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Doubled patterns with reversal and square-free doubled patterns

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Abstract

In combinatorics on words, a word w over an alphabet Σ is said to avoid a pattern p over an alphabet Δ if there is no factor f of w such that $f = h(p)$ where $h : \Delta^* \rightarrow \Sigma^*$ is a non-erasing morphism. A pattern p is said to be k -avoidable if there exists an infinite word over a k -letter alphabet that avoids p . A pattern is *doubled* if every variable occurs at least twice. Doubled patterns are known to be 3-avoidable. Currie, Mol, and Rampersad have considered a generalized notion which allows variable occurrences to be reversed. That is, $h(V^R)$ is the mirror image of $h(V)$ for every $V \in \Delta$. We show that doubled patterns with reversal are 3-avoidable. We also conjecture that (classical) doubled patterns that do not contain a square are 2-avoidable. We confirm this conjecture for patterns with at most 4 variables. This implies that for every doubled pattern p , the growth rate of ternary words avoiding p is at least the growth rate of ternary square-free words. A previous version of this paper containing only the first result has been presented at WORDS 2021.

1 Introduction

The *mirror image* of the word $w = w_1 w_2 \dots w_n$ is the word $w^R = w_n w_{n-1} \dots w_1$. A pattern with reversal p is a non-empty word over an alphabet $\Delta = \{A, A^R, B, B^R, C, C^R, \dots\}$ such that $\{A, B, C, \dots\}$ are the *variables* of p . An *occurrence* of p in a word w is a non-erasing morphism $h : \Delta^* \rightarrow \Sigma^*$ satisfying $h(X^R) = (h(X))^R$ for every variable X and such that $h(p)$ is a factor of w . The avoidability index $\lambda(p)$ of a pattern with reversal p is the size of the smallest alphabet Σ such that there exists an infinite word w over Σ containing no occurrence of p . A pattern p such that $\lambda(p) \leq k$ is said to be k -avoidable. To emphasize that a pattern is without reversal (i.e., it contains no X^R), it is said to be *classical*. A pattern is *doubled* if every variable occurs at least twice.

Our aim is to strengthen the following result.

Theorem 1. [1, 7, 8] *Every doubled pattern is 3-avoidable.*

First, we extend it to patterns with reversal.

Theorem 2. *Every doubled pattern with reversal is 3-avoidable.*

Then, we notice that all the known classical doubled patterns that are 2-unavoidable contain a square, such as $AABB$, $ABAB$, or $ABCCBADD$.

Conjecture 3. Every square-free doubled pattern is 2-avoidable.

Notice that Conjecture 3 is related to but independent of the following conjecture.

Conjecture 4. [8, 10] There exist only finitely many 2-unavoidable doubled patterns.

The proof of Conjecture 3 for patterns up to 3 variables follows from the 2-avoidability of $ABACBC$, $ABCBABC$, $ABCACB$ and $ABCBAC$. We were able to verify it for patterns up to 4 variables.

Theorem 5. *Every square-free doubled pattern with at most 4 variables is 2-avoidable.*

Finally, we obtain a lower bound on the number of ternary words avoiding a doubled pattern. The factor complexity of a factorial language L over Σ is $f(n) = |L \cap \Sigma^n|$. The growth rate of L over Σ is $\lim_{n \rightarrow \infty} f(n)^{\frac{1}{n}}$. We denote by $GR_3(p)$ the growth rate of ternary words avoiding the doubled pattern p .

Theorem 6. *For every doubled pattern p , $GR_3(p) \geq GR_3(AA)$.*

Let $v(p)$ be the number of distinct variables of the pattern p . In the proof of Theorem 1, the set of doubled patterns is partitioned as follows:

1. Patterns with $v(p) \leq 3$: the avoidability index of every ternary pattern has been determined [7].
2. Patterns shown to be 3-avoidable with the so-called power series method:
 - Patterns with $v(p) \geq 6$ [1]
 - Patterns with $v(p) = 5$ and prefix ABC or length at least 11 [8]
 - Patterns with $v(p) = 4$ and prefix $ABCD$ or length at least 9 [8]
3. Ten sporadic patterns with $4 \leq v(p) \leq 5$ whose 3-avoidability cannot be deduced from the previous results: they have been shown to be 2-avoidable [8] using the method in [7].

