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A novel approach to stability analysis of a wide class of irrational linear systems

Vukan Turkulov¹ · Milan R. Rapaic¹ · Rachid Malti²

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Abstract This paper presents a general methodology for finding stability equivalence regions, of a wide class of linear time-invariant systems with irrational transfer functions, inside a parametric space. The proposed methodology can be applied to distributed-parameter, time-delay and fractional systems. Unlike rational transfer functions which have only a finite number of poles, irrational transfer functions may generally possess an infinite number of poles, branch points and even essential singularities. Due to this, stability of such systems is more difficult to analyze. Two variants of the new methodology are presented. The first one analyzes stability equivalence along a curve in the parametric space, starting from a given parametric point. The second one finds the maximal stability equivalence region in the parametric space around a given parametric point. Both methodologies are based on iterative application of Rouché's theorem. They are illustrated on several examples, including heat diffusion equation and generalized time-fractional telegrapher's equation, which exhibit special functions such as \sinh and \cosh of \sqrt{s} , the Laplace variable of order 0.5.

Keywords stability · distributed parameter systems · irrational transfer functions · fractional systems

1 Introduction

This paper proposes a methodology for analyzing stability of linear, time-invariant (LTI) systems with (generally) irrational transfer functions. Such systems are infinite dimensional, with examples including distributed parameter, time-delay and fractional

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systems. Naturally, behavioral analysis of infinite dimensional systems is challenging compared to their finite dimensional counterparts, resulting in several different analysis methodologies developed over the last decades.

One of the major ideas in the existing literature is to avoid directly dealing with the fact that the singularities are not just poles, and/or their count is infinite. This can be achieved in several ways. For example, an irrational transfer function can be approximated with a rational one, effectively reducing the system order [1, 31, 4, 5, 10, 2, 14]. This allows application of conventional system analysis and controller synthesis methods, at the cost of potential approximation errors. Alternatively, some systems allow coprime matrix decomposition, with behavioral tests based on the resulting coprime factors [21, 39, 43, 38, 26]. Contrary to this family of approaches, the methodology proposed in this paper directly operates on irrational transfer functions without any approximations.

Distributed parameter systems are, in particular, often modelled in Hilbert state-spaces [30, 36, 48, 16, 19, 49, 13, 14, 44]. Consequently, results from the semigroup theory are used for behavioral analysis of such models. The resulting methods are remarkably similar to their finite-dimensional counterparts. For example, generalizations of small-gain theorem and Nyquist criterion for such models have been developed [20]. Compared to the aforementioned approaches, the methodology proposed in the present work assumes that the system is modelled by a parametrized transfer function in the spectral (Laplace) domain.

A number of Lyapunov-Krasovskii methods has been adapted to distributed parameter systems [15, 45, 33, 3]. Such methods directly analyze system stability in time domain, usually yielding sufficient stability tests. Our proposed methodology operates in the spectral domain and provides conditions for determining the entire connected region of parameters in which the number of unstable poles is invariant.

Several proposed stability tests are specific to certain system classes. For example, [29, 8] propose various stability tests for transfer functions that are rational with regards to the square root of the Laplace variable. Various stability tests for different fractional-order systems have been proposed in [24, 40, 51, 47]. The stability of distributed-order fractional systems with distributed delays is analyzed in [7]. Furthermore, a number of methods is developed for analyzing stability of several specific important irrational systems such as heat [6, 9], wave [17, 50], vibrating string [46] and telegrapher's [12] equations. The methodology presented in this paper is uniformly applicable to all of the aforementioned irrational systems.

A considerable amount of research has been conducted on control of irrational systems. The resulting methods often deal with synthesis of (robust) stabilizing controllers, without directly analyzing the stability of the irrational plant itself [4, 5, 18, 25]. The H^∞ control is an especially popular control method for distributed parameter plants [28, 27, 41, 11]. Our methodology does not explicitly tackle control problems. However, the stability of (generally irrational) closed-loop control systems can be analyzed using the presented methodology which extends [42, 35] to a more general set of LTI systems with irrational transfer functions.

Stability of linear irrational systems is treated, in a general framework under some hypotheses in [23], based on the determination of stability crossing sets, i.e. a set of surfaces for which the system contains at least one pole on the imaginary axis.

The authors formulate the problem as a constraint satisfaction and solve it using algorithms from interval arithmetics which have an exponential complexity with respect to the number of parameters. This paper deals with a similar problem and proposes algorithms on the basis of Rouché's theorem from complex analysis. The proposed algorithms allow concluding on stability in a given direction and/or in a neighborhood of a parametric point.

1.1 Notation and definitions

The Laplace variable is denoted as s . Angled brackets $\langle \cdot, \cdot \rangle$ represent the dot product. The p -norm of a vector \mathbf{x} is denoted as $\|\mathbf{x}\|_p$. The expressions “left-hand side” and “right-hand side” are abbreviated to LHS and RHS, respectively. The set of real numbers is denoted as \mathbb{R} and the set of non-negative real numbers is denoted as \mathbb{R}^+ . The field of complex numbers is denoted as \mathbb{C} , the open right-half plane $\{s \in \mathbb{C} : \Re(s) > 0\}$ is denoted as \mathbb{C}^+ , and the closed right-half plane $\{s \in \mathbb{C} : \Re(s) \geq 0\}$ is denoted as $\overline{\mathbb{C}^+}$. A complex function f is defined to be analytic in a closed set $\overline{\mathcal{Y}}$ if and only if there exists an open set \mathcal{Y}^* such that f is analytic in \mathcal{Y}^* and $\overline{\mathcal{Y}} \subset \mathcal{Y}^*$.

1.1.1 Characteristic function

The transfer function of a linear, time-invariant dynamical system is denoted as $G(s, \eta)$, where η is a vector of real-valued parameters belonging to a compact and connected admissible domain $\eta \in \mathcal{D} \subset \mathbb{R}^n, n \in \mathbb{N}$.

Definition 1 $f(s, \eta)$ is a characteristic function of a given transfer function $G(s, \eta)$ if $\forall \eta \in \mathcal{D}$:

- (1) f has no poles.
- (2) f has branch points and/or essential singularities everywhere G has branch points and/or essential singularities.
- (3) All finite zeros of f match poles of G in both location and multiplicity.

The definition of characteristic function is adopted from [23], and generalizes the concept of a characteristic polynomial to irrational transfer functions. There exist multiple valid characteristic functions f for any given transfer function G , all sharing the same stability properties. Likewise, a given characteristic function f is valid for multiple transfer functions G .

