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# NOTE ON LINEAR INDEPENDENCE OF ALTERNATING MULTIPLE ZETA VALUES IN POSITIVE CHARACTERISTIC

BO-HAE IM, HOJIN KIM, KHAC NHUAN LE, TUAN NGO DAC, AND LAN HUONG PHAM

*Dedicated to Professor Ngô Việt Trung with admiration*

ABSTRACT. We discuss certain results related to the linear independence of alternating multiple zeta values introduced by Harada in 2021.

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## 1. INTRODUCTION

The present paper was initially conceived in January 2021 as an extension of some results of [22] to the setting of alternating multiple zeta values (AMZV's) introduced by Harada [14]. At that time, we endeavored to explore Zagier-Hoffman's conjectures in positive characteristic for both MZV's and AMZV's, by establishing certain results related to the linear independence of specific sets of AMZV's. Subsequently, in their work [17], the authors discovered a more successful approach by substituting these sets with those of alternating Carlitz multiple polylogarithms (ACMPL's). For the MZV's version of Zagier-Hoffman's conjectures in positive characteristic, we would like to refer the reader to the work [9] which adopts the same approach and yields the same results. Consequently, we made the decision to briefly reference the findings of this paper in Proposition 4.6 of the aforementioned work (*loc. cit.*), deferring the complete exposition of the details to the present article.

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### 1.1. Background.

Let  $A = \mathbb{F}_q[\theta]$  be the polynomial ring in the variable  $\theta$  over a finite field  $\mathbb{F}_q$  of  $q$  elements of characteristic  $p > 0$ . We denote by  $A_+$  the set of monic polynomials in  $A$ . Let  $K = \mathbb{F}_q(\theta)$  be the fraction field of  $A$  equipped with the rational point  $\infty$ . Let  $K_\infty$  be the completion of  $K$  at  $\infty$  and  $\mathbb{C}_\infty$  be the completion of a fixed algebraic closure  $\overline{K}$  of  $K$  at  $\infty$ . We denote by  $v_\infty$  the discrete valuation on  $K$  corresponding to the place  $\infty$  normalized such that  $v_\infty(\theta) = -1$ , and by  $|\cdot|_\infty = q^{-v_\infty}$  the associated absolute value on  $K$ . The unique valuation of  $\mathbb{C}_\infty$  which extends  $v_\infty$  will still be denoted by  $v_\infty$ .

Let  $\mathbb{N} = \{1, 2, \dots\}$  be the set of positive integers and  $\mathbb{Z}^{\geq 0} = \{0, 1, 2, \dots\}$  be the set of non-negative integers. In [6] Carlitz introduced the Carlitz zeta values  $\zeta_A(n)$  for  $n \in \mathbb{N}$  given by

$$\zeta_A(n) := \sum_{a \in A_+} \frac{1}{a^n} \in K_\infty$$

which are analogues of classical special zeta values in the function field setting. For any tuple of positive integers  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ , Thakur [24] defined the characteristic  $p$  multiple zeta value (MZV for short)  $\zeta_A(\mathfrak{s})$  or  $\zeta_A(s_1, \dots, s_r)$  by

$$\zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \dots a_r^{s_r}} \in K_\infty$$

where the sum runs through the set of tuples  $(a_1, \dots, a_r) \in A_+^r$  with  $\deg a_1 > \dots > \deg a_r$ . We call  $r$  the depth and  $w(\mathfrak{s}) = s_1 + \dots + s_r$  the weight of  $\mathfrak{s}$  and  $\zeta_A(\mathfrak{s})$ . We note that Carlitz zeta values are exactly depth one MZV's. Thakur [25] showed that all the MZV's do not vanish.

For  $w \in \mathbb{N}$  we denote by  $\mathcal{Z}_w$  the  $K$ -vector space spanned by the MZV's of weight  $w$ . We denote by  $\mathcal{J}_w$  the set of tuples  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$  of weight  $w$  with  $1 \leq s_i \leq q$  for  $1 \leq i \leq r-1$  and  $s_r < q$ , and by  $\mathcal{T}_w$  is the set of  $\zeta_A(\mathfrak{s})$  with  $\mathfrak{s} \in \mathcal{J}_w$ . We also define  $\mathcal{J}_w^0$  to be the set of tuples  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$  of weight  $w$  such that  $1 \leq s_i < q$  for all  $1 \leq i \leq r$  and  $\mathcal{T}_w^0$  to be the set of MZV's  $\zeta_A(\mathfrak{s})$  such that  $\mathfrak{s} \in \mathcal{J}_w^0$ . Note that  $\mathcal{T}_w^0$  is a subset of  $\mathcal{T}_w$ . We now recall two results of [22]:

**Theorem 1.1** ([22, Theorems B]). *Let  $w \in \mathbb{N}$ . Then the MZV's in  $\mathcal{T}_w^0$  are linearly independent over  $K$ .*

**Theorem 1.2** ([22, Theorems D]). *Let  $w \leq 2q - 2$ . Then the MZV's in  $\mathcal{T}_w$  are linearly independent over  $K$ .*

### 1.2. Statement of the results.

In [14] Harada introduced the alternating MZV's (AMZV's) in positive characteristic. For any tuple of positive integers  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$  and  $\epsilon = (\epsilon_1, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$ , we consider the array  $\begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix}$  and the corresponding AMZV is given by

$$\zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} := \sum \frac{\epsilon_1^{\deg a_1} \dots \epsilon_r^{\deg a_r}}{a_1^{s_1} \dots a_r^{s_r}} \in K_\infty$$

where the sum runs through the set of tuples  $(a_1, \dots, a_r) \in A_+^r$  with  $\deg a_1 > \dots > \deg a_r$ . We call  $r$  the depth,  $w(\mathfrak{s}) = s_1 + \dots + s_r$  the weight and  $\chi(\epsilon) = \epsilon_1 \dots \epsilon_r$  the character of  $\begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix}$  and  $\zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix}$ . We note all the AMZV's do not vanish [14]. For  $w \in \mathbb{N}$  we denote by  $\mathcal{AZ}_w$  the  $K$ -vector space spanned by the AMZV's of weight

$w$ . We refer the reader to [12, 14] for more details about these objects. Note that in the classical setting the reader could consult [16, p. 536] for a discussion about the dimensions of the  $\mathbb{Q}$ -vector space spanned by all the AMZV's of fixed weight.

We now introduce several sets of AMZV's. First, we define the sets  $\mathcal{J}_w$  and  $\mathcal{AT}_w$  by

$$\mathcal{J}_w := \left\{ \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} : \mathfrak{s} \in \mathcal{J}_w \right\}, \quad \mathcal{AT}_w := \left\{ \zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} : \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} \in \mathcal{J}_w \right\}.$$

Second, we introduce the following sets

$$\begin{aligned} \mathcal{J}_w^0 &:= \left\{ \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} : \mathfrak{s} \in \mathcal{J}_w^0 \right\}, & \mathcal{AT}_w^0 &:= \left\{ \zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} : \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} \in \mathcal{J}_w^0 \right\}, \\ \mathcal{J}_w^1 &:= \left\{ \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} : \mathfrak{s} \in \mathcal{J}_w, \text{ and } \epsilon_i = 1 \text{ whenever } s_i = q \right\}, \\ \mathcal{AT}_w^1 &:= \left\{ \zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} : \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} \in \mathcal{J}_w^1 \right\}. \end{aligned}$$

We remark that in general,

$$\mathcal{AT}_w^0 \subsetneq \mathcal{AT}_w^1 \subsetneq \mathcal{AT}_w.$$

Finally, for  $1 \leq w \leq 3q-2$ , we denote by  $\mathcal{J}'_w$  the set of tuples  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$  of weight  $w$  as follows:

- For  $1 \leq w \leq 2q-2$ ,  $\mathcal{J}'_w$  consists of tuples  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$  of weight  $w$  where  $s_i \neq q$  for all  $i$ .
- For  $2q-1 \leq w \leq 3q-3$ ,  $\mathcal{J}'_w$  consists of tuples  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$  of weight  $w$  of the form
  - either  $s_i \neq q, 2q-1, 2q$  for all  $i$ ,
  - or there exists a unique integer  $1 \leq i < r$  such that  $(s_i, s_{i+1}) = (q-1, q)$ .
- For  $w = 3q-2$  and  $q > 2$ ,  $\mathcal{J}'_w$  consists of tuples  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$  of weight  $w$  of the form
  - either  $s_i \neq q, 2q-1, 2q, 3q-2$  for all  $i$ ,
  - or there exists a unique integer  $1 \leq i < r$  such that  $(s_i, s_{i+1}) \in \{(q-1, q), (2q-2, q)\}$ , but  $\mathfrak{s} \neq (q-1, q-1, q)$ ,
  - or  $\mathfrak{s} = (q-1, 2q-1)$ .
- For  $q = 2$  and  $w = 3q-2 = 4$ ,  $\mathcal{J}'_w$  consists of the following tuples:  $(2, 1, 1)$ ,  $(1, 2, 1)$  and  $(1, 3)$ .

Then we denote by  $\mathcal{J}'_w$  the set given by

$$\mathcal{J}'_w := \left\{ \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} : \mathfrak{s} \in \mathcal{J}'_w, \text{ and } \epsilon_i = 1 \text{ whenever } s_i \in \{q, 2q-1\} \right\}$$

and by  $\mathcal{AT}'_w$  the subset of AMZV's given by

$$\mathcal{AT}'_w := \left\{ \zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} : \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} \in \mathcal{J}'_w \right\}.$$

We mention that as already noticed in [10, 19], the case  $q = 2$  is somewhat special and needs a careful examination.

Here are some basic properties of these sets that are crucial for linear independence of AMZV's.

**Proposition 1.3.** *We define a Fibonacci-like sequence  $s(w)$  as follows. We put*

$$s(w) = \begin{cases} (q-1)q^{w-1} & \text{if } 1 \leq w < q, \\ (q-1)(q^{w-1} - 1) & \text{if } w = q, \end{cases}$$

and for  $w > q$ ,  $s(w) = (q-1) \sum_{i=1}^{q-1} s(w-i) + s(w-q)$ . Then for all  $w \in \mathbb{N}$ ,

$$|\mathcal{AT}_w^1| = s(w).$$

**Proposition 1.4.** *Let  $1 \leq w \leq 3q-2$ . Then*

1) *If  $1 \leq w \leq 3q-3$ , then we have*

$$|\mathcal{AT}'_w| = s(w).$$

2) *For  $q > 2$  and any  $(\mathfrak{s}; \epsilon) = (s_1, \dots, s_r; \epsilon_1, \dots, \epsilon_r) \in \mathbb{N}^r \times (\mathbb{F}_q^\times)^r$ , if  $\zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} \in$*

$$\mathcal{AT}'_w, \text{ then } \zeta_A \begin{pmatrix} \epsilon_1 & \cdots & \epsilon_{r-1} \\ s_1 & \cdots & s_{r-1} \end{pmatrix} \text{ belongs to } \mathcal{AT}'_{w-s_r}.$$

We are now ready to state some results for linear independence of AMZV's which extend Theorems 1.1 and 1.2:

**Theorem 1.5.** *Let  $w \in \mathbb{N}$ . Then the AMZV's in  $\mathcal{AT}_w^0$  are linearly independent over  $K$ .*

**Theorem 1.6.** *Let  $1 \leq w \leq 3q-2$ . Then the AMZV's in  $\mathcal{AT}'_w$  are all linearly independent over  $K$ . In particular, if  $1 \leq w \leq 3q-3$ , then*

$$\dim_K \mathcal{AZ}_w \geq s(w).$$

We would like to mention several comments. First, Theorem 1.6 is exactly [17, Proposition 4.6]. Second, the proofs of both previous theorems are based on the theory of dual  $t$ -motives introduced by Anderson [1] and the Anderson-Brownawell-Papanikolas criterion [2]. Finally, it is proved in [17] that  $\dim_K \mathcal{AZ}_w = s(w)$  for all weights  $w \in \mathbb{N}$  (see [17, Theorem A]). However, the approach given in *loc. cit.* goes beyond the scope of this paper as we have mentioned in the beginning of the introduction.

### 1.3. Organization of the paper.

In §2 we recall the notion of dual  $t$ -motives following Anderson and the Anderson-Brownawell-Papanikolas criterion for linear independence. Then we state some applications for linear independence of MZV's and AMZV's (see Theorems 2.4 and 2.5). In §3 we prove Propositions 1.3 and 1.4. The main theorems 1.5 and 1.6 will be shown in §4 and §5 respectively.

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2. DUAL  $t$ -MOTIVES AND LINEAR INDEPENDENCE

We continue with the notation given in the Introduction and follow closely the presentation given as in [17, 22] (see also [9]). In particular, a large part of this section is line-by-line parallel to [17, §2].

## 2.1. Notation.

In what follows, letting  $t$  be another independent variable, we denote by  $\mathbb{T}$  the Tate algebra in the variable  $t$  with coefficients in  $\mathbb{C}_\infty$  equipped with the Gauss norm  $\|\cdot\|_\infty$ .

We denote by  $\mathcal{E}$  the ring of series  $\sum_{n \geq 0} a_n t^n \in \overline{K}[[t]]$  such that  $\lim_{n \rightarrow +\infty} \sqrt[n]{|a_n|_\infty} = 0$  and  $[K_\infty(a_0, a_1, \dots) : K_\infty] < \infty$ . Then any  $f \in \mathcal{E}$  is an entire function.

For  $a \in A = \mathbb{F}_q[\theta]$ , we set  $a(t) := a|_{\theta=t} \in \mathbb{F}_q[t]$ .

2.2. Dual  $t$ -motives and Anderson-Brownawell-Papanikolas (ABP) criterion.

We recall the notion of dual  $t$ -motives due to Anderson (see [5, §4] and [15, §5] for more details). We refer the reader to [1] for the related notion of  $t$ -motives.

For  $i \in \mathbb{Z}$  we consider the  $i$ -fold twisting of  $\mathbb{C}_\infty((t))$  defined by

$$\begin{aligned} \mathbb{C}_\infty((t)) &\rightarrow \mathbb{C}_\infty((t)) \\ f = \sum_j a_j t^j &\mapsto f^{(i)} := \sum_j a_j^{q^i} t^j. \end{aligned}$$

We extend  $i$ -fold twisting to matrices with entries in  $\mathbb{C}_\infty((t))$  by twisting entry-wise.

Let  $\overline{K}[t, \sigma]$  be the non-commutative  $\overline{K}[t]$ -algebra generated by the new variable  $\sigma$  subject to the relation  $\sigma f = f^{(-1)} \sigma$  for all  $f \in \overline{K}[t]$ .

**Definition 2.1.** An effective dual  $t$ -motive is a  $\overline{K}[t, \sigma]$ -module  $\mathcal{M}'$  which is free and finitely generated over  $\overline{K}[t]$  such that for  $\ell \gg 0$  we have

$$(t - \theta)^\ell (\mathcal{M}' / \sigma \mathcal{M}') = \{0\}.$$

We mention that effective dual  $t$ -motives are called Frobenius modules in [10, 12, 14, 18]. Note that Hartl and Juschka [15, §4] introduced a more general notion of dual  $t$ -motives. In particular, effective dual  $t$ -motives are always dual  $t$ -motives.

Throughout this paper, we will always work with effective dual  $t$ -motives. Therefore, we will sometimes drop the word “effective” where there is no confusion.

Let  $\mathcal{M}$  and  $\mathcal{M}'$  be two effective dual  $t$ -motives. Then a morphism of effective dual  $t$ -motives  $\mathcal{M} \rightarrow \mathcal{M}'$  is just a homomorphism of left  $\overline{K}[t, \sigma]$ -modules. We denote by  $\mathcal{F}$  the category of effective dual  $t$ -motives equipped with the trivial object  $\mathbf{1}$ .

We say that an object  $\mathcal{M}$  of  $\mathcal{F}$  is given by a matrix  $\Phi \in \text{Mat}_r(\overline{K}[t])$  if  $\mathcal{M}$  is a  $\overline{K}[t]$ -module free of rank  $r$  and the action of  $\sigma$  is represented by the matrix  $\Phi$  on a given  $\overline{K}[t]$ -basis for  $\mathcal{M}$ . We say that an object  $\mathcal{M}$  of  $\mathcal{F}$  is uniformizable or rigid analytically trivial if there exists a matrix  $\Psi \in \text{GL}_r(\mathbb{T})$  satisfying  $\Psi^{(-1)} = \Phi \Psi$ . The matrix  $\Psi$  is called a rigid analytic trivialization of  $\mathcal{M}$ .

We now recall the Anderson-Brownawell-Papanikolas criterion which is crucial in the sequel (see [2, Theorem 3.1.1]).

**Theorem 2.2** (Anderson-Brownawell-Papanikolas). *Let  $\Phi \in \text{Mat}_\ell(\overline{K}[t])$  be a matrix such that  $\det \Phi = c(t - \theta)^s$  for some  $c \in \overline{K}^\times$  and  $s \in \mathbb{Z}^{\geq 0}$ . Let  $\psi \in \text{Mat}_{\ell \times 1}(\mathcal{E})$*

be a vector satisfying  $\psi^{(-1)} = \Phi\psi$  and  $\rho \in \text{Mat}_{1 \times \ell}(\overline{K})$  such that  $\rho\psi(\theta) = 0$ . Then there exists a vector  $P \in \text{Mat}_{1 \times \ell}(\overline{K}[t])$  such that

$$P\psi = 0 \quad \text{and} \quad P(\theta) = \rho.$$

### 2.3. First construction of dual $t$ -motives.

We briefly review some constructions of dual  $t$ -motives introduced in [10] (see also [7, 12, 14]). Let  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$  be a tuple of positive integers and  $\Omega = (Q_1, \dots, Q_r) \in \overline{K}[t]^r$  satisfying the condition

$$(2.1) \quad (\|Q_1\|_\infty / |\theta|_\infty^{\frac{qs_1}{q-1}})^{q^{i_1}} \dots (\|Q_r\|_\infty / |\theta|_\infty^{\frac{qs_r}{q-1}})^{q^{i_r}} \rightarrow 0$$

as  $0 \leq i_r < \dots < i_1$  and  $i_1 \rightarrow \infty$ .

We consider the dual  $t$ -motives  $\mathcal{M}_{\mathfrak{s};\Omega}$  and  $\mathcal{M}'_{\mathfrak{s};\Omega}$  attached to  $(\mathfrak{s}; \Omega)$  given by the matrices

$$\Phi_{\mathfrak{s};\Omega} = \begin{pmatrix} (t-\theta)^{s_1+\dots+s_r} & 0 & 0 & \dots & 0 \\ Q_1^{(-1)}(t-\theta)^{s_1+\dots+s_r} & (t-\theta)^{s_2+\dots+s_r} & 0 & \dots & 0 \\ 0 & Q_2^{(-1)}(t-\theta)^{s_2+\dots+s_r} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{s_r} & 0 \\ 0 & \dots & 0 & Q_r^{(-1)}(t-\theta)^{s_r} & 1 \end{pmatrix} \\ \in \text{Mat}_{r+1}(\overline{K}[t]),$$

and  $\Phi'_{\mathfrak{s};\Omega} \in \text{Mat}_r(\overline{K}[t])$  is the upper left  $r \times r$  sub-matrix of  $\Phi_{\mathfrak{s};\Omega}$ .

