Thin Ferromagnetic plates with Magnetostriction

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ABSTRACT
In this paper, we establish the existence of global-in-time weak solutions for the Landau-Lifschitz-Gilbert equation with magnetostriction in the case of mixed boundary conditions. From this model, we derive by asymptotic method a two-dimensional model for thin ferromagnetic plates taking into account magnetostrictive effects.

KEYWORDS
ferromagnetism, magnetostriction, weak solutions, thin plates

1. Introduction

Ferromagnetic materials are characterized by a spontaneous magnetization, even in the absence of an external magnetic field. This magnetization influences the shape of the materials, and conversely, a deformation of the material leads to a change in magnetization. This phenomenon, called magnetostriction, is used for industrial applications, for example in magnetostrictive pressure sensors or ultrasonic transducers.

In this paper, starting from the 3D model describing the time behavior of the magnetization of a material taking into account the magnetostriction, our goal is to obtain and justify by asymptotic method a two-dimensional model of thin ferromagnetic plate. We first describe the three-dimensional model coupling Landau-Lifschitz-Gilbert with elasticity equations (see [15]).

1.1. Three dimensional model

We denote by $\mathbf{\Omega}$, the domain occupied by the ferromagnetic material, and by $\mathbf{m}(\mathbf{t}, \mathbf{x}) \in \mathbb{R}^3$ the magnetic moment at time $\mathbf{t}$ and at point $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in \mathbf{\Omega}$. We assume that the material is saturated, so that $\mathbf{m}$ satisfies the saturation constraint $|\mathbf{m}(\mathbf{t}, \mathbf{x})| = \boldsymbol{m}_s$ a.e., where $\boldsymbol{m}_s$ is a constant expressed in $\mathrm{A.m}^{-1}$. The dynamics of $\mathbf{m}$ is described by

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the Landau-Lifschitz-Gilbert equation (see [1], [5], [8] and [14]):

$$\begin{cases} \dfrac{\partial \mathbf{m}}{\partial \mathbf{t}} - \dfrac{\alpha}{\boldsymbol{m}_s}\mathbf{m} \times \dfrac{\partial \mathbf{m}}{\partial \mathbf{t}} = -(1+\alpha^2)\gamma\, \mathbf{m} \times \mathbf{H}_{\text{eff}} \quad \text{in } \mathbb{R}^+ \times \boldsymbol{\Omega}, \\ \\ \partial_{\mathbf{n}}\mathbf{m} = 0 \quad \text{on } \mathbb{R}^+ \times \partial\boldsymbol{\Omega}, \end{cases} \tag{1}$$

where $\times$ is the cross product in $\mathbb{R}^3$, γ is the gyromagnetic ratio (expressed in A.s.kg^{-1}), α is the dimensionless damping coefficient and $\mathbf{n}$ the outward unit normal vector on $\partial\boldsymbol{\Omega}$. The effective field $\mathbf{H}_{\text{eff}}$ (expressed in $\text{T} = \text{kg.s}^{-2}.\text{A}^{-1}$) is given by:

$$\mathbf{H}_{\text{eff}} = \frac{A}{\boldsymbol{m}_s^2}\Delta\mathbf{m} + \mu_0 \mathbf{h}_d(\mathbf{m}) + \mu_0 \Psi(\mathbf{m}) + (\boldsymbol{\Lambda}^{\text{m}} : \boldsymbol{\sigma})\mathbf{m},$$

where

- A is the exchange constant (expressed in J.m^{-1}),
- $\mu_0 = 4\pi.10^{-7}\text{kg.m.s}^{-2}.\text{A}^{-2}$ is the vacuum permeability,
- Ψ is an anisotropic linear term satisfying: $\Psi(m) = -\nabla\Phi(m)$, where $\Phi : \mathbb{R}^3 \to \mathbb{R}$ is a non negative quadratic form,
- the demagnetizing field $\mathbf{h}_d(\mathbf{m})$ is calculated from $\mathbf{m}$ solving the static Maxwell equation coupled with the law of Faraday:

$$\text{curl } \mathbf{h}_d(\mathbf{m}) = 0 \quad \text{and} \quad \text{div } (\mathbf{h}_d(\mathbf{m}) + \overline{\mathbf{m}}) = 0, \tag{2}$$

  where $\overline{\mathbf{m}}$ is the extension of $\mathbf{m}$ by zero outside $\boldsymbol{\Omega}$,
- in the magnetostrictive term, $\boldsymbol{\sigma}$ is the stress tensor (2-tensor expressed in $\text{kg.m}^{-1}.\text{s}^{-2}$), $\boldsymbol{\Lambda}^{\text{m}}$ is a 4-tensor expressed in $\text{m}^2.\text{A}^{-2}$, and : is the contraction operator (see below).

Notation 1. Let Λ be a 4-tensor in $\mathbb{R}^3$. Let ξ be a 3×3 matrix. We denote $\Lambda : \xi$ the 3×3 matrix which entries are given by:

$$(\Lambda : \xi)_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} \Lambda_{ijkl}\xi_{kl}.$$

Let ξ and ζ be two 3×3 matrices. We denote $\xi : \zeta$ the scalar given by:

$$\xi : \zeta = \sum_{i=1}^{3}\sum_{j=1}^{3} \xi_{ij}\zeta_{ij}.$$

The Landau-Lifschitz-Gilbert Equation (1) is coupled with the wave elasticity equation:

$$\boldsymbol{\rho}\frac{\partial^2 \mathbf{u}}{\partial \mathbf{t}^2} - \text{div } \boldsymbol{\sigma} = 0, \tag{3}$$

where $\boldsymbol{\rho}$ is the mass density (expressed in kg.m^{-3}) and where the stress tensor $\boldsymbol{\sigma}$ is given by $\boldsymbol{\sigma} = \boldsymbol{\Lambda}^{\text{e}} : \boldsymbol{\varepsilon}^{\text{e}}$, with

- $\mathbf{\Lambda}^{\mathrm{e}}$ is a 4-tensor (expressed in $\mathrm{kg.m^{-1}s^{-2}}$),
- the total strain $\boldsymbol{\varepsilon}^{\mathrm{e}}$ is a 3×3 matrix obtained from the linearized strain tensor $\boldsymbol{\varepsilon}(\mathbf{u})$ and the magnetostrictive strain tensor $\boldsymbol{\varepsilon}^{\mathrm{m}}$ by the relation $\boldsymbol{\varepsilon}(\mathbf{u}) = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{m}}$, with

$$\boldsymbol{\varepsilon}_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j} + \frac{\partial \mathbf{u}_j}{\partial \mathbf{x}_i}\right) \quad \text{and} \quad \boldsymbol{\varepsilon}^{\mathrm{m}} = \mathbf{\Lambda}^{\mathrm{m}} : \mathbf{m} \otimes \mathbf{m},$$

where $\mathbf{m} \otimes \mathbf{m} = \mathbf{m} \cdot {}^t\mathbf{m}$ is the 3×3 matrix which entries are given by $(\mathbf{m} \otimes \mathbf{m})_{ij} = \mathbf{m}_i \mathbf{m}_j$.

The 4-tensors $\mathbf{\Lambda}^{\mathrm{e}}$ and $\mathbf{\Lambda}^{\mathrm{m}}$ are supposed to be symmetric and positive, as defined below:

Definition 1.1. Let Λ be a 4-tensor in $\mathbb{R}^3$. This tensor is said to be symmetric if:

$$\forall (i,j,k,l) \in \{1,2,3\}^4, \quad \Lambda_{ijkl} = \Lambda_{jikl} = \Lambda_{ijlk} = \Lambda_{klij}.$$

This tensor is said to be positive if there exists $\lambda^* \in \mathbb{R}^{*+}$ such that for all symmetric matrix ξ of entries ξ_{ij}, we have:

$$\sum_{ijkl} \Lambda_{ijkl} \xi_{ij} \xi_{kl} \geq \lambda^* \sum_{ij} |\xi_{ij}|^2.$$

We assume that the material is clamped on $\mathbf{\Gamma_1} \subset \partial\mathbf{\Omega}$ and that a surface force $\boldsymbol{f}$ (expressed in $\mathrm{N.m^2}$) is applied on $\mathbf{\Gamma_2} = \partial\mathbf{\Omega} \setminus \mathbf{\Gamma_1}$, so that we have the following boundary conditions:

$$\begin{cases} \mathbf{u} = 0 \ \text{ on } \mathbf{\Gamma_1}, \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \boldsymbol{f} \ \text{ on } \mathbf{\Gamma_2}, \end{cases} \tag{4}$$

In order to obtain a dimensionless model, we write $\mathbf{m}(\mathbf{t}, \mathbf{x}) = \boldsymbol{m_s} m(\frac{\mathbf{t}}{\tau}, \frac{\mathbf{x}}{\ell})$ and $\mathbf{u}(\mathbf{t}, \mathbf{x}) = \ell u(\frac{\mathbf{t}}{\tau}, \frac{\mathbf{x}}{\ell})$, where the characteristic time τ and the exchange length ℓ are respectively given by:

$$\tau = \frac{1}{\gamma \mu_0 \boldsymbol{m_s}} \quad \text{and} \quad \ell^2 = \frac{A}{\mu_0 \boldsymbol{m_s}^2}.$$

We denote by $t = \mathbf{t}/\tau$ the dimensionless time and by $x = (x_1, x_2, x_3) = \mathbf{x}/\ell \in \Omega = \mathbf{\Omega}/\ell$ the dimensionless position. We denote by $\Gamma_1 = \mathbf{\Gamma_1}/\ell$ and $\Gamma_2 = \mathbf{\Gamma_2}/\ell$. We introduce the dimensionless tensors λ^{m} and λ^{e} such that:

$$\mathbf{\Lambda}^{\mathrm{m}} = \frac{1}{\boldsymbol{m_s}^2} \lambda^{\mathrm{m}} \quad \text{and} \quad \mathbf{\Lambda}^{\mathrm{e}} = \mu_0 \boldsymbol{m_s}^2 \lambda^{\mathrm{e}},$$

and we define the dimensionless applied force $f \in L^2(\Gamma_2)$ by $\boldsymbol{f} = \mu_0 \boldsymbol{m_s}^2 f$. We denote then by σ and ρ the dimensionless stress tensor and density given respectively by:

$$\sigma(t,x) = \frac{1}{\boldsymbol{m_s}^2} \boldsymbol{\sigma}(\tau t, \ell x) = \left(\lambda^{\mathrm{e}} : \varepsilon(u) - \lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : m \otimes m)\right)(t,x) \quad \text{and} \quad \rho = \frac{\boldsymbol{\rho} \gamma^2 A}{\boldsymbol{m_s}^2}.$$

We obtain then the following dimensionless model: $m(t,x)$ and $u(t,x)$ are defined for $t \in \mathbb{R}^+$ and $x \in \Omega$ and satisfy:

$$\begin{cases} \dfrac{\partial m}{\partial t} - \alpha m \times \dfrac{\partial m}{\partial t} = -(1+\alpha^2) m \times H_{\text{eff}} \text{ in } \mathbb{R}^+ \times \Omega, \\ H_{\text{eff}} = \Delta m + h_d(m) + \Psi(m) + (\lambda^{\text{m}} : \sigma) m, \\ \sigma = \lambda^{\text{e}} : \varepsilon^{\text{e}} \quad \text{with} \quad \varepsilon^{\text{e}} = \varepsilon(u) - \lambda^{\text{m}} : m \otimes m, \\ \rho \dfrac{\partial^2 u}{\partial t^2} - \text{div } \sigma = 0 \text{ in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{5}$$

with the following boundary conditions

$$\begin{cases} \partial_{\mathbf{n}} m = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega, \\ u(t,x) = 0 \text{ on } \mathbb{R}^+ \times \Gamma_1, \\ \sigma \cdot n = f \text{ on } \mathbb{R}^+ \times \Gamma_2, \end{cases} \tag{6}$$

We remark that from the assumptions on $\mathbf{\Lambda}^{\text{e}}$, there exists a constant $\lambda^{\text{e}*} > 0$ such that for all symmetric 2-tensor ξ:

$$\sum_{ijkl} \lambda^{\text{e}}_{ijkl} \xi_{ij} \xi_{kl} \geq \lambda^{\text{e}*} \sum_{ij} |\xi_{ij}|^2. \tag{7}$$

Note that α and ρ are dimensionless constants whose values do not affect the mathematical analysis of the equations. We therefore take them equal to 1. We define the space

$$V(\Omega) = \{v \in H^1(\Omega; \mathbb{R}^3);\ v = 0 \text{ on } \Gamma_1\}.$$

For $m \in H^1(\Omega; S^2)$, $v \in V(\Omega)$ and $w \in L^2(\Omega; \mathbb{R}^3)$, we define the energy $\mathcal{E}(m,v,w)$ by:

$$\begin{aligned} \mathcal{E}(m,v,w) = & \frac{1}{2}\int_\Omega |\nabla m(x)|^2 dx + \int_\Omega \Phi(m(x)) dx + \frac{1}{2}\int_{\mathbb{R}^3} |h_d(m)(x)|^2 dx \\ + & \frac{1}{4}\int_\Omega (\lambda^{\text{e}} : \varepsilon^{\text{e}}(x)) : \varepsilon^{\text{e}}(x) dx + \frac{1}{4}\int_\Omega |w(x)|^2 dx - \frac{1}{2}\int_{\Gamma_2} f(s) \cdot u(s)\, d\Gamma_s, \end{aligned} \tag{8}$$

where $\varepsilon^{\text{e}}(x) = \varepsilon(v(x)) - \lambda^{\text{m}} : m(x) \otimes m(x)$.

