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Abstract

Experiment design is an important topic in system identification. It enables choosing the best input signal that allows computing parameters with minimum variance. Experiment design for system identification with fractional models is treated in this paper. Elementary fractional models of the second kind are considered, extending the previous results regarding models of the first kind. It allows choosing the best frequency(ies) of a sine input signal by minimizing the D-optimality criterion of the covariance-matrix. The case of a single unknown parameter is treated which can be either of the following: gain, pseudo-damping factor, natural frequency, or even the differentiation order. Then, all the combinations of two unknown parameters are considered. Finally, three and four unknown parameters are considered and the covariance matrix computed. It is shown that its optimum is difficult to obtain analytically in the general case. The particular case of second order (rational) systems, when the commensurate differentiation order equals one, is pointed out, as a special case in this study.

Keywords: Experiment planning, fractional system, system identification, D-optimality, Fisher Information Matrix.

1. Introduction

The field of optimal experiment design dates back to the seventies. The main idea consists in finding the best input signal that provides the most precise estimation of unknown parameters, where multi-sine approaches gave rise to important results still used today [1, 2]. Important developments emerged in the late eighties based on Ljung's asymptotic covariance formula [3]. Some research focus on optimizing the whole input signal spectrum in relation with desired constraints [4, 5].

Fractional calculus has gained more and more importance in the last decades with a rapid growth of the number of applications in widespread fields of science. This is due to the ability of fractional differential equations to elegantly model diffusive phenomena. In electrochemistry, for example, it is proven that the diffusion of charges in acid batteries is governed by Randles' models [6] that involve a half order integrator. Since then, multiple references regarding modeling and identification of various batteries have been proposed in the literature [7, 8, 9]. In semi-infinite thermal systems, Battaglia et al. [10] have shown that the exact solution of the heat equation links the thermal flux applied at one edge of a homogenous medium to a half order derivative of the temperature. Consequently, [11, 12] use fractional models for estimating thermal parameters and controlling the temperature along a thermal rod. This fact has been extended to large temperature variation in [13]. Biological systems [14, 15], medical systems [16, 17], and many others can elegantly be expressed in terms of fractional-order differential equations. System identification using fractional models was initiated in the late nineties and the beginning of this century [18, 19]. It continued gaining interest in the last few years and multiple journal references were published on this subject [20, 21, 22, 23, 24, 25, 26, 27].

Experiment design of elementary transfer functions of the first kind was treated in [28, 29]. In [28], D-optimality of the covariance matrix is minimized, whereas in [29] the problem is formulated in a convex finite dimensional form, the input spectrum is decomposed in Laguerre polynomial basis and an LMI solution is proposed. This method enables synthesizing optimal input spectrum of LTI continuous-time systems that can also be fractional ones. It is well known in the literature that the optimal experiment design depends on the optimal parameters which are unknown. This chicken-and-egg problem can be solved by pre-estimating system parameters using a non optimal input signal and then by using an optimal signal with a prescribed power. When a bounded interval of the estimated parameters is

known, an alternative way consists in performing robust experiment design as described in [28] for transfer functions of the first kind.

This paper focuses on fractional systems of the second kind and answers the question what is the best input signal which allows computing any of the system parameters by minimizing the D-optimality criterion of the covariance matrix, for a prescribed input energy. The results developed in this paper assume implicitly that time-domain responses of fractional systems are correctly simulated with negligible simulation errors [30, 31]. A preliminary version of this paper, published in the SYSID'2018 conference [32], focuses on a single unknown parameter. This paper goes beyond the latter case, in section 3, by explaining the discontinuity issue when the differentiation orders are unknown. Then, results regarding two unknown parameters is treated in section 4. Finally, the covariance matrix for the case of three and four unknown parameters is computed in section 5. It is shown that the analytical computation of the optimal input signal is very complex in these latter cases. Finally, a conclusion is drawn at the end of the paper.

Mathematical background

Fractional systems are usually defined in the Laplace domain using a transfer function representation:

$$G(s) = \frac{B(s)}{A(s)} = \frac{\sum_{i=0}^{M} b_i s^{\beta_i}}{1 + \sum_{j=1}^{N} a_j s^{\gamma_j}}, \tag{1}$$

where s is the Laplace variable, $(a_j, b_i) \in \mathbb{R}^2$ $\forall i \in \{0, 1, \ldots, M\}$ and $\forall j \in \{1, \ldots, N\}$ and the differentiation orders (γ_j, β_i) are positive real numbers that are ordered for identifiability purposes:

$$\begin{cases} 0 < \gamma_1 < \gamma_2 < \ldots < \gamma_N \\ 0 \leq \beta_0 < \beta_1 < \ldots < \beta_M. \end{cases} \tag{2}$$

Elementary transfer functions of the first kind involve a simple and real s^ν-pole:

$$F_1(s) = \frac{K}{1 + (\tau s)^\nu}, \tag{3}$$

which are straightforward generalization of first order transfer functions, when $\nu = 1$. Their stability conditions write:

$$\begin{cases} \tau > 0 \\ 0 < \nu < 2. \end{cases} \tag{4}$$

When two s^ν-complex-conjugate poles are involved, transfer functions of the second kind are used, as they have only real-valued coefficients:

$$F_2(s) = \frac{K}{\left(\frac{s}{\omega_0}\right)^{2\nu} + 2\zeta\left(\frac{s}{\omega_0}\right)^{\nu} + 1}. \tag{5}$$

Fractional transfer functions of the second kind, as in (5), were studied extensively in [33]. It was shown that their stability conditions must satisfy (6):

$$\begin{cases} \omega_0 > 0 \\ 0 < \nu < 2 \\ \zeta > -\cos(\nu\frac{\pi}{2}), \end{cases} \tag{6}$$

which generalizes the stability conditions of rational transfer functions $\zeta > 0$, when $\nu = 1$. Consequently, stable fractional transfer functions may be obtained for negative values of ζ, provided $\zeta > -\cos\left(\nu\frac{\pi}{2}\right)$.

This result was determined by applying Matignon's stability theorem [34]:

$$|\arg(s)| > \nu\frac{\pi}{2} \tag{7}$$

to the s^ν-roots of (5). Further, some new stability results have been established for incommensurate systems in [35, 36].

2. Basics on experiment design and problem formulation

Let a single input single output (SISO) system be defined as:

$$y(t) = F(p)u(t) + \epsilon(t), \tag{8}$$

where p is the differential operator $\frac{\mathbf{d}}{\mathbf{d}t}$, $F(p)$ is a fractional transfer function and $\epsilon(t)$ is a Wiener process independent and identically distributed that follows a normal distribution $N(0;\sigma^2)$. All parameters of $F(p)$ are gathered in a vector θ. The objective in system identification is to estimate the vector θ with a maximum possible precision which can be assessed by minimizing a norm of the covariance matrix which is the inverse of Fisher Information Matrix (FIM) [1, 3]:

$$\overline{P}^{-1}(\Phi_u(\omega)) = P^{-1}(\Phi_u(\omega)) + R_0, \tag{9}$$

with

$$P^{-1}(\Phi_u(\omega)) = \frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty} \mathscr{R}e\left(\left[\frac{\partial F(\mathrm{j}\omega)}{\partial\theta}\right]^T \Phi_u(\omega)\left[\frac{\partial F(-\mathrm{j}\omega)}{\partial\theta}\right]\right)d\omega, \tag{10}$$

where σ^2 is the noise variance, $\Phi_u(\omega)$ is the spectrum of the input signal $u(t)$, and R_0 a constant independent of $\Phi_u(\omega)$. Moreover, for simplicity, a unit power with $\sigma^2 = 1$ is further assumed.

