Transparent scatterers and transmission eigenvalues of infinite multiplicity

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We give a short review of old and recent results on scatterers with transmission eigenvalues of infinite multiplicity, including transparent scatterers. Our examples include potentials from the Schwartz class and multipoint potentials of Bethe - Peierls type.

This talk is based, in particular, on the works:

Grinevich P.G., Novikov R.G. "Transparent Potentials at Fixed Energy in Dimension Two. Fixed-Energy Dispersion Relations for the Fast Decaying Potentials" - *Commun. Math. Phys.,* 1995, v.174, pp. 409-446.

Grinevich P.G., Novikov R.G. "Transmission eigenvalues for multipoint scatterers" - *Eurasian Journal of Mathematical and Computer Applications,* 2021, v.9:4, pp. 17-25.

Grinevich P.G., Novikov R.G. "Spectral inequality for Schrödinger's equation with multipoint potential" - *Russian Mathematical Surveys,* 2022, v.77:6, pp. 1021-1028.

Basic definitions

We consider the stationary Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d = 1, 2, 3, \tag{1}$$

We assume that potential $v(x)$ decays sufficiently fast as $|x| \to \infty$.

The scattering at potential $v(x)$ is described by the following solutions of Equation (1):

$$\psi^+ = e^{ikx} + f^+\left(k, |k|\frac{x}{|x|}\right)\frac{e^{i|k||x|}}{|x|^{(d-1)/2}} + O\left(\frac{1}{|x|^{(d+1)/2}}\right), \tag{2}$$

as $|x| \to \infty$, for *a priori* unknown $f^+ = f^+(k, l)$. Here $k, l \in \mathbb{R}^d$, $k^2 = l^2 = E > 0$.

The functions $f^+ = f^+(k, l)$ is called **the scattering amplitude.**

It is also convenient to rewrite $f^+(k,l)$ as:

$$f^+(k,l) = c(d,|k|)f(k,l), \quad \text{where} \tag{3}$$
$$c(d,|k|) = -\pi i(-2\pi i)^{(d-1)/2}|k|^{(d-3)/2}, \quad \text{where } \sqrt{-2\pi i} = \sqrt{2\pi}e^{-i\pi/4}.$$

The scattering operator $\hat{S} = \hat{S}_E$ at a fixed energy level $E = \varkappa^2$ is defined by:

$$(\hat{S}_E u)(\theta) = u(\theta) - i\pi\varkappa^{d-2}\int_{\mathbb{S}^{d-1}} f(\varkappa\theta', \varkappa\theta)u(\theta')d\theta', \tag{4}$$

where $u(\theta)$ is a test function, $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$, θ, $\theta' \in \mathbb{S}^{d-1}$, $d\theta'$ is the standard volume element at $\mathbb{S}^{d-1}$.

Transparent potentials

Definition

Potential $v = v(x)$ is called **transparent at a fixed energy $E > 0$** if

$$\hat{S}_E \equiv \hat{1},$$

that is

$$f(k, l) = 0, \quad \text{for all} \quad k, l \in \mathbb{R}^d \quad \text{such that} \quad k^2 = l^2 = E.$$

Historically, studies on transparent potentials in multidimensions go back to
Regge, T.,"Introduction to complex orbital moments", *Nuovo Cimento*, 1959, v.14, pp. 951-976.

For the case $d = 1$, these studies go back to multisoliton ponentials, see, for example, review paper:
Faddeev L.D., "The inverse problem in the quantum theory of scattering. II", *J. Soviet Math.*,, 1976, v. 5:3, 334-396.

Transparent potentials

In particular, in works of Tulio Regge (1959), Roger Newton (1962), and Pierre Sabatier (1966), spherically symmetric ponentials transparent at a fixed positive energy E were constructed for $d = 3$. These potentials decay at infinity as $|x|^{-3/2}$.

In turn, for $d = 1$, N-soliton potentials are reflectionsless for all positive energies, and are transparent for $N - 1$ energies. These potentials decay at infinity exponentially.