We aim to solve the Cauchy problem coupling (5)-(6) with the following initial

conditions:

$$\begin{cases} m(t=0) = m_0 \text{ in } \Omega, \\ \\ u(t=0) = u_0 \text{ in } \Omega, \\ \\ \dfrac{\partial u}{\partial t}(t=0) = u_1 \text{ in } \Omega, \end{cases} \tag{9}$$

where $m_0 \in H^1(\Omega; S^2)$, $u_0 \in V(\Omega)$ and $u_1 \in L^2(\Omega; \mathbb{R}^3)$. We define the notion of weak solutions for the previous Cauchy problem:

Definition 1.2. We say that (m, u) is a weak solution for (5)-(6)-(9) if

(1) $m \in L^\infty(\mathbb{R}^+; H^1(\Omega; \mathbb{R}^3))$, $\dfrac{\partial m}{\partial t} \in L^2(\mathbb{R}^+; L^2(\Omega; \mathbb{R}^3))$ and $m(0, \cdot) = m_0$,

(2) m satisfies the saturation constraint

$$|m(t,x)| = 1 \text{ for almost every } (t,x) \in \mathbb{R}^+ \times \Omega, \tag{10}$$

(3) $u \in L^\infty(\mathbb{R}^+; V(\Omega))$, $\dfrac{\partial u}{\partial t} \in L^\infty(\mathbb{R}^+; L^2(\Omega; \mathbb{R}^3))$ and $u(0, \cdot) = u_0$,

(4) for all $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^+; H^1(\Omega; \mathbb{R}^3))$,

$$\begin{aligned} &\int_{\mathbb{R}^+\times\Omega} \left(\frac{\partial m}{\partial t} - m \times \frac{\partial m}{\partial t} \right) \cdot \chi = 2 \int_{\mathbb{R}^+\times\Omega} \sum_{i=1}^{3} m \times \frac{\partial m}{\partial x_i} \cdot \frac{\partial \chi}{\partial x_i} \\ &-2 \int_{\mathbb{R}^+\times\Omega} m \times (h_d(m) + \Psi(m) + (\lambda^{\mathrm{m}} : \sigma) m) \cdot \chi, \\ &\text{with } \sigma = \lambda^{\mathrm{e}} : \varepsilon(u) - \lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : m \otimes m), \end{aligned} \tag{11}$$

(5) for all $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^+; V(\Omega))$,

$$\begin{aligned} \rho \int_{\mathbb{R}^+\times\Omega} \frac{\partial u}{\partial t} \cdot \frac{\partial \chi}{\partial t} - \int_{\mathbb{R}^+\times\Omega} (\lambda^{\mathrm{e}} : \varepsilon(u)) : \varepsilon(\chi) + \int_{\mathbb{R}^+\times\Gamma_2} f \cdot \chi \\ - \int_\Omega u_1 \chi(0,x) = - \int_{\mathbb{R}^+\times\Omega} (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : m \otimes m)) : \varepsilon(\chi), \end{aligned} \tag{12}$$

(6) we have the following energy inequality: for all $t \geq 0$,

$$\mathcal{E}(m(t,\cdot), u(t,\cdot), \frac{\partial u}{\partial t}(t,\cdot)) + \int_0^t \int_\Omega \left| \frac{\partial m}{\partial t}(\tau, x) \right|^2 d\tau\, dx \leq \mathcal{E}(m_0, u_0, u_1) \tag{13}$$

where $\mathcal{E}$ is defined by (8).

1.2. Statement of the results

First, we establish the existence of global-in-time weak solutions for the Cauchy problem (5)-(6)-(9). Such a result is proved in [8] for homogeneous Dirichlet boundary conditions for the deformation. With the same method, we address mixed boundary conditions and we obtain the following theorem:

Theorem 1.3. *Let $m_0 \in H^1(\Omega; S^2)$, $u_0 \in V(\Omega)$, $u_1 \in L^2(\Omega)$ and $f \in L^2(\Gamma_2)$. Then, there exists a weak solution (m, u) for (5)-(6)-(9).*

In a second time, we aim to obtain a reduced two-dimensional model for thin ferromagnetic plate. We consider a thin plate Ω_η of the form $\Omega_\eta = \omega \times]-\eta, \eta[$, where ω is a smooth bounded domain of $\mathbb{R}^2$.

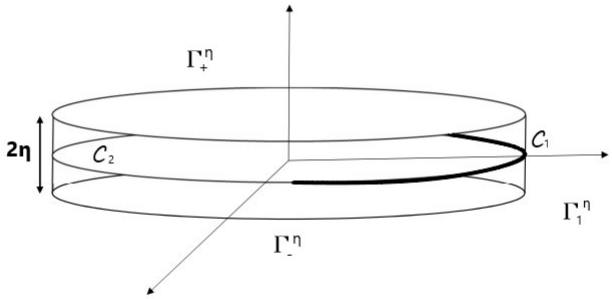

Figure 1. $\Omega_\eta = \omega \times]-\eta, \eta[$

We assume that this plate is clamped on $\Gamma_1^\eta = \mathcal{C}_1 \times]-\eta, \eta[$, (where $\mathcal{C}_1 \subset \partial\omega$ such that the one-dimensional measure of $\mathcal{C}_1$ is non vanishing). We denote by $\mathcal{C}_2 = \partial\omega \setminus \mathcal{C}_1$, $\Gamma_b^\eta = \mathcal{C}_2 \times]-\eta, \eta[$, $\Gamma_+^\eta = \bar{\omega} \times \{+\eta\}$ and $\Gamma_-^\eta = \bar{\omega} \times \{-\eta\}$, so that $\Gamma_1^\eta \cup \Gamma_2^\eta$ (with $\Gamma_2^\eta = \Gamma_b^\eta \cup \Gamma_+^\eta \cup \Gamma_-^\eta$) is a partition of $\partial\Omega_\eta$.

We define the spaces:

$$V(\Omega_\eta) = \{v \in H^1(\Omega_\eta; \mathbb{R}^3) \ ; \ v = 0 \text{ on } \Gamma_1^\eta\},$$

$$\mathcal{W}(\omega) = \big\{v = (v_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) \, ; \, v_i = 0 \text{ on } \mathcal{C}_1 \text{ and } \partial_1 v_3 = \partial_2 v_3 = 0 \text{ on } \mathcal{C}_1\big\}.$$

Notation: for $y = (y_1, y_2, y_3) \in \mathbb{R}^3$, we denote $y_T = (y_1, y_2)$.

Let $h \in L^2(\mathcal{C}_2; \mathbb{R}^3)$ and g^+ and g^- in $L^2(\omega; \mathbb{R}^3)$. We define the η-depending boundary data on Γ_2^η by:

$$\begin{cases} \forall\, x_T \in \omega, \ \forall\, \alpha \in \{1,2\}, \ f_\alpha^\eta(x_T, \pm\eta) = g_\alpha^\pm(x_T) \quad \text{and} \quad f_3^\eta(x_T, \pm\eta) = \eta^2 g_3^\pm(x_T), \\ \forall\, (x_T, x_3) \in \mathcal{C}_2 \times]-\eta, \eta[, \ \forall \alpha \in \{1,2\}, \ f_T^\eta(x_T, x_3) = h_T(x_T), \\ \forall\, (x_T, x_3) \in \mathcal{C}_2 \times]-\eta, \eta[, \ \forall \alpha \in \{1,2\}, \ f_3^\eta(x_T, x_3) = \eta h_3(x_T). \end{cases} \tag{14}$$

Let $m_0 \in H^1(\omega; S^2(0,1))$, $\tilde{u}_0 \in \mathcal{W}(\omega)$ and $u_1 \in L^2(\omega; \mathbb{R}^3)$. We assume that the third

component of u_1 vanishes: $u_{1,3} = 0$ on ω. We define the η-depending initial data by: for $x = (x_T, x_3) \in \Omega_\eta$,

$$\begin{cases} m_0^\eta(x_T, x_3) = m_0(x_T), \\ \forall\, \alpha \in \{1,2\},\ u_{0,\alpha}^\eta(x_T, x_3) = \tilde{u}_{0,\alpha}(x_T) - \dfrac{x_3}{\eta}\partial_\alpha \tilde{u}_{0,3}(x_T), \\ u_{0,3}^\eta(x_T, x_3) = \frac{1}{\eta}\tilde{u}_{0,3}(x_T), \\ u_1^\eta(x_T, x_3) = u_1(x_T). \end{cases} \tag{15}$$

We consider (m^η, u^η) the weak solution of (5)-(6)-(9), given by Theorem 1.3, with initial data m_0^η, u_0^η, u_1^η and external force f^η previously defined. We rescale this solution in order to work on the fixed domain $\Omega_1 = \omega \times]-1, 1[$: for $(x_T, x_3) \in \Omega_1$, we set:

$$\begin{aligned} &M^\eta(t, x_T, x_3) = m^\eta(t, x_T, \eta x_3), \quad H^\eta(t, x_T, x_3) = (h_d(m^\eta))(t, x_T, \eta x_3), \\ &U_T^\eta(t, x_T, x_3) = u_T^\eta(t, x_T, \eta x_3), \quad U_3^\eta(t, x_T, x_3) = \eta u_3^\eta(t, x_T, \eta x_3). \end{aligned} \tag{16}$$

We obtain the following convergence result, announced in [13] in the isotropic case:

Theorem 1.4. *Using the notations above, when η tends to zero, there exists a subsequence still denoted $(M^\eta, H^\eta, U^\eta)_\eta$ such that M^η tends to M in $L^\infty(0,T; H^1(\Omega_1))$ weak *, H^η tends to H in $L^\infty(0,T; L^2(\Omega_1))$ strongly, U^η tends to U in $L^\infty(0,T; V(\Omega_1))$ weak *. In addition, M does not depend on its third variable and there exists $\tilde{u} \in \mathcal{W}(\omega)$ such that:*

$$U_\alpha(x_T, x_3) = \tilde{u}_\alpha(x_T) - x_3 \partial_\alpha \tilde{u}_3 \text{ for } \alpha \in \{1,2\}, \text{ and } U_3(x_T, x_3) = \tilde{u}_3(x_T).$$

The limit $(M, \tilde{u})$ is a weak solution for the following initial and boundary value problem:

$$\begin{cases} \dfrac{\partial M}{\partial t} - M \times \dfrac{\partial M}{\partial t} = -2M \times H_{\text{eff}}(M) \quad \text{in } \mathbb{R}^+ \times \omega \\ \qquad \text{with } H_{\text{eff}}(M) = \Delta M - (M, \vec{e_3})\vec{e_3} + \Psi(M) + (\lambda^{\text{m}} : \widetilde{\sigma})M, \\ \dfrac{\partial^2 \tilde{u}_T}{\partial t^2} - \widetilde{\text{div}}\, \widetilde{\sigma} = -\dfrac{1}{2}(g_T^+ + g_T^-) \text{ in } \mathbb{R}^+ \times \omega, \\ \widetilde{\sigma} = \lambda^{\text{eq}} : \varepsilon(\tilde{u}_T) - \lambda^{\text{eq}} : (\lambda^{\text{m}} : M \otimes M), \\ M(0, \cdot) = m_0, \tilde{u}_T(0, \cdot) = \tilde{u}_{0,T} \text{ and } \dfrac{\partial \tilde{u}_T}{\partial t}(0, \cdot) = u_{1,T}, \\ \partial_n M = 0 \text{ on } \partial\omega, \quad \tilde{u}_T = 0 \text{ on } \mathcal{C}_1 \text{ and } \widetilde{\sigma}.n = h_T \text{ on } \mathcal{C}_2, \end{cases} \tag{17}$$

where λ^{eq} is a 4-tensor only depending on the coefficients of λ^{e} (see below), and where $\widetilde{\text{div}}\, S$ is defined by $\left(\widetilde{\text{div}}\, S\right)_\alpha = \sum_{\beta=1}^{2} \partial_\beta S_{\alpha\beta}$.

