

Maximal Martingale Wasserstein Inequality

Benjamin Jourdain* Kexin Shao†

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Abstract

In this note, we complete the analysis of the Martingale Wasserstein Inequality started in [5] by checking that this inequality fails in dimension $d \geq 2$ when the integrability parameter ρ belongs to $[1, 2)$ while a stronger Maximal Martingale Wasserstein Inequality holds whatever the dimension d when $\rho \geq 2$.

1 Introduction

The present paper elaborates on the convergence to 0 as $n \rightarrow \infty$ of $\inf_{M \in \Pi^M(\mu_n, \nu_n)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^\rho M(dx, dy)$ with the Wasserstein distance $\mathcal{W}_\rho(\mu_n, \nu_n)$ when for each $n \in \mathbb{N}$, μ_n and ν_n belong to the set $\mathcal{P}_\rho(\mathbb{R}^d)$ of probability measures on \mathbb{R}^d with a finite moment of order $\rho \in [1, +\infty)$ and the former is smaller than the latter in the convex order. The convex order between $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$ which is denoted $\mu \leq_{cx} \nu$ amounts to

$$\int_{\mathbb{R}^d} f(x) \mu(dx) \leq \int_{\mathbb{R}^d} f(y) \nu(dy) \text{ for each convex function } f : \mathbb{R}^d \rightarrow \mathbb{R}, \quad (1)$$

and, by Strassen's theorem [7], is equivalent to the non emptiness of the set of martingale couplings between μ and ν defined by

$$\Pi^M(\mu, \nu) = \left\{ M(dx, dy) = \mu(dx) m(x, dy) \in \Pi(\mu, \nu) \mid \mu(dx)\text{-a.e.}, \int_{\mathbb{R}^d} y m(x, dy) = x \right\} \text{ where}$$
$$\Pi(\mu, \nu) = \{ \pi \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d) \mid \forall A \in \mathcal{B}(\mathbb{R}^d), \pi(A \times \mathbb{R}^d) = \mu(A) \text{ and } \pi(\mathbb{R}^d \times A) = \nu(A) \}.$$

The Wasserstein distance with index ρ is defined by

$$\mathcal{W}_\rho(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^\rho \pi(dx, dy) \right)^{1/\rho}$$

and we also introduce $\underline{\mathcal{M}}_\rho(\mu, \nu)$ and $\overline{\mathcal{M}}_\rho(\mu, \nu)$ respectively defined by

$$\underline{\mathcal{M}}_\rho^\rho(\mu, \nu) = \inf_{M \in \Pi^M(\mu, \nu)} \int_{\mathbb{R}^{2d}} |x - y|^\rho M(dx, dy), \quad \overline{\mathcal{M}}_\rho^\rho(\mu, \nu) = \sup_{M \in \Pi^M(\mu, \nu)} \int_{\mathbb{R}^{2d}} |x - y|^\rho M(dx, dy). \quad (2)$$

In dimension $d = 1$, the optimization problems defining $\underline{\mathcal{M}}_\rho$ and $\overline{\mathcal{M}}_\rho$ are the respective subjects of [3] and [4] when $\rho = 1$, while the general case $\rho \in (0, +\infty)$ is studied in [6].

The question of interest is related to the stability of Martingale Optimal Transport problems with respect to the marginal distributions μ and ν established in dimension $d = 1$ in [1, 8] while

*CERMICS, Ecole des Ponts, INRIA, Marne-la-Vallée, France. E-mail: benjamin.jourdain@enpc.fr - This research benefited from the support of the "Chaire Risques Financiers", Fondation du Risque.

†INRIA Paris, 2 rue Simone Iff, CS 42112, 75589 Paris Cedex 12, France, Université Paris-Dauphine, Ecole des Ponts ParisTech. E-mail: kexin.shao@inria.fr. This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 945322.

it fails in higher dimension according to [2]. A quantitative answer is given in dimension $d = 1$ by the Martingale Wasserstein inequality established in [5, Proposition 1] for $\rho \in [1, +\infty)$,

$$\exists \underline{C}_{(\rho,\rho),1} < \infty, \forall \mu, \nu \in \mathcal{P}_\rho(\mathbb{R}) \text{ with } \mu \leq_{cx} \nu, \underline{M}_\rho^\rho(\mu, \nu) \leq \underline{C}_{(\rho,\rho),1} \mathcal{W}_\rho(\mu, \nu) \sigma_\rho^{\rho-1}(\nu), \quad (3)$$

where the central moment $\sigma_\rho(\nu)$ of ν is defined by

$$\sigma_\rho(\nu) = \inf_{c \in \mathbb{R}^d} \left(\int_{\mathbb{R}^d} |y - c|^\rho \nu(dy) \right)^{1/\rho} \text{ when } \rho \in [1, +\infty) \text{ and } \sigma_\infty(\nu) = \inf_{c \in \mathbb{R}^d} \nu - \text{ess sup}_{y \in \mathbb{R}^d} |y - c|.$$

