & x \overset{\varphi}{\dashrightarrow} z \\ \partial_\sigma \downarrow & & \downarrow \quad \downarrow \\ \mathcal{S}(A) & & \partial_\sigma x \xrightarrow{\psi} y \end{array} \quad (4)$$

In contrast, thin strategies satisfy a different lifting property: a unique lifting exists, up to a positive symmetry.

Lemma 3. *Let $\sigma : A$ be a strategy as in Definition 7.*

For all $x \in \mathcal{C}(\sigma)$ and $\psi : \partial_\sigma x \cong_A y$, there are unique $\varphi : x \cong_\sigma z$ and $\theta^+ : \partial_\sigma z \cong_A^+ y$ such that $\theta^+ \circ \partial_\sigma \varphi = \psi$:

$$\begin{array}{ccc} \mathcal{S}(\sigma) & & x \overset{\varphi}{\dashrightarrow} z \\ \partial_\sigma \downarrow & & \downarrow \quad \downarrow \\ \mathcal{S}(A) & & \partial_\sigma x \xrightarrow{\psi} y \xleftarrow{\theta^+} \partial_\sigma z \end{array}$$

Sketch. *Existence* is proved first for ψ positive, by induction on x using condition (1) of Definition 7 and properties of isomorphism families; and then generalized to arbitrary ψ . *Uniqueness* follows from condition (2) of Definition 7. \square

Below, we will use this to construct a distributor from a thin strategy. We note that saturated strategies are closer to distributors, because the saturation property (4) directly induces a functorial action of the groupoid $\mathcal{S}(A)$. However, saturated strategies are more difficult to understand operationally, and have not achieved the precision of thin strategies for languages with state, concurrency, or non-determinism.

B. Strategies to distributors

For tcgs A and B , we show how to construct a distributor

$$\|\sigma\| : \mathcal{S}(A)^{\text{op}} \times \mathcal{S}(B) \rightarrow \mathbf{Set}$$

from a strategy $\sigma : A \vdash B$. The key idea is to use witnesses “up to positive symmetry” and use the lifting property in Lemma 3.

For $x_A \in \mathcal{C}(A)$ and $x_B \in \mathcal{C}(B)$ we define the set of **positive witnesses** of (x_A, x_B) , written $\|\sigma\|(x_A, x_B)$, as the set of all triples $(\theta_A^-, x^\sigma, \theta_B^+)$ such that $x^\sigma \in \mathcal{C}^+(\sigma)$ and

$$\theta_A^- : x_A \cong_A^- x_A^\sigma \quad \theta_B^+ : x_B^\sigma \cong_B^+ x_B$$

are positive symmetries on A^\perp and B . The groupoid actions of A and B on this set are determined by the uniqueness result:

Proposition 3. *Consider $(\theta_A^-, x^\sigma, \theta_B^+) \in \|\sigma\|(x_A, x_B)$.*

For each $\Omega_A : y_A \cong_A x_A$ and $\Omega_B : x_B \cong_B y_B$, there are unique $\varphi : x^\sigma \cong_\sigma y^\sigma$ and $\vartheta_A^- : y_A \cong_A^- x_A^\sigma$, $\vartheta_B^- : y_B^\sigma \cong_B^- x_B$ such that the following two diagrams commute:

$$\begin{array}{ccc} x_A \xrightarrow{\theta_A^-} x_A^\sigma & & x_B^\sigma \xrightarrow{\theta_B^+} x_B \\ \Omega_A \uparrow & \varphi_A^\sigma \downarrow & \varphi_B^\sigma \downarrow \\ y_A \xrightarrow{\vartheta_A^-} y_A^\sigma & & y_B^\sigma \xrightarrow{\vartheta_B^-} y_B \\ & & \Omega_B \downarrow \end{array}$$

Proof. *Existence* by Lem. 3, *uniqueness* by (2) of Def. 7. \square

Thus we may set $\|\sigma\|(\Omega_A, \Omega_B)(\theta_A^-, x^\sigma, \theta_B^+)$ as the positive witness $(\vartheta_A^-, y^\sigma, \vartheta_B^+)$ above. This is a distributor.

Proposition 4. *We have $\|\sigma\| : \mathcal{S}(A)^{\text{op}} \times \mathcal{S}(B) \rightarrow \mathbf{Set}$.*

C. The bicategory of thin concurrent games

We now define the bicategory **TCG** of thin concurrent games – note that we have already defined the objects (Definition 6) and the morphisms (Definition 7).

Morphisms of strategies: The 2-cells of **TCG** are more liberal than those in **CG**, because there should be an isomorphism between two strategies which play symmetric moves. Recall the 2-dimensional structure in **ESS**, given by the equivalence relation \sim on morphisms (§IV-A1). For two maps $f, g : E \rightarrow A$ into a tcg, we write $f \sim^+ g$ if $f \sim g$ and for every $x \in \mathcal{C}(E)$ the symmetry $\theta^x : fx \cong_A gx$ is positive.

For strategies $\sigma, \tau : A$ on a tcg, a **positive morphism of strategies** $f : \sigma \Rightarrow \tau$ is a rigid map of *ess* s.t. $\partial_\tau \circ f \sim^+ \partial_\sigma$.

Composition and identity: The composition of thin strategies $\sigma : A \vdash B$ and $\tau : B \vdash C$ is obtained by equipping $\tau \circ \sigma$ (Proposition 2) with an adequate isomorphism family. If $\mathcal{S}^+(\sigma)$ is the restriction of $\mathcal{S}(\sigma)$ to $+$ -covered configurations, then we can write $\mathbf{CC}(\mathcal{S}^+(\sigma), \mathcal{S}^+(\tau))$ for the pairs $(\varphi^\sigma, \varphi^\tau)$ of symmetries which are matching, i.e. $\varphi_B^\sigma = \varphi_B^\tau$ and whose domain (and necessarily, codomain) are causally compatible.