The new tensor λ^{eq} arising in the two-dimensional model (17) is defined as follows: we denote by $\mathcal{G}$ the set of symmetric real 3×3 matrices and by $\mathcal{G}_n$ the set of the matrices $D = (d_{ij}) \in \mathcal{G}$ such that $d_{\alpha\beta} = 0$ for $(\alpha, \beta) \in \{1, 2\}^2$. We define the linear operator $\mathcal{K} : \mathcal{G} \longrightarrow \mathcal{G}_n$ by:

$$\forall\, D \in \mathcal{G}, (\mathcal{K}(D))_{ij} = \begin{cases} 0 \quad \text{if } (i,j) \in \{1,2\}^2, \\ \\ (\lambda^{\mathrm{e}} : D)_{ij} = \displaystyle\sum_{(k,l)\in\{1,2,3\}^2} \lambda^{\mathrm{e}}_{ijkl} d_{kl} \quad \text{if } i = 3 \text{ or } j = 3. \end{cases}$$

We claim that the restriction of $\mathcal{K}$ to $\mathcal{G}_n$ is a bijection from $\mathcal{G}_n$ to itself (see the proof in subsection 3.4). We denote by $\mathcal{T} : \mathcal{G}_n \longrightarrow \mathcal{G}_n$ the inverse of $\mathcal{K}_{|\mathcal{G}_n}$. Then λ^{eq} is given by:

$$\forall\, D \in \mathcal{G},\ \lambda^{\mathrm{eq}} : D = \lambda^{\mathrm{e}} : \big(D - \mathcal{T}(\mathcal{K}(D))\big). \tag{18}$$

We remark that the resulting model (17) is indeed two-dimensional, but it does not describe the behavior of the normal deformations. We remark also that in this regime, the magnetization is not influenced by the normal deformations, so that our model is closed.

This paper is organized as follows: Theorem 1.3 is proved in section 2. We follows the method due to Alouges and Soyeur [2] and generalized in [9]. First, we prove the existence of solution for a penalized system, in which the saturation constraint is relaxed. Then, we take the limit when the penalization constant tends toward zero. In [8], global existence for (5)-(9) is obtained in the case of a clamped sample, that is with $u = 0$ on $\partial\Omega$. Our proof is very close to the proof in [8]. We reproduce it for the convenience of the reader in order to present a self-contained paper.

Theorem 1.4 is established in section 3. In order to avoid working on a domain depending on η, we perform a rescaling inspired both by [7] for the magnetization and by [12] for the deformation. The thickness parameter η appears then as a stiff term in the equations. Nevertheless, we are able to obtain a limit model thanks to the energy inequality and by choosing convenient test functions in the rescaled weak formulation.

2. Weak solutions for the Landau-Lifschitz-Gilbert equation with magnestostriction

Let $m_0 \in H^1(\Omega)$ with values in the unit sphere S^2, $u_0 \in V(\Omega)$, $u_1 \in L^2(\Omega)$ and $f \in L^2(\Gamma_2)$.

2.1. Penalized system

As in [2] and [8], we relax the saturation contraint and for $\zeta > 0$, we consider the following penalized system:

$$\begin{cases} \dfrac{\partial m^\zeta}{\partial t} + m^\eta \times \dfrac{\partial m^\zeta}{\partial t} - 2H^\zeta_{\text{eff}} + \dfrac{1}{\zeta}(|m^\zeta|^2 - 1)m^\zeta = 0 \text{ in } \mathbb{R}^+ \times \Omega, \\ H^\zeta_{\text{eff}} = \Delta m^\zeta + h_d(m^\zeta) + \Psi(m^\zeta) + (\lambda^{\text{m}} : \sigma^\zeta)m^\zeta, \\ \sigma^\zeta = \lambda^{\text{e}} : \varepsilon(u^\zeta) - \lambda^{\text{e}} : (\lambda^{\text{m}} : m^\zeta \otimes m^\zeta), \\ \dfrac{\partial^2 u^\zeta}{\partial t^2} - \text{div } \sigma^\zeta = 0 \text{ in } \mathbb{R}^+ \times \Omega, \\ m^\zeta(t=0) = m_0, \quad u^\zeta(t=0) = u_0, \quad \dfrac{\partial u^\zeta}{\partial t}(t=0) = u_1, \\ \partial_{\mathbf{n}} m^\zeta = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega, \quad u^\zeta(t,x) = 0 \text{ on } \mathbb{R}^+ \times \Gamma_1, \quad \sigma^\zeta.n = f \text{ on } \mathbb{R}^+ \times \Gamma_2. \end{cases} \tag{19}$$

For a fixed penalization parameter $\zeta > 0$, we construct by Galerkine method a weak solution for (19) such that

(1) $m^\zeta \in L^\infty(\mathbb{R}^+; H^1(\Omega;\mathbb{R}^3))$ and $\dfrac{\partial m^\zeta}{\partial t} \in L^2(\mathbb{R}^+; L^2(\Omega;\mathbb{R}^3))$,

(2) $u^\zeta \in L^\infty(\mathbb{R}^+; V(\Omega)))$ and $\dfrac{\partial u^\zeta}{\partial t} \in L^\infty(\mathbb{R}^+; L^2(\Omega;\mathbb{R}^3))$,

(3) for all $t \geq 0$, we have the following energy inequality:

$$\mathcal{E}^\zeta(t) + \frac{1}{4\zeta}\int_\Omega \Big(|m^\zeta(t,x)|^2 - 1\Big)^2 dx + \int_0^t \int_\Omega \left|\frac{\partial m^\zeta(\tau,x)}{\partial t}\right|^2 d\tau dx \leq \mathcal{E}^\zeta(0)$$

with $\mathcal{E}^\zeta(t) = \mathcal{E}(m^\zeta(t,\,\cdot\,), u^\zeta(t,\,\cdot\,), \dfrac{\partial u^\zeta}{\partial t}(t,\,\cdot\,))$.

We recall that since the surface measure of Γ_1 is positive, the Korn inequality below is valid (see [11], Theorem 6.3-4 page 292): there exists a constant $c(\Omega)$ such that for all $v \in V(\Omega)$,

$$\int_\Omega \varepsilon(v) : \varepsilon(v) \geq c(\Omega) \int_\Omega |v|^2. \tag{20}$$

This inequality yields that $\|\cdot\|_V = \big(\int_\Omega \varepsilon(\,\cdot\,) : \varepsilon(\,\cdot\,)\big)^{\frac{1}{2}}$ is a norm on $V(\Omega)$ equivalent to the norm $\|\cdot\|_{H^1(\Omega)} = \big(\int_\Omega |\cdot|^2 + |\nabla\cdot|^2\big)^{\frac{1}{2}}$, and then it is also equivalent to the norm $|\cdot|_{1,\Omega} = \big(\int_\Omega |\nabla\cdot|^2\big)^{\frac{1}{2}}$.

2.1.1. First step: Galerkin approximation

For m, we use a Galerkin basis $(e_1, e_2, \ldots)$ of eigenvectors of $-\Delta$ with homogeneous Neumann conditions at the boundary.

$$\begin{cases} -\Delta e_i = \alpha_i e_i \text{ in } \Omega, \\ \partial_{\mathbf{n}} e_i = 0 \text{ on } \partial\Omega. \end{cases}$$

We denote by V_N =span$(e_1, \ldots, e_N)$ and by P_N the orthogonal projection map onto V_N.

For u, we use a the Galerkin basis $(\phi_1, \phi_2, \ldots)$ of eigenvectors of $-\text{div}\,(\lambda^{\text{e}} : \varepsilon)$ with homogeneous mixed conditions at the boundary:

$$\begin{cases} -\text{div}\,(\lambda^{\text{e}} : \varepsilon(\phi_j)) = b_j \phi_j \text{ in } \Omega, \\ \phi_j = 0 \text{ on } \Gamma_1, \\ (\lambda^{\text{e}} : \varepsilon(\phi_j)) \cdot n = 0 \text{ on } \Gamma_2. \end{cases} \tag{21}$$

We denote by W_N =span$(\phi_1, \ldots, \phi_N)$ and by Π_N the orthogonal projection map into W_N.

We consider for N fixed, the following Galerkin problem: Find $m_N^\zeta : [0, T_N[\to V_N$ and $u_N^\zeta : [0, T_N[\to W_N$, such that $\forall\, g_1 \in V_N$ and $g_2 \in W_N$, we have

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega m_N^\zeta \cdot g_1 + \int_\Omega m_N^\zeta \times \frac{\partial m_N^\zeta}{\partial t} \cdot g_1 - 2\int_\Omega H_{\text{eff}}^N \cdot g_1 + \frac{1}{\zeta}\int_\Omega (|m_N^\zeta|^2 - 1) m_N^\zeta \cdot g_1 = 0 \text{ in } \mathbb{R}^+, \\ H_{\text{eff}}^N = \Delta m_N^\zeta + h_d(m_N^\zeta) + \Psi(m_N^\zeta) + (\lambda^{\text{m}} : \sigma_N^\zeta) m_N^\zeta, \\ \sigma_N^\zeta = \lambda^{\text{e}} : \varepsilon_N^{\text{e},\zeta} \text{ with } \varepsilon_N^{\text{e},\zeta} = \varepsilon(u_N^\zeta) - \lambda^{\text{m}} : m_N^\zeta \otimes m_N^\zeta, \\ \dfrac{d^2}{dt^2}\displaystyle\int_\Omega u_N^\zeta \cdot g_2 + \int_\Omega (\lambda^{\text{e}} : \sigma_N^\zeta) : \varepsilon(g_2) - \int_{\Gamma_2} f \cdot g_2 = 0 \text{ in } \mathbb{R}^+, \\ m_N^\zeta(t=0) = P_N(m_0), \quad u_N^\zeta(t=0) = \Pi_N(u_0), \quad \dfrac{\partial u_N^\zeta}{\partial t}(t=0) = \Pi_N(u_1). \end{cases} \tag{22}$$

By the Cauchy-Lipschitz theorem, as in subsection 2.1 in [8], there exists a unique solution (m_N^ζ, u_N^ζ) for (22) whose maximal existence time is denoted by T_N.

2.1.2. Energy estimate on the Galerkin approximation

We denote by $\mathcal{E}_N^\zeta(t)$ the following quantity:

$$\begin{aligned} \mathcal{E}_N^\zeta(t) := \;& \mathcal{E}\Big(m_N^\zeta(t,\cdot), u_N^\zeta(t,\cdot), \frac{\partial u_N^\zeta}{\partial t}(t,\cdot)\Big) \\ =\;& \frac{1}{2}\int_\Omega |\nabla m_N^\zeta(t,x)|^2 dx + \int_\Omega \Phi(m_N^\zeta(t,x)) dx + \frac{1}{2}\int_{\mathbb{R}^3} |h_d(m_N^\zeta(t,\cdot))(x)|^2 dx \\ +\;& \frac{1}{4}\int_\Omega \Big[\lambda^{\text{e}} : \varepsilon_N^{\text{e},\zeta}(t,x)\Big] : \varepsilon_N^{\text{e},\zeta}(t,x) dx + \frac{1}{4}\int_\Omega |\frac{\partial u_N^\zeta}{\partial t}(t,x)|^2 dx \\ -\;& \frac{1}{2}\int_{\Gamma_2} f(s) \cdot u_N^\zeta(t,s)\, d\Gamma_s. \end{aligned}$$

Using the symmetry of λ^{e}, we have:

$$\frac{d}{dt}\mathcal{E}_N^\zeta = \int_\Omega \nabla m_N^\zeta : \nabla \frac{\partial m_N^\zeta}{\partial t} + \int_\Omega \nabla \Phi(m_N^\zeta) \cdot \frac{\partial m_N^\zeta}{\partial t} + \int_{\mathbb{R}^3} h_d(m_N^\zeta) \cdot h_d(\frac{\partial m_N^\zeta}{\partial t})$$
$$+\frac{1}{2}\int_\Omega (\lambda^{\mathrm{e}} : \varepsilon_N^{\mathrm{e},\zeta}) : \frac{\partial \varepsilon_N^{\mathrm{e},\zeta}}{\partial t} + \frac{1}{2}\int_\Omega \frac{\partial^2 u_N^\zeta}{\partial t^2} \cdot \frac{\partial u_N^\zeta}{\partial t} - \frac{1}{2}\int_{\Gamma_2} f \cdot \frac{\partial u_N^\zeta}{\partial t}.$$

Since $-h_d$ is an orthogonal projection for the $L^2(\mathbb{R}^3)$-inner product, we have:

$$\int_{\mathbb{R}^3} h_d(m_N^\zeta) \cdot h_d(\frac{\partial m_N^\zeta}{\partial t}) = -\int_\Omega h_d(m_N^\zeta) \cdot \frac{\partial m_N^\zeta}{\partial t}.$$

In addition, using the symmetry of λ^{m}, we have:

$$\frac{\partial \varepsilon_N^{\mathrm{e},\zeta}}{\partial t} = \varepsilon(\frac{\partial u_N^\zeta}{\partial t}) - 2\lambda^{\mathrm{m}} : m_N^\zeta \otimes \frac{\partial m_N^\zeta}{\partial t}.$$

Using that $\sigma_N^\zeta = \lambda^{\mathrm{e}} : \varepsilon_N^{\mathrm{e},\zeta}$, by the symmetry of λ^{e} and λ^{m}, we have:

$$\sigma_N^\zeta : (\lambda^{\mathrm{m}} : m_N^\zeta \otimes \frac{\partial m_N^\zeta}{\partial t}) = \left[(\lambda^{\mathrm{m}} : \sigma_N^\zeta) m_N^\zeta\right] \cdot \frac{\partial m_N^\zeta}{\partial t}.$$