The objective in experimental design is to compute a parametrized optimal input signal that maximizes a norm of the inverse covariance matrix (10). For this chicken-and-egg problem, parameters are generally pre-estimated and then used for optimal experiment design.

Obviously, the covariance matrix depends on the input signal spectrum $\Phi_u(\omega)$ and also on system parameters. Different matrix norms can be used to assess this performance. Among them, the D-optimality criterion is used in this paper. It corresponds to the volume of the confidence region of θ:

$$J_D(\Phi_u(\omega)) = \det(P(\Phi_u(\omega))). \tag{11}$$

Furthermore, the input signal is assumed to be multi-cosine:

$$u(t) = \sum_{i=1}^{N} A_i\cos(\omega_i t), \quad \omega_i \geq 0, \tag{12}$$

where each magnitude A_i allows setting the total energy on the corresponding frequency component ω_i, with a constraint allowing to fix the total power of the input signal $\sum_{i=1}^{N} A_i = A$. The required number of cosine components N depends on the number of unknown parameters. It is shown in [1] that the optimal choice for estimating one or two unknown parameters is to design an input signal with a single cosine component $N = 1$:

$$u(t) = A\cos(\omega t), \quad \omega \geq 0. \tag{13}$$

However, when 3 or 4 parameters are unknown, two cosine components are necessary to have a persistent excitation:

$$u(t) = A(\alpha\cos(\omega_1 t) + (1-\alpha)\cos(\omega_2 t)), \quad \omega_{1,2} \geq 0. \tag{14}$$

By introducing the parameter α, the total amount of energy A of the input signal may be optimized so that it splits into an optimal way on every cosine component. Moreover, since the spectrum of each cosine function is a Dirac distribution, (10) can be rewritten as (note that $P^{-1}(\underline{\omega})$ is used as a shorter notation of $P^{-1}(\Phi_u(\underline{\omega}))$):

$$P^{-1}(\underline{\omega}) = \frac{1}{2\pi}\sum_{i=1}^{N} \mathscr{R}e\left(\left[\frac{\partial F(j\omega_i)}{\partial\theta}\right]^T \left[\frac{\partial F(-j\omega_i)}{\partial\theta}\right]\right), \tag{15}$$

where N depends on the number of used cosine components and $\underline{\omega} = [\omega_1, \omega_2, ..., \omega_N]$.

Remark: As indicated previously, the Fisher Information Matrix (FIM) is the inverse of the covariance matrix. Hence the bigger the FIM, the smaller the covariance matrix and the more precise the estimation.

Problem formulation

Hence the problem treated in this paper can be formulated as finding the best frequency(ies) of the cosine input signal (12), for estimating the parameters K, ω_0, ζ, and the differentiation order ν of the transfer function of the second kind (5). For that purpose, the D-optimality criterion of the inverse covariance matrix is maximized for a prescribed power of the input signal A.

When one or two parameters are unknown a single cosine component is considered and the problem resumes in finding

$$\omega_{opt} = \arg\max_{\omega} \det(P^{-1}(\omega)). \tag{16}$$

For three and four unknown parameters, the amount of energy associated to each cosine component (parameter α in (14)) can be optimized along with the frequencies. The problem then resumes in:

$$[\underline{\omega}_{opt}, \alpha_{opt}] = \arg\max_{\underline{\omega},\alpha} \det(P^{-1}(\underline{\omega})). \tag{17}$$

A general matrix is given when the number of parameters equals three and four. It is shown how complex the computation of the maximum is, in that case.

3. Fractional systems of the second kind with a single unknown parameter

Experiment planning of elementary fractional transfer functions of the second kind (5) is considered in this section. Hence, a single parameter is supposed to be unknown among K, ζ, ω_0, and ν. In this case, the input signal is composed of a single frequency as in (13), the amplitude A is fixed and the optimal frequency maximizing (16) is computed in each case. Consequently, the obtained optimal frequency allows to estimate the unknown parameter with a minimum variance.

In this section, the elementary fractional transfer function of the second kind (5) is considered to be subject to the input signal (13) with a fixed amplitude A. First, a single parameter among K, ζ, ω_0, and ν is considered to be unknown and the optimal frequency ω of the input signal (13) is computed allowing to estimate the unknown parameter with a minimum variance (15).

Some preliminary results of this section were previously published in the conference paper [32]. These results are recalled, for the sake of completeness, and afterwards augmented.

3.1. Case K is unknown

In this section, parameters ζ, ω_0 and ν are supposed to be known and only the parameter K is supposed to be unknown. It is interesting to start by examining the case of a rational second order system ($\nu = 1$).

3.1.1. Rational second order system with $\nu = 1$

Define, all along the paper, the normalized frequency $\omega_n = \dfrac{\omega}{\omega_0}$. Then, the FIM computed by (15) is given by:

$$P_K^{-1}(\omega_n) = \frac{A}{2\pi} \times \frac{1}{D_1(\omega_n)} \tag{18}$$

where:

$$D_1(\omega_n) = 4\zeta^2\omega_n^2 + (1 - \omega_n^2)^2. \tag{19}$$

The maximum of the FIM (18) is obtained for the minimum of the denominator:

$$\frac{\mathbf{d}D_1(\omega_n)}{\mathbf{d}\omega_n} = 0 \Rightarrow 4\omega_n(\omega_n^2 + 2\zeta^2 - 1) = 0. \tag{20}$$

There is a single non-negative solution at

$$\omega_n = 0, \quad \text{when} \quad \zeta > \frac{\sqrt{2}}{2} \tag{21}$$

and two non-negative solutions at

$$\begin{cases} \omega_n = 0 \\ \omega_n = \sqrt{1-2\zeta^2} \end{cases} \quad \text{when} \quad 0 < \zeta < \frac{\sqrt{2}}{2}. \tag{22}$$

It is easy to verify, that the global optimum in (22) is at $\omega_{n,opt} = \sqrt{1-2\zeta^2}$.

For non-resonant systems, $\zeta > \frac{\sqrt{2}}{2}$, the optimum frequency is zero which corresponds to a step signal. Intuitively, estimating the gain of a transfer function is usually done by a step signal. It turns out to be the optimum signal for non-resonant systems. However, for resonant systems, $0 < \zeta < \frac{\sqrt{2}}{2}$, the optimum frequency is the resonant one. This can easily be explained by the fact that sensitivity to gain variations is the most important at the resonant frequency.

3.1.2. General (fractional) system with $\nu \in (0, 2)$

The FIM, computed from (15), is given by

$$P_K^{-1}(\omega_n) = \frac{A}{2\pi} \times \frac{1}{D_2(\omega_n)} \tag{23}$$

where

$$D_2(\omega_n) = 1 + \omega_n^{4\nu} + 4\zeta^2\omega_n^{2\nu} + 2\omega_n^{2\nu}\cos(\nu\pi) + 4\zeta\omega_n^{\nu}\left(1+\omega_n^{2\nu}\right)\cos\left(\nu\frac{\pi}{2}\right). \tag{24}$$

The optimum of the FIM (15) is not easy to obtain, as it requires solving the nonlinear equation obtained after differentiating the FIM with respect to ω_n, $\frac{\mathbf{d}P_K^{-1}(\omega_n)}{\mathbf{d}\omega_n} = 0$. Except the trivial solution at $\omega_n = 0$, no other analytical solution can be found. Consequently, numerical solutions are computed and plotted for different values of ν in Fig.1-left. Hence, the optimal signal in the sense of (16), depends on whether the system is resonant or not[1]:

$$\omega_n = 0, \quad \text{when the system is non-resonant} \tag{25}$$

$$\omega_n = \omega_R \quad \text{otherwise} \tag{26}$$

where, ω_R is the resonant frequency, computed numerically, and plotted in Fig.1 versus ζ for different values of ν.