Proposition 5. *There is a unique symmetry on $\tau \circ \sigma$ with*

$$(- \circ -) : \mathbf{CC}(\mathcal{S}^+(\sigma), \mathcal{S}^+(\tau)) \simeq \mathcal{S}^+(\tau \circ \sigma)$$

a bijection commuting with dom and cod, and compatible with display maps, i.e. $(\varphi^\tau \circ \varphi^\sigma)_A = \varphi_A^\sigma$ and $(\varphi^\tau \circ \varphi^\sigma)_C = \varphi_C^\tau$.

For the identity in **TCG**, we equip *copycat* $\mathbf{c}_A : A \vdash A$ (Proposition 1) with the unique symmetry that has an iso

$$\mathbf{c}_{(-)} : \mathcal{S}(A) \simeq \mathcal{S}^+(\mathbf{c}_A)$$

commuting with dom and cod, such that $\partial_{\mathbf{c}_A}(\mathbf{c}_\theta) = \theta \vdash \theta$.

Proposition 6 ([46]). *There is a bicategory **TCG**, and an embedding $\mathbf{CG} \hookrightarrow \mathbf{TCG}$ that preserves all structure.*

Proof note. Most of the effort is spent on the difficulty of composing 2-cells “horizontally”, i.e. proving that if $\sigma, \sigma' : A \vdash B$ and $\tau : B \vdash C$, then we can turn positive morphisms

$f : \sigma \Rightarrow \sigma'$ and $g : \tau \Rightarrow \tau'$ into a positive map $\tau \odot f : \tau \odot \sigma \Rightarrow \tau \odot \sigma'$, using an inductive construction based on thin-ness. \square

In summary, we have defined a dynamic model with symmetries. This model embeds the basic model **CG** from §III-A, and supports an exponential modality (§IV-E).

D. An oplax functor $\mathbf{TCG} \rightarrow \mathbf{Dist}$

We give the components of a pseudofunctor $\mathbf{TCG} \rightarrow \mathbf{Dist}$. We have already explained the action on objects and morphisms, by turning every strategy into a distributor (§IV-B).

From positive morphisms to natural transformations: We show that $f : \sigma \Rightarrow \tau : A \vdash B$ gives a natural transformation of distributors $\|\sigma\| \Rightarrow \|\tau\|$. Its components are the functions

$$\begin{aligned} \|f\|_{x_A, x_B} : \|\sigma\|(x_A, x_B) &\rightarrow \|\tau\|(x_A, x_B) \\ (\theta_A^-, x^\sigma, \theta_B^+) &\mapsto (\theta_A^x \circ \theta_A^-, f(x^\sigma), \theta_B^+ \circ \theta_B^{x^{-1}}) \end{aligned}$$

for $x_A \in \mathcal{C}(A)$ and $x_B \in \mathcal{C}(B)$, where $\theta_A^x : x_A^\sigma \cong_A^- f(x^\sigma)_A$ and $\theta_B^x : x_B^\sigma \cong_B^+ f(x^\sigma)_B$ come from $\partial_\tau \circ f \sim^+ \partial_\sigma$ (§IV-C). This is natural, as an application of Proposition 3.

The unitor and compositor: We now explain in what sense the operation $\|-\|$ is functorial, by giving the appropriate structural 2-cells for an oplax functor. We start with the **unitor**:

Proposition 7. *Consider A a tcg. Then, there is a natural iso*

$$\text{pid}^A : \|\mathbf{c}_A\| \xrightarrow{\cong} \mathcal{S}(A)[-,-] : \mathcal{S}(A)^{\text{op}} \times \mathcal{S}(A) \rightarrow \mathbf{Set}.$$

Proof. Consider $(\theta^-, \mathbf{c}_z, \theta^+) \in \|\mathbf{c}_A\|(x, y)$, with $\theta^- : x \cong_A^- z$ and $\theta^+ : z \cong_A^+ y$. We set $\text{pid}^A(\theta^-, \mathbf{c}_z, \theta^+) = \theta^+ \circ \theta^-$; naturality and invertibility follow from Lemma 2. \square

Now, we focus on the preservation of composition. For two strategies $\sigma : A \vdash B$ and $\tau : B \vdash C$, we have the **compositor**:

Proposition 8. *There is a natural transformation:*

$$\text{pcomp}^{\sigma, \tau} : \|\tau \odot \sigma\| \Rightarrow \|\tau\| \bullet \|\sigma\| : \mathcal{S}(A)^{\text{op}} \times \mathcal{S}(B) \rightarrow \mathbf{Set}.$$

Proof. Consider $(\theta_A^-, x^\tau \odot x^\sigma, \theta_C^+) \in \|\tau \odot \sigma\|(x_A, x_C)$; this is sent by $\text{pcomp}^{\sigma, \tau}_{x_A, x_C}$ to (the equivalence class of) the pair

$$((\theta_A^-, x^\sigma, \text{id}_{x_B}), (\text{id}_{x_B}, x^\tau, \theta_C^+)) \in (\|\tau\| \bullet \|\sigma\|)(x_A, x_C)$$

for $x_B^\sigma = x_B^\tau = x_B$. \square

Altogether, the operation $\|-\|$ equipped with this satisfies:

Theorem 3. *We have an oplax functor $\|-\| : \mathbf{TCG} \rightarrow \mathbf{Dist}$.*

E. Difficulties with the exponential modality

1) *An exponential modality in polarized **TCG**:* We show how to construct an exponential modality on **TCG**. For an ess E , the ess $!E$ is an infinitary symmetric tensor product, where the elements of the indexing set are called **copy indices**:

Definition 8. *Consider E an ess.*

Then $!E$ has: events, $|!E| = \mathbb{N} \times |E|$; causality and conflict inherited transparently. The isomorphism family comprises all

$$\theta : \Sigma_{i \in \mathbb{N}} x_i \simeq \Sigma_{i \in \mathbb{N}Y} y_i$$

such that there is a bijection $\pi : \mathbb{N} \simeq \mathbb{N}$ and for every $i \in \mathbb{N}$, a symmetry $\theta_i : x_i \cong_A y_{\pi(i)}$, such that $\theta(i, a) = (\pi(i), \theta_i(a))$.