The proof of Theorems 2 and 6 use the same partition. Sections 3 to 5 are each devoted to one type of doubled pattern with reversal. Theorem 5 is proved in Section 6 Theorem 6 is proved in Section 7

2 Preliminaries

A word w is *d-directed* if for every factor f of w of length d , the word f^R is not a factor of w .

Remark 7. If a d -directed word contains an occurrence h of XX^R for some variable X , then $|h(X)| \leq d - 1$.

A variable that appears only once in a pattern is said to be *isolated*. The *formula* f associated to a pattern p is obtained by replacing every isolated variable in p by a dot. The factors between the dots are called *fragments*. An occurrence of a formula f in a word w is a non-erasing morphism h such that the h -image of every fragment of f is a factor of w . As for patterns, the avoidability index $\lambda(f)$ of a formula f is the size of the smallest alphabet allowing the existence of an infinite word containing no occurrence of f . Recently, the avoidability of formulas with reversal has been considered by Currie, Mol, and Rampersad [4, 5] and Ochem [9].

Recall that a formula is *nice* if every variable occurs at least twice in the same fragment. In particular, a doubled pattern is a nice formula with exactly one fragment.

The *avoidability exponent* $AE(f)$ of a formula f is the largest real x such that every x -free word avoids f . Every nice formula f with $v(f) \geq 3$ variables is such that $AE(f) \geq 1 + \frac{1}{2v(f)-3}$ [12].

Let \simeq be the equivalence relation on words defined by $w \simeq w'$ if $w' \in \{w, w^R\}$. Avoiding a pattern up to \simeq has been investigated for every binary formulas [3]. Remark that for a given classical pattern or formula p , avoiding p up to \simeq implies avoiding simultaneously all the variants of p with reversal.

Recall that a word is (β^+, n) -free if it contains no repetition with exponent strictly greater than β and period at least n .

3 Formulas with at most 3 variables

For classical doubled patterns with at most 3 variables, all the avoidability indices are known. There are many such patterns, so it would be tedious to consider all their variants with reversal.

However, we are only interested in their 3-avoidability, which follows from the 3-avoidability of nice formulas with at most 3 variables [11].

Thus, to obtain the 3-avoidability of doubled patterns with reversal with at most 3 variables, we show that every minimally nice formula with at most 3 variables is 3-avoidable up to \simeq .

The minimally nice formulas with at most 3 variables, up to symmetries, are determined in [11] and listed in the following table. Every such formula f is avoided by the image by a q -uniform morphism of either any infinite $\left(\frac{5^+}{4}\right)$ -free word w_5 over Σ_5 or any infinite $\left(\frac{7^+}{5}\right)$ -free word w_4 over Σ_4 , depending on whether the avoidability exponent of f is smaller than $\frac{7}{5}$.