The results in the fractional case are comparable to the rational case: when the fractional system (5) is non-resonant, the optimal frequency is zero. When it is resonant, the optimal frequency is the resonant one, as it offers the highest sensitivity to gain variations.

3.2. Case ζ is unknown

In this section, parameters K, ω_0 and ν are supposed to be known and only the parameter ζ is supposed to be unknown. Again, it is interesting to start by examining the case of a rational second order system ($\nu = 1$).

3.2.1. Rational second order system with $\nu = 1$

The FIM, computed from (15), is:

$$P_\zeta^{-1}(\omega_n) = \frac{2AK^2\omega_n^2}{\pi D_1^2(\omega_n)}. \tag{27}$$

Analytical solutions of $\frac{\mathbf{d}P_\zeta^{-1}(\omega_n)}{\mathbf{d}\omega_n} = 0$ are difficult to obtain for any ζ. To get an analytical solution, the problem must be even more simplified by setting, for example $\zeta = 1$, which yields:

$$\frac{\mathbf{d}P_\zeta^{-1}(\omega_n)}{\mathbf{d}\omega_n} = \frac{4AK^2}{\pi} \times \frac{\omega_n(3\omega_n^2-1)}{(\omega_n^2+1)^5}. \tag{28}$$

This function has two roots at:

$$\begin{cases} \omega_n = 0 \\ \omega_n = \frac{\sqrt{3}}{3} \simeq 0.577. \end{cases} \tag{29}$$

[1] Resonance conditions are computed numerically in [33] and plotted in Fig.1 in the aforementioned reference.

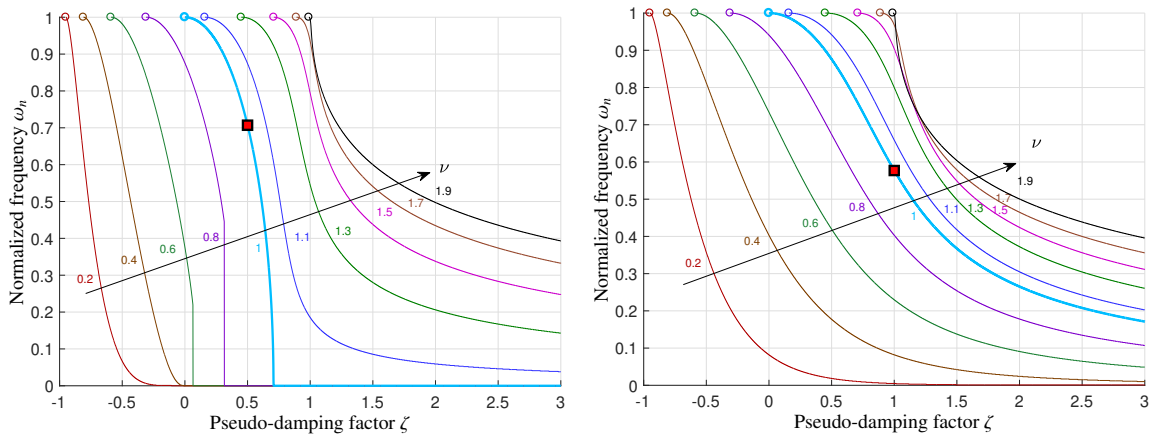

Figure 1: Optimal frequency, ω_n, versus ζ, for different values of ν in different cases. **Left**: only K is unknown; the rational case ($\nu = 1, \zeta = 0.5$) corresponding to $\omega_n = \frac{\sqrt{2}}{2}$, from (22), is highlighted by a square. **Right**: only ζ is unknown; the rational case ($\nu = 1, \zeta = 1$) corresponding to $\omega_n = \frac{\sqrt{3}}{3}$, from (29), is highlighted by a square.

Hence, for $\nu = 1$ and $\zeta = 1$, the maximum of the FIM which would allow estimating (an unknown) ζ with a maximum precision is: $\omega_{n,opt} = \dfrac{\sqrt{3}}{3}$. However, in the general case of any ζ and $\nu = 1$, a numerical solution is plotted in Fig.1-right (with $\nu = 1$).

3.2.2. General (fractional) system with ($\nu \in (0, 2)$)

Consider the second order elementary fractional transfer function (5) with stability conditions (6) satisfied. Consider that the only unknown system parameter is the pseudo-damping factor ζ and that the input signal is given by (13) with a prescribed A. Then, the FIM, calculated from (15), is given by

$$P_\zeta{}^{-1}(\omega_n) = \frac{2AK^2\omega_n^{2\nu}}{\pi D_2^2(\omega_n)}, \tag{30}$$

and the optimal frequency of the input signal in the sense of (16), solution of $\frac{\mathbf{d}P_\zeta^{-1}(\omega_n)}{\mathbf{d}\omega_n} = 0$, is computed numerically, and plotted in Fig.1-right versus ζ for different values of ν.

Due to the stability conditions (6), for each value of ν in Fig.1, the starting point of $\zeta = -\cos(\nu\frac{\pi}{2})$, corresponds to the critical stability. In that case, the optimal input frequency equals the natural frequency and hence the normalized frequency equals one. The higher the pseudo-damping factor ζ, the lesser the system damping, and the smaller the natural frequency ω_n, which tends to 0 as ζ tends to ∞ for all ν.

3.3. Case ω_0 is unknown

In this section, parameters K, ζ and ν are supposed to be known and only the parameter ω_0 is supposed to be unknown. Again, it is interesting to start by examining the case of a rational second order system.

Rational second order system with $\nu = 1$. The FIM, computed from (15) is given by:

$$P_{\omega_0}^{-1}(\omega_n) = \frac{2AK\omega_n^2}{\pi\omega_0^2} \times \frac{\omega_n^2 + \zeta^2}{D_1^2(\omega_n)}. \tag{31}$$

Again, solving $\frac{\mathbf{d}P_{\omega_0}^{-1}(\omega_n)}{\mathbf{d}\omega_n} = 0$ analytically for all values of ζ is not an easy task. However, by setting $\zeta = 1$, (31) simplifies to:

$$\frac{\mathbf{d}P_{\omega_0}^{-1}(\omega_n)}{\mathbf{d}\omega_n} = \frac{-4AK^2}{\pi\omega_0} \times \frac{\omega_n(2\omega_n^2 - 1)}{(\omega_n^2 + 1)^4}, \tag{32}$$

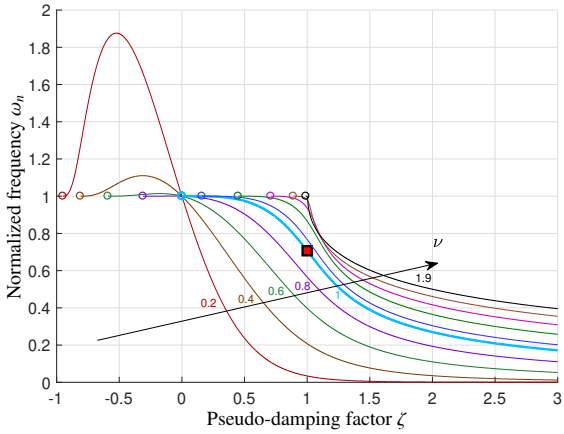

Figure 2: Optimal frequency versus ζ, for different values of ν, in the case only ω_0 is unknown. The rational case ($\nu = 1, \zeta = 1$) corresponding to an optimal frequency given by $\omega_n = \frac{\sqrt{2}}{2}$ from (33) is highlighted by a square.

which solutions are:

$$\begin{cases} \omega_{n,opt} = 0 \\ \omega_{n,opt} = \frac{\sqrt{2}}{2} \simeq 0.7. \end{cases} \tag{33}$$

However, in the general case of any ζ and $\nu = 1$, a numerical solution is plotted in Fig.2 ($\nu = 1$ curve).