To extend this to tcgs, we must treat the positive and negative symmetries. Intuitively, symmetries that only change the copy indices of negative moves should be negative, and likewise for positive moves – but this naive definition does not yield a tcg in general [3]. We must restrict to a polarized setting in which tcgs are **negative**, meaning that all minimal events are negative. For a negative tcg A , a symmetry $\theta \in \mathcal{S}(!A)$ is in the sub-family $\mathcal{S}_-(!A)$ if each θ_i in Definition 8 is negative in $\mathcal{S}(A)$, whereas θ is in $\mathcal{S}_+(!A)$ if each θ_i is in $\mathcal{S}_+(A)$ and *additionally* π is the identity bijection. This extends to a pseudo-comonad, see [3] for details.

2) *Our functor does not preserve the modality:* For a negative tcg A , the two groupoids $\mathcal{S}(!A)$ and $\mathbf{Sym}(\mathcal{S}(A))$ are not equivalent in general. This can be seen even if A is the empty game: then $!A$ is empty and $\mathcal{S}(!A)$ is the singleton groupoid, while $\mathbf{Sym}(\mathcal{S}(A))$ has countably many non-isomorphic objects $\emptyset, \emptyset\emptyset, \emptyset\emptyset\emptyset$, and so on. Intuitively, like the relational model, **Esp** records how many times we “do nothing”, whereas **TCG** only records when we do something.

Thus, although one can construct a cartesian closed Kleisli bicategory from the restriction of **TCG** to negative games [46], the functor $\|-\|$ will not preserve cartesian closed structure – we shall resolve this in the next section.

V. A CARTESIAN CLOSED PSEUDOFUNCTOR

TCG is fairly agnostic to the programming language or system being represented, but to close up the distance to **Esp** we must specialise the games and strategies to those involved in the interpretation of pure functional languages. We construct a refined bicategory **Vis** (§V-A), and a pseudofunctor $\mathbf{Vis} \rightarrow \mathbf{Dist}$ (Theorem 4) that we extend to a cartesian closed pseudofunctor between the Kleisli bicategories (§V-B-V-D).

A. The Bicategory **Vis** of Winning Visible Strategies

1) *Arenas:* The objects of our refined model are called *arenas*. Arenas are defined to achieve two goals: firstly, narrow down the causal structure to an *alternating forest*, required for the definition of visible (deadlock-free) strategies later on; secondly, introduce a notion of *payoff* to distinguish between incomplete and complete executions in a game, since only the latter are represented in **Dist**. As we will see, complete executions have payoff 0, and if the payoff is 1 (resp. -1) then the execution is incomplete because Opponent (resp. Player) is stalling. We adapt definitions from [25] (see also [13]):

Definition 9. *An arena is a tcg A such that*

- (1) *if $a_1, a_2 \leq_A a_3$ then $a_1 \leq_A a_2$ or $a_2 \leq_A a_1$,*
- (2) *if $a_1 \rightarrow_A a_2$, then $\text{pol}_A(a_1) \neq \text{pol}_A(a_2)$,*

*equipped with a function $\kappa_A : \mathcal{C}(A) \rightarrow \{-1, 0, +1\}$ called the **payoff**, preserved by all symmetries.*

*Moreover, A is called a **--arena** if A is negative as a tcg and $\kappa_A(\emptyset) \geq 0$. It is **strict** if it is negative, $\kappa_A(\emptyset) = 1$ and all its minimal events are in pairwise conflict.*

\otimes	-1	0	1
-1	-1	-1	-1
0	-1	0	1
1	-1	1	1

\wp	-1	0	1
-1	-1	-1	1
0	-1	0	1
1	1	1	1

Fig. 2. Payoff tables for operations on arenas, with $A \wp B = (A^\perp \otimes B^\perp)^\perp$.

The **dual** of an arena A has $\kappa_{A^\perp}(x_A) = -\kappa_A(x_A)$. The **tensor** of tcgs extends to arenas with $\kappa_{A \otimes B}(x_A \otimes x_B) = \kappa_A(x_A) \otimes \kappa_B(x_B)$ with \otimes described in Figure 2. Its De Morgan dual, the **par** $A \wp B$ of arenas, is also based on the tensor tcg (written $A \wp B$ for disambiguation, with action on configurations written $x_A \wp x_B \in \mathcal{C}(A \wp B)$), and $\kappa_{A \wp B}(x_A \wp x_B) = \kappa_A(x_A) \wp \kappa_B(x_B)$. Finally, the **hom** of A and B is defined as $A \vdash B = A^\perp \wp B$.

2) *Winning strategies*: As mentioned above, configurations with null payoff are those appearing in **Dist**. The others are intermediate stages, that only appear in the dynamic model. A strategy is *winning* if Player never stalls:

Definition 10. Consider A an arena. A strategy $\sigma : A$ is *winning* if for all $x^\sigma \in \mathcal{C}^+(\sigma)$, $\kappa_A(\partial_\sigma x^\sigma) \geq 0$.

Winning strategies compose, and copycat is winning [25].

3) *Visible strategies*: Visibility captures a property of purely-functional parallel programs, in which threads may fork and join but each should be a well-formed stand-alone sequential execution. In an event structure E , a thread is formalized as a **grounded causal chain (gcc)**, i.e. a finite set $\rho \subseteq_f |E|$ on which \leq_E is a total order, forming a sequence

$$\rho_1 \rightarrow_E \dots \rightarrow_E \rho_n$$

where ρ_1 is minimal in E . We write $\text{gcc}(E)$ for the set of gccs. A gcc need not be a configuration (although it will always be if the strategy interprets a sequential program). A strategy is *visible* if gccs only reach valid states of the arena:

Definition 11 ([47]). Consider A a --arena, and $\sigma : A$. Then σ is *visible* if it is **negative**, i.e. all minimal events of σ display to a negative event, and for all $\rho \in \text{gcc}(\sigma)$, $\partial_\sigma \rho \in \mathcal{C}(A)$.

This definition is analogous to visibility in Hyland-Ong games [6]. The key property of visible strategies for this paper is that their composition is always deadlock-free [26]:

Lemma 4. Consider visible $\sigma : A \vdash B$ and $\tau : B \vdash C$.

If $x^\sigma \in \mathcal{C}^+(\sigma)$, $x^\tau \in \mathcal{C}^+(\tau)$ are matching, then they are necessarily also causally compatible (§III-B2).

