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# Maximal Martingale Wasserstein Inequality

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## Abstract

In this note, we complete the analysis of the Martingale Wasserstein Inequality started in [5] by checking that this inequality fails in dimension  $d \geq 2$  when the integrability parameter  $\rho$  belongs to  $[1, 2)$  while a stronger Maximal Martingale Wasserstein Inequality holds whatever the dimension  $d$  when  $\rho \geq 2$ .

## 1 Introduction

The present paper elaborates on the convergence to 0 as  $n \rightarrow \infty$  of  $\inf_{M \in \Pi^M(\mu_n, \nu_n)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^\rho M(dx, dy)$  with the Wasserstein distance  $\mathcal{W}_\rho(\mu_n, \nu_n)$  when for each  $n \in \mathbb{N}$ ,  $\mu_n$  and  $\nu_n$  belong to the set  $\mathcal{P}_\rho(\mathbb{R}^d)$  of probability measures on  $\mathbb{R}^d$  with a finite moment of order  $\rho \in [1, +\infty)$  and the former is smaller than the latter in the convex order. The convex order between  $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$  which is denoted  $\mu \leq_{cx} \nu$  amounts to

$$\int_{\mathbb{R}^d} f(x) \mu(dx) \leq \int_{\mathbb{R}^d} f(y) \nu(dy) \text{ for each convex function } f : \mathbb{R}^d \rightarrow \mathbb{R}, \quad (1)$$

and, by Strassen's theorem [7], is equivalent to the non emptiness of the set of martingale couplings between  $\mu$  and  $\nu$  defined by

$$\begin{aligned} \Pi^M(\mu, \nu) &= \left\{ M(dx, dy) = \mu(dx) m(x, dy) \in \Pi(\mu, \nu) \mid \mu(dx)\text{-a.e.}, \int_{\mathbb{R}^d} y m(x, dy) = x \right\} \text{ where} \\ \Pi(\mu, \nu) &= \{ \pi \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d) \mid \forall A \in \mathcal{B}(\mathbb{R}^d), \pi(A \times \mathbb{R}^d) = \mu(A) \text{ and } \pi(\mathbb{R}^d \times A) = \nu(A) \}. \end{aligned}$$

The Wasserstein distance with index  $\rho$  is defined by

$$\mathcal{W}_\rho(\mu, \nu) = \left( \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^\rho \pi(dx, dy) \right)^{1/\rho}$$

and we also introduce  $\underline{\mathcal{M}}_\rho(\mu, \nu)$  and  $\overline{\mathcal{M}}_\rho(\mu, \nu)$  respectively defined by

$$\underline{\mathcal{M}}_\rho^\rho(\mu, \nu) = \inf_{M \in \Pi^M(\mu, \nu)} \int_{\mathbb{R}^{2d}} |x - y|^\rho M(dx, dy), \quad \overline{\mathcal{M}}_\rho^\rho(\mu, \nu) = \sup_{M \in \Pi^M(\mu, \nu)} \int_{\mathbb{R}^{2d}} |x - y|^\rho M(dx, dy). \quad (2)$$

In dimension  $d = 1$ , the optimization problems defining  $\underline{\mathcal{M}}_\rho$  and  $\overline{\mathcal{M}}_\rho$  are the respective subjects of [3] and [4] when  $\rho = 1$ , while the general case  $\rho \in (0, +\infty)$  is studied in [6].

The question of interest is related to the stability of Martingale Optimal Transport problems with respect to the marginal distributions  $\mu$  and  $\nu$  established in dimension  $d = 1$  in [1, 8] while

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it fails in higher dimension according to [2]. A quantitative answer is given in dimension  $d = 1$  by the Martingale Wasserstein inequality established in [5, Proposition 1] for  $\rho \in [1, +\infty)$ ,

$$\exists \underline{C}_{(\rho,\rho),1} < \infty, \forall \mu, \nu \in \mathcal{P}_\rho(\mathbb{R}) \text{ with } \mu \leq_{cx} \nu, \underline{M}_\rho^\rho(\mu, \nu) \leq \underline{C}_{(\rho,\rho),1} \mathcal{W}_\rho(\mu, \nu) \sigma_\rho^{\rho-1}(\nu), \quad (3)$$

where the central moment  $\sigma_\rho(\nu)$  of  $\nu$  is defined by

$$\sigma_\rho(\nu) = \inf_{c \in \mathbb{R}^d} \left( \int_{\mathbb{R}^d} |y - c|^\rho \nu(dy) \right)^{1/\rho} \text{ when } \rho \in [1, +\infty) \text{ and } \sigma_\infty(\nu) = \inf_{c \in \mathbb{R}^d} \nu - \text{ess sup}_{y \in \mathbb{R}^d} |y - c|.$$