The presented methodology operates on characteristic functions. Hence, the stability analysis results of f are applicable to the set of transfer functions for which f satisfies Definition 1.

1.1.2 Notion of stability

The presented methodology finds regions in which the number of poles with positive real part is invariant. Therefore, throughout this paper, a pole is considered to be unstable if its real part is strictly positive, and stable otherwise. The actual relationship

between the invariance in number of such poles and invariance in stability depends on the stability notion being used. For example, if each pole p_i of an irrational system $G(s, \eta)$ satisfies $\Re\{p_i\} < -\gamma$ for some $\gamma > 0$, then G is exponentially stable (or more precisely, γ -exponentially stable). Apart from exponential stabilities, there are other useful definitions such as L_p stability [22], BIBO stability, H^2 stability, H^∞ stability [32]. An in-depth analysis of these notions is beyond the scope of our present work, but we would like to stress the fact that it is possible for an infinite-dimensional system to have all poles in the left-hand side of the complex plane and still be unstable in a certain sense, as demonstrated in [32]. It is also worth pointing out that in order to claim H^∞ stability, it is required to additionally check that $G(s, \eta)$ is bounded on the imaginary axis.

1.1.3 Stability equivalence region

The notion of stability equivalence region is partially adopted from [42]. The number of zeros of $f(s, \eta)$ belonging to \mathbb{C}^+ is denoted as $NU_f(\eta)$, where each zero is counted as many times as its multiplicity. The set of all parameter points $\eta \in \mathcal{D}$ for which $NU_f(\eta) = NU_f(\eta_0)$ for a chosen fixed point $\eta_0 \in \mathcal{D}$ is denoted as

$$\mathcal{M}_f^\#(\eta_0) = \{\eta \in \mathcal{D} : NU_f(\eta) = NU_f(\eta_0)\}. \quad (1.1)$$

Define the maximum surrounding stability equivalence region of η_0 , $\mathcal{M}_f(\eta_0)$, as a set of points satisfying the following conditions:

1. $\eta \in \mathcal{M}_f^\#(\eta_0) \subset \mathcal{D}$
2. There exists a path \mathcal{P} which connects η_0 with η , such that $\mathcal{P} \subset \text{int}(\mathcal{M}_f^\#(\eta_0))$.

1.2 Assumptions

Consider a system described by a characteristic function f , as in Definition 1. The following assumptions are imposed on f .

Assumption 1 *There exists a finite set $\mathcal{Z}^f \subset \mathbb{R}$, independent of \mathcal{D} , such that $s \mapsto f(s, \eta)$ is analytic on $\overline{\mathbb{C}^+} \setminus \mathcal{Z}^f$ for all $\eta \in \mathcal{D}$.*

Assumption 2 *It holds that*

$$\lim_{\omega \rightarrow \infty} |f(j\omega, \eta)|^{-1} < \infty, \quad \forall \eta \in \mathcal{D}. \quad (1.2)$$

Assumption 2 is not required for proving the ensuing theorems. However, it is necessary to conclude on the H^∞ stability equivalence, as discussed in Section 1.1.2.

Assumption 3 *Function $\eta \mapsto f(s, \eta)$ is continuously differentiable on \mathcal{D} for every $s \in \overline{\mathbb{C}^+} \setminus \mathcal{Z}^f$.*

Assumption 4 *For all $\eta, \eta' \in \mathcal{D}$ and for some $p \geq 1$, the following implication holds*

$$\inf_{\omega \in \mathbb{R} \setminus \mathcal{Z}^f} \frac{|f(j\omega, \eta)|}{\|\nabla f(j\omega, \eta')\|_p} = 0 \implies \inf_{\omega \in \mathbb{R} \setminus \mathcal{Z}^f} |f(j\omega, \eta)| = 0. \quad (1.3)$$

Intuitively, Assumption 4 assures that if the infimum of the ratio of the magnitude of the characteristic function and its gradient for a pair of parametric points vanishes, the infimum of the function itself vanishes as well. The assumption prevents the case in which the ratio vanishes due to the gradient becoming unbounded. Because the gradient is a finite-dimensional vector, all p -norms are equivalent. Hence, if the assumption holds for one p , then it holds for all $p \geq 1$.

Before formulating the final assumption, let us define the following auxiliary values

$$R_*^f = \max\{|\omega^*| : \omega^* \in \mathcal{Z}^f\},$$

$$\rho_*^f = \frac{1}{2} \min\{|\omega_i^* - \omega_j^*| : \omega_i^*, \omega_j^* \in \mathcal{Z}^f, \omega_i^* \neq \omega_j^*\}.$$

In other words, R_*^f is the largest absolute singularity of f on the imaginary axis, while ρ_*^f is the half of the smallest distance between any pair of purely imaginary singularities. Further, given $0 < \rho < \rho_*^f$ and $R > R_*^f + \rho_*^f$, let us also introduce the following sets

$$\begin{aligned} \rho, R \mathcal{K}_1^f &= \{j\omega : |\omega| < R\} \setminus \bigcup_{\omega^* \in \mathcal{Z}^f} \{j\omega : |\omega - \omega^*| < \rho\} \\ \rho \mathcal{K}_2^f &= \bigcup_{\omega^* \in \mathcal{Z}^f} \left\{ \omega^* + \rho e^{j\varphi} : \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\} \\ R \mathcal{K}_3^f &= \left\{ R e^{j\varphi} : \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\}. \end{aligned}$$

The union of those sets, $\rho, R \mathcal{K}^f = \rho, R \mathcal{K}_1^f \cup \rho \mathcal{K}_2^f \cup R \mathcal{K}_3^f$ is a contour lying completely in the closed RHS of the complex plane. In fact, in the limit when ρ vanishes and R grows indefinitely, the interior spans the entire open RHS of the complex plane, while the boundary lies on the imaginary axis, excluding singular points of f . The contour $\rho, R \mathcal{K}^f$ is shown in Figure 1.

Given two sets $\mathcal{A} \subset \mathbb{C}$ and $\mathcal{B} \subset \mathcal{D}$, a characteristic function f , a value $1 \leq p \leq \infty$ and a parametric point $\eta \in \mathcal{B}$, define further

$${}_p B_\eta^f(\mathcal{A}, \mathcal{B}) = \inf_{s \in \mathcal{A}} \frac{|f(s, \eta)|}{\sup_{\eta' \in \mathcal{B}} \|\nabla f(s, \eta')\|_p}. \quad (1.4)$$

We are now ready to formulate the final assumption.