Throughout this paper, we work with the Carlitz period  $\tilde{\pi}$  which is a fundamental period of the Carlitz module (see [13, 24]). We fix a choice of  $(q-1)$ -th root of  $(-\theta)$  and set

$$\Omega(t) := (-\theta)^{-q/(q-1)} \prod_{i \geq 1} \left(1 - \frac{t}{\theta q^i}\right) \in \mathbb{T}^\times$$

so that

$$\Omega^{(-1)} = (t-\theta)\Omega \quad \text{and} \quad \frac{1}{\Omega(\theta)} = \tilde{\pi}.$$

Given  $(\mathfrak{s}; \Omega)$  as above, Chang introduced the following series (see [7, Lemma 5.3.1] and also [10, Eq. (2.3.2)])

$$(2.2) \quad \mathfrak{L}(\mathfrak{s}; \Omega) = \mathfrak{L}(s_1, \dots, s_r; Q_1, \dots, Q_r) := \sum_{i_1 > \dots > i_r \geq 0} (\Omega^{s_r} Q_r)^{(i_r)} \dots (\Omega^{s_1} Q_1)^{(i_1)}.$$

Recall that  $\mathcal{E}$  denotes the ring of series  $\sum_{n \geq 0} a_n t^n \in \overline{K}[[t]]$  such that  $\lim_{n \rightarrow +\infty} \sqrt[n]{|a_n|_\infty} = 0$  and  $[K_\infty(a_0, a_1, \dots) : K_\infty] < \infty$ . It is proved that  $\mathfrak{L}(\mathfrak{s}; \Omega) \in \mathcal{E}$  (see [7, Lemma 5.3.1]). In the sequel, we will use the following crucial property of this series (see [7, Lemma 5.3.5] and [10, Proposition 2.3.3]): for all  $j \in \mathbb{Z}^{\geq 0}$ , we have

$$(2.3) \quad \mathfrak{L}(\mathfrak{s}; \Omega) \left( \theta^{q^j} \right) = (\mathfrak{L}(\mathfrak{s}; \Omega)(\theta))^{q^j}.$$

Then the matrix given by

$$\Psi_{\mathfrak{s};\Omega} = \begin{pmatrix} \Omega^{s_1+\dots+s_r} & 0 & 0 & \dots & 0 \\ \mathfrak{L}(s_1; Q_1)\Omega^{s_2+\dots+s_r} & \Omega^{s_2+\dots+s_r} & 0 & \dots & 0 \\ \vdots & \mathfrak{L}(s_2; Q_2)\Omega^{s_3+\dots+s_r} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \mathfrak{L}(s_1, \dots, s_{r-1}; Q_1, \dots, Q_{r-1})\Omega^{s_r} & \mathfrak{L}(s_2, \dots, s_{r-1}; Q_2, \dots, Q_{r-1})\Omega^{s_r} & \dots & \Omega^{s_r} & 0 \\ \mathfrak{L}(s_1, \dots, s_r; Q_1, \dots, Q_r) & \mathfrak{L}(s_2, \dots, s_r; Q_2, \dots, Q_r) & \dots & \mathfrak{L}(s_r; Q_r) & 1 \end{pmatrix} \\ \in \mathrm{GL}_{r+1}(\mathbb{T})$$

satisfies

$$\Psi_{\mathfrak{s};\Omega}^{(-1)} = \Phi_{\mathfrak{s};\Omega} \Psi_{\mathfrak{s};\Omega}.$$

Thus  $\Psi_{\mathfrak{s};\Omega}$  is a rigid analytic trivialization associated to the dual  $t$ -motive  $\mathcal{M}_{\mathfrak{s};\Omega}$ .

We also denote by  $\Psi'_{\mathfrak{s};\Omega}$  the upper  $r \times r$  sub-matrix of  $\Psi_{\mathfrak{s};\Omega}$ . It is clear that  $\Psi'_{\mathfrak{s}}$  is a rigid analytic trivialization associated to the dual  $t$ -motive  $\mathcal{M}'_{\mathfrak{s};\Omega}$ .

Further, combined with Eq. (2.3), the above construction of dual  $t$ -motives implies that  $\tilde{\pi}^w \mathfrak{L}(\mathfrak{s}; \Omega)(\theta)$  where  $w = s_1 + \dots + s_r$  has the MZ (multizeta) property in the sense of [7, Definition 3.4.1]. By [7, Proposition 4.3.1], we get

**Proposition 2.3** ([17, Proposition 2.3]). *Let  $(\mathfrak{s}_i; \Omega_i)$  as before for  $1 \leq i \leq n$ . We suppose that all the tuples of positive integers  $\mathfrak{s}_i$  have the same weight, says  $w$ . Then the following assertions are equivalent:*

- i)  $\mathfrak{L}(\mathfrak{s}_1; \Omega_1)(\theta), \dots, \mathfrak{L}(\mathfrak{s}_n; \Omega_n)(\theta)$  are  $K$ -linearly independent.
- ii)  $\mathfrak{L}(\mathfrak{s}_1; \Omega_1)(\theta), \dots, \mathfrak{L}(\mathfrak{s}_n; \Omega_n)(\theta)$  are  $\overline{K}$ -linearly independent.

#### 2.4. Second construction of dual $t$ -motives.

Based on the first construction, we present another construction which will be useful when working with linear combinations of MZV's or related objects.

Let  $w \in \mathbb{N}$  be a positive integer and  $n \in \mathbb{N}$ . Let  $\{(\mathfrak{s}_i; \Omega_i)\}_{1 \leq i \leq n}$  be a collection of pairs satisfying Condition (2.1) such that  $\mathfrak{s}_i$  always has weight  $w$ . We write  $\mathfrak{s}_i = (s_{i1}, \dots, s_{i\ell_i}) \in \mathbb{N}^{\ell_i}$  and  $\Omega_i = (Q_{i1}, \dots, Q_{i\ell_i}) \in \overline{K}[t]^{\ell_i}$  so that  $s_{i1} + \dots + s_{i\ell_i} = w$ . We introduce the set of tuples

$$I(\mathfrak{s}_i; \Omega_i) := \{\emptyset, (s_{i1}; Q_{i1}), \dots, (s_{i1}, \dots, s_{i(\ell_i-1)}; Q_{i1}, \dots, Q_{i(\ell_i-1)})\},$$

and set

$$I := \cup_i I(\mathfrak{s}_i; \Omega_i).$$

Let  $a_i \in A \setminus \{0\}$  for all  $1 \leq i \leq n$ . For all  $(\mathfrak{t}; \Omega) \in I$ , we denote by  $J(\mathfrak{t}; \Omega)$  the set of indices  $i$  such that  $(\mathfrak{t}; \Omega) = (s_{i1}, \dots, s_{ik}; Q_{i1}, \dots, Q_{ik})$  for some  $0 \leq k \leq \ell_i - 1$  and set

$$(2.4) \quad f_{\mathfrak{t};\Omega} := \sum_i a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i}; Q_{i(k+1)}, \dots, Q_{i\ell_i}),$$

where the sum runs through the set  $J(\mathfrak{t}; \Omega)$ . In particular,  $f_{\emptyset} = \sum_i a_i(t) \mathfrak{L}(\mathfrak{s}_i; \Omega_i)$ .

We now construct a new matrix  $\Phi'$  by merging the same rows of  $\Phi'_{\mathfrak{s}_1; \Omega_1}, \dots, \Phi'_{\mathfrak{s}_n; \Omega_n}$  as follows. Then the matrix  $\Phi'$  will be a matrix indexed by elements of  $I$ , says  $\Phi' = \left( \Phi'_{(\mathfrak{t}; \Omega), (\mathfrak{t}'; \Omega')} \right)_{(\mathfrak{t}; \Omega), (\mathfrak{t}'; \Omega') \in I} \in \mathrm{Mat}_{|I|}(\overline{K}[t])$ . For the row which corresponds to



the empty pair  $\emptyset$  we put

$$\Phi'_{\emptyset, (\mathfrak{t}; \Omega')} = \begin{cases} (t - \theta)^w & \text{if } (\mathfrak{t}; \Omega') = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

For the row indexed by  $(\mathfrak{t}; \Omega) = (s_{i1}, \dots, s_{ij}; Q_{i1}, \dots, Q_{ij})$  for some  $i$  and  $1 \leq j \leq \ell_i - 1$  we put

$$\Phi'_{(\mathfrak{t}; \Omega), (\mathfrak{t}'; \Omega')} = \begin{cases} (t - \theta)^{w-w(\mathfrak{t}')} & \text{if } (\mathfrak{t}'; \Omega') = (\mathfrak{t}; \Omega), \\ Q_{ij}^{(-1)} (t - \theta)^{w-w(\mathfrak{t}')} & \text{if } (\mathfrak{t}'; \Omega') = (s_{i1}, \dots, s_{i(j-1)}; Q_{i1}, \dots, Q_{i(j-1)}), \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $\Phi'_{\mathfrak{s}_i; \Omega_i} = \left( \Phi'_{(\mathfrak{t}; \Omega), (\mathfrak{t}'; \Omega')} \right)_{(\mathfrak{t}; \Omega), (\mathfrak{t}'; \Omega') \in I(\mathfrak{s}_i; \Omega_i)}$  for all  $i$ .

We define  $\Phi \in \text{Mat}_{|I|+1}(\overline{K}[t])$  by

$$\Phi = \begin{pmatrix} \Phi' & 0 \\ \mathbf{v} & 1 \end{pmatrix} \in \text{Mat}_{|I|+1}(\overline{K}[t]), \quad \mathbf{v} = (v_{\mathfrak{t}; \Omega})_{(\mathfrak{t}; \Omega) \in I} \in \text{Mat}_{1 \times |I|}(\overline{K}[t]),$$

where

$$v_{\mathfrak{t}; \Omega} = \sum_i a_i(t) Q_{i\ell_i}^{(-1)} (t - \theta)^{w-w(\mathfrak{t})}.$$

Here the sum runs through the set of indices  $i$  such that  $(\mathfrak{t}; \Omega) = (s_{i1}, \dots, s_{i(\ell_i-1)}; Q_{i1}, \dots, Q_{i(\ell_i-1)})$  and the empty sum is defined to be zero.

We now introduce a rigid analytic trivialization matrix  $\Psi$  for  $\Phi$ . We define  $\Psi' = \left( \Psi'_{(\mathfrak{t}; \Omega), (\mathfrak{t}'; \Omega')} \right)_{(\mathfrak{t}; \Omega), (\mathfrak{t}'; \Omega') \in I} \in \text{GL}_{|I|}(\mathbb{T})$  as follows. For the row which corresponds to the empty pair  $\emptyset$  we define

$$\Psi'_{\emptyset, (\mathfrak{t}'; \Omega')} = \begin{cases} \Omega^w & \text{if } (\mathfrak{t}'; \Omega') = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

For the row indexed by  $(\mathfrak{t}; \Omega) = (s_{i1}, \dots, s_{ij}; Q_{i1}, \dots, Q_{ij})$  for some  $i$  and  $1 \leq j \leq \ell_i - 1$  we put

$$\Psi'_{(\mathfrak{t}; \Omega), (\mathfrak{t}'; \Omega')} = \begin{cases} \mathfrak{L}(\mathfrak{t}; \Omega) \Omega^{w-w(\mathfrak{t}')} & \text{if } (\mathfrak{t}'; \Omega') = \emptyset, \\ \mathfrak{L}(s_{i(k+1)}, \dots, s_{ij}; Q_{i(k+1)}, \dots, Q_{ij}) \Omega^{w-w(\mathfrak{t}')} & \text{if } (\mathfrak{t}'; \Omega') = (s_{i1}, \dots, s_{ik}; Q_{i1}, \dots, Q_{ik}) \text{ for some } 1 \leq k \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $\Psi'_{\mathfrak{s}_i; \Omega_i} = \left( \Psi'_{(\mathfrak{t}; \Omega), (\mathfrak{t}'; \Omega')} \right)_{(\mathfrak{t}; \Omega), (\mathfrak{t}'; \Omega') \in I(\mathfrak{s}_i; \Omega_i)}$  for all  $i$ . Here we agree that  $\mathfrak{L}(\emptyset) = 1$ .

We define  $\Psi \in \text{GL}_{|I|+1}(\mathbb{T})$  by

$$\Psi = \begin{pmatrix} \Psi' & 0 \\ \mathbf{f} & 1 \end{pmatrix} \in \text{GL}_{|I|+1}(\mathbb{T}), \quad \mathbf{f} = (f_{\mathfrak{t}; \Omega})_{\mathfrak{t} \in I} \in \text{Mat}_{1 \times |I|}(\mathbb{T}).$$

Here we recall (see Eq. (2.4))

$$f_{\mathfrak{t}; \Omega} = \sum_i a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i}; Q_{i(k+1)}, \dots, Q_{i\ell_i})$$

where the sum runs through the set of indices  $i$  such that  $(\mathbf{t}; \mathfrak{Q}) = (s_{i1}, \dots, s_{ik}; Q_{i1}, \dots, Q_{ik})$  for some  $0 \leq k \leq \ell_i - 1$ . In particular,

$$f_\emptyset = \sum_i a_i(t) \mathfrak{L}(\mathfrak{s}_i; \mathfrak{Q}_i).$$

By construction and by §2.3, we get  $\Psi^{(-1)} = \Phi\Psi$ , that means  $\Psi$  is a rigid analytic trivialization for  $\Phi$ .

### 2.5. A result for linear independence.

We are now ready to state an application of ABP criterion for linear independence (see [17, 22] and also [9]).

**Theorem 2.4.** *We keep the above notation. We suppose further that  $\{(\mathfrak{s}_i; \mathfrak{Q}_i)\}_{1 \leq i \leq n}$  satisfies the following conditions:*

- (LW) *For any weight  $w' < w$ , the values  $\mathfrak{L}(\mathbf{t}; \mathfrak{Q})(\theta)$  with  $(\mathbf{t}; \mathfrak{Q}) \in I$  and  $w(\mathbf{t}) = w'$  are all  $K$ -linearly independent. In particular,  $\mathfrak{L}(\mathbf{t}; \mathfrak{Q})(\theta)$  is always nonzero.*
- (LD) *There exist  $a_i \in A \setminus \{0\}$  for  $1 \leq i \leq n$  such that*

$$\sum_{i=1}^n a_i \mathfrak{L}(\mathfrak{s}_i; \mathfrak{Q}_i)(\theta) = 0.$$

Then

- $f_{\mathbf{t}; \mathfrak{Q}}(\theta)$  belongs to  $K$  where  $f_{\mathbf{t}; \mathfrak{Q}}$  is given as in (2.4) for all  $(\mathbf{t}; \mathfrak{Q}) \in I$ .
- There exist  $b \in \mathbb{F}_q[t] \setminus \{0\}$  and  $\delta := (\delta_{\mathbf{t}; \mathfrak{Q}})_{(\mathbf{t}; \mathfrak{Q}) \in I} \in \text{Mat}_{1 \times |I|}(\overline{K}[t])$  such that

$$\begin{pmatrix} \text{Id} & 0 \\ \delta & 1 \end{pmatrix}^{(-1)} \begin{pmatrix} \Phi' & 0 \\ b\mathbf{v} & 1 \end{pmatrix} = \begin{pmatrix} \Phi' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Id} & 0 \\ \delta & 1 \end{pmatrix}.$$

Here we note that LW stands for Lower Weights and LD for Linear Dependence. We mention below some comments about the above result.

- (1) In [10] Chang, Papanikolas and Yu studied zeta-like MZV's and proved Theorem 2.4 in the case where  $n = 2$ ,  $\mathfrak{s}_1$  is of weight  $w$ , and  $\mathfrak{s}_2 = (w)$  (see Theorem 2.5.2 of *loc. cit.*).
- (2) In [18, 21], the authors apply the previous theorem of Chang-Papanikolas-Yu to obtain several results concerning zeta-like MZV's.
- (3) In [8, 12], the authors generalized the aforementioned theorem of Chang-Papanikolas-Yu and obtain some particular cases of Theorem 2.4. In all these works, Condition (LW) is replaced by a stronger condition: one requires that for any weight  $w' < w$ , there exists at most one pair  $(\mathbf{t}; \mathfrak{Q}) \in I$  such that  $w(\mathbf{t}) = w'$  and further  $\mathfrak{L}(\mathbf{t}; \mathfrak{Q})(\theta)$  is always nonzero for all  $(\mathbf{t}; \mathfrak{Q}) \in I$ .
- (4) In [22], the author proved a version of Theorem 2.4 for MZV's (see Theorem 2.5) and applied it to obtain some results towards Zagier-Hoffman's conjectures in positive characteristic. Theorem 2.4 is a generalization of the previous result and is proved in [17, Theorem 2.4] (see also [9]).
- (5) The curious reader is invited to read the previous works for more details and an exhaustive list of related works.

### 2.6. Example: dual $t$ -motives connected to MZV's and AMZV's.

### 2.6.1. First construction.

Following Anderson and Thakur [3, 4] we introduce dual  $t$ -motives connected to MZV's and AMZV's. We briefly review Anderson-Thakur polynomials introduced in the seminal work of Anderson and Thakur [3]. For  $k \geq 0$ , we set  $[k] := \theta^{q^k} - \theta$  and  $D_k := \prod_{\ell=1}^k [\ell]^{q^{k-\ell}}$ . For  $n \in \mathbb{N}$  we write  $n-1 = \sum_{j \geq 0} n_j q^j$  with  $0 \leq n_j \leq q-1$  and define

$$\Gamma_n := \prod_{j \geq 0} D_j^{n_j}.$$

We set  $\gamma_0(t) := 1$  and  $\gamma_j(t) := \prod_{\ell=1}^j (\theta^{q^\ell} - t^{q^\ell})$  for  $j \geq 1$ . Then Anderson-Thakur polynomials  $\alpha_n(t) \in A[t]$  are given by the generating series

$$\sum_{n \geq 1} \frac{\alpha_n(t)}{\Gamma_n} x^n := x \left( 1 - \sum_{j \geq 0} \frac{\gamma_j(t)}{D_j} x^{q^j} \right)^{-1}.$$

Finally, we define  $H_n(t)$  by switching  $\theta$  and  $t$

$$(2.5) \quad H_n(t) = \alpha_n(t) \Big|_{t=\theta, \theta=t}.$$

By [3, Eq. (3.7.3)] we get

$$(2.6) \quad \deg_{\theta} H_n \leq \frac{(n-1)q}{q-1} < \frac{nq}{q-1}.$$

Let  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$  be a tuple and  $\epsilon = (\epsilon_1, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$ . For all  $1 \leq i \leq r$  we fix a fixed  $(q-1)$ -th root  $\gamma_i$  of  $\epsilon_i \in \mathbb{F}_q^\times$  and set  $Q_{s_i; \epsilon_i} := \gamma_i H_{s_i}$ . Then we set  $\mathfrak{Q}_{\mathfrak{s}; \epsilon} := (Q_{s_1; \epsilon_1}, \dots, Q_{s_r; \epsilon_r})$  and put  $\mathfrak{L}(\mathfrak{s}; \epsilon) := \mathfrak{L}(\mathfrak{s}; \mathfrak{Q}_{\mathfrak{s}; \epsilon})$ . By (2.6) we

know that  $\|H_n\|_{\infty} < |\theta|_{\infty}^{\frac{nq}{q-1}}$  for all  $n \in \mathbb{N}$ , thus  $\mathfrak{Q}_{\mathfrak{s}; \epsilon}$  satisfies Condition (2.1). Thus we can define the dual  $t$ -motives  $\mathcal{M}_{\mathfrak{s}; \epsilon} = \mathcal{M}_{\mathfrak{s}; \mathfrak{Q}_{\mathfrak{s}; \epsilon}}$  and  $\mathcal{M}'_{\mathfrak{s}; \epsilon} = \mathcal{M}'_{\mathfrak{s}; \mathfrak{Q}_{\mathfrak{s}; \epsilon}}$  attached to  $\mathfrak{s}$  whose matrices and rigid analytic trivializations will be denoted by  $(\Phi_{\mathfrak{s}; \epsilon}, \Psi_{\mathfrak{s}; \epsilon})$  and  $(\Phi'_{\mathfrak{s}; \epsilon}, \Psi'_{\mathfrak{s}; \epsilon})$ , respectively. These dual  $t$ -motives are connected to MZV's and AMZV's by the following result (see [12, Proposition 2.12] for more details):

$$(2.7) \quad \mathfrak{L}(\mathfrak{s}; \epsilon)(\theta) = \frac{\gamma_1 \dots \gamma_r \Gamma_{s_1} \dots \Gamma_{s_r} \zeta_A \left( \begin{matrix} \epsilon \\ \mathfrak{s} \end{matrix} \right)}{\tilde{\pi}^{s_1 + \dots + s_r}}.$$

By a result of Thakur [26], one can show (see [14, Theorem 2.1]) that  $\zeta_A \left( \begin{matrix} \epsilon \\ \mathfrak{s} \end{matrix} \right) \neq 0$ .