| Formula f | $= f^R$ | $AE(f)$ | Word | q | d | freeness |
|------------------|------------------|-------------|--------------|-----|-----|-------------------------------------|
| $ABA.BAB$ | yes | 1.5 | $g_a(w_4)$ | 9 | 9 | $\left(\frac{131}{90}^+, 28\right)$ |
| $ABCA.BCAB.CABC$ | yes | 1.333333333 | $g_b(w_5)$ | 6 | 8 | $\left(\frac{4}{3}^+, 25\right)$ |
| $ABCBA.CBABC$ | yes | 1.333333333 | $g_c(w_5)$ | 4 | 9 | $\left(\frac{30}{23}^+, 18\right)$ |
| $ABCA.BCAB.CBC$ | no | 1.381966011 | $g_d(w_5)$ | 9 | 4 | $\left(\frac{62}{45}^+, 37\right)$ |
| $ABA.BCB.CAC$ | yes | 1.5 | $g_e(w_4)^1$ | 9 | 4 | $\left(\frac{67}{45}^+, 37\right)$ |
| $ABCA.BCAB.CBAC$ | yes ² | 1.333333333 | $g_f(w_5)$ | 6 | 6 | $\left(\frac{31}{24}^+, 31\right)$ |
| $ABCA.BAB.CAC$ | yes | 1.414213562 | $g_g(w_4)$ | 6 | 8 | $\left(\frac{89}{63}^+, 61\right)$ |
| $ABCA.BAB.CBC$ | no | 1.430159709 | $g_h(w_4)$ | 6 | 7 | $\left(\frac{17}{12}^+, 61\right)$ |
| $ABCA.BAB.CBAC$ | no | 1.381966011 | $g_i(w_5)$ | 8 | 7 | $\left(\frac{127}{96}^+, 41\right)$ |
| $ABCBA.CABC$ | no | 1.361103081 | $g_j(w_5)$ | 6 | 8 | $\left(\frac{4}{3}^+, 25\right)$ |
| $ABCBA.CAC$ | yes | 1.396608253 | $g_k(w_5)$ | 6 | 13 | $\left(\frac{4}{3}^+, 25\right)$ |

In the table above, the columns indicate respectively, the considered minimally nice formula f , whether f is equivalent to its reversed formula, the avoidability exponent of f , the infinite ternary word avoiding f , the value q such that the corresponding morphism is q -uniform, the value such that the avoiding word is d -directed, the suitable property of (β^+, n) -freeness used in the proof that f is avoided. We list below the corresponding morphisms.

| g_a | g_b | g_c | g_d | g_e |
|-----------|--------|-------|-----------|-----------|
| 002112201 | 021221 | 2011 | 020112122 | 001220122 |
| 001221122 | 021121 | 1200 | 020101112 | 001220112 |
| 001220112 | 020001 | 1120 | 020001222 | 001120122 |
| 001122012 | 011102 | 0222 | 010121222 | 001120112 |
| | 010222 | 0012 | 000111222 | |

¹The formula $ABA.BCB.CAC$ seems also avoided up to \simeq by the Hall-Thue word, i.e., the fixed point of $0 \rightarrow 012; 1 \rightarrow 02; 2 \rightarrow 1$.

²We mistakenly said in [11] that $ABCA.BCAB.CBAC$ is different from its reverse.

| g_f | g_g | g_h | g_i | g_j | g_k |
|--------|--------|--------|----------|--------|--------|
| 012220 | 021210 | 011120 | 01222112 | 021121 | 022110 |
| 012111 | 011220 | 002211 | 01112022 | 012222 | 021111 |
| 012012 | 002111 | 002121 | 01100022 | 011220 | 012222 |
| 011222 | 001222 | 001222 | 01012220 | 011112 | 012021 |
| 010002 | | | 01012120 | 000102 | 011220 |

As an example, we show that $ABCBA.CAC$ is avoided by $g_k(w_5)$. First, we check that $g_k(w_5)$ is $\left(\frac{4}{3}^+, 25\right)$ -free using the main lemma in [7], that is, we check the $\left(\frac{4}{3}^+, 25\right)$ -freeness of the g_k -image of every $\left(\frac{5}{4}^+\right)$ -free word of length at most $\frac{2 \times \frac{4}{3}}{\frac{4}{3} - \frac{5}{4}} = 32$. Then we check that $g_k(w_5)$ is 13-directed by inspecting the factors of $g_k(w_5)$ of length 13. For contradiction, suppose that $g_k(w_5)$ contains an occurrence h of $ABCBA.CAC$ up to \simeq . Let us write $a = |h(A)|$, $b = |h(B)|$, $c = |h(C)|$.

Suppose that $a \geq 25$. Since $g_k(w_5)$ is 13-directed, all occurrences of $h(A)$ are identical. Then $h(ABCBA)$ is a repetition with period $|h(ABCBA)| \geq 25$. So the $\left(\frac{4}{3}^+, 25\right)$ -freeness implies the bound $\frac{2a+2b+c}{a+2b+c} \leq \frac{4}{3}$, that is, $a \leq b + \frac{1}{2}c$.