3.3.1. General (fractional) system with $\nu \in (0, 2)$

Consider the second order elementary fractional transfer function (5) with stability conditions (6) satisfied. Consider that the only unknown system parameter is the natural frequency ω_n and that the input signal is given by (13) with a prescribed A. Then, the FIM, calculated from (15), is given by

$$P^{-1}_{\omega_0}(\omega_n) = \frac{2AK^2\nu^2\omega_n^{2\nu}}{\pi\omega_0^2} \times \frac{\omega_n^{2\nu} + 2\zeta\omega_n^{\nu}\cos(\nu\frac{\pi}{2}) + \zeta^2}{D_2^2(\omega_n)}, \tag{34}$$

and the optimal frequency of the input signal in the sense of (16), solution of $\frac{\mathbf{d}P^{-1}_{\omega_0}(\omega_n)}{\mathbf{d}\omega_n} = 0$, is computed numerically, and plotted in Fig.2 versus ζ for different values of ν.

3.4. Case ν is unknown

Consider the second order elementary fractional transfer function (5) with stability conditions (6) satisfied. Consider that the only unknown system parameter is the differentiation order ν_n and that the input signal is given by (13) with a prescribed A. Then, the FIM, calculated from (15), is given by

$$P^{-1}_{\nu}(\omega_n) = \frac{AK^2\omega_n^{2\nu}}{2\pi} \times \frac{(4\ln^2(\omega_n) + \pi^2)(\omega_n^{2\nu} + 2\zeta\cos(\nu\frac{\pi}{2})\omega_n^{\nu} + \zeta^2)}{D_2^2(\omega_n)}, \tag{35}$$

and the optimal frequency of the input signal in the sense of (16), solution of $\frac{\mathbf{d}P^{-1}_{\nu}(\omega_n)}{\mathbf{d}\omega_n} = 0$, is computed numerically, and plotted in Fig.3 versus ζ for different values of ν. Fig.3 upper left and right, show an apparent discontinuity at $\zeta = 0$ when $\nu < 0.38$. This is due to the presence of two global maximum points in the FIM as illustrated in the Fig.3 lower left. The Fig.3 lower right shows the presence of two global optima of the normalized frequency when $\nu < 0.38$ and a single global optimum when $\nu \geq 0.38$. The presence of two global maxima of the FIM was also pointed out in fractional systems of the first kind in [28].

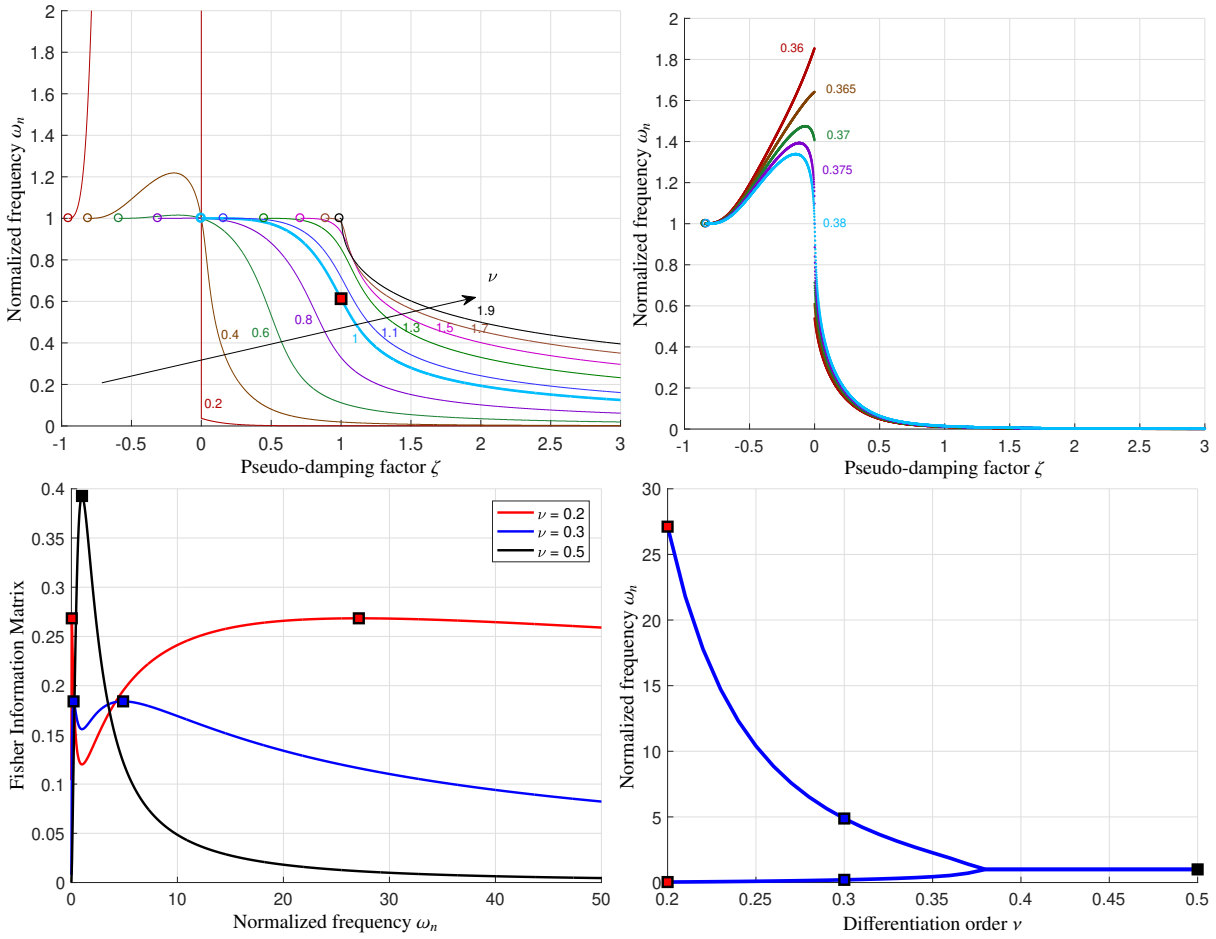

Figure 3: Optimal frequency versus ζ, for different values of ν, in the case only ν is unknown. **Upper-left**: values of ν varying from 0.2 to 1.9, with a peak for $\nu = 0.2$ above $\omega_n = 25$ (truncated in the Fig.); in the case ($\nu = 1, \zeta = 1$), the optimal frequency, $\omega_n = 0.6$, is highlighted by a square. **Upper-right**: a zoom for $0.36 \leq \nu \leq 0.38$, highlighting a discontinuity for small values of ν, around $\zeta = 0$. **Lower-left**: Fisher Information Matrix versus ω_n, for $\zeta = 0$, illustrating the presence of two global maxima points for $\nu = 0.2$ and $\nu = 0.3$, highlighted by squares, and only one maximum point for $\nu = 0.5$. **Lower-right**: optimum normalized frequency (frequencies) versus ν, for $\zeta = 0$; presence of two optima points when $\nu \leq 0.38$; the squares highlight the maximum points of the lower left plot.