4) *A pseudofunctor*: The results of this section take place in the bicategory **Vis** with objects: --arenas; morphisms from A to B : winning visible strategies on $A \vdash B$; and 2-cells: positive morphisms $f : \sigma \Rightarrow \tau$.

We define a pseudofunctor $\mathbf{Vis} \rightarrow \mathbf{Dist}$ by restricting the collapse functor $\mathbf{TCG} \rightarrow \mathbf{Dist}$ (§IV-D) to complete configurations: if A is an arena, we write $\mathcal{G}(A)$ for the full sub-groupoid of $\mathcal{S}(A)$ whose objects are restricted to the $x \in \mathcal{C}(A)$ with null payoff, i.e. such that $\kappa_A(x) = 0$.

It is straightforward that for $\sigma \in \mathbf{Vis}[A, B]$, the distributor $\|\sigma\|$ restricts to a distributor $\mathcal{G}(A)^{\text{op}} \times \mathcal{G}(B) \rightarrow \mathbf{Set}$, which we still write $\|\sigma\|$. By the deadlock-freeness property, this gives a functor which is not oplax but *pseudo*:

Theorem 4. There is a pseudofunctor $\|_ \| : \mathbf{Vis} \rightarrow \mathbf{Dist}$, with $\|A\| = \mathcal{G}(A)$ the configurations of null payoff.

Proof. The natural transformation $\text{pcomp}^{\sigma, \tau} : \|\tau \circ \sigma\| \Rightarrow \|\tau\| \bullet \|\sigma\|$ for preservation of composition is still valid, as witnesses of complete configurations in $\tau \circ \sigma$ must synchronize on complete configurations.

We show that $\text{pcomp}^{\sigma, \tau}(x_A, x_C)$ is surjective. Consider

$$\begin{aligned} w^\sigma &= (\theta_A^-, x^\sigma, \theta_B^+) \in \|\sigma\|(x_A, x_B) \\ w^\tau &= (\theta_B^-, x^\tau, \theta_C^+) \in \|\tau\|(x_B, x_C) \end{aligned}$$

composable witnesses. By Lemma 4, $(x^\sigma, \theta_B^- \circ \theta_B^+, x^\tau)$ is causally compatible. By well-known properties of synchronization in thin concurrent games, there are unique $y^\tau \circ y^\sigma \in \mathcal{C}^+(\tau \circ \sigma)$ along with $\varphi^\sigma, \varphi^\tau, \vartheta_A^-, \vartheta_C^+$ such that:

$$\begin{array}{ccccccc} & & \theta_A^- & & x_A^\sigma & & \theta_B^+ \\ & & \nearrow & & \downarrow \varphi_A^\sigma & & \\ x_A & & & & y_A^\sigma & & \\ & & \searrow \vartheta_A^- & & & & \\ & & & & y_B^\sigma = y_B & = & y_B^\tau \\ & & & & \downarrow \varphi_B^\sigma & & \downarrow \varphi_B^\tau \\ & & & & & & y_C^\tau \\ & & & & & & \downarrow \varphi_C^\tau \\ & & & & & & x_C \\ & & & & & & \uparrow \vartheta_C^+ \end{array}$$

which, writing $\Theta_B = \varphi_B^\sigma \circ \theta_B^+{}^{-1} = \varphi_B^\tau \circ \theta_B^-$, entails

$$\begin{aligned} v^\sigma &= (\vartheta_A^-, y^\sigma, \text{id}_{y_B}) = \Theta_B \cdot (\theta_A^-, x^\sigma, \theta_B^+) \\ v^\tau &= (\text{id}_{y_B}, y^\tau, \vartheta_C^+) = (\theta_B^-, x^\tau, \theta_C^+) \cdot \Theta_B \end{aligned}$$

so $(v^\sigma, v^\tau) = (\Theta_B \cdot w^\sigma, v^\tau) \sim (w^\sigma, v^\tau \cdot \Theta_B) = (w^\sigma, w^\tau)$. Now $(v^\sigma, v^\tau) = \text{pcomp}^{\sigma, \tau}(\vartheta_A^-, y^\tau \circ y^\sigma, \vartheta_C^+)$, showing surjectivity.

Injectivity follows from the *uniqueness* clause above. \square

B. Kleisli bicategories, and relating them

Next, we compare the exponential modalities in **Vis** (written **!**) and in **Dist** (written **Sym**).

1) *The exponential modality for arenas*: The construction of $!A$ as a countable symmetric tensor of copies of A (§IV-E) can be extended to arenas. Note that any configuration of $!A$ has a representation as $\sum_{i \in I} x_i$ for $I \subseteq_f \mathbb{N}$, and this representation is unique if we insist that every x_i is non-empty.

Using that, we set (all x_i below are non-empty):

$$\begin{aligned} \kappa_{!A} &: \mathcal{C}(!A) \rightarrow \{-1, 0, +1\} \\ \sum_{i \in I} x_i &\mapsto \bigotimes_{i \in I} \kappa_A(x_i) \end{aligned}$$

that is well-defined because \otimes is associative on $\{-1, 0, +1\}$.

2) *Strict arenas*: To precisely capture the relationship between **!** and **Sym** we use *strict* arenas (Def. 9), where \emptyset has payoff 1 and is not considered complete. The situation with the empty configuration was at the heart of the issue in §IV-E2.

We now state the following key property: for strict arenas, the two constructions **Sym** and **!** are equivalent:

Proposition 9. Consider a strict arena A .

There is an adjoint equivalence of categories:

$$L_A^! : \mathcal{G}(!A) \simeq \mathbf{Sym}(\mathcal{G}(A)) : R_A^!$$

Proof. For a strict arena A , we can identify the objects of $\mathcal{G}(!A)$ with families $(x_i)_{i \in I}$ of objects of $\mathcal{G}(A)$, where I is a finite subset of natural numbers.

Thus, from left to right, $L_A^!$ sends $(x_i)_{i \in I}$ to the sequence $x_{i_1} \dots x_{i_n}$ for $I = \{i_1, \dots, i_n\}$ sorted in increasing order. From right to left, $R_A^!$ sends $x_0 \dots x_n$ to $(x_i)_{i \in \{0..n\}}$. \square

Although this equivalence only holds for *strict* arenas, it is all we need for a cartesian closed pseudofunctor.