Thus  $\mathfrak{L}(\mathfrak{s}; \epsilon)(\theta) \neq 0$ .

### 2.6.2. Second construction.

Let  $w \in \mathbb{N}$  be a positive integer and  $n \in \mathbb{N}$ . Let  $\{(\mathfrak{s}_i; \epsilon_i)\}_{1 \leq i \leq n}$  be a collection of pairs such that  $\mathfrak{s}_i$  always has weight  $w$  and  $a_i \in A \setminus \{0\}$  for all  $1 \leq i \leq n$ . Recall from §2.6 that the collection  $\{(\mathfrak{s}_i; \mathfrak{Q}_{\mathfrak{s}_i; \epsilon_i})\}_{1 \leq i \leq n}$  satisfies Condition (2.1). We write  $\mathfrak{s}_i = (s_{i1}, \dots, s_{i\ell_i}) \in \mathbb{N}^{\ell_i}$  and  $\epsilon_i = (\epsilon_{i1}, \dots, \epsilon_{i\ell_i}) \in (\mathbb{F}_q^\times)^{\ell_i}$  so that  $s_{i1} + \dots + s_{i\ell_i} = w$ .

We introduce the following sets

$$\begin{aligned} I(\mathfrak{s}_i; \epsilon_i) &:= \{\emptyset, (s_{i1}; \epsilon_{i1}), \dots, (s_{i1}, \dots, s_{i(\ell_i-1)}; \epsilon_{i1}, \dots, \epsilon_{i(\ell_i-1)})\}, \\ I^+(\mathfrak{s}_i; \epsilon_i) &:= I(\mathfrak{s}_i; \epsilon_i) \cup \{(\mathfrak{s}_i; \epsilon_i)\}, \end{aligned}$$

and set

$$I := \cup_i I(\mathfrak{s}_i; \epsilon_i), \quad I^+ := \cup_i I^+(\mathfrak{s}_i; \epsilon_i).$$

For all  $(\mathbf{t}; \boldsymbol{\epsilon}) \in I$ , we denote by  $J(\mathbf{t}; \boldsymbol{\epsilon})$  the set of indices  $i$  such that  $(\mathbf{t}; \boldsymbol{\epsilon}) = (s_{i1}, \dots, s_{ik}; \epsilon_{i1}, \dots, \epsilon_{ik})$  for some  $0 \leq k \leq \ell_i - 1$  and set

$$(2.8) \quad f_{\mathbf{t}; \boldsymbol{\epsilon}} := \sum_i a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i}; \epsilon_{i(k+1)}, \dots, \epsilon_{i\ell_i}),$$

where the sum runs through the set  $J(\mathbf{t}; \boldsymbol{\epsilon})$ . In particular,

$$f_{\emptyset} = \sum_i a_i(t) \mathfrak{L}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i).$$

By §2.4 we get a new matrix  $\Phi'$  by merging the same rows of  $\Phi'_{\mathfrak{s}_1; \boldsymbol{\epsilon}_1}, \dots, \Phi'_{\mathfrak{s}_n; \boldsymbol{\epsilon}_n}$ . Then we obtain  $\Phi \in \text{Mat}_{|I|+1}(\overline{K}[t])$  by

$$(2.9) \quad \Phi = \begin{pmatrix} \Phi' & 0 \\ \mathbf{v} & 1 \end{pmatrix} \in \text{Mat}_{|I|+1}(\overline{K}[t]), \quad \mathbf{v} = (v_{\mathbf{t}; \boldsymbol{\epsilon}})_{(\mathbf{t}; \boldsymbol{\epsilon}) \in I} \in \text{Mat}_{1 \times |I|}(\overline{K}[t]),$$

where

$$v_{\mathbf{t}; \boldsymbol{\epsilon}} = \sum_i a_i(t) Q_{s_{i\ell_i}; \epsilon_{i\ell_i}}^{(-1)} (t - \theta)^{w - w(\mathbf{t})}.$$

Here the sum runs through the set of indices  $i$  such that  $(\mathbf{t}; \boldsymbol{\epsilon}) = (s_{i1}, \dots, s_{ik}; \epsilon_{i1}, \dots, \epsilon_{ik})$  and the empty sum is defined to be zero.

Next we get a rigid analytic trivialization matrix  $\Psi \in \text{GL}_{|I|+1}(\mathbb{T})$  for  $\Phi$  which is of the form

$$\Psi = \begin{pmatrix} \Psi' & 0 \\ \mathbf{f} & 1 \end{pmatrix} \in \text{GL}_{|I|+1}(\mathbb{T})$$

where  $\Psi' = \left( \Psi'_{(\mathbf{t}; \boldsymbol{\epsilon}), (\mathbf{t}'; \boldsymbol{\epsilon}')} \right)_{(\mathbf{t}; \boldsymbol{\epsilon}), (\mathbf{t}'; \boldsymbol{\epsilon}')} \in \text{GL}_{|I|}(\mathbb{T})$  and  $\mathbf{f} = (f_{\mathbf{t}; \boldsymbol{\epsilon}})_{\mathbf{t}; \boldsymbol{\epsilon}} \in \text{Mat}_{1 \times |I|}(\mathbb{T})$  with  $f_{\mathbf{t}; \boldsymbol{\epsilon}}$  given by Eq. (2.8).

### 2.6.3. Linear independence for AMZV's.

Theorem 2.4 applied to this case yields

**Theorem 2.5.** *We keep the above notation. We suppose further that  $\{(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)\}_{1 \leq i \leq n}$  satisfies the following conditions:*

(LW) *For any weight  $w' < w$ , the values  $\mathfrak{L}(\mathbf{t}; \boldsymbol{\epsilon})(\theta)$  with  $(\mathbf{t}; \boldsymbol{\epsilon}) \in I$  and  $w(\mathbf{t}) = w'$  are all  $K$ -linearly independent.*

(LD) *There exist  $a_i \in A \setminus \{0\}$  for  $1 \leq i \leq n$  such that*

$$\sum_{i=1}^n a_i \mathfrak{L}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0.$$

Then

- 1)  $f_{\mathbf{t}; \boldsymbol{\epsilon}}(\theta)$  belongs to  $K$  where  $f_{\mathbf{t}; \boldsymbol{\epsilon}}$  is given as in (2.8) for all  $(\mathbf{t}; \boldsymbol{\epsilon}) \in I$ .
- 2) There exist  $b \in \mathbb{F}_q[t] \setminus \{0\}$  and  $\delta := (\delta_{\mathbf{t}; \boldsymbol{\epsilon}})_{(\mathbf{t}; \boldsymbol{\epsilon}) \in I} \in \text{Mat}_{1 \times |I|}(\overline{K}[t])$  such that

$$(2.10) \quad \begin{pmatrix} \text{Id} & 0 \\ \delta & 1 \end{pmatrix}^{(-1)} \begin{pmatrix} \Phi' & 0 \\ b\mathbf{v} & 1 \end{pmatrix} = \begin{pmatrix} \Phi' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Id} & 0 \\ \delta & 1 \end{pmatrix}.$$

From now on, as we will work with MZV's and AMZV's, we will use freely the notation given as in this section.

## 3. PROOF OF PROPOSITIONS 1.3 AND 1.4

## 3.1. Proof of Proposition 1.3.

We see that  $|\mathcal{AT}_w^1| = |\mathcal{J}_w^1|$ . By induction on  $w$ , we prove that  $|\mathcal{J}_w^1| = s(w)$  for all  $w \in \mathbb{N}$ . The proposition follows.

## 3.2. Proof of Proposition 1.4.

Part 2 follows immediately from the construction of  $\mathcal{AT}'_w$ .

For Part 1, following [17, §1.4], we consider the set  $\mathcal{J}_w^2$  consisting of  $\begin{pmatrix} \epsilon_1 & \cdots & \epsilon_n \\ s_1 & \cdots & s_n \end{pmatrix}$  of weight  $w$  such that for all  $i$ ,  $s_i \in \mathbb{N}$ ,  $q \nmid s_i$  and  $\epsilon_i \in \mathbb{F}_q^\times$  and define a map

$$\varphi: \mathcal{J}_w^2 \longrightarrow \mathcal{J}_w^1$$

as follows: for  $\begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \epsilon_1 & \cdots & \epsilon_n \\ s_1 & \cdots & s_n \end{pmatrix} \in \mathcal{J}_w^2$ , since  $q \nmid s_i$ , we can write  $s_i = h_i q + r_i$

where  $0 < r_i < q$  and  $h_i \in \mathbb{Z}^{\geq 0}$ . The image  $\varphi\left(\begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix}\right)$  is the array

$$\varphi\left(\begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix}\right) = \left( \underbrace{\begin{pmatrix} 1 & \cdots & 1 \\ q & \cdots & q \end{pmatrix}}_{h_1 \text{ times}} \begin{pmatrix} \epsilon_1 \\ r_1 \end{pmatrix} \cdots \underbrace{\begin{pmatrix} 1 & \cdots & 1 \\ q & \cdots & q \end{pmatrix}}_{h_n \text{ times}} \begin{pmatrix} \epsilon_n \\ r_n \end{pmatrix} \right).$$

Then one sees that  $\varphi$  is a bijection, hence  $|\mathcal{J}_w^2| = |\mathcal{J}_w^1| = s(w)$  by the proof of Proposition 1.3.

Thus, to finish the proof, it suffices to construct of a bijection

$$\psi: \mathcal{J}_w^2 \rightarrow \mathcal{J}'_w$$

for  $1 \leq w \leq 3q - 3$ . For  $\begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \epsilon_1 & \cdots & \epsilon_n \\ s_1 & \cdots & s_n \end{pmatrix} \in \mathcal{J}_w^2$ , we know that for all  $i$ ,  $q \nmid s_i$ , which implies  $s_i \neq q, 2q$ . We distinguish two cases:

- If  $s_i \neq q, 2q - 1, 2q$  for all  $i$ , then

$$\psi\left(\begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix}\right) = \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix}.$$

- If there exists a unique integer  $1 \leq i < n$  such that  $s_i = 2q - 1$ , then

$$\psi\left(\begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix}\right) = \begin{pmatrix} \epsilon_1 & \cdots & \epsilon_{i-1} & \epsilon_i & 1 & \epsilon_{i+1} & \cdots & \epsilon_n \\ s_1 & \cdots & s_{i-1} & q-1 & q & s_{i+1} & \cdots & s_n \end{pmatrix}.$$

Then we see that  $\psi$  is a bijection and Part 1 follows.

## 4. PROOF OF THEOREM 1.5

## 4.1. Preliminaries.

We begin this section by proving several auxiliary lemmas which will be useful in the sequel. Here we can refer to the readers [17, §3].

**Lemma 4.1** ([17, Lemma 3.1]). *Let  $\epsilon_1, \dots, \epsilon_n$  be different elements in  $\mathbb{F}_q^\times$ . We denote by  $\gamma_i \in \mathbb{F}_q$  a  $(q-1)$ -th root of  $\epsilon_i$  for all  $1 \leq i \leq n$ . Then  $\gamma_1, \dots, \gamma_n$  are all  $\mathbb{F}_q$ -linearly independent.*

*Proof.* We refer the reader to [17, Lemma 3.1] for a proof of this lemma. We give here another proof using the Moore determinant (see [13, Section 1.3]). Suppose that  $\epsilon_1, \dots, \epsilon_n$  are distinct elements in  $\mathbb{F}_q^\times$ . The Moore determinant associated to  $\gamma_1, \dots, \gamma_n$  is defined as

$$\Delta(\gamma_1, \dots, \gamma_n) := \det \begin{pmatrix} \gamma_1 & \cdots & \gamma_n \\ \gamma_1^q & \cdots & \gamma_n^q \\ \vdots & & \vdots \\ \gamma_1^{q^{n-1}} & \cdots & \gamma_n^{q^{n-1}} \end{pmatrix}.$$

Since  $\gamma_i$  is a  $(q-1)$ -th root of  $\epsilon_i$ , one verifies by induction that  $\gamma_i^{q^k} = \gamma_i \epsilon_i^k$  for  $0 \leq k \leq n-1$ . It follows that

$$\begin{aligned} \Delta(\gamma_1, \dots, \gamma_n) &= \det \begin{pmatrix} \gamma_1 & \cdots & \gamma_n \\ \gamma_1 \epsilon_1 & \cdots & \gamma_n \epsilon_n \\ \vdots & & \vdots \\ \gamma_1 \epsilon_1^{n-1} & \cdots & \gamma_n \epsilon_n^{n-1} \end{pmatrix} = \gamma_1 \cdots \gamma_n \det \begin{pmatrix} 1 & \cdots & 1 \\ \epsilon_1 & \cdots & \epsilon_n \\ \vdots & & \vdots \\ \epsilon_1^{n-1} & \cdots & \epsilon_n^{n-1} \end{pmatrix} \\ &= \gamma_1 \cdots \gamma_n \prod_{1 \leq j < i \leq n} (\epsilon_i - \epsilon_j). \end{aligned}$$

The last equality follows from the Vandermonde determinant. We thus deduce that  $\Delta(\gamma_1, \dots, \gamma_n) \neq 0$  since  $\epsilon_1, \dots, \epsilon_n$  are distinct elements in  $\mathbb{F}_q^\times$ . From [13, Lemma 1.3.3], we conclude that  $\gamma_1, \dots, \gamma_n$  are linearly independent over  $\mathbb{F}_q$ , as desired.  $\square$

Recall that for any tuple  $\epsilon = (\epsilon_1, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$ , we recall that the character of  $\epsilon$  is given by  $\chi(\epsilon) = \epsilon_1 \cdots \epsilon_r \in \mathbb{F}_q^\times$  (see §1.2).

**Lemma 4.2.** *Let  $\zeta_A \begin{pmatrix} \epsilon_i \\ \mathfrak{s}_i \end{pmatrix} \in \mathcal{AT}_w$  and  $a_i \in A \setminus \{0\}$  satisfying*

$$\sum_i a_i \mathfrak{L}(\mathfrak{s}_i; \epsilon_i)(\theta) = 0.$$

*For  $\epsilon \in \mathbb{F}_q^\times$  we denote by  $I(\epsilon) = \{i : \chi(\epsilon_i) = \epsilon\}$  the set of indices  $i$  such that the corresponding character equals  $\epsilon$ . Then for all  $\epsilon \in \mathbb{F}_q^\times$ ,*

$$\sum_{i \in I(\epsilon)} a_i \mathfrak{L}(\mathfrak{s}_i; \epsilon_i)(\theta) = 0.$$

*Proof.* We adapt the proof given as in [17, Lemma 3.2] for the convenience of the reader. Suppose that we have a relation

$$\sum_i \gamma_i a_i = 0$$

with  $a_i \in K_\infty$ . By Lemma 4.1 and the fact that  $K_\infty = \mathbb{F}_q((1/\theta))$ , we deduce that  $a_i = 0$  for all  $i$ .

By (2.7) the relation  $\sum_i a_i \mathfrak{L}(\mathfrak{s}_i; \epsilon_i)(\theta) = 0$  is equivalent to the following one

$$\sum_i a_i \gamma_{i1} \cdots \gamma_{i\ell_i} \zeta_A \begin{pmatrix} \epsilon_i \\ \mathfrak{s}_i \end{pmatrix} = 0.$$

By the previous discussion, since  $a_i \zeta_A \begin{pmatrix} \epsilon_i \\ \mathfrak{s}_i \end{pmatrix} \in K_\infty$  for all  $i$ , we deduce that for all  $\epsilon \in \mathbb{F}_q^\times$ ,

$$\sum_{i \in I(\epsilon)} a_i \gamma_{i1} \dots \gamma_{i\ell_i} \zeta_A \begin{pmatrix} \epsilon_i \\ \mathfrak{s}_i \end{pmatrix} = 0.$$

By (2.7) again we deduce the desired relation

$$\sum_{i \in I(\epsilon)} a_i \mathfrak{L}(\mathfrak{s}_i; \epsilon_i)(\theta) = 0.$$

□

**Lemma 4.3.** *Let  $m \in \mathbb{N}$ ,  $\delta \in \overline{K}[t]$  and  $F(t, \theta) \in \overline{\mathbb{F}}_q[t, \theta]$  (resp.  $F(t, \theta) \in \mathbb{F}_q[t, \theta]$ ) satisfying*

$$\delta = \delta^{(-1)}(t - \theta)^m + F^{(-1)}(t, \theta).$$

*Then  $\delta \in \overline{\mathbb{F}}_q[t, \theta]$  (resp.  $\delta \in \mathbb{F}_q[t, \theta]$ ) and*

$$\deg_\theta \delta \leq \max \left\{ \frac{qm}{q-1}, \frac{\deg_\theta F(t, \theta)}{q} \right\}.$$

*Proof.* The proof follows the same line as that of [18, Theorem 2]. See also [17, Lemma 3.3]. □

#### 4.2. Proof of Theorem 1.5.

The proof of Theorem 1.5 follows the same line as that of [22, Theorem B].

By (2.7) it suffices to show that all the values  $\mathfrak{L}(\mathfrak{s}; \epsilon)(\theta)$  such that  $\zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} \in \mathcal{AT}_w^0$  are  $K$ -linearly independent. The proof is by induction on the weight  $w \in \mathbb{N}$ .

For  $w = 1$  suppose that there exist  $\zeta_A \begin{pmatrix} \epsilon_i \\ 1 \end{pmatrix} \in \mathcal{AT}_1^0$  and  $a_i \in A \setminus \{0\}$  such that

$$\sum_i a_i \mathfrak{L}(1; \epsilon_i)(\theta) = 0.$$

By Lemma 4.2, for all  $i$ , we have  $a_i \mathfrak{L}(1; \epsilon_i)(\theta) = 0$ . As  $\mathfrak{L}(1; \epsilon_i)(\theta) \neq 0$ ,  $a_i = 0$  and we get a contradiction.

Now suppose that for  $w' < w$  the values  $\mathfrak{L}(\mathfrak{s}; \epsilon)(\theta)$  such that  $\zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} \in \mathcal{AT}_{w'}^0$  are all linearly independent over  $K$ . We claim that the values  $\mathfrak{L}(\mathfrak{s}; \epsilon)(\theta)$  such that  $\zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} \in \mathcal{AT}_w^0$  are all linearly independent over  $K$ . Suppose that there exist  $\zeta_A \begin{pmatrix} \epsilon_i \\ \mathfrak{s}_i \end{pmatrix} \in \mathcal{AT}_w^0$  and  $a_i \in A \setminus \{0\}$  such that

$$(4.1) \quad \sum_i a_i \mathfrak{L}(\mathfrak{s}_i; \epsilon_i)(\theta) = 0.$$

We write  $\mathfrak{s}_i = (s_{i1}, \dots, s_{i\ell_i}) \in \mathbb{N}^{\ell_i}$  and  $\epsilon_i = (\epsilon_{i1}, \dots, \epsilon_{i\ell_i}) \in (\mathbb{F}_q^\times)^{\ell_i}$ . By Lemma 4.2 we can suppose further that  $\epsilon_i$  has the same character, i.e., there exists  $\epsilon \in \mathbb{F}_q^\times$  such that for all  $i$ ,

$$(4.2) \quad \chi(\epsilon_i) = \epsilon_{i1} \dots \epsilon_{i\ell_i} = \epsilon.$$

We show that the relation (4.1) leads to a contradiction.

