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Quantification of Time-Domain Truncation Errors for the Reinitialization of Fractional Integrators

Andreas Rauh* and Rachid Malti^a

Abstract

In recent years, fractional differential equations have received a significant increase in their use for modeling a wide range of engineering applications. In such cases, they are mostly employed to represent non-standard dynamics that involve long-term memory effects or to represent the dynamics of system models that are identified from measured frequency response data in which magnitude and phase variations are observed that could be captured either by low-order fractional models or high-order rational ones. Fractional models arise also when synthesizing CRONE (Commande Robuste d'Ordre Non Entier) and/or fractional PID controllers for rational or fractional systems. In all these applications, it is frequently required to transform the frequency domain representation into time domain. When doing so, it is necessary to carefully address the issue of the initialization of the pseudo state variables of the time domain system model. This issue is discussed in this article for the reinitialization of fractional integrators which arises among others when solving state estimation tasks for continuous-time systems with discrete-time measurements. To quantify the arising time-domain truncation errors due to integrator resets, a novel interval observer-based approach is presented and, finally, visualized for a simplified battery model.

Keywords: Fractional differential equations (FDEs), Observer design, Uncertain cooperative dynamics, Temporal truncation errors, State estimation

1 Introduction

Fractional differential equations (FDEs) are powerful modeling tools in many engineering applications in which non-standard dynamics, characterized by infinite horizon states, can be observed. An example for such applications is modeling the charging and discharging dynamics of batteries [?]. Previous work for an interval-based state estimation of such systems has accounted for a cooperativity preserving

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or cooperativity enforcing design of observers [?, ?]. These interval observers exploit specific monotonicity properties of positive dynamic systems¹ and provide lower and upper bounding trajectories for all pseudo state variables¹ as soon as suitable initialization functions for the fractional dynamic system model are specified.

Moreover, FDEs arise naturally if the CRONE design methodology for efficient shaping of frequency response characteristics of linear control systems and/or if fractional PID controllers [2] are employed. FDEs then arise independently of whether the plant to be controlled is represented by a classical integer-order or fractional system model, and if the resulting closed-loop system dynamics are subsequently represented in the time domain.

In contrast to the case of integer-order models, the time responses of fractional systems significantly depend on the initialization of the pseudo state. This is shown exemplarily in this paper with the help of the Grünwald-Letnikov definition of fractional derivatives to illustrate further that the Caputo initialization corresponds to the special case that an FDE model is initialized with an initial condition that also represents a perfectly constant, infinitely long history of the pseudo states in the past. Although this may hold (at least in good approximation) for the initialization of a dynamic system which is fully in rest, this is obviously not true when resetting integrators after a finitely long time interval.

Apart from the discussions above, interval-valued iteration procedures have been developed in [?] for a verified simulation of FDE models. These iteration procedures, based on Mittag-Leffler function parameterizations of the pseudo-state enclosures, are not *a priori* restricted to cooperative models but are applicable also to nonlinear systems with interval parameters. So far, this procedure assumes that — for the initialization — a fractional derivative definition according to Caputo is used. This verified simulation, however, allows for resetting the integration after a finite time span by applying (to our knowledge, for the first time in a verified simulation of FDEs) an error quantification originally published in the book ... by Podlubny.

In this paper, we aim at improving this error quantification scheme by a novel interval observer-based approach that allows for estimating guaranteed interval bounds for time-domain truncation errors in scenarios in which fractional integrators need to be reset. Such cases occur when state estimation for continuous-time FDE models with discrete-time measurements is considered. So far, the state-of-the-art in the evaluation of observer-based pseudo-state estimation procedures for continuous-time FDE models supposes that measurements are also available in a continuous-time form or at least at each sampling period [?]. If measurements are available only at discrete time instants, continuous-time pseudo-state predictions need to be performed between the measurements' sampling instants.

Then, the measured pseudo-state information (described by intervals to represent bounded measurement errors) can be intersected with the predicted state

¹The notion *pseudo state* is used throughout this manuscript to indicate the existence of the infinite memory problem of FDEs in contrast to the classical notion of *state variables* that only need to be specified at distinct points in time to unambiguously solve initial or boundary value problems for classical integer-order dynamic system models.

information to enhance the knowledge of the actual system dynamics. However, this intersection demands reinitializing the integration of the fractional model. As already mentioned above, a similar requirement is discussed in [?], where temporal sub-slices were considered to reduce the overestimation of interval-based simulation approaches. Moreover, such integrator resets also help to limit memory demands that would grow continuously with increasing integration times if solutions to FDEs were approximated with the help of series expansion techniques based on the Grünwald-Letnikov definition of fractional derivatives.

Due to the infinite horizon memory property of fractional systems, the reinitialization of time-domain simulations requires a rigorous consideration of the arising truncation errors. Although guaranteed outer bounds for these errors were derived by Podlubny in [?], they may be unnecessarily conservative due to an assumption of the time invariance of these bounds for all future points after the integrator reset. We aim at using Podlubny's initial bounds as a basis for a novel error refinement strategy between discrete reinitialization points in an observer-based setting.

After an introduction into the infinite memory problem of FDE models in Sec. ..., an approach that accounts for handling non-constant pseudo-state initializations from a bounded past time window in terms of uncertain initial conditions at a single point is derived. This approach is based on a conservative interval-valued correction of the FDE model. It forms the basis for implementing an observer-based quantification of truncation errors for simulations of FDEs in which a periodic reinitialization is employed in Sec ... This approach is then applied in Sec ... to an academic benchmark example as well as to the interval contractor-based state estimation of a continuous-time battery model [?] with discrete-time measurements, before the paper is concluded with an outlook on future work in Sec.

2 Influence of the Initialization of FDE Models

To visualize the influence of the initialization of the pseudo state of FDE models on their future behavior, consider the representation of the solution of a commensurate autonomous FDE

$$\mathbf{x}^{(\nu)}(t) = \mathbf{f}(\mathbf{x}(t)) \quad , \quad \mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n \quad , \quad (1)$$

in terms of the infinite series

$$\mathbf{x}(t_{k+1}) = \nu \mathbf{I} \cdot \mathbf{x}(t_k) + \Delta T_k^\nu \cdot \mathbf{f}(\mathbf{x}(t_k)) - \sum_{i=2}^{\infty} c_i \cdot \mathbf{x}(t_{k+1-i}) \quad (2)$$

with the sufficiently short step size $\Delta T_k = t_{k+1} - t_k$. This series expansion results from the Grünwald-Letnikov definition² of a non-integer derivative of order $0 < \nu \leq 1$ with the coefficients

$$c_i = (-1)^i \cdot \binom{\nu}{i} = (-1)^i \cdot \frac{\Gamma(\nu + 1)}{\Gamma(i + 1) \cdot \Gamma(\nu - i + 1)} \quad , \quad (3)$$