4. Fractional systems of the second kind with two unknown parameters

4.1. Case K and ζ are unknown

In this section, K and ζ are assumed unknown whereas parameters ω_0 and ν are assumed known. Start by examining the rational case.

4.1.1. Rational second order system with $\nu = 1$

The FIM is given by:

$$P^{-1}_{K,\zeta}(\omega_n) = \frac{A}{2\pi}\begin{bmatrix} \frac{1}{D_1(\omega_n)} & \frac{-4K\zeta\omega_n^2}{D_1^2(\omega_n)} \\ \frac{-4K\zeta\omega_n^2}{D_1^2(\omega_n)} & \frac{4K^2\omega_n^2}{D_1^2(\omega_n)} \end{bmatrix}, \tag{36}$$

the determinant of which is:

$$\det(P^{-1}_{K,\zeta}(\omega_n)) = \frac{A^2K^2\omega_n^2}{\pi^2} \times \frac{(1-\omega_n^2)^2}{D_1^4(\omega_n)}. \tag{37}$$

Finding the maximum of (37), requires solving the equation $\frac{\mathbf{d}\det(P^{-1}_{K,\zeta}(\omega_n))}{\mathbf{d}\omega_n} = 0$. The analytical solution is difficult to obtain. Instead a numerical solution is computed and plotted in Fig. 4-left for $\nu = 1$.

4.1.2. General (fractional) system with ($\nu \in (0, 2)$)

Consider the second order elementary fractional transfer function (5) with stability conditions (6) satisfied. Consider that the two parameters K and ζ are unknown and that the input signal is given by (13) with a prescribed A. Then, the FIM, calculated from (15), is given by:

$$P^{-1}_{K,\zeta}(\omega_n) = \frac{A}{2\pi}\begin{bmatrix} \frac{1}{D_2(\omega_n)} & \frac{-2K\omega_n^\nu\left(\cos(\nu\frac{\pi}{2})\omega_n^{2\nu}+2\omega_n^\nu\zeta+\cos(\nu\frac{\pi}{2})\right)}{D_2^2(\omega_n)} \\ \frac{-2K\omega_n^\nu\left(\cos(\nu\frac{\pi}{2})\omega_n^{2\nu}+2\omega_n^\nu\zeta+\cos(\nu\frac{\pi}{2})\right)}{D_2^2(\omega_n)} & \frac{4K^2\omega_n^{2\nu}}{D_2^2(\omega_n)} \end{bmatrix}, \tag{38}$$

and its determinant by

$$\det(P^{-1}_{K,\zeta}(\omega_n)) = \frac{A^2K^2\omega_n^{2\nu}}{\pi^2} \times \frac{D_2(\omega_n) - \left(\cos(\nu\frac{\pi}{2})\omega_n^{2\nu} + 2\omega_n^\nu\zeta + \cos(\nu\frac{\pi}{2})\right)^2}{D_2^4(\omega_n)}. \tag{39}$$

It allows computing numerically the optimal frequency of the input signal in the sense of (16), by solving $\frac{\mathbf{d}\det(P^{-1}_{K,\zeta}(\omega_n))}{\mathbf{d}\omega_n} = 0$. The maximum of the FIM is plotted in Fig.4 versus ζ for multiple values of ν.

4.2. Case K and ω_0 are unknown

In this section, K and ω_0 are assumed unknown whereas parameters ζ and ν are assumed known. Start by examining the rational case.

4.2.1. Rational second order system with $\nu = 1$

The FIM is given by:

$$P^{-1}_{K,\omega_0}(\omega_n) = \frac{A}{2\pi} \times \begin{bmatrix} \frac{1}{D_1(\omega_n)} & \frac{2K\omega_n^2(\omega_n^2+2\zeta^2-1)}{\omega_0 D_1^2(\omega_n)} \\ \frac{2K\omega_n^2(\omega_n^2+2\zeta^2-1)}{\omega_0 D_1^2(\omega_n)} & \frac{4K^2\omega_n^2(\omega_n^2+\zeta^2)}{\omega_0^2 D_1^2(\omega_n)} \end{bmatrix}, \tag{40}$$

and its determinant by:

$$\det(P^{-1}_{K,\omega_0}(\omega_n)) = \frac{A^2K^2\omega_n^2}{\pi^2\omega_0^2} \times \frac{D_1(\omega_n)(\omega_n^2+\zeta^2) - (\omega_n^2+2\zeta^2-1)^2}{D_1^4(\omega_n)}. \tag{41}$$

Finding the maximum of (41), requires solving the equation $\frac{\mathbf{d}\det(P^{-1}_{K,\omega_0}(\omega_n))}{\mathbf{d}\omega_n} = 0$. The numerical solution is plotted in Fig. 4-right for $\nu = 1$.

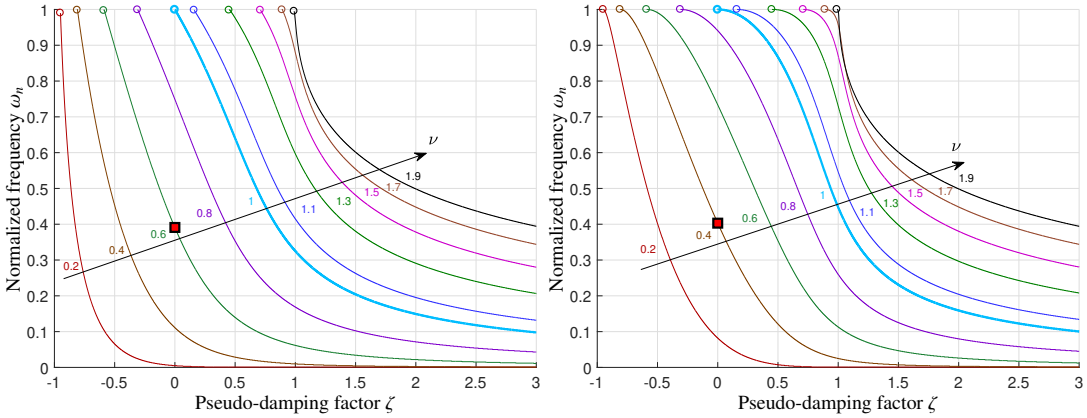

Figure 4: Optimal frequency versus ζ, when two parameters are unknown. **Left**: K and ζ are unknown; in the case ($\nu = 0.6, \zeta = 0$), the corresponding optimal frequency, given by $\omega_n = 0.4$, is highlighted by a square. **Right**: K and ω_0 are unknown; in the case ($\nu = 0.4, \zeta = 0$), the corresponding optimal frequency, given by $\omega_n = 0.4$, is highlighted by a square.

