Stability analysis of sampled-data control systems with input saturation

Arthur Scolari Fagundes, João Manoel Gomes da Silva Jr., Marc Jungers

Stability analysis of sampled-data control systems with input saturation: a hybrid system approach*

Arthur Scolari Fagundes[1], João Manoel Gomes da Silva Jr.[1], and Marc Jungers[2]

[1]PPGEE, Universidade Federal do Rio Grande do Sul (UFRGS), Porto Alegre, Brazil, (e-mails:arthur.fagundes@ufrgs.br; jmgomes@ufrgs.br)
[2]Université de Lorraine, CNRS, CRAN, F-54000 Nancy, France , (e-mail: marc.jungers@univ-lorraine.fr)

October 13, 2023

Abstract:

This work addresses the stability analysis of sampled-data control systems subject to variable sampling intervals and input saturation. From a hybrid system representation, stability conditions based on quadratic timer-dependent Lyapunov functions are proposed. Considering affine and generic polynomial timer-dependence, these conditions are cast in semidefinite and sum-of-squares optimization problems to provide maximized estimates of the region of attraction or to provide a lower bound for the maximum allowable intersampling time considering a given set of admissible initial conditions.

Keywords: Input saturation; Sampled-data systems; Hybrid systems;

1 Introduction

With the spread of digital controllers and networked control in the industry, to the point that one can suppress the word "digital" without change of meaning, the literature has accumulated in the last decades a great amount of publications concerning the subject of sampled-data systems (SDS). One of the aspects of the SDS receiving considerable attention is the aperiodic sampling, when there is no guarantee that the digital controller operates with a constant period between samples (see [21] for a comprehensive overview). There is a number of different approaches that have been proposed to address aperiodic SDS, each one with advantages and drawbacks. The input delay approach models the sampling effect as a time-varying delay affecting the control input and uses Lyapunov-Krasovskii functionals to derive stability conditions [14],[13]. Combining the ideas of the delay approach and the lifting technique [2], the looped-functional approach [30]

*This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior Pessoal de Nível Superior (CAPES), Brazil - Finance Code 001, Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil - Grant PQ 307449/2019-0 and SWE scholarship, and ANR, France - project HANDY, number ANR-18-CE40-0010.

relaxes the need of positive definite functionals. Some studies have tackled the problem considering an uncertain discrete-time model and the use of convex embeddings of the transition matrix, which results, in the case of aperiodic sampling, in an equivalent linear parameter-varying model [15][20][8]. In [12], based on a partition of the interval of possible intersampling times and polytopic embeddings, set-induced polyhedral Lyapunov functions are used to certify the stability of SDS. SDS can also be seen as impulsive systems [5] and cast in a hybrid system framework [17].

On the other hand, the input saturation, ubiquitous in real control systems, is a field of study on its own. Different strategies have been developed over the years to study the stability, stabilization and the anti-windup problems for systems with input saturation (see for instance [23], [34], [36]). In particular, for closed-loop systems with saturating inputs, it may not be possible to achieve the global stabilization of the origin [33]. In this case, it is relevant to characterize estimates of the region of attraction to the origin (RAO). Until the turn of the century and some years after, the early literature studied the RAO estimation problem in the case of purely continuous or discrete-time systems (see [23], [34] and references therein). More recently, with the growth of interest in the already mentioned networked control systems, some studies have modified the basic approaches mentioned in the last paragraph to contemplate the analysis of a sampled-data system subject to input saturation. In [31], the looped-functional approach from [30] was combined with the generalized sector that applies to deadzone nonlinearities [34] to obtain a method of stability analysis and stabilization, based on LMI-constrained optimization problems. In [11] and [24], methods inspired in the approach of [12] and set-invariance results have been proposed to compute estimates of the RAO based on quadratic Lyapunov functions. In [10], considering the hybrid system framework proposed in [17] to represent the sampled-data system, preliminary results to compute estimates of the RAO have been presented, where the conditions are based on the application of Finsler's Lemma and the use of quadratic functions depending affinely on the timer variable.

In this paper, the stability analysis of linear systems subject to sampled-data control and input saturation is revisited. Differently from the previous works mentioned above that deal with the same problem ([31], [11] and [24]), we consider a formal hybrid system approach and the framework proposed in [17]. In this case, the actual sampled-data closed-loop system is represented by a linear flow and a nonlinear jump dynamics, which allows to model the evolution of the states with a frozen control between two consecutive sampling instants and the impulsive update of the control signal at the sampling instants. The state of the hybrid (impulsive) system is composed by the plant states, the control inputs and an auxiliary timer variable, that counts the time elapsed from the last sampling instant and that is reset to zero at the sampling instants. Thus, using quadratic timer-dependent Lyapunov functions and a timer-dependent version of the generalized sector relation proposed in [19], conditions to assess the local stability of the origin of the closed-loop are formally stated along with a characterization of estimates of the RAO of the resulting hybrid system. The link between this estimate, which regards the extended hybrid system state space, and the one of the actual system, defined only by the states of the plant, is established. In particular, it is shown that the level sets of the Lyapunov function for the hybrid system with extended state lead to estimates of the RAO defined by piecewise quadratic functions in the state space of the plant. The approach to obtain the conditions is different from our previous preliminary work [10], leading to potentially less conservative ones. Furthermore, considering not only an affine timer-dependence, but also a generic polynomial timer-dependence, these conditions are cast in semidefinite and sum-of-squares optimization problems to provide maximized estimates of the region of attraction or of the maximum allowable intersampling time considering a given set of admissible initial conditions. Although the results focus mainly on the regional stability, conditions for the certification of the global asymptotic stability of the origin (when possible) can be straightforwardly obtained as particular cases.

The paper is organized as follows. The problems of interest are stated in Section 2. The hybrid system representation for the SDS and generic stability conditions are presented in Section 3. Based on quadratic timer-dependent functions, infinite dimensional LMI conditions are proposed in Section 4. Considering affine and generic polynomial timer-dependence, optimization problems based on semidefinite and sum-of-squares programming are formulated in Section 5 to maximize the region of attraction or the bound on the maximum allowable intersampling time. Numerical examples illustrating the effectiveness of the approach and some concluding remarks are presented in Sections 6 and 7, respectively.

Notation

$\mathbb{N}$ is the set of natural numbers, $\mathbb{R}$ is the set of real numbers, and $\mathbb{R}_+$ is the set of nonnegative real numbers. The ith element of a vector v is denoted by $v_{(i)}$, the element in the ith row and jth column of a matrix M is denoted by $M_{(i,j)}$, while $M_{(i)}$ denotes the ith row of M. $\mathbb{S}^n$ is the set of symmetric matrices of size n, and for a symmetric matrix $S \in \mathbb{S}^n$, $S > 0$ ($S \geq 0$) means that S is positive (semi) definite. $\mathbb{D}^n$ is the set of diagonal matrices of size n. M' denotes the transpose of M. I denotes an identity matrix of appropriate size. $|v|$ denotes either the Euclidean norm, when x is a vector, or the absolute value, when v is a scalar.

2 Problem Formulation

Consider a continuous linear time-invariant plant:

$$\dot{x}_p(t) = Ax_p(t) + Bu(t) \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the state vector of the plant, $u \in \mathbb{R}^m$ is the input vector, $A \in \mathbb{R}^{n_p \times n_p}$ and $B \in \mathbb{R}^{n_p \times m}$.

We consider the following saturated sampled-data state feedback control law:

$$u(t) = \text{sat}(Kx_p(t_k)), \qquad \forall t \in [t_k,\ t_{k+1}) \tag{2}$$

where the vector-valued function $\text{sat}(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ accounts for normalized actuators amplitude limitations:

$$\text{sat}(v)_{(i)} = \text{sign}(v_{(i)})\min(|v_{(i)}|, 1) \quad \forall i = 1, \cdots, m \tag{3}$$

The values t_k, $k \in \mathbb{N}$, are the sampling instants. By definition, we set $t_0 = 0$. We consider an aperiodic sampling policy and assume that the time interval between two successive sampling instants, given by $t_{k+1} - t_k$, is bounded as follows:

$$0 < \underline{\tau} \leq t_{k+1} - t_k \leq \overline{\tau}. \tag{4}$$

From the definition of the control law (2), we assume that the value of $u(t)$ is kept constant (by a zero-order holder) between two successive sampling instants and is updated at the sampling instants.