²This representation corresponds to the one discussed in ..., except for the correction of a small typo in the quoted previous work of the first author.

in which the term $\binom{\nu}{i}$ is the Newton binomial coefficient with the gamma function

$$\Gamma(\nu) = \int_0^{\infty} \xi^{\nu-1} e^{-\xi} d\xi \quad (4)$$

as a generalization of the factorial to the case of non-integer arguments. To avoid excessive numerical errors when evaluating the coefficients c_i , they are typically computed in a recursive manner according to

$$c_i = c_{i-1} \cdot \left(1 + \frac{1+\nu}{i}\right) \quad \text{with } c_0 = 0 \quad \text{and } i \in \mathbb{N} . \quad (5)$$

As it can be seen already in Eq. (2), future pseudo states $\mathbf{x}(t_{k+1})$ do not only depend on the current state $\mathbf{x}(t_k)$ as a kind of initialization (as it would be the case for integer-order system models), but they also depend on an infinite horizon of pseudo states from previous points of time $t < t_k$. Note, stability properties of this series expansion and properties of its convergence toward the true solution of a fractional system models have been analyzed in detail in ... In fact, the Grünwald-Letnikov discretization can be interpreted as a generalization of the well-known Euler discretization scheme for integer-order models so that the true state evolution can be approximated accurately for sufficiently small values of ΔT_k . For methods that allow a rigorous quantification of time discretization errors, the reader is referred to ..., where an exponential state enclosure technique is generalized to fractional models by using an iteration scheme exploiting an interval extension of Mittag-Leffler functions, or to ... where series expansion approaches and Picard iteration schemes were generalized to the fractional case.

For linear FDEs of Caputo type, typically only initial conditions $\mathbf{x}(t_k)$ are specified explicitly at a point of time t_0 that is set to $t_0 = 0$ without loss of generality in the remainder of this paper. As shown in the following example, this specification implicitly imposes that the pseudo state of the system showed an exactly constant behavior for an infinitely long time window in the past.

To perform this investigation, consider the FDE

$$x^{(0.5)}(t) = -x(t) + u(t) \quad (6)$$

with the pseudo state initialization

$$x(t) = x_0 \quad \text{for } t \leq 0 \quad (7)$$

and the constant external control input

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ u_0 & \text{for } t \leq 0 . \end{cases} \quad (8)$$

Due to the fact that fractional derivatives of constant values in the Caputo sense are zero, the linear change of variables

$$x(t) = y(t) + u_0 \quad \text{with } x^{(0.5)}(t) = y^{(0.5)}(t) \quad (9)$$

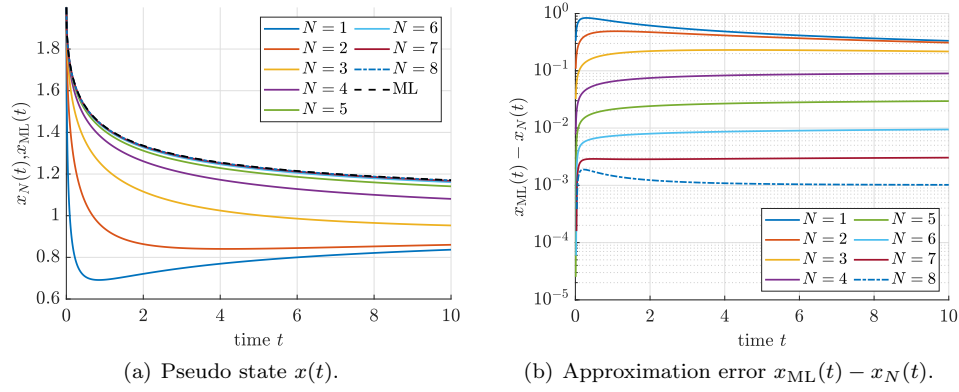


Figure 1: Influence of the memory length of the pseudo state initialization.

yields the equivalent FDE

$$y^{(0.5)}(t) = -y(t) \quad (10)$$

with

$$y(t) = x_0 - u_0 \quad \text{for } t \leq 0 \quad (11)$$

for which the exact solution is given by

$$y(t) = (x_0 - u_0) \cdot E_{0.5,1}(-t^{0.5}) \quad \text{for } t \geq 0 \quad (12)$$

being equivalent to

$$x(t) = x_{ML}(t) = (x_0 - u_0) \cdot E_{0.5,1}(-t^{0.5}) + u_0 \quad \text{for } t \geq 0. \quad (13)$$

In (12) and (13), $E_{\nu,1}(\cdot)$ is the Mittag-Leffler function with the fractional derivative order $\nu = 0.5$ as parameter.

In Fig. 1 different approximations of the solution to the FDE model (6)–(8) are computed by using the Grünwald-Letnikov approximation with the constant discretization step size $\Delta T_k = 0.01$ and approximations of the infinitely long constant initialization of the pseudo state in (7). These latter approximations are defined by

$$x(t) = x_0 \quad \text{for } t \in [-T \cdot 10^N ; 0] \quad (14)$$

with $N \in \{1, 2, \dots, 8\}$, where the corresponding approximations to the true solution $x(t)$ are denoted by $x_N(t)$. As described above, the exact solution $x_{ML}(t)$ corresponds to a solution representation in terms of the Mittag-Leffler function according to (13), which has been evaluated in Fig. 1 by the MATLAB implementation by R. Garrappa in [1].

It can be seen that insufficiently long memory lengths in the state initialization lead to large deviations between the Grünwald-Letnikov approximation and the

true solution. These deviations due to an insufficiently long initialization window are much larger than the influence of the finitely long discretization step size ΔT_k .

In the following section, time domain truncation errors resulting from resetting a numerical integration of an FDE model after a finitely long time span are quantified in a rigorous manner under the assumption that the system behavior for $t \leq 0$ is known in advance. For the sake of simplicity, we rely on the temporally constant initialization according to (7) in the following.

Remark 1. Due to the fact that the numerical solution of an FDE with non-constant initialization functions is influenced by the change of coordinates in (9), this linear shift of the coordinate system will form a potential basis for identifying the pseudo states' history in future work.

3 Interval Observer Technique for the Identification of Improved Bounds of Time-Domain Truncation Errors

3.1 Constant Bounds for Time-Domain Truncation Errors

So far, we have assumed that initial conditions for the pseudo state of the FDE system model (1) are specified at the instant $t = 0$ with a temporally constant past for all times $t < 0$ in accordance to the Caputo definition of fractional derivatives. To allow for a notation denoting the influence of the point of time at which the derivative operator ${}_0\mathcal{D}_t^{(\nu)}$ is initialized, the notation of Eq. (1) is changed to

$${}_0\mathcal{D}_t^{(\nu)} \mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t)) \quad (15)$$

in the following, where the left subscript of ${}_0\mathcal{D}_t^{(\nu)}$ specifies the temporal initialization point (in the case above, $t = 0$).