Therefore, we have:

$$\frac{d}{dt}\mathcal{E}_N^\zeta = \int_\Omega \nabla m_N^\zeta : \nabla \frac{\partial m_N^\zeta}{\partial t} - \int_\Omega \Psi(m_N^\zeta) \cdot \frac{\partial m_N^\zeta}{\partial t} - \int_\Omega h_d(m_N^\zeta) \cdot \frac{\partial m_N^\zeta}{\partial t}$$
$$-\int_\Omega \left[(\lambda^{\mathrm{m}} : \sigma_N^\zeta) m_N^\zeta\right] \cdot \frac{\partial m_N^\zeta}{\partial t} + \frac{1}{2}\int_\Omega \frac{\partial^2 u_N^\zeta}{\partial t^2} \cdot \frac{\partial u_N^\zeta}{\partial t} + \frac{1}{2}\int_\Omega \sigma_N^\zeta : \varepsilon(\frac{\partial u_N^\zeta}{\partial t})$$
$$-\frac{1}{2}\int_{\Gamma_2} f \cdot \frac{\partial u_N^\zeta}{\partial t}.$$

Taking $g_1 = \frac{\partial m_n^\zeta}{\partial t}$ and $g_2 = \frac{\partial u_N^\zeta}{\partial t}$ in (22), we obtain that

$$\frac{d}{dt}\left(\mathcal{E}_N^\zeta + \frac{1}{4\zeta}\int_\Omega (|m_N^\zeta|^2 - 1)^2\right) + \int_\Omega \left|\frac{\partial m_N^\zeta}{\partial t}\right|^2 = 0,$$

and integrating from $t = 0$ to $t = T$, we get that for all $T < T_N$

$$\mathcal{E}_N^\zeta(T) + \frac{1}{4\zeta}\int_\Omega (|m_N^\zeta|^2 - 1)^2 + \int_0^T \int_\Omega \left|\frac{\partial m_N^\zeta}{\partial t}\right|^2 = \mathcal{E}_N^\zeta(0) + \frac{1}{4\zeta}\int_\Omega (|P_N(m_0)|^2 - 1)^2, \quad (23)$$

We establish the following lemma:

Lemma 2.1. *There exists a constant C such that for all $u \in V(\Omega)$ and for all $m \in H^1(\Omega;\mathbb{R}^3)$,*

$$\frac{1}{2}\int_\Omega [\lambda^{\mathrm{e}} : (\varepsilon(u) - \lambda^{\mathrm{m}} : m\otimes m)] : (\varepsilon(u) - \lambda^{\mathrm{m}} : m\otimes m) \geq \frac{\lambda^{\mathrm{e}*}}{4}\|\varepsilon(u)\|^2_{L^2(\Omega)} - C\|m\|^4_{L^4(\Omega)}.$$

where $\lambda^{\mathrm{e}*}$ is the coercivity constant appearing in (7).

Proof. From (7), we have:

$$\begin{aligned}
[\lambda^{\mathrm{e}} : (\varepsilon(u) - \lambda^{\mathrm{m}} : m\otimes m)] : (\varepsilon(u) - \lambda^{\mathrm{m}} : m\otimes m) &\geq \lambda^{\mathrm{e}*}\left|\varepsilon(u) - \lambda^{\mathrm{m}} : m\otimes m\right|^2, \\
&\geq \lambda^{\mathrm{e}*}\left||\varepsilon(u)| - |\lambda^{\mathrm{m}} : m\otimes m|\right|^2, \\
&\geq \tfrac{\lambda^{\mathrm{e}*}}{2}|\varepsilon(u)|^2 - \lambda^{\mathrm{e}*}|\lambda^{\mathrm{m}} : m\otimes m|^2
\end{aligned}$$

using that $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$. By integrating on Ω and using that there exists a constant K such that $|\lambda^{\mathrm{m}} : m\otimes m|^2 \leq K|m|^4$, we conclude the proof of Lemma 2.1. □

By the previous lemma, we obtain that

$$\begin{aligned}
\frac{1}{2}\int_\Omega \left[\lambda^{\mathrm{e}} : (\varepsilon(u_N^\zeta) - \lambda^{\mathrm{m}} : m_N^\zeta\otimes m_N^\zeta)\right] : (\varepsilon(u_N^\zeta) - \lambda^{\mathrm{m}} : m_N^\zeta\otimes m_N^\zeta) \\
\geq \frac{1}{2}\lambda^{\mathrm{e}*}\|\varepsilon(u_N^\zeta)\|^2_{L^2(\Omega)} - C\|m_N^\zeta\|^4_{L^4(\Omega)}.
\end{aligned}$$

Moreover, since the trace application $H^1(\Omega)\to H^{\frac{1}{2}}(\Gamma)$ is linear continuous and with the Cauchy-Schwarz inequality, there exist $c<0$ such that

$$\int_{\Gamma_2} f\cdot u_N^\zeta d\Gamma \leq c\|f\|_{L^2(\Gamma_2)} \parallel u_N^\zeta \parallel_{H^1(\Omega)}.$$

Thus, if ζ is small enough, by using of $(|\xi|^2-1)^2 \geq \frac{1}{2}|\xi|^4 - 1$, we obtain that

$$\begin{aligned}
\mathcal{E}_N^\zeta + \frac{1}{4\zeta}\int_\Omega (|m_N^\zeta|^2-1)^2 \geq \|\nabla m_N^\zeta\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\frac{\partial u_N^\zeta}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{1}{4}\lambda^{\mathrm{e}*}\|\varepsilon(u_N^\zeta)\|^2_{L^2(\Omega)} \\
+\frac{1}{8\zeta}\int_\Omega (|m_N^\zeta|^2-1)^2 - 2C\mathrm{mes}(\Omega) - c\|f\|_{L^2(\Gamma_2)}\|\varepsilon(u_N^\zeta)\|_{L^2(\Omega)}.
\end{aligned} \tag{24}$$

2.1.3. Limit in the Galerkin Approximation

Let us prove that the right-hand side of the energy estimate (23) is uniformly bounded with respect to N. We will follow the calculation made in [8] section 2.3 and the following.

Take the term $\mathcal{E}_N^{\zeta}(0)$, then the following calculation holds

$$\begin{aligned}\int_\Omega (\lambda^{\mathrm{e}} : \varepsilon(\Pi_N(u_0))) : \varepsilon(\Pi_N(u_0)) = & -\int_\Omega \mathrm{div}\ (\lambda^{\mathrm{e}} : \varepsilon(\Pi_N(u_0)))\Pi_N(u_0) \\ & +\int_{\partial\Omega} (\lambda^{\mathrm{e}} : \varepsilon(\Pi_N(u_0)))\Pi_N(u_0)\cdot n \\ = & -\int_\Omega \mathrm{div}\ \left(\lambda^{\mathrm{e}} : \varepsilon(\Pi_N(u_0))\right)(u_0),\end{aligned}$$

since $\Pi_N(u_0) \in W_N = \mathrm{spam}(\phi_1, \ldots, \phi_N)$ and ϕ_j satisfy (21), then $\Pi_N(u_0) = 0$ on Γ_1 and $(\lambda^{\mathrm{e}} : \varepsilon(\Pi_N(u_0)))n = 0$ on Γ_2. Furthermore, W_N is stable by $v \mapsto \mathrm{div}\ (\lambda^{\mathrm{e}} : \varepsilon(v))$ because $-\mathrm{div}\ (\lambda^{\mathrm{e}} : \varepsilon(\phi_j)) = -b_j\phi_j$, and Π_N is self-adjoint. Then, by Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\int_\Omega (\lambda^{\mathrm{e}} : \varepsilon(\Pi_N(u_0))) : & \quad \varepsilon(\Pi_N(u_0)) = \int_\Omega (\lambda^{\mathrm{e}} : \varepsilon(\Pi_N(u_0))) : \varepsilon(u_0) \\ & \leq \int_\Omega ((\lambda^{\mathrm{e}} : \varepsilon(\Pi_N(u_0))) : \varepsilon(\Pi_N(u_0)))^{\frac{1}{2}} \cdot ((\lambda^{\mathrm{e}} : \varepsilon(u_0)) : \varepsilon(u_0))^{\frac{1}{2}} \\ & \leq \left(\int_\Omega (\lambda^{\mathrm{e}} : \varepsilon(\Pi_N(u_0))) : \varepsilon(\Pi_N(u_0))\right)^{\frac{1}{2}} \left(\int_\Omega (\lambda^{\mathrm{e}} : \varepsilon(u_0)) : \varepsilon(u_0)\right)^{\frac{1}{2}}.\end{aligned}$$

This last inequality implies

$$\int_\Omega (\lambda^{\mathrm{e}} : \varepsilon(\Pi_N(u_0)))\, \varepsilon(\Pi_N(u_0)) \leq \int_\Omega (\lambda^{\mathrm{e}} : \varepsilon(u_0))\, \varepsilon(u_0). \tag{25}$$

With (25), we prove the density of $\underset{N\geq 1}{\cup} W_N$ in $V(\Omega)$. For this, if $v \in V(\Omega)$, then $\Pi_N(v)$ is also bounded in $V(\Omega)$ since $f_j = 0$ on Γ_1. In addition,

$$\int_\Omega (\lambda^{\mathrm{e}} : \varepsilon(\Pi_N(v))) : \varepsilon(\Pi_N(v)) \leq \int_\Omega (\lambda^{\mathrm{e}} : \varepsilon(v)) : \varepsilon(v),$$

so $(\varepsilon(\Pi_N(v)))_N$ is bounded in $L^2(\Omega)$, hence $\varepsilon(\Pi_N(v)) \rightharpoonup \varepsilon(v)$ weakly in $L^2(\Omega)$, because $\Pi_N(v) \to v$ strongly in $L^2(\Omega)$. Then

$$\overline{\lim} \int_\Omega \varepsilon(\Pi_N(v)) : \varepsilon(\Pi_N(v)) \leq \int_\Omega \varepsilon(v) : \varepsilon(v) \leq \underline{\lim} \int_\Omega \varepsilon(\Pi_N(v)) : \varepsilon(\Pi_N(v)),$$

then,

$$\underset{N\to+\infty}{lim} \int_\Omega \varepsilon(\Pi_N(v)) : \varepsilon(\Pi_N(v)) = \int_\Omega \varepsilon(v) : \varepsilon(v).$$

It follows that there exists a positive constant K^{ζ}, uniformly bounded with respect to N of the right side of (23). Hence, and from (24) together with the energy estimate

(23), we obtain that for ζ small enough

$$\begin{aligned}&\|\nabla m_N^\zeta\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\frac{\partial u_N^\zeta}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{1}{4}\lambda^{e*}\|\varepsilon(u_N^\zeta)\|^2_{L^2(\Omega)} + \frac{1}{8\zeta}\int_\Omega (|m_N^\zeta|^2-1)^2 \\ &-c\|g\|_{L^2(\Gamma_2)}\|\varepsilon(u_N^\zeta)\|_{L^2(\Omega)} + \int_0^T\int_\Omega \left|\frac{\partial m_N^\zeta}{\partial t}\right|^2 \leq K^\zeta + 2C\text{mes}(\Omega).\end{aligned} \tag{26}$$

The uniform bound (26) implies uniform bounds of the following quantities:

- $\dfrac{\partial m_N^\zeta}{\partial t}$ in $L^2(0,T_N;L^2(\Omega))$,
- ∇m_N^ζ in $L^\infty(0,T_N;L^2(\Omega))$,
- m_N^ζ in $L^\infty(0,T_N;L^4(\Omega))$,
- $\dfrac{\partial u_N^\zeta}{\partial t}$ in $L^\infty(0,T_N;L^2(\Omega))$,
- $\varepsilon(u_N^\zeta)$ in $L^\infty(0,T_N;L^2(\Omega))$.

This implies directly that $T_N = +\infty$. In addition, since the bounds do not depend on N, we can assume that there exists a sub-sequence of $(m_N^\zeta, u_N^\zeta)_N$, still noted $(m_N^\zeta, u_N^\zeta)_N$, and there exist (m^ζ, u^ζ) such that for all $T>0$:

- $m_N^\zeta \rightharpoonup m^\zeta$ in $L^\infty(0,T;H^1(\Omega))$ weak $*$,
- $m_N^\zeta \to m^\zeta$ in $L^\infty(0,T;L^4(\Omega))$ strong (by applying the Aubin-Simon lemma [3] Theorem II.5.16),
- $\dfrac{\partial m_N^\zeta}{\partial t} \rightharpoonup \dfrac{\partial m^\zeta}{\partial t}$ in $L^2(0,T;L^2(\Omega))$ weak,
- $u_N^\zeta \rightharpoonup u^\zeta$ in $L^\infty(0,T;V(\Omega))$ weak $*$,
- $\dfrac{\partial u_N^\zeta}{\partial t} \rightharpoonup \dfrac{\partial u^\zeta}{\partial t}$ in $L^\infty(0,T;L^2(\Omega))$ weak $*$.