Since the closed-loop system (1)-(2) is nonlinear, it may not be possible to guarantee the global asymptotic stability of the origin (e.g. when A is not Hurwitz) [34]. Hence, it becomes necessary to define a region of attraction of the origin.

Definition 1 (Region of attraction of the closed-loop system)**.** *Supposing that the control law $u(t)$ given in* (2) *ensures the asymptotic stability of the origin of the nonlinear closed-loop system* (1)-(2)*, the region of attraction of the origin (RAO) is defined as*

$$R_{a,0} = \{\xi \in \mathbb{R}^n \ :\ \lim_{t\to\infty} x_p(t) = 0 \text{ for } x_p(0) = \xi\}$$

From the setup above, we can state the following problems of interest:

P1. Given the closed-loop system (1)-(2) and the sampling interval limits $\underline{\tau}$ and $\overline{\tau}$, estimate the region of attraction to the origin $x = 0$ of the closed-loop sampled-data system.

P2. Given the closed-loop system (1)-(2), a set of admissible initial conditions $\mathcal{X}_0 \in \mathbb{R}^{n_p}$ and $\underline{\tau}$, estimate the upper bound $\overline{\tau}$ such that, for all $x(0) \in \mathcal{X}_0$, the corresponding trajectories $x_p(t)$ converge asymptotically to the origin.

To deal with the stability problems stated above, a hybrid (impulsive) system is used in the next section as the framework for modeling the closed-loop system behavior [17].

3 Hybrid System Framework

3.1 Sampled-data system representation

The closed-loop system (1)-(2) can be represented in the hybrid system framework given in [17] by a system $\mathcal{H}$ with state given by $\eta \triangleq \begin{bmatrix} x \\ \tau \end{bmatrix}$, where $x \triangleq \begin{bmatrix} x_p \\ u \end{bmatrix}$ and $\tau \in \mathbb{R}_+$ is a timer that counts the time elapsed since the last sampling, as follows:

$$\mathcal{H} \begin{cases} \dot{\eta} = \begin{bmatrix} \dot{x} \\ \dot{\tau} \end{bmatrix} = \mathcal{F}(\eta), & \forall \eta \in C \\ \eta^+ = \begin{bmatrix} x^+ \\ \tau^+ \end{bmatrix} = \mathcal{G}(\eta), & \forall \eta \in D \end{cases} \tag{5}$$

The sets $C = \mathbb{R}^n \times [0, \overline{\tau}]$ and $D = \mathbb{R}^n \times [\underline{\tau}, \overline{\tau}]$, with $n = n_p + m$, are the *flow* and the *jump sets*, respectively, while the function $\mathcal{F} : \mathbb{R}^h \to \mathbb{R}^h$, with $h = n + 1$, is the *flow map* and the function $\mathcal{G} : \mathbb{R}^h \to \mathbb{R}^h$ is the *jump map*. The functions $\mathcal{F}$ and $\mathcal{G}$ are defined from (1)-(2) as follows:

$$\mathcal{F}(\eta) = \begin{bmatrix} A_f x \\ 1 \end{bmatrix} \tag{6}$$

$$\mathcal{G}(\eta) = \begin{bmatrix} A_j x + B_j \text{sat}(K_j x) \\ 0 \end{bmatrix} \tag{7}$$

with:

$$A_f = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \quad A_j = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad B_j = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad K_j = \begin{bmatrix} K & 0 \end{bmatrix}.$$

The solutions of system $\mathcal{H}$ defined above are given by an hybrid arc $\eta(t,k) \triangleq \begin{bmatrix} x(t,k) \\ \tau(t,k) \end{bmatrix}$ with hybrid domain dom $\eta = \cup_{k=0}^{\infty}([t_k, t_{k+1}], k)$. Note that the state variable τ is a timer that has time derivative equal to one and is reset to zero at each jump instant (which corresponds to the sampling instants), that is $\tau(t,k) \triangleq t - t_k$. Furthermore, $x(t,k) \triangleq x(t_k + \tau)$ for $t \in [t_k, t_{k+1})$. The system $\mathcal{H}$ satisfies the *hybrid basic conditions* (see [17], chapter 6): the sets C and D are closed and the functions $\mathcal{F}$ and $\mathcal{G}$ are continuous. For any $\eta(0,0) \in C \cup D$, the solution starting from $\eta(0,0)$ is well posed and maximal. In particular, dom $\eta(t,k)$ is complete and, from (4), without Zeno behavior. Since we assume $t_0 = 0$, it follows that $\tau(0,0) = 0$. Hence, we are interested in the behavior of system (5) with respect to the initial conditions $\eta(0,0) = \begin{bmatrix} x(0,0) \\ 0 \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$.

3.2 Stability of the hybrid system

The notion of stability for system $\mathcal{H}$ is characterized by the stability of a closed set containing the origin of the plant state space and the domain of the timer variable defined as follows:

$$\mathcal{A} = \{0\} \times [0,\ \overline{\tau}].$$

In this case, the following definitions are considered.

Definition 2 (Distance to a closed set)**.** *Given a vector $\eta \in \mathbb{R}^h$ and a closed set $\mathcal{A} \subset \mathbb{R}^h$, the distance of η to $\mathcal{A}$ is denoted $|\eta|_{\mathcal{A}}$ and is defined by*

$$|\eta|_{\mathcal{A}} = \inf_{y \in \mathcal{A}} |\eta - y|.$$

Definition 3 (Local asymptotic stability)**.** *[17, p.139] Consider the hybrid system $\mathcal{H}$. The set $\mathcal{A}$ is said to be*

a) ***locally stable*** *for $\mathcal{H}$ if for every ϵ there exists δ such that every solution η to $\mathcal{H}$ with $|\eta(0,0)|_{\mathcal{A}} \leq \delta$ satisfies $|\eta(t,k)|_{\mathcal{A}} \leq \epsilon$ for all $(t,k) \in \text{dom } \eta$;*

b) ***locally attractive for*** *$\mathcal{H}$ if there exists μ such that every solution η to $\mathcal{H}$ with $|\eta(0,0)|_{\mathcal{A}} \leq \mu$ is bounded and $\lim_{(t+k)\to\infty} |\eta(t,k)|_{\mathcal{A}} = 0$.*

c) ***locally asymptotically stable for*** *$\mathcal{H}$ if it is both locally stable and locally attractive.*

Definition 4 (Region of attraction of the hybrid system)**.** *[17, p.141] Once the set $\mathcal{A}$ is asymptotically stable for the system $\mathcal{H}$, the region (or basin) of attraction of $\mathcal{A}$ is defined as*

$$R_{a,\mathcal{A}} = \{\zeta \in \mathbb{R}^h \ :\ \lim_{(t+k)\to\infty} |\eta(t,k)|_{\mathcal{A}} = 0 \text{ for } \eta(0,0) = \zeta\}$$

In order to state a generic stability result, it is convenient to re-write the jump map (7) using a deadzone vector-valued function defined as follows:

$$\text{dz}(v) = \text{sat}(v) - v \tag{8}$$

that is, $\text{dz}(v)_{(i)} = \text{sign}(v_{(i)})\left(1 - \max(|v_{(i)}|, 1)\right)$, $i = 1, \ldots, m$.

Hence, we have that

$$\mathcal{G}(\eta) = \begin{bmatrix} (A_j + B_j K_j)x + B_j \text{dz}(K_j x) \\ 0 \end{bmatrix} = \tilde{A}_j \eta + \tilde{B}_j \text{dz}(\tilde{K}_j \eta) \tag{9}$$

with

$$\tilde{A}_j = \begin{bmatrix} (A_j + B_j K_j) & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_j = \begin{bmatrix} B_j \\ 0 \end{bmatrix}, \quad \tilde{K}_j = \begin{bmatrix} K_j & 0 \end{bmatrix}.$$

The following Lemma is an extension of the generalized sector condition for deadzone nonlinearities introduced in [19]:

Lemma 1. *Consider a vector function $\tilde{g}_j : \mathbb{R}^h \to \mathbb{R}^m$ and a vector-dependent diagonal matrix $T : \mathbb{R}^h \to \mathbb{D}^{m\times m}$. If*

$$|\tilde{K}_{j(i)}\eta - \tilde{g}_{j(i)}(\eta)| \leq 1,\ i = 1, \cdots, m \tag{10}$$

then relation

$$r(T(\eta), \tilde{g}_j(\eta)) = \text{dz}(\tilde{K}_j\eta)' T(\eta)(\text{dz}(\tilde{K}_j\eta) + \tilde{g}_j(\eta)) \leq 0 \tag{11}$$

is verified for any matrix $T(\eta) > 0$.