By the previous discussion, we can apply Theorem 2.5 for the above tuples  $(\mathfrak{s}_i; \epsilon_i)$ . With the notation of Theorem 2.5, we deduce that for all  $(\mathfrak{t}; \epsilon) \in I$ ,  $f_{\mathfrak{t}; \epsilon}(\theta)$  belongs to  $K$ . Here we recall that  $J(\mathfrak{t}; \epsilon)$  denotes the set of indices  $i$  such that  $(\mathfrak{t}; \epsilon) = (s_{i1}, \dots, s_{ik}; \epsilon_{i1}, \dots, \epsilon_{ik})$  for some  $0 \leq k \leq \ell_i - 1$ , and  $f_{\mathfrak{t}; \epsilon}$  is given by

$$f_{\mathfrak{t}; \epsilon} := \sum_i a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i}; \epsilon_{i(k+1)}, \dots, \epsilon_{i\ell_i}),$$

where the sum runs through the set  $J(\mathfrak{t}; \epsilon)$ .

We now deduce a contradiction from this fact. For  $(\mathfrak{s}_i; \epsilon_i)$  we denote by  $m_i \in \mathbb{Z}^{\geq 0}$  the biggest index such that  $(s_{i1}, \dots, s_{im_i}; \epsilon_{i1}, \dots, \epsilon_{im_i})$  belongs to  $I(\mathfrak{s}_j; \epsilon_j)$  for some  $j \neq i$ . Note that  $s_{i(m_i+1)} \neq 0$  since the weights of  $s_i$  and  $s_j$  ( $j \neq i$ ) are the same. For  $m_i + 1 \leq k < \ell_i$  we consider  $(\mathfrak{t}; \epsilon) = (s_{i1}, \dots, s_{ik}; \epsilon_{i1}, \dots, \epsilon_{ik})$ . Thus (2.8) gives  $f_{\mathfrak{t}; \epsilon} = a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i}; \epsilon_{i(k+1)}, \dots, \epsilon_{i\ell_i})$  and we have seen that  $f_{\mathfrak{t}; \epsilon}$  belongs to  $K$ . By (2.7) we get

$$a_i \gamma_{i(k+1)} \dots \gamma_{i\ell_i} \Gamma_{i(k+1)} \dots \Gamma_{i\ell_i} \zeta_A \begin{pmatrix} \epsilon_{i(k+1)} & \dots & \epsilon_{i\ell_i} \\ s_{i(k+1)} & \dots & s_{i\ell_i} \end{pmatrix} / \tilde{\pi}^{s_{i(k+1)} + \dots + s_{i\ell_i}} \in K.$$

Since  $a_i \in K^\times$ , this forces that  $s_{i(k+1)} + \dots + s_{i\ell_i}$  is divisible by  $q - 1$  and  $\epsilon_{i(k+1)} \dots \epsilon_{i\ell_i} = 1$  by Lemma 4.2. Since this condition holds for all  $m_i + 1 \leq k < \ell_i$ , it follows that  $s_{i(k+1)}$  is divisible by  $q - 1$  and  $\epsilon_{i(k+1)} = 1$  for all  $m_i + 1 \leq k < \ell_i$ . Since  $1 \leq s_{i(k+1)} \leq q - 1$ , we conclude that

$$s_{i(k+1)} = q - 1, \quad \epsilon_{i(k+1)} = 1, \quad \text{for all } m_i + 1 \leq k < \ell_i.$$

Let  $m := \max_i m_i$ . Then we can find two different tuples, says  $(\mathfrak{s}_1; \epsilon_1)$  and  $(\mathfrak{s}_2; \epsilon_2)$  be such that  $m_1 = m_2 = m$ . Thus

$$(s_{11}, \dots, s_{1m}; \epsilon_{11}, \dots, \epsilon_{1m}) = (s_{21}, \dots, s_{2m}; \epsilon_{21}, \dots, \epsilon_{2m})$$

and  $(s_{1(m+1)}, \epsilon_{1(m+1)}) \neq (s_{2(m+1)}, \epsilon_{2(m+1)})$ . The previous discussion shows that  $s_{1(m+2)} = \dots = q - 1$ ,  $\epsilon_{1(m+2)} = \dots = 1$  and  $s_{2(m+2)} = \dots = q - 1$ ,  $\epsilon_{2(m+2)} = \dots = 1$ . Combining with the fact that  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  have the same weight yields  $s_{1(m+1)} \equiv s_{2(m+1)} \pmod{q - 1}$ . Since  $1 \leq s_{1(m+1)}, s_{2(m+1)} \leq q - 1$ , we deduce  $s_{1(m+1)} = s_{2(m+1)}$ . Finally, combining with the fact that  $\epsilon_1$  and  $\epsilon_2$  have the same character (see (4.2)), we obtain  $\epsilon_{1(m+1)} = \epsilon_{2(m+1)}$ . Thus we get a contradiction since  $(s_{1(m+1)}; \epsilon_{1(m+1)}) = (s_{2(m+1)}; \epsilon_{2(m+1)})$ .

The proof of Theorem 1.5 is finished.

## 5. PROOF OF THEOREM 1.6

### 5.1. Strategy of the proof.

Recall that  $w \leq 3q - 2$ . We use Proposition 1.4 and prove Theorem 1.6 by induction on  $w$ .

For  $w \leq q$ , Theorem 1.6 follows immediately from Theorem 1.5 (see §5.2).

We suppose that for  $w' < w$  AMZV's in  $\mathcal{AT}'_{w'}$  are all linearly independent over  $K$ . We claim that AMZV's in  $\mathcal{AT}'_w$  are all linearly independent over  $K$ .

By (2.7) and Proposition 2.3, it suffices to show that all the values  $\mathfrak{L}(\mathfrak{s}; \epsilon)(\theta)$  such that  $\zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} \in \mathcal{AT}'_w$  are all linearly independent over  $K$ . Suppose that there



exist  $\zeta_A \begin{pmatrix} \epsilon_i \\ \mathfrak{s}_i \end{pmatrix} \in \mathcal{AT}'_w$  and  $a_i \in A \setminus \{0\}$  such that

$$\sum_i a_i \mathfrak{L}(\mathfrak{s}_i; \epsilon_i)(\theta) = 0.$$

We write  $\mathfrak{s}_i = (s_{i1}, \dots, s_{i\ell_i}) \in \mathbb{N}^{\ell_i}$  and  $\epsilon_i = (\epsilon_{i1}, \dots, \epsilon_{i\ell_i}) \in (\mathbb{F}_q^\times)^{\ell_i}$ . By Lemma 4.2 we can suppose further that  $\epsilon_i$  has the same character, i.e., there exists  $\epsilon \in \mathbb{F}_q^\times$  such that for all  $i$ ,

$$\chi(\epsilon_i) = \epsilon_{i1} \dots \epsilon_{i\ell_i} = \epsilon.$$

The proof of Theorem 1.6 will be divided into two steps.

- **Step 1:** as Proposition 1.4 implies that Condition (LW) holds, we apply Theorem 2.5 for the tuples  $(\mathfrak{s}_i; \epsilon_i)$ . Part 1 of Theorem 2.5 gives a shortlist of tuples  $(\mathfrak{s}_i; \epsilon_i)$  which often contains few elements.
- **Step 2:** we write down explicitly the system of equations (2.10) given in Part 2 of Theorem 2.5 (we note that in this system we can suppose that  $b = 1$ ) and manage to show that either  $a_i = 0$  for all  $i$  or it does not have a nontrivial solution  $\delta \neq 0$ . This completes the proof.

In what follows we carry out this strategy in detail.

### 5.2. Proof of Theorem 1.6 for $w \leq q$ .

We suppose that  $w \leq q$ . As a direct consequence of Theorem 1.5, we deduce that Theorem 1.6 holds when  $w \leq q$ . In fact, we note that  $\mathcal{AT}'_w = \mathcal{AT}^1_w = \mathcal{AT}^0_w$  when  $w \leq q$ .

### 5.3. Proof of Theorem 1.6 for $q + 1 \leq w \leq 2q - 2$ .

We suppose that  $q > 2$  and  $q + 1 \leq w \leq 2q - 2$ . We want to prove that AMZV's in  $\mathcal{AT}'_w$  are all linearly independent over  $K$ .

The proof is done by induction on  $w$ . For  $q + 1 \leq w \leq 2q - 2$ , we suppose that for  $w' < w$  AMZV's in  $\mathcal{AT}'_{w'}$  are all linearly independent over  $K$ . We claim that AMZV's in  $\mathcal{AT}'_w$  are all linearly independent over  $K$ .

By (2.7) and Proposition 2.3, it suffices to show that all the values  $\mathfrak{L}(\mathfrak{s}; \epsilon)(\theta)$  such that  $\zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} \in \mathcal{AT}'_w$  are all linearly independent over  $K$ . Suppose that there exist  $\zeta_A \begin{pmatrix} \epsilon_i \\ \mathfrak{s}_i \end{pmatrix} \in \mathcal{AT}'_w$  and  $a_i \in A \setminus \{0\}$  such that

$$\sum_i a_i \mathfrak{L}(\mathfrak{s}_i; \epsilon_i)(\theta) = 0.$$

We write  $\mathfrak{s}_i = (s_{i1}, \dots, s_{i\ell_i}) \in \mathbb{N}^{\ell_i}$  and  $\epsilon_i = (\epsilon_{i1}, \dots, \epsilon_{i\ell_i}) \in (\mathbb{F}_q^\times)^{\ell_i}$ . By Lemma 4.2, we can suppose further that  $\epsilon_i$  has the same character, i.e., there exists  $\epsilon \in \mathbb{F}_q^\times$  such that for all  $i$ ,

$$\chi(\epsilon_i) = \epsilon_{i1} \dots \epsilon_{i\ell_i} = \epsilon.$$

We show that this relation leads to a contradiction.

### 5.3.1. Step 1.

We apply Theorem 2.5 for the above tuples  $(\mathfrak{s}_i; \epsilon_i)$ . With the notation of Theorem 2.5, we deduce that for all  $(\mathfrak{t}; \epsilon) \in I$ ,  $f_{\mathfrak{t}; \epsilon}(\theta)$  belongs to  $K$ . Here we recall that  $J(\mathfrak{t}; \epsilon)$  denotes the set of indices  $i$  such that  $(\mathfrak{t}; \epsilon) = (s_{i1}, \dots, s_{ik}; \epsilon_{i1}, \dots, \epsilon_{ik})$  for some  $0 \leq k \leq \ell_i - 1$ , and  $f_{\mathfrak{t}; \epsilon}$  is given by

$$f_{\mathfrak{t}; \epsilon} := \sum_i a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i}; \epsilon_{i(k+1)}, \dots, \epsilon_{i\ell_i}),$$

where the sum runs through the set  $J(\mathfrak{t}; \epsilon)$ .

We claim that for all  $(\mathfrak{t}; \epsilon) \in I \setminus \{\emptyset\}$ ,  $(q-1) \mid (w - w(\mathfrak{t}))$ . In fact, suppose that  $q-1$  does not divide  $w - w(\mathfrak{t})$ . Then  $\tilde{\pi}^{w-w(\mathfrak{t})} \notin K_\infty$ , so

$$\sum_{i \in J(\mathfrak{t}; \epsilon)} a_i(t) \gamma_{i(k+1)} \cdots \gamma_{i\ell_i} \Gamma_{s_{i(k+1)}} \cdots \Gamma_{s_{i\ell_i}} \zeta_A \begin{pmatrix} \epsilon_{i(k+1)} & \cdots & \epsilon_{i\ell_i} \\ s_{i(k+1)} & \cdots & s_{i\ell_i} \end{pmatrix} = 0,$$

which contradicts to the induction hypothesis on  $\mathcal{A}\mathcal{T}'_{w-w(\mathfrak{t})}$ .

Thus, we have  $(q-1) \mid s_{ij}$  for all  $2 \leq j \leq \ell_i$ . With our assumption  $w \leq 2q-2$ , we get the shortlist of  $(\mathfrak{s}_i; \epsilon_i)$  as follows:

$$(w; \epsilon), \quad (k, q-1; \epsilon_1, \epsilon_1^{-1})$$

where  $\epsilon_1 \in \mathbb{F}_q^\times$  and  $k := w - (q-1)$ . For  $(\mathfrak{t}; \epsilon) = (k; \epsilon_1)$ , it follows that  $f_{k; \epsilon_1}(\theta) = a_{(k, q-1; \epsilon_1, \epsilon_1^{-1})} \mathfrak{L}(q-1; \epsilon_1^{-1})(\theta) \in K$ . By (2.7) and Proposition 2.3, if  $\epsilon_1^{-1} \neq 1$ , then  $a_{(k, q-1; \epsilon_1, \epsilon_1^{-1})} = 0$ . Hence the first shortlist of  $(\mathfrak{s}_i; \epsilon_i)$  is reduced to the following list

$$(w; \epsilon), \quad (k, q-1; \epsilon, 1)$$

Thus we have the following relation,

$$(5.1) \quad a_1 \zeta_A \begin{pmatrix} \epsilon \\ w \end{pmatrix} + a_2 \zeta_A \begin{pmatrix} \epsilon & 1 \\ k & q-1 \end{pmatrix} = 0 \text{ for some } a_1, a_2 \in A \setminus \{0\}.$$

### 5.3.2. Step 2.

We claim that (5.1) leads to a contradiction. In fact, we put  $s_1 = k$  and  $s_2 = q-1$ . Then  $s_1 \leq q-1$ . Also fix  $\gamma$  a  $(q-1)$ -th root of  $\epsilon$ . By Theorem 2.5, we recall that the matrix  $\Phi'$  is given by

$$\Phi' = \begin{pmatrix} (t-\theta)^w & 0 \\ (\gamma H_{s_1})^{(-1)}(t-\theta)^w & (t-\theta)^{s_2} \end{pmatrix}$$

for  $I = \{\emptyset, (k; \epsilon)\}$ . For some  $\delta_1, \delta_2 \in \overline{K}[t]$ , Eq. (2.10) induces

$$(5.2) \quad \delta_1 = \left( \delta_1^{(-1)} + \delta_2^{(-1)} (\gamma H_{s_1})^{(-1)} + a_1(t) \cdot (\gamma H_w)^{(-1)} \right) (t-\theta)^w,$$

$$(5.3) \quad \delta_2 = \left( \delta_2^{(-1)} + a_2(t) \cdot H_{s_2}^{(-1)} \right) (t-\theta)^{s_2}.$$

We know that  $H_{s_1} = H_{s_2} = 1$  and  $H_w = (t^q - t) + (k-1)(t^q - \theta^q)$ . So (5.3) is reduced to

$$\delta_2 = \left( \delta_2^{(-1)} + a_2(t) \right) (t-\theta)^{s_2}.$$

By Lemma 4.3 we deduce that  $\delta_2 \in \mathbb{F}_q[t, \theta]$  and  $\deg_\theta \delta_2 \leq \frac{q(q-1)}{q-1} = q$ . We put  $\delta_2 = (a\theta + b)(t-\theta)^{q-1}$  with  $a, b \in \mathbb{F}_q[t]$ . This yields

$$\delta_2 = f(\theta^q - t^q), \quad a_2(t) = f(t^q - t) \text{ for some } f \in \mathbb{F}_q[t] \setminus \{0\}$$

after calculations (twisting once and comparing the coefficients of  $\theta^{q-1}$ -terms).

Now for  $\delta_1$ , by Lemma 4.3 again,  $\delta_1 \in \overline{\mathbb{F}}_q[t, \theta]$  and  $\deg_\theta \delta_1 \leq \frac{qw}{q-1}$ . We treat the cases when  $1 < k < q-1$  and  $k = q-1$  separately.

If  $1 < k < q-1$ , we have  $\deg_\theta \delta_1 \leq w+1$ , i.e.,  $\delta_1 = (a\theta + b)(t - \theta)^w$  for some  $a, b \in \overline{\mathbb{F}}_q[t]$ . After twisting, comparing the coefficients of  $\theta^{k-1}$ - and  $\theta^k$ - terms yields  $a = b = 0$ , and then comparing the coefficients of  $\theta^q$ -terms gives  $a_1(t)(t^q - t) = 0$ . Since  $a_1 \neq 0$ , we have a contradiction.

Next consider when  $k = q-1$ , i.e., when  $w = 2q-2$ . Since  $\deg_\theta \delta_1 \leq w+2$ , we let

$$\delta_1 = (a\theta^2 + b\theta + c)(t - \theta)^w \quad \text{for some } a, b, c \in \overline{\mathbb{F}}_q[t].$$

Then twisting once and comparing the coefficients of  $\theta^{2q-1}$ - and  $\theta^{2q-2}$  terms give  $a\theta^2 + b\theta + c = a(t - \theta)^2$ , i.e.,

$$a^{(1)}(t - \theta^q)^2 = a(t - \theta)^{2q} + \gamma f(\theta^q - t^q) + a_1(t)\gamma((t^q - t) - 2(t^q - \theta^q)).$$

Comparing the coefficients of  $\theta^{2q}$  yields  $a^{(1)} = a$ , which implies  $a \in \mathbb{F}_q[t]$ . Hence

$$a(t - \theta^q)^2 = a(t - \theta)^{2q} + \gamma f(\theta^q - t^q) + a_1(t)\gamma((t^q - t) - 2(t^q - \theta^q)).$$

Multiplying by  $(t^q - t)$ , we get

$$\gamma f(\theta^q - t^q)(t^q - t) = ((t - \theta^q)^2 - (t - \theta)^{2q})(a(t^q - t) - a_1(t)\gamma)$$

by using the identity  $(t^q - t)[(t^q - t) - 2(t^q - \theta^q)] = (t - \theta^q)^2 - (t^q - \theta^q)^2$ . Since the right-hand side is not divided by  $t - \theta$  unless it is zero, we have  $f = 0$ , which is a contradiction.

As a direct consequence of the previous proof, we obtain

**Corollary 5.1.** *Suppose that  $q > 2$ . Then  $\zeta_A \begin{pmatrix} 1 & 1 \\ q-1 & q-1 \end{pmatrix} / \zeta_A \begin{pmatrix} 1 \\ 2q-2 \end{pmatrix} \notin K$ .*

#### 5.4. Proof of Theorem 1.6 for $2q-1 \leq w \leq 3q-2$ : auxiliary lemmas.

In this section we consider the case when  $2q-1 \leq w \leq 3q-2$ . We recall that  $\mathcal{J}'_w$  denotes the set of tuples  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$  of weight  $w$  as follows:

- For  $2q-1 \leq w \leq 3q-3$ ,  $\mathcal{J}'_w$  consists of tuples  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$  of weight  $w$  of the form
  - either  $s_i \neq q, 2q-1, 2q$  for all  $i$ ,
  - or there exists a unique integer  $1 \leq i < r$  such that  $(s_i, s_{i+1}) = (q-1, q)$ .
- For  $w = 3q-2$  and  $q > 2$ ,  $\mathcal{J}'_w$  consists of tuples  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$  of weight  $w$  of the form
  - either  $s_i \neq q, 2q-1, 2q, 3q-2$  for all  $i$ ,
  - or there exists a unique integer  $1 \leq i < r$  such that  $(s_i, s_{i+1}) \in \{(q-1, q), (2q-2, q)\}$ , but  $\mathfrak{s} \neq (q-1, q-1, q)$ ,
  - or  $\mathfrak{s} = (q-1, 2q-1)$ .
- For  $q = 2$  and  $w = 3q-2 = 4$ ,  $\mathcal{J}'_w$  consists of the following tuples:  $(2, 1, 1)$ ,  $(1, 2, 1)$  and  $(1, 3)$ .