Let us study the limit in the initial condition. On the one hand, using Aubin-Simon Lemma [3], we have

$$m_N^\zeta \to m^\zeta \text{ and } u_N^\zeta \to u^\zeta \text{ in } \mathcal{C}^0([0,T];L^2(\Omega)).$$

On the other hand, using Hilbert basis properties, we have

$$P_N(m_0) \to m_0 \text{ and } \Pi_N(u_0) \to u_0 \text{ in } L^2(\Omega).$$

Therefore, by the uniqueness of the limit,

$$m^\zeta(t=0) = m_0 \text{ and } u^\zeta(t=0) = u_0.$$

By the third equation of (22), $\frac{\partial^2 u_N^\zeta}{\partial t^2}$ is bounded uniformly with respect to N in $L^\infty(0,T;L^2(\Omega))$, then $(\frac{\partial u_N^\zeta}{\partial t})_N$ is bounded in $W^{1,\infty}(0,T;L^2(\Omega))$, so there exists a sub-

sequence, still noted $(\frac{\partial u_N^\zeta}{\partial t})_N$, such that

$$\frac{\partial u_N^\zeta}{\partial t} \rightharpoonup \frac{\partial u^\zeta}{\partial t} \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak-*}.$$

The application tr defined by

$$\begin{array}{rccc} tr: & W^{1,\infty}(0,T;L^2(\Omega)) & \longrightarrow & L^2(\Omega) \\ & h & \longmapsto & h(0) \end{array}$$

is linear continuous for the weak $*$ topology, then

$$\frac{\partial u_N^\zeta}{\partial t}(t=0) \to \frac{\partial u^\zeta}{\partial t}(t=0) \text{ in } L^2(\Omega).$$

Additionally, we have

$$\Pi_N(u_1) \to u_1 \text{ in } L^2(\Omega).$$

Therefore

$$\frac{\partial u^\zeta}{\partial t}(t=0) = u_1.$$

Finally, taking the limit of the Galerkin approximation (22) and the lower limit in the energy formula (23), we obtain, for a fixed ζ small enough, that the weak limit (m^ζ, u^ζ) is a weak solution of the penalized system (19) and satisfies the following energy formula

$$\mathcal{E}^\zeta(t) + \frac{1}{4\zeta}\int_\Omega (|m^\zeta|-1)^2 + \int_0^t \int_\Omega \left|\frac{\partial m^\zeta}{\partial t}\right|^2 \leq \mathcal{E}^\zeta(0), \tag{27}$$

where $\mathcal{E}^\zeta$ is defined by $\mathcal{E}^\zeta(t) = \mathcal{E}(m^\zeta(t,\,\cdot\,), u^\zeta(t,\,\cdot\,), \frac{\partial u^\zeta}{\partial t}(t,\,\cdot\,))$.

2.2. Limit when ζ tends to zero

We remark that the right side of the estimate of energy (27) does not depend on ζ, because the initial data m_0 satisfies $|m_0| = 1$ a.e., then, by using the same arguments as in the previous section, we obtain uniform bounds with respect to ζ for the following quantities:

- $\frac{\partial m^\zeta}{\partial t}$ in $L^2(\mathbb{R}^+;L^2(\Omega))$,
- ∇m^ζ in $L^\infty(\mathbb{R}^+;L^2(\Omega))$,
- m^ζ in $L^\infty(\mathbb{R}^+;L^4(\Omega))$,
- $\varepsilon(u^\zeta)$ in $L^\infty(\mathbb{R}^+;L^2(\Omega))$,
- $\frac{\partial u^\zeta}{\partial t}$ in $L^\infty(\mathbb{R}^+;L^2(\Omega))$.

Therefore, there exists a subsequence still noted $(m^\zeta, u^\zeta)_\zeta$, such that for all $T > 0$,

- $m^\zeta \rightharpoonup m$ in $L^\infty(0,T;H^1(\Omega))$ weak $*$,
- $m^\zeta \to m$ in $L^\infty(0,T;L^4(\Omega))$ strong (by applying the Aubin-Simon lemma [3] Theorem II.5.16),
- $\dfrac{\partial m^\zeta}{\partial t} \rightharpoonup \dfrac{\partial m}{\partial t}$ in $L^2(0,T;L^2(\Omega))$ weak,
- $u^\zeta \rightharpoonup u$ in $L^\infty(0,T;V(\Omega))$ weak $*$,
- $\dfrac{\partial u^\zeta}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t}$ in $L^\infty(0,T;L^2(\Omega))$ weak $*$.

If we apply the same calculation as in (26) and (27), we obtain that $\dfrac{1}{4\zeta}\displaystyle\int_\Omega (|m^\zeta|^2-1)^2$ is bounded uniformly with respect to ζ, so $\displaystyle\int_\Omega (|m^\zeta|^2-1)^2 \to 0$ as ζ tends to zero, and since $m^\zeta \to m$ in $L^\infty(0,T;L^4(\Omega))$ strong, then we obtain that m satisfies the saturation constraint (10).

Now, using the strong and weak convergence and by taking the limit of (27) as $\zeta \to 0$, we obtain that for all $T > 0$

$$\mathcal{E}(m(T), u(T), \frac{\partial u}{\partial t}(t)) + \int_0^T \int_\Omega \left| \frac{\partial m}{\partial t}(s,x) \right|^2 dx \leq \mathcal{E}(m_0, u_0, u_1). \tag{28}$$

In order to show that (m,u) satisfies the Landau-Lifschitz-Gilbert equation, we take the inner product of the first equation of (19) and the test function $(t,x) \mapsto m^\zeta(t,x) \times \chi(t,x)$, where $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^+; H^1(\Omega))$ compactly supported in $[0,T[$ (as in [8], [2] and [9]), then by making the limit as ζ goes to zero, we obtain

$$\begin{aligned}\int_{\mathbb{R}^+\times\Omega} \left(\frac{\partial m}{\partial t} - m \times \frac{\partial m}{\partial t}\right)\chi(t,x)dt\,dx = 2\int_{\mathbb{R}^+\times\Omega} \sum_{i=1}^3 m \times \frac{\partial m}{\partial x_i} \cdot \frac{\partial \chi}{\partial x_i} \\ -2\int_{\mathbb{R}^+\times\Omega} m \times (h_d(m) + \Psi(m)) \cdot \chi - 2\int_{\mathbb{R}^\times+\Omega} m \times \Big((\lambda^{\mathrm{m}} : (\lambda^{\mathrm{e}} : \varepsilon(u)))m\Big) \cdot \chi \\ -2\int_{\mathbb{R}^\times+\Omega} m \times \Big((\lambda^{\mathrm{m}} : (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : m \otimes m)))m\Big) \cdot \chi.\end{aligned}$$

Furthermore, if we take the inner product of the third equation of (19), and a test function $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^+; V(\Omega))$, and by passing to the limit as ζ tends to zero, we obtain

$$\begin{aligned}\int_{\mathbb{R}^+\times\Omega} \frac{\partial u}{\partial t} \cdot \frac{\partial \chi}{\partial t} - \int_{\mathbb{R}^+\times\Omega} (\lambda^{\mathrm{e}} : \varepsilon(u)) : \varepsilon(\chi) - \int_\Omega u_1 \chi(0,x)dx + \int_{\mathbb{R}^+\times\Gamma_2} f \cdot \chi \\ = -\int_{\mathbb{R}^+\times\Omega} (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : m \otimes m)) : \varepsilon(\chi).\end{aligned}$$

Consequently, (m,u), is a global in time weak solution of (5)-(6)-(9), which concludes the proof of Theorem 1.3 .

3. Asymptotic model for plates: proof of Theorem 1.4

Let ω be a regular bounded domain in $\mathbb{R}^2$. We assume that its boundary $\partial\omega$ is divided in two parts of empty intersection: $\partial\omega = \mathcal{C}_1 \cup \mathcal{C}_2$, such that the one dimensional measure of $\mathcal{C}_1$ is positive.
Given $\eta > 0$, we denote:

- $\Omega_\eta = \omega \times]-\eta, \eta[$,
- $\Gamma_1^\eta = \mathcal{C}_1 \times]-\eta, \eta[$,
- $\Gamma_b^\eta = \mathcal{C}_2 \times]-\eta, \eta[$, $\quad \Gamma_-^\eta = \bar{\omega} \times \{-\eta\}$, $\quad \Gamma_+^\eta = \bar{\omega} \times \{\eta\}$, $\quad \Gamma_2^\eta = \Gamma_b^\eta \cup \Gamma_-^\eta \cup \Gamma_+^\eta$.

We recall the definitions of the following spaces:

$$V(\Omega_\eta) = \{v \in H^1(\Omega_\eta)\ ;\ v = 0 \text{ on } \Gamma_1^\eta\},$$

$$V(\Omega_1) = \{v \in H^1(\Omega_1)\ ;\ v = 0 \text{ on } \Gamma_1^1\},$$

$$\mathcal{V}(\omega) = \{v \in H^1(\omega)\ ;\ v = 0 \text{ on } \mathcal{C}_1\},$$

$$\mathcal{W}(\omega) = \big\{v = (v_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)\,;\, v_i = 0 \text{ on } \mathcal{C}_1 \text{ and } \partial_1 v_3 = \partial_2 v_3 = 0 \text{ on } \mathcal{C}_1\big\},$$

$$V_{KL}(\Omega_1) = \big\{\xi \in H^1(\Omega_1)\,;\, \xi = 0 \text{ on } \Gamma_1^1 \text{ and } \varepsilon_{i3}(\xi) = 0 \text{ in } \Omega_1 \text{ for } i \in \{1,2,3\}\big\}.$$

We recall without proof the following result, proved in details in [10] (of Th. 1.4.1.).

Lemma 3.1. *The application I defined as follows:*

$$\begin{array}{rcl} I : \mathcal{W}(\omega) & \longrightarrow & V_{KL}(\Omega_1) \\ v & \longmapsto & \big((v_\alpha - x_3\partial_\alpha v_3), v_3\big), \end{array}$$

is an isomorphism.

Let $m_0 \in H^1(\omega\,; S^2(0,1))$, $\tilde{u}_0 \in \mathcal{W}(\omega)$ and $u_1 \in L^2(\omega; \mathbb{R}^3)$ such that $(u_1)_3 = 0$. We define $m_0^\eta \in H^1(\Omega_\eta; S^2)$, $u_0^\eta \in V(\Omega_\eta)$ and $u_1^\eta \in L^2(\Omega_\eta; \mathbb{R}^3)$ by:

$$\left\{\begin{array}{l} m_0^\eta(x_T, x_3) = m_0(x_T), \\ \\ \forall\, \alpha \in \{1,2\},\ u_{0,\alpha}^\eta(x_T, x_3) = \tilde{u}_{0,\alpha}(x_T) - \dfrac{x_3}{\eta}\partial_\alpha \tilde{u}_{0,3}(x_T), \\ \\ u_{0,3}^\eta(x_T, x_3) = \frac{1}{\eta}\tilde{u}_{0,3}(x_T), \\ \\ u_1^\eta(x_T, x_3) = u_1(x_T). \end{array}\right. \tag{29}$$

Let g^+ and $g^- \in L^2(\omega)$ and $h \in L^2(\mathcal{C}_2)$. From these data, we define on Γ_2^η the

η-depending boundary data f^η by:

$$\begin{cases} \forall\, x_T \in \omega,\ \forall\, \alpha \in \{1,2\},\ f^\eta_\alpha(x_T, \pm\eta) = g^\pm_\alpha(x_T) \quad \text{and} \quad f^\eta_3(x_T, \pm\eta) = \eta^2 g^\pm_3(x_T), \\ \forall\, (x_T, x_3) \in \mathcal{C}_2 \times]-\eta, \eta[,\ \forall \alpha \in \{1,2\},\ f^\eta_T(x_T, x_3) = h_T(x_T), \\ \forall\, (x_T, x_3) \in \mathcal{C}_2 \times]-\eta, \eta[,\ \forall \alpha \in \{1,2\},\ f^\eta_3(x_T, x_3) = \eta h_3(x_T). \end{cases} \tag{30}$$

We consider then the solution for the initial and boundary value problem (5)-(6)-(9) with these initial and boundary data, and we define the rescaled variables M^η, U^η and H^η by:

$$\begin{aligned} &M^\eta(t, x_T, x_3) = m^\eta(t, x_T, \eta x_3), \quad H^\eta(t, x_T, x_3) = (h_d(m^\eta))(t, x_T, \eta x_3), \\ &U^\eta_T(t, x_T, x_3) = u^\eta_T(t, x_T, \eta x_3), \quad U^\eta_3(t, x_T, x_3) = \eta u^\eta_3(t, x_T, \eta x_3). \end{aligned} \tag{31}$$