The proposition also states that  $\mathcal{W}_\rho(\mu, \nu)$  and  $\sigma_\rho(\nu)$  have the right exponent in this inequality in the sense that for  $1 < s < \rho$ ,  $\sup_{\substack{\mu, \nu \in \mathcal{P}_\rho(\mathbb{R}) \\ \mu \leq_{cx} \nu, \mu \neq \nu}} \frac{\underline{M}_\rho^\rho(\mu, \nu)}{\mathcal{W}_\rho^s(\mu, \nu) \sigma_\rho^{\rho-s}(\nu)} = +\infty$ . The generalization of (3) to higher dimensions  $d$  is also investigated in [5] where it is proved that for any  $d \geq 2$ ,

$$\underline{C}_{(\rho,\rho),d} := \sup_{\substack{\mu, \nu \in \mathcal{P}_\rho(\mathbb{R}^d) \\ \mu \leq_{cx} \nu, \mu \neq \nu}} \frac{\underline{M}_\rho^\rho(\mu, \nu)}{\mathcal{W}_\rho(\mu, \nu) \sigma_\rho^{\rho-1}(\nu)}$$

is infinite when  $\rho \in [1, \frac{1+\sqrt{5}}{2})$ , while the one-dimensional constant  $\underline{C}_{(\rho,\rho),1}$  is preserved when  $\mu$  and  $\nu$  are products of one-dimensional probability measures or when, for  $X$  distributed according to  $\mu$ , the conditional expectation of  $X$  given the direction of  $X - \mathbb{E}[X]$  is a.s. equal to  $\mathbb{E}[X]$  and  $\nu$  is the distribution of  $X + \lambda(X - \mathbb{E}[X])$  for some  $\lambda \geq 0$ . The present paper answers the question of the finiteness of  $\underline{C}_{(\rho,\rho),d}$  when  $\rho \in [\frac{1+\sqrt{5}}{2}, +\infty)$  and  $d \geq 2$ , which remained open. It turns out that  $\underline{C}_{(\rho,\rho),d} = +\infty$  for  $d \geq 2$  when  $\rho \in [1, 2)$  while for  $\rho \in [2, +\infty)$  the inequality (3) generalizes in any dimension  $d$  into a Maximal Martingale Wasserstein inequality with the left-hand side  $\underline{M}_\rho^\rho(\mu, \nu)$  replaced by the larger  $\overline{M}_\rho^\rho(\mu, \nu)$ . We even replace conjugate exponents  $\rho$  and  $\frac{\rho}{\rho-1}$  leading to the respective indices  $\rho = \rho \times 1$  and  $\rho = \frac{\rho}{\rho-1} \times (\rho-1)$  in the factors  $\mathcal{W}$  and  $\sigma$  in (3) by general conjugate exponents  $q \in [1, +\infty]$  and  $\frac{q}{q-1} \in [1, +\infty]$  leading to indices  $q$  and  $\frac{q(\rho-1)}{q-1}$  (equal to  $+\infty$  and  $\rho-1$  when  $q$  is respectively equal to 1 and  $+\infty$ ) and define

$$\underline{C}_{(\rho,q),d} := \sup_{\substack{\mu, \nu \in \mathcal{P}_{q \vee \frac{(\rho-1)q}{q-1}}(\mathbb{R}^d) \\ \mu \leq_{cx} \nu, \mu \neq \nu}} \frac{\underline{M}_\rho^\rho(\mu, \nu)}{\mathcal{W}_q(\mu, \nu) \sigma_{\frac{q(\rho-1)}{q-1}}^{\rho-1}(\nu)} \text{ and } \overline{C}_{(\rho,q),d} := \sup_{\substack{\mu, \nu \in \mathcal{P}_{q \vee \frac{(\rho-1)q}{q-1}}(\mathbb{R}^d) \\ \mu \leq_{cx} \nu, \mu \neq \nu}} \frac{\overline{M}_\rho^\rho(\mu, \nu)}{\mathcal{W}_q(\mu, \nu) \sigma_{\frac{q(\rho-1)}{q-1}}^{\rho-1}(\nu)},$$

with  $\mathcal{W}_\infty(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \pi - \text{ess sup}_{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d} |x - y|$ . Since  $\underline{M}_\rho \leq \overline{M}_\rho$ , one has  $\underline{C}_{(\rho,q),d} \leq \overline{C}_{(\rho,q),d}$ . These constants of course depend on the norm  $|\cdot|$  on  $\mathbb{R}^d$  (even if we do not make this dependence explicit) but, by equivalence of the norms, their finiteness does not. Since the Euclidean norm plays a particular role, we will denote it by  $\|\cdot\|$  rather than  $|\cdot|$ .