3) *Relative pseudocomonads:* The pseudofunctor $\mathbf{Vis} \rightarrow \mathbf{Dist}$ does not preserve the exponential modality as a pseudocomonad on \mathbf{Vis} , but as a pseudocomonad *relative* to the sub-bicategory of *strict* arenas. We recall the categorical notions.

Recall that a monad on category \mathcal{C} relative to a functor $J : \mathcal{D} \rightarrow \mathcal{C}$ is a functor $T : \mathcal{D} \rightarrow \mathcal{C}$ with a restricted monadic structure, which we can use to form a Kleisli category \mathcal{C}_T with objects those of \mathcal{D} . (Often, \mathcal{D} is a sub-bicategory of \mathcal{C} and J is the inclusion functor.) This generalizes to relative pseudomonads [48] and pseudocomonads:

Definition 12. Consider $J : \mathcal{C} \rightarrow \mathcal{D}$ a pseudofunctor between bicategories. A **relative pseudocomonad** T over J consists of:

- (1) an object $TX \in \mathcal{D}$, for every $X \in \mathcal{C}$,
- (2) a family of functors $(-)^*_{X,Y} : \mathcal{D}[TX, JY] \rightarrow \mathcal{D}[TX, TY]$,
- (3) a family of morphisms $i_X \in \mathcal{D}[TX, JX]$,
- (4) a natural family of invertible 2-cells,

$$\mu_{f,g} : (g \circ f^*)^* \xrightarrow{\cong} g^* \circ f^*$$

for $f \in \mathcal{D}[TX, JY]$ and $g \in \mathcal{D}[TY, JZ]$,

- (5) a natural family of invertible 2-cells, for $f \in \mathcal{D}[TX, JY]$:

$$\eta_f : f \xrightarrow{\cong} i_X \circ f^*$$

- (6) a family of invertible 2-cells $\theta_X : i_X^* \xrightarrow{\cong} \text{id}_{TX}$, where X, Y, Z range over objects of \mathcal{C} . This is subject to coherence conditions that are omitted here.

Those conditions are exactly what is needed to form a Kleisli bicategory written \mathcal{D}_T , with *objects* those of \mathcal{C} , *morphisms* and *2-cells* from X to Y the category $\mathcal{D}[TX, JY]$. We can compose $f \in \mathcal{D}[TX, JY]$ and $g \in \mathcal{D}[TY, JZ]$ as $g \circ_T f = g \circ f^*$, and the *identity* on X is i_X .

4) *The exponential relative pseudocomonad:* Here, \mathcal{C} is the sub-bicategory \mathbf{Vis}_s of *strict* arenas, and $J : \mathbf{Vis}_s \hookrightarrow \mathbf{Vis}$ the embedding. Note that even if A is strict, $!A$ is not strict, and so $! : \mathbf{Vis}_s \rightarrow \mathbf{Vis}$.

We now outline the components in Definition 12. For the component (2), we must introduce some additional notions. Fix an injection $\langle -, - \rangle : \mathbb{N}^2 \rightarrow \mathbb{N}$. If $I \subseteq_f \mathbb{N}$ and $J_i \subseteq_f \mathbb{N}$ for all $i \in I$, write $\Sigma_{i \in I} J_i \subseteq_f \mathbb{N}$ for the set of all $\langle i, j \rangle$ for $i \in I$ and $j \in J_i$. Then, we may define:

Definition 13. Consider $\sigma \in \mathbf{Vis}[!A, B]$. Its **promotion** $\sigma^!$ has ess $!\sigma$, and display map the unique map of ess such that

$$\partial_{\sigma^!}((x^{\sigma,i})_{i \in I}) = (x_{A,j}^{\sigma,i})_{\langle i,j \rangle \in \Sigma_{i \in I} J_i} \vdash (x_B^{\sigma,i})_{i \in I} \quad (5)$$

where $\partial_{\sigma}(x^{\sigma,i}) = (x_{A,j}^{\sigma,i})_{j \in J_i} \vdash x_B^{\sigma,i}$.

For (3), the **dereliction** $\text{der}_A \in \mathbf{Vis}[!A, A]$ on strict A has ess \mathbf{c}_A , and display map $\partial_{\text{der}_A}(\mathbf{c}_x) = (x)_{\{0\}} \vdash x$. Then

$$\text{join}_{\sigma,\tau} : (\tau \circ \sigma^!)^! \xrightarrow{\cong} \tau^! \circ \sigma^!$$

sends $(x_i^{\tau} \circ (x_{i,j}^{\sigma}))_{j \in J_i}$ to $(x_i^{\tau})_{i \in I} \circ (x_{i,j}^{\sigma})_{\langle i,j \rangle \in \Sigma_{i \in I} J_i}$, a positive iso providing (4). For (5), given B strict and $\sigma \in \mathbf{Vis}[!A, B]$ we have a positive iso $\text{runit}_{\sigma} : \sigma \cong \text{der}_B \circ \sigma^!$ sending $x^{\sigma} \in \mathcal{C}^+(x)$ to $\mathbf{c}_{x_B} \circ (x^{\sigma})_{\{0\}} \in \mathcal{C}^+(\text{der}_B \circ \sigma^!)$. Finally, for (6) we have a positive iso $\text{lunit}_A : \text{der}_A^! \cong \mathbf{c}_{!A}$ sending $(\mathbf{c}_{x_i})_{i \in I}$ to $\mathbf{c}_{(x_i)_{i \in I}}$. Altogether, this gives us:

Theorem 5. The components described above define a pseudocomonad $!$ relative to the embedding of \mathbf{Vis}_s into \mathbf{Vis} .

In particular, there is a Kleisli bicategory $\mathbf{Vis}_!$ whose objects are strict arenas. (In the next section we show this is a cartesian closed bicategory.)