We define the 2-tensor $\varepsilon(\eta, \xi)$ for $\xi \in \mathcal{C}^\infty(\mathbb{R}^+; V(\Omega_1))$ by

$$\begin{aligned} \varepsilon_{\alpha\beta}(\eta, \xi) &= \varepsilon_{\alpha\beta}(\xi) \text{ for } \alpha, \beta \in \{1,2\}, \\ \varepsilon_{\alpha 3}(\eta, \xi) &= \frac{1}{\eta}\varepsilon_{\alpha 3}(\xi) \text{ for } \alpha \in \{1,2\}, \\ \varepsilon_{33}(\eta, \xi) &= \frac{1}{\eta^2}\varepsilon_{33}(\xi), \end{aligned} \tag{32}$$

So that we have:

$$\varepsilon(u^\eta) = \varepsilon(\eta, U^\eta). \tag{33}$$

Now we rewrite the properties satisfied by m^η and u^η (see Definition 1.2) for the rescaled variables. We obtain that:

- $M^\eta \in L^\infty(\mathbb{R}^+; H^1(\Omega_1; \mathbb{R}^3))$, $\dfrac{\partial M^\eta}{\partial t} \in L^2(\mathbb{R}^+; L^2(\Omega_1; \mathbb{R}^3))$, and $|M^\eta(t,x)| = 1$ for a.e. $(t,x) \in \mathbb{R}^+ \times \Omega_1$,
- $M^\eta(0, x_1, x_2, x_3) = m^\eta_0(0, x_1, x_2, \eta x_3) = m_0(x_1, x_2)$ in the trace sense,
- $U^\eta \in L^\infty(\mathbb{R}^+; V(\Omega_1))$ and $\dfrac{\partial U^\eta}{\partial t} \in L^\infty(\mathbb{R}^+; L^(\Omega_1))$,
- $U^\eta(0, x_1, x_2, x_3) = u_0(x_1, x_2)$ in the trace sense,
- for all $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^+; H^1(\Omega_1; \mathbb{R}^3))$, we define $\chi^\eta \in \mathcal{C}^\infty_c(\mathbb{R}^+; H^1(\Omega_\eta; \mathbb{R}^3))$ by:

$$\chi^\eta(t, x^\eta_1, x^\eta_2, x^\eta_3) = \chi(t, x_1, x_2, \frac{x_3}{\eta})$$

Taking χ^η as a test function in (11), we obtain that:

$$\begin{aligned}\int_{\mathbb{R}^+\times\Omega_1}\left(\frac{\partial M^\eta}{\partial t}-M^\eta\times\frac{\partial M^\eta}{\partial t}\right)\chi &= 2\int_{\mathbb{R}^+\times\Omega_1}\sum_{\alpha=1}^{2}M^\eta\times\partial_\alpha M^\eta\cdot\partial_\alpha\chi\\ +\frac{2}{\eta^2}\int_{\mathbb{R}^+\times\Omega_1}M^\eta\times\partial_3 M^\eta\cdot\partial_3\chi &- 2\int_{\mathbb{R}^+\times\Omega_1}M^\eta\times\left(H^\eta+\Psi(M^\eta)\right)\cdot\chi\\ -2\int_{\mathbb{R}^+\times\Omega_1}&\left(M^\eta\times(\lambda^{\mathrm{m}}:(\lambda^{\mathrm{e}}:\varepsilon(\eta,U^\eta)))M^\eta\right)\cdot\chi\\ +2\int_{\mathbb{R}^+\times\Omega_1}&M^\eta\times\left((\lambda^{\mathrm{m}}:(\lambda^{\mathrm{e}}:(\lambda^{\mathrm{m}}:M^\eta\otimes M^\eta)))M^\eta\right)\cdot\chi,\end{aligned}\tag{34}$$

- for all $\xi\in\mathcal{C}^\infty(\mathbb{R}^+;V(\Omega_1))$, we define $\xi^\eta\in\mathcal{C}^\infty(\mathbb{R}^+;V(\Omega_\eta))$ by:

$$\begin{aligned}\xi^\eta_\alpha(t,x_1,x_2,x_3) &= \xi_\alpha(t,x_1,x_2,\frac{x_3}{\eta})\text{ for }\alpha=1,2,\\ \xi^\eta_3(t,x_1,x_2,x_3) &= \frac{1}{\eta}\xi_3(t,x_1,x_2,\frac{x_3}{\eta}).\end{aligned}\tag{35}$$

We remark then that

$$\varepsilon(\xi^\eta)=\varepsilon(\eta,\xi).\tag{36}$$

Taking ξ^η as a test function in (12), using that the third component of $u_1=0$, we obtain that:

$$\begin{aligned}\int_{\mathbb{R}^+\times\Omega_1}\sum_{\alpha=1}^{2}\frac{\partial U^\eta_\alpha}{\partial t}\frac{\partial\xi_\alpha}{\partial t}+\frac{1}{\eta^2}\int_{\mathbb{R}^+\times\Omega_1}\frac{\partial U^\eta_3}{\partial t}\frac{\partial\xi_3}{\partial t}-\int_{\mathbb{R}^+\times\Omega_1}(\lambda^{\mathrm{e}}:\varepsilon(\eta,U^\eta)):\varepsilon(\eta,\xi)\\ +\int_{\mathbb{R}^+\times\Gamma^1_-}g^+\cdot\xi-\int_{\mathbb{R}^+\times\Gamma^1_+}g^+\cdot\xi+\int_{\mathbb{R}^+\times\Gamma^1_2}h\cdot\xi-\int_{\Omega_1}u_1\cdot\xi(0,x)\\ =-\int_{\mathbb{R}^+\times\Omega_1}(\lambda^{\mathrm{e}}:(\lambda^{\mathrm{m}}:M^\eta\otimes M^\eta)):\varepsilon(\eta,\xi).\end{aligned}\tag{37}$$

- for all $t>0$, from (13), we have

$$\mathcal{E}^\eta(t)+\int_0^t\int_{\Omega_1}\left|\frac{\partial M^\eta}{\partial t}\right|^2\leq\mathcal{E}^\eta(0),\tag{38}$$

where

$$\begin{aligned}
\mathcal{E}^\eta(t) :=\ & \frac{1}{\eta}\mathcal{E}(m^\eta, u^\eta, \frac{\partial u^\eta}{\partial t}) \\
=\ & \int_{\Omega_1}\sum_{\alpha=1}^{2}|\partial_\alpha M^\eta|^2 + \frac{1}{\eta^2}\int_{\Omega_1}|\partial_3 M^\eta|^2 + \int_{\Omega_1}\Phi(M^\eta) + \int_{\mathbb{R}^3}|H^\eta|^2 \\
+\ & \frac{1}{2}\int_{\Omega_1}[\lambda^{\mathrm{e}} : (\varepsilon(\eta, U^\eta) - \lambda^{\mathrm{m}} : M^\eta \otimes M^\eta)] : (\varepsilon(\eta, U^\eta) - \lambda^{\mathrm{m}} : M^\eta \otimes M^\eta) \\
+\ & \frac{1}{2}\int_{\Omega_1}\left(\sum_{\alpha=1}^{2}\left|\frac{\partial U^\eta_\alpha}{\partial t}\right| + \frac{1}{\eta^2}\left|\frac{\partial U^\eta_3}{\partial t}\right|^2\right) - \int_{\Gamma^1_-} g^- \cdot U^\eta - \int_{\Gamma^1_+} g^+ \cdot U^\eta \\
-\ & \int_{\Gamma^1_2} h \cdot U^\eta,
\end{aligned}$$

3.1. Uniform bound and limit when η tends to zero

Since $M^\eta(0)$ does not depend on x_3, we have:

$$\begin{aligned}
\mathcal{E}^\eta(0) :=\ & \frac{1}{\eta}\mathcal{E}(m^\eta_0, u^\eta_0, u^\eta_1) \\
=\ & 2\int_\omega\left[\sum_{\alpha=1}^{2}|\partial_\alpha m_0|^2 + \Phi(m_0)\right]dx_T + \int_{\mathbb{R}^3}|H^\eta(0)|^2 dx \\
+\ & \frac{1}{2}\int_{\Omega_1}[\lambda^{\mathrm{e}} : (\varepsilon(\eta, U^\eta(0)) - \lambda^{\mathrm{m}} : m_0 \otimes m_0)] : (\varepsilon(\eta, U^\eta(0)) - \lambda^{\mathrm{m}} : m_0 \otimes m_0) \\
+\ & \int_\omega |u_{1,T}|^2 dx_T - \int_\omega \left(g^- + g^+\right)dx_T \cdot u_0 - \int_{\Gamma^1_2} h \cdot u_0.
\end{aligned}$$

Since $\varepsilon_{i3}(U^\eta(0)) = 0$ for $i \in \{1, 2, 3\}$, $\varepsilon(\eta, U^\eta(0))$ does not depend on η. In addition,

$$\int_{\mathbb{R}^3}|H^\eta(0)|^2 dx = \frac{1}{\eta}\int_{\mathbb{R}^3}|h_d(m^\eta(0))|^2 \leq \frac{1}{\eta}\int_{\Omega_\eta}|m^\eta(0)|^2 \leq 2\mathrm{meas}(\omega).$$

Therefore, there exists a constant C_1, independent of η, such that:

$$\forall \eta > 0,\ \mathcal{E}^\eta(0) \leq C_1. \tag{39}$$

Using (39) and the same arguments as in the proof of Lemma 2.1, using also the saturation constraint satisfied by M^η, we obtain that there exists a constant C such that for all η,

$$\begin{aligned}
\frac{1}{2}\int_{\Omega_1}[\lambda^{\mathrm{e}} : (\varepsilon(\eta, U^\eta) - \lambda^{\mathrm{m}} : M^\eta \otimes M^\eta)] : (\varepsilon(\eta, U^\eta) - \lambda^{\mathrm{m}} : M^\eta \otimes M^\eta) \\
\geq \frac{\lambda^{\mathrm{e}*}}{4}\|\varepsilon(\eta, U^\eta)\|^2_{L^2(\Omega_1)} - C.
\end{aligned} \tag{40}$$

By continuity of the trace from $H^1(\Omega)$ into $H^{\frac{1}{2}}(\partial\Omega)$, we have

$$\int_{\Gamma_-^1} g^- \cdot U^\eta + \int_{\Gamma_+^1} g^+ \cdot U^\eta + \int_{\Gamma_2^1} h \cdot U^\eta \leq C_2 \|U^\eta\|_{H^1(\Omega_1)}, \tag{41}$$

where C_2 is a constant depending only on Ω_1.
Now, using the energy inequality (38), and inequalities (40), (41) and the fact that $\Phi(M^\eta)$ is non negative, for all $t > 0$, we have

$$\begin{aligned}
&\sum_{\alpha=1}^{2} \|\partial_\alpha M^\eta\|^2_{L^2(\Omega_1)} + \frac{1}{\eta^2}\|\partial_3 M^\eta\|^2_{L^2(\Omega_1)} + \|H^\eta\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \int_{\Omega_1} \left|\frac{\partial M^\eta}{\partial t}\right|^2 \\
&+ \frac{\lambda^{e*}}{4}\|\varepsilon(\eta, U^\eta)\|^2_{L^2(\Omega_1)} - C_1 - C_2\|U^\eta\|_{H^1(\Omega_1)} + \sum_{\alpha=1}^{2} \left\|\frac{\partial U_\alpha^\eta}{\partial t}\right\|^2_{L^2(\Omega_1)} + \frac{1}{\eta^2}\left\|\frac{\partial U_3^\eta}{\partial t}\right\|^2_{L^2(\Omega_1)} \\
&\leq \mathcal{E}^\eta(0).
\end{aligned} \tag{42}$$

Using Definition (32) and Korn inequality (20) we can prove that

$$\|\varepsilon(\eta, \xi)\|_{L^2(\Omega_1)} \geq \|\varepsilon(\xi)\|_{L^2(\Omega_1)} \geq c\|\xi\|_{H^1(\Omega)} \text{ for all } \xi \in V(\Omega_1). \tag{43}$$

Hence, applying Inequality (43) on (42), we obtain that, for all $T > 0$ and η in a neighborhood of zero, there exists a constant C independent of η such that

- $\|\frac{\partial M^\eta}{\partial t}\|_{L^2(0,T;L^2(\Omega_1))} \leq C$,
- $\|\partial_1 M^\eta\|_{L^\infty(0,T;L^2(\Omega_1))} + \|\partial_2 M^\eta\|_{L^\infty(0,T;L^2(\Omega_1))} \leq C$,
- $\frac{1}{\eta}\|\partial_3 M^\eta\|_{L^\infty(0,T;L^2(\Omega_1))} \leq C$,
- $\|U^\eta\|_{L^\infty(0,T;L^2(\Omega_1))} \leq C$,
- $\|\frac{\partial U_1^\eta}{\partial t}\|_{L^\infty(0,T;L^2(\Omega_1))} + \|\frac{\partial U_2^\eta}{\partial t}\|_{L^\infty(0,T;L^2(\Omega_1))} \leq C$,
- $\frac{1}{\eta}\|\frac{\partial U_3^\eta}{\partial t}\|_{L^\infty(0,T;L^2(\Omega_1))} \leq C$,
- $\|\varepsilon(\eta, U^\eta)\|_{L^2(\Omega_1)} \leq C$,
- $\|H^\eta\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq C$.