**Theorem 1.** (i) Let  $\rho \in [1, 2)$ . For  $q \in [1, \frac{1}{2-\rho}]$  (and even  $q \in [1, +\infty]$  when  $\rho = 1$ ), one has  $\underline{C}_{(\rho,q),1} \leq K_\rho < +\infty$  where the constant  $K_\rho$  is studied in [5, Proposition 1] while, for  $q \in [1, +\infty]$ ,  $\overline{C}_{(\rho,q),1} = +\infty$  and  $\underline{C}_{(\rho,q),d} = +\infty$  for  $d \geq 2$ .

(ii) Let  $\rho \in [2, +\infty)$  and  $q \in [1, +\infty]$ . One has  $\overline{C}_{(\rho,q),d} < +\infty$  whatever  $d$ . Moreover, when  $\mathbb{R}^d$  (resp. each  $\mathbb{R}^d$ ) is endowed with the Euclidean norm,  $\overline{C}_{(2,q),d} = 2$  and  $\sup_{d \geq 1} \overline{C}_{(\rho,q),d} < +\infty$ .

**Remark 2.** • The fact that  $\rho = 2$  appears as a threshold is related to the equality  $\int_{\mathbb{R}^d \times \mathbb{R}^d} \|y - x\|^2 M(dx, dy) = \int_{\mathbb{R}^d} \|y\|^2 \nu(dy) - \int_{\mathbb{R}^d} \|x\|^2 \mu(dx)$  for  $M \in \Pi^M(\mu, \nu)$  when  $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$  are such that  $\mu \leq_{cx} \nu$ , which implies that when  $\mathbb{R}^d$  is endowed with the Euclidean norm

$$\underline{M}_2^2(\mu, \nu) = \overline{M}_2^2(\mu, \nu) = \int_{\mathbb{R}^d} \|y\|^2 \nu(dy) - \int_{\mathbb{R}^d} \|x\|^2 \mu(dx).$$

• For  $\rho \in [1, 2)$ , one has  $\overline{C}_{(\rho,q),d} = +\infty$  while  $\sup_{\substack{\mu, \nu \in \mathcal{P}_{q \vee \frac{q}{q-1}}(\mathbb{R}^d) \\ \mu \leq_{cx} \nu, \mu \neq \nu}} \frac{\overline{M}_\rho^2(\mu, \nu)}{\mathcal{W}_q(\mu, \nu) \sigma_{\frac{q}{q-1}}(\nu)} \leq \overline{C}_{(2,q),d} < +\infty$  since  $\overline{M}_\rho \leq \overline{M}_2$ .

## 2 Proof

The proof of Theorem 1 (ii) relies on the next lemma, the proof of the lemma is postponed after the proof of the theorem. In what follows, to avoid making distinctions in case  $q \in \{1, +\infty\}$ , we use the convention that for any probability measure  $\gamma$  and any measurable function  $f$  on the same probability space  $(\int |f(z)|^q \gamma(dz))^{1/q}$  (resp.  $\left(\left(\int |f(z)|^{\frac{q}{q-1}} \gamma(dz)\right)^{(q-1)/q}, \left(\int |f(z)|^{\frac{q(\rho-1)}{q-1}} \gamma(dz)\right)^{(q-1)/q}\right)$ ) is equal to  $\gamma - \text{ess sup}_z |f(z)|$  (resp.  $(\gamma - \text{ess sup}_z |f(z)|, \gamma - \text{ess sup}_z |f(z)|^{\rho-1})$ ) when  $q = +\infty$  (resp.  $q = 1$ ).

**Lemma 3.** *Given  $\rho \in [2, +\infty)$ , there exist constants  $\kappa_\rho, \tilde{\kappa}_\rho \in [0, +\infty)$  such that for all  $d \geq 1$  and  $x, y \in \mathbb{R}^d$ ,*

$$\|x - y\|^\rho \leq \kappa_\rho ((\rho - 1)\|x\|^\rho + \|y\|^\rho - \rho\|x\|^{\rho-2}\langle x, y \rangle), \quad (4)$$

$$\|y\|^\rho - \|x\|^\rho \leq \tilde{\kappa}_\rho \|y - x\| \left( \|x\|^{\rho-1} + \|y\|^{\rho-1} \right). \quad (5)$$

**Remark 4.** *When  $\rho = 2$ , then (4) holds as an equality with  $\kappa_\rho = 1$  while, by the Cauchy-Schwarz and the triangle inequalities,*

$$\|y\|^2 - \|x\|^2 \leq \langle y - x, y + x \rangle \leq \|y - x\| \times \|y + x\| \leq \|y - x\| (\|x\| + \|y\|)$$

so that (5) holds with  $\tilde{\kappa}_\rho = 1$ .