According to the work of Podlubny, guaranteed bounds for the influence of shifting this initialization point from the time instant t_k to another point $t_k + T$, $T > 0$, can be computed component-wise according to

$$\left| {}_{t_k}\mathcal{D}_t^{(\nu)} \mathbf{x}(t) - {}_{t_k+T}\mathcal{D}_t^{(\nu)} \mathbf{x}(t) \right| \leq \frac{\mathcal{X}T^{-\nu}}{|\Gamma(1-\nu)|} =: \boldsymbol{\mu} \quad (16)$$

with

$$\mathcal{X} = [\mathcal{X}_1 \quad \dots \quad \mathcal{X}_n]^T \quad (17)$$

comprising the suprema

$$\mathcal{X}_i = \sup_{t \in [t_k; t_k+T]} |x_i(t)|, \quad i \in \{1, \dots, n\} \quad (18)$$

of the reachable pseudo states over the time interval $t \in [t_k; t_k + T]$ for each element of the vector $\mathbf{x}(t)$.

As shown in ..., these error bounds can be employed to reset interval-based verified solution procedures for FDE models after a certain time span and to use the solution enclosures determined after the reset to reduce overestimation arising due to pessimism that is introduced by long integration time intervals. For that purpose, the right-hand side of the system model (15) is inflated by the error bound interval $[-\boldsymbol{\mu}; \boldsymbol{\mu}]$ to obtain the uncertain system model

$$\tilde{\mathbf{f}}(\mathbf{x}(t)) \in \mathbf{f}(\mathbf{x}(t)) + [-\boldsymbol{\mu}; \boldsymbol{\mu}] . \quad (19)$$

Using this modification, the simulation is continued after at the point $t = t_k + T$ for the differential inclusion model defined by the expression $\tilde{\mathbf{f}}(\mathbf{x}(t))$ and the pseudo state values $\mathbf{x}(t_k)$ as initial condition, while the entire past for $t < t_k$ is no longer required for a further system simulation.

Under the assumption of cooperativity of the state equations, see ... for further details, independent lower and upper bounding trajectories can be extracted from the modified system model (19) so that set-based integration routines such as the one based on interval extensions of the Mittag-Leffler function from ... can be avoided when solving the corresponding initial value problem for the differential inclusion problem (19) after the inflation of the original system's right-hand side $\mathbf{f}(\mathbf{x}(t))$.

Remark 2. To limit the pessimism introduced by the additive error bounds in (19), the following two aspects should be accounted for:

- Define the pseudo state $\mathbf{x}(t)$ in such a way that $\mathbf{x} = \mathbf{0}$ corresponds to the equilibrium of an asymptotically stable FDE. If $\mathbf{x} = \mathbf{0}$ is not the corresponding steady state after a first-principle modeling, perform a shift of coordinates as inspired by Eq. (11) so that the absolute values for the bounds $\boldsymbol{\mathcal{X}}$ do not increase for sufficiently large values of $t_k + T$.
- Set the initial point t_k in (16) to $t_k = 0$. Together with the first aspect in this remark, this allows for a computation of values for the error bounds $\boldsymbol{\mu}$ that decrease after sufficiently long integration times and thus lead to less conservative system models than always recomputing the bounds $\boldsymbol{\mu}$ with respect to a previous reset point $t_k > 0$.

Even though these two aspects can be accounted for in many practical situations, the bounds $\boldsymbol{\mu}$ given in (16) remain conservative due to the fact that they are temporally constant. This property does not explicitly account for the observation that shifting the initialization point of the fractional derivative operator becomes less important for increasing integration times³. Therefore, an observer-based refinement of the bounds $\boldsymbol{\mu}$ — to our knowledge not yet considered in any other publication — is presented in the following subsection.

³This observation is denoted as *short memory principle* in [Podlubny].

3.2 Observer-Based Enhancement of the Bounds for Time-Domain Truncation Errors

For the observer-based enhancement of the time-domain truncation error bounds when resetting fractional integrators, we restrict ourselves to the case of cooperative system models in this paper.

As a generator for *virtual measurements* of a cooperative dynamic system model, we compute pseudo state enclosures

$$\mathbf{x}(t) \in [\mathbf{v}(t) ; \mathbf{w}(t)] \quad (20)$$

for the FDE model (15) with the temporally constant initialization

$$\mathbf{x}(t) \in [\mathbf{x}_0] , \quad \dot{\mathbf{x}}(t) = \mathbf{0} \quad \text{for } t < 0 . \quad (21)$$

This setting corresponds to uncertain initial conditions in the sense of Caputo, while the influence of temporally varying initializations is taken into account as soon as the first integrator reset has been performed.

Cooperativity of the system model (15) is guaranteed as a sufficient condition if all off-diagonal elements of the Jacobian of the system model's right-hand side with respect to the pseudo state vector $\mathbf{x}(t)$ ($i, j \in \{1, \dots, n\}$, $i \neq j$) satisfy the inequalities

$$\frac{\partial f_i(\mathbf{x})}{\partial x_j} \leq 0 . \quad (22)$$

Then, all reachable pseudo states can be enclosed by the lower and upper bounding systems

$$\begin{aligned} {}_0\mathcal{D}_t^{(\nu)} \mathbf{v}(t) &= \mathbf{f}_v(\mathbf{v}(t)) , \quad \mathbf{v}(t \leq 0) = \underline{\mathbf{x}}_0 \quad \text{and} \\ {}_0\mathcal{D}_t^{(\nu)} \mathbf{w}(t) &= \mathbf{f}_w(\mathbf{w}(t)) , \quad \mathbf{w}(t \leq 0) = \bar{\mathbf{x}}_0 , \end{aligned} \quad (23)$$

respectively, where the inequalities

$$v_j(t) \leq x_j(t) \leq w_j(t) \quad (24)$$

hold for all $j \in \{1, \dots, n\}$.

If the integration of a cooperative FDE model is reinitialized at a point $t = T > 0$, an observer-based approach

$${}_T\mathcal{D}_t^{(\nu)} \mathbf{z}(t) = \begin{bmatrix} \mathbf{f}_v(\tilde{\mathbf{v}}(t)) + \boldsymbol{\mu}_v(t) \\ \mathbf{f}_w(\tilde{\mathbf{w}}(t)) + \boldsymbol{\mu}_w(t) \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \mathbf{H} \cdot \begin{bmatrix} \mathbf{v}(t) - \tilde{\mathbf{v}}(t) \\ \mathbf{w}(t) - \tilde{\mathbf{w}}(t) \end{bmatrix} \quad (25)$$

with the augmented state vector

$$\mathbf{z}(t) = \begin{bmatrix} \tilde{\mathbf{v}}(t) \\ \tilde{\mathbf{w}}(t) \\ \boldsymbol{\mu}_v(t) \\ \boldsymbol{\mu}_w(t) \end{bmatrix} \in \mathbb{R}^{4n} \quad (26)$$

can be used to enhance the pseudo state enclosures and the truncation error bounds in comparison with the ones obtained by the integrator resetting approach according to the previous subsection that only employs temporally constant truncation error bounds.