So, we can extract subsequences, still denoted (M^η, U^η) and H^η, such that when η tends to zero:

- $M^\eta \rightharpoonup M$ in $L^\infty(0,T;H^1(\Omega_1))$ weak $*$,
- $\frac{\partial M^\eta}{\partial t} \rightharpoonup \frac{\partial M}{\partial t}$ in $L^2(0,T;L^2(\Omega_1))$ weak,
- $U^\eta \rightharpoonup U$ in $L^\infty(0,T;V(\Omega_1))$ weak $*$,
- $\varepsilon(\eta, U^\eta) \rightharpoonup A$ in $L^\infty(0,T;L^2(\Omega_1))$ weak $*$,
- $\frac{\partial U^\eta}{\partial t} \rightharpoonup \frac{\partial U}{\partial t}$ in $L^\infty(0,T;L^2(\Omega_1))$ weak $*$,
- $\frac{\partial U_3^\eta}{\partial t} \to 0$ in $L^\infty(0,T;L^2(\Omega_1))$
- $H^\eta \rightharpoonup H$ in $L^\infty(0,T;L^2(\mathbb{R}^3))$ weak $*$.

Using the Aubin-Simon lemma (see [3], Theorem II.5.16), we can prove that $M^\eta \to M$ in $L^\infty(0,T;L^r(\Omega_1)) \cap \mathcal{C}^0([0,T];L^2(\Omega_1))$ strong for $r < 6$. In addition, we can extract

a subsequence, still denoted by $(M^\eta)_\eta$, such that $M^\eta \to M$ almost everywhere in $[0,T]\times\Omega_1$. Hence M verifies the saturation constraint

$$|M| = 1 \ \textit{a.e.} \text{ in } \mathbb{R}^+ \times \Omega_1,$$

and by continuity in time with values in $L^2(\Omega_1)$, $M(0,x) = m_0(x)$ in the trace sense.

Furthermore, we have $\partial_3 M^\eta \to 0$ in $L^\infty(0,T;L^2(\Omega_1))$ strong, so M only depends on $(t,x_1,x_2) \in \mathbb{R}^+ \times \omega$ and does not depend on x_3. In addition, $\dfrac{\partial U_3}{\partial t} = 0$.

3.2. Limit in the Landau-Lifschitz Equation

In order to characterize H, the weak limit of H^η, we apply Lemma 2.A in [6] to obtain:

$$H(t,x,y,z) = \begin{cases} -\begin{pmatrix} 0 \\ 0 \\ M_3(t,x,y) \end{pmatrix} & \text{for } (x,y,z) \in \Omega_1, \\ \\ 0 & \text{for } x \notin \mathbb{R}^3 \setminus \Omega_1. \end{cases} \tag{44}$$

Let $\tilde\chi \in \mathcal{C}_c^\infty(\mathbb{R}^+;\mathcal{D}(\bar\omega))$, and define χ in $\mathbb{R}^+\times\Omega_1$ by $\chi(t,x_1,x_2,x_3) = \tilde\chi(t,x_1,x_2)$. Then, $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^+;H^1(\Omega))$ and $\partial_3\chi = 0$, so we can take χ as a test function in (34). Using the strong convergence of M^η in $L^\infty(0,T;L^4(\Omega_1))$ and the weak convergence of $\frac{\partial M}{\partial t}^\eta$ and ∇M^η in $L^2(0,T\times\Omega_1)$, by weak $*$ convergence of $\mathcal{E}(\eta;U^\eta)$ in $L^\infty(0,T;L^2(\Omega_1))$, we obtain that

$$\begin{aligned} \int_{\mathbb{R}^+\times\Omega_1} \left(\frac{\partial M}{\partial t} - M\times\frac{\partial M}{\partial t}\right)\tilde\chi = -2\int_{\mathbb{R}^+\times\Omega_1}\sum_{\alpha=1}^{2} M\times\partial_\alpha M\cdot\partial_\alpha\tilde\chi \\ -2\int_{\mathbb{R}^+\times\Omega_1} M\times(-M_3e_3+\Psi(M))\cdot\tilde\chi - 2\int_{\mathbb{R}^+\times\Omega_1} M\times(\lambda^{\mathrm{m}}:(\lambda^{\mathrm{e}}:A))M\cdot\tilde\chi \\ +2\int_{\mathbb{R}^+\times\Omega_1} M\times\big((\lambda^{\mathrm{m}}:(\lambda^{\mathrm{e}}:(\lambda^{\mathrm{m}}:M\otimes M)))M\big)\cdot\tilde\chi. \end{aligned}$$

We denote $\bar A := \dfrac{1}{2}\displaystyle\int_{-1}^{1} A\,dx_3$, so that $\bar A$ is a symmetric 2-tensor. We define $\widetilde\sigma$ by:

$$\widetilde\sigma = \lambda^{\mathrm{e}} : \epsilon^e, \text{ with } \epsilon^e = \bar A - \lambda^{\mathrm{m}} : M\otimes M. \tag{45}$$

Since $\tilde\chi$ and M do not depend on x_3, we obtain that:

$$\begin{aligned} \int_{\mathbb{R}^+\times\omega} \left(\frac{\partial M}{\partial t} - M\times\frac{\partial M}{\partial t}\right)\tilde\chi = -2\int_{\mathbb{R}^+\times\omega}\sum_{\alpha=1}^{2} M\times\partial_\alpha M\cdot\partial_\alpha\tilde\chi \\ -2\int_{\mathbb{R}^+\times\omega} M\times(-M_3e_3+\Psi(M))\cdot\tilde\chi - 2\int_{\mathbb{R}^+\times\omega} M\times(\lambda^{\mathrm{m}}:(\lambda^{\mathrm{e}}:\widetilde\sigma))M\cdot\tilde\chi. \end{aligned} \tag{46}$$

3.3. Limit in the elasticity wave equation

We remark that $(\frac{1}{\eta}\varepsilon_{\alpha 3}(U^\eta))_\eta$ for $\alpha \in \{1,2\}$ and $(\frac{1}{\eta^2}\varepsilon_{33}(U^\eta))_\eta$ are bounded in $L^\infty(0,T;L^2(\Omega))$, because $(\varepsilon(\eta,U^\eta))_\eta$ is bounded, this gives that $\varepsilon_{i3}(U^\eta)\to 0$ strongly in $L^\infty(0,T;L^2(\Omega))$, for all $T>0$, i.e. $\varepsilon_{i3}(U)=0$ for $i\in\{1,2,3\}$. Therefore, $U\in L^\infty(\mathbb{R}^+;V_{KL}(\Omega_1))$, then by Lemma 3.1, there exist $\tilde{u}\in L^\infty(\mathbb{R}^+;V(\omega))$ such that

$$\begin{array}{ll} U_\alpha(t,x_1,x_2,x_3) & = \tilde{u}_\alpha(t,x_1,x_2) - x_3\partial_\alpha\tilde{u}_3 \text{ for } \alpha\in\{1,2\},\\ U_3(t,x_1,x_2,x_3) & = \tilde{u}_3(t,x_1,x_2). \end{array} \tag{47}$$

For abbreviation, we write $\tilde{u}_T=(\tilde{u}_1,\tilde{u}_2)$. Using that $U^\eta \rightharpoonup U$ in $L^\infty(0,T;V(\Omega_1))$ weak $*$, since for all $(\alpha,\beta)\in\{1,2\}^2$, $(\varepsilon(\eta,U^\eta))_{\alpha\beta}=(\varepsilon(U^\eta))_{\alpha\beta}$, we obtain at the weak limit that:

$$A_{\alpha\beta}=\varepsilon_{\alpha\beta}(U) \quad \text{for } (\alpha,\beta)\in\{1,2\}^2. \tag{48}$$

Taking the average in the variable $x_3\in[-1,1]$ and using (47), we obtain that:

$$\bar{A}_{\alpha\beta}=\varepsilon_{\alpha\beta}(\tilde{u}_T) \quad \text{for } (\alpha,\beta)\in\{1,2\}^2. \tag{49}$$

For $v\in\mathcal{C}^\infty(\mathbb{R}^+;\mathcal{W}(\omega))$, we define $\xi\in\mathcal{C}^\infty(\mathbb{R};V_{KL}(\Omega_1))$ by

$$\begin{array}{ll} \xi_\alpha(x_1,x_2,x_3) & = v_\alpha(x_1,x_2) - x_3\partial_\alpha v_3(x_1,x_2) \text{ for } \alpha=1,2,\\ \xi_3(x_1,x_2,x_3) & = v_3(x_1,x_2). \end{array} \tag{50}$$

Then, $\varepsilon_{i3}(\xi)=0$ for $i\in\{1,2,3\}$, so that $\varepsilon(\eta,\xi)=\varepsilon(\xi)$. With ξ as a test function in the weak formulation (37), we obtain

$$\begin{aligned} \int_{\mathbb{R}^+\times\Omega_1}\sum_{\alpha=1}^{2}\frac{\partial U_\alpha^\eta}{\partial t}\frac{\partial\xi_\alpha}{\partial t}+\frac{1}{\eta^2}\int_{\mathbb{R}^+\times\Omega_1}\frac{\partial U_3^\eta}{\partial t}\frac{\partial\xi_3}{\partial t}-\int_{\mathbb{R}^+\times\Omega_1}(\lambda^{\mathrm{e}}:\varepsilon(\eta,U^\eta)):\varepsilon(\xi)\\ +\int_{\mathbb{R}^+\times\Gamma_-^1}g^+\cdot\xi-\int_{\mathbb{R}^+\times\Gamma_+^1}g^+\cdot\xi+\int_{\mathbb{R}^+\times\Gamma_2^1}h\cdot\xi-\int_{\Omega_1}u_1\cdot\xi(0,x)\\ =\int_{\mathbb{R}^+\times\Omega_1}(\lambda^{\mathrm{e}}:(\lambda^{\mathrm{m}}:M^\eta\otimes M^\eta)):\varepsilon(\xi). \end{aligned} \tag{51}$$

Taking $v=(v_1,v_2,0)$ in (51), where v_1 and v_2 are in $\mathcal{C}^\infty(\mathbb{R}^+;\mathcal{V}(\omega))$, we obtain:

$$\begin{aligned} \int_{\mathbb{R}^+\times\Omega_1}\sum_{\alpha=1}^{2}\frac{\partial U_\alpha^\eta}{\partial t}\frac{\partial v_\alpha}{\partial t}-\int_{\mathbb{R}^+\times\Omega_1}(\lambda^{\mathrm{e}}:\varepsilon(\eta,U^\eta)):\varepsilon(\xi)\\ +\int_{\mathbb{R}^+\times\omega}(g^+-g^-)_T\cdot v_T+\int_{\mathbb{R}^+\times\Gamma_2^1}h_T\cdot v_T-\int_{\Omega_1}u_{1,T}\cdot v_T(0,x)\\ =\int_{\mathbb{R}^+\times\Omega_1}\sum_{(\alpha,\beta)\in\{1,2\}^2}(\lambda^{\mathrm{e}}:(\lambda^{\mathrm{m}}:M^\eta\otimes M^\eta))_{\alpha\beta}:\varepsilon_{\alpha\beta}(v). \end{aligned}$$

By taking the limit when η tends to zero, we obtain that, for all $v_T \in \mathcal{C}^\infty(\mathbb{R}^+;\mathcal{V}(\omega))$:

$$\int_{\mathbb{R}^+\times\omega} \frac{\partial \tilde{u}_T}{\partial t}\frac{\partial v_T}{\partial t} - \int_{\mathbb{R}^+\times\omega} \sum_{\alpha\beta=1}^{2} (\lambda^{\mathrm{e}} : \bar{A})_{\alpha\beta}\varepsilon_{\alpha\beta}(v_T) + \frac{1}{2}\int_{\mathbb{R}^+\times\omega} (g_T^+ + g_T^-)\cdot v_T$$
$$+ \int_{\mathbb{R}^+\times\mathcal{C}_2} h_T \cdot v_T - \int_{\mathbb{R}^+\times\omega} u_{1T} v_T(0,x) = -\int_{\mathbb{R}^+\times\omega} \sum_{\alpha\beta=1}^{2} (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : M\otimes M))_{\alpha\beta}\varepsilon_{\alpha\beta}(v),$$

and therefore, using notation (45), we have:

$$\int_{\mathbb{R}^+\times\omega} \frac{\partial \tilde{u}_T}{\partial t}\frac{\partial v_T}{\partial t} - \int_{\mathbb{R}^+\times\omega} \sum_{\alpha\beta=1}^{2} \widetilde{\sigma}_{\alpha\beta}\varepsilon_{\alpha\beta}(v) + \frac{1}{2}\int_{\mathbb{R}^+\times\omega} (g_T^+ + g_T^-)\cdot v_T + \int_{\mathbb{R}^+\times\mathcal{C}_2} h_T \cdot v_T$$
$$- \int_{\mathbb{R}^+\times\omega} u_{1T} v_T(0,x) = 0. \tag{52}$$