*Proof of Theorem 1.* (i) In dimension  $d = 1$ , one has  $\underline{\mathcal{M}}_1 \leq K_1 \mathcal{W}_1$  with  $K_1 = 2$  according to [5, Proposition 1] and we deduce that  $\underline{C}_{(1,q),1} \leq K_1$  for  $q \in [1, +\infty]$  since  $\mathcal{W}_1 \leq \mathcal{W}_q$ . Let now  $\rho \in (1, 2)$  and  $q \in [1, \frac{1}{2-\rho}]$ . One has  $\frac{q(\rho-1)}{q-1} \geq 1$  since, when  $q > 1$ ,  $\frac{q}{q-1} = 1 + \frac{1}{q-1} \geq 1 + \frac{2-\rho}{\rho-1} = \frac{1}{\rho-1}$ . For  $\mu, \nu \in \mathcal{P}_{q \vee \frac{q(\rho-1)}{q-1}}(\mathbb{R})$  with respective quantile functions  $F_\mu^{-1}$  and  $F_\nu^{-1}$ , one has by optimality of the comonotonic coupling and Hölder's inequality

$$\begin{aligned} \mathcal{W}_\rho^\rho(\mu, \nu) &= \int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)| \times |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^{\rho-1} du \\ &\leq \left( \int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^q du \right)^{1/q} \left( \left( \int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^{\frac{q(\rho-1)}{q-1}} du \right)^{\frac{q-1}{q(\rho-1)}} \right)^{\rho-1}. \end{aligned}$$

Since, by the triangle inequality and  $\mu \leq_{cx} \nu$ , one has for  $c \in \mathbb{R}$

$$\begin{aligned} \left( \int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^{\frac{q(\rho-1)}{q-1}} du \right)^{\frac{q-1}{q(\rho-1)}} &\leq \left( \int_0^1 |F_\nu^{-1}(u) - c|^{\frac{q(\rho-1)}{q-1}} du \right)^{\frac{q-1}{q(\rho-1)}} + \left( \int_0^1 |F_\mu^{-1}(u) - c|^{\frac{q(\rho-1)}{q-1}} du \right)^{\frac{q-1}{q(\rho-1)}} \\ &\leq 2 \left( \int_0^1 |F_\nu^{-1}(u) - c|^{\frac{q(\rho-1)}{q-1}} du \right)^{\frac{q-1}{q(\rho-1)}}, \end{aligned}$$

we deduce by minimizing over the constant  $c$  that

$$\mathcal{W}_\rho^\rho(\mu, \nu) \leq \mathcal{W}_q(\mu, \nu) \times 2^{\rho-1} \sigma_{\frac{q(\rho-1)}{q-1}}^{\rho-1}(\nu).$$

With this inequality replacing (30) in the proof of Proposition 1 [5] and the general inequality

$$\int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)| |F_\nu^{-1}(u) - c|^{\rho-1} du \leq \mathcal{W}_q(\mu, \nu) \left( \int_0^1 |F_\nu^{-1}(u) - c|^{\frac{q(\rho-1)}{q-1}} du \right)^{\frac{q-1}{q}},$$

replacing the special case  $q = \rho$  in the second equation p840 in this proof, we deduce that  $\mathcal{W}_\rho^\rho(\mu, \nu) \leq K_\rho \mathcal{W}_q(\mu, \nu) \sigma_{\frac{q(\rho-1)}{q-1}}^{\rho-1}(\nu)$ .