The observer (25) is initialized with the pseudo state vector

$$\mathbf{z}(T) = \begin{bmatrix} \mathbf{v}(T) \\ \mathbf{w}(T) \\ -\boldsymbol{\mu}_T \\ \boldsymbol{\mu}_T \end{bmatrix}, \quad (27)$$

where the truncation error bounds $[-\boldsymbol{\mu}_T ; \boldsymbol{\mu}_T]$ are computed as described in the previous subsection. Due to the inclusion of the truncation error bounds $\boldsymbol{\mu}_v(t)$ and $\boldsymbol{\mu}_w(t)$ by means of so-called integrator disturbance models in (25), leading to constant values if $\mathbf{H} = \mathbf{0}$, the case of the error quantification according to the previous subsection is included as a special case in this formulation. If $\mathbf{H} \neq \mathbf{0}$, the bounds $\boldsymbol{\mu}_v(t)$ and $\boldsymbol{\mu}_w(t)$ are enhanced in such a way that the *virtual measurements* and the enhanced bounds for the pseudo state variables approach each other as close as possible. For that purpose, the augmented system model (25) must be a valid interval observer.

To make the augmented system model (25) with the estimated lower and upper bounding trajectories $\tilde{\mathbf{v}}(t)$ and $\tilde{\mathbf{w}}(t)$ a valid observer, the gain matrix \mathbf{H} needs to be chosen so that the error dynamics associated with the bounding trajectories remain asymptotically stable with

$$\|\mathbf{v}(t) - \tilde{\mathbf{v}}(t)\| \rightarrow 0 \quad \text{and} \quad \|\mathbf{w}(t) - \tilde{\mathbf{w}}(t)\| \rightarrow 0 \quad \text{for } t \rightarrow \infty \quad (28)$$

and that

$$[\mathbf{v}(t) ; \mathbf{w}(t)] \subseteq [\tilde{\mathbf{v}}(t) ; \tilde{\mathbf{w}}(t)] \quad (29)$$

is ensured for all $t \geq T$.

For corresponding stability criteria for linear fractional differential equations, see the eigenvalue domains derived in ... Due to the fact that these domains can be expressed effectively by linear matrix inequality constraints and that nonlinear models can be bounded by quasi-linear system models with polytopic uncertainty representations, stability requirements for the gain matrix \mathbf{H} cannot only be obtained for linear system models. They can also be obtained from the existing literature for nonlinear ones as it has been shown, for example in ..., for the design of robust state estimation schemes for FDEs.

To ensure the enclosure property (29) and to verify the decoupled nature of the equations in (25) with respect to $\tilde{\mathbf{v}}(t)$ and $\tilde{\mathbf{w}}(t)$ as well as $\boldsymbol{\mu}_v(t)$ and $\boldsymbol{\mu}_w(t)$, respectively, the Jacobian of the right-hand side of the augmented model (25) with respect to the pseudo state vector $\mathbf{z}(t)$ needs to satisfy the sign property presented in the inequalities (22) with now $4n$ as the dimension of the augmented model.

3.3 Periodic Reset of Fractional Integrators and Their Application to Predictor–Corrector State Estimation

The observer approach from the previous subsection is the basis for a predictor–corrector approach for state estimation if measurements

$$\mathbf{y}(t_{m,k}) \in [\mathbf{y}](t_{m,k}) = \left[\underline{\mathbf{y}}(t_{m,k}) ; \bar{\mathbf{y}}(t_{m,k}) \right] \quad (30)$$

(including interval uncertainty to represent bounded measurement errors with unknown distributions) are available at the time instants $t_{m,k}$ in the form

$$\mathbf{g}(\mathbf{x}(t_{m,k})) \in \left[\underline{\mathbf{y}}(t_{m,k}) ; \bar{\mathbf{y}}(t_{m,k}) \right] . \quad (31)$$

Then, the same observer (25) as in the previous subsection is employed with *virtual measurements* obtained from a simulation of the original system dynamics. The *actual state measurements* $[\mathbf{y}](t_{m,k})$ are then used to tighten the bounds included in the pseudo state initialization $\mathbf{z}(T)$ at each point $T = t_{m,k}$.

This tightening is either obtained by a direct intersection of the measured intervals $[\mathbf{y}](t_{m,k})$ with the already computed state bounds $[\tilde{\mathbf{v}}(t_{m,k}) ; \tilde{\mathbf{w}}(t_{m,k})]$ in the case of a direct pseudo state measurement or by applying a suitable contraction scheme (forward–backward contractor or Krawczyk-type contractor) to the relation (31), where the bounds for $\mathbf{x}(t_m)$ are initialized with the interval $[\tilde{\mathbf{v}}(t_{m,k}) ; \tilde{\mathbf{w}}(t_{m,k})]$ as in the first case. After this tightening step, the procedure is continued as described in the previous subsection, where the modification of the pseudo state reinitialization is the only modification in comparison with the previous subsection.

This approach allows for directly handling the continuous-time dynamics of the system model between two subsequent discrete time instants at which measured data are available. In such a way, the sampling times both for the numerical evaluation of the FDE model and the measurements can be decoupled.

Remark 3. Cases in which the measurement step size is not an integer multiple of the numerical integration step or in which the measurement times themselves are uncertain, can be handled with the same procedure as in ...

Remark 4. Future work will aim at removing the precondition of cooperativity of the original as well as the observed system dynamics in (15) and (25). To solve this task, the so-called TNL approach for the parameterization of interval observers as derived in ... is a promising solution which — due to its direct applicability to descriptor models — can extend the approach presented in this paper also to cases in which only some of the pseudo state variables are described by the explicit FDE models studied in (15) and other are expressed implicitly by using algebraic constraints.

4 Illustrating Example: Observer Approach

In this section, the observer-based identification of bounds for time-domain truncation errors of FDE models is presented for both a nonlinear academic simulation scenario and for a close-to-life quasi-linear model for the charging and discharging dynamics of Lithium-ion batteries.