3.4. Characterization of $\widetilde{\sigma}$

For all $\xi \in \mathcal{C}^\infty(\mathbb{R}^+;V(\Omega_1))$, multiplying (37) by η^2 and taking the limit when η tends to zero, we obtain that

$$\int_{\mathbb{R}^+\times\Omega_1} \frac{\partial U_3}{\partial t}\cdot\frac{\partial \xi}{\partial t} - \int_{\mathbb{R}^+\times\Omega_1} (\lambda^{\mathrm{e}} : A)_{33}\varepsilon_{33}(\xi) = -\int_{\mathbb{R}^+\times\Omega_1} (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : M\otimes M))_{33}\varepsilon_{33}(\xi). \tag{53}$$

We remark that $\dfrac{\partial U_3}{\partial t} = 0$, so the first integral in (53) vanishes. For $\varphi \in \mathbb{D}(\mathbb{R}^+\times\omega)$, we choose ξ such that:

$$\begin{aligned} &\xi_\alpha = 0 \text{ for } \alpha = 1,2\\ &\xi_3 = x_3\,\varphi, \end{aligned}$$

then $e_{33}(\xi) = \varphi$, and (53) implies

$$\int_{\mathbb{R}^+\times\omega} (\lambda^{\mathrm{e}} : \bar{A})_{33}\varphi = \int_{\mathbb{R}^+\times\omega} (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : M\otimes M))_{33}\varphi,$$

since M and φ are independent of x_3. Then

$$(\lambda^{\mathrm{e}} : \bar{A})_{33} = (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : M\otimes M))_{33} \text{ in } L^2(\mathbb{R}^+\times\omega).$$

Fix $\alpha \in \{1,2\}$ and choose ξ such that $\xi_\alpha = x_3\varphi$, and $\xi_i = 0$ for $i \neq \alpha$, where $\varphi \in \mathbb{D}(\mathbb{R}^+\times\omega)$, then $e_{\alpha 3}(\xi) = \frac{1}{2}\varphi$ and $\varepsilon_{33}(\xi) = 0$. Replacing ξ by its value in (37), multiplying the obtained equation by η and $\eta \to 0$, we get:

$$\int_{\mathbb{R}^+\times\Omega_1} (\lambda^{\mathrm{e}} : A)_{\alpha 3}\varphi = \int_{\mathbb{R}^+\times\Omega_1} (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : M\otimes M))_{\alpha 3}\varphi.$$

This is true for all $\varphi \in \mathbb{D}(\mathbb{R}^+ \times \omega)$, then by using the fact that φ and M are independent of x_3, we obtain

$$(\lambda^{\mathrm{e}} : \bar{A})_{\alpha 3} = (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : M \otimes M))_{\alpha 3} \text{ in } L^2(\mathbb{R}^+ \times \omega). \tag{54}$$

Therefore, we obtain that:

$$\forall i \in \{1,2,3\}, \quad \widetilde{\sigma}_{i3} = \widetilde{\sigma}_{3i} = 0. \tag{55}$$

We define I and J by:

$$I = \{(i,j) \in \{1,2\}^2\} \quad \text{and} \quad J = \{(i,j) \in \{1,2,3\}^2,\ i = 3 \text{ or } j = 3\}.$$

We denote by $\mathcal{G}$ the set of symmetric real 3×3 matrices endowed with the scalar product :. We denote by $\mathcal{G}_n$ the set of the matrices $D = (d_{ij}) \in \mathcal{G}$ such that $d_{\alpha\beta} = 0$ for $(\alpha, \beta) \in I$, and by $\mathcal{G}_T$ the set of the matricies $D = (d_{ij}) \in \mathcal{G}$ such that $d_{ij} = 0$ for $(i,j) \in J$, so that

$$\mathcal{G} = \mathcal{G}_T \oplus^{\perp} \mathcal{G}_n$$

.

For $D \in \mathcal{G}$, we define $\mathcal{K}(D) \in \mathcal{G}_n$ by:

$$(\mathcal{K}(D))_{ij} = \begin{cases} 0 \text{ if } (i,j) \in I, \\ \\ (\lambda^{\mathrm{e}} : D)_{ij} = \displaystyle\sum_{(k,l) \in \{1,2,3\}^2} \lambda^{\mathrm{e}}_{ijkl} d_{kl} \text{ if } (i,j) \in J. \end{cases}$$

We remark that $\mathcal{K}$ is linear and that its restriction to $\mathcal{G}_n$ is a bijection from $\mathcal{G}_n$ to itself. Indeed, if $D \in \mathcal{G}_n$ with $\mathcal{K}(D) = 0$, we have:

$$\begin{aligned} 0 = & \sum_{(i,j) \in J} (\mathcal{K}(D))_{ij} D_{ij} \text{ since } \mathcal{K}(D) = 0, \\ = & \sum_{(i,j) \in J} (\lambda^{\mathrm{e}} : D)_{ij} D_{ij}, \\ = & \sum_{(i,j) \in \{1,2,3\}^2} (\lambda^{\mathrm{e}} : D)_{ij} D_{ij} \text{ since } D_{ij} = 0 \text{ if } (i,j) \in I, \\ = & \ (\lambda^{\mathrm{e}} : D) : D. \end{aligned}$$

So by positivity of λ^{e}, if $\mathcal{K}(D) = 0$ with $D \in \mathcal{G}_n$, then $D = 0$. So $\mathcal{K}$ is bijective from $\mathcal{G}_n$ into $\mathcal{G}_n$. We denote by $\mathcal{T} : \mathcal{G}_n \longrightarrow \mathcal{G}_n$ the inverse of $\mathcal{K}_{|\mathcal{G}_n}$. We split ϵ^e as $\epsilon^e = \epsilon^e_T + \epsilon^e_n$ where:

$$(\epsilon^e_T)_{ij} = \begin{cases} 0 \text{ if } (i,j) \in J, \\ \\ \varepsilon(\tilde{u}_T)_{ij} - (\lambda^{\mathrm{m}} : M \otimes M)_{ij} \text{ if } (i,j) \in I, \end{cases}$$

and

$$(\epsilon_n^e)_{ij} = \begin{cases} 0 \text{ if } (i,j) \in I, \\ \\ \bar{A}_{ij} - (\lambda^{\mathrm{m}} : M \otimes M)_{ij} \text{ if } (i,j) \in J. \end{cases}$$

By (55), $\mathcal{K}(\epsilon_T^e + \epsilon_n^e) = 0$, so $\mathcal{K}(\epsilon_n^e) = -\mathcal{K}(\epsilon_T^e)$. Since $\epsilon_n^e \in \mathcal{G}$, we obtain that:

$$\epsilon_n^e = -\mathcal{T}(\mathcal{K}(\epsilon_T^e)).$$

Therefore,

$$\widetilde{\sigma} = \lambda^{\mathrm{e}} : (\epsilon_T^e - \mathcal{T}(\mathcal{K}(\epsilon_T^e))).$$

We denote by λ^{eq} the 4-tensor defined by:

$$\text{for } D \in \mathcal{G},\ \lambda^{\mathrm{eq}} : D = \lambda^{\mathrm{e}} : \big(D - \mathcal{T}(\mathcal{K}(D))\big),$$

so that we have:

$$\widetilde{\sigma} = \lambda^{\mathrm{eq}} : \big(\varepsilon(\tilde{u}_T) - \lambda^{\mathrm{m}} : M \otimes M\big).$$

Remark 1. In the isotropic case, that is for λ^{e} given by

$$\forall\, S \in \mathcal{G},\ (\lambda^{\mathrm{e}} : S)_{ij} = \frac{E}{1+\nu}\left(S_{ij} + \frac{\nu}{1-2\nu}(\operatorname{tr} S)\delta_{ij}\right),$$

where E is the young modulus and $\nu \in [0, \frac{1}{2}[$ is the Poisson coefficient, we obtain that $\widetilde{\sigma}$ is deduced from ϵ_T^e by:

$$\widetilde{\sigma}_{ij} = \begin{cases} 0 \text{ if } i = 3 \text{ or } j = 3, \\ \\ \frac{E}{1+\nu}\left((\epsilon_T^e)_{ij} + \frac{\nu}{1-\nu}(\widetilde{\operatorname{tr}}\,\epsilon_T^e)\delta_{ij}\right) \text{ if } (i,j) \in \{1,2\}^2, \end{cases}$$

where $\widetilde{\operatorname{tr}}\,\epsilon_T^e = (\epsilon_T^e)_{11} + (\epsilon_T^e)_{22}$. This particular case was studied in [13].

4. Conclusion

We have obtained the following 2d-model for thin ferromagnetic plates with magnetostriction:

- $M \in L^\infty(\mathbb{R}^+; H^1(\omega; S^2))$, $\dfrac{\partial M}{\partial t} \in L^2(\mathbb{R}^+ \times \omega)$,
- $\tilde{u}_T \in L^\infty(\mathbb{R}^+; H^1(\omega; \mathbb{R}^2))$ with $\tilde{u}_T = 0$ on $\mathbb{R}^+ \times \mathcal{C}_1$ and $\dfrac{\partial \tilde{u}_T}{\partial t} \in L^\infty(\mathbb{R}^+; L^2(\omega; \mathbb{R}^2))$

- For all $\tilde{\chi} \in \mathcal{C}^\infty(\mathbb{R}^+; H^1(\omega; \mathbb{R}^3))$,

$$\int_{\mathbb{R}^+\times\omega} \left(\frac{\partial M}{\partial t} - M \times \frac{\partial M}{\partial t} \right) \tilde{\chi} = -2 \int_{\mathbb{R}^+\times\omega} \sum_{\alpha=1}^{2} M \times \partial_\alpha M \cdot \partial_\alpha \tilde{\chi}$$

$$-2 \int_{\mathbb{R}^+\times\omega} M \times (-M_3 e_3 + \Psi(M) + (\lambda^{\mathrm{m}} : \widetilde{\sigma}) M) \cdot \tilde{\chi}.$$

- For all $v_T \in \mathcal{C}^\infty(\mathbb{R}^+; \mathcal{V}(\Omega; \mathbb{R}^2))$,

$$\int_{\mathbb{R}^+\times\omega} \frac{\partial \tilde{u}_T}{\partial t} \frac{\partial v_T}{\partial t} - \int_{\mathbb{R}^+\times\omega} \sum_{\alpha\beta=1}^{2} \widetilde{\sigma}_{\alpha\beta} \varepsilon_{\alpha\beta}(v_T) + \frac{1}{2} \int_{\mathbb{R}^+\times\omega} (g_T^+ + g_T^-) \cdot v_T$$
$$+ \int_{\mathbb{R}^+\times\mathcal{C}_2} h_T \cdot v_T - \int_{\mathbb{R}^+\times\omega} u_{1T} v_T(0, x) = 0.$$

- $\widetilde{\sigma} = \lambda^{\mathrm{eq}} \epsilon_T^e$ with $\epsilon_T^e = \varepsilon(\tilde{u}_T) - \lambda^{\mathrm{m}} : M \otimes M$.

We remark that the equation in v_T is hyperbolic since the equivalent tensor λ^{eq} is positive on $\mathcal{G}_T$. Indeed, if $D \in \mathcal{G}_T$, then:

$$(\lambda^{\mathrm{eq}} : D) : D = (\lambda^{\mathrm{e}} : (D - \mathcal{T}(\mathcal{K}(D)))) : D.$$

We remark that $\lambda^{\mathrm{e}} : (D - \mathcal{T}(\mathcal{K}(D))) \in \mathcal{G}_T$ and $\mathcal{T}(\mathcal{K}(D)) \in \mathcal{G}_n$. Thus,

$$\begin{aligned}
(\lambda^{\mathrm{eq}} : D) : D = & \quad (\lambda^{\mathrm{e}} : (D - \mathcal{T}(\mathcal{K}(D)))) : (D - \mathcal{T}(\mathcal{K}(D))) \\
\geq & \quad \lambda^{e*} \|D - \mathcal{T}(\mathcal{K}(D))\|^2 \\
\geq & \quad \lambda^{e*} \|D\|^2 \text{ since } \mathcal{G}_T \perp \mathcal{G}_n.
\end{aligned}$$

It is well known that in 3d, the weak solutions for the Landau Lifschitz equation are not unique (see [2]). The uniqueness is only proved for regular solutions (at least with H^2 regularity in the space variable). For the 2D-model coupling , the uniqueness of weak solutions remains an open problem. Although, using J L Lions theorem, we can show that if M is fixed, then the solution of the initial and boundary value hyperbolic problem satisfied by $\tilde{u}_T$ is unique (see Theorem 10.14 in [4] and [16]).

In conclusion, the model we obtain will be easier to study and to simulate since it is bi-dimensional and since the 2d demagnetizing field is local. It would be interesting to characterize the normal deformations in the 2d model, even though they have no influence on magnetization in the regime we studied.

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