To check that  $\overline{C}_{(\rho,q),1} = +\infty$  for  $\rho \in [1, +\infty)$  and  $q \in [1, +\infty]$ , let us introduce for  $n \geq 2$  and  $z > 0$ ,

$$\mu_{n,z} = \frac{1}{2((n-1)z+1)} \left( (1+z)(\delta_1 + \delta_n) + 2z \sum_{i=2}^{n-1} \delta_i \right)$$

$$\text{and } \nu_{n,z} = \frac{1}{2((n-1)z+1)} \left( \delta_{1-z} + \delta_{n+z} + z(\delta_1 + \delta_n) + 2z \sum_{i=2}^{n-1} \delta_i \right).$$

This example generalizes the one introduced by Brücknerhoff and Juillet in [2] which corresponds to the choice  $z = 1$ . Since

$$M_{n,z} = \frac{1}{2((n-1)z+1)} \left( \delta_{(1,1-z)} + z\delta_{(1,2)} + z\delta_{(n,n-1)} + \delta_{(n,n+z)} + z \sum_{i=2}^{n-1} (\delta_{(i,i-1)} + \delta_{(i,i+1)}) \right)$$

belongs to  $\Pi^M(\mu_{n,z}, \nu_{n,z})$ , we have

$$\overline{\mathcal{M}}_\rho^\rho(\mu_{n,z}, \nu_{n,z}) \geq \int_{\mathbb{R} \times \mathbb{R}} |y-x|^\rho M_{n,z}(dx, dy) = \frac{(n-1)z + z^\rho}{(n-1)z + 1}.$$

On the other hand, by optimality of the comonotonic coupling  $\mathcal{W}_\rho^\rho(\mu_{n,z}, \nu_{n,z}) = \frac{z^\rho}{(n-1)z+1}$  for  $\rho \in [1, +\infty)$  and  $\mathcal{W}_\infty(\mu_{n,z}, \nu_{n,z}) = z$ . Last  $\sigma_\infty(\nu_{n,z}) = \frac{n-1+2z}{2}$  and, when  $\rho \in [1, +\infty)$ ,

$$\sigma_\rho^\rho(\nu_{n,z}) = \frac{1}{2^\rho((n-1)z+1)} \left( (n-1+2z)^\rho + z(n-1)^\rho + 2z \sum_{i=2}^{\lfloor \frac{n+1}{2} \rfloor} (n+1-2i)^\rho \right).$$

Let  $\alpha \in [0, 1)$ . The sequence  $n^{1-\alpha}$  goes to  $\infty$  with  $n$  and for  $\rho \in [1, +\infty)$  and  $q \in [1, +\infty]$ , we have

$$\int_{\mathbb{R} \times \mathbb{R}} |y-x|^\rho M_{n,n^{-\alpha}}(dx, dy) \rightarrow 1, \quad \mathcal{W}_q(\mu_{n,n^{-\alpha}}, \nu_{n,n^{-\alpha}}) \sim n^{\alpha \frac{(1-q)}{q} - \frac{1}{q}}$$

and  $\sigma_{\frac{q(\rho-1)}{q-1}}^{\rho-1}(\nu_{n,n^{-\alpha}}) \sim \frac{n^{\rho-1}}{2^{\rho-1}(1+\frac{q(\rho-1)}{q-1})^{\frac{q-1}{q}}}$  where  $\left(1 + \frac{q(\rho-1)}{q-1}\right)^{\frac{q-1}{q}} = 1$  by convention when  $q = 1$  so that

$$\frac{\int_{\mathbb{R} \times \mathbb{R}} |y-x|^\rho M_{n,n^{-\alpha}}(dx, dy)}{\mathcal{W}_q(\mu_{n,n^{-\alpha}}, \nu_{n,n^{-\alpha}}) \sigma_{\frac{q(\rho-1)}{q-1}}^{\rho-1}(\nu_{n,n^{-\alpha}})} \sim 2^{\rho-1} \left(1 + \frac{q(\rho-1)}{q-1}\right)^{\frac{q-1}{q}} n^{\frac{q-1}{q}\alpha + \frac{1}{q} + 1 - \rho}.$$

Let  $\rho \in [1, 2)$ . For  $q = 1$ , the exponent of  $n$  in the equivalent of the ratio is equal to  $2 - \rho > 0$  so that the right-hand side goes to  $+\infty$  with  $n$ . For  $q \in (1, +\infty]$ , we may choose  $\alpha \in \left(\frac{q(\rho-1)-1}{q-1}, 1\right)$  (with left boundary equal to  $\rho-1$  when  $q = +\infty$ ) so that  $\frac{q-1}{q}\alpha + \frac{1}{q} + 1 - \rho > 0$  and the right-hand side still goes to  $+\infty$  with  $n$ . Therefore  $\overline{C}_{(\rho,q),1} = +\infty$ . To prove that  $\underline{C}_{(\rho,q),d} = +\infty$  for  $d \geq 2$  it is enough by [5, Lemma 1] to deal with the case  $d = 2$ , in which we use the rotation argument in [2]. For  $n \geq 2$  and  $\theta \in (0, \pi)$ ,  $M_n^\theta$  defined as  $\frac{1}{2((n-1)n^{-\alpha}+1)}$  times