4.2.2. General (fractional) system with $\nu \in (0, 2)$

Consider the second order elementary fractional transfer function (5) with stability conditions (6) satisfied. Consider that the two parameters K and ω_0 are unknown and that the input signal is given by (13) with a prescribed A. Then, the FIM, calculated from (15), is given by:

$$P^{-1}_{K,\omega_0}(\omega_n) = \frac{A}{2\pi}\begin{bmatrix} \frac{1}{D_2(\omega_n)} & \frac{2K\nu\omega_n^{\nu}(\omega_n^{3\nu}+3\zeta\cos(\nu\frac{\pi}{2})\omega_n^{2\nu}+(2\zeta^2+\cos(\pi\nu))\omega_n^{\nu}+\zeta\cos(\nu\frac{\pi}{2}))}{\omega_0 D_2^2(\omega_n)} \\ \frac{2K\nu\omega_n^{\nu}(\omega_n^{3\nu}+3\zeta\cos(\nu\frac{\pi}{2})\omega_n^{2\nu}+(2\zeta^2+\cos(\pi\nu))\omega_n^{\nu}+\zeta\cos(\nu\frac{\pi}{2}))}{\omega_0 D_2^2(\omega_n)} & \frac{4K^2\nu^2\omega_n^{2\nu}(\omega_n^{2\nu}+2\zeta\omega_n^{\nu}\cos(\nu\frac{\pi}{2})+\zeta^2)}{\omega_0^2 D_2^2(\omega_n)} \end{bmatrix}, \quad (42)$$

and its determinant, by

$$\det(P^{-1}_{K,\omega_0}(\omega_n)) = \frac{A^2K^2\nu^2\omega_n^{2\nu}}{\pi^2\omega_0^2}\times \frac{D_2(\omega_n)(\omega_n^{2\nu}+2\zeta\omega_n^{\nu}\cos(\nu\frac{\pi}{2})+\zeta^2) - (\omega_n^{3\nu}+3\zeta\cos(\nu\frac{\pi}{2})\omega_n^{2\nu}+(2\zeta^2+\cos(\pi\nu))\omega_n^{\nu}+\zeta\cos(\nu\frac{\pi}{2}))^2}{D_2^4(\omega_n)}. \quad (43)$$

The solution of $\frac{\mathbf{d}\det(P^{-1}_{K,\omega_0}(\omega_n))}{\mathbf{d}\omega_n} = 0$ allows obtaining optimal frequencies as plotted in Fig.4 for multiple values of ν.

4.3. Case K and ν are unknown

Consider the second order elementary fractional transfer function (5) with stability conditions (6) satisfied. Consider that the two parameters K and ω_0 are unknown and that the input signal is given by (13) with a prescribed A. Then, the FIM, calculated from (15), is given by

$$P^{-1}_{K,\nu}(\omega_n) = \frac{A}{2\pi}\times\begin{bmatrix} \frac{1}{D_2(\omega_n)} & \frac{-2K\omega_n^{\nu}\ln(\omega_n)\left(\omega_n^{3\nu}+3\zeta\cos(\nu\frac{\pi}{2})\omega_n^{2\nu}+(2\zeta^2+\cos(\pi\nu))\omega_n^{\nu}+\zeta\cos(\nu\frac{\pi}{2})\right)}{D_2^2(\omega_n)} \\ \frac{-2K\omega_n^{\nu}\ln(\omega_n)\left(\omega_n^{3\nu}+3\zeta\cos(\nu\frac{\pi}{2})\omega_n^{2\nu}+(2\zeta^2+\cos(\pi\nu))\omega_n^{\nu}+\zeta\cos(\nu\frac{\pi}{2})\right)}{D_2^2(\omega_n)} & \frac{K^2\omega_n^{2\nu}(4\ln^2(\omega_n)+\pi^2)(\omega_n^{2\nu}+2\zeta\omega_n^{\nu}\cos(\nu\frac{\pi}{2})+\zeta^2)}{D_2^2(\omega_n)} \end{bmatrix} \quad (44)$$

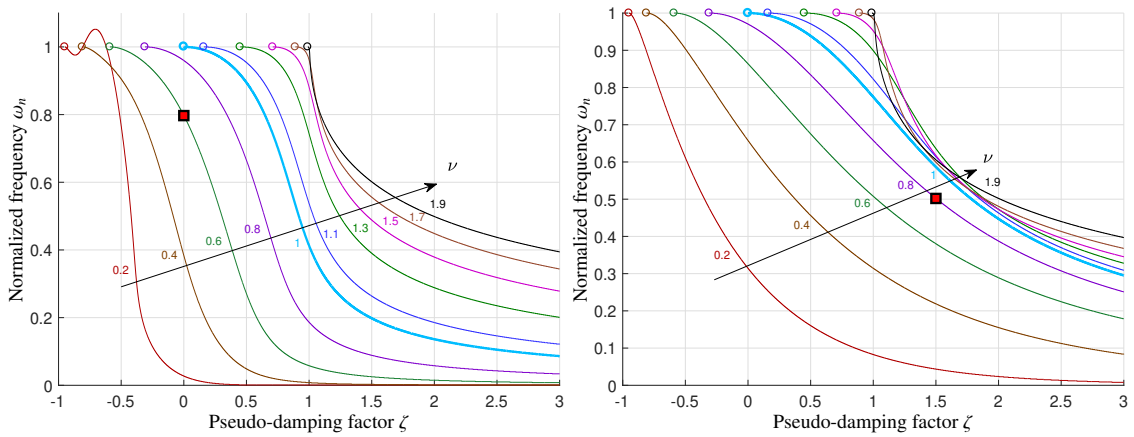

Figure 5: Optimal frequency versus ζ, when two parameters are unknown. **Left**: K and ν are unknown; in the case ($\nu = 0.6, \zeta = 0$), the corresponding optimal frequency, given by $\omega_n = 0.8$, is highlighted by a square. **Right**: ζ and ω_0 are unknown; in the case ($nu = 0.8, \zeta = 1.5$), the corresponding optimal frequency, given by $\omega_n = 0.5$, is highlighted by a square.

and its determinant, by

$$\det(P^{-1}_{K,\nu}(\omega_n)) = \frac{A^2K^2\omega_n^{2\nu}}{4\pi^2} \times \left[\frac{D_2(\omega_n)\left(4\ln^2(\omega_n)+\pi^2\right)\left(\omega_n^{2\nu}+2\zeta\omega_n^{\nu}\cos(\nu\frac{\pi}{2})+\zeta^2\right)}{D_2^4(\omega_n)} - \frac{\left(2\ln(\omega_n)\left(\omega_n^{3\nu}+3\zeta\cos(\nu\frac{\pi}{2})\omega_n^{2\nu}+(2\zeta^2+\cos(\pi\nu))\omega_n^{\nu}+\zeta\cos(\nu\frac{\pi}{2})\right)\right)^2}{D_2^4(\omega_n)} \right]. \quad (45)$$

The solution of $\frac{\mathbf{d}\det(P^{-1}_{K,\nu}(\omega_n))}{\mathbf{d}\omega_n} = 0$ allows obtaining optimal frequencies, plotted in Fig.5-left for multiple values of ν.

4.4. Case ζ and ω_0 are unknown

In this section, ζ and ω_0 are assumed unknown whereas parameters K and ν are assumed known. Start by examining the rational case.

4.4.1. Rational second order system with $\nu = 1$

The FIM is given by:

$$P^{-1}_{\zeta,\omega_0}(\omega_n) = \frac{A}{2\pi} \times \begin{bmatrix} \frac{4K^2\omega_n^2}{D_1^2(\omega_n)} & \frac{-4K^2\zeta\omega_n^2}{\omega_0 D_1^2(\omega_n)} \\ \frac{-4K^2\zeta\omega_n^2}{\omega_0 D_1^2(\omega_n)} & \frac{K^2\omega_n^2(\omega_n^2+\zeta^2)}{\omega_0^2 D_1^2(\omega_n)} \end{bmatrix}, \quad (46)$$

and its determinant by:

$$\det(P^{-1}_{\zeta,\omega_0}(\omega_n)) = \frac{4A^2K^4\omega_n^6}{\pi^2\omega_0^2 D_1^4(\omega_n)}. \quad (47)$$

Finding the maximum of (47), requires solving the equation $\frac{\mathbf{d}\det(P^{-1}_{\zeta,\omega_0}(\omega_n))}{\mathbf{d}\omega_n} = 0$. The numerical solution is plotted in Fig. 5 for $\nu = 1$.