5) *Lifting of $\|-\|$ to the Kleisli bicategories:* Recall that we write \mathbf{Esp} for the Kleisli bicategory $\mathbf{Dist}_{\mathbf{Sym}}$. We show how to lift $\|-\|$ to a pseudofunctor $\|-\|_! : \mathbf{Vis}_! \rightarrow \mathbf{Esp}$. (We give a direct proof, although one could write down a notion of pseudofunctor between relative pseudocomonads that lifts to the Kleisli bicategories [49].)

Theorem 6. We have a pseudofunctor $\|-\|_! : \mathbf{Vis}_! \rightarrow \mathbf{Esp}$.

Proof. For A a strict arena, we have $\|A\|_! = \|A\| = \mathcal{G}(A)$.

If A, B are strict and $\sigma \in \mathbf{Vis}[!A, B]$, we $\|\sigma\|_!$ to be

$$\mathbf{Sym}(\mathcal{G}(A))^{\text{op}} \times \mathcal{G}(B) \xrightarrow{R_A^{\text{op}} \times \mathcal{G}(B)} \mathcal{G}(!A)^{\text{op}} \times \mathcal{G}(B) \xrightarrow{\|\sigma\|} \mathbf{Set}$$

using $R_A^! : \mathbf{Sym}(\mathcal{G}(A))^{\text{op}} \rightarrow \mathcal{G}(!A)$ from Proposition 9. For preservation of identities, from the definition we have

$$\|\text{der}_A\|_!((x_A^0 \dots x_A^n), y_A) = \|\text{der}_A\|((x_A^i)_{\{0 \leq i \leq n\}}, y_A)$$

i.e. the set comprising all triples $(\theta_A^-, \mathbf{c}_{x_A}, \theta_A^+)$ such that

$$\theta_A^- : (x_A^i)_{\{0 \leq i \leq n\}} \cong_{!A}^- (x_A)_{\{0\}}, \quad \theta_A^+ : x_A \cong_{!A}^+ y_A$$

but by definition of negative symmetries on $!A$, θ_A^- forces $n = 0$ and boils down to a symmetry $x_A^0 \cong_{!A}^- x_A$ so that such triples are in bijection with $\mathcal{G}(A)[x_A^0, y_A]$ as in \mathbf{Esp} .

The analysis for preservation of composition is a more elaborate version of Theorem 4. \square

C. Cartesian closed structure

We show the bicategory $\mathbf{Vis}_!$ is cartesian closed, using typical constructions for cartesian closed structure in concurrent games, in sufficient detail to keep the paper self-contained. For a precise definition of the structure of cartesian closed bicategories we refer to [50].

1) *Cartesian products in $\mathbf{Vis}_!$:* The empty arena \top , with $\kappa_{\top}(\emptyset) = 1$, is strict, and it is direct that \top is a terminal object in $\mathbf{Vis}_!$, since any negative strategy $A \vdash \top$ must be empty.

Now if A, B are strict, the **product arena** $A \& B$ is defined as $A \otimes B$, except that all events of A are in conflict with events of B . This means that configurations of $A \& B$ are

either empty, or of the form $\{1\} \times x$ for $x \in \mathcal{C}(A)$ (written $i_1(x)$) or $\{2\} \times x$ for $x \in \mathcal{C}(B)$ (written $i_2(x)$). For the payoff, we set $\kappa_{A \& B}(\emptyset) = 1$ and $\kappa_{A_1 \& A_2}(i_i(x)) = \kappa_{A_i}(x)$, making $A \& B$ a strict arena. By strictness, we have an isomorphism

$$L_{A,B}^{\&} : \mathcal{G}(A \& B) \cong \mathcal{G}(A) + \mathcal{G}(B) : R_{A,B}^{\&}$$

which reflects the definition of binary products in **Esp**.

The **first projection** $\pi_A \in \mathbf{Vis}_![(A \& B), A]$ has ess \mathbf{c}_A , with display map the map of ess characterized by

$$\partial(\mathbf{c}_x) = (i_1(x))_{\{0\}} \vdash x,$$

with the **second projection** defined symmetrically.

If $\sigma \in \mathbf{Vis}_![\Gamma, A]$ and $\tau \in \mathbf{Vis}_![\Gamma, B]$, their **pairing** has ess $\sigma \& \tau$ and display map the unique such that

$$\partial(i_1(x^\sigma)) = x_{\Gamma}^{\sigma} \vdash i_1(x_A^{\sigma}) \in \mathcal{C}(!\Gamma \vdash A \& B),$$

and likewise for $\partial(i_2(x^\tau))$, yielding $\langle \sigma, \tau \rangle \in \mathbf{Vis}_![\Gamma, A \& B]$. This extends to a functor $\langle -, - \rangle : \mathbf{Vis}_![\Gamma, A] \times \mathbf{Vis}_![\Gamma, B] \rightarrow \mathbf{Vis}_![\Gamma, A \& B]$ in a straightforward way.

Proposition 10. *For any strict arenas Γ, A and B , there is*

$$\begin{array}{ccc} & \xrightarrow{(\pi_A \odot (-)^!, \pi_B \odot (-)^!)} & \\ \mathbf{Vis}_![\Gamma, A \& B] & \simeq & \mathbf{Vis}_![\Gamma, A] \times \mathbf{Vis}_![\Gamma, B] \\ & \xleftarrow{\langle -, - \rangle} & \end{array}$$

an adjoint equivalence providing the data to turn $A \& B$ into a cartesian product in the bicategorical sense.

2) *Closed structure in $\mathbf{Vis}_!$:* Recall that the objects of $\mathbf{Vis}_!$ are strict arenas, and observe that any strict arena B is isomorphic to $\&_{i \in I} B_i$ where the B_i are **pointed**, meaning that they have exactly one minimal event. We first define a **linear arrow** for a $-$ -arena A and a pointed strict arena B , by setting $A \multimap B$ to be $A \vdash B$ with a stricter dependency order, so that all events in A causally depend on the unique minimal move in B . This is generalized for any strict B as

$$A \multimap \&_{i \in I} B_i = \&_{i \in I} (A \multimap B_i)$$

whose configurations have a convenient description:

Lemma 5. *For any $-$ -arena A , and strict arena B , we have*

$$(- \multimap -) : \mathcal{C}(A) \times \mathcal{C}^{\neq \emptyset}(B) \cong \mathcal{C}^{\neq \emptyset}(A \multimap B)$$

where $\mathcal{C}^{\neq \emptyset}(E)$ is the set of non-empty configurations.