In every case, we have

$$a \leq \max \left\{ b + \frac{1}{2}c, 24 \right\}.$$

Similarly, the factors $h(BCB)$ and $h(CAC)$ imply

$$b \leq \max \left\{ \frac{1}{2}c, 24 \right\}$$

and

$$c \leq \max \left\{ \frac{1}{2}a, 24 \right\}.$$

Solving these inequalities gives $a \leq 36$, $b \leq 24$, and $c \leq 24$. Now we can check exhaustively that $g_k(w_5)$ contains no occurrence up to \simeq satisfying these bounds.

Except for $ABCBA.CBABC$, the avoidability index of the nice formulas in the above table is 3. So the results in this section extend their 3-avoidability up to \simeq .

4 The power series method

The so-called power series method has been used [1, 8] to prove the 3-avoidability of many classical doubled patterns with at least 4 variables and every doubled pattern with at least 6 variables, as mentioned in the introduction.

Let p be such a classical doubled pattern and let p' be a doubled pattern with reversal obtained by adding some $-^R$ to p . Without loss of generality, the leftmost appearance of every variable X of p remains free of $-^R$ in p' . Then we will see that p' is also 3-avoidable. The power series method is a counting argument that relies on the following observation. If the h -image of the leftmost appearance of the variable X of p is fixed, say $h(X) = w_X$, then there is exactly one possibility for the h -image of the other appearances of X , namely $h(X) = w_X$. This observation can be extended to p' , since there is also exactly one possibility for $h(X^R)$, namely $h(X^R) = w_X^R$.

Notice that this straightforward generalization of the power series method from classical doubled patterns to doubled patterns with reversal cannot be extended to avoiding a doubled pattern up to \simeq . Indeed, if $h(X) = w_X$ for the leftmost appearance of the variable X and w_X is not a palindrome, then there exist two possibilities for the other appearances of X , namely w_X and w_X^R .

5 Sporadic patterns

Up to symmetries, there are ten doubled patterns whose 3-avoidability cannot be deduced by the previous results. They have been identified in [8] and are listed in the following table.

Let w_5 be any infinite $\left(\frac{5^+}{4}\right)$ -free word over Σ_5 and let h be the following 9-uniform morphism.

$$\begin{aligned} h(0) &= 020022221 \\ h(1) &= 011111221 \\ h(2) &= 010202110 \\ h(3) &= 010022112 \\ h(4) &= 000022121 \end{aligned}$$

Table 1: The seven sporadic patterns on 4 variables and the three sporadic patterns on 5 variables

| Doubled pattern | Avoidability exponent |
|--|-----------------------|
| <i>ABACBD\overline{CD}</i> | 1.381966011 |
| <i>ABACDB\overline{DC}</i> | 1.333333333 |
| <i>ABACDC\overline{BD}</i> | 1.340090632 |
| <i>ABCADB\overline{DC}</i> | 1.292893219 |
| <i>ABCADC\overline{BD}</i> | 1.295597743 |
| <i>ABCADC\overline{DB}</i> | 1.327621756 |
| <i>ABCBDAD\overline{C}</i> | 1.302775638 |
| <i>ABACBD\overline{CDEDE}</i> | 1.366025404 |
| <i>ABACDB\overline{CEDE}</i> | 1.302775638 |
| <i>ABACDB\overline{DECE}</i> | 1.320416579 |

First, we check that $h(w_5)$ is 7-directed and $\left(\frac{139}{108}^+, 46\right)$ -free. Then, using the same method as in Section 3, we show that $h(w_5)$ avoids up to \simeq these ten sporadic patterns simultaneously.

6 Square-free doubled patterns with at most 4 variables

Here we show Theorem 5, that is, every square-free doubled pattern with at most 4 variables is 2-avoidable. We list them as follows:

- Among patterns that are equal up to letter permutation, we only list the lexicographically least.
- If a pattern is distinct from its mirror image, we only list the lexicographically least among the pattern and its mirror image.
- We do not list patterns that contain a square-free doubled pattern as a strict factor.
- We do not list patterns that contain an occurrence of *ABACBC*, *ABCACB*, *ABCBABC*, *ABCD \overline{BD} ABC*, *ABCD \overline{BD} AC*, *ABACDC \overline{BD}* , or their mirror image.