The proposition also states that $\mathcal{W}_\rho(\mu, \nu)$ and $\sigma_\rho(\nu)$ have the right exponent in this inequality in the sense that for $1 < s < \rho$, $\sup_{\substack{\mu, \nu \in \mathcal{P}_\rho(\mathbb{R}) \\ \mu \leq_{cx} \nu, \mu \neq \nu}} \frac{\underline{M}_\rho^\rho(\mu, \nu)}{\mathcal{W}_\rho^s(\mu, \nu) \sigma_\rho^{\rho-s}(\nu)} = +\infty$. The generalization of (3) to higher dimensions d is also investigated in [5] where it is proved that for any $d \geq 2$,

$$\underline{C}_{(\rho,\rho),d} := \sup_{\substack{\mu, \nu \in \mathcal{P}_\rho(\mathbb{R}^d) \\ \mu \leq_{cx} \nu, \mu \neq \nu}} \frac{\underline{M}_\rho^\rho(\mu, \nu)}{\mathcal{W}_\rho(\mu, \nu) \sigma_\rho^{\rho-1}(\nu)}$$

is infinite when $\rho \in [1, \frac{1+\sqrt{5}}{2})$, while the one-dimensional constant $\underline{C}_{(\rho,\rho),1}$ is preserved when μ and ν are products of one-dimensional probability measures or when, for X distributed according to μ , the conditional expectation of X given the direction of $X - \mathbb{E}[X]$ is a.s. equal to $\mathbb{E}[X]$ and ν is the distribution of $X + \lambda(X - \mathbb{E}[X])$ for some $\lambda \geq 0$. The present paper answers the question of the finiteness of $\underline{C}_{(\rho,\rho),d}$ when $\rho \in [\frac{1+\sqrt{5}}{2}, +\infty)$ and $d \geq 2$, which remained open. It turns out that $\underline{C}_{(\rho,\rho),d} = +\infty$ for $d \geq 2$ when $\rho \in [1, 2)$ while for $\rho \in [2, +\infty)$ the inequality (3) generalizes in any dimension d into a Maximal Martingale Wasserstein inequality with the left-hand side $\underline{M}_\rho^\rho(\mu, \nu)$ replaced by the larger $\overline{M}_\rho^\rho(\mu, \nu)$. We even replace conjugate exponents ρ and $\frac{\rho}{\rho-1}$ leading to the respective indices $\rho = \rho \times 1$ and $\rho = \frac{\rho}{\rho-1} \times (\rho - 1)$ in the factors \mathcal{W} and σ in (3) by general conjugate exponents $q \in [1, +\infty]$ and $\frac{q}{q-1} \in [1, +\infty]$ leading to indices q and $\frac{q(\rho-1)}{q-1}$ (equal to $+\infty$ and $\rho - 1$ when q is respectively equal to 1 and $+\infty$) and define

$$\underline{C}_{(\rho,q),d} := \sup_{\substack{\mu, \nu \in \mathcal{P}_{q \vee \frac{(\rho-1)q}{q-1}}(\mathbb{R}^d) \\ \mu \leq_{cx} \nu, \mu \neq \nu}} \frac{\underline{M}_\rho^\rho(\mu, \nu)}{\mathcal{W}_q(\mu, \nu) \sigma_{\frac{q(\rho-1)}{q-1}}^{\rho-1}(\nu)} \text{ and } \overline{C}_{(\rho,q),d} := \sup_{\substack{\mu, \nu \in \mathcal{P}_{q \vee \frac{(\rho-1)q}{q-1}}(\mathbb{R}^d) \\ \mu \leq_{cx} \nu, \mu \neq \nu}} \frac{\overline{M}_\rho^\rho(\mu, \nu)}{\mathcal{W}_q(\mu, \nu) \sigma_{\frac{q(\rho-1)}{q-1}}^{\rho-1}(\nu)},$$

with $\mathcal{W}_\infty(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \pi - \text{ess sup}_{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d} |x - y|$. Since $\underline{M}_\rho \leq \overline{M}_\rho$, one has $\underline{C}_{(\rho,q),d} \leq \overline{C}_{(\rho,q),d}$. These constants of course depend on the norm $|\cdot|$ on \mathbb{R}^d (even if we do not make this dependence explicit) but, by equivalence of the norms, their finiteness does not. Since the Euclidean norm plays a particular role, we will denote it by $\|\cdot\|$ rather than $|\cdot|$.