Then we recall that  $\mathcal{J}'_w$  denotes the set given by

$$\mathcal{J}'_w := \left\{ \binom{\epsilon}{\mathfrak{s}} : \mathfrak{s} \in \mathcal{J}'_w, \text{ and } \epsilon_i = 1 \text{ whenever } s_i \in \{q, 2q-1\} \right\}.$$

and by  $\mathcal{AT}'_w$  the subset of AMZV's given by

$$\mathcal{AT}'_w := \left\{ \zeta_A \begin{pmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{s} \end{pmatrix} : \begin{pmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{s} \end{pmatrix} \in \mathcal{J}'_w \right\}.$$

We want to prove that AMZV's in  $\mathcal{AT}'_w$  are all linearly independent over  $K$ . Suppose that there exist  $\zeta_A \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \boldsymbol{s}_i \end{pmatrix} \in \mathcal{AT}'_w$  and  $a_i \in A \setminus \{0\}$  such that

$$(5.4) \quad \sum_i a_i \mathfrak{L}(\boldsymbol{s}_i; \boldsymbol{\epsilon}_i) = 0.$$

By Lemma 4.2, we can suppose further that  $\boldsymbol{\epsilon}_i$  has the same character, i.e., there exists  $\epsilon \in \mathbb{F}_q^\times$  such that for all  $i$ ,

$$\chi(\boldsymbol{\epsilon}_i) = \epsilon.$$

We will proceed by induction to lead a contradiction. We write  $\boldsymbol{s}_i = (s_{i1}, \dots, s_{i\ell_i})$  and  $\boldsymbol{\epsilon}_i = (\epsilon_{i1}, \dots, \epsilon_{i\ell_i})$ , and recall

$$\begin{aligned} I(\boldsymbol{s}_i; \boldsymbol{\epsilon}_i) &= \{\emptyset, (s_{i1}; \epsilon_{i1}), \dots, (s_{i1}, \dots, s_{i(\ell_i-1)}; \epsilon_{i1}, \dots, \epsilon_{i(\ell_i-1)})\}, \\ I^+(\boldsymbol{s}_i; \boldsymbol{\epsilon}_i) &= I(\boldsymbol{s}_i; \boldsymbol{\epsilon}_i) \cup \{(\boldsymbol{s}_i; \boldsymbol{\epsilon}_i)\}, \end{aligned}$$

and

$$\begin{aligned} I &= \cup_i I(\boldsymbol{s}_i; \boldsymbol{\epsilon}_i), \\ I^+ &= \cup_i I^+(\boldsymbol{s}_i; \boldsymbol{\epsilon}_i). \end{aligned}$$

For all  $(\mathbf{t}; \boldsymbol{\epsilon}) \in I$ , we recall

$$f_{\mathbf{t}; \boldsymbol{\epsilon}} = \sum_i a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i}; \epsilon_{i(k+1)}, \dots, \epsilon_{i\ell_i})$$

where the sum runs over the set  $J(\mathbf{t}; \boldsymbol{\epsilon})$  consisting of indices  $i$  such that  $(\mathbf{t}; \boldsymbol{\epsilon}) = (s_{i1}, \dots, s_{ik}; \epsilon_{i1}, \dots, \epsilon_{ik})$  for some  $0 \leq k \leq \ell_i - 1$ ,

By Theorem 2.5 and referring to (2.7) and (2.8),

$$(5.5) \quad f_{\mathbf{t}; \boldsymbol{\epsilon}}(\theta) = \sum_{i \in J(\mathbf{t}; \boldsymbol{\epsilon})} a_i \gamma_{i(k+1)} \dots \gamma_{i\ell_i} \Gamma_{s_{i(k+1)}} \dots \Gamma_{s_{i\ell_i}} \zeta_A \begin{pmatrix} \epsilon_{i(k+1)} & \dots & \epsilon_{i\ell_i} \\ s_{i(k+1)} & \dots & s_{i\ell_i} \end{pmatrix} / \tilde{\pi}^{w-w(\mathbf{t})} \in K.$$

We now prove several lemmas that will be useful in the sequel.

**Lemma 5.2.** *Let  $(\mathbf{t}; \boldsymbol{\epsilon}) \in I$  and  $\mathbf{t} \neq \emptyset$ . Recall that  $J(\mathbf{t}; \boldsymbol{\epsilon})$  denotes the set of indices  $i$  such that  $(\mathbf{t}; \boldsymbol{\epsilon}) = (s_{i1}, \dots, s_{ik}; \epsilon_{i1}, \dots, \epsilon_{ik})$  for some  $0 \leq k \leq \ell_i - 1$ . Then*

- either  $(\mathbf{t}, q; \boldsymbol{\epsilon}, 1)$  or  $(\mathbf{t}, 2q - 1; \boldsymbol{\epsilon}, 1)$  belongs to  $I^+$ ,
- or  $w - w(\mathbf{t})$  is divisible by  $q - 1$  and there exists  $a \in K$  such that

$$\begin{aligned} & \sum_{i \in J(\mathbf{t}; \boldsymbol{\epsilon})} a_i \gamma_{i(k+1)} \dots \gamma_{i\ell_i} \Gamma_{s_{i(k+1)}} \dots \Gamma_{s_{i\ell_i}} \zeta_A \begin{pmatrix} \epsilon_{i(k+1)} & \dots & \epsilon_{i\ell_i} \\ s_{i(k+1)} & \dots & s_{i\ell_i} \end{pmatrix} \\ & + a \Gamma_{w-w(\mathbf{t})} \zeta_A \begin{pmatrix} 1 \\ w - w(\mathbf{t}) \end{pmatrix} = 0. \end{aligned}$$

*Proof.* Suppose that the tuples  $(\mathbf{t}, q; \boldsymbol{\epsilon}, 1)$  and  $(\mathbf{t}, 2q - 1; \boldsymbol{\epsilon}, \epsilon')$  for all  $\epsilon' \in \mathbb{F}_q^\times$  do not belong to  $I^+$  and  $w - w(\mathbf{t})$  is not divisible by  $q - 1$ . Then  $\tilde{\pi}^{w-w(\mathbf{t})} \notin K_\infty$  as  $w - w(\mathbf{t})$

is not divisible by  $q - 1$ . Referring to (2.7), since  $f_{\mathbf{t};\epsilon}(\theta) \in K$  by (5.5), this implies that

$$(5.6) \quad \sum_{i \in J(\mathbf{t}; \epsilon)} a_i \gamma_{i(k+1)} \cdots \gamma_{i\ell_i} \Gamma_{s_{i(k+1)}} \cdots \Gamma_{s_{i\ell_i}} \zeta_A \begin{pmatrix} \epsilon_{i(k+1)} & \cdots & \epsilon_{i\ell_i} \\ s_{i(k+1)} & \cdots & s_{i\ell_i} \end{pmatrix} = 0.$$

Since  $(\mathbf{t}, q; \epsilon, 1)$  and  $(\mathbf{t}, 2q - 1; \epsilon, \epsilon')$  for all  $\epsilon' \in \mathbb{F}_q^\times$  do not belong to  $I^+$ , the AMZV's appearing in (5.6) belong to  $\mathcal{AJ}'_{w-w(\mathbf{t})}$ . Then, recalling that  $a_i \neq 0$  for  $i \in J(\mathbf{t}; \epsilon)$ , we obtain a contradiction by the induction hypothesis. Hence, we have shown that if  $(\mathbf{t}, q; \epsilon, 1)$  and  $(\mathbf{t}, 2q - 1; \epsilon, \epsilon')$  for all  $\epsilon' \in \mathbb{F}_q^\times$  do not belong to  $I^+$ , then  $w - w(\mathbf{t})$  is divisible by  $q - 1$ .

If  $q - 1$  divides  $w - w(\mathbf{t})$ , then since  $f_{\mathbf{t};\epsilon}(\theta) \in K$  and  $\zeta_A \begin{pmatrix} 1 \\ w - w(\mathbf{t}) \end{pmatrix} / \tilde{\pi}^{w-w(\mathbf{t})} \in K^\times$ , we conclude that there exists some  $a \in K$  such that

$$\begin{aligned} & \sum_{i \in J(\mathbf{t}; \epsilon)} a_i \gamma_{i(k+1)} \cdots \gamma_{i\ell_i} \Gamma_{s_{i(k+1)}} \cdots \Gamma_{s_{i\ell_i}} \zeta_A \begin{pmatrix} \epsilon_{i(k+1)} & \cdots & \epsilon_{i\ell_i} \\ s_{i(k+1)} & \cdots & s_{i\ell_i} \end{pmatrix} \\ & + a \Gamma_{w-w(\mathbf{t})} \zeta_A \begin{pmatrix} 1 \\ w - w(\mathbf{t}) \end{pmatrix} = 0. \end{aligned}$$

□

As a direct consequence of the proof of the previous lemma, we obtain

**Lemma 5.3.** *Let  $(\mathbf{t}; \epsilon) \in I$  and  $\mathbf{t} \neq \emptyset$  satisfying both conditions:*

- *the tuples  $(\mathbf{t}, q; \epsilon, 1)$  and  $(\mathbf{t}, 2q - 1; \epsilon, 1)$  belongs to  $I^+$ ,*
- *$w - w(\mathbf{t})$  is divisible by  $q - 1$ .*

*Then there exists  $a \in K^\times$  such that*

$$\begin{aligned} & \sum_{i \in J(\mathbf{t}; \epsilon)} a_i \gamma_{i(k+1)} \cdots \gamma_{i\ell_i} \Gamma_{s_{i(k+1)}} \cdots \Gamma_{s_{i\ell_i}} \zeta_A \begin{pmatrix} \epsilon_{i(k+1)} & \cdots & \epsilon_{i\ell_i} \\ s_{i(k+1)} & \cdots & s_{i\ell_i} \end{pmatrix} \\ & + a \Gamma_{w-w(\mathbf{t})} \zeta_A \begin{pmatrix} 1 \\ w - w(\mathbf{t}) \end{pmatrix} = 0. \end{aligned}$$

*Further, for all  $i \in J(\mathbf{t}; \epsilon)$ , we have  $\epsilon_{i(k+1)} \cdots \epsilon_{i\ell_i} = 1$ .*

*Proof.* As  $q - 1$  divides  $w - w(\mathbf{t})$ , then since  $f_{\mathbf{t};\epsilon}(\theta) \in K$  and  $\zeta_A \begin{pmatrix} 1 \\ w - w(\mathbf{t}) \end{pmatrix} / \tilde{\pi}^{w-w(\mathbf{t})} \in K^\times$ , we conclude that there exists some  $a \in K$  such that

$$\begin{aligned} & \sum_{i \in J(\mathbf{t}; \epsilon)} a_i \gamma_{i(k+1)} \cdots \gamma_{i\ell_i} \Gamma_{s_{i(k+1)}} \cdots \Gamma_{s_{i\ell_i}} \zeta_A \begin{pmatrix} \epsilon_{i(k+1)} & \cdots & \epsilon_{i\ell_i} \\ s_{i(k+1)} & \cdots & s_{i\ell_i} \end{pmatrix} \\ & + a \Gamma_{w-w(\mathbf{t})} \zeta_A \begin{pmatrix} 1 \\ w - w(\mathbf{t}) \end{pmatrix} = 0. \end{aligned}$$

Since  $(\mathbf{t}, q; \epsilon, 1)$  and  $(\mathbf{t}, 2q - 1; \epsilon, \epsilon')$  for all  $\epsilon' \in \mathbb{F}_q^\times$  do not belong to  $I^+$ , the AMZV's appearing in the first sum belong to  $\mathcal{AJ}'_{w-w(\mathbf{t})}$ . By the induction hypothesis, we deduce that  $a \neq 0$ . Further, by Lemma 4.2, it follows that for all  $i \in J(\mathbf{t}; \epsilon)$ , we have  $\epsilon_{i(k+1)} \cdots \epsilon_{i\ell_i} = 1$ . □

**Remark 5.4.** 1) Note that if  $w = 2q - 1$ , then  $(\mathbf{t}, 2q - 1; \epsilon, 1) \notin I^+$ , and the condition  $(\mathbf{t}, q; \epsilon, 1) \in I^+$  forces  $s_{ik} = q - 1$  and so  $(\mathbf{t}; \epsilon) = (q - 1; \epsilon_{i1})$  and  $(\mathbf{s}_i; \epsilon_i) = (q - 1, q; \epsilon_{i1}, 1)$ .

2) If  $2q \leq w \leq 3q - 3$ , then  $(\mathbf{t}, 2q - 1; \epsilon, 1) \notin I^+$  since  $w \leq 3q - 3$ , and the condition  $(\mathbf{t}, q; \epsilon, 1) \in I^+$  forces  $s_{ik} = q - 1$  and  $\sum_{j=1}^{k-1} s_{jk} \leq w - (2q - 1) \leq q - 2$ .

3) If  $w = 3q - 2$  and  $q > 2$ , then the condition  $(\mathbf{t}, 2q - 1; \epsilon, 1) \in I^+$  forces  $s_{ik} = q - 1$ , so  $(\mathbf{t}; \epsilon) = (q - 1; \epsilon_{i1})$  and  $(\mathbf{s}_i; \epsilon_i) = (q - 1, 2q - 1; \epsilon_{i1}, 1)$ . The condition  $(\mathbf{t}, q; \epsilon, 1) \in I^+$  forces  $s_{ik} = q - 1$  or  $2q - 2$ .

**Lemma 5.5.** *Let  $(\mathbf{t}; \epsilon) \in I$ . Suppose that  $\mathbf{t} = (s_{i1}, \dots, s_{ik}) \neq \emptyset$  for some  $i$  with  $k < \ell_i$  and  $w - w(\mathbf{t})$  is divisible by  $q - 1$ . Then we have  $\epsilon_{i(k+1)} = \dots = \epsilon_{i\ell_i} = 1$ .*

*Proof.* We claim that for all  $k < j \leq \ell_i$ ,  $s_{ij}$  is divisible by  $q - 1$ . In fact, suppose that there exists  $k < j \leq \ell_i$  such that  $s_{i(k+1)}, \dots, s_{i(j-1)}$  are divisible by  $q - 1$ , but  $s_{ij}$  is not divisible by  $q - 1$ . Since  $w - w(\mathbf{t})$  is divisible by  $q - 1$ , it follows that  $(\mathbf{t}'; \epsilon') = (s_{i1}, \dots, s_{ij}; \epsilon_{i1}, \dots, \epsilon_{ij})$ . Then  $w - w(\mathbf{t}')$  is not divisible by  $q - 1$ . It follows that  $(\mathbf{t}'; \epsilon')$  belongs to  $I$ . Combining the definition of  $\mathcal{A}\mathcal{T}'_w$  and the fact that  $s_{ij}$  is not divisible by  $q - 1$  implies that the tuples  $(\mathbf{t}', q; \epsilon', 1)$  and  $(\mathbf{t}', 2q - 1; \epsilon', \epsilon')$  for all  $\epsilon' \in \mathbb{F}_q^\times$  do not belong to  $I^+$ . Thus Lemma 5.2 applied to  $(\mathbf{t}'; \epsilon') \in I$  yields a contradiction.

Now, for all  $j$  such that  $k < j < \ell_i$ , we consider  $(\mathbf{t}'; \epsilon') = (s_{i1}, \dots, s_{ij}; \epsilon_{i1}, \dots, \epsilon_{ij})$  which verifies the hypothesis of Lemma 5.3. By Lemma 5.3, we get  $\epsilon_{i(j+1)} \dots \epsilon_{i\ell_i} = 1$ . It follows that  $\epsilon_{i(k+1)} = \dots = \epsilon_{i\ell_i} = 1$ .  $\square$

### 5.5. Proof of Theorem 1.6 for $q > 2$ and $w = 2q - 1$ .

In this section we keep the notation of §5.4 and finish the proof of Theorem 1.6 for  $q > 2$  and  $w = 2q - 1$  by following the strategy given in §5.1. Further, by Lemma 4.2, we can suppose further that for all tuples  $(\mathbf{s}_i; \epsilon_i)$ ,  $\epsilon_i$  has the same character, i.e., there exists  $\epsilon \in \mathbb{F}_q^\times$  such that for all  $i$ ,

$$\chi(\epsilon_i) = \epsilon.$$

#### 5.5.1. Step 1.

By Lemma 5.2 and Lemma 5.5, we have the first shortlist as follows:

$$\begin{aligned} &(1, 2q - 2; \epsilon, 1), \quad (1, q - 1, q - 1; \epsilon, 1, 1), \\ &(q - 1, q; \epsilon, 1), \quad (q - 1, 1, q - 1; \epsilon_1, \epsilon_2, 1), \end{aligned}$$

with  $\epsilon_1, \epsilon_2 \in \mathbb{F}_q^\times$  such that  $\epsilon_1 \epsilon_2 = \epsilon$ .

Applying Lemma 5.2 with  $(\mathbf{t}; \epsilon) = (1; \epsilon)$ , there exists some  $a \in K$  such that

$$\begin{aligned} &a_{(1, 2q-2; \epsilon, 1)} \Gamma_{2q-2} \zeta_A \begin{pmatrix} 1 \\ 2q-2 \end{pmatrix} + a_{(1, q-1, q-1; \epsilon, 1, 1)} \Gamma_{q-1} \Gamma_{q-1} \zeta_A \begin{pmatrix} 1 & 1 \\ q-1 & q-1 \end{pmatrix} \\ &+ a \Gamma_{2q-2} \zeta_A \begin{pmatrix} 1 \\ 2q-2 \end{pmatrix} = 0. \end{aligned}$$

By Corollary 5.1, we deduce that  $a_{(1, q-1, q-1; \epsilon, 1, 1)} = 0$ . So we reduce to the second shortlist as follows:

$$\begin{aligned} &(1, 2q - 2; \epsilon, 1), \quad (q - 1, q; \epsilon, 1), \\ &(q - 1, 1, q - 1; \epsilon_1, \epsilon_2, 1) \end{aligned}$$

with  $\epsilon_1, \epsilon_2 \in \mathbb{F}_q^\times$  such that  $\epsilon_1 \epsilon_2 = \epsilon$ .

5.5.2. *Step 2.*

We proceed as done in the discussion preceding Theorem 2.5 to examine the linear independence. We construct  $\Phi'$  with respect to the following tuples in  $I$ :

$$\emptyset, \quad (1; \epsilon), \quad (q-1; \epsilon_1), \quad (q-1, 1; \epsilon_1, \epsilon_2),$$

with  $\epsilon_1, \epsilon_2 \in \mathbb{F}_q^\times$  such that  $\epsilon_1 \epsilon_2 = \epsilon$ . Also  $\gamma, \gamma_1, \gamma_2$  are  $(q-1)$ -th roots of  $\epsilon, \epsilon_1, \epsilon_2$ , respectively as chosen in Lemma 5.2.

Recalling the constructions of  $\Phi'$  and  $\mathbf{v}$  in Theorem 2.5, we get

$$\begin{aligned} \Phi'_{\emptyset, \emptyset} &= (t - \theta)^{2q-1}, \\ \Phi'_{(1; \epsilon), \emptyset} &= (\gamma H_1)^{(-1)} (t - \theta)^{2q-1}, \\ \Phi'_{(1; \epsilon), (1; \epsilon)} &= (t - \theta)^{2q-2}, \\ \Phi'_{(q-1; \epsilon_1), \emptyset} &= (\gamma_1 H_{q-1})^{(-1)} (t - \theta)^{2q-1}, \\ \Phi'_{(q-1; \epsilon_1), (q-1; \epsilon_1)} &= (t - \theta)^q, \\ \Phi'_{(q-1, 1; \epsilon_1, \epsilon_2), (q-1; \epsilon_1)} &= (\gamma_2 H_1)^{(-1)} (t - \theta)^q, \\ \Phi'_{(q-1, 1; \epsilon_1, \epsilon_2), (q-1, 1; \epsilon_1, \epsilon_2)} &= (t - \theta)^{q-1}, \\ \text{others} &= 0, \end{aligned}$$

and

$$\begin{aligned} v_\emptyset &= 0, \\ v_{(1; \epsilon)} &= a_{(1, 2q-2; \epsilon, 1)}(t) H_{2q-2}^{(-1)} (t - \theta)^{2q-2}, \\ v_{(q-1; \epsilon)} &= a_{(q-1, q; \epsilon, 1)}(t) H_q^{(-1)} (t - \theta)^q, \\ v_{(q-1; \epsilon_1)} &= 0, \quad \text{for } \epsilon_1 \neq \epsilon, \\ v_{(q-1, 1; \epsilon_1, \epsilon_2)} &= a_{(q-1, 1, q-1; \epsilon_1, \epsilon_2, 1)}(t) H_{q-1}^{(-1)} (t - \theta)^{q-1}. \end{aligned}$$

Note that if  $(1 \leq n \leq q)$ ,  $H_n = 1$ , and if  $q+1 \leq n \leq q^2$ , then

$$H_n(t) = \sum_{j=0}^k \binom{(n-1) - jq + j}{j} (t^q - t)^{k-j} (t^q - \theta^q)^j$$

where  $k = \lfloor \frac{n-1}{q} \rfloor$  (see [21]). Thus  $H_{2q-2} = 2\theta^q - t - t^q$ .