The next theorem follows from the results in [16, 17] and provides sufficient conditions to ensure that $\mathcal{A}$ is locally asymptotically stable for the system $\mathcal{H}$, along with a characterization of an estimate of its region of attraction.

Theorem 1. *If there exists a function $V : \mathbb{R}^h \to \mathbb{R}_+$, a vector function $\tilde{g}_j : \mathbb{R}^h \to \mathbb{R}^m$, a vector dependent diagonal matrix $T : \mathbb{R}^h \to \mathbb{D}^{m\times m}$, class $\mathcal{K}_\infty$ functions α_1 and α_2, and a positive definite function ρ such that*

$$\alpha_1(|\eta|_{\mathcal{A}}) \leq V(\eta) \leq \alpha_2(|\eta|_{\mathcal{A}}), \quad \forall \eta \in (C \cup D) \tag{12}$$

$$\langle \nabla V(\eta), \mathcal{F}(\eta)\rangle \leq -\rho(|\eta|_{\mathcal{A}}), \quad \forall \eta \in C\backslash\mathcal{A} \tag{13}$$

$$V(\mathcal{G}(\eta)) - V(\eta) - r(T(\eta), \tilde{g}(\eta)) \leq -\rho(|\eta|_{\mathcal{A}}), \quad \forall \eta \in D\backslash\mathcal{A} \tag{14}$$

$$|\tilde{K}_{j(i)}\eta - \tilde{g}_{j(i)}(\eta)|^2 \leq \frac{1}{\mu}V(\eta), \qquad \forall \eta \in D\backslash\mathcal{A},\ i \in \{1, \cdots, m\} \tag{15}$$

then

1. *the set $\mathcal{A}$ is locally asymptotically stable for $\mathcal{H}$.*
2. *for any initial condition $\eta(0,0) \in \mathcal{L}_V(\mu) = \{\eta \in \mathbb{R}^h \ : \ V(\eta) \leq \mu\}$, it follows that $\eta(t_k, k) \to \mathcal{A}$ as $(t + k) \to \infty$, that is $\mathcal{L}_V(\mu) \subset R_{a,\mathcal{A}}$.*

Proof. Suppose that $\eta(t_0, 0) = \eta(0, 0) \in \mathcal{L}_V(\mu)$. If condition (13) holds, it follows that

$$V(\eta(t, 0)) < V(\eta(0, 0)) \leq \mu, \quad \forall t \in [0, t_1]. \tag{16}$$

On the other hand, if (15) and (14) are verified, one has that

$$|\tilde{K}_{j(i)}\eta(t_1, 0) - \tilde{g}_{j(i)}(\eta(t_1, 0))|^2 \leq \frac{1}{\mu}V(\eta(t_1, 0)) < 1, \tag{17}$$

$$V(\eta(t_1, 1)) - V(t_1, 0) - r(T(\eta(t_1, 0)), \tilde{g}(\eta(t_1, 0))) < -\rho(|\eta|_{\mathcal{A}}). \tag{18}$$

Since (17) is verified, it follows from Lemma 1 that $r(T(\eta(t_1, 0)), \tilde{g}(\eta(t_1, 0))) \leq 0$. Hence, from (18) we conclude that

$$V(\eta(t_1, 1)) < V(\eta(t_1, 0)) \tag{19}$$

The relation in (16) combined with (19), results that

$$V(\eta(t_1, 1)) < V(\eta(t, 0)) \leq V(\eta(0, 0)) \leq \mu, \quad \forall t \in [0, t_1].$$

Repeating the reasoning, we can generalize the relation above to:

$$V(\eta(t_{k+1}, k + 1)) < V(\eta(t, k)) < V(\eta(t_k, k)) \leq \mu, \quad \forall t \in [t_k, t_{k+1}] \tag{20}$$

for all $(t, k) \in \text{dom } \eta$.

From (12) and (20), it follows that for all $\eta(0, 0) \in \mathcal{L}_V(\tilde{\mu})$, with $\tilde{\mu} \leq \mu$, there exist $r_1(\tilde{\mu})$ such that $|\eta(t, k)|_{\mathcal{A}} \leq r_1(\tilde{\mu}) \leq r_1(\mu)$, $\forall (t, k)$. Furthermore, there exists $r_2(\tilde{\mu})$ such that $|\eta(0, 0)|_{\mathcal{A}} \leq r_2(\tilde{\mu})$ implies that $\eta(0, 0) \in \mathcal{L}_V(\tilde{\mu})$. Hence, for every $\epsilon \geq r_1(\mu)$, we can conclude that $|\eta(t, k)|_{\mathcal{A}} \leq \epsilon$ provided $|\eta(0, 0)|_{\mathcal{A}} \leq r_2(\mu)$. On the other hand for every $\epsilon < r_1(\mu)$ there exists $\tilde{\mu}$ such that $\epsilon = r_1(\tilde{\mu})$ and $|\eta(t, k)|_{\mathcal{A}} \leq \epsilon$ provided $|\eta(0, 0)|_{\mathcal{A}} \leq r_2(\tilde{\mu})$. Thus the local stability of $\mathcal{A}$ follows.

The local attractivity of $\mathcal{A}$ also comes directly from (12) and (20). Note that (20) implies that $\lim_{(t+k)\to\infty} V(\eta) = 0$ provided that $\eta(0, 0) \in \mathcal{L}_V(\mu)$, which from (12) implies that $\lim_{(t+k)\to\infty} |\eta(t, k)|_{\mathcal{A}} = 0$, which concludes the proof.

□

Theorem 1 provides an estimate to the region of attraction of the set $\mathcal{A}$ (see Definition 4) of the hybrid system (5), with $\mathcal{F}(\eta)$ and $\mathcal{G}(\eta)$ given in (6) and (9), respectively. Although this is an equivalent representation of the closed-loop dynamics, it is worth noticing that it considers an extended state space η composed by the plant states, the inputs and the timer. Hence, it is important to formally characterize an estimate of the region of attraction to the origin of system (1)-(2) (see Definition 1), which should be given in terms of the plant states x_p. With this aim recall that $\eta(0,0) = \begin{bmatrix} x(0,0) \\ 0 \end{bmatrix}$ and that $x(0,0)$ is given by

$$x(0,0) = \begin{bmatrix} x_p(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} x_p(0) \\ \text{sat}(Kx_p(0)) \end{bmatrix}.$$

We can therefore state the following Corollary that regards the regional stability of the original sampled-data system (1)-(2).

Corollary 1. *If the conditions of Theorem 1 are satisfied, the set*

$$\mathcal{L}_V^{x_p}(\mu) = \{x_p \in \mathbb{R}^{n_p} \ : \ V\left(\begin{bmatrix} x_p \\ \text{sat}(Kx_p) \\ 0 \end{bmatrix}\right) \leq \mu\}$$

is included in the region of attraction of the origin of the closed-loop system (1)-(2) *, i.e.* $\mathcal{L}_V^{x_p}(\mu) \subset R_{a,0}$.