Theorem 1. (i) Let $\rho \in [1, 2)$. For $q \in [1, \frac{1}{2-\rho}]$ (and even $q \in [1, +\infty]$ when $\rho = 1$), one has $\underline{C}_{(\rho,q),1} \leq K_\rho < +\infty$ where the constant K_ρ is studied in [5, Proposition 1] while, for $q \in [1, +\infty]$, $\overline{C}_{(\rho,q),1} = +\infty$ and $\underline{C}_{(\rho,q),d} = +\infty$ for $d \geq 2$.

(ii) Let $\rho \in [2, +\infty)$ and $q \in [1, +\infty]$. One has $\overline{C}_{(\rho,q),d} < +\infty$ whatever d . Moreover, when \mathbb{R}^d (resp. each \mathbb{R}^d) is endowed with the Euclidean norm, $\overline{C}_{(2,q),d} = 2$ and $\sup_{d \geq 1} \overline{C}_{(\rho,q),d} < +\infty$.

Remark 2. • The fact that $\rho = 2$ appears as a threshold is related to the equality $\int_{\mathbb{R}^d \times \mathbb{R}^d} \|y - x\|^2 M(dx, dy) = \int_{\mathbb{R}^d} \|y\|^2 \nu(dy) - \int_{\mathbb{R}^d} \|x\|^2 \mu(dx)$ for $M \in \Pi^M(\mu, \nu)$ when $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ are such that $\mu \leq_{cx} \nu$, which implies that when \mathbb{R}^d is endowed with the Euclidean norm

$$\underline{M}_2^2(\mu, \nu) = \overline{M}_2^2(\mu, \nu) = \int_{\mathbb{R}^d} \|y\|^2 \nu(dy) - \int_{\mathbb{R}^d} \|x\|^2 \mu(dx).$$

• For $\rho \in [1, 2)$, one has $\overline{C}_{(\rho,q),d} = +\infty$ while $\sup_{\substack{\mu, \nu \in \mathcal{P}_{q \vee \frac{q}{q-1}}(\mathbb{R}^d) \\ \mu \leq_{cx} \nu, \mu \neq \nu}} \frac{\overline{M}_\rho^2(\mu, \nu)}{\mathcal{W}_q(\mu, \nu) \sigma_{\frac{q}{q-1}}(\nu)} \leq \overline{C}_{(2,q),d} < +\infty$ since $\overline{M}_\rho \leq \overline{M}_2$.

2 Proof

The proof of Theorem 1 (ii) relies on the next lemma, the proof of the lemma is postponed after the proof of the theorem. In what follows, to avoid making distinctions in case $q \in \{1, +\infty\}$, we use the convention that for any probability measure γ and any measurable function f on the same probability space $(\int |f(z)|^q \gamma(dz))^{1/q}$ (resp. $\left(\left(\int |f(z)|^{\frac{q}{q-1}} \gamma(dz)\right)^{(q-1)/q}, \left(\int |f(z)|^{\frac{q(\rho-1)}{q-1}} \gamma(dz)\right)^{(q-1)/q}\right)$) is equal to γ -ess sup $_z |f(z)|$ (resp. $(\gamma$ -ess sup $_z |f(z)|, \gamma$ -ess sup $_z |f(z)|^{\rho-1})$) when $q = +\infty$ (resp. $q = 1$).

Lemma 3. *Given $\rho \in [2, +\infty)$, there exist constants $\kappa_\rho, \tilde{\kappa}_\rho \in [0, +\infty)$ such that for all $d \geq 1$ and $x, y \in \mathbb{R}^d$,*

$$\|x - y\|^\rho \leq \kappa_\rho ((\rho - 1)\|x\|^\rho + \|y\|^\rho - \rho\|x\|^{\rho-2}\langle x, y \rangle), \quad (4)$$

$$\|y\|^\rho - \|x\|^\rho \leq \tilde{\kappa}_\rho \|y - x\| \left(\|x\|^{\rho-1} + \|y\|^{\rho-1} \right). \quad (5)$$

Remark 4. *When $\rho = 2$, then (4) holds as an equality with $\kappa_\rho = 1$ while, by the Cauchy-Schwarz and the triangle inequalities,*

$$\|y\|^2 - \|x\|^2 \leq \langle y - x, y + x \rangle \leq \|y - x\| \times \|y + x\| \leq \|y - x\| (\|x\| + \|y\|)$$

so that (5) holds with $\tilde{\kappa}_\rho = 1$.