4.1 Nonlinear Academic Benchmark System

4.1.1 Observer-Based Quantification of Time-Domain Truncation Errors

As a first example, consider the uncertain FDE model

$$x^{(0.5)}(t) = -x(t) - p \cdot x^3(t) \quad (32)$$

with the interval-based Caputo initialization

$$x(t) \in [x_0] = [0.9 ; 1.0] \quad \text{for } t \leq 0 \quad (33)$$

and the uncertain, time-invariant parameter $p \in [0.1 ; 0.2]$. This system model is simulated over the time interval $t \in [0 ; 10]$ with integrator resets at the time instants

$$T \in \{T', 2T', 3T', \dots\}, \quad \text{where } T' = 1. \quad (34)$$

Due to its scalar nature, this system models satisfies the property of cooperativity, so that (without integrator resetting) the true pseudo state enclosure $x(t) \in [v(t) ; w(t)]$ according to (23) can be determined by means of the craps system models

$${}_0\mathcal{D}_t^{(0.5)}v(t) = -v(t) - \bar{p} \cdot v^3(t), \quad v(t \leq 0) = \underline{x}_0 \quad (35)$$

and

$${}_0\mathcal{D}_t^{(0.5)}w(t) = -w(t) - \underline{p} \cdot w^3(t), \quad w(t \leq 0) = \bar{x}_0. \quad (36)$$

These bounds, computed with the help of the numerical integration routine `fd12` ..., are visualized by the solid lines in Figs. 2 and 3.

To investigate the observer approach given in Eq. (25), the time- and state-independent gain matrix

$$\mathbf{H} = 20 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (37)$$

is chosen which ensures stability of the estimation error dynamics and cooperativity of the augmented pseudo state equations.

Fig. 2(a) presents a comparison between the integrator resetting in combination with piecewise constant bounds μ_T for each time slice $t \in [(i-1)T' ; iT']$, $i \in \{1, 2, \dots, 10\}$ (determined according to (16), where $t_k = 0$ is chosen in each reinitialization point T defined in (34)), while Fig. 2(b) shows the observer-based enhancement of the pseudo state enclosures due to the temporal adaptation of the truncation error bounds according to Eq. (25) in Sec. 3.2.

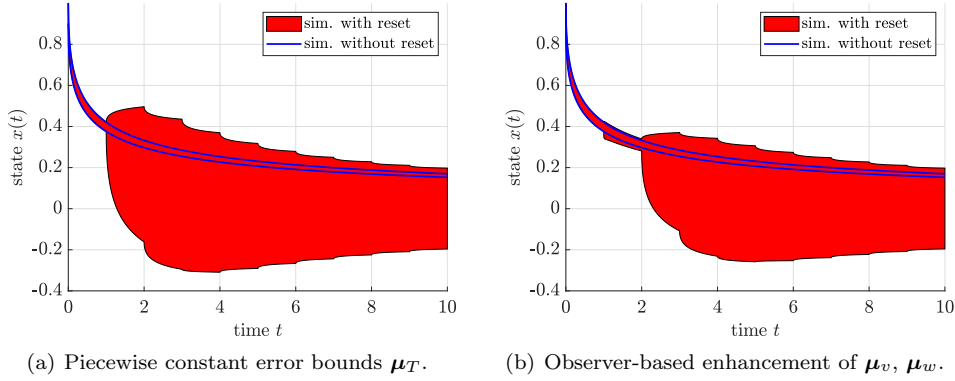


Figure 2: Simulation of the uncertain, nonlinear benchmark system (32).

4.1.2 Predictor–Corrector State Estimation

For the implementation of the predictor–corrector state estimator according to Sec. 3.3, we assume that pseudo state measurements are available at the time instants $t_{m,k} = T$ listed in (34).

The results in Fig. 3(a) distinguish the following two cases:

- The measured pseudo state information at the time instants $t_{m,k}$ corresponds to the enclosures from (35) and (36) with

$$y(t_{m,k}) \in [v(t_{m,k}); w(t_{m,k})] . \quad (38)$$

This scenario is depicted in Figs. 3(a) and 3(b).

- The measured pseudo state information at the time instants $t_{m,k}$ is obtained as

$$y(t_{m,k}) \in \hat{x}(t_{m,k}) + 0.001 \cdot [-1; 1] \quad (39)$$

with $\hat{x}(t)$ as the simulation of a nominal parameter model

$$\hat{x}^{(0.5)}(t) = -\hat{x}(t) - 0.15 \cdot \hat{x}^3(t) \quad (40)$$

with $\hat{x}(t) = 0.95$ for $t \leq 0$, see Figs. 3(c) and 3(d).

From a comparison of Figs. 3(a) and 3(b), it is obvious that the observer-based approach in combination with resetting the pseudo state to the measured data leads to significantly tighter enclosures of the solutions than the use of piecewise constant error bounds μ_T .

A further tightening of the simulated bounds becomes possible if the uncertainty in the measured data is reduced in Figs. 3(c) and 3(d) in accordance with the second case above. Then, the new solution approach is capable of determining

pseudo state enclosures that are partially tighter (in this case for the upper bounding trajectory) than a pure simulation of the uncertain nonlinear model (32) that still serves as the *virtual measurement* generator between the points T at which the *actual discrete-time measurements* are available. In such a way, the proposed observer-based enhancement of the time-domain truncation error bounds as well as the predictor–corrector state estimation scheme form the basis for the development of set-based parameter identification schemes that are part of our ongoing research activities.

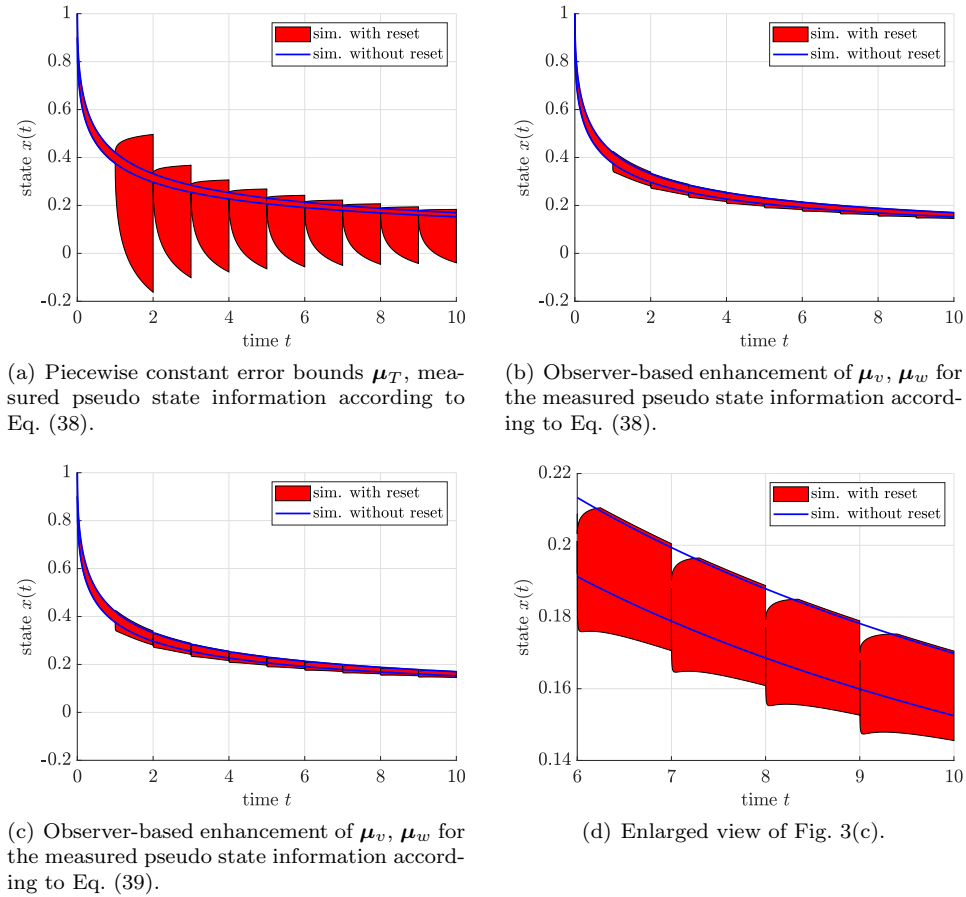


Figure 3: Predictor–corrector state estimation for the uncertain, nonlinear benchmark system (32).