Proof. It follows directly from the fact that if $x_p(0) \in \mathcal{L}_V^{x_p}(\mu)$ then $\eta(0,0) \in \mathcal{L}_V(\mu)$. In this case, from Theorem 1, we conclude that $\eta(t_k,k) \to \mathcal{A}$ as $(t+k) \to \infty$, which implies that $x_p(t_k,k) \to 0$ as $(t+k) \to \infty$ or, equivalently, $x_p(t) \to 0$ as $t \to \infty$. □

Remark 1. *Theorem 1 and Corollary 1 focus on the regional stability characterization and provide estimates of the region of attraction. In the case that the linear plant* (1) *is asymptotically stable (i.e. it is Hurwitz), it may be possible to ensure the global asymptotic stability of* $\mathcal{A}$ *(or equivalently, of the origin of the system* (1)-(2)*) depending on the admissible intersampling interval* $[\underline{\tau}, \overline{\tau}]$*. In fact, taking into account that the deadzone function globally satisfies the sector relation [34]*

$$r(T(\eta), \tilde{K}_j\eta) = \text{dz}(\tilde{K}_j\eta)'T(\eta)(\text{dz}(\tilde{K}_j\eta) + \tilde{K}_j\eta) \leq 0, \tag{21}$$

for any $T(\eta)$ *diagonal and positive definite, Theorem 1 can therefore be adapted to provide a sufficient condition for the global asymptotic stability of* $\mathcal{A}$*. In this case, it suffices to replace condition* (14) *by:*

$$V(\mathcal{G}(\eta)) - V(\eta) - r(T(\eta), \tilde{K}_j\eta) \leq -\rho(|\eta|_{\mathcal{A}}), \quad \forall \eta \in D\backslash\mathcal{A} \tag{22}$$

and condition (15) *is no longer needed.*

Note that the asymptotic stability of the plant is a necessary (but not sufficient) condition to ensure the exponential global asymptotic stability of the origin under a saturating state feedback control law (see [34] and references therein), which can be designed for instance to improve performance and robustness.

4 Main results

Based on Theorem 1, which provides generic Lyapunov-based conditions, in this section we derive testable conditions to assess the stability of the sampled-data closed-loop system and to

address problems **P1** and **P2** stated in Section 2. With this aim, we consider a class of quadratic timer-dependent (also named clock-dependent [7]) Lyapunov functions :

$$V(\eta) = V(x,\tau) = x'P(\tau)x = x'W^{-1}(\tau)x. \tag{23}$$

Note that, if $W(\tau)$ is positive definite $\forall \tau \in [0,\ \overline{\tau}]$, the quadratic form defined above satisfies condition (12) of Theorem 1 with $\alpha_2(|\eta|_{\mathcal{A}}) = \overline{\lambda}|\eta|^2_{\mathcal{A}}$ and $\alpha_1(|\eta|_{\mathcal{A}}) = \underline{\lambda}|\eta|^2_{\mathcal{A}}$, where $\overline{\lambda}$ and $\underline{\lambda}$ are, respectively, the maximal and the minimal eigenvalue of $P(\tau)$ for $\tau \in [0, \bar{\tau}]$. It should also be noted that, as $\tau \in [0, \bar{\tau}]$, $|\eta|^2_{\mathcal{A}} = |x|^2$.

Hence, the following theorem providing linear matrix inequalities (LMI) conditions can be stated.

Theorem 2. *If there exist matrix functions $W : [0,\ \overline{\tau}] \to \mathbb{S}^n$, $Y : [0,\ \overline{\tau}] \to \mathbb{R}^{m\times n}$ and $S : [0,\ \overline{\tau}] \to \mathbb{D}^m$ that satisfy the following linear matrix inequalities*

$$W(\tau) > 0, \quad \forall \tau \in [0,\ \overline{\tau}] \tag{24}$$

$$W(\tau)A_f' + A_f W(\tau) - \dot{W}(\tau) < 0, \quad \forall \tau \in [0,\ \overline{\tau}) \tag{25}$$

$$\begin{bmatrix} W(\tau) & Y'(\tau) & W(\tau)(A_j + B_j K_j)' \\ \star & 2S(\tau) & S(\tau)B_j' \\ \star & \star & W(0) \end{bmatrix} > 0, \quad \forall \tau \in [\underline{\tau},\ \overline{\tau}] \tag{26}$$

$$\begin{bmatrix} W(\tau) & W(\tau)K_{j(i)}' - Y_{(i)}'(\tau) \\ \star & 1 \end{bmatrix} \geq 0, \quad \forall \tau \in [\underline{\tau},\ \overline{\tau}], \quad i = 1, \cdots, m \tag{27}$$

then, considering $P(\tau) = W^{-1}(\tau)$, provided that $x_p(0) \in \mathcal{L}_P^{x_p}$, where

$$\mathcal{L}_P^{x_p} \triangleq \{x_p \in \mathbb{R}^{n_p} \ : \ \begin{bmatrix} x_p \\ \mathrm{sat}(Kx_p) \end{bmatrix}' P(0) \begin{bmatrix} x_p \\ \mathrm{sat}(Kx_p) \end{bmatrix} \leq 1\},$$

the trajectories of the closed-loop system (1)-(2)*, with $(t_{k+1} - t_k) \in [\underline{\tau}, \overline{\tau}]$, converge asymptotically to the origin as $t \to +\infty$.*

Proof. From (24), it follows that $W(\tau)$ is nonsingular $\forall \tau \in [0,\ \overline{\tau}]$. Consider $P(\tau) = W^{-1}(\tau)$. Taking into account that $\frac{\mathrm{d}P(\tau)W(\tau)}{\mathrm{d}\tau} = P(\tau)\frac{\mathrm{d}W(\tau)}{\mathrm{d}\tau} + \frac{\mathrm{d}P(\tau)}{\mathrm{d}\tau}W(\tau) = 0$, $\dot{W}(\tau) \triangleq \frac{\mathrm{d}W(\tau)}{\mathrm{d}\tau}\dot{\tau}$ and the fact that $\dot{\tau} = 1$ for $\eta \in C$ it follows that:

$$\dot{W}(\tau) = -W(\tau)\dot{P}(\tau)W(\tau). \tag{28}$$

Left and right multiplying (25) by $x'P(\tau)$ and $P(\tau)x$, respectively, it follows that

$$x'\left(P(\tau)A_f + A_f'P(\tau) + \dot{P}(\tau)\right)x < 0, \quad \forall x \in \mathbb{R}^n, \forall \tau \in [0,\ \overline{\tau}] \tag{29}$$

and thus (25) implies that condition (13) of Theorem 1 is satisfied with $V(\eta) = x'P(\tau)x$.

Applying Schur's complement to (27), it follows that:

$$W(\tau) - (W(\tau)K_{j(i)}' - Y_{(i)}'(\tau))(K_{j(i)}W(\tau) - Y_{(i)}(\tau)) \geq 0, \quad \forall \tau \in [\underline{\tau},\ \overline{\tau}],\ i = 1, ..., m \tag{30}$$

left and right multiplying (30) by $x'P(\tau)$ and $P(\tau)x$, respectively, and defining $G_j(\tau) = Y(\tau)P(\tau)$, it follows that

$$x'(K_{j(i)} - G_{j(i)}(\tau))'(K_{j(i)} - G_{j(i)}(\tau))x \leq x'P(\tau)x, \quad \forall \tau \in [\underline{\tau},\ \overline{\tau}], \quad i = 1, ..., m \tag{31}$$

and thus, recalling that $\tilde{K}_j \triangleq [K_j \;\; 0]$ and $\eta \triangleq \begin{bmatrix} x \\ \tau \end{bmatrix}$, we conclude from (31) that

$$|(\tilde{K}_{j(i)} - \tilde{G}_{j(i)}(\tau))\eta|^2 \leq V(\eta), \quad \forall \tau \in [\underline{\tau},\ \overline{\tau}], \quad i = 1, ..., m \tag{32}$$

with $\tilde{G}_j(\tau) = \begin{bmatrix} G_j(\tau) & 0 \end{bmatrix}$, which implies that condition (15) of Theorem 1 with $\tilde{g}(\eta) = \tilde{G}_j(\tau)\eta$ is satisfied with $\mu = 1$.