Hence, for some  $\delta_i$ 's in  $\overline{K}[t]$  for corresponding indices  $i$ , Eq. (2.10) induces

$$(5.7) \quad \delta_\emptyset = \left( \delta_\emptyset^{(-1)} + \delta_{(1; \epsilon)}^{(-1)} (\gamma H_1)^{(-1)} + \sum_{\epsilon_1} \delta_{(q-1; \epsilon_1)}^{(-1)} (\gamma_1 H_{q-1})^{(-1)} \right) (t - \theta)^{2q-1}$$

$$(5.8) \quad \delta_{(1; \epsilon)} = \left( \delta_{(1; \epsilon)}^{(-1)} + a_{(1, 2q-2; \epsilon, 1)}(t) H_{2q-2}^{(-1)} \right) (t - \theta)^{2q-2}$$

$$(5.9) \quad \delta_{(q-1; \epsilon)} = \left( \delta_{(q-1; \epsilon)}^{(-1)} + \delta_{(q-1, 1; \epsilon, 1)}^{(-1)} H_1^{(-1)} + a_{(q-1, q; \epsilon, 1)}(t) H_q^{(-1)} \right) (t - \theta)^q$$

(5.10)

$$\delta_{(q-1; \epsilon_1)} = \left( \delta_{(q-1; \epsilon_1)}^{(-1)} + \delta_{(q-1, 1; \epsilon_1, \epsilon_2)}^{(-1)} (\gamma_2 H_1)^{(-1)} \right) (t - \theta)^q, \quad \text{for } \epsilon_1 \neq \epsilon$$

(5.11)

$$\delta_{(q-1, 1; \epsilon_1, \epsilon_2)} = \left( \delta_{(q-1, 1; \epsilon_1, \epsilon_2)}^{(-1)} + a_{(q-1, 1, q-1; \epsilon_1, \epsilon_2, 1)}(t) H_{q-1}^{(-1)} \right) (t - \theta)^{q-1}$$

First, we consider (5.11). Lemma 4.3 shows that  $\delta_{(q-1,1;\epsilon_1,\epsilon_2)} \in \mathbb{F}_q[t, \theta]$  and  $\deg_\theta \delta_{(q-1,1;\epsilon_1,\epsilon_2)} \leq q$ . We can put  $\delta_{(q-1,1;\epsilon_1,\epsilon_2)} = (f\theta + g)(t - \theta)^{q-1}$  with  $f, g \in \mathbb{F}_q[t]$ , and plug it into (5.11). Canceling terms and twisting once yield

$$f\theta^q + g = (f\theta + g)(t - \theta)^{q-1} + a_{(q-1,1,q-1;\epsilon_1,\epsilon_2,1)}(t).$$

Comparing the  $\theta^{q-1}$ -terms yields  $g = -tf$ , i.e.,  $\delta_{(q-1,1;\epsilon_1,\epsilon_2)} = f(\theta^q - t^q)$  and  $a_{(q-1,1,q-1;\epsilon_1,\epsilon_2,1)}(t) = f(t^q - t)$ . Since  $f$  here is parametrized by  $\epsilon_1$ , we write  $f_{\epsilon_1} = f \in \mathbb{F}_q[t]$ , to have

$$\delta_{(q-1,1;\epsilon_1,\epsilon_2)} = f_{\epsilon_1}(\theta^q - t^q), \quad \text{and} \quad a_{(q-1,1,q-1;\epsilon_1,\epsilon_2,1)}(t) = f_{\epsilon_1}(t^q - t).$$

Second, for (5.10), Lemma 4.3 shows that  $\delta_{(q-1;\epsilon_1)} \in \overline{\mathbb{F}}_q[t, \theta]$  and  $\deg_\theta \delta_{(q-1;\epsilon_1)} \leq \frac{q^2}{q-1} = (q+1) + \frac{1}{q-1}$ , i.e.,  $\delta_{(q-1;\epsilon_1)} = (f'\theta + g')(t - \theta)^q$  with  $f', g' \in \overline{\mathbb{F}}_q[t]$ . We plug it into (5.10), cancel terms, twist and compare the  $\theta^{q+1}$ -terms to have  $f' = 0$ , and compare the  $\theta^q$ -terms to have  $g' = \gamma_2 f_{\epsilon_1}$ . Further, comparing the  $\theta^0$ -terms yields  $g'^{(1)} = (g' - \gamma_2 f_{\epsilon_1})t^q = 0$ , i.e.,  $g' = f_{\epsilon_1} = 0$ , and

$$(5.12) \quad \delta_{(q-1;\epsilon_1)} = 0 \quad \text{for } \epsilon_1 \neq \epsilon.$$

Third, for (5.9), we proceed similarly to have

$$\delta_{(q-1;\epsilon)} = f'_\epsilon(t - \theta)^q, \quad \text{and} \quad a_{(q-1,q;\epsilon,1)}(t) = f'_\epsilon, \quad \text{for some } f'_\epsilon \in \mathbb{F}_q[t].$$

Next, for (5.8), recall  $H_{2q-2}^{(-1)} = \frac{(t-\theta)^2 - (t^q - \theta)^2}{t^q - t} = -t - t^q + 2\theta$ . Then, Lemma 4.3 shows that  $\delta_{(1;\epsilon)} \in \mathbb{F}_q[t, \theta]$  with  $\deg_\theta \delta_{(1;\epsilon)} \leq 2q$ . Let  $\delta_{(1;\epsilon)} = (c\theta^2 + d\theta + e)(t - \theta)^{2q-2}$  for some  $c, d, e \in \mathbb{F}_q[t]$ ; in a similar process, we have  $\delta_{(1;\epsilon)} = c(t - \theta)^{2q}$  for some  $c \in \mathbb{F}_q[t]$ . This  $c$  is parametrized by  $\epsilon$ , so we write  $c_\epsilon = c \in \mathbb{F}_q[t]$  to have

$$\delta_{(1;\epsilon)} = c_\epsilon(t - \theta)^{2q} = c_\epsilon(t^q - \theta^q)^2, \quad \text{and} \quad a_{(1,2q-2;\epsilon,1)}(t) = c_\epsilon(t^q - t).$$

Finally for (5.7), Lemma 4.3 shows that  $\delta_\theta \in \overline{\mathbb{F}}_q[t, \theta]$  with  $\deg_\theta \delta_\theta \leq 2q + 1$ , i.e.,  $\delta_\theta = (c'\theta^2 - d'\theta + e')(t - \theta)^{2q-1}$  for some  $c', d', e' \in \overline{\mathbb{F}}_q[t]$ . Proceeding similarly, comparing the  $\theta^{2q+1}$ - and the  $\theta^{2q-1}$ - coefficients yields  $c' = 0$  and  $e' = d't$ , i.e.,  $\delta_\theta = d'(t - \theta)^{2q}$  for some  $d' \in \overline{\mathbb{F}}_q[t]$ . Thus, recalling (5.12),

$$d'^{(1)}(t - \theta^q) = d'(t - \theta)^{2q} + \gamma c_\epsilon(t - \theta)^{2q} + \gamma(t - \theta)^q f'_\epsilon.$$

The right hand side is divisible by  $t - \theta$ , which forces  $d'^{(1)} = 0$ , i.e.,  $d' = 0$ . By canceling common factors of the remaining terms, we have

$$f'_\epsilon = -c_\epsilon(t - \theta)^q.$$

Again the right hand side of this is divisible by  $t - \theta$ , i.e.,  $f'_\epsilon = c_\epsilon = 0$ . Thus we have  $a_{(1,2q-2;\epsilon,1)}(t) = c_\epsilon(t^q - t) = 0$ ,  $a_{(q-1,q;\epsilon,1)}(t) = f'_\epsilon = 0$ ,  $a_{(q-1,1,q-1;\epsilon_1,\epsilon_2,1)}(t) = f_{\epsilon_1}(t^q - t) = 0$ . This completes the proof.

### 5.6. Proof of Theorem 1.6 for $q = 2$ and $w = 2q - 1$ .

In this section we keep the notation of §5.4 and finish the proof of Theorem 1.6 for  $q = 2$  and  $w = 2q - 1 = 3$ .



5.6.1. *Step 1.*

As  $q = 2$  and  $w = 2q - 1 = 3$ , all characters are trivial and the set  $\mathcal{J}'_w$  consists of two tuples

$$(1, 1, 1; 1, 1, 1), \quad (1, 2; 1, 1).$$

We can skip Step 1.

5.6.2. *Step 2.*

We verify that these tuples satisfy the condition (LW) of Theorem 2.5. We construct  $\Phi'$  with respect to the following tuples in  $I$ :

$$\emptyset, \quad (1; 1), \quad (1, 1; 1, 1),$$

Then,

$$\Phi' = \begin{pmatrix} (t - \theta)^3 & 0 & 0 \\ H_1^{(-1)}(t - \theta)^3 & (t - \theta)^2 & 0 \\ 0 & H_1^{(-1)}(t - \theta)^2 & (t - \theta) \end{pmatrix},$$

and

$$\mathbf{v} = \left( 0, \quad a_{(1,2;1,1)}(t)H_2^{(-1)}(t - \theta)^2, \quad a_{(1,1,1;1,1,1)}(t)H_1^{(-1)}(t - \theta) \right).$$

Since  $q = 2$ , we recall that  $H_1 = H_2 = 1$ . For some  $\delta_i$ 's in  $\overline{K}[t]$  for corresponding indices  $i$ , Eq. (2.10) induces

$$(5.13) \quad \delta_\emptyset = \left( \delta_\emptyset^{(-1)} + \delta_{(1;1)}^{(-1)}H_1^{(-1)} \right) (t - \theta)^3$$

$$(5.14) \quad \delta_{(1;1)} = \left( \delta_{(1;1)}^{(-1)} + \delta_{(1,1;1,1)}^{(-1)}H_1^{(-1)} + a_{(1,2;1,1)}(t)H_2^{(-1)} \right) (t - \theta)^2$$

$$(5.15) \quad \delta_{(1,1;1,1)} = \left( \delta_{(1,1;1,1)}^{(-1)} + a_{(1,1,1;1,1,1)}(t)H_1^{(-1)} \right) (t - \theta).$$

For (5.15), it follows from Lemma 4.3 that  $\delta_{(1,1;1,1)} \in \mathbb{F}_q[t, \theta]$  and  $\deg_\theta \delta_{(1,1;1,1)} \leq 2$ , hence we can write  $\delta_{(1,1;1,1)} = (b\theta + c)(t - \theta)$ , where  $b, c \in \mathbb{F}_q[t]$ . From (5.15), we have

$$b\theta^2 + c = (b\theta + c)(t - \theta) + a_{(1,1,1;1,1,1)}(t).$$

Consider the coefficients of  $\theta$  and  $\theta^0$ , we deduce that  $c = -bt$  and

$$\begin{aligned} \delta_{(1,1;1,1)} &= b(\theta - t)^2 \\ a_{(1,1,1;1,1,1)}(t) &= b(t^2 - t). \end{aligned}$$

For (5.14), it follows from Lemma 4.3 that  $\delta_{(1;1)} \in \mathbb{F}_q[t, \theta]$  and  $\deg_\theta \delta_{(1;1)} \leq 4$ , hence we can write  $\delta_{(1;1)} = F(t - \theta)^2$ , where  $F \in \mathbb{F}_q[t, \theta]$  and  $\deg_\theta F \leq 2$ . From (5.14), we have

$$F^{(1)} = F(t - \theta)^2 + b(\theta^2 - t^2) + a_{(1,2;1,1)}(t).$$

It follows that  $F \in \mathbb{F}_q[t, \theta^2]$ , hence we can write  $F = f\theta^2 + g$ , where  $f, g \in \mathbb{F}_q[t]$ . From the above equation, we have

$$f\theta^4 + g = (f\theta^2 + g)(t^2 - \theta^2) + b(\theta^2 - t^2) + a_{(1,2;1,1)}(t).$$

Consider the coefficients of  $\theta^2$  and  $\theta^0$ , we deduce that  $g = b - ft^2$  and  $a_{(1,2;1,1)}(t) = g + ft^4$ , hence

$$\delta_{(1;1)} = f(\theta - t)^4 + b(\theta - t)^2$$

$$a_{(1,2;1,1)}(t) = f(t^4 - t^2) + b.$$

For (5.13), it follows from Lemma 4.3 that  $\delta_\emptyset \in \mathbb{F}_q[t, \theta]$  and  $\deg_\theta \delta_\emptyset \leq 6$ , hence we can write  $\delta_{(1;1)} = F(t - \theta)^3$ , where  $F \in \mathbb{F}_q[t, \theta]$  and  $\deg_\theta F \leq 3$ . From (5.13), we have

$$F^{(1)} = F(t - \theta)^3 + f(\theta^4 - t^4) + b(\theta^2 - t^2).$$

It follows that  $F(t - \theta)^3 \in \mathbb{F}_q[t, \theta^2]$ , hence we can write  $F = (t - \theta)(u\theta^2 + v)$ , where  $u, v \in \mathbb{F}_q[t]$ . From the above identity, we have

$$(t - \theta^2)(u\theta^4 + v) = (t - \theta)^4(u\theta^2 + v) + f(\theta^4 - t^4) + b(\theta^2 - t^2).$$

Consider the coefficients of  $(t - \theta^2)$  and  $(t^2 - \theta^2)$ , we deduce that  $u = v = 0$  and  $f = b = 0$  hence  $a_{(1,1,1;1,1,1)}(t) = a_{(1,2;1,1)}(t) = 0$ . This completes the proof.

### 5.7. Proof of Theorem 1.6 for $w = 2q$ .

In this section we keep the notation of §5.4 and finish the proof of Theorem 1.6 for  $w = 2q$  by following the strategy given in §5.1. As we are interested in the case  $w = 2q \leq 3q - 3$ , we can suppose that  $q > 2$ .

#### 5.7.1. Step 1.

By Lemma 5.2 and Lemma 5.5, the first shortlist of tuples  $(\mathfrak{s}_i; \epsilon_i)$  appearing (5.4) is

$$(2, q - 1, q - 1; \epsilon, 1, 1), \quad (2, 2q - 2; \epsilon, 1), \quad (q + 1, q - 1; \epsilon, 1).$$

Note that any component of integer parts in the above list is not  $q$  as  $q > 2$ .

Applying Lemma 5.2 with  $(\mathfrak{t}; \epsilon) = (w - (2q - 2); \epsilon)$ , we obtain that there exists some  $a \in K$  such that

$$\begin{aligned} & a_{(w-(2q-2), q-1, q-1; \epsilon, 1, 1)} \Gamma_{q-1} \Gamma_{q-1} \zeta_A \begin{pmatrix} 1 & 1 \\ q-1 & q-1 \end{pmatrix} \\ & + a_{(w-(2q-2), 2q-2; \epsilon, 1)} \Gamma_{2q-2} \zeta_A \begin{pmatrix} 1 \\ 2q-2 \end{pmatrix} + a \Gamma_{2q-2} \zeta_A \begin{pmatrix} 1 \\ 2q-2 \end{pmatrix} = 0, \end{aligned}$$

By Corollary 5.1, we deduce that  $a_{(w-(2q-2), q-1, q-1; \epsilon, 1, 1)} = 0$ . Thus we reduce to the following second shortlist

$$(2, 2q - 2; \epsilon, 1), \quad (q + 1, q - 1; \epsilon, 1).$$

#### 5.7.2. Step 2.

We proceed as done in the discussion preceding Theorem 2.5 to examine the linear independence. We construct  $\Phi'$  with respect to the following tuples in  $I$ :

$$\emptyset, \quad (2; \epsilon), \quad (q + 1; \epsilon).$$

Then,

$$\Phi' = \begin{pmatrix} (t - \theta)^{2q} & 0 & 0 \\ (\gamma H_2)^{(-1)}(t - \theta)^w & (t - \theta)^{2q-2} & 0 \\ (\gamma H_{q+1})^{(-1)}(t - \theta)^w & 0 & (t - \theta)^{q-1} \end{pmatrix},$$

and

$$\mathbf{v} = \left( 0, \quad a_{(2, 2q-2; \epsilon, 1)}(t) H_{2q-2}^{(-1)}(t - \theta)^{2q-2}, \quad a_{(q+1, q-1; \epsilon, 1)}(t) H_{q-1}^{(-1)}(t - \theta)^{q-1} \right).$$

Recall  $H_2 = H_{q-1} = 1$ ,  $H_{q+1} = 2t^q - t - \theta^q$  and  $H_{2q-2} = 2\theta^q - t - t^q$ .

For some  $\delta_i$ 's in  $\overline{K}[t]$  for corresponding indices  $i$ , Eq. (2.10) induces

$$(5.16) \quad \delta_\emptyset = \left( \delta_\emptyset^{(-1)} + \delta_{(2;\epsilon)}^{(-1)}(\gamma H_2)^{(-1)} + \delta_{(q+1;\epsilon)}^{(-1)}(\gamma H_{q+1})^{(-1)} \right) (t - \theta)^w$$

$$(5.17) \quad \delta_{(2;\epsilon)} = \left( \delta_{(2;\epsilon)}^{(-1)} + a_{(2,2q-2;\epsilon,1)}(t) H_{2q-2}^{(-1)} \right) (t - \theta)^{2q-2}$$

$$(5.18) \quad \delta_{(q+1;\epsilon)} = \left( \delta_{(q+1;\epsilon)}^{(-1)} + a_{(q+1,q-1;\epsilon,1)}(t) H_{q-1}^{(-1)} \right) (t - \theta)^{q-1}.$$

First, for (5.18) and (5.17), we obtain

$$\begin{aligned} \delta_{(q+1;\epsilon)} &= f_\epsilon(\theta - t)^q, & \text{and} & \quad a_{(q+1,q-1;\epsilon,1)}(t) = f_\epsilon(t^q - t), \\ \delta_{(2;\epsilon)} &= f'_\epsilon(\theta - t)^{2q}, & \text{and} & \quad a_{(2,2q-2;\epsilon,1)}(t) = f'_\epsilon(t^q - t), \end{aligned}$$

for some  $f_\epsilon, f'_\epsilon \in \mathbb{F}_q[t]$ .

Next, for (5.16), we have  $\delta_\emptyset \in \overline{\mathbb{F}}_q[t]$  with  $\deg_\theta \delta_\emptyset \leq 2q+3$ . Let  $\delta_\emptyset = F(t, \theta) \cdot (t - \theta)^{2q}$  with  $F := F(t, \theta) \in \overline{\mathbb{F}}_q[t, \theta]$  with  $\deg_\theta F \leq 3$ . Then by direct calculation,

$$\begin{aligned} F^{(1)} &= (t - \theta)^{2q} F + \gamma f'_\epsilon(\theta - t)^{2q} H_2 + \gamma f_\epsilon(\theta - t)^q H_{q+1} \\ &= (\theta - t)^{2q} F + \gamma f'_\epsilon(\theta - t)^{2q} + \gamma f_\epsilon(\theta - t)^q (2t^q - t - \theta^q) \end{aligned}$$

We claim that  $F \neq 0$ ; if  $F = 0$  then comparing the  $\theta^q$ - and  $\theta^{2q}$ -coefficients yield  $f_\epsilon = f'_\epsilon = 0$ , i.e.,  $a_{(q+1,q-1;\epsilon,1)}(t) = a_{(2,2q-2;\epsilon,1)}(t) = 0$ , which is a contradiction.