4.2 Simplified Fractional Battery Model

As a final application scenario, consider the fractional-order equivalent circuit model for the charging and discharging dynamics of Lithium-ion batteries depicted in Fig. 4.

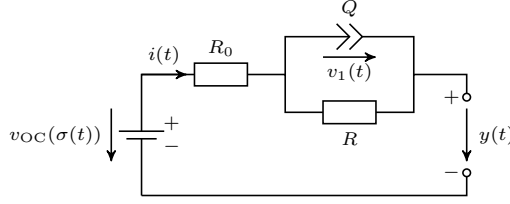


Figure 4: Equivalent circuit representation of a simplified fractional battery model.

Using the parameter values identified experimentally in [3], continuous-time state equations

$${}_0\mathcal{D}_t^{(0.5)} \mathbf{x}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{b} \cdot i(t) \quad (41)$$

with the system and input matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\eta_1 \cdot \text{sign}(i(t))}{3600C_N} & 0 & 0 \\ 0 & 0 & -\frac{1}{RQ} \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ -\frac{\eta_0}{3600C_N} \\ \frac{1}{Q} \end{bmatrix} \quad (42)$$

as well as the pseudo state vector

$$\mathbf{x}(t) = \left[\sigma(t) \quad {}_0\mathcal{D}_t^{(0.5)} \sigma(t) \quad v_1(t) \right]^T \in \mathbb{R}^3 \quad (43)$$

can be derived by applying Kirchhoff's voltage and current laws. In (43), $\sigma(t)$ denotes the state of charge of the battery, its fractional derivative is included in the vector $\mathbf{x}(t)$ to represent long-term memory phenomena, and $v_1(t)$ is the voltage across a non-integer constant phase element Q serving as a generalization of capacitors that are typically employed to represent polarization effects and the transportation of charge carriers in Thevenin equivalent circuit models of batteries.

For state estimation purposes, the terminal voltage (given in a quasi-linear representation)

$$y(t) = g(\mathbf{x}(t)) = \left[\sum_{k=0}^4 c_k \sigma^{k-1}(t) \quad 0 \quad -1 \right] \cdot \mathbf{x}(t) + \left(-R_0 + d_0 e^{d_1 \sigma(t)} \right) \cdot i(t) \quad (44)$$

is assumed to be available as a measured system output at specific discrete points in time.

To obtain further a cooperative system model, we consider the special case with $\eta_1 = 0$ and a controlled discharging process of the battery with the terminal current

$$i(t) = -\mathbf{k}^T \cdot \mathbf{x}(t) \quad (45)$$

as the system input in which the controller gain vector \mathbf{k}^T is determined by pole assignment so that the eigenvalues of the closed-loop system are located at the points $\lambda \in \{0; -0.0002; -\frac{1}{RQ}\}$.

This leads to the linear autonomous system model

$${}_0\mathcal{D}_t^{(0.5)} \mathbf{x}(t) = \mathbf{A}_C \cdot \mathbf{x}(t) \quad (46)$$

with

$$\mathbf{A}_C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \quad (47)$$

in which the entries a_{22} , a_{32} , and a_{33} are converted into interval parameters according to

$$\begin{aligned} a_{22} &\in [-0.00022000000000; -0.00017999999999] \\ a_{32} &\in [0.09755718517445; 0.11923655965767] \\ a_{33} &\in [-0.53155619600024; -0.43490961490928] \end{aligned} \quad (48)$$

to account for independent uncertainties of each of these quantities in the intervals of $\pm 10\%$ around the respective nominal values obtained with the help of the parameters given in [3] and ...

The initial conditions of the system are assumed to be uncertain according to the Caputo definition

$$\mathbf{x}(t) = [0.5 \quad 0.01 \quad 0.1]^T \cdot [0.9; 1.1], \quad t \leq 0. \quad (49)$$

To set up the observer-based enhancement of the time-domain truncation error bounds according to Eq. (25) with the resetting time instants

$$T \in \{T', 2T', 3T', \dots\}, \quad \text{where } T' = 60 \text{ s}, \quad (50)$$

the gain matrix

$$\mathbf{H} = 5 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (51)$$

is chosen in the following subsections. According to the description in Sec. 3.2, this matrix is specified so that stability and cooperativity of the estimation error dynamics are ensured.

4.2.1 Observer-Based Quantification of Time-Domain Truncation Errors

Fig. 5 summarizes a simulation of the fractional battery model in terms of a direct evaluation of the pseudo state equations (46)–(48) by directly exploiting the property of cooperativity. These results are shown by solid lines, indicating the lower