Proof of Theorem 1. (i) In dimension $d = 1$, one has $\underline{\mathcal{M}}_1 \leq K_1 \mathcal{W}_1$ with $K_1 = 2$ according to [5, Proposition 1] and we deduce that $\underline{C}_{(1,q),1} \leq K_1$ for $q \in [1, +\infty]$ since $\mathcal{W}_1 \leq \mathcal{W}_q$. Let now $\rho \in (1, 2)$ and $q \in [1, \frac{1}{2-\rho}]$. One has $\frac{q(\rho-1)}{q-1} \geq 1$ since, when $q > 1$, $\frac{q}{q-1} = 1 + \frac{1}{q-1} \geq 1 + \frac{2-\rho}{\rho-1} = \frac{1}{\rho-1}$. For $\mu, \nu \in \mathcal{P}_{q\sqrt{\frac{q(\rho-1)}{q-1}}}(\mathbb{R})$ with respective quantile functions F_μ^{-1} and F_ν^{-1} , one has by optimality of the comonotonic coupling and Hölder's inequality

$$\begin{aligned} \mathcal{W}_\rho^\rho(\mu, \nu) &= \int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)| \times |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^{\rho-1} du \\ &\leq \left(\int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^q du \right)^{1/q} \left(\left(\int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^{\frac{q(\rho-1)}{q-1}} du \right)^{\frac{q-1}{q(\rho-1)}} \right)^{\rho-1}. \end{aligned}$$

Since, by the triangle inequality and $\mu \leq_{cx} \nu$, one has for $c \in \mathbb{R}$

$$\begin{aligned} \left(\int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)|^{\frac{q(\rho-1)}{q-1}} du \right)^{\frac{q-1}{q(\rho-1)}} &\leq \left(\int_0^1 |F_\nu^{-1}(u) - c|^{\frac{q(\rho-1)}{q-1}} du \right)^{\frac{q-1}{q(\rho-1)}} + \left(\int_0^1 |F_\mu^{-1}(u) - c|^{\frac{q(\rho-1)}{q-1}} du \right)^{\frac{q-1}{q(\rho-1)}} \\ &\leq 2 \left(\int_0^1 |F_\nu^{-1}(u) - c|^{\frac{q(\rho-1)}{q-1}} du \right)^{\frac{q-1}{q(\rho-1)}}, \end{aligned}$$

we deduce by minimizing over the constant c that

$$\mathcal{W}_\rho^\rho(\mu, \nu) \leq \mathcal{W}_q(\mu, \nu) \times 2^{\rho-1} \sigma_{\frac{q(\rho-1)}{q-1}}^{\rho-1}(\nu).$$

With this inequality replacing (30) in the proof of Proposition 1 [5] and the general inequality

$$\int_0^1 |F_\nu^{-1}(u) - F_\mu^{-1}(u)| |F_\nu^{-1}(u) - c|^{\rho-1} du \leq \mathcal{W}_q(\mu, \nu) \left(\int_0^1 |F_\nu^{-1}(u) - c|^{\frac{q(\rho-1)}{q-1}} du \right)^{\frac{q-1}{q}},$$

replacing the special case $q = \rho$ in the second equation p840 in this proof, we deduce that $\mathcal{W}_\rho^\rho(\mu, \nu) \leq K_\rho \mathcal{W}_q(\mu, \nu) \sigma_{\frac{q(\rho-1)}{q-1}}^{\rho-1}(\nu)$.