More recently, in our work Grinevich, Novikov (CMP, 1995), mentioned at the beginning, spherically symmetric ponentials transparent at a fixed positive energy E were constructed for $d = 2$. These potentials belong to the Schwartz class.

All these works were fulfilled before more recent studies on invisibility using cloaking!

Transparent potentials

The Regge-Newton-Sabatier construction is based on Gelfand-Levitan type equations for spherically symmetric potentials at fixed energy.

The construction used in our work (Grinevich, Novikov, 1995) also goes back historically to Gelfand-Levitan type equations. More precisely, this construction is based on the $\bar{\partial}$-method for the complex geometric optics Faddeev's solutions of the Schrödinger equation at fixed energy.

Note that inverse scattering for the two-dimensional Schrödinger equation at fixed energy was essentially developed in the framework of the soliton theory in dimension 2+1. This connection was discovered in (Dubrovin, Krichever, S.P. Novikov, 1976) and (Manakov, 1976).

Transmission eigenvalues

Definition

Energy level E is called **transmission eigenvalue**, if the operator

$$\hat{S}_E - \hat{1}$$

has nontrivial kernel in $L^2(\mathbb{S}^{d-1})$. Dimension of this kernel is called **multiplicity of the transmission eigenvalue**. Here $\hat{1}$ denotes the identity operator.

We also consider interior transmission eigenvalues for potentials $v = v(x)$ such that

$$\text{supp} \quad v \subset \mathcal{D}.$$

where $\mathcal{D}$ is a connected bounded domain in $\mathbb{R}^d$ with C^2 boundary, such that $\mathbb{R}^d \backslash \overline{\mathcal{D}}$ is also connected.

Definition

Energy level E is called **interior transmission eigenvalue** for equation (5) if the exists a pair of non-zero functions $\psi(x)$, $\phi(x)$ such that

$$-\Delta\psi(x) + v(x)\psi(x) = E\psi(x), \quad x \in \mathcal{D}, \tag{5}$$

$$-\Delta\phi(x) = E\phi(x), \quad x \in \mathcal{D},$$

and

$$\psi(x) \equiv \phi(x), \quad \frac{\partial}{\partial\nu}\psi(x) \equiv \frac{\partial}{\partial\nu}\phi(x) \quad \text{for all} \quad x \in \partial\mathcal{D},$$

where $\frac{\partial}{\partial\nu}$ denotes the normal derivative.

Transmission eigenvalues

Historically, studies on transmission eigenvalues for scatterers with compact support go back to

Kirsch, A., "The denseness of the far field patterns for the transmission problem", *IMA J. Appl. Math.*, 1986, v.37, pp. 213-223.

Colton, D., Monk, P., "Transmission eigenvalues", *Q. J. Mech. Appl. Math.*, 1988, v.41, pp. 97-125.

In connection with more recent results in this direction, see, e.g.

Cakoni, F., Haddar, H., "Transmission eigenvalues in inverse scattering theory", *Inside Out II, MSRI Publications*, 2012, v.60, pp. 529-580.

Cakoni, F., Haddar, H., "Transmission eigenvalues", *Inverse Problems*, 2013, v.29, 100201 (3pp).

Nguyen, H-M., Nguyen, Q-H. "Discreteness of interior transmission eigenvalues revisited", *Calculus of Variations and Partial Differential Equations*, **56**:2, article: 51 (38pp) (2017).

Transmission eigenvalues

A typical result of these studies is as follows

Theorem: For sufficiently regular compactly supported isotropic scatterers the transmission eigenvalues are discrete and have finite multiplicity.

This result is not valid for potentials from the Schwartz class in view of our aforementioned result (Grinevich, Novikov, 1995)! The energy of transparency for 2D potentials in (Grinevich, Novikov, 1995) is a transmission eigenvalue of infinite multiplicity, moreover, the kernel of $\hat{S}_E - \hat{1}$ coincides with the full space $L^2(\mathbb{S}^1)$.