Assumption 5 *Function f is such that*

$$\inf_{\eta \in \mathcal{D}} \lim_{\substack{\rho \rightarrow 0 \\ R \rightarrow \infty}} {}_1 B_\eta^f(\rho \mathcal{K}_2^f \cup R \mathcal{K}_3^f, \mathcal{D}) = \zeta^f,$$

where $\zeta^f \in \mathbb{R}^+ \cup \{\infty\}$ and $\zeta^f > 0$.

The assumption ensures that, on $\rho \mathcal{K}_2^f$ and $R \mathcal{K}_3^f$, small perturbation of η results in a sufficiently small change to f .

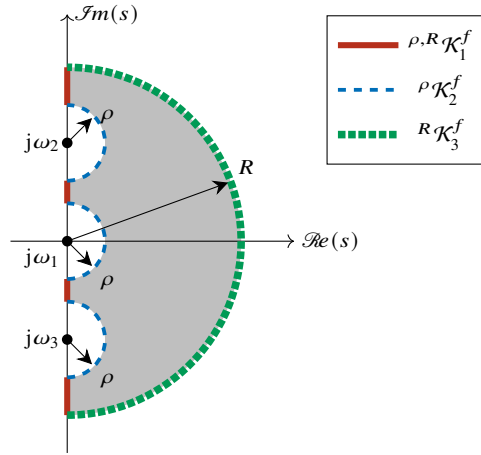


Fig. 1: Graphical representation of contour $\rho, R \mathcal{K}^f$

1.3 Paper organization

The paper is organized as follows. Section 2 presents a method for finding the maximal stability equivalence segment along a curve in the space of parameters, starting from a given parametric point. Section 3 presents a method for finding the maximal stability equivalence region around that point. Section 4 shows the results of applying both methodologies on illustrative examples. Section 5 summarizes the paper with several closing comments.

2 Stability along a curve

Consider a continuously differentiable curve

$$\eta : \Omega \rightarrow \mathcal{D}, \quad \Omega = [\theta_{\min}, \theta_{\max}] \quad (2.1)$$

with $-\infty < \theta_{\min} < \theta_{\max} \leq +\infty$, where

$$\left| \frac{\partial \eta_i}{\partial \theta} \right| \leq 1 \quad (2.2)$$

for each vector component η_i , $i = 1, \dots, n$. Within this section, consider only parametric points belonging to this curve; the characteristic function is abbreviated for convenience as

$$f(s, \eta(\theta)) \equiv f(s, \theta). \quad (2.3)$$

Choose a $\theta_0 \in \Omega$, corresponding to a starting parametric point $\eta(\theta_0)$. Given an arbitrary other value $\theta' \in \Omega$, $\theta' > \theta_0$, we say that θ_0 and θ' are stability equivalent along η if and only if

$$NU_f(\eta(\theta_0)) = NU_f(\eta(\theta)), \quad \forall \theta \in [\theta_0, \theta'] . \quad (2.4)$$

Introducing $\mathbb{T}(\theta_0) \subset \Omega$ as the set of all values which are stability equivalent to θ_0 , we may finally define the stability-limiting value as

$$\theta_{lim} = \sup \mathbb{T}(\theta_0) . \quad (2.5)$$

2.1 Sufficient stability equivalence conditions

Theorem 1 *Let f be a characteristic function (see Definition 1) of an LTI system, satisfying Assumptions 1-5. Consider an arbitrary continuously differentiable curve within the parametric space (2.1), and select an arbitrary $\theta_0 \in \underline{\Omega}$ such that $f(j\omega, \theta_0) \neq 0, \forall \omega \in \mathbb{R} \setminus \mathcal{Z}^f$. Then, $NU_f(\theta_0 + \Delta) = NU_f(\theta_0)$ if $0 < \Delta < \bar{\Delta}(\theta_0)$, with*

$$\bar{\Delta}(\theta_0) = \min \left\{ {}_1B_{\eta(\theta_0)}^f(\mathcal{A}, \mathcal{B}), \zeta^f \right\} \quad (2.6)$$

where $\mathcal{A} = \{j\omega : \omega \in \mathbb{R} \setminus \mathcal{Z}^f\}$ and $\mathcal{B} = \{\eta(\theta) : \theta_0 \leq \theta \leq \theta_0 + \bar{\Delta}(\theta_0)\}$.

Proof Assume all the premises of the Theorem true, and select an arbitrary $\Delta < \bar{\Delta}(\theta_0)$. By fundamental properties of the integral operator, for an arbitrary $s \in \mathbb{C}^+ \setminus j\mathcal{Z}^f$,

$$\left| \int_{\theta_0}^{\theta_0 + \Delta} \frac{\partial f}{\partial \theta}(s, \theta) d\theta \right| \leq \Delta \max_{\theta_0 \leq \theta \leq \theta_0 + \Delta} \left| \frac{\partial f}{\partial \theta}(s, \theta) \right| . \quad (2.7)$$

Due to (1.4) and (2.6), for arbitrary $s = j\omega, \omega \in \mathbb{R} \setminus \mathcal{Z}^f$ and arbitrary $\theta_0 \leq \theta \leq \theta_0 + \bar{\Delta}(\theta_0)$

$$\bar{\Delta}(\theta_0) \leq \min \left\{ \frac{|f(s, \theta_0)|}{\left| \frac{\partial f}{\partial \theta}(s, \theta) \right|}, \zeta^f \right\} . \quad (2.8)$$

However, due to Assumption 5, one can always find $\rho_{\max} > 0$ and $R_{\min} > 0$ such that the last inequality holds also on ${}^\rho\mathcal{K}_2^f \cup {}^R\mathcal{K}_3^f$ for all $\rho < \rho_{\max}$ and all $R > R_{\min}$. Combining the last two inequalities yields

$$\left| \int_{\theta_0}^{\theta_0 + \Delta} \frac{\partial f}{\partial \theta}(s, \theta) d\theta \right| < |f(s, \theta_0)| , \quad (2.9)$$

for all $s \in {}^{\rho, R}\mathcal{K}^f$. Furthermore, due to Assumption 3 the Fundamental Theorem of Calculus can be applied, so it may be concluded that

$$|f(s, \theta_0 + \Delta) - f(s, \theta_0)| < |f(s, \theta_0)|, \quad \forall s \in {}^{\rho, R}\mathcal{K}^f . \quad (2.10)$$

Finally, Assumption 1 allows applying the Rouché's theorem, guaranteeing that $f(s, \theta_0)$ and $f(s, \theta_0 + \Delta)$ have the same number of zeros in ${}^{\rho, R}\mathcal{K}^f$.