Now, right and left-multiplying (26) by the matrix $\text{diag}(P(\tau), T(\tau), I)$, with $T(\tau) = S(\tau)^{-1}$, it follows that (26) is equivalent to

$$\begin{bmatrix} P(\tau) & G_j'(\tau)T(\tau) & (A_j + B_jK_j)' \\ \star & 2T(\tau) & B_j' \\ \star & \star & W(0) \end{bmatrix} > 0. \tag{33}$$

Applying now the Schur's complement to (33) and thus right and left multiplying by $\begin{bmatrix} x \\ \text{dz}(K_j x) \end{bmatrix}$ and its transpose, respectively, one gets

$$\begin{bmatrix} x \\ \text{dz}(K_j x) \end{bmatrix}' \left(\begin{bmatrix} P(\tau) & G_j'(\tau)T(\tau) \\ \star & 2T(\tau) \end{bmatrix} - \begin{bmatrix} (A_j + B_jK_j)' \\ B_j' \end{bmatrix} P(0) \begin{bmatrix} (A_j + B_jK_j)' \\ B_j' \end{bmatrix}' \right) \begin{bmatrix} x \\ \text{dz}(K_j x) \end{bmatrix} > 0 \tag{34}$$

which implies that condition (14) of Theorem 1 is satisfied.

Hence, the satisfaction of conditions (24), (25), (26) and (27) imply the satisfaction of the conditions of Theorem 1[1] with $V(\eta) = x'P(\tau)x$ and $\tilde{g}(\eta) = \tilde{G}_j(\tau)\eta$, which concludes the proof. □

It should be noticed that the stability conditions provided by Theorem 2 result in infinite dimensional LMIs. It is then important to consider particular structures for $W(\tau)$, $S(\tau)$ and $Y(\tau)$ to obtain finite dimensional test conditions. This is done in the following subsections by considering affine and polynomial dependences on τ [3].

4.1 Affine dependence on τ

In this subsection, the matrix functions $W(\tau)$, $S(\tau)$ and $Y(\tau)$ in Theorem 2 are considered affinely dependent on τ. Generically, a matrix $M(\tau)$ is affine on τ if it can be expressed as

$$M(\tau) = M_0 + \tau M_1, \tag{35}$$

where M_0 and M_1 are constant matrices. Note that in this case, since $P(\tau) = W(\tau)^{-1}$, the Lyapunov function will present a rational dependence on τ. Lyapunov functions with affine dependence on τ can be found in the literature, though more commonly in the non-inverted form, where $P(\tau)$ is affine on τ instead of $W(\tau)$ [4], [1] and [22]. A previous result, considering this type of candidate, can be found in [10]. Here we adopted the inversion of $P(\tau)$ because, through convexity arguments, the stability conditions can be cast directly as true LMIs, that is, without the need of fixing some scalar variables as in [10]. This candidate underpins the development of the next theorem.

[1]Note that $|\eta|_{\mathcal{A}} = |x|$ and since the LMIs (25) and (26) are strict, it is always possible to determine a scalar $\tilde{\rho}$ such that $\rho(|\eta|_{\mathcal{A}}) = \tilde{\rho}|x|^2$.

Theorem 3. *If there exist matrices $W_0 \in \mathbb{S}^n$, $W_1 \in \mathbb{S}^n$, $Y_0, Y_1 \in \mathbb{R}^{m\times n}$, $S_0, S_1 \in \mathbb{D}^m$ that satisfy the following LMIs*

$$W_0 > 0 \tag{36}$$

$$W_0 + \overline{\tau} W_1 > 0 \tag{37}$$

$$W_0 A_f' + A_f W_0 - W_1 < 0 \tag{38}$$

$$(W_0 + \overline{\tau} W_1) A_f' + A_f (W_0 + \overline{\tau} W_1) - W_1 < 0 \tag{39}$$

$$\begin{bmatrix} (W_0 + \underline{\tau} W_1) & (Y_0 + \underline{\tau} Y_1)' & (W_0 + \underline{\tau} W_1)(A_j + B_j K_j)' \\ \star & 2(S_0 + \underline{\tau} S_1) & (S_0 + \underline{\tau} S_1) B_j' \\ \star & \star & W_0 \end{bmatrix} \geq 0 \tag{40}$$

$$\begin{bmatrix} (W_0 + \overline{\tau} W_1) & (Y_0 + \overline{\tau} Y_1)' & (W_0 + \overline{\tau} W_1)(A_j + B_j K_j)' \\ \star & 2(S_0 + \overline{\tau} S_1) & (S_0 + \overline{\tau} S_1) B_j' \\ \star & \star & W_0 \end{bmatrix} \geq 0 \tag{41}$$

$$\begin{bmatrix} (W_0 + \underline{\tau} W_1) & (W_0 + \underline{\tau} W_1) K_{j(i)}' - (Y_{0(i)} + \underline{\tau} Y_{1(i)})' \\ \star & 1 \end{bmatrix} \geq 0, \quad i = 1, \cdots, m \tag{42}$$

$$\begin{bmatrix} (W_0 + \overline{\tau} W_1) & (W_0 + \overline{\tau} W_1) K_{j(i)}' - (Y_{0(i)} + \overline{\tau} Y_{1(i)})' \\ \star & 1 \end{bmatrix} \geq 0, \quad i = 1, \cdots, m \tag{43}$$

then, considering $P(\tau) = (W_0 + \tau W_1)^{-1}$, provided that $x_p(0) \in \mathcal{L}_P^{x_p}$, where

$$\mathcal{L}_P^{x_p} \triangleq \{x_p \in \mathbb{R}^{n_p} \ : \ \begin{bmatrix} x_p \\ \text{sat}(Kx_p) \end{bmatrix}' P(0) \begin{bmatrix} x_p \\ \text{sat}(Kx_p) \end{bmatrix} \leq 1\},$$

the trajectories of the closed-loop system (1)-(2)*, with $(t_{k+1} - t_k) \in [\underline{\tau}, \overline{\tau}]$, converge asymptotically to the origin as $t \to +\infty$.*

Proof. Consider $V(\eta) \triangleq x' W^{-1}(\tau) x$, where $W(\tau) = W_0 + \tau W_1$. Hence, since $W(\tau)$ is affine on τ and $\tau \in [0, \overline{\tau}]$, it follows that $W(\tau)$ can be expressed as a convex combination of W_0 and $W_0 + \overline{\tau} W_1$, that is

$$W(\tau) = \lambda_0 W_0 + \lambda_1 (W_0 + \overline{\tau} W_1)$$

with $\lambda_0 = \left(1 - \dfrac{\tau}{\overline{\tau}}\right)$ and $\lambda_1 = \dfrac{\tau}{\overline{\tau}}$. Thus, provided that (36) and (37) are verified, we conclude that $W(\tau) > 0$, $\forall \tau \in [0, \overline{\tau}]$, i.e. (24) is satisfied. Applying now this same convex combination to (38) and (39), we can conclude that these two matrix inequalities are equivalent to (25).

On the other hand, for $\tau \in [\underline{\tau}, \overline{\tau}]$, $W(\tau)$ can be expressed by the following convex combination:

$$W(\tau) = \gamma_0 (W_0 + \underline{\tau} W_1) + \gamma_1 (W_0 + \overline{\tau} W_1)$$

with $\gamma_0 = \left(\dfrac{\overline{\tau} - \tau}{\overline{\tau} - \underline{\tau}}\right)$ and $\gamma_1 = \left(\dfrac{\tau - \underline{\tau}}{\overline{\tau} - \underline{\tau}}\right)$. Hence, applying this same convex combination to (40)-(41) and to (42)-(43), we conclude that these pairs of matrix inequalities are equivalent to (26) and (27), respectively, which concludes the proof. □

4.2 Polynomial dependence on τ

To reduce conservatism, we consider now $W(\tau)$, $S(\tau)$ and $Y(\tau)$ with a more generic polynomial dependence on τ, characterized by a finite sum of monomials, or an univariate matrix polynomial (UMP). Generically, a matrix $M(\tau)$ is expressed as an UMP of degree d as follows:

$$M(\tau) = \sum_{i=0}^{d} M_i \tau^i = M_0 + \tau M_1 + \cdots + \tau^d M_d. \tag{44}$$

The degree of the polynomial above is denoted by $\deg(M(\tau)) = d$. When we choose $\deg(M(\tau)) = 1$, as in the last subsection, it is possible to write a finite number of stability conditions directly as LMIs. When the stability conditions of Theorem 2 are written with $\deg(M(\tau)) > 1$, however, this approach is not valid. The alternative is thus to use the sum of squares framework [25], [26], [27].