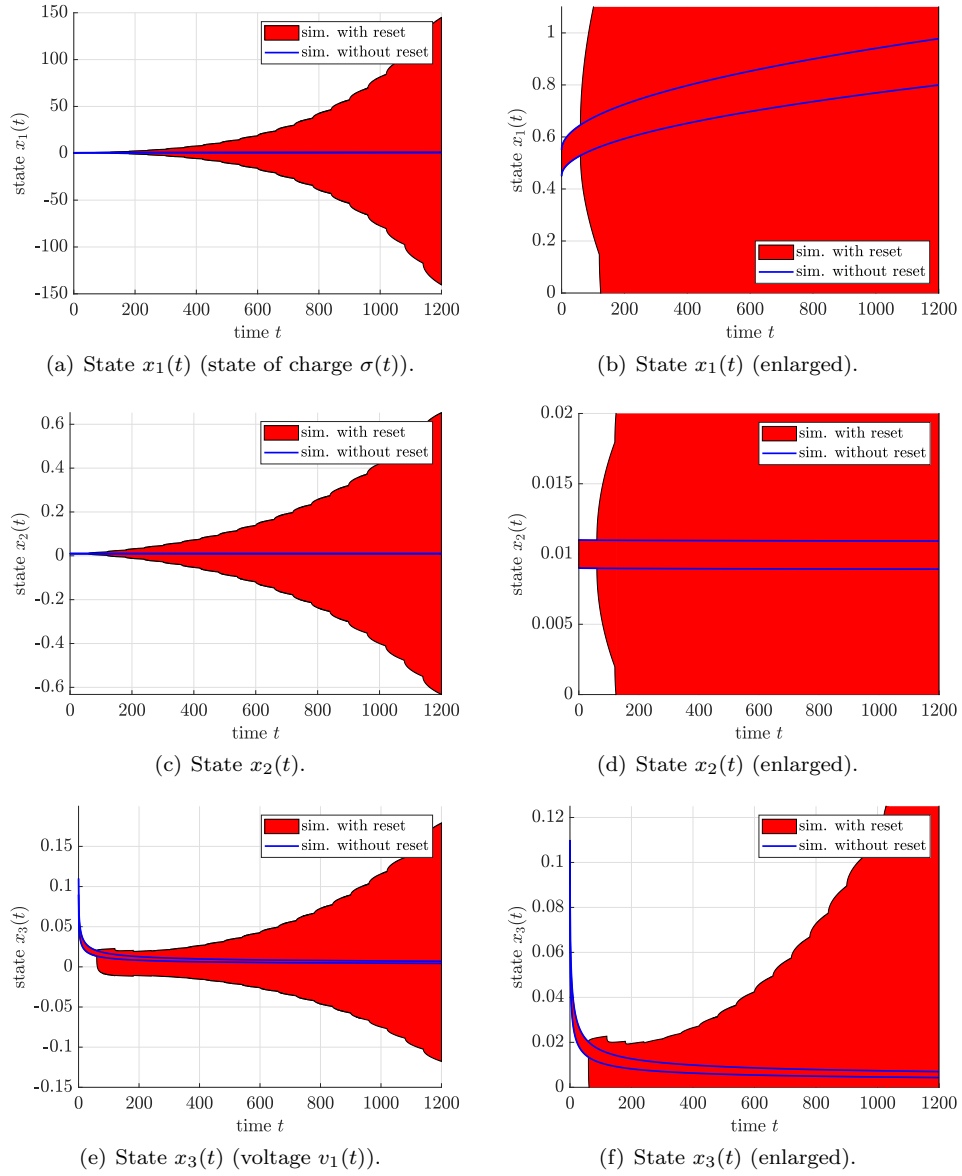


Figure 5: Simulation of the simplified battery model with piecewise constant error bounds μ_T ; integrator reset at the points T defined in (50).

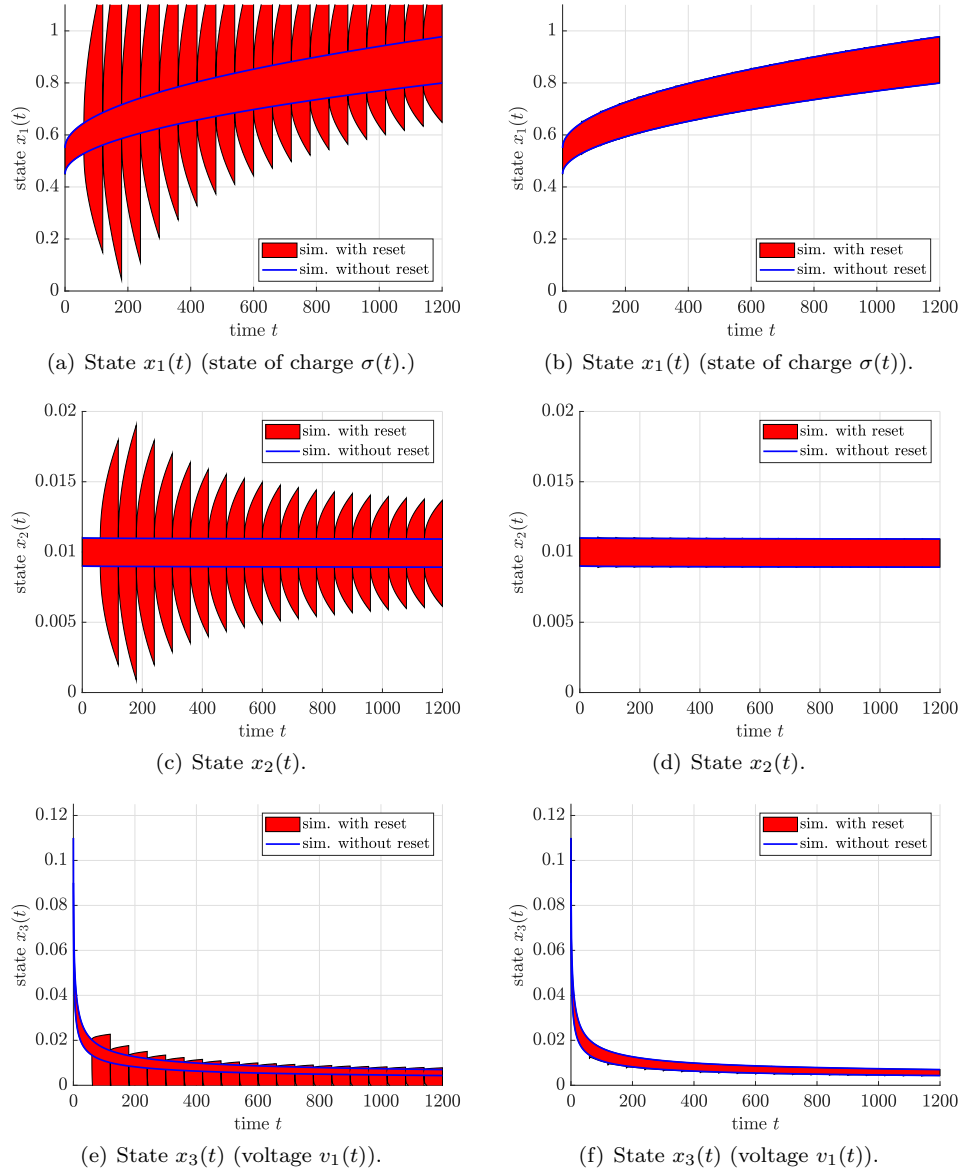


Figure 6: Comparison of piecewise constant error bounds μ_T (left column) with the observer-based enhancement μ_v, μ_w (right column) according to Sec. 3.2; resetting to the true state enclosures $[\mathbf{v}(t); \mathbf{w}(t)]$ at each time instant T defined in (50).

and upper bounds of each state variable, respectively. As for the previous academic example, the numerical solver `fdel2` has been used for this purpose.

When resetting the fractional integrator at the time instants (50) and using constant bounds for the time-domain truncation errors, a rapid inflation of the pseudo state enclosures can be observed. This inflation is reduced in Fig. 6, where it has been assumed that the bounds, resulting from the cooperative system simulation, are available as initial conditions for each of the time slices. This resetting of the integrator, together with a reinitialization of the pseudo state for each point in time T is shown in the left column of Fig. 6 for piecewise constant bounds of the time-domain truncation errors.