The proof is completed by noting that ρ can be chosen arbitrary small and R arbitrary large, so that in the limit ${}^{\rho, R}\mathcal{K}^f$ covers the entire closed right-half complex plane, apart from the singular points on the imaginary axis. However, the characteristic function cannot be zero in those points by the assumptions of the Theorem, which concludes the proof. \square

Remark 1 Note that the RHS of (2.6) is non-increasing w.r.t. $\bar{\Delta}(\theta_0)$. Thus, the circular equation (2.6) may be efficiently solved by bisection.

2.2 Stability limit

It is possible to extend the stability equivalence interval around some given θ_0 by applying (2.6) iteratively, so that

$$\theta_{k+1} = \theta_k + \Delta_k, \quad \forall k \in \mathbb{N}_0, \quad (2.11)$$

with $\Delta_k \leq \bar{\Delta}(\theta_k)$. It will now be proven that by applying $\lim_{k \rightarrow \infty}$ to (2.11), the proposed iterative sequence will converge to the stability boundary along the curve $\eta(\theta)$, assuming that such a boundary indeed exists. In other words, a point $\theta > \theta_0$ is stability equivalent to θ_0 along $\eta(\theta)$ if and only if $\theta_{lim} > \theta$. The aforementioned claim is formalized and proven in the following theorem.

Theorem 2 *Let all the hypotheses of Theorem 1 hold, let θ_{lim} be defined in (2.5) and assume that θ_{lim} exists. Consider a sequence θ_k obtained by (2.11), with increments $\Delta_k = \xi \bar{\Delta}(\theta_k)$ for some $\xi \in (0, 1)$. Then, θ_k is non-decreasing and*

$$\lim_{k \rightarrow \infty} \theta_k = \theta_{lim}.$$

Proof It is easy to see that the sequence θ_k is monotonous by construction. By definition of θ_{lim} (see (2.5)), it is impossible that $\theta_k > \theta_{lim}$ for some k , and therefore

$$\lim_{k \rightarrow \infty} \theta_k \leq \theta_{lim}.$$

It will now be proven that this inequality reduces to an equality.

Assume that θ_k converges to some finite $\theta_{\#} < \theta_{lim}$. Since the sequence is converging, the increments Δ_k must get arbitrarily small as $k \rightarrow \infty$. This, combined with (2.6) and Assumption 4 implies that the value of

$$\inf_{\omega \in \mathbb{R} \setminus \mathcal{Z}_f} |f(j\omega, \theta_k)| \quad (2.12)$$

becomes arbitrarily small as well. This contradicts the definition of θ_{lim} in (2.5). Therefore, θ_k converges to θ_{lim} . \square

Theorem 3 *Let all the hypotheses of Theorem 1 hold, let θ_{lim} be defined in (2.5) and assume that θ_{lim} does not exist. Consider a sequence θ_k obtained by (2.11), with increments $\Delta_k = \xi \bar{\Delta}(\theta_k)$ for some $\xi \in (0, 1)$. Then, θ_k is non-decreasing and divergent.*

Proof It is easy to see that the sequence θ_k is monotonous by construction. The remainder of the theorem shall be proven by contradiction.

Assume that θ_k is convergent. Combined with Assumption 4, this implies that

$$\lim_{k \rightarrow \infty} \inf_{\omega \in \mathbb{R} \setminus \mathcal{Z}_f} |f(j\omega, \theta_k)| = 0. \quad (2.13)$$

This further implies that θ_{lim} exists, contradicting the theorem assumption. \square

The complete procedure for determining the stability limit along a curve is shown as Algorithm 1. The parameter δ is introduced as a termination criterion. Different termination criteria could be used as well (for example, imposing an upper limit on k).

Algorithm 1 Asymptotic computation of θ_{lim}

Require: $\delta > 0, 0 < \theta_{min} < \theta_{max}, \theta_0 \in [\theta_{min}, \theta_{max}), \xi \in (0, 1), \mathcal{A} = \{j\omega : \omega \in \mathbb{R} \setminus \mathcal{Z}^f\}$
 $\theta_k := \theta_0$
 $\overline{\Delta}_k := \infty$
while $\xi \overline{\Delta}_k > \delta$ and $\theta_k < \theta_{max}$ **do**
 $\overline{\Delta}_k := \Delta$, where Δ satisfies $\Delta = \min \left\{ {}_1B_{\eta(\theta_k)}^f(\mathcal{A}, \mathcal{B}), \zeta^f \right\}$, for $\mathcal{B} = \{\eta(\theta) : \theta_k \leq \theta \leq \theta_k + \Delta\}$
 $\theta_k := \theta_k + \xi \overline{\Delta}_k$
 $k := k + 1$
end while
 $\theta_{lim} := \theta_k$

3 Stability inside a region

This section builds towards determining $\mathcal{M}_f(\eta_0)$ for a given starting parametric point η_0 . For convenience, denote

$${}_q\mathcal{W}_\varepsilon^\xi(\eta) = \{\eta + \mathbf{v} : \|\mathbf{v}\|_q \leq \xi\varepsilon\}. \quad (3.1)$$

3.1 Sufficient stability equivalence conditions

Theorem 4 Let f be as in Definition 1, satisfying Assumptions 1-5. Let $\eta_0 \in \mathcal{D}$ be any parameter point satisfying $f(j\omega, \eta_0) \neq 0, \forall \omega \in \mathbb{R} \setminus \mathcal{Z}^f$. Let p and q be real numbers satisfying

$$\frac{1}{p} + \frac{1}{q} = 1, \quad 1 \leq p, q \leq \infty. \quad (3.2)$$

Then,

$$NU_f(\eta_0) = NU_f(\eta), \quad \forall \eta \in {}_q\mathcal{W}_{\varepsilon_{p,q}^1(\eta_0)}^1(\eta_0). \quad (3.3)$$

holds if

$$\varepsilon_{p,q}(\eta_0) < \min \left\{ \zeta^f, {}_pB_{\eta_0}^f \left(\mathcal{A}, {}_q\mathcal{W}_{\varepsilon_{p,q}^1(\eta_0)}^1(\eta_0) \right) \right\}, \quad (3.4)$$

where $\mathcal{A} = \{j\omega : \omega \in \mathbb{R} \setminus \mathcal{Z}^f\}$.