A sum of squares decomposition of the UMP in (44) is given by

$$M(\tau) = H'(\tau)H(\tau). \tag{45}$$

When a polynomial can be decomposed into a sum of squares, it is abbreviated here that the polynomial is a sum of squares or, simply, that it is SOS. The idea employed here is that if a UMP given by (44) is SOS, then it is semidefinite positive $\forall \tau$.

The next theorem casts the inequalities of Theorem 2 in the sum-of-squares framework.

Theorem 4. *If there exist matrix polynomials $W : [0, \overline{\tau}] \to \mathbb{S}^n$, Q_1, Q_2, Q_3 and $Q_4 : [0, \overline{\tau}] \to \mathbb{S}^n$, $Y : [0, \overline{\tau}] \to \mathbb{R}^{m \times n}$ and $S : [0, \overline{\tau}] \to \mathbb{D}^m$ and a scalar $\gamma > 0$ that satisfy the following conditions:*

$$Q_1(\tau), Q_2(\tau), Q_3(\tau), Q_4(\tau) \text{ are SOS} \tag{46}$$

$$W(\tau) - \gamma I - Q_1(\tau)\tau(\overline{\tau} - \tau) \text{ is SOS} \tag{47}$$

$$-(W(\tau)A_f' + A_f W(\tau) - \dot{W}(\tau)) - \gamma I - Q_2(\tau)\tau(\overline{\tau} - \tau) \text{ is SOS} \tag{48}$$

$$\begin{bmatrix} W(\tau) & Y'(\tau) & W(\tau)(A_j + B_j K_j)' \\ \star & 2S(\tau) & S(\tau)B_j' \\ \star & \star & W(0) \end{bmatrix} - \gamma I - Q_3(\tau)(\tau - \underline{\tau})(\overline{\tau} - \tau) \text{ is SOS} \tag{49}$$

$$\begin{bmatrix} W(\tau) & W(\tau)K_{j(i)}' - Y_{(i)}'(\tau) \\ \star & 1 \end{bmatrix} - \; Q_4(\tau)(\tau - \underline{\tau})(\overline{\tau} - \tau) \text{ is SOS} \quad i = 1, \cdots, m \tag{50}$$

then, considering $P(\tau) = W^{-1}(\tau)$, provided that $x_p(0) \in \mathcal{L}_P^{x_p}$, where

$$\mathcal{L}_P^{x_p} \triangleq \{x_p \in \mathbb{R}^{n_p} \; : \; \begin{bmatrix} x_p \\ \text{sat}(Kx_p) \end{bmatrix}' P(0) \begin{bmatrix} x_p \\ \text{sat}(Kx_p) \end{bmatrix} \leq 1\},$$

the trajectories of the closed-loop system (1)-(2), with $(t_{k+1} - t_k) \in [\underline{\tau}, \overline{\tau}]$, converge asymptotically to the origin as $t \to +\infty$.

Proof. If the conditions in (46) are verified, then

$$Q_1(\tau) \geq 0, \; Q_2(\tau) \geq 0, \; Q_3(\tau) \geq 0, \; Q_4(\tau) \geq 0, \quad \forall \tau.$$

Hence, noting that $\tau(\overline{\tau} - \tau) \geq 0$ for $\tau \in [0, \; \overline{\tau}]$, provided that $Q_1(\tau) \geq 0 \; \forall \tau$, and $\gamma > 0$, it follows that (47) ensures that

$$W(\tau) \geq \gamma I > 0 \quad \tau \in [0, \overline{\tau}]. \tag{51}$$

Likewise, as $Q_2(\tau) \geq 0$, $\forall \tau$, the condition (47) implies that

$$-(W(\tau)A_f' + A_f W(\tau) - \dot{W}(\tau)) > 0, \forall \tau \in [0,\ \overline{\tau}] \tag{52}$$

and therefore (25) in Theorem 2 is verified.

On the other hand, as $(\tau - \underline{\tau})(\overline{\tau} - \tau) \geq 0$, $\forall \tau \in [\underline{\tau}, \overline{\tau}]$ and $Q_3(\tau), Q_4(\tau) \geq 0$, it follows that (48) and (49) ensure (26) and (27) in Theorem 2, which concludes the proof. □

The terms involving $Q_i(\tau)$, $i = 1, 2, 3, 4$ in the conditions of Theorem 4 correspond in fact to the application of the the *Positivstellensatz* [29] and can be seen as a generalized S-procedure ([35], Appendix A). Indeed, these terms are used to relax the validity of the LMIs in Theorem 2 only for $\tau \in [0, \bar{\tau}]$ or for $\tau \in [\underline{\tau}, \bar{\tau}]$ [6], [7].

5 Optimization problems

Theorems 3 and 4 provide conditions to assess the regional asymptotic stability of the origin of the closed-loop system (1)-(2) and to address problems **P1** and **P2** stated in Section 2.

First of all, it should be noted that $\mathcal{L}_P^{x_p}$ is a sub-level set of a piecewise quadratic function. Indeed, depending on x_p, the term $\text{sat}(K_{(i)}x_p)$ can assume the value 1, -1 or $K_{(i)}x_p$, for $i = 1, \ldots, m$. This leads to 3^m quadratic forms in x_p for the function $\begin{bmatrix} x_p \\ \text{sat}(Kx_p) \end{bmatrix}' P(0) \begin{bmatrix} x_p \\ \text{sat}(Kx_p) \end{bmatrix}$, each one associated to a region in the state space [18].

Regarding **P1**, the objective is to use the conditions given by Theorems 3 and 4 to obtain an estimate of the RAO, described by $\mathcal{L}_P^{x_p}$, as large as possible considering a size criterion. To this end, we can consider a piecewise quadratic shape set, described as follows

$$\mathcal{L}_I^{x_p}(\varepsilon) = \{x_p \in \mathbb{R}^{n_p} \ : \ \begin{bmatrix} x_p \\ \text{sat}(Kx_p) \end{bmatrix}' I \begin{bmatrix} x_p \\ \text{sat}(Kx_p) \end{bmatrix} \leq \varepsilon\}. \tag{53}$$

The idea is therefore to maximize the scaling factor ε such that $\mathcal{L}_I^{x_p} \subseteq \mathcal{L}_P^{x_p}$. Note that this inclusion is satisfied if $W_0 - \varepsilon I \geq 0$, since this implies that $W_0^{-1} = P(0) \leq \varepsilon^{-1} I$. Thus, a new constraint given by $W_0 - \varepsilon I \geq 0$ is considered and the problem **P1** is posed as the optimization problems (54) and (55) below, corresponding to the application of Theorems 3 and 4, respectively:

$$\begin{aligned} &\max \varepsilon \\ &\text{subject to:} \\ &(36),\ (37),\ (38),\ (39),\ (40),\ (41),\ (42),\ (43) \\ &\text{and } W_0 - \varepsilon I \geq 0 \end{aligned} \tag{54}$$

$$\begin{aligned} &\max \varepsilon \\ &\text{subject to:} \\ &(46),\ (47),\ (48),\ (49),\ (50) \\ &\text{and } W_0 - \varepsilon I \geq 0 \end{aligned} \tag{55}$$

Regarding problem **P2**, the set of admissible initial conditions $\mathcal{X}_0$ can be defined as a piecewise quadratic shape set as follows:

$$\mathcal{X}_0 \triangleq \mathcal{L}_R^{x_p} = \{x_p \in \mathbb{R}^{n_p} \ : \ \begin{bmatrix} x_p \\ \text{sat}(Kx_p) \end{bmatrix}' R \begin{bmatrix} x_p \\ \text{sat}(Kx_p) \end{bmatrix} \leq 1\}. \tag{56}$$

In this case, the inclusion of $\mathcal{L}_R^{x_p} \subseteq \mathcal{L}_P^{x_p}$ holds if $R - P(0) = R - W_0^{-1} \geq 0$, which is equivalently expressed by the following LMI :

$$\begin{bmatrix} R & I \\ \star & W_0 \end{bmatrix} \geq 0. \tag{57}$$

Then, to solve **P2**, we can consider the optimization problems (58) and (59) below, corresponding to the conditions in Theorem 3 and 4, respectively:

$$\begin{aligned} &\max \overline{\tau} \\ &\text{subject to:} \\ &\text{(36), (37), (38), (39), (40), (41), (42), (43) and (57)} \end{aligned} \tag{58}$$

$$\begin{aligned} &\max \overline{\tau} \\ &\text{subject to:} \\ &\text{(46), (47), (48), (49), (50) and (57)} \end{aligned} \tag{59}$$

Observe that the constraints in (58) and (59) are not LMIs or SOS expressions if $\overline{\tau}$ is considered as a variable (i.e. if $\overline{\tau}$ is not a priori given). Thus, to solve these two optimization problems, we can consider an iterative bisection procedure, where in each step we fix $\overline{\tau}$ and test an LMI or SOS feasibility problem: if the problem is feasible we increase $\overline{\tau}$, otherwise we decrease it.