Since  $(\theta - t)^q \mid F^{(1)}$ , we have  $F = (\theta - t^q)G(t, \theta)$  for some  $G(t, \theta) \in \overline{\mathbb{F}}_q[t]$ . Note that  $\deg_\theta G \leq 2$ . Then we have

$$G^{(1)} = (\theta - t^q)(\theta - t)^q G + \gamma f'_\epsilon(\theta - t)^q + \gamma f_\epsilon(2t^q - t - \theta^q).$$

Let  $d = \deg_\theta G$ . Then  $dq = q + 1 + d$ , and  $0 \leq d \leq 2$ , which leads to that  $d = 2$  and  $q = 3$ . Let  $G(t, \theta) = a\theta^2 + b\theta + c$  with  $a, b, c \in \overline{\mathbb{F}}_q[t]$ . Then comparing the  $\theta^{2q-1}$ -,  $\theta^{2q-2}$ -,  $\theta^0$ -coefficients yields  $c^{(1)} = ct^{2q}$ ,  $c = bt^q$ ,  $b = at^q$ , i.e.,  $a = b = c = 0$ . This leads to  $F = 0$ , which is a contradiction, so this completes the proof.

### 5.8. Proof of Theorem 1.6 for $2q + 1 \leq w \leq 3q - 3$ .

In this section we keep the notation of §5.4 and finish the proof of Theorem 1.6 for  $2q + 1 \leq w \leq 3q - 3$  by following the strategy given in §5.1. As we are interested in the case  $2q + 1 \leq w \leq 3q - 3$ , we can assume that  $q > 3$ .

#### 5.8.1. Step 1.

By Lemma 5.2 and Lemma 5.5, the first shortlist of tuples  $(\mathfrak{s}_i; \epsilon_i)$  appearing (5.4) is

$$\begin{aligned} &(w - (2q - 2), q - 1, q - 1; \epsilon, 1, 1), \quad (w - (2q - 2), 2q - 2; \epsilon, 1), \\ &(w - (q - 1), q - 1; \epsilon, 1), \quad (w; \epsilon), \end{aligned}$$

Note that any component of integer parts in the above list is not  $q$  as  $2q + 1 \leq w \leq 3q - 3$ .

Applying Lemma 5.2 with  $(\mathfrak{t}; \epsilon) = (w - (2q - 2); \epsilon)$ , we obtain that there exists some  $a \in K$  such that

$$\begin{aligned} &a_{(w-(2q-2), q-1, q-1; \epsilon, 1, 1)} \Gamma_{q-1} \Gamma_{q-1} \zeta_A \begin{pmatrix} 1 & 1 \\ q-1 & q-1 \end{pmatrix} \\ &+ a_{(w-(2q-2), 2q-2; \epsilon, 1)} \Gamma_{2q-2} \zeta_A \begin{pmatrix} 1 \\ 2q-2 \end{pmatrix} + a \Gamma_{2q-2} \zeta_A \begin{pmatrix} 1 \\ 2q-2 \end{pmatrix} = 0, \end{aligned}$$

By Corollary 5.1, we deduce that  $a_{(w-(2q-2),q-1,q-1;\epsilon_1,1,1)} = 0$ . Thus we reduce to the following second shortlist

$$(w - (2q - 2), 2q - 2; \epsilon, 1), \quad (w - (q - 1), q - 1; \epsilon, 1), \quad (w; \epsilon).$$

5.8.2. *Step 2.*

We proceed as done in the discussion preceding Theorem 2.5 to examine the linear independence. We construct  $\Phi'$  with respect to the following tuples in  $I$ :

$$\emptyset, \quad (w - (2q - 2); \epsilon), \quad (w - (q - 1); \epsilon).$$

Then, we get

$$\Phi' = \begin{pmatrix} (t - \theta)^w & 0 & 0 \\ (\gamma H_{w-(2q-2)})^{(-1)}(t - \theta)^w & (t - \theta)^{2q-2} & 0 \\ (\gamma H_{w-(q-1)})^{(-1)}(t - \theta)^w & 0 & (t - \theta)^{q-1} \end{pmatrix},$$

and

$$\mathbf{v} = \left( a_{(w;\epsilon)}(t)(\gamma H_w)^{(-1)}(t - \theta)^w, \quad a_{(w-(2q-2),2q-2;\epsilon,1)}(t)H_{2q-2}^{(-1)}(t - \theta)^{2q-2}, \quad a_{(w-(q-1),q-1;\epsilon,1)}(t)H_{q-1}^{(-1)}(t - \theta)^{q-1} \right)$$

Recall  $H_{q-1} = 1$ ,  $H_{2q-2} = 2\theta^q - t - t^q$ , and for  $1 \leq k \leq q$  (noting that  $q \geq 3$ ),  $H_{q+k} = (t^q - t) + k(t^q - \theta^q)$ , and

$$H_{2q+k} = (t^q - t)^2 + k(t^q - t)(t^q - \theta^q) + \frac{k(k+1)}{2}(t^q - \theta^q)^2.$$

For some  $\delta_i$ 's in  $\overline{K}[t]$  for corresponding indices  $i$ , Eq. (2.10) induces:

$$(5.19) \quad \delta_\emptyset = \left( \delta_\emptyset^{(-1)} + \delta_{(w-(2q-2);\epsilon)}^{(-1)}(\gamma H_{w-(2q-2)})^{(-1)} \right. \\ \left. + \delta_{(w-(q-1);\epsilon)}^{(-1)}(\gamma H_{w-(q-1)})^{(-1)} + a_{(w;\epsilon)}(t)(\gamma H_w)^{(-1)} \right) (t - \theta)^w$$

(5.20)

$$\delta_{(w-(2q-2);\epsilon)} = \left( \delta_{(w-(2q-2);\epsilon)}^{(-1)} + a_{(w-(2q-2),2q-2;\epsilon,1)}(t)H_{2q-2}^{(-1)} \right) (t - \theta)^{2q-2}$$

(5.21)

$$\delta_{(w-(q-1);\epsilon)} = \left( \delta_{(w-(q-1);\epsilon)}^{(-1)} + a_{(w-(q-1),q-1;\epsilon,1)}(t)H_{q-1}^{(-1)} \right) (t - \theta)^{q-1}$$

First, we consider (5.21). Lemma 4.3 shows that  $\delta_{(w-(q-1);\epsilon)} \in \mathbb{F}_q[t, \theta]$  with  $\deg_\theta \delta_{(w-(q-1);\epsilon)} \leq q$ . Let  $\delta_{(w-(q-1);\epsilon)} = (a\theta + b)(t - \theta)^{q-1}$  for some  $a, b \in \mathbb{F}_q[t]$ , cancel terms and twist once to have

$$(a\theta^q + b) = (a\theta + b)(t - \theta)^{q-1} + a_{(w-(q-1),q-1;\epsilon,1)}(t).$$

Comparing the  $\theta^q$ -coefficients yields  $at + b = 0$ , so rewriting  $a$  as  $f_\epsilon \in \mathbb{F}_q[t]$ , we have

$$\delta_{(w-(q-1);\epsilon)} = f_\epsilon(\theta - t)^q, \quad \text{and} \quad a_{(w-(q-1),q-1;\epsilon,1)}(t) = f_\epsilon(t^q - t).$$

Second, for (5.20), by Lemma 4.3 we have  $\delta_{(w-(2q-2);\epsilon)} \in \mathbb{F}_q[t]$  with  $\deg_\theta \delta_{(w-(2q-2);\epsilon)} \leq 2q$ . Let  $\delta_{(w-(2q-2);\epsilon)} = (a'\theta^2 + b'\theta + c')(t - \theta)^{2q-2}$  then it is reduced to

$$\delta_{(w-(2q-2);\epsilon)} = f'_\epsilon(\theta - t)^{2q}, \quad \text{and} \quad a_{(w-(2q-2),2q-2;\epsilon,1)}(t) = f'_\epsilon(t^q - t)$$

for some  $a' = f'_{w;\epsilon} \in \mathbb{F}_q[t]$ .

Finally, for (5.19), we have  $\delta_\emptyset \in \overline{\mathbb{F}}_q[t]$  with  $\deg_\theta \delta_\emptyset \leq w + 3$  by Lemma 4.3. Let  $\delta_\emptyset = (t - \theta)^w F$  for some  $F \in \overline{\mathbb{F}}_q[t, \theta]$  with  $\deg_\theta F \leq 3$ . Then by direct calculation,

$$F^{(1)} = (t - \theta)^w F + \gamma f'_\epsilon(\theta - t)^{2q} H_{w-(2q-2)} + \gamma f_\epsilon(\theta - t)^q H_{w-(q-1)} + \gamma a_{(w;\epsilon)}(t) H_w$$

$$\begin{aligned}
&= (t - \theta)^w F + \gamma f'_\epsilon (\theta - t)^{2q} + \gamma f_\epsilon (\theta - t)^q H_{q+(w-2q+1)} + \gamma a_{(w;\epsilon)}(t) H_{2q+(w-2q)} \\
&= (t - \theta)^w F + \gamma f'_\epsilon (\theta^q - t^q)^2 \\
&+ \gamma f_\epsilon (\theta^q - t^q) (t^q - t + (w+1)(t^q - \theta^q)) \\
&+ \gamma a_{(w;\epsilon)}(t) \left( (t^q - t)^2 + w(t^q - t)(t^q - \theta^q) + \frac{(w-2q)(w-2q+1)}{2} (t^q - \theta^q)^2 \right)
\end{aligned}$$

It follows that that  $(t - \theta)^w F \in \overline{\mathbb{F}}_q[t, \theta^q]$ . We claim that  $F \neq 0$ ; otherwise, by comparing the coefficients of powers of the  $(t - \theta)$ -term,  $F = 0$  implies  $a_{(w;\epsilon)}(t) = 0$ , and then  $f_\epsilon = f'_\epsilon = 0$ , i.e.,  $a_{(w-(q-1), q-1; \epsilon, 1)}(t) = a_{(w-(2q-2), 2q-2; \epsilon, 1)}(t) = 0$ , so  $a_{(w;\epsilon)}(t) = 0$  by (5.4), which is a contradiction, so  $F \neq 0$ . Thus, the only possibility arises for  $w = 3q - 3$ ,  $F = \gamma(t - \theta)^3 g$  for some  $g \in \overline{\mathbb{F}}_q[t]$ . Since  $w = 3q - 3$ , we have

$$\begin{aligned}
F^{(1)} &= F(t - \theta)^{3q-3} + f'_\epsilon (\theta^q - t^q)^2 \gamma \\
&+ f_\epsilon (\theta^q - t^q) (t^q - t - 2(t^q - \theta^q)) \gamma \\
&+ a_{(3q-3; \epsilon)} \left( (t^q - t)^2 - 3(t^q - t)(t^q - \theta^q) + 3(t^q - \theta^q)^2 \right) \gamma, \\
\epsilon(t - \theta^q)^3 g^{(1)} &= (t^q - \theta^q)^3 g + f'_\epsilon (\theta^q - t^q)^2 + f_\epsilon (\theta^q - t^q) (2\theta^q - t - t^q) \\
&+ a_{(3q-3; \epsilon)}(t) \frac{(t^q - \theta^q)^3 - (t - \theta^q)^3}{t^q - t}
\end{aligned}$$

and by comparing the coefficients of powers of the  $(t^q - \theta^q)$ -term, we have  $a_{(3q-3; \epsilon)}(t) = -(t^q - t)g$ , and  $f_\epsilon = f'_\epsilon = 0$ . Thus,  $a_{(2q-2, q-1; \epsilon, 1)}(t) = a_{(q-1, 2q-2; \epsilon, 1)}(t) = 0$ , so  $a_{(3q-3; \epsilon)}(t) = 0$ , which is a contradiction.

### 5.9. Proof of Theorem 1.6 for $q > 2$ and $w = 3q - 2$ .

In this section we keep the notation of §5.4 and finish the proof of Theorem 1.6 for  $w = 3q - 2$  and  $q > 2$  by following the strategy given in §5.1.

#### 5.9.1. Step 1.

By Lemma 5.2, and Lemma 5.5, the first shortlist of tuples  $(\mathfrak{s}_i; \epsilon_i)$  appearing (5.4) is

$$\begin{array}{ll}
(2q - 2, q; \epsilon, 1), & (1, q - 1, q - 1, q - 1; \epsilon, 1, 1, 1), \\
(1, q - 1, 2q - 2; \epsilon, 1, 1), & (1, 2q - 2, q - 1; \epsilon, 1, 1), \\
(1, 3q - 3; \epsilon, 1), & (q - 1, 2q - 1; \epsilon, 1), \\
(q - 1, 1, q - 1, q - 1; \epsilon_3, \epsilon'_3, 1, 1), & (q - 1, 1, 2q - 2; \epsilon_1, \epsilon'_1, 1), \\
(q - 1, q, q - 1; \epsilon, 1, 1), & (2q - 2, 1, q - 1; \epsilon_2, \epsilon'_2, 1),
\end{array}$$

with  $\epsilon_1 \epsilon'_1 = \epsilon_2 \epsilon'_2 = \epsilon_3 \epsilon'_3 = \epsilon$ .

Applying Lemma 5.2 with  $(\mathfrak{t}; \epsilon) = (1, q - 1; \epsilon, 1)$ , there exists  $a \in K$  such that

$$\begin{aligned}
&a_{(1, q-1, q-1, q-1; \epsilon, 1, 1, 1)} \Gamma_1 \Gamma_1 \zeta_A \begin{pmatrix} 1 & 1 \\ q-1 & q-1 \end{pmatrix} + a_{(1, q-1, 2q-2; \epsilon, 1, 1)} \Gamma_1 \zeta_A \begin{pmatrix} 1 \\ 2q-2 \end{pmatrix} \\
&+ a \Gamma_{2q-2} \zeta_A \begin{pmatrix} 1 \\ 2q-2 \end{pmatrix} = 0.
\end{aligned}$$

By Corollary 5.1, we deduce that  $a_{(1, q-1, q-1, q-1; \epsilon, 1, 1, 1)} = 0$ .

Similarly applying Lemma 5.2 with  $(\mathfrak{t}; \epsilon) = (q - 1, 1; \epsilon_3, \epsilon'_3)$  where  $\epsilon_3 \epsilon'_3 = \epsilon$ , we deduce that  $a_{(q-1, 1, q-1, q-1; \epsilon_3, \epsilon'_3, 1, 1)} = 0$ . Since the choice of  $\epsilon_3$  is free in  $\mathbb{F}_q^\times$ , all candidates of the form can be removed from the list.

With  $(\mathbf{t}; \epsilon) = (1; \epsilon)$ , Lemma 5.2 asserts that there exists  $b \in K$  such that

$$\begin{aligned} & a_{(1, q-1, 2q-2; \epsilon, 1, 1)} \Gamma_{q-1} \Gamma_{2q-2} \zeta_A \begin{pmatrix} 1 & 1 \\ q-1 & 2q-2 \end{pmatrix} \\ & + a_{(1, 2q-2, q-1; \epsilon, 1, 1)} \Gamma_{2q-2} \Gamma_{q-1} \zeta_A \begin{pmatrix} 1 & 1 \\ 2q-2 & q-1 \end{pmatrix} \\ & + a_{(1, 3q-3; \epsilon, 1)} \Gamma_{3q-3} \zeta_A \begin{pmatrix} 1 \\ 3q-3 \end{pmatrix} + b \Gamma_{3q-3} \zeta_A \begin{pmatrix} 1 \\ 3q-3 \end{pmatrix} = 0, \end{aligned}$$

Since the AMZV's  $\zeta_A \begin{pmatrix} 1 & 1 \\ q-1 & 2q-2 \end{pmatrix}$ ,  $\zeta_A \begin{pmatrix} 1 & 1 \\ 2q-2 & q-1 \end{pmatrix}$ ,  $\zeta_A \begin{pmatrix} 1 \\ 3q-3 \end{pmatrix}$  belong to  $\mathcal{AT}'_{3q-3}$ , the induction hypothesis implies that  $a_{(1, q-1, 2q-2; \epsilon, 1, 1)} = a_{(1, 2q-2, q-1; \epsilon, 1, 1)} = 0$ .

Next, we consider  $(\mathbf{t}; \epsilon) = (q-1; \epsilon)$ . By (5.5),  $f_{(q-1; \epsilon)}(\theta) \in K$ , so

$$\begin{aligned} & a_{(q-1, 2q-1; \epsilon, 1)} \Gamma_{2q-1} \zeta_A \begin{pmatrix} 1 \\ 2q-1 \end{pmatrix} + a_{(q-1, 1, 2q-2; \epsilon, 1, 1)} \Gamma_1 \Gamma_{2q-2} \zeta_A \begin{pmatrix} 1 & 1 \\ 1 & 2q-2 \end{pmatrix} \\ & + a_{(q-1, q, q-1; \epsilon, 1, 1)} \Gamma_q \Gamma_{q-1} \zeta_A \begin{pmatrix} 1 & 1 \\ q & q-1 \end{pmatrix} = 0. \end{aligned}$$

Since  $\zeta_A \begin{pmatrix} 1 \\ 2q-1 \end{pmatrix}$  and  $\zeta_A \begin{pmatrix} 1 & 1 \\ 1 & 2q-2 \end{pmatrix}$  are colinear, but  $\zeta_A \begin{pmatrix} 1 \\ 2q-1 \end{pmatrix}$  and  $\zeta_A \begin{pmatrix} 1 & 1 \\ q & q-1 \end{pmatrix}$  are not colinear (see for example [21]), we have  $a_{(q-1, q, q-1; \epsilon, 1, 1)} = 0$ .

Thus we reduce to the following second shortlist:

$$\begin{aligned} & (2q-2, q; \epsilon, 1), \quad (1, 3q-3; \epsilon, 1), \quad (q-1, 2q-1; \epsilon, 1), \\ & (q-1, 1, 2q-2; \epsilon_1, \epsilon \epsilon_1^{-1}, 1), \quad (2q-2, 1, q-1; \epsilon_2, \epsilon \epsilon_2^{-1}, 1), \end{aligned}$$

with  $\epsilon_1, \epsilon_2 \in \mathbb{F}_q^\times$ .