To check that $\overline{C}_{(\rho,q),1} = +\infty$ for $\rho \in [1, +\infty)$ and $q \in [1, +\infty]$, let us introduce for $n \geq 2$ and $z > 0$,

$$\mu_{n,z} = \frac{1}{2((n-1)z+1)} \left((1+z)(\delta_1 + \delta_n) + 2z \sum_{i=2}^{n-1} \delta_i \right)$$

and $\nu_{n,z} = \frac{1}{2((n-1)z+1)} \left(\delta_{1-z} + \delta_{n+z} + z(\delta_1 + \delta_n) + 2z \sum_{i=2}^{n-1} \delta_i \right).$

This example generalizes the one introduced by Brückerohoff and Juillet in [2] which corresponds to the choice $z = 1$. Since

$$M_{n,z} = \frac{1}{2((n-1)z+1)} \left(\delta_{(1,1-z)} + z\delta_{(1,2)} + z\delta_{(n,n-1)} + \delta_{(n,n+z)} + z \sum_{i=2}^{n-1} (\delta_{(i,i-1)} + \delta_{(i,i+1)}) \right)$$

belongs to $\Pi^M(\mu_{n,z}, \nu_{n,z})$, we have

$$\overline{\mathcal{M}}_\rho^\rho(\mu_{n,z}, \nu_{n,z}) \geq \int_{\mathbb{R} \times \mathbb{R}} |y-x|^\rho M_{n,z}(dx, dy) = \frac{(n-1)z + z^\rho}{(n-1)z + 1}.$$

On the other hand, by optimality of the comonotonic coupling $\mathcal{W}_\rho^\rho(\mu_{n,z}, \nu_{n,z}) = \frac{z^\rho}{(n-1)z+1}$ for $\rho \in [1, +\infty)$ and $\mathcal{W}_\infty(\mu_{n,z}, \nu_{n,z}) = z$. Last $\sigma_\infty(\nu_{n,z}) = \frac{n-1+2z}{2}$ and, when $\rho \in [1, +\infty)$,

$$\sigma_\rho^\rho(\nu_{n,z}) = \frac{1}{2^\rho((n-1)z+1)} \left((n-1+2z)^\rho + z(n-1)^\rho + 2z \sum_{i=2}^{\lfloor \frac{n+1}{2} \rfloor} (n+1-2i)^\rho \right).$$

Let $\alpha \in [0, 1)$. The sequence $n^{1-\alpha}$ goes to ∞ with n and for $\rho \in [1, +\infty)$ and $q \in [1, +\infty]$, we have

$$\int_{\mathbb{R} \times \mathbb{R}} |y-x|^\rho M_{n,n^{-\alpha}}(dx, dy) \rightarrow 1, \quad \mathcal{W}_q(\mu_{n,n^{-\alpha}}, \nu_{n,n^{-\alpha}}) \sim n^{\alpha \frac{(1-q)}{q} - \frac{1}{q}}$$

and $\sigma_{\frac{q(\rho-1)}{q-1}}^{\rho-1}(\nu_{n,n^{-\alpha}}) \sim \frac{n^{\rho-1}}{2^{\rho-1} \left(1 + \frac{q(\rho-1)}{q-1}\right)^{\frac{q-1}{q}}}$ where $\left(1 + \frac{q(\rho-1)}{q-1}\right)^{\frac{q-1}{q}} = 1$ by convention when $q = 1$ so that

$$\frac{\int_{\mathbb{R} \times \mathbb{R}} |y-x|^\rho M_{n,n^{-\alpha}}(dx, dy)}{\mathcal{W}_q(\mu_{n,n^{-\alpha}}, \nu_{n,n^{-\alpha}}) \sigma_{\frac{q(\rho-1)}{q-1}}^{\rho-1}(\nu_{n,n^{-\alpha}})} \sim 2^{\rho-1} \left(1 + \frac{q(\rho-1)}{q-1}\right)^{\frac{q-1}{q}} n^{\frac{q-1}{q}\alpha + \frac{1}{q} + 1 - \rho}.$$

Let $\rho \in [1, 2)$. For $q = 1$, the exponent of n in the equivalent of the ratio is equal to $2 - \rho > 0$ so that the right-hand side goes to $+\infty$ with n . For $q \in (1, +\infty]$, we may choose $\alpha \in \left(\frac{q(\rho-1)-1}{q-1}, 1\right)$ (with left boundary equal to $\rho - 1$ when $q = +\infty$) so that $\frac{q-1}{q}\alpha + \frac{1}{q} + 1 - \rho > 0$ and the right-hand side still goes to $+\infty$ with n . Therefore $\overline{C}_{(\rho,q),1} = +\infty$. To prove that $\underline{C}_{(\rho,q),d} = +\infty$ for $d \geq 2$ it is enough by [5, Lemma 1] to deal with the case $d = 2$, in which we use the rotation argument in [2]. For $n \geq 2$ and $\theta \in (0, \pi)$, M_n^θ defined as $\frac{1}{2((n-1)n^{-\alpha}+1)}$ times