- We do not include the seven sporadic patterns on 4 variables from Table 1, which are 2-avoidable.

Table 2 contains every pattern p in this list with an infinite binary word avoiding p . Let us detail how to read Table 2:

- A morphism is m given in the format $m(0)/m(1)/\dots$
- We denote by b_2, b_4, b_5 the famous morphisms $01/10, 01/21/03/23, 01/23/4/21/0$, respectively.
- We denote by w_k any infinite $RT(k)^+$ -free word over Σ_k .
- If the avoiding word is a pure morphic word $m^\omega(0)$, then m is given.
- If the avoiding word is a morphic word $f(m^\omega(0))$, then we write $m; f$.
- If the avoiding word is of the form $f(w_k)$, then we write $w_k; f$.

The proofs that a (pure) morphic word avoids a pattern use Cassaigne's algorithm [2] and the proofs that a morphic image word a Dejean word avoids a pattern use the technique described in Section 3.

7 Growth rate of ternary words avoiding a doubled pattern

Theorem 6 obviously holds for $p = AA$. Without loss of generality, we do not need to consider a doubled pattern p that contains an occurrence of another doubled pattern. In particular, p is square-free. So we need to show that $GR_3(p)$ is at least $GR_3(AA)$, which is close to 1.30176 [13].

If p is 2-avoidable, then p is avoided by sufficiently many ternary words. By Lemma 4.1 in [7], $\lambda(p) = 2$ implies that $GR_3(p) \geq 2^{\frac{1}{2}} > GR_3(AA)$. Thus, Conjecture 3 implies Theorem 6. By Theorem 5, we can assume that $v(p) \geq 5$. We can also rule out the three sporadic patterns on 5 variables from Table 1, which are 2-avoidable.

According to the partition of the set of doubled patterns mentioned in the introduction, there remains to consider the doubled patterns p whose 3-avoidability has been obtained via the power series method. In that case, we even get $GR_3(p) > 2 > GR_3(AA)$.