Now for A, B strict, we define the **arrow** $A \Rightarrow B$ as $!A \multimap B$. This is equipped with an **evaluation** strategy

$$\mathbf{ev}_{A,B} \in \mathbf{Vis}_![!(A \multimap B) \& A, B]$$

consisting of the ess $\mathbf{c}_{!A \multimap B}$, and where $\partial(\mathbf{c}_{(x_A^i)_{i \in I} \multimap x_B})$ is \emptyset if $x_B = \emptyset$, and $\gamma \vdash x_B$ otherwise, with

$$\gamma = (i_1((x_A^i)_{i \in I} \multimap x_B))_{\{(0,0)\}} \uplus (i_2(x_A^i))_{\{(1,i) \in \Sigma_{\{1\}} I\}}.$$

Likewise, the **currying** of $\sigma \in \mathbf{Vis}_![\Gamma \& A, B]$ is a strategy $\Lambda(\sigma)$ with ess σ and display map

$$\partial_{\Lambda(\sigma)}(x^\sigma) = (x_{\Gamma}^i)_{i \in I} \vdash (x_A^j)_{j \in J} \multimap x_B$$

for $x^\sigma \neq \emptyset$, where $\partial_\sigma(x^\sigma) = (i_1(x_{\Gamma}^i))_{i \in I} \uplus (i_2(x_A^j))_{j \in J} \vdash x_B$. This gives a functor $\Lambda : \mathbf{Vis}_![\Gamma \& A, B] \rightarrow \mathbf{Vis}_![\Gamma, A \Rightarrow B]$.

Proposition 11. *For any strict Γ, A and B , there is*

$$\begin{array}{ccc} & \xrightarrow{\mathbf{ev}_{A,B} \odot \langle - \circ \pi_{\Gamma}^!, \pi_A \rangle^!} & \\ \mathbf{Vis}_![\Gamma, A \Rightarrow B] & \perp & \mathbf{Vis}_![\Gamma \& A, B] \\ & \xleftarrow{\Lambda(-)} & \end{array}$$

an adjoint equivalence, providing the data to turn $A \Rightarrow B$ into an exponential object in the bicategorical sense.

D. *A cartesian closed pseudofunctor*

We show that the pseudofunctor $\|-\|_! : \mathbf{Vis}_! \rightarrow \mathbf{Esp}$ preserves cartesian closed structure.

The terminal object is preserved in a strict sense, since $\mathcal{G}(\top)$ is empty. For preservation of the binary product, note that for A and B strict arenas, the map

$$\langle \|\pi_A\|_!, \|\pi_B\|_! \rangle \in \mathbf{Esp}[\mathcal{G}(A \& B), \mathcal{G}(A) + \mathcal{G}(B)]$$

is naturally isomorphic to $\widehat{L_{A,B}^{\&}}$, and is thus easily completed with $\mathbf{q}_{A,B}^{\times} = \widehat{R_{A,B}^{\&}} \in \mathbf{Esp}[\mathcal{G}(A) + \mathcal{G}(B), \mathcal{G}(A \& B)]$ forming an equivalence. This establishes:

Proposition 12. *Equipped with those equivalences, the pseudofunctor $\|-\|_! : \mathbf{Vis}_! \rightarrow \mathbf{Esp}$ preserves finite products.*

1) *Preservation:* Observe that we have an equivalence

$$L_{A,B}^{\rightarrow} : \mathcal{G}(A \Rightarrow B) \simeq \mathbf{Sym}(\mathcal{G}(A))^{\text{op}} \times \mathcal{G}(B) : R_{A,B}^{\rightarrow}$$

using first Lemma 5 as since B is strict, its complete configurations are non-empty, and observing that this decomposition also holds for symmetries; followed by Proposition 9.

Now, for A and B strict we consider $\Lambda(\|\mathbf{ev}_{A,B}\|_! \bullet \mathbf{q}^{\times})$ in

$$\mathbf{Esp}[\mathcal{G}(A \Rightarrow B), \mathbf{Sym}(\mathcal{G}(A))^{\text{op}} \times \mathcal{G}(B)]$$

and verify it is naturally isomorphic to $\widehat{L_{A,B}^{\rightarrow}}$; thus completed to an equivalence in **Esp** with $\mathbf{q}_{A,B}^{\rightarrow} = \widehat{R_{A,B}^{\rightarrow}}$.

Altogether this completes the proof of our main theorem:

Theorem 7. *We have a cartesian closed pseudofunctor*

$$\|-\|_! : \mathbf{Vis}_! \rightarrow \mathbf{Esp}.$$

VI. SOME CONSEQUENCES FOR THE λ -CALCULUS

We illustrate this pseudofunctor by relating a dynamic and a static model of the pure (untyped) λ -calculus.

A. *Two models of the pure λ -calculus*

1) *A reflexive object in $\mathbf{Vis}_!$:* Our bicategory of games contains a **universal arena** \mathbf{U} with an isomorphism of arenas

$$\mathbf{unf}_{\mathbf{U}} : \mathbf{U} \cong \mathbf{U} \Rightarrow \mathbf{U} : \mathbf{fld}_{\mathbf{U}}$$

making \mathbf{U} an extensional reflexive object [51]. Concretely, \mathbf{U} is constructed corecursively as $\otimes_{\mathbb{N}} !\mathbf{U} \multimap o$, where o is the arena with one negative move $*$, $\kappa_o(\emptyset) = 1$ and $\kappa_o(\{*\}) = 0$.

Following the interpretation of the λ -calculus in a reflexive object, we have, for every closed λ -term M , a strategy $\llbracket M \rrbracket : \mathbf{U}$. This strategy has a clear interpretation: it is a representation of the *Nakajima tree* of M (see e.g. [52], [53]).