4.4.2. General (fractional) system with $\nu \in (0, 2)$

Consider the second order elementary fractional transfer function (5) with stability conditions (6) satisfied. Consider that the two parameters ζ and ω_0 are unknown and that the input signal is given by (13) with a prescribed A.

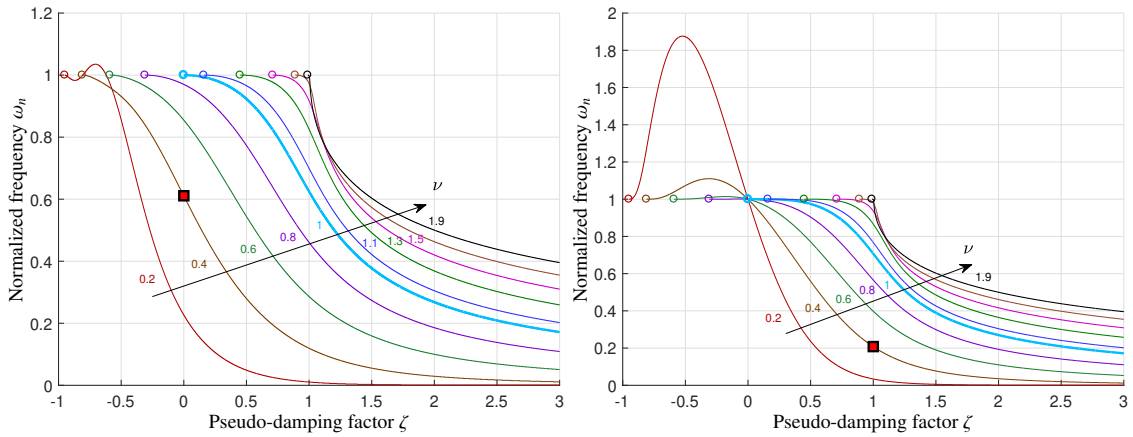

Figure 6: Optimal frequency versus ζ, when two parameters are unknown. **Left**: ζ and ν are unknown; in the case ($\nu = 0.4, \zeta = 0$), the corresponding optimal frequency, given by $\omega_n = 0.6$, is highlighted by a square. **Right**: ω_0 and ν are unknown; in the case ($\nu = 0.4, \zeta = 1$), the corresponding optimal frequency, given by $\omega_n = 0.2$, is highlighted by a square.

Then, the FIM, calculated from (15), is given by

$$P^{-1}_{\zeta,\omega_0}(\omega_n) = \frac{A}{2\pi} \times \begin{bmatrix} \frac{4K^2\omega_n^{2\nu}}{D_2^2(\omega_n)} & \frac{-4K^2\nu\omega_n^{2\nu}(\zeta+\omega_n^\nu\cos(\nu\frac{\pi}{2}))}{\omega_0 D_2^2(\omega_n)} \\ \frac{-4K^2\nu\omega_n^{2\nu}(\zeta+\omega_n^\nu\cos(\nu\frac{\pi}{2}))}{\omega_0 D_2^2(\omega_n)} & \frac{4K^2\nu^2\omega_n^{2\nu}(\omega_n^{2\nu}+2\zeta\omega_n^\nu\cos(\nu\frac{\pi}{2})+\zeta^2)}{\omega_0^2 D_2^2(\omega_n)} \end{bmatrix}, \tag{48}$$

and its determinant by:

$$\det(P^{-1}_{\zeta,\omega_0}(\omega_n)) = \frac{4A^2K^4\nu^2\omega_n^{6\nu}}{\pi^2\omega_0^2} \times \frac{\sin^2(\nu\frac{\pi}{2})}{D_2^4(\omega_n)}. \tag{49}$$

The solution of $\frac{\mathbf{d}\det(P^{-1}_{\zeta,\omega_0}(\omega_n))}{\mathbf{d}\omega_n} = 0$ allows obtaining optimal frequencies, plotted in Fig.5-right for multiple values of ν.

4.5. Case ζ and ν are unknown

Consider the second order elementary fractional transfer function (5) with stability conditions (6) satisfied. Consider that the two parameters ζ and ν are unknown and that the input signal is given by (13) with a prescribed A. Then, the FIM, calculated from (15), is given by:

$$P^{-1}_{\zeta,\nu}(\omega_n) = \frac{A}{2\pi} \times \begin{bmatrix} \frac{4K^2\omega_n^{2\nu}}{D_2^2(\omega_n)} & \frac{4K^2\omega_n^{2\nu}\ln(\omega_n)(\zeta+\omega_n^\nu\cos(\nu\frac{\pi}{2}))}{D_2^2(\omega_n)} \\ \frac{4K^2\omega_n^{2\nu}\ln(\omega_n)(\zeta+\omega_n^\nu\cos(\nu\frac{\pi}{2}))}{D_2^2(\omega_n)} & \frac{K^2\omega_n^{2\nu}(4\ln^2(\omega_n)+\pi^2)(\omega_n^{2\nu}+2\zeta\omega_n^\nu\cos(\nu\frac{\pi}{2})+\zeta^2)}{D_2^2(\omega_n)} \end{bmatrix}, \tag{50}$$

and its determinant by

$$\det(P^{-1}_{\zeta,\nu}(\omega_n)) = \frac{A^2K^4\omega_n^{4\nu}}{\pi^2} \times \frac{(4\ln^2(\omega_n)+\pi^2)(\omega_n^{2\nu}+2\zeta\omega_n^\nu\cos(\nu\frac{\pi}{2})+\zeta^2) - \ln(\omega_n)^2(\zeta+\omega_n^\nu\cos(\nu\frac{\pi}{2}))^2}{D_2^4(\omega_n)}. \tag{51}$$

The solution of $\frac{\mathbf{d}\det(P^{-1}_{\zeta,\nu}(\omega_n))}{\mathbf{d}\omega_n} = 0$ allows obtaining optimal frequencies, plotted in Fig.6-left for multiple values of ν.

4.6. Case ω_0 and ν are unknown

Consider the second order elementary fractional transfer function (5) with stability conditions (6) satisfied. Consider that the two parameters ω_0 and ν are unknown and that the input signal is given by (13) with a prescribed A.

Then, the FIM, calculated from (15), is given by

$$P^{-1}_{\omega_0,\nu}(\omega_n) = \frac{A}{2\pi} \times \begin{bmatrix} \frac{4K^2\nu^2\omega_n^{2\nu}(\omega_n^{2\nu}+2\zeta\omega_n^{\nu}\cos(\nu\frac{\pi}{2})+\zeta^2)}{\omega_0^2 D_2^2(\omega_n)} & \frac{-4K^2\nu\omega_n^{2\nu}\ln(\omega_n)(\omega_n^{2\nu}+2\zeta\omega_n^{\nu}\cos(\nu\frac{\pi}{2})+\zeta^2)}{\omega_0 D_2^2(\omega_n)} \\ \frac{-4K^2\nu\omega_n^{2\nu}\ln(\omega_n)(\omega_n^{2\nu}+2\zeta\omega_n^{\nu}\cos(\nu\frac{\pi}{2})+\zeta^2)}{\omega_0 D_2^2(\omega_n)} & \frac{K^2\omega_n^{2\nu}(4\ln^2(\omega_n)+\pi^2)(\omega_n^{2\nu}+2\zeta\omega_n^{\nu}\cos(\nu\frac{\pi}{2})+\zeta^2)}{D_2^2(\omega_n)} \end{bmatrix} \tag{52}$$

and its determinant by

$$\det(P^{-1}_{\omega_0,\nu}(\omega_n)) = \frac{A^2K^4\nu^2\omega_n^{4\nu}}{\omega_0^2} \times \frac{(\omega_n^{2\nu}+2\zeta\omega_n^{\nu}\cos(\nu\frac{\pi}{2})+\zeta^2)^2}{D_2^4(\omega_n)}. \tag{53}$$

The solution of $\frac{\mathbf{d}\det(P^{-1}_{\zeta,\nu}(\omega_n))}{\mathbf{d}\omega_n} = 0$ allows obtaining optimal frequencies, plotted in Fig.6 for multiple values of ν.