$$\delta_{((1,0),(1-n^{-\alpha}\cos\theta, -n^{-\alpha}\sin\theta))} + n^{-\alpha}\delta_{((1,0),(1+\cos\theta, \sin\theta))} + n^{-\alpha}\delta_{((n,0),(n-\cos\theta, -\sin\theta))}$$

$$+ \delta_{((n,0),(n+n^{-\alpha}\cos\theta, n^{-\alpha}\sin\theta))} + n^{-\alpha} \sum_{i=2}^{n-1} (\delta_{((i,0),(i-\cos\theta, -\sin\theta))} + \delta_{((i,0),(i+\cos\theta, \sin\theta))})$$

which is a martingale coupling between the image μ_n of $\mu_{n,n^{-\alpha}}$ by $\mathbb{R} \ni x \mapsto (x, 0) \in \mathbb{R}^2$ and its second marginal ν_n^θ which, as $\theta \rightarrow 0$, converges in any \mathcal{W}_q with $q \in [1, +\infty]$ to the image of $\nu_{n,n^{-\alpha}}$

by the same mapping. According to the proof of [2, Lemma 1.1], $\Pi^M(\mu_n, \nu_n^\theta) = \{M_n^\theta\}$ so that $\underline{\mathcal{M}}_\rho^\theta(\mu_n, \nu_n^\theta) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} |y - x|^\rho M_n^\theta(dx, dy)$ and

$$\lim_{\theta \rightarrow 0} \frac{\underline{\mathcal{M}}_\rho^\theta(\mu_n, \nu_n^\theta)}{\mathcal{W}_q(\mu_n, \nu_n^\theta) \sigma_{\frac{q(\rho-1)}{q-1}}(\nu_n^\theta)} = \frac{\int_{\mathbb{R} \times \mathbb{R}} |y - x|^\rho M_{n, n^{-\alpha}}(dx, dy)}{\mathcal{W}_q(\mu_{n, n^{-\alpha}}, \nu_{n, n^{-\alpha}}) \sigma_{\frac{q(\rho-1)}{q-1}}(\nu_{n, n^{-\alpha}})}.$$

With the above analysis of the asymptotic behaviour of the right-hand side as $n \rightarrow \infty$, we conclude that $\underline{C}_{(\rho, q), d} = +\infty$.

(ii) Now, let $\rho \in [2, +\infty)$ and $M \in \Pi^M(\mu, \nu)$. Applying Equation (4) in Lemma 3 for the inequality and then using the martingale property of M , we obtain that for $c \in \mathbb{R}^d$, we have

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^\rho M(dx, dy) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \|(x - c) - (y - c)\|^\rho M(dx, dy) \\ &\leq \kappa_\rho \int_{\mathbb{R}^d \times \mathbb{R}^d} ((\rho - 1)\|x - c\|^\rho + \|y - c\|^\rho - \rho\|x - c\|^{\rho-2} \langle x - c, y - c \rangle) M(dx, dy) \\ &= \kappa_\rho \left(\int_{\mathbb{R}^d} \|y - c\|^\rho \nu(dy) - \int_{\mathbb{R}^d} \|x - c\|^\rho \mu(dx) \right). \end{aligned} \quad (6)$$

Denoting by $\pi \in \Pi(\mu, \nu)$ an optimal coupling for $\mathcal{W}_q(\mu, \nu)$, we have using Equation (5) in Lemma 3 for the inequality

$$\begin{aligned} \int_{\mathbb{R}^d} \|y - c\|^\rho \nu(dy) - \int_{\mathbb{R}^d} \|x - c\|^\rho \mu(dx) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} (\|y - c\|^\rho - \|x - c\|^\rho) \pi(dx, dy) \\ &\leq \tilde{\kappa}_\rho \int_{\mathbb{R}^d \times \mathbb{R}^d} \|y - x\| \left(\|x - c\|^{\rho-1} + \|y - c\|^{\rho-1} \right) \pi(dx, dy). \end{aligned} \quad (7)$$

By the fact that all norms are equivalent in finite dimensional vector spaces, there exists $\lambda \in [1, \infty)$ such that for all $z \in \mathbb{R}^d$, we have

$$\frac{\|z\|}{\lambda} \leq |z| \leq \lambda \|z\|.$$

Therefore, using (6) and (7) for the second inequality, Hölder's inequality for the fourth, the triangle inequality for the fifth and $\mu \leq_{cx} \nu$ for the sixth, we get that for $c \in \mathbb{R}^d$,