$$\delta_{((1,0),(1-n^{-\alpha}\cos\theta, -n^{-\alpha}\sin\theta))} + n^{-\alpha}\delta_{((1,0),(1+\cos\theta, \sin\theta))} + n^{-\alpha}\delta_{((n,0),(n-\cos\theta, -\sin\theta))}$$

$$+ \delta_{((n,0),(n+n^{-\alpha}\cos\theta, n^{-\alpha}\sin\theta))} + n^{-\alpha} \sum_{i=2}^{n-1} (\delta_{((i,0),(i-\cos\theta, -\sin\theta))} + \delta_{((i,0),(i+\cos\theta, \sin\theta))})$$

which is a martingale coupling between the image  $\mu_n$  of  $\mu_{n,n^{-\alpha}}$  by  $\mathbb{R} \ni x \mapsto (x, 0) \in \mathbb{R}^2$  and its second marginal  $\nu_n^\theta$  which, as  $\theta \rightarrow 0$ , converges in any  $\mathcal{W}_q$  with  $q \in [1, +\infty]$  to the image of  $\nu_{n,n^{-\alpha}}$

by the same mapping. According to the proof of [2, Lemma 1.1],  $\Pi^M(\mu_n, \nu_n^\theta) = \{M_n^\theta\}$  so that  $\underline{\mathcal{M}}_\rho^\rho(\mu_n, \nu_n^\theta) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} |y - x|^\rho M_n^\theta(dx, dy)$  and

$$\lim_{\theta \rightarrow 0} \frac{\underline{\mathcal{M}}_\rho^\rho(\mu_n, \nu_n^\theta)}{\mathcal{W}_q(\mu_n, \nu_n^\theta) \sigma_{\frac{q(\rho-1)}{q-1}}^{\rho-1}(\nu_n^\theta)} = \frac{\int_{\mathbb{R} \times \mathbb{R}} |y - x|^\rho M_{n, n^{-\alpha}}(dx, dy)}{\mathcal{W}_q(\mu_{n, n^{-\alpha}}, \nu_{n, n^{-\alpha}}) \sigma_{\frac{q(\rho-1)}{q-1}}^{\rho-1}(\nu_{n, n^{-\alpha}})}.$$

With the above analysis of the asymptotic behaviour of the right-hand side as  $n \rightarrow \infty$ , we conclude that  $\underline{C}_{(\rho, q), d} = +\infty$ .

(ii) Now, let  $\rho \in [2, +\infty)$  and  $M \in \Pi^M(\mu, \nu)$ . Applying Equation (4) in Lemma 3 for the inequality and then using the martingale property of  $M$ , we obtain that for  $c \in \mathbb{R}^d$ , we have

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^\rho M(dx, dy) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \|(x - c) - (y - c)\|^\rho M(dx, dy) \\ &\leq \kappa_\rho \int_{\mathbb{R}^d \times \mathbb{R}^d} ((\rho - 1)\|x - c\|^\rho + \|y - c\|^\rho - \rho\|x - c\|^{\rho-2} \langle x - c, y - c \rangle) M(dx, dy) \\ &= \kappa_\rho \left( \int_{\mathbb{R}^d} \|y - c\|^\rho \nu(dy) - \int_{\mathbb{R}^d} \|x - c\|^\rho \mu(dx) \right). \end{aligned} \quad (6)$$

Denoting by  $\pi \in \Pi(\mu, \nu)$  an optimal coupling for  $\mathcal{W}_q(\mu, \nu)$ , we have using Equation (5) in Lemma 3 for the inequality

$$\begin{aligned} \int_{\mathbb{R}^d} \|y - c\|^\rho \nu(dy) - \int_{\mathbb{R}^d} \|x - c\|^\rho \mu(dx) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} (\|y - c\|^\rho - \|x - c\|^\rho) \pi(dx, dy) \\ &\leq \tilde{\kappa}_\rho \int_{\mathbb{R}^d \times \mathbb{R}^d} \|y - x\| \left( \|x - c\|^{\rho-1} + \|y - c\|^{\rho-1} \right) \pi(dx, dy). \end{aligned} \quad (7)$$

By the fact that all norms are equivalent in finite dimensional vector spaces, there exists  $\lambda \in [1, \infty)$  such that for all  $z \in \mathbb{R}^d$ , we have

$$\frac{\|z\|}{\lambda} \leq |z| \leq \lambda \|z\|.$$

Therefore, using (6) and (7) for the second inequality, Hölder's inequality for the fourth, the triangle inequality for the fifth and  $\mu \leq_{cx} \nu$  for the sixth, we get that for  $c \in \mathbb{R}^d$ ,