Table 2: Binary words avoiding doubled patterns

| Doubled pattern | Avoiding word |
|-----------------|--|
| ABCABDCBD | w_5 ; 0010101110/0010011000/0001111110/0001110101/0000011001 |
| ABCACDCBD | w_5 ; 000101010111/000100110111/000011001111/000001011111/000000111111 |
| ABCBAABDCBD | b_4 ; 01/00/10/11 |
| ABCBAACBCD | b_4 ; 0000/0011/1111/1010 |
| ABCBDACBCD | b_4 ; 01/00/10/11 |
| ABCBDACBCD | b_2 |
| ABCBDCAACBCD | b_4 ; 1000/0111/0110/0010 |
| ABCDACBD | w_5 ; 00100110111111000/00100110111011000/ 00011110110101010/00001111111011010/0000101010101111 |
| ABCDEACBD | w_6 ; 010101111100/010010100000/001001110111/000111111101/000101010111/000100011011 |
| ABCDEBADC | w_5 ; 10001000101111101010110/00000110110101000111111/ 00000101011100100111111/00000011101010010011111/00000011011000101011111 |
| ABCDECBACBD | 001/011 |
| ABCDACBD | b_5 ; 0011110110000/0011010100110/0001111100111/0001110001000/0001101101111 |
| ABCDABACBD | avoided by every $(\frac{3}{2}^+, 4)$ -free binary word [6] |
| ABCDACBCBD | b_5 ; 00/01/10/110/111 |
| ABACDCBCD | w_5 ; 10011011000/01011111000/00111010100/00100100111/00001111111 |
| ABCABDBCD | w_5 ; 0010111111/0010011110/0010011100/0000010101/0000001101 |
| ABCADBCBD | w_5 ; 001011010000/0010011111000/000110011001/000011101010/000010111111 |
| ABCADCBCD | w_5 ; 001101111000/001101101000/001001111111/000101110101/000001100101 |
| ABCBAABDC | w_5 ; 001111110110/0001010111100/0000101101110/0000011010111/0000001011111 |
| ABCBDABCD | w_4 ; 1111/1101/0010/0000 |
| ABCBDABDC | w_5 ; 101110000001/101100100001/011111110100/010001111110/010001101110 |
| ABCBDACBD | b_5 ; 00/01/10/110/111 |
| ABCBDADBC | w_5 ; 00110111010010/00110000000010/0001111111011/00011110101000/00010101100011 |
| ABCBDABDC | b_5 ; 111/101/000/011/001 |
| ABCBDABDC | b_5 ; 00/01/10/110/111 |
| ABCBDABDC | b_5 ; 000/011/001/111/101 |
| ABCBDACBCD | b_4 ; 01/00/10/11 |
| ABCDEBACD | w_5 ; 0001111101010/0001110111000/0001011111111/0000111001111/0000011011001 |
| ABCBDABDC | 011/100 |
| ABCBDABDC | 00111101110000/00111011000010/00111010100000/00011001001111/00010101111111 |
| ABCBDACBCD | b_5 ; 00/01/10/110/111 |
| ABCBDACBCD | b_5 ; 00/01/10/110/111 |
| ABCBDACAD | w_5 ; 00011110110011/00011101101001/00011011010100/0001011111110/00000011111010 |
| ABCBDACBACD | b_4 ; 000/111/10/01 |
| ABCBDACBACD | b_2 |
| ABCBDACBACD | b_4 ; 00/01/10/11 |
| ABCBDACBACD | 001/110 |
| ABCBDACBACD | b_5 ; 00/10/111/01/011 |
| ABCBDACBACD | w_5 ; 10000000011/01111010010/01101100010/0101111110/00001010101 |
| ABCBDACBACD | b_5 ; 111/101/000/100/110 |
| ABCBDACBACD | b_5 ; 00/01/10/110/111 |
| ABCBDACBACD | b_5 ; 00/01/10/110/111 |
| ABCBDACBACD | w_5 ; 00011110110011/00011101101001/00011011010100/0001011111110/00000011111010 |
| ABCBDACBACD | b_4 ; 000/111/10/01 |
| ABCBDACBACD | b_2 |
| ABCBDACBACD | b_4 ; 00/01/10/11 |
| ABCBDACBACD | w_5 ; 1111100/1100110/0110101/0010010/0000101 |
| ABCBDACBACD | w_5 ; 0000010001111110101000100111110111/0000010001111100100001100101101111/ 000000100111111101000011010111011/0000001001111110110100010101111011/ 0000000101110010000111111010010111 |
| ABCBDABDC | w_5 ; 0011111110101/0010110111010/0010101110000/0000111111001/0000110110001 |
| ABCBDABDC | w_5 ; 0101111111/01001000111/00101000011/00011110101/00000001011 |
| ABCBDACBACD | b_5 ; 101/000/110/111/100 |
| ABCBDACBACD | w_5 ; 0110101/0100000/0011110/0001111/0000111 |
| ABCBDACBACD | w_5 ; 00010111001010/00001111010101/00001110001010/00001100111111/00001100010110 |
| ABCBDACBACD | w_5 ; 0000001101101100111000111101101110000101111010100100101110111/ 000000110111011000010011111010101000010101111010100100101110111/ 000000101100111101010011000111000010101111010100100101110111/ 00000010101101101000100011111101000010101111010100100101110111/ 000000101011001110001111010011000010101111010100100101110111 |
| ABCBDACBACD | w_5 ; 01111101/00111100/00111001/00110110/00000101 |
| ABCBDACBACD | w_5 ; 00110001000110/0010101111110/00011111010011/0001010101111/00000001010011 |

8 Conclusion

Unlike classical formulas, we know that there exist avoidable formulas with reversal of arbitrarily high avoidability index [9]. Maybe doubled patterns and nice formulas are easier to avoid. We propose the following open problems.