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^\rho M(dx, dy) &\leq \lambda^\rho \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^\rho M(dx, dy) \\ &\leq \kappa_\rho \tilde{\kappa}_\rho \lambda^\rho \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\| \left(\|x - c\|^{\rho-1} + \|y - c\|^{\rho-1} \right) \pi(dx, dy) \\ &\leq \kappa_\rho \tilde{\kappa}_\rho \lambda^{2\rho} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| \left(|x - c|^{\rho-1} + |y - c|^{\rho-1} \right) \pi(dx, dy) \\ &\leq \kappa_\rho \tilde{\kappa}_\rho \lambda^{2\rho} \mathcal{W}_q(\mu, \nu) \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} \left(|x - c|^{\rho-1} + |y - c|^{\rho-1} \right)^{\frac{q}{q-1}} \pi(dx, dy) \right)^{\frac{q-1}{q}} \\ &\leq \kappa_\rho \tilde{\kappa}_\rho \lambda^{2\rho} \mathcal{W}_q(\mu, \nu) \left(\left(\int_{\mathbb{R}^d} |x - c|^{\frac{q(\rho-1)}{q-1}} \mu(dx) \right)^{(q-1)/q} + \left(\int_{\mathbb{R}^d} |y - c|^{\frac{q(\rho-1)}{q-1}} \nu(dy) \right)^{(q-1)/q} \right) \\ &\leq 2\kappa_\rho \tilde{\kappa}_\rho \lambda^{2\rho} \mathcal{W}_q(\mu, \nu) \left(\int_{\mathbb{R}^d} |y - c|^{\frac{q(\rho-1)}{q-1}} \nu(dy) \right)^{\frac{q-1}{q}}. \end{aligned}$$

By taking the infimum with respect to $c \in \mathbb{R}^d$, we conclude that the statement holds with $\overline{C}_{(\rho, q), d} \leq 2\kappa_\rho \tilde{\kappa}_\rho \lambda^{2\rho}$. Finally, let us suppose that \mathbb{R}^d is endowed with the Euclidean norm. Then we can choose $\lambda = 1$, so that $\overline{C}_{(\rho, q), d} \leq 2\kappa_\rho \tilde{\kappa}_\rho$ with the right-hand side not depending on d according to Lemma 3. Moreover, by Remark 4, $\overline{C}_{(2, q), d} \leq 2$ and since for $\alpha \in [0, 1)$,

$$\lim_{n \rightarrow \infty} \frac{\overline{\mathcal{M}}_2(\mu_{n, n^{-\alpha}}, \nu_{n, n^{-\alpha}})}{\sqrt{\mathcal{W}_1(\mu_{n, n^{-\alpha}}, \nu_{n, n^{-\alpha}}) \sigma_\infty(\nu_{n, n^{-\alpha}})}} = 2,$$

we have $\overline{C}_{(2, q), d} = 2$. □

Proof of Lemma 3. We suppose that $\rho > 2$ since the case $\rho = 2$ has been addressed in Remark 4.

Suppose $x \neq 0$ and $y \neq x$ and set $e = \frac{x}{\|x\|}$ and $z = \frac{\langle y, x \rangle}{\|x\|^2}$. The vector $\frac{y}{\|x\|} - ze$ is orthogonal to e and can be rewritten as ωe^\perp with $\omega \geq 0$ and $e^\perp \in \mathbb{R}^d$ such that $\|e^\perp\| = 1$ and $\langle e, e^\perp \rangle = 0$. One then has $\frac{y}{\|x\|} = ze + \omega e^\perp$ and since $y \neq x$, $(z, \omega) \neq (1, 0)$.

The first inequality (4) divided by $\|x\|^\rho$ writes:

$$\left((1-z)^2 + \omega^2 \right)^{\frac{\rho}{2}} \leq \kappa_\rho \left((\rho-1) + (z^2 + \omega^2)^{\frac{\rho}{2}} - \rho z \right).$$

Let us define $\varphi(z, \omega) = \rho - 1 + (z^2 + \omega^2)^{\frac{\rho}{2}} - \rho z = -\rho(z-1) - 1 + (1 + 2(z-1) + (z-1)^2 + \omega^2)^{\frac{\rho}{2}}$ as the second factor in the right-hand side. Applying a Taylor's expansion at $t = 0$ to $t \mapsto (1+t)^{\frac{\rho}{2}}$, we obtain

$$\varphi(z, \omega) = \frac{\rho}{2}\omega^2 + \frac{\rho}{2}(\rho-1)(z-1)^2 + o((z-1)^2 + \omega^2).$$