Proof Due to (3.4) and Assumption 5, one can find ρ_{max} and R_{min} such that

$$\varepsilon_{p,q}(\eta_0) < \frac{|f(s, \eta_0)|}{\sup_{\eta \in {}_q\mathcal{W}_{\varepsilon_{p,q}^1(\eta_0)}^1(\eta_0)} \|\nabla f(s, \eta)\|_p}, \quad \forall s \in {}^{\rho,R}\mathcal{K}^f, \quad (3.5)$$

for all $\rho < \rho_{min}$ and $R > R_{max}$. By rearranging (3.5), using the fundamental properties of the integral operator and applying the Hölder's inequality, one readily obtains that

$$\int_0^1 |\langle \nabla f(s, \eta_0 + \beta \mathbf{v}), \mathbf{v} \rangle| d\beta < |f(s, \eta_0)| \quad (3.6)$$

$\forall \mathbf{v} \in {}_q\mathcal{W}_{\varepsilon_{p,q}^1(\eta_0)}^1(\mathbf{0})$ and $\forall s \in {}^{\rho,R}\mathcal{K}^f$. Applying the Fundamental Theorem of Calculus and Assumption 3 to (3.6) yields

$$|f(s, \eta) - f(s, \eta_0)| < |f(s, \eta_0)| \quad (3.7)$$

for all $\forall \eta \in {}_q\mathcal{W}_{\varepsilon_{p,q}^1}^1(\eta_0)$ and $\forall s \in {}^{\rho,R}\mathcal{K}^f$. Assumption 1 allows applying Rouché's theorem to (3.7), guaranteeing that $f(s, \eta_0)$ and $f(s, \eta)$, $\forall \eta \in {}_q\mathcal{W}_{\varepsilon_{p,q}^1}^1(\eta_0)$ have the same number of poles inside ${}^{\rho,R}\mathcal{K}^f$.

The claim of the theorem is proven by letting $\rho \rightarrow 0$ and $R \rightarrow \infty$. \square

3.2 Stability limit

Analogously to the line-based version of the method, the maximal region surrounding η_0 is established in which the number of unstable zeroes of f is invariant.

Choose a starting point η_0 and define a set \mathcal{S}_0 as

$$\mathcal{S}_0 = \{\eta_0\}. \quad (3.8)$$

Choose $\xi \in (0, 1)$ and construct a monotonously growing sequence of sets

$$\mathcal{S}_{k+1} = \mathcal{S}_k \cup \bigcup_{\eta \in \partial \mathcal{S}_k} {}_q\mathcal{W}_{\varepsilon_{p,q}^\xi}^\xi(\eta), \quad \forall k \in \mathbb{N}_0, \quad (3.9)$$

where $\varepsilon_{p,q}(\eta)$ satisfies (3.4). It is now established in the following theorem that \mathcal{S}_k converges to $\mathcal{M}_f(\eta_0)$.

Theorem 5 *Let f be as in Definition 1, satisfying Assumptions 1-5. Let p and q be real numbers satisfying (3.2). Let $\eta_0 \in \mathcal{D}$ be any parameter point satisfying $f(j\omega, \eta_0) \neq 0, \forall \omega \in \mathbb{R} \setminus \mathcal{Z}^f$. Define \mathcal{S}_0 as in (3.8), \mathcal{S}_k as in (3.9), and $\varepsilon_{p,q}(\eta)$ as in (3.4). Then,*

$$\limsup_{k \rightarrow \infty} \mathcal{S}_k = \mathcal{M}_f(\eta_0). \quad (3.10)$$

Proof The claim trivially holds true if

$$\forall \eta_* \in \mathcal{M}_f(\eta_0), \quad \exists k(\eta_*) \in \mathbb{N} \text{ such that } \eta_* \in \mathcal{S}_{k(\eta_*)}. \quad (3.11)$$

The proof will therefore focus on (3.11).

For each $\eta_* \in \mathcal{M}_f(\eta_0)$, $\exists \mathcal{P}$ defined by $g : [0, 1] \rightarrow \mathcal{P} \subset \text{int}(\mathcal{M}_f(\eta_0))$ such that $g(0) = \eta_0$, $g(1) = \eta_*$ and g is continuous. Define the sequence

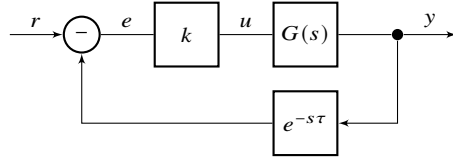
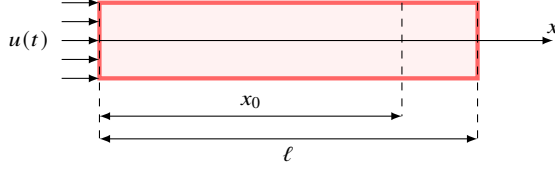
$$m_k = \max \{x : g(x) \in \mathcal{S}_k\}. \quad (3.12)$$

and notice that claim (3.11) holds if

$$\exists k(\eta_*) \text{ such that } m_{k(\eta_*)} = 1. \quad (3.13)$$

At each iteration k for which $m_k \neq 1$, (3.4) is evaluated on $\eta_k = g(m_k)$. Since for each such η_k the value of $|f(j\omega, \eta_k)|$ is positive $\forall \omega \in \mathbb{R} \setminus \mathcal{Z}^f$, then $m_{k+1} > m_k$.

The parametric point $g(m_k)$ belongs to $\text{int}(\mathcal{M}_f(\eta_0))$, $\forall k \in \mathbb{N}$. Combined with Assumption 4, this implies $\exists \alpha > 0$ such that $|f(j\omega, g(m_k))| > \alpha, \forall m_k \in [0, 1], \forall \omega \in \mathbb{R} \setminus \mathcal{Z}^f$. Hence, the sequence m_k is strictly increasing and converging to 1 (it does not converge to any other value in the interval $[0, 1]$). Therefore, (3.13) holds which concludes the proof. \square

Fig. 2: A feedback control system for the plant $G(s)$ Fig. 3: Heating a thin rod of length $\ell > 0$.

4 Examples

Consider the control system shown in Figure 2, with the plant transfer function $G(s)$, a proportional controller k and a delayed sensor $e^{-s\tau}$. The closed-loop transfer function is given by

$$F(s) = \frac{Y(s)}{R(s)} = \frac{kG(s)}{1 + ke^{-s\tau}G(s)}. \quad (4.1)$$

Stability of $F(s)$ is analyzed for three different plants in the remainder of the section.

4.1 Heat diffusion in a thin rod

Consider a one-dimensional heat diffusion in a thin rod, as shown in Figure 3. The rod is thermally isolated, except at its boundary cross-sections. At the left end $x = 0$, the rod is subjected to an input thermal flux $u(t, 0) = -\lambda \frac{\partial \theta}{\partial x}(t, 0)$, where λ is the thermal conductivity. The temperature of the rod is measured at the cross section $0 < x_0 < \ell$.