In addition, recently in (Grinevich, Novikov, 2021) we observed that the result about discreetness and finite multiplicity of transmission eigenvalues is not valid also for some physically relevant compactly supported singular scatterers!

Multipoint scatterers

More precisely, in (Grinevich, Novikov, 2021) we consider the multipoint scatterers of the Bethe-Peierls-Thomas-Fermi type in dimensions $d = 2$ and $d = 3$. We show that for these scatterers:

- Each positive energy E is a transmission eigenvalue (in the strong sense) of infinite multiplicity!
- Each complex E is an interior transmission eigenvalue of infinite multiplicity for domains $\mathcal{D}$ containing support of v!

Point scatterers

For the first time the point scatterers in question were introduced in (Bethe, Peierls, 1935) and (Thomas, 1935) for $d = 3$, to describe the interaction between neutrons and protons. Subsequently, point and multipoint scatterers were considered by many authors, including (Fermi, 1936), (Zel'dovich, 1960), (Berezin and Faddeev, 1961); also see the monograph (Albeverio, Gesztesy, Høegh-Krohn, Holden, 1988). In particular, Fermi used such a model to explain strong interaction of slow neutrons with nuclei, Nobel prize 1938.

The most intuitive definition of point scatterers is as follows. Consider a sequence of regular compactly supported potentials

$$v_N(x), \quad N \in \mathbb{N},$$

such that for $N \to \infty$:

- Diameter of supp $v_N(x)$ converges to 0;
- at any compact interval of energies the scattering amplitude $f_N^+(k, l)$ converges to well-defined non-trivial limit $f_\infty^+(k, l)$.

Point scatterers

The simplest scatterers of such a type are the Bethe-Peierls-Thomas-Fermi scatterers.

- For $d = 1$ these scatterers are standard δ-functions with some coefficients.
- For $d = 2, 3$ the sequence $v_N(x)$ converges to 0 in the sense of distributions.

One can start with

$$v_N(x) = \phi(N) v_1(Nx),$$

$v_1(r)$ is the characteristic function of unit ball in $\mathbb{R}^d$, $d = 1, 2, 3$.

Then to obtain a good limit one has to assume, in particular, that

$$\phi(N) \sim \begin{cases} N, & d = 1, \\ N^2/\log(N), & d = 2, \\ N^2 & d = 3, \end{cases}$$

up to coefficients and lower terms.

Point scatterers

By specifying properly the behavior of $\phi(N)$ we obtain for each $d = 1, 2, 3$ a family of Bethe-Peierls-Thomas-Fermi scatterers $\delta_\alpha(x)$, supported at $x = 0$ and parameterized by $\alpha \in \mathbb{R} \cup \infty$. The number $1/\alpha$ can be interpreted as the **strength of the scatterer**.

Sometimes, for $d = 2, 3$, these $\delta_\alpha(x)$ are called "renormalized δ-functions".

For $d \geq 4$ the point potentials are not defined.

In addition, ψ satisfies the stationary Schrödiger equation

$$-\Delta\psi + v(x)\psi = E\psi,$$

with

$$v(x) = \sum_{j=1}^{n} \delta_{\alpha_j}(x - y_j)$$

iff

Multipoint scatterers

I
$$-\Delta\psi = E\psi, \quad \text{for all} \quad x \neq y_j, \quad j = 1, \ldots, n,$$

II and near the points $y_1, \ldots, y_n$:

1. If $d = 1$, then $\psi(x)$ is continuous at $x = y_j$, and its first derivative has a jump
$$-\alpha_j \left[\psi'(y_j + 0) - \psi'(y_j - 0)\right] = \psi(y_j);$$
2. If $d = 2$, then
$$\psi(x) = \psi_{j,-1} \ln|x - y_j| + \psi_{j,0} + O(|x - y_j|) \quad \text{as} \quad x \to y_j,$$
and
$$\left[-2\pi\alpha_j - \ln 2 + \gamma\right] \psi_{j,-1} = \psi_{j,0},$$
where $\gamma = 0.577\ldots$ is the Euler's constant.
3. If $d = 3$, then
$$\psi(x) = \frac{\psi_{j,-1}}{|x - y_j|} + \psi_{j,0} + O(|x - y_j|) \quad \text{as} \quad x \to y_j,$$
and
$$4\pi\alpha_j\psi_{j,-1} = \psi_{j,0}.$$

The Schrödinger equation (1) with multipoint potentials as above is exactly solvable! In particular, it is known that

$$\psi^+(x,k) = e^{ikx} + \sum_{j=1}^{n} q_j(k) G^+(x - y_j, |k|^2),$$

$$f(k,l) = \frac{1}{(2\pi)^d} \sum_{j=1}^{n} q_j(k) e^{-ily_j},$$

where

Exact solvability

$$G^+(x,E) = \frac{e^{i|k||x|}}{2i|k|},\ d=1,$$

$$G^+(x,E) = -\frac{i}{4}H_0^1(|x||k|),\ d=2,$$

where H_0^1 is the Hankel function of the first kind,

$$G^+(x,E) = \frac{1}{2\pi}\left[\ln|x| + \ln|k| - \ln 2 + \gamma - \frac{\pi i}{2}\right] + O(|x|^2|\ln|x||),$$

as $|x| \to 0$, $\gamma = 0.577\ldots$ is the Euler constant,

$$G^+(x,k) = -\frac{e^{i|k||x|}}{4\pi|x|},\ d=3,$$

(6)

Exact solvability

The vector of constants $q(k) = (q_1(k), \ldots, q_n(k))^t$ is defined as the solution of the linear system:

$$A(|k|)q(k) = b(k),$$

with the $n \times n$ matrix A:

$$\begin{aligned}
A_{j,j}(|k|) &= \alpha_j - i(4\pi)^{-1}|k|, && d = 3,\\
A_{j,j}(|k|) &= \alpha_j - (4\pi)^{-1}(\pi i - 2\ln(|k|)), && d = 2,\\
A_{j,j}(|k|) &= \alpha_j + (2i|k|)^{-1}, && d = 1,\\
A_{j,j'}(|k|) &= G^+(y_j - y_{j'}, |k|^2), && j' \neq j,
\end{aligned}$$

and the right-hand side vector $b(k)$:

$$b_j(k) = -e^{iky_j}.$$

Already the formula

$$f(k,l) = \frac{1}{(2\pi)^d} \sum_{j=1}^{n} q_j(k) e^{-ily_j},$$

implies that each positive energy E for the aforementioned multipoint potentials is a transmission eigenvalue of infinity multiplicity, since the operator

$$\hat{S}_E - \hat{1},$$

has rank at most n.

Multipoint scatterers

We continued studies on multipoint scatterers in (Grinevich, Novikov, 2022) by considering Schrödinger's equation with potential that is a sum of a regular function and a finite number of point scatterers of Bethe-Peierls-Thomas-Fermi type.

For this equation we considered the spectral problem with homogeneous linear boundary conditions, which covers the Dirichlet, Neumann, and Robin cases.

We showed that if the energy E is an eigenvalue with multiplicity m for the regular potential, it remains an eigenvalue with multiplicity at least $m - n$ after adding $n < m$ point scatterers.

As a consequence, because for the zero potential all energies are transmission eigenvalues with infinite multiplicity, this property also holds for n-point scatterings as we mentioned above.

Multipoint scatterers

Very recently we also observed that a converse inequality also holds.

We showed that if the energy E is an eigenvalue with multiplicity m for a sum of a regular potential and a n-point potential, $n < m$, then this energy is an eigenvalue with multiplicity at least $m - n$ for the regular potential.

These spectral inequalities are our most recent contributions into the direct and inverse eigenvalue problems.

Thank you for your attention!