5. Fractional systems of the second kind with three or four unknown parameters

When three or four parameters are unknown, then the input signal should have at least two cosine terms, as in (14), to fulfill the requirements for persistent excitation. Consequently, the problem becomes even more difficult to solve because two frequencies and the weighting factor α in (14) need to be determined by finding the maximum of the determinant of a 3×3 or a 4×4-matrix with respect to both frequencies and the weighting factor. In the case of 4 unknown parameters, the FIM writes:

$$P^{-1}_{K,\zeta,\omega_0,\nu}(\omega_{n_1},\omega_{n_2},\alpha) = \frac{A}{2\pi}\Big(\alpha M_1 + (1-\alpha)M_2\Big) \tag{54}$$

with

$$M_i = \begin{bmatrix} \frac{1}{D_2(\omega_{n_i})} & \frac{-2K\omega_{n_i}^{\nu}\left(\cos(\nu\frac{\pi}{2})\omega_{n_i}^{2\nu}+2\omega_{n_i}^{\nu}\zeta+\cos(\nu\frac{\pi}{2})\right)}{D_2^2(\omega_{n_i})} & \cdots \\ \frac{-2K\omega_{n_i}^{\nu}\left(\cos(\nu\frac{\pi}{2})\omega_{n_i}^{2\nu}+2\omega_{n_i}^{\nu}\zeta+\cos(\nu\frac{\pi}{2})\right)}{D_2^2(\omega_{n_i})} & \frac{4K^2\omega_{n_i}^{2\nu}}{D_2^2(\omega_{n_i})} & \\ \frac{2K\nu\omega_{n_i}^{\nu}(\omega_{n_i}^{3\nu}+3\zeta\cos(\nu\frac{\pi}{2})\omega_{n_i}^{2\nu}+(2\zeta^2+\cos(\pi\nu))\omega_{n_i}^{\nu}+\zeta\cos(\nu\frac{\pi}{2}))}{\omega_0 D_2^2(\omega_{n_i})} & \frac{-4K^2\nu\omega_{n_i}^{2\nu}(\zeta+\omega_{n_i}^{\nu}\cos(\nu\frac{\pi}{2}))}{\omega_0 D_2^2(\omega_{n_i})} & \\ \frac{-2K\omega_{n_i}^{\nu}\ln(\omega_{n_i})\left(\omega_{n_i}^{3\nu}+3\zeta\cos(\nu\frac{\pi}{2})\omega_{n_i}^{2\nu}+(2\zeta^2+\cos(\pi\nu))\omega_{n_i}^{\nu}+\zeta\cos(\nu\frac{\pi}{2})\right)}{D_2^2(\omega_{n_i})} & \frac{(4K^2\omega_{n_i}^{2\nu}\ln(\omega_{n_i}))(\zeta+\omega_{n_i}^{\nu}\cos(\nu\frac{\pi}{2}))}{D_2^2(\omega_{n_i})} & \end{bmatrix}$$

$$\begin{bmatrix} \cdots & \frac{2K\nu\omega_{n_i}^{\nu}(\omega_{n_i}^{3\nu}+3\zeta\cos(\nu\frac{\pi}{2})\omega_{n_i}^{2\nu}+(2\zeta^2+\cos(\pi\nu))\omega_{n_i}^{\nu}+\zeta\cos(\nu\frac{\pi}{2}))}{\omega_0 D_2^2(\omega_{n_i})} & \frac{-2K\omega_{n_i}^{\nu}\ln(\omega_{n_i})\left(\omega_{n_i}^{3\nu}+3\zeta\cos(\nu\frac{\pi}{2})\omega_{n_i}^{2\nu}+(2\zeta^2+\cos(\pi\nu))\omega_{n_i}^{\nu}+\zeta\cos(\nu\frac{\pi}{2})\right)}{D_2^2(\omega_{n_i})} \\ & \frac{-4K^2\nu\omega_{n_i}^{2\nu}(\zeta+\omega_{n_i}^{\nu}\cos(\nu\frac{\pi}{2}))}{\omega_0 D_2^2(\omega_{n_i})} & \frac{(4K^2\omega_{n_i}^{2\nu}\ln(\omega_{n_i}))(\zeta+\omega_{n_i}^{\nu}\cos(\nu\frac{\pi}{2}))}{D_2^2(\omega_{n_i})} \\ & \frac{4K^2\nu^2\omega_{n_i}^{2\nu}(\omega_{n_i}^{2\nu}+2\zeta\omega_{n_i}^{\nu}\cos(\nu\frac{\pi}{2})+\zeta^2)}{\omega_0^2 D_2^2(\omega_{n_i})} & \frac{-4K^2\nu\omega_{n_i}^{2\nu}\ln(\omega_{n_i})(\omega_{n_i}^{2\nu}+2\zeta\omega_{n_i}^{\nu}\cos(\nu\frac{\pi}{2})+\zeta^2)}{\omega_0 D_2^2(\omega_{n_i})} \\ & \frac{-4K^2\nu\omega_{n_i}^{2\nu}\ln(\omega_{n_i})(\omega_{n_i}^{2\nu}+2\zeta\omega_{n_i}^{\nu}\cos(\nu\frac{\pi}{2})+\zeta^2)}{\omega_0 D_2^2(\omega_{n_i})} & \frac{K^2\omega_{n_i}^{2\nu}(4\ln(\omega_{n_i})^2+\pi^2)(\omega_{n_i}^{2\nu}+\zeta^2+2\zeta\omega_{n_i}^{\nu}\cos(\nu\frac{\pi}{2}))}{D_2^2(\omega_{n_i})} \end{bmatrix}. \tag{55}$$

Obviously, solving $\frac{\mathbf{d}\det P^{-1}_{K,\zeta,\omega_0,\nu}(\omega_{n_1},\omega_{n_2},\alpha)}{\mathbf{d}[\omega_{n_1},\omega_{n_2},\alpha]} = [0,0,0]$ can only be done numerically.

In the case of any combination of 3 unknown parameters, the FIM can straightforwardly be deduced from (54) and (55), by simply eliminating the line and the column corresponding to the supposed known parameter, i.e. the first line/column if K is known, the second line/column if ζ is know, the third line/column if ω_0 is known, and the fourth line/column if ν is known.

6. Conclusion

In this paper, experiment design for system identification using fractional transfer functions of the second kind has been considered. First of all, a proposed method computes the optimal frequency allowing to identify a single parameter (gain, pseudo-damping factor, natural frequency, or the commensurate differentiation order) by maximizing the Fisher Information criterion. Then, the proposed method computes the optimal frequency allowing to identify

a couple of these parameters by maximizing the Fisher Information criterion. It is shown that the obtained results allow to extend straightforwardly the optimality of rational systems. Finally, the procedure is extended to three and four unknown parameters, though requiring some tedious computations for the determinant and even more for the computation of its optimality conditions. It might be interesting to extend the results of this paper to elementary fractional transfer functions with time delays. Moreover, it is well known that fractional systems are infinite dimensional and that they have polynomial convergence, as compared to the exponential convergence of rational systems. Hence, further insights could be oriented towards setting up time-bounded-criteria in the framework of experiment planning.

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