The corresponding model describing the spatio-temporal distribution of the temperature along the rod is given by the heat equation

$$\frac{\partial^2 \theta(t, x)}{\partial x^2} = \sigma^{-1} \frac{\partial \theta(t, x)}{\partial t}, \quad (4.2)$$

where σ denotes the thermal diffusivity of the rod's material. The resulting plant transfer function is given by

$$G(s) = \frac{\sinh((\ell - x)\sqrt{\frac{s}{\sigma}})}{\lambda\sqrt{\frac{s}{\sigma}} \cosh(\ell\sqrt{\frac{s}{\sigma}})}, \quad (4.3)$$

yielding the closed-loop transfer function

$$F(s) = \frac{k \sinh\left((\ell - x)\sqrt{\frac{s}{\sigma}}\right)}{\lambda\sqrt{\frac{s}{\sigma}} \cosh\left(\ell\sqrt{\frac{s}{\sigma}}\right) + ke^{-s\tau} \sinh\left((\ell - x)\sqrt{\frac{s}{\sigma}}\right)}. \quad (4.4)$$

Theorems 2 and 5 are hence applied to the characteristic function

$$f(s, \tau, k) = \lambda\sqrt{\frac{s}{\sigma}} \cosh\left(\ell\sqrt{\frac{s}{\sigma}}\right) + ke^{-s\tau} \sinh\left((\ell - x)\sqrt{\frac{s}{\sigma}}\right). \quad (4.5)$$

Due to the multivalued function \sqrt{s} , a branch-cut is chosen along the negative real axis from 0 to $-\infty$. Hence, $f(s, \tau, k)$ is holomorphic in $\mathbb{C} \setminus \mathbb{R}^-$, and the following restriction is imposed on the arguments of s

$$|\arg(s)| < \pi. \quad (4.6)$$

Therefore, (4.5) fulfills Assumptions 1-5, with $\mathcal{Z}^f = \{0\}$ and $\zeta^f = 1$.

The rod is considered to be of aluminum, with $\sigma = 98.8 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$ and $\lambda = 237 \text{ W m}^{-1} \text{ K}^{-1}$. Its length is set to $\ell = 0.2 \text{ m}$ and the temperature is measured at $x_0 = 0.15 \text{ m}$.

The curve algorithm, applied to straight lines, and the region algorithm are initialized at the parametric points indicated by crosses on Figures 4a and 4b and yield the results shown therein. They confirm the results of [23, section 4]. The starting points are chosen in a way that illustrates effectiveness of both curve and region methodologies. Since (4.5) satisfies Assumption 2, the obtained regions are H^∞ stability equivalent.

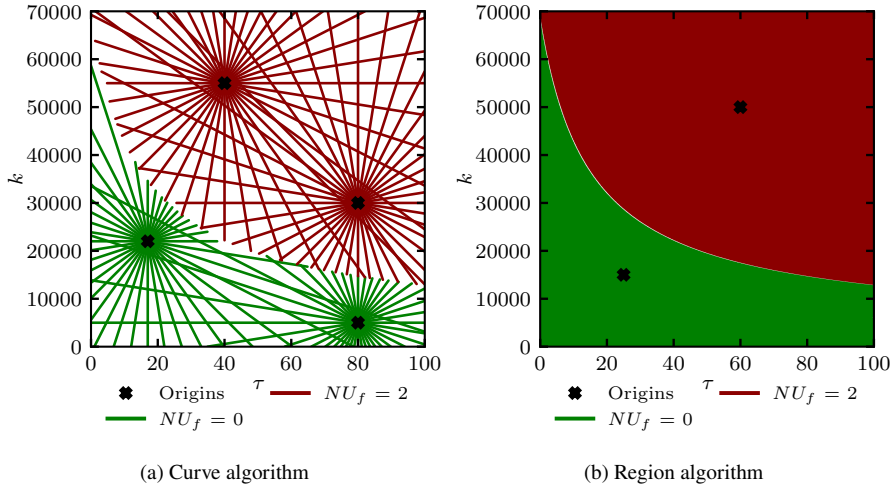


Fig. 4: Stability analysis of a heated rod described from 4.1

4.2 Voltage propagation in paired transmission lines

Consider a pair of semi-infinite transmission lines, modelled by the telegrapher's equations. In this case, we denote the transmission line length as $\ell = \infty$. The voltage $v(x, t)$ in spatio-temporal domain is governed by

$$\frac{\partial^2}{\partial x^2} v(x, t) = \left[LC \frac{\partial^2}{\partial t^2} + (RC + GL) \frac{\partial}{\partial t} + GR \right] v(x, t), \quad (4.7)$$

where L is the inductance, C the capacitance, R the parallel resistance and G the serial conductance, all specified per unit of length. At the finite end, input voltage is applied as $v(0, t) = u(t)$ which represents the first boundary condition, whereas the second boundary condition is given by $v(\ell = \infty, t) = 0$. The voltage is measured and controlled at distance x_0 , yielding plant output $y(t) = v(x_0, t)$.

By letting $\ell \rightarrow \infty$, one readily obtains [34] the plant transfer function

$$G(s) = \frac{Y(s)}{U(s)} = e^{-x_0 \sqrt{(Cs+G)(Ls+R)}}, \quad (4.8)$$

and the closed-loop transfer function

$$F(s) = \frac{k e^{-x_0 \sqrt{(Cs+G)(Ls+R)}}}{1 + k e^{-s\tau} e^{-x_0 \sqrt{(Cs+G)(Ls+R)}}}. \quad (4.9)$$

The presented methodology is applied to the corresponding characteristic function

$$f(s, \tau, k) = 1 + k e^{-s\tau} e^{-x_0 \sqrt{(Cs+G)(Ls+R)}}, \quad (4.10)$$

investigating system stability with respect to $\tau > 0$ and $k > 0$.

Since the function $\sqrt{(Cs+G)(Ls+R)}$ is multivalued, a branch cut is chosen as a segment between the branch points $-\frac{G}{C}$ and $-\frac{R}{L}$. However, the polynomial $(Cs+G)(Ls+R)$ is Hurwitz for positive values of C , G , R and L , meaning that the branch cut belongs strictly to the LHS of the complex plane, not influencing system stability. Hence, (4.10) readily fulfills Assumptions 1-5, with $\mathcal{Z}^f = \emptyset$ and $\zeta^f = \infty$.