- Are there infinitely many doubled patterns up to \simeq that are not 2-avoidable?
- Is there a nice formula up to \simeq that is not 3-avoidable?

A first step would be to improve Theorem 2 by generalizing the 3-avoidability of doubled patterns with reversal to doubled patterns up to \simeq . Notice that the results in Sections 3 and 5 already consider avoidability up to \simeq . However, the power series method gives weaker results. Classical doubled patterns with at least 6 variables are 3-avoidable because

$$1 - 3x + \left(\frac{3x^2}{1 - 3x^2} \right)^v$$

has a positive real root for $v \geq 6$. The (basic) power series for doubled patterns up to \simeq with v variables would be

$$1 - 3x + \left(\frac{6x^2}{1 - 3x^2} - \frac{3x^2 + 3x^4}{1 - 3x^4} \right)^v.$$

The term $\frac{6x^2}{1-3x^2}$ counts for twice the term $\frac{3x^2}{1-3x^2}$ in the classical setting, for $h(V)$ and $h(V)^R$. The term $\frac{3x^2+3x^4}{1-3x^4}$ corrects for the case of palindromic $h(V)$, which should not be counted twice. This power series has a positive real root only for $v \geq 10$. This leaves many doubled patterns up to \simeq whose 3-avoidability must be proved with morphisms.

Looking at the proof of Theorem 2, we may wonder if a doubled pattern with reversal is always easier to avoid than the corresponding classical pattern. This is not the case: backtracking shows that $\lambda(ABCA^RC^RB) = 3$, whereas $\lambda(ABCACB) = 2$ [7].

To get a more precise version of both conjectures 3 and 4, we plan to obtain the (conjectured) list of all 2-unavoidable doubled patterns, which should be a finite list containing no square-free pattern.

References

- [1] J. Bell, T. L. Goh. Exponential lower bounds for the number of words of uniform length avoiding a pattern. *Inform. and Comput.* **205** (2007), 1295-1306.
- [2] J. Cassaigne. *An Algorithm to Test if a Given Circular HD0L-Language Avoids a Pattern*. *IFIP Congress*, pages 459–464, 1994.
- [3] J. Currie and L. Mol. The undirected repetition threshold and undirected pattern avoidance. *Theor. Comput. Sci.* **866** (2021), 56–69.
- [4] J. Currie, L. Mol, and N. Rampersad. A family of formulas with reversal of high avoidability index. *International Journal of Algebra and Computation* **27(5)** (2017), 477–493.
- [5] J. Currie, L. Mol, and N. Rampersad. Avoidance bases for formulas with reversal. *Theor. Comput. Sci.* **738** (2018), 25–41.
- [6] L. Ilie, P. Ochem, and J.O. Shallit. A generalization of repetition threshold. *Theoret. Comput. Sci.* **92(2)** (2004), 71–76.
- [7] P. Ochem. A generator of morphisms for infinite words. *RAIRO: Theoret. Informatics Appl.* **40** (2006), 427–441.
- [8] P. Ochem. Doubled patterns are 3-avoidable. *Electron. J. Combinatorics*. **23(1)** (2016), #P1.19.
- [9] P. Ochem. A family of formulas with reversal of arbitrarily high avoidability index. *Theoret. Comput. Sci.* **896** (2021), 168–170.
- [10] P. Ochem and A. Pinlou. Application of entropy compression in pattern avoidance. *Electron. J. Combinatorics*. **21(2)** (2014), #RP2.7.
- [11] P. Ochem and M. Rosenfeld. On some interesting ternary formulas. *Electron. J. Combinatorics*. **26(1)** (2019), #P1.12.
- [12] P. Ochem and M. Rosenfeld. Avoidability of palindrome patterns. *Electron. J. Combinatorics*. **28(1)** (2021), #P1.4.
- [13] A. Shur. Growth rates of complexity of power-free languages. *Theoret. Comput. Sci.* **411(34-36)** (2010), 3209–3223.