Since $\rho > 2$, we conclude that

$$\lim_{(z, \omega) \rightarrow (1, 0)} \frac{\left((1-z)^2 + \omega^2 \right)^{\frac{\rho}{2}}}{\varphi(z, \omega)} = 0.$$

As $|(z, \omega)| \rightarrow +\infty$, $\varphi(z, \omega) \sim (z^2 + \omega^2)^{\frac{\rho}{2}} \sim ((z-1)^2 + \omega^2)^\rho$. Therefore,

$$\lim_{|(z, \omega)| \rightarrow +\infty} \frac{\left((z-1)^2 + \omega^2 \right)^{\frac{\rho}{2}}}{\varphi(z, \omega)} = 1.$$

The function $(z, \omega) \mapsto \frac{\left((z-1)^2 + \omega^2 \right)^{\frac{\rho}{2}}}{\varphi(z, \omega)}$ being continuous on $\mathbb{R}^2 \setminus \{(1, 0)\}$, we deduce that

$$1 \leq \sup_{(z, \omega) \neq (1, 0)} \frac{\left((z-1)^2 + \omega^2 \right)^{\frac{\rho}{2}}}{\varphi(z, \omega)} < +\infty.$$

Since when $x = 0$ or $y = x$, (4) holds with κ_ρ replaced by 1, we conclude that the optimal constant is $\kappa_\rho = \sup_{(z, \omega) \neq (1, 0)} \frac{\left((z-1)^2 + \omega^2 \right)^{\frac{\rho}{2}}}{\varphi(z, \omega)}$.

For the second inequality (5), we can apply the same approach: divided by $\|x\|^\rho$, it writes

$$\left(z^2 + \omega^2 \right)^{\frac{\rho}{2}} - 1 \leq \tilde{\kappa}_\rho \left((z-1)^2 + \omega^2 \right)^{\frac{1}{2}} \left((z^2 + \omega^2)^{\frac{\rho-1}{2}} + 1 \right).$$

As $(z, \omega) \rightarrow (1, 0)$, $\left(z^2 + \omega^2 \right)^{\frac{\rho}{2}} - 1 = (1 + 2(z-1) + (z-1)^2 + \omega^2)^{\frac{\rho}{2}} - 1 \sim \frac{\rho}{2} (2(z-1) + \omega^2)$

$$\limsup_{(z, \omega) \rightarrow (1, 0)} \frac{\left(z^2 + \omega^2 \right)^{\frac{\rho}{2}} - 1}{\left((z-1)^2 + \omega^2 \right)^{\frac{1}{2}} \left(1 + (z^2 + \omega^2)^{\frac{\rho-1}{2}} \right)} = \limsup_{z \rightarrow 1} \frac{\rho(z-1)}{2|z-1|} = \frac{\rho}{2}.$$

On the other hand,

$$\lim_{|(z, \omega)| \rightarrow +\infty} \frac{\left(z^2 + \omega^2 \right)^{\frac{\rho}{2}} - 1}{\left((z-1)^2 + \omega^2 \right)^{\frac{1}{2}} \left(1 + (z^2 + \omega^2)^{\frac{\rho-1}{2}} \right)} = 1.$$

By continuity of the considered function over $\mathbb{R}^2 \setminus \{(1, 0)\}$, we deduce that

$$\frac{\rho}{2} \vee 1 \leq \sup_{(z, \omega) \neq (1, 0)} \frac{\left(z^2 + \omega^2 \right)^{\frac{\rho}{2}} - 1}{\left((z-1)^2 + \omega^2 \right)^{\frac{1}{2}} \left(1 + (z^2 + \omega^2)^{\frac{\rho-1}{2}} \right)} < +\infty.$$

Since when $x = 0$ or $y = x$, (5) holds with $\tilde{\kappa}_\rho$ replaced by 1, we conclude that the optimal constant

is $\tilde{\kappa}_\rho = \sup_{(z, \omega) \neq (1, 0)} \frac{\left(z^2 + \omega^2 \right)^{\frac{\rho}{2}} - 1}{\left((z-1)^2 + \omega^2 \right)^{\frac{1}{2}} \left(1 + (z^2 + \omega^2)^{\frac{\rho-1}{2}} \right)}$. \square

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