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^\rho M(dx, dy) &\leq \lambda^\rho \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^\rho M(dx, dy) \\ &\leq \kappa_\rho \tilde{\kappa}_\rho \lambda^\rho \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\| \left( \|x - c\|^{\rho-1} + \|y - c\|^{\rho-1} \right) \pi(dx, dy) \\ &\leq \kappa_\rho \tilde{\kappa}_\rho \lambda^{2\rho} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| \left( |x - c|^{\rho-1} + |y - c|^{\rho-1} \right) \pi(dx, dy) \\ &\leq \kappa_\rho \tilde{\kappa}_\rho \lambda^{2\rho} \mathcal{W}_q(\mu, \nu) \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |x - c|^{\rho-1} + |y - c|^{\rho-1} \right)^{\frac{q}{q-1}} \pi(dx, dy) \right)^{\frac{q-1}{q}} \\ &\leq \kappa_\rho \tilde{\kappa}_\rho \lambda^{2\rho} \mathcal{W}_q(\mu, \nu) \left( \left( \int_{\mathbb{R}^d} |x - c|^{\frac{q(\rho-1)}{q-1}} \mu(dx) \right)^{(q-1)/q} + \left( \int_{\mathbb{R}^d} |y - c|^{\frac{q(\rho-1)}{q-1}} \nu(dy) \right)^{(q-1)/q} \right) \\ &\leq 2\kappa_\rho \tilde{\kappa}_\rho \lambda^{2\rho} \mathcal{W}_q(\mu, \nu) \left( \int_{\mathbb{R}^d} |y - c|^{\frac{q(\rho-1)}{q-1}} \nu(dy) \right)^{\frac{q-1}{q}}. \end{aligned}$$

By taking the infimum with respect to  $c \in \mathbb{R}^d$ , we conclude that the statement holds with  $\overline{C}_{(\rho, q), d} \leq 2\kappa_\rho \tilde{\kappa}_\rho \lambda^{2\rho}$ . Finally, let us suppose that  $\mathbb{R}^d$  is endowed with the Euclidean norm. Then we can choose  $\lambda = 1$ , so that  $\overline{C}_{(\rho, q), d} \leq 2\kappa_\rho \tilde{\kappa}_\rho$  with the right-hand side not depending on  $d$  according to Lemma 3. Moreover, by Remark 4,  $\overline{C}_{(2, q), d} \leq 2$  and since for  $\alpha \in [0, 1)$ ,

$$\lim_{n \rightarrow \infty} \frac{\overline{\mathcal{M}}_2^2(\mu_{n, n^{-\alpha}}, \nu_{n, n^{-\alpha}})}{\sqrt{\mathcal{W}_1(\mu_{n, n^{-\alpha}}, \nu_{n, n^{-\alpha}}) \sigma_\infty(\nu_{n, n^{-\alpha}})}} = 2,$$

we have  $\overline{C}_{(2, q), d} = 2$ . □

*Proof of Lemma 3.* We suppose that  $\rho > 2$  since the case  $\rho = 2$  has been addressed in Remark 4.

Suppose  $x \neq 0$  and  $y \neq x$  and set  $e = \frac{x}{\|x\|}$  and  $z = \frac{\langle y, x \rangle}{\|x\|^2}$ . The vector  $\frac{y}{\|x\|} - ze$  is orthogonal to  $e$  and can be rewritten as  $\omega e^\perp$  with  $\omega \geq 0$  and  $e^\perp \in \mathbb{R}^d$  such that  $\|e^\perp\| = 1$  and  $\langle e, e^\perp \rangle = 0$ . One then has  $\frac{y}{\|x\|} = ze + \omega e^\perp$  and since  $y \neq x$ ,  $(z, \omega) \neq (1, 0)$ .