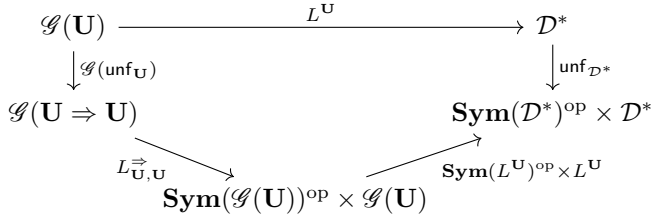


Fig. 3. Compatibility with unfoldings

2) *The Pure λ -Calculus in \mathbf{Esp}* : Likewise, one can construct models of the λ -calculus in \mathbf{Esp} [4], [19], [22].

We consider a groupoid \mathcal{D}^* , defined as a “categorified graph model” in [19], equipped with an equivalence

$$\text{unf}_{\mathcal{D}^*} : \mathcal{D}^* \simeq \mathbf{Sym}(\mathcal{D}^*)^{\text{op}} \times \mathcal{D}^* : \text{fld}_{\mathcal{D}^*}$$

in \mathbf{Esp} [19, Theorem 5.2]. The interpretation of a closed λ -term M in \mathcal{D}^* is a presheaf $\llbracket M \rrbracket_{\mathbf{Esp}}^{\mathcal{D}^*} : \mathcal{D}^* \rightarrow \mathbf{Set}$, which can be explained using an intersection type system. More precisely, objects of \mathcal{D}^* may be presented as intersection types, and for each $a \in \mathcal{D}^*$, $\llbracket M \rrbracket_{\mathbf{Esp}}^{\mathcal{D}^*}(a)$ is (up to a canonical isomorphism) the set of derivations of the judgement $\vdash M : a$ in the type system. The construction of this intersection type system reflects the corresponding operations on distributors: in particular derivations carry explicit symmetries, $\llbracket M \rrbracket_{\mathbf{Esp}}^{\mathcal{D}^*}(a)$ is really the set of derivations *quotiented* by an equivalence relation letting symmetries flow through the concrete derivation.

B. Relating the Interpretations

Using a straightforward analysis based on the above descriptions, we can define an equivalence of groupoids

$$L^{\mathbf{U}} : \mathcal{G}(\mathbf{U}) \simeq \mathcal{D}^* : R^{\mathbf{U}}$$

which is compatible with unfoldings in the sense that the diagram of Figure 3 commutes, and compatible with folding in the same way. Using our Theorem 7, it follows that:

Theorem 8. *For any closed term M , we have a natural iso*

$$\llbracket M \rrbracket_{\mathbf{Esp}}^{\mathcal{D}^*} \cong \|\llbracket M \rrbracket_{\mathbf{Vis}_!}^{\mathbf{U}}\|_! \circ R^{\mathbf{U}}$$

This shows that, for $a \in \mathbf{U}$, the set $\llbracket M \rrbracket_{\mathbf{Esp}}^{\mathcal{D}^*}(a)$ described by Olimpieri as a set of derivations up to congruence may be equivalently described, up to canonical isomorphism, as a set of positive witnesses of the form

$$(x \in \mathcal{C}^+(\llbracket M \rrbracket_{\mathbf{Vis}_!}^{\mathbf{U}}), \theta^+ : \partial(x) \cong_{\mathbf{U}}^+ R^{\mathbf{U}}(a)).$$

In other words, the interpretation of pure λ -terms as species computes the set of $+$ -covered configurations *equipped with a positive symmetry*. Interestingly, this set is constructed *without quotient*, and thus provides canonical representatives for the equivalence classes of derivations in Olimpieri’s model.

In what follows, we show how our cartesian closed pseudofunctor allows us to transfer results from game semantics to generalized species. It is known that the game semantics of the λ -calculus captures the maximal sensible λ -theory \mathcal{H}^* ,

because strategies coincide with the corresponding normal forms, Nakajima trees [52]. Using Theorem 8, we can deduce a result on the λ -theory induced by \mathcal{D}^* . Given a model \mathcal{D} of λ -calculus in an arbitrary bicategory, the *theory* induced by \mathcal{D} is the λ -theory induced by the following relation on λ -terms:

$$\{(M, N) \mid M, N \in \Lambda \text{ s.t. } \llbracket M \rrbracket^{\mathcal{D}} \cong \llbracket N \rrbracket^{\mathcal{D}}.\}$$

The correspondence established in this paper allows us to derive the following new result:

Corollary 1. *The theory of \mathcal{D}^* is \mathcal{H}^* .*

Proof. Consider two λ -terms M and N . If $M \equiv_{\mathcal{H}^*} N$, then $\llbracket M \rrbracket_{\mathbf{Vis}_!}^{\mathbf{U}} \cong \llbracket N \rrbracket_{\mathbf{Vis}_!}^{\mathbf{U}}$, and by Theorem 8, $\llbracket M \rrbracket_{\mathbf{Esp}}^{\mathcal{D}^*} \cong \llbracket N \rrbracket_{\mathbf{Esp}}^{\mathcal{D}^*}$.

Reciprocally, isomorphisms in \mathbf{Esp} form a sensible λ -theory (see [19], Corollary 6.14). As \mathcal{H}^* is the greatest sensible λ -theory, it must include isomorphisms in \mathbf{Esp} . \square

This is a simple application, but recent work suggests that there is much to explore in the bicategorical semantics of the λ -calculus [19].

VII. CONCLUSION

In this paper, we have mapped out the links between thin concurrent games and generalized species of structures, two bicategorical models of linear logic and programming languages. By giving a proof-relevant and bicategorical extension of the relationship between dynamic and static models, we have established the new state of the art in this line of work.

This bridges previously disconnected semantic realms. In the past, such bridges have proved fruitful for transporting results between dynamic and static semantics ([21], [25], [53]). This opens up many perspectives: bicategorical models are a very active field, and several recent developments may be re-examined in light of this connection ([22], [25], [54]).

Moreover, this work exposes fundamental phenomena regarding symmetries. Symmetries lie at the heart of both thin concurrent games and generalized species, but they are treated completely differently: in \mathbf{Esp} , witnesses referring to multiple copies of a resource are closed under the action of all symmetries (“saturated”), whereas \mathbf{TCG} relies on a mechanism for choreographing a choice of copy indices, providing an address for individual resources (“thin”). Beyond semantics, this approach to managing symmetries hints at an alternative to Joyal’s species [36] for representing combinatorial objects.

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