The primary line constants, taken from [37], are set to $R = 17.224 \text{ m}\Omega \text{ m}^{-1}$, $C = 51.57 \text{ pF m}^{-1}$, $L = 612.9 \text{ nH m}^{-1}$, $G = 1 \text{ n}\Omega^{-1} \text{ m}^{-1}$, and the output voltage is evaluated at $x_0 = 20 \text{ km}$. The curve algorithm, applied to straight lines, and the region algorithm are initialized at the parametric points indicated by crosses in Figures 5a and 5b and yield the results shown therein. Since (4.10) satisfies Assumption 2, the obtained regions are H^∞ stability equivalent.

4.3 Generalized time-fractional transmission line

Consider a pair of semi-infinite transmission lines, modelled by the generalized time-fractional telegrapher's equation

$$\left(mLC \frac{\partial^{\alpha+\beta+\gamma}}{\partial t^{\alpha+\beta+\gamma}} + mLG \frac{\partial^{\alpha+\beta}}{\partial t^{\alpha+\beta}} + LC \frac{\partial^{\alpha+\gamma}}{\partial t^{\alpha+\gamma}} + LG \frac{\partial^\alpha}{\partial t^\alpha} + RC \frac{\partial^\gamma}{\partial t^\gamma} + RG \right) v(x, t) =$$

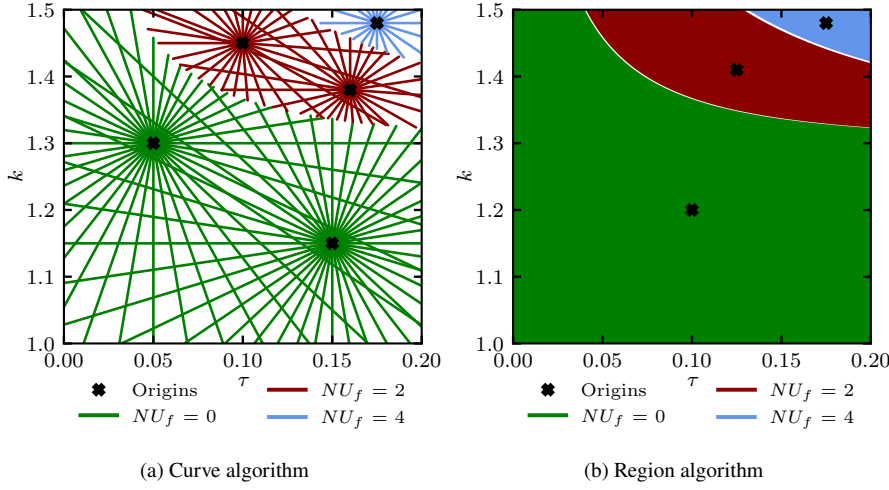


Fig. 5: Stability analysis of transmission lines from Section 4.2

$$= \left(m \frac{\partial^\beta}{\partial t^\beta} + 1 \right) \frac{\partial^2}{\partial x^2} v(x, t), \quad (4.11)$$

presenting a generalization of the classical telegrapher's equation where both capacitive and inductive phenomena are considered to be of fractional order and the phenomena of charge accumulation along the line are taken into account. In (4.11), symbols R , L , G and C denote the line serial resistance and inductance, and parallel conductance and capacitance, respectively, per unit of length. The voltage between line conductors at position x and time t is denoted $u(x, t)$, while m represents a positive constant related to charge accumulation effects. Values α , β and γ represent the fractional orders of the capacitive and diffusive phenomena.

By assuming $C = G$ and taking $a = \frac{1}{m}$ and $b = \frac{R}{mL}$, the corresponding transfer function can be obtained

$$G(s) = e^{-x\sqrt{\psi(s)}}, \quad \psi(s) = \frac{(s^{\alpha+\beta} + as^\alpha + b)(s^\gamma + 1)}{s^\beta + a}. \quad (4.12)$$

Assuming no transmission delay ($\tau = 0$), the closed-loop transfer function of the system is given by

$$F(s) = \frac{ke^{-x\sqrt{\psi(s)}}}{1 + ke^{-x\sqrt{\psi(s)}}} \quad (4.13)$$

with characteristic function

$$f(s) = 1 + ke^{-x\sqrt{\psi(s)}} \quad (4.14)$$

The curve and region algorithms are both applied in two parametric spaces:

- The parametric space $[x, k] \in \mathcal{D}_1 = [0, 4] \times [4, 8]$, while taking $\alpha = \frac{5}{6}$ and $\gamma = \frac{2}{3}$.
- The parametric space $[\alpha, \gamma] \in \mathcal{D}_2 = [0.2, 0.9]^2$, while taking $x = 0.8$ and $k = 30$.

The remaining parameters are chosen as $\beta = \frac{2}{3}$, $a = 4.16$ and $b = 3.5$. In both parametric spaces, (4.14) fulfills Assumptions 1-5 with $\mathcal{Z}^f = \{0\}$ and $\zeta^f = \infty$. Due to the presence of a branch point at the origin, a branch cut is introduced along the real axis. A rigorous proof that (4.14) contains no additional branch points in \mathbb{C}^+ , nor singularities on the imaginary axis can be found in [12].

The curve algorithm, applied to straight lines, and the region algorithm are initialized at the parametric points indicated by crosses on Figures 6a, 6b, 6c and 6d, and yield the results shown therein. Since (4.14) satisfies Assumption 2, the obtained regions are H^∞ stability equivalent.

Figure 7 shows the step-by-step iterative process of obtaining one such stability equivalent region, starting from parametric point $(\alpha, \gamma) = (0.7, 0.7)$. Each subfigure shows the stability equivalent set \mathcal{S}_k for a particular algorithm iteration k .

5 Summary

A methodology for stability analysis of irrational transfer functions satisfying assumptions of Section 1.2 has been presented. The required assumptions are mild, resulting in a vast number of applicable systems, some of which are illustrated in Section 4. Although the paper focuses on irrational transfer functions, the methodology is applicable to rational transfer functions as well. The complexity of the curve algorithm is independent of the number of parameters. The presented methodology has previously been applied on time-delay systems [42] and fractional order systems [35], and is further generalized in this paper to a broader class of LTI systems with irrational transfer functions.

Two general algorithms have been presented, analyzing stability along a curve or inside a region. These general algorithms can be adapted and combined to specific needs of the problem being solved. For example, the stability can be analyzed along an operational curve of a plant, while applying the region algorithm along the operational curve to analyze robustness in presence of small parameter perturbations.

Conflict of interest

The authors declare that they have no conflict of interest.