The first inequality (4) divided by  $\|x\|^\rho$  writes:

$$((1-z)^2 + \omega^2)^{\frac{\rho}{2}} \leq \kappa_\rho \left( (\rho-1) + (z^2 + \omega^2)^{\frac{\rho}{2}} - \rho z \right).$$

Let us define  $\varphi(z, \omega) = \rho - 1 + (z^2 + \omega^2)^{\frac{\rho}{2}} - \rho z = -\rho(z-1) - 1 + (1 + 2(z-1) + (z-1)^2 + \omega^2)^{\frac{\rho}{2}}$  as the second factor in the right-hand side. Applying a Taylor's expansion at  $t = 0$  to  $t \mapsto (1+t)^{\frac{\rho}{2}}$ , we obtain

$$\varphi(z, \omega) = \frac{\rho}{2}\omega^2 + \frac{\rho}{2}(\rho-1)(z-1)^2 + o((z-1)^2 + \omega^2).$$

Since  $\rho > 2$ , we conclude that

$$\lim_{(z, \omega) \rightarrow (1, 0)} \frac{((1-z)^2 + \omega^2)^{\frac{\rho}{2}}}{\varphi(z, \omega)} = 0.$$

As  $|(z, \omega)| \rightarrow +\infty$ ,  $\varphi(z, \omega) \sim (z^2 + \omega^2)^{\frac{\rho}{2}} \sim ((z-1)^2 + \omega^2)^\rho$ . Therefore,

$$\lim_{|(z, \omega)| \rightarrow +\infty} \frac{((z-1)^2 + \omega^2)^{\frac{\rho}{2}}}{\varphi(z, \omega)} = 1.$$

The function  $(z, \omega) \mapsto \frac{((z-1)^2 + \omega^2)^{\frac{\rho}{2}}}{\varphi(z, \omega)}$  being continuous on  $\mathbb{R}^2 \setminus \{(1, 0)\}$ , we deduce that

$$1 \leq \sup_{(z, \omega) \neq (1, 0)} \frac{((z-1)^2 + \omega^2)^{\frac{\rho}{2}}}{\varphi(z, \omega)} < +\infty.$$

Since when  $x = 0$  or  $y = x$ , (4) holds with  $\kappa_\rho$  replaced by 1, we conclude that the optimal constant is  $\kappa_\rho = \sup_{(z, \omega) \neq (1, 0)} \frac{((z-1)^2 + \omega^2)^{\frac{\rho}{2}}}{\varphi(z, \omega)}$ .

For the second inequality (5), we can apply the same approach: divided by  $\|x\|^\rho$ , it writes

$$(z^2 + \omega^2)^{\frac{\rho}{2}} - 1 \leq \tilde{\kappa}_\rho ((z-1)^2 + \omega^2)^{\frac{1}{2}} \left( (z^2 + \omega^2)^{\frac{\rho-1}{2}} + 1 \right).$$

As  $(z, \omega) \rightarrow (1, 0)$ ,  $(z^2 + \omega^2)^{\frac{\rho}{2}} - 1 = (1 + 2(z-1) + (z-1)^2 + \omega^2)^{\frac{\rho}{2}} - 1 \sim \frac{\rho}{2} (2(z-1) + \omega^2)$

$$\limsup_{(z, \omega) \rightarrow (1, 0)} \frac{(z^2 + \omega^2)^{\frac{\rho}{2}} - 1}{((z-1)^2 + \omega^2)^{\frac{1}{2}} \left( 1 + (z^2 + \omega^2)^{\frac{\rho-1}{2}} \right)} = \limsup_{z \rightarrow 1} \frac{\rho(z-1)}{2|z-1|} = \frac{\rho}{2}.$$

On the other hand,

$$\lim_{|(z, \omega)| \rightarrow +\infty} \frac{(z^2 + \omega^2)^{\frac{\rho}{2}} - 1}{((z-1)^2 + \omega^2)^{\frac{1}{2}} \left( 1 + (z^2 + \omega^2)^{\frac{\rho-1}{2}} \right)} = 1.$$

By continuity of the considered function over  $\mathbb{R}^2 \setminus \{(1, 0)\}$ , we deduce that

$$\frac{\rho}{2} \vee 1 \leq \sup_{(z, \omega) \neq (1, 0)} \frac{(z^2 + \omega^2)^{\frac{\rho}{2}} - 1}{((z-1)^2 + \omega^2)^{\frac{1}{2}} \left( 1 + (z^2 + \omega^2)^{\frac{\rho-1}{2}} \right)} < +\infty.$$

Since when  $x = 0$  or  $y = x$ , (5) holds with  $\tilde{\kappa}_\rho$  replaced by 1, we conclude that the optimal constant

$$\text{is } \tilde{\kappa}_\rho = \sup_{(z, \omega) \neq (1, 0)} \frac{(z^2 + \omega^2)^{\frac{\rho}{2}} - 1}{((z-1)^2 + \omega^2)^{\frac{1}{2}} \left( 1 + (z^2 + \omega^2)^{\frac{\rho-1}{2}} \right)}. \quad \square$$

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