Activating the observer-based quantification of the truncation according to Sec. 3.2 additionally, as illustrated in the right column of Fig. 6, leads to significantly tight outer enclosures that satisfy the relation (29) with certainty.

4.2.2 Predictor–Corrector State Estimation

In practical situations, the resetting of the fractional integrator is often combined with a pseudo state estimation approach as presented in Sec. 3.3. To visualize the applicability of this technique for the model of a controlled Lithium-ion battery, it is assumed that uncertain measurements of the terminal voltage of the battery are available at each time instant $t_{m,k} = T$. These uncertain measurements are chosen as the intervals

$$[y](t_{m,k}) = \text{mid} \left([g] \left([\mathbf{v}(t_{m,k}); \mathbf{w}(t_{m,k})] \right) \right) + [-10; 10] \text{ mV} , \quad (52)$$

where $\text{mid}([x]) = \frac{1}{2} \cdot (\underline{x} + \bar{x})$ defines the midpoint of an interval $[x]$.

At each measurement instant T , the new pseudo state bounds are then initialized with the enclosure $[\tilde{\mathbf{v}}(T); \tilde{\mathbf{w}}(T)]$ obtained at the end of the previous time slice. From these bounds, an axis-aligned interval box is extracted by a SIVIA-like state reconstruction for the measured system output (44) that eliminates subboxes that are incompatible with the measurement intervals (52). To continue the simulation further, a tight axis-aligned interval hull around the not eliminated boxes is formed, so that the pseudo state enclosures shown in Fig. 7 are obtained.

Future work will make use of these bounds for an identification of interval parameters included in both the FDE model and the algebraic output equation of a dynamic system.

5 Conclusions

In this paper, a novel observer-based approach for the quantification of time-domain truncation errors of FDE models has been presented. These errors arise inevitably when resetting fractional integrators. Integrator resets are necessary for the numerical evaluation of FDE models both to restrict the growth of memory demands when evaluating FDEs over long time spans and to take into account measured

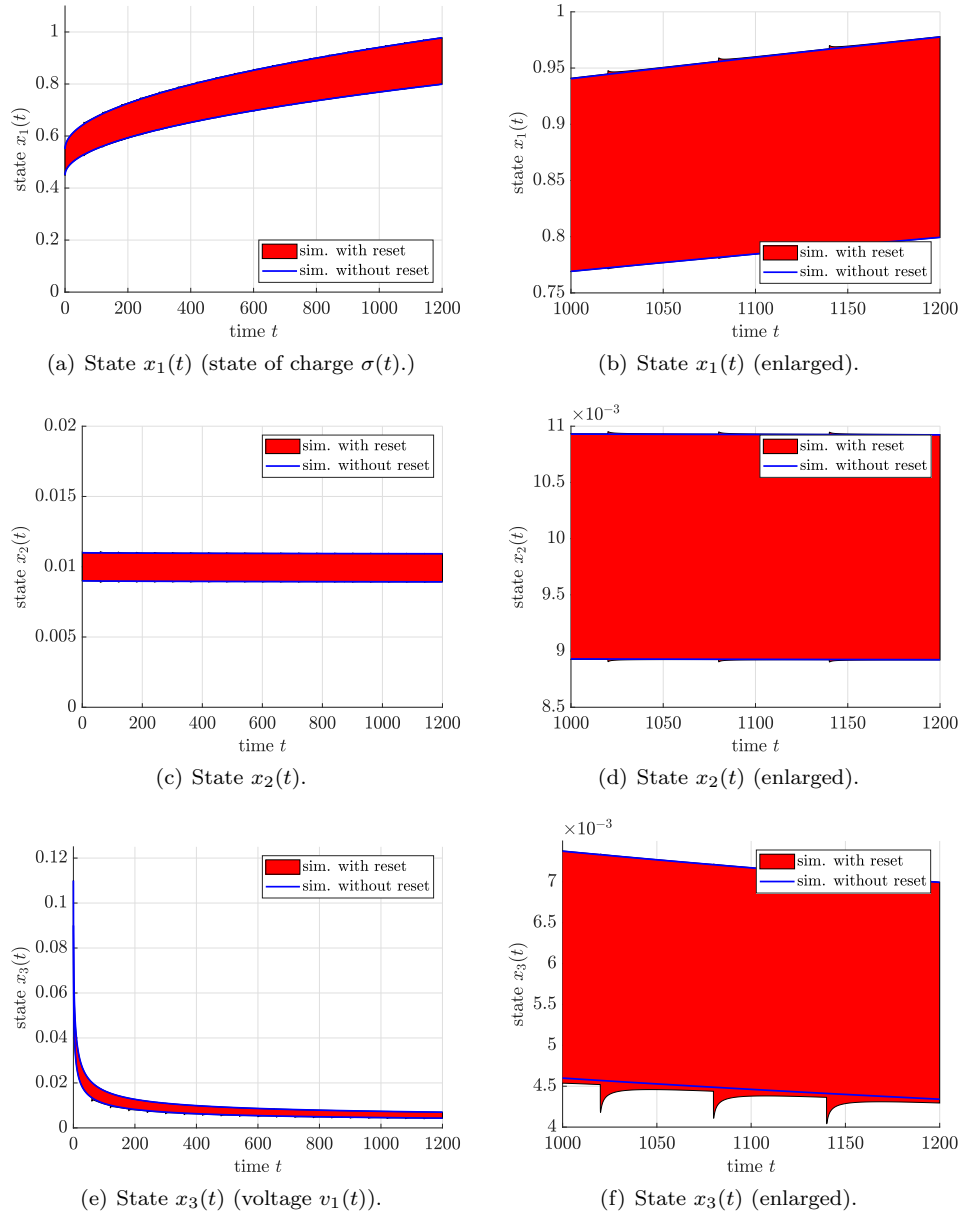


Figure 7: Simulation using the observer-based enhancement μ_v , μ_w according to Sec. 3.2 and contractor-based resetting of the state variables at each measurement instant T according to Sec. 3.3.

state information at distinct points in time between which the system dynamics are evolving continuously.

Future work will make use of the presented approach to solve the tasks of identifying past pseudo state information from an observed evolution of these quantities into the future and to identify uncertain system parameters on the basis of predictor-corrector state estimators. Moreover, the TNL interval observer design approach for non-cooperative system models will be taken into consideration to avoid the currently existing necessity to transform non-cooperative models into cooperative ones by using the approaches presented in Although these approaches are useful for many practical applications, they always lead to conservative state enclosures due to the wrapping effect that is inevitable when transforming the state equations and the domains of uncertain initial conditions with the help of (static) similarity transformations. This pessimism can be reduced by the TNL approach due to the introduction of further degrees of freedom for the observer parameterization. Moreover, this approach will also make the proposed methodology applicable to fractional descriptor systems.

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