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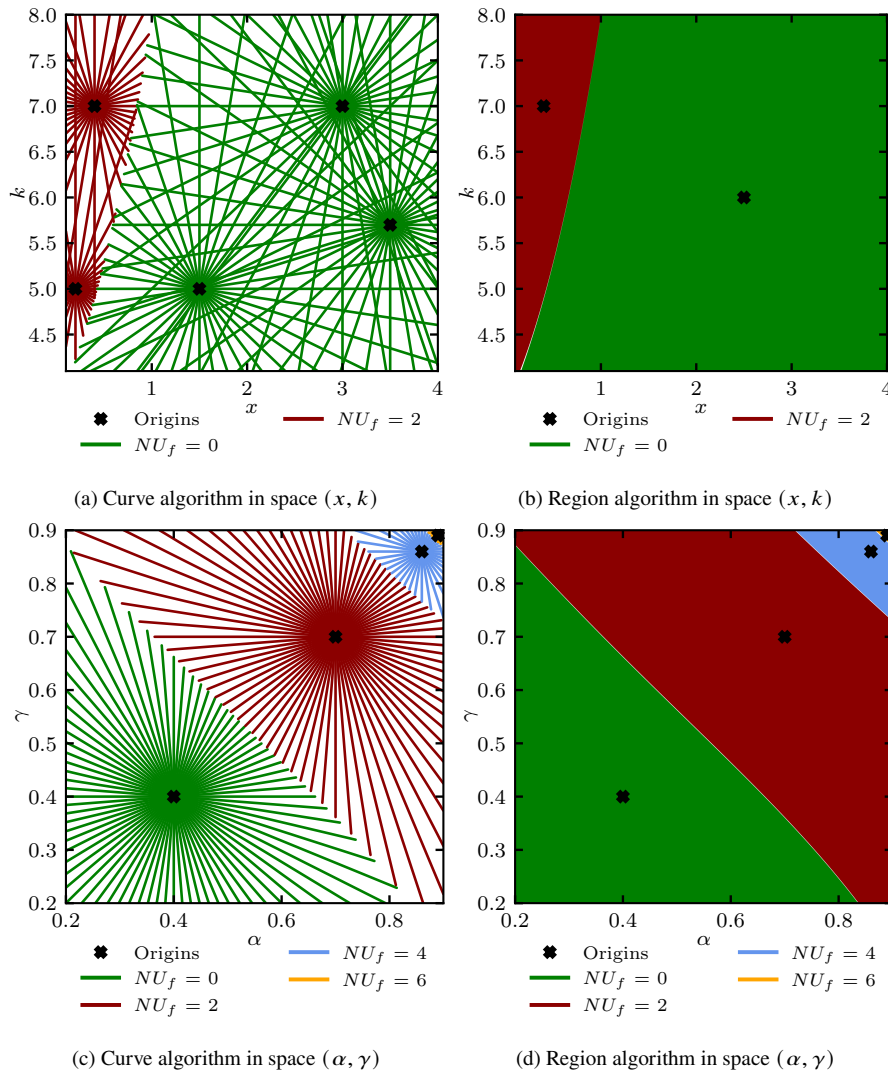


Fig. 6: Stability analysis of the generalized telegrapher's equation from Section 4.3

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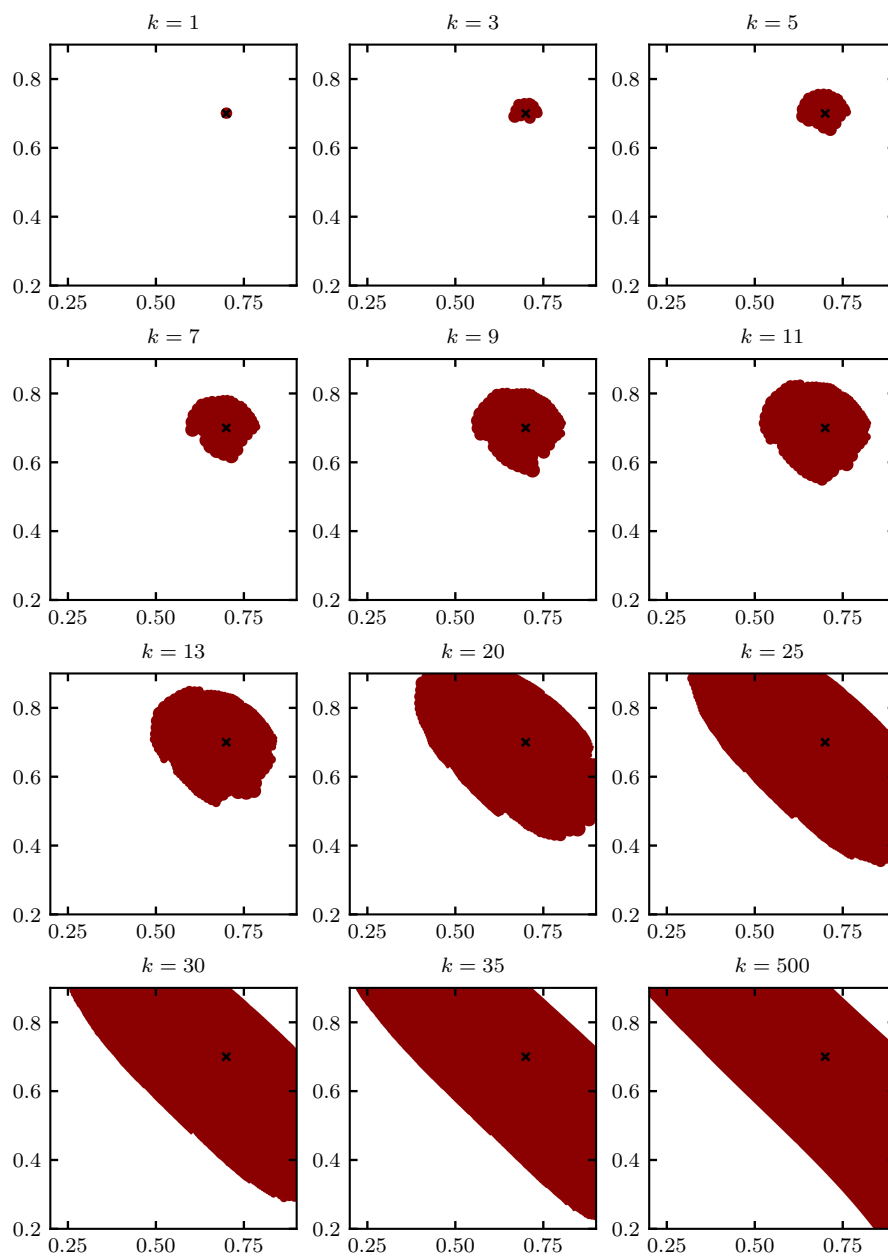


Fig. 7: Iterative application of Theorem 4 to (4.14)

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