In the next section, some illustrative examples are provided. The problems (54) and (58) are solved with SeDuMi [32], and the problems (55) and (59) with SOSTools [28].

6 Numerical examples

6.1 Example 1

Consider the system (1) and the control law (2) borrowed from [11], where:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};\ B = \begin{bmatrix} 0 \\ -5 \end{bmatrix};\ K = \begin{bmatrix} 2.6 & 1.4 \end{bmatrix};\ [\underline{\tau}, \overline{\tau}] = [0.05, 0.1]\,. \tag{60}$$

We address first the problem **P1**, regarding the determination of estimates of the RAO. In this context, the optimization problem (54), regarding an affine dependence on the timer, is solved and results in $\varepsilon_{\max} = 0.0128$. Before solving problem (55), which considers a polynomial timer dependence, note that in this case (and also when solving optimization problem (59)) the degree of the polynomial matrices variables must be fixed a priori. In principle, with a greater degree d, the results tend to be less conservative, that is, the final value of $\varepsilon_{\max}$ tends to be greater. Since an increase in d also entails a more complex problem from a numerical perspective, the trade-off between conservatism and computational burden should be mindfully balanced. Results of optimization problem (55) for different values of d are shown in Table 1. The polynomial degrees in the column of d are respective to all polynomials of the SOS constraints, meaning that $\deg(W(\tau)) = \deg(Y(\tau)) = \deg(S(\tau)) = \deg(\tau^2 Q_i(\tau)) = d$, and consequently that $\deg(Q_i(\tau)) = d - 2$ for $i = 1, ..., 4$. We can observe that, as expected, the use of a polynomial dependence greatly improve the maximum value of ε, that is, larger estimates of the RAO are obtained. On the other hand, there is little advantage in choosing d greater than 4 for this system.

In Figure 1 the obtained estimates from problem (54) and (55) (with $d = 4$) are compared to the ones obtained with the conditions of references [11], [31] and [10]. Regarding [11], the

Table 1: Results of problem (55).

d	$\varepsilon_{\max}$
2	1.183
4	1.694
6	1.701
8	1.701

method is applied with a number of interval partitions $J = 27$, and decrease factor $\lambda = 0.98$, as in the numerical example of that paper. It can be observed that the estimate obtained from (54) is relatively conservative, as expected. On the other hand, the one obtained with the matrices with polynomial dependence on τ, that is from (55), are clearly less conservative than the ones obtained from the conditions in [31] (which uses a looped-functional approach) and [10] (which considers a hybrid system approach, but with different conditions). Furthermore, the estimate obtained from (55) is slightly better than the one obtained with the approach in [11], with the advantage that it is not necessary to search a good partition for the interval $[\underline{\tau}, \overline{\tau}]$. The region of linearity, which corresponds to the region where the control is not saturated, is colored in light blue in the figure.

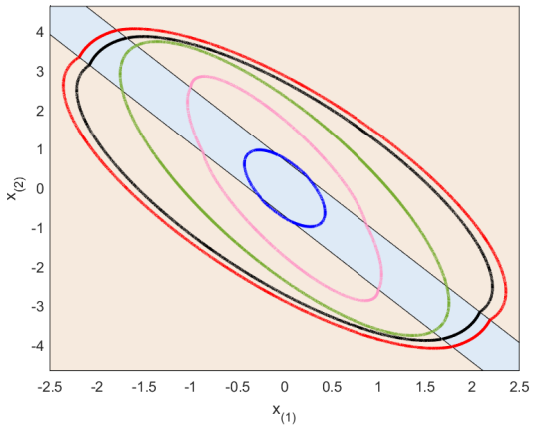

Figure 1: Estimates obtained from problem (55) (red), from [11] (black), from [31] (green), from [10] (pink), and from problem (54) (blue). The region of linearity is colored in light blue.

6.2 Example 2

In this example, the problem **P2** is tackled from the solution of (58) and (59) for system (1)-(2) described with

$$A = \begin{bmatrix} -0.25 & 1 \\ 1 & -0.25 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; K = \begin{bmatrix} -1.5 & -1 \end{bmatrix} \text{ and } \underline{\tau} = 0.05. \tag{61}$$

For this problem, we define a set of admissible initial conditions $\mathcal{X}_0$, which is given by a piecewise quadratic shape $\mathcal{L}_R^{x_p}$ parameterized by a matrix R, as in (56).

Adopting again $d = 4$, the obtained results for **P2** considering different values for matrix R are shown in Figures 2 and 3, and in Table 2. As expected, the solution of problem (59),

which is based on a polynomial dependence on τ, provides a higher bound for the maximum admissible $\overline{\tau}$ when compared to the one of problem (58), which is based on an affine dependence on τ. Furthermore, for a larger region $\mathcal{X}_0$, a smaller value of $\overline{\tau}$ is obtained. This characterizes an expected trade-off between the RAO and the maximum bound on the intersampling interval.

Table 2: Results of problems (58) and (59) for different sets of admissible initial state, as defined in (56)

R	$\overline{\tau}$ prob. (58)	$\overline{\tau}$ prob. (59)
$4I$	0.14	0.46
$0.5I$	unfeasible	0.12

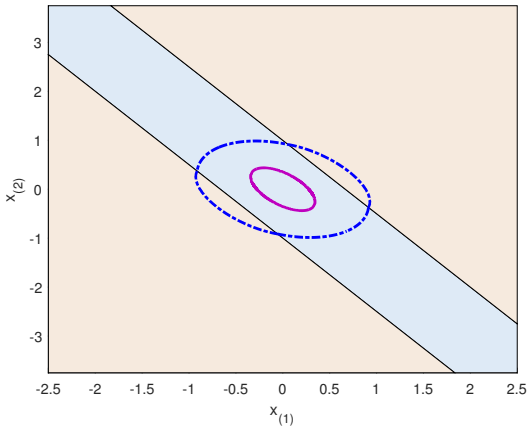

Figure 2: $\mathcal{L}_R^{x_p}$ in purple, and $\mathcal{L}_P^{x_p}$ obtained from (58) in blue, for $R = 4I$. The region of linearity is colored in light blue.

7 Conclusion

In this work, a new method of stability analysis for linear plants subject to saturating sampled-data control was proposed. Considering the hybrid systems framework in [17] and a quadratic timer-dependent Lyapunov function, the stability conditions were first derived with matrices having affine dependence on the timer, leading to LMI conditions, and then extended to the case of polynomial dependence, leading to SOS conditions. In both cases, it was shown how to incorporate these conditions in optimization problems to provide maximized estimates of the RAO or of the maximum bound on the intersampling time for which it is possible to ensure that a given set of admissible initial states is included in the RAO.

Compared to the looped functional approach [31], it has been shown that the proposed method leads to less conservative results. Although the estimates obtained with the approach in [24] can be in some cases better than the proposed one, it depends on the choice of a parameter λ, and also a sufficiently fine partition of the possible intersampling interval is needed, leading in general to a high number of LMIs and variables. On the other hand, the proposed approach allows to compute both the estimate of the RAO or the maximum allowable intersampling time in a rather direct way, without the need of iterative procedures.

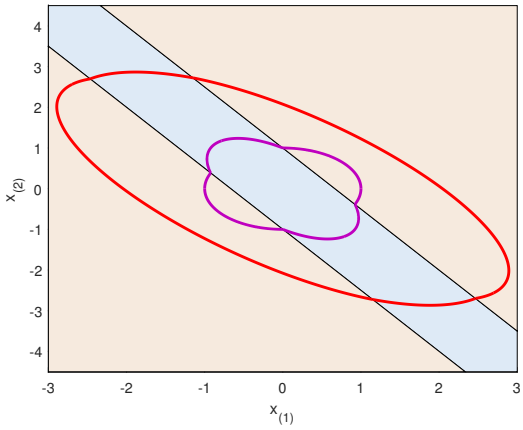

Figure 3: $\mathcal{L}_R^{x_p}$ in purple, and $\mathcal{L}_P^{x_p}$ obtained from (59) in red, for $R = 0.5I$. The region of linearity is colored in light blue.

In the present work, only the stability analysis has been addressed, i.e. we have considered that the control law has been previously designed. The extension of the proposed results to address the synthesis of the control law aiming either at the maximization of the allowable inter-sampling time or at the maximization of an estimate of the RAO has to be further investigated. The challenge in this case is to obtain constant gains and keep the degrees of freedom introduced by a timer-dependent Lyapunov function. Note that classical change of variables, applied to the case of continuous or discrete-time cases [34], can not be used here without generating timer-dependent gains. To overcome this issue, some preliminary results have been proposed in [9].

References

[1] Liron I Allerhand and Uri Shaked. Robust stability and stabilization of linear switched systems with dwell time. *IEEE Transactions on Automatic Control*, 56(2):381–386, 2011.

[2] B. Bamieh, J.B. Pearson, B.A. Francis, and Tannenbaum A. A lifting technique for linear periodic systems with applications to sampled-data control. *Systems & Control Letters*, 17(2):79–88, 1991.

[3] Pierre-Alexandre Bliman. An existence result for polynomial solutions of parameter-dependent LMIs. *Systems & Control Letters*, 51:165–169, 2004.

[4] Shmuel Boyarski and Uri Shaked. Time-convexity and time-gain-scheduling in finite-horizon robust H_∞-control. In *48th IEEE Conference on Decision and Control*, page 2765–2770, Shanghai, 2009.

[5] C. Briat. Convex conditions for robust stability analysis and stabilization of linear aperiodic impulsive and sampled-data systems under dwell-time constraints. *Automatica*, 49(11):3449–3457, 2013.

[6] Corentin Briat. Stability analysis and control of a class of LPV systems with piecewise constant parameters. *Systems & Control Letters*, 82:10–17, 2015.

[7] Corentin Briat. Theoretical and numerical comparisons of looped functionals and clock-dependent Lyapunov functions—the case of periodic and pseudo-periodic systems with impulses. *International Journal of Robust and Nonlinear Control*, 26:2232–2255, 2015.

[8] M.B.G. Cloosterman, L. Hetel, N. Van De Wouw, W.P.M.H. Heemels, J. Daafouz, and H. Nijmeijer. Controller synthesis for networked control systems. *Automatica*, 46(10):1584–1594, 2010.

[9] Arthur Scolari Fagundes and João Manoel Gomes da Silva Jr. Design of saturating aperiodic sampled-data control laws for linear plants: a hybrid system approach. In *20th European Control Conference (ECC)*, London, UK, 687-692.

[10] Arthur Scolari Fagundes, João Manoel Gomes da Silva Jr., and Marc Jungers. Stability of control systems under aperiodic sampling and input saturation with clock-dependent Lyapunov functions. In *14th Simpósio Brasileiro de Automação Inteligente (SBAI)*, Ouro Preto, Brazil, 2019.

[11] Mirko Fiacchini and João Manoel Gomes da Silva Jr. Stability of sampled-data control systems under aperiodic sampling and input saturation. In *57th IEEE Conference on Decision and Control*, page 1286–1293, Miami Beach, 2018.

[12] Mirko Fiacchini and Irinel-Constantin Morărescu. Constructive necessary and sufficient condition for the stability of quasi-periodic linear impulsive systems. *IEEE Transactions on Automatic Control*, 61(9):2512–2517, 2016.

[13] E. Fridman. A refined input delay approach to sampled-data control. *Automatica*, 46(2):421–427, 2010.

[14] Emilia Fridman, Alexandre Seuret, and Jean-Pierre Richard. Robust sampled-data stabilization of linear systems - an input delay approach introduction. *Automatica*, 40:1441–1446, 2004.

[15] H. Fujioka. A discrete-time approach to stability analysis of systems with aperiodic sample-and-hold devices. *IEEE Transactions on Automatic Control*, 54(10):2440–2445, 2009.

[16] R. Goebel, R. G. Sanfelice, and A. R. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29(2):28–93, 2009.

[17] R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: Modeling, Stability and Robustness*. Princeton University Press, Princeton, NJ, 2012.

[18] João Manoel Gomes da Silva Jr. and S. Tarbouriech. Polyhedral regions of local stability for linear discrete-time systems with saturating controls. *IEEE Transactions on Automatic Control*, 44(11):2081–2085, 1999.

[19] João Manoel Gomes da Silva Jr. and Sophie Tarbouriech. Antiwindup design with guaranteed regions of stability: An LMI-based approach. *IEEE Transactions on Automatic Control*, 50(1):106–111.

[20] L. Hetel, J. Daafouz, J.P. Richard, and M. Jungers. Delay-dependent sampled-data control based on delay estimates. *Systems & Control Letters*, 60(2):146 – 150, 2011.

[21] L. Hetel, C. Fiter, H. Omran, A. Seuret, E. Fridman, J.P. Richard, and S.I. Niculescu. Recent developments on the stability of systems with aperiodic sampling: An overview. *Automatica*, 76:309–335, 2016.

[22] Li-Sheng Hu, James Lam, Yong-Yan Cao, and Hui-He Sha. A linear matrix inequality (LMI) approach to robust H_2 sampled-data control for linear uncertain systems. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, 33(1):149–155, 2003.

[23] T. Hu and Z. Lin. *Control systems with actuator saturation: analysis and design*. Birkhauser, Boston, MA, 2001.

[24] D. D. Huff, M. Fiacchini, and J. M. Gomes da Silva Jr. Stability and stabilization of aperiodic sampled-data systems subject to control input saturation: a set invariant approach. *IEEE Transactions on Automatic Control*, 67(3):1423–1429, 2022.

[25] Pablo A. Parrilo. *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. PhD thesis, California Institute of Technology, Pasadena, Califorina, 2000.

[26] Pablo A. Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Mathematical Programming*, 96(2):293–320, 2003.

[27] Helfried Peyrl and Pablo A. Parrilo. Computing sum of squares decompositions with rational coefficients. *Theoretical Computer Science*, 4:269–281, 2008.

[28] S. Prajna, A. Papachristodoulou, J. Valmorbida, J. Anderson, P. Seiler, and P. Parrilo. SOSTOOLS: sum of squares optimization toolbox for Matlab. *GNU General Public License*, 2004.

[29] Mihai Putinar. Positive polynomials on compact semi-algebraic sets. *Indiana University Mathematics Journal*, 42(3):969–984, 1993.

[30] A. Seuret. A novel stability analysis of linear systems under asynchrounous samplings. *Automatica*, 48(1):177–182, 2012.

[31] A. Seuret and João Manoel Gomes da Silva Jr. Taking into account period variations and actuator saturation in sampled-data systems. *Systems & Control Letters*, 61:1286–1293, 2012.

[32] Jos F. Sturm. Using sedumi 1.02, a Matlab Toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11, 1999.

[33] H. J. Sussmann, E. D. Sontag, and Y. Yang. A general result on the stabilization of linear systems using bounded controls. *IEEE Transactions on Automatic Control*, 39(12):2411–2425, 1994.

[34] Sophie Tarbouriech, Germain Garcia, João Manoel Gomes da Silva Jr., and Isabelle Queinnec. *Stability and Stabilization of Linear Systems with Saturating Actuators*. Springer, Berlin, 2011.

[35] Giorgio Valmorbida, Sophie Tarbouriech, and Germain Garcia. Design of polynomial control laws for polynomial systems subject to actuator saturation. *IEEE Transactions on Automatic Control*, 58(7):1758–1770, 2013.

[36] Luca Zaccarian and Andrew R. Teel. *Modern Anti-windup Synthesis: Control Augmentation for Actuator Saturation*. Princeton University Press, Princeton, NJ, 2011.