### 5.9.2. Step 2.

From here, we proceed as done in the discussion preceding Theorem 2.5 to examine the linear independence. We construct  $\Phi'$  with respect to the following tuples in  $I$ . Note that  $\epsilon \in \mathbb{F}_q^\times$  is given throughout the remaining part of the proof, and  $\epsilon_1, \epsilon_2 \in \mathbb{F}_q^\times$  are free.

$$\begin{aligned} & \emptyset, \quad (1; \epsilon), \quad (q-1; \epsilon_1), \quad (q-1, 1; \epsilon_1, \epsilon \epsilon_1^{-1}), \\ & (2q-2; \epsilon_2), \quad (2q-2, 1; \epsilon_2, \epsilon \epsilon_2^{-1}). \end{aligned}$$

Recalling the construction of  $\Phi'$  and  $\mathbf{v}$ , we get

$$\begin{aligned} \Phi'_{\emptyset, \emptyset} & = (t - \theta)^{3q-2}, \\ \Phi'_{(1; \epsilon), \emptyset} & = (\gamma H_1)^{(-1)} (t - \theta)^{3q-2}, \\ \Phi'_{(1; \epsilon), (1; \epsilon)} & = (t - \theta)^{3q-3}, \\ \Phi'_{(q-1; \epsilon_1), \emptyset} & = (\gamma_1 H_{q-1})^{(-1)} (t - \theta)^{3q-2}, \\ \Phi'_{(q-1; \epsilon_1), (q-1; \epsilon_1)} & = (t - \theta)^{2q-1}, \\ \Phi'_{(q-1, 1; \epsilon_1, \epsilon \epsilon_1^{-1}), (q-1; \epsilon_1)} & = (\gamma \gamma_1^{-1} H_1)^{(-1)} (t - \theta)^{2q-1}, \\ \Phi'_{(q-1, 1; \epsilon_1, \epsilon \epsilon_1^{-1}), (q-1, 1; \epsilon_1, \epsilon \epsilon_1^{-1})} & = (t - \theta)^{2q-2}, \end{aligned}$$

$$\begin{aligned}
\Phi'_{(2q-2;\epsilon_2),\emptyset} &= (\gamma_2 H_{2q-2})^{(-1)} (t-\theta)^{3q-2}, \\
\Phi'_{(2q-2;\epsilon_2),(2q-2;\epsilon_2)} &= (t-\theta)^q, \\
\Phi'_{(2q-2,1;\epsilon_2,\epsilon\epsilon_2^{-1}),(2q-2;\epsilon_2)} &= (\gamma\gamma_2^{-1}H_1)^{(-1)}(t-\theta)^q, \\
\Phi'_{(2q-2,1;\epsilon_2,\epsilon\epsilon_2^{-1}),(2q-2,1;\epsilon_2,\epsilon\epsilon_2^{-1})} &= (t-\theta)^{q-1}, \\
\text{others} &= 0,
\end{aligned}$$

and

$$\begin{aligned}
v_{(1;\epsilon)} &= a_{(1,3q-3;\epsilon,1)}(t)H_{3q-3}^{(-1)}(t-\theta)^{3q-3} \\
v_{(q-1;\epsilon)} &= a_{(q-1,2q-1;\epsilon,1)}(t)H_{2q-1}^{(-1)}(t-\theta)^{2q-1} \\
v_{(q-1,1;\epsilon_1,\epsilon\epsilon_1^{-1})} &= a_{(q-1,1,2q-2;\epsilon_1,\epsilon\epsilon_1^{-1},1)}(t)H_{2q-2}^{(-1)}(t-\theta)^{2q-2} \\
v_{(2q-2;\epsilon)} &= a_{(2q-2,q;\epsilon,1)}(t)H_q^{(-1)}(t-\theta)^q \\
v_{(2q-2,1;\epsilon_2,\epsilon\epsilon_2^{-1})} &= a_{(2q-2,1,q-1;\epsilon_2,\epsilon\epsilon_2^{-1},1)}(t)H_{q-1}^{(-1)}(t-\theta)^{q-1}, \\
\text{others} &= 0,
\end{aligned}$$

where  $\gamma$  and  $\gamma_j$  are  $(q-1)$ -th roots of unity of  $\epsilon$  and  $\epsilon_j$ , for  $j = 1, 2$ , respectively. Recall again that  $H_1 = H_{q-1} = H_q = 1$ ,  $H_{2q-1} = \theta^q - t$ ,  $H_{2q-2} = 2\theta^q - t - t^q$ , and

$$H_{3q-3} = \begin{cases} (t-t^q)^2 + 3(t^q - \theta^q)(t-\theta^q), & \text{if } q > 3, \\ (t^q - t), & \text{if } q = 3. \end{cases}$$

For some  $\delta_i$ 's in  $\overline{K}[t]$  for corresponding indices  $i$ , Eq. (2.10) induces: for  $\epsilon_1, \epsilon_2 \in \mathbb{F}_q^\times$ ,

$$\begin{aligned}
(5.22) \quad \delta_\emptyset &= \left( \delta_\emptyset^{(-1)} + \sum_{\epsilon_2 \in \mathbb{F}_q^\times} (\gamma_2 H_{2q-2})^{(-1)} \delta_{(2q-2;\epsilon_2)}^{(-1)} + \sum_{\epsilon_1 \in \mathbb{F}_q^\times} (\gamma_1 H_{q-1})^{(-1)} \delta_{(q-1;\epsilon_1)}^{(-1)} \right. \\
&\quad \left. + (\gamma H_1)^{(-1)} \delta_{(1;\epsilon)}^{(-1)} \right) (t-\theta)^{3q-2}
\end{aligned}$$

(5.23)

$$\delta_{(1;\epsilon)} = \left( \delta_{(1;\epsilon)}^{(-1)} + a_{(1,3q-3;\epsilon,1)}(t)H_{3q-3}^{(-1)} \right) (t-\theta)^{3q-3}$$

(5.24)

$$\delta_{(2q-2;\epsilon)} = \left( \delta_{(2q-2;\epsilon)}^{(-1)} + \delta_{(2q-2,1;\epsilon,1)}^{(-1)} H_1^{(-1)} + a_{(2q-2,q;\epsilon,1)}(t)H_q^{(-1)} \right) (t-\theta)^q$$

(5.25)

$$\delta_{(2q-2;\epsilon_2)} = \left( \delta_{(2q-2;\epsilon_2)}^{(-1)} + \delta_{(2q-2,1;\epsilon_2,\epsilon\epsilon_2^{-1})}^{(-1)} (\gamma\gamma_2^{-1}H_1)^{(-1)} \right) (t-\theta)^q \quad \text{for } \epsilon_2 \neq \epsilon$$

(5.26)

$$\delta_{(q-1;\epsilon)} = \left( \delta_{(q-1;\epsilon)}^{(-1)} + \delta_{(q-1,1;\epsilon,1)}^{(-1)} H_1^{(-1)} + a_{(q-1,2q-1;\epsilon,1)}(t)H_{2q-1}^{(-1)} \right) (t-\theta)^{2q-1}$$

(5.27)

$$\delta_{(q-1;\epsilon_1)} = \left( \delta_{(q-1;\epsilon_1)}^{(-1)} + \delta_{(q-1,1;\epsilon_1,\epsilon\epsilon_1^{-1})}^{(-1)} (\gamma\gamma_1^{-1}H_1)^{(-1)} \right) (t-\theta)^{2q-1} \quad \text{for } \epsilon_1 \neq \epsilon$$

(5.28)

$$\delta_{(q-1,1;\epsilon_1,\epsilon\epsilon_1^{-1})} = \left( \delta_{(q-1,1;\epsilon_1,\epsilon\epsilon_1^{-1})}^{(-1)} + a_{(q-1,1,2q-2;\epsilon_1,\epsilon\epsilon_1^{-1},1)}(t)H_{2q-2}^{(-1)} \right) (t-\theta)^{2q-2}$$

(5.29)

$$\delta_{(2q-2,1;\epsilon_2,\epsilon\epsilon_2^{-1})} = \left( \delta_{(2q-2,1;\epsilon_2,\epsilon\epsilon_2^{-1})}^{(-1)} + a_{(2q-2,1,q-1;\epsilon_2,\epsilon\epsilon_2^{-1},1)}(t)H_{q-1}^{(-1)} \right) (t-\theta)^{q-1}$$

First, we consider (5.29). Lemma 4.3 shows that  $\delta_{(2q-2,1;\epsilon_2,\epsilon\epsilon_2^{-1})} \in \mathbb{F}_q[t, \theta]$  and  $\deg_\theta \delta_{(2q-2,1;\epsilon_2,\epsilon\epsilon_2^{-1})} \leq q$ . We let  $\delta_{(2q-2,1;\epsilon_2,\epsilon\epsilon_2^{-1})} = (f_{\epsilon_2}\theta + g_{\epsilon_2})(t-\theta)^{q-1}$  with  $f_{\epsilon_2}, g_{\epsilon_2} \in \mathbb{F}_q[t]$ . Then we plug it into (5.29) and compare the coefficients of powers of  $\theta$ -terms, to obtain

$$\begin{aligned} \delta_{(2q-2,1;\epsilon_2,\epsilon\epsilon_2^{-1})} &= f_{\epsilon_2}(\theta-t)^q, \\ a_{(2q-2,1,q-1;\epsilon_2,\epsilon\epsilon_2^{-1},1)}(t) &= f_{\epsilon_2}(t^q-t). \end{aligned}$$

Second, for (5.28),  $\delta_{(q-1,1;\epsilon_1,\epsilon\epsilon_1^{-1})} \in \mathbb{F}_q[t, \theta]$  and  $\deg_\theta \delta_{(q-1,1;\epsilon_1,\epsilon\epsilon_1^{-1})} \leq 2q$ . We let  $\delta_{(q-1,1;\epsilon_1,\epsilon\epsilon_1^{-1})} = (f'_{\epsilon_1}\theta^2 + g'_{\epsilon_1}\theta + h'_{\epsilon_1})(t-\theta)^{2q-2}$  with  $f'_{\epsilon_1}, g'_{\epsilon_1}, h'_{\epsilon_1} \in \mathbb{F}_q[t]$ . Then, we plug it into (5.28) and compare the coefficients of powers of  $\theta$ -terms, to obtain

$$\begin{aligned} \delta_{(q-1,1;\epsilon_1,\epsilon\epsilon_1^{-1})} &= f'_{\epsilon_1}(\theta-t)^{2q}, \\ a_{(q-1,1,2q-2;\epsilon_1,\epsilon\epsilon_1^{-1},1)}(t) &= f'_{\epsilon_1}(t^q-t). \end{aligned}$$

Third, for (5.27), Lemma 4.3 shows that  $\delta_{(q-1;\epsilon_1)} \in \overline{\mathbb{F}}_q[t, \theta]$  with  $\deg_\theta \delta_{(q-1;\epsilon_1)} \leq 2q+1$ . We let  $\delta_{(q-1;\epsilon_1)} = (a\theta^2 + b\theta + c)(t-\theta)^{2q-1}$  for  $a, b, c \in \overline{\mathbb{F}}_q[t]$  to obtain

$$a^{(1)}\theta^{2q} + b^{(1)}\theta^q + c^{(1)} = (a\theta^2 + b\theta + c)(t-\theta)^{2q-1} + \gamma'_1 f'_{\epsilon_1}(\theta^q - t^q)^2.$$

Here  $\gamma'_1 = \gamma\gamma_1^{-1}$ . Comparing the coefficients of the  $\theta^{2q+1}$ -,  $\theta^{2q}$ - and  $\theta^{2q-1}$ -terms yields  $a = 0$ ,  $b = \gamma'_1 f'_{\epsilon_1}$  and  $c = -bt = -\gamma'_1 f'_{\epsilon_1} t$ .

Thus for  $\epsilon_1 \neq \epsilon$ ,

$$\delta_{(q-1;\epsilon_1)} = (\gamma'_1 f'_{\epsilon_1} \theta - \gamma'_1 f'_{\epsilon_1} t)(t-\theta)^{2q-1} = \gamma'_1 f'_{\epsilon_1}(\theta-t)(t-\theta)^{2q-1} = -\gamma'_1 f'_{\epsilon_1}(\theta-t)^{2q}.$$

Now we replace  $a = 0$ ,  $b = \gamma'_1 f'_{\epsilon_1}$ ,  $c = -\gamma'_1 f'_{\epsilon_1} t$  and  $\delta_{(q-1;\epsilon_1)} = -\gamma'_1 f'_{\epsilon_1}(\theta-t)^{2q}$  in (5.27). Then we have

$$\gamma_1^{(1)} f'_{\epsilon_1}(\theta^q - t) = -\gamma'_1 f'_{\epsilon_1}(\theta-t)^{2q} + \gamma_1 f'_{\epsilon_1}(\theta-t)^{2q} = 0,$$

hence  $f'_{\epsilon_1} = 0$  for all  $\epsilon_1 \neq \epsilon$ , which shows that for all  $\epsilon_1 \neq \epsilon$ ,

$$\delta_{(q-1;\epsilon_1)} = 0.$$

Moreover, for all  $\epsilon_1 \neq \epsilon$ , we have

$$a_{(q-1,1,2q-2;\epsilon_1,\epsilon\epsilon_1^{-1},1)}(t) = f'_{\epsilon_1}(t^q-t) = 0.$$

Fourth, for (5.26), Lemma 4.3 shows that  $\delta_{(q-1;\epsilon)} \in \overline{\mathbb{F}}_q[t, \theta]$  with  $\deg_\theta \delta_{(q-1;\epsilon)} \leq 2q+1$ . We let  $\delta_{(q-1;\epsilon)} = (a\theta^2 + b\theta + c)(t-\theta)^{2q-1}$  for  $a, b, c \in \overline{\mathbb{F}}_q[t]$  to obtain

$$\begin{aligned} a^{(1)}\theta^{2q} + b^{(1)}\theta^q + c^{(1)} &= (a\theta^2 + b\theta + c)(t-\theta)^{2q-1} \\ &\quad + f'_\epsilon(\theta^q - t^q)^2 + a_{(q-1,2q-1;\epsilon,1)}(\theta^q - t). \end{aligned}$$

Comparing the coefficients of the  $\theta^{2q+1}$ -,  $\theta^{2q}$ - and  $\theta^{2q-1}$ -terms yields  $a = 0$ ,  $b = f'_\epsilon$ ,  $c = -tf'_\epsilon$ . Thus

$$(5.30) \quad \begin{aligned} \delta_{(q-1;\epsilon)} &= -f'_\epsilon(\theta-t)^{2q}, \\ a_{(q-1,2q-1;\epsilon,1)}(t) &= f'_\epsilon. \end{aligned}$$



Fifth, for (5.25), Lemma 4.3 shows that when  $\epsilon_2 \neq \epsilon$ ,  $\delta_{(2q-2;\epsilon_2)} \in \overline{\mathbb{F}}_q[t, \theta]$  with  $\deg_\theta \delta_{(2q-2;\epsilon_2)} \leq q+1$ . Letting  $\delta_{(2q-2;\epsilon_2)} = (a\theta + b)(t - \theta)^q$  for  $a, b \in \overline{\mathbb{F}}_q[t]$ , cancelling, twisting once and comparing the  $\theta^q$ - and  $\theta^{q+1}$ -terms yields  $a = 0$ ,  $b = \gamma\gamma_2^{-1}f_{\epsilon_2}$ , i.e.,  $\delta_{(2q-2;\epsilon_2)} = -\gamma\gamma_2^{-1}f_{\epsilon_2}(\theta - t)^q$ , and plugging it into (5.25) yields  $f_{\epsilon_2} = 0$ , for  $\epsilon_2 \neq \epsilon$ . Note that this implies that for  $\epsilon_2 \neq \epsilon$ ,

$$(5.31) \quad \delta_{(2q-2;\epsilon_2)} = 0, \quad a_{(2q-2,1,q-1;\epsilon_2,\epsilon\epsilon_2^{-1},1)}(t) = 0.$$

Next, for (5.24), Lemma 4.3 shows that  $\delta_{(2q-2;\epsilon)} \in \mathbb{F}_q[t, \theta]$  with  $\deg_\theta \delta_{(2q-2;\epsilon)} \leq q+1$ . Letting  $\delta_{(2q-2;\epsilon)} = (a\theta + b)(t - \theta)^q$  for  $a, b \in \mathbb{F}_q[t]$ , cancelling, twisting once and comparing the  $\theta^q$ - and  $\theta^{q+1}$ -terms yield  $a = 0$ ,  $b = f_\epsilon$ , i.e.,

$$(5.32) \quad \delta_{(2q-2;\epsilon)} = -f_\epsilon(\theta - t)^q, \quad a_{(2q-2,q;\epsilon,1)}(t) = f_\epsilon.$$

For (5.23), Lemma 4.3 shows that  $\delta_{(1;\epsilon)} \in \mathbb{F}_q[t, \theta]$  with  $\deg_\theta \delta_{(1;\epsilon)} \leq 3q$ . Letting  $\delta_{(1;\epsilon)} = (a\theta^3 + b\theta^2 + c\theta + d)(t - \theta)^{3q-3}$  with some  $a, b, c, d \in \mathbb{F}_q[t]$ , cancelling, twisting once and comparing the  $\theta^{3q-2}$ - and  $\theta^{3q-1}$ -terms yield  $b = -3at$ ,  $c = 3at^2$ . When  $q > 3$ , comparing the  $\theta^{3q-3}$ -terms yields  $d = -at^3$ , i.e.,  $\delta_{(1;\epsilon)} = a(\theta - t)^{3q}$ ; when  $q = 3$ , comparing the  $\theta^{3q-3}$ -terms yield  $d = -at^3$ , i.e.,  $\delta_{(1;\epsilon)} = a(\theta - t)^{3q}$ . So in any case  $\delta_{(1;\epsilon)} = f''(\theta - t)^{3q}$  for  $f'' = a \in \mathbb{F}_q[t]$ . Again, when  $q > 3$ , one can put this into (5.23), cancel terms, twist once and compare the  $\theta^0$ -terms to have  $a_{(1,3q-3;\epsilon,1)}(t) = f''(t^q - t)$ ; when  $q = 3$ , one have  $a_{(1,3q-3;\epsilon,1)}(t) = f''(t^q - t)^2$ . In summary,

$$\delta_{(1;\epsilon)} = f''(\theta - t)^{3q}, \text{ and } a_{(1,3q-3;\epsilon,1)} = \begin{cases} f''(t^q - t), & \text{if } q > 3, \\ f''(t^q - t)^2, & \text{if } q = 3. \end{cases}$$

Finally, with (??), (5.30), (5.31) and (5.32), (5.22) is written as

$$\delta_\emptyset = \left( \delta_\emptyset^{(-1)} + (\gamma(2\theta^q - t - t^q)\delta_{(2q-2;\epsilon)})^{(-1)} + (\gamma\delta_{(q-1;\epsilon)})^{(-1)} + (\gamma\delta_{(1;\epsilon)})^{(-1)} \right) (t - \theta)^{3q-2}.$$

Lemma 4.3 shows that  $\delta_\emptyset \in \overline{\mathbb{F}}_q[t, \theta]$  with  $\deg_\theta \delta_\emptyset \leq 3q+1$ . Let  $\delta_\emptyset = \gamma(t - \theta)^{3q-2}G$  for some  $G \in \overline{\mathbb{F}}_q[t, \theta]$  with  $\deg_\theta G \leq 3$ , then canceling terms and twisting once yields

$$\epsilon G^{(1)} = (t - \theta)^{3q-2}G - (\theta - t)^q (f_\epsilon(2\theta^q - t - t^q) + f'_\epsilon(\theta - t)^q) + f''(\theta - t)^{3q},$$

so  $(t - \theta)^{3q-2}G \in \overline{\mathbb{F}}_q[t, \theta^q]$ , i.e.,  $G = (t - \theta)^2g$  with  $g \in \overline{\mathbb{F}}_q[t]$ . Then

$$\begin{aligned} & (t - \theta)^{3q}(g - f'') - \epsilon(t - \theta^q)^2g^{(1)} \\ &= (\theta - t)^q (f_\epsilon(2\theta^q - t - t^q) + f'_\epsilon(\theta - t)^q) \\ &= (\theta - t)^q(\theta^q - t)f_\epsilon + (\theta - t)^{2q}f_\epsilon + (\theta - t)^{2q}f'_\epsilon. \end{aligned}$$

Considering the factors of powers of the  $(t - \theta^q)$ - and  $(t - \theta)$ -terms, we have  $g^{(1)} = 0$ , i.e.,  $g = 0$ ,  $f_\epsilon = 0$ ,  $f'_\epsilon = 0$ , and  $f'' = 0$ , so  $a_{(2q-2,q;\epsilon,1)}(t) = a_{(1,3q-3;\epsilon,1)}(t) = a_{(2q-2,1,q-1;\epsilon_2,\epsilon\epsilon_2^{-1},1)}(t) = a_{(q-1,2q-1;\epsilon,1)}(t) = a_{(q-1,1,2q-2;\epsilon_1,\epsilon\epsilon_1^{-1},1)}(t) = 0$  for all  $\epsilon_1, \epsilon_2 \in \mathbb{F}_q^\times$ , which is a contradiction. This completes the proof.

### 5.10. Proof of Theorem 1.6 for $q = 2$ and $w = 3q - 2$ .

In this section we keep the notation of §5.4 and finish the proof of Theorem 1.6 for  $q = 2$  and  $w = 3q - 2 = 4$ .

In what follows, we assume  $q = 2$  and  $w = 4$ . Then all characters are trivial and therefore, all AMZV's are MZV's. The set  $\mathcal{J}'_w$  consists of three tuples

$$(2, 1, 1; 1, 1, 1), \quad (1, 2, 1; 1, 1, 1), \quad (1, 3; 1, 1).$$

#### 5.10.1. Step 1.

We will keep all three tuples so that we can skip Step 1 of the strategy given in §5.1.

#### 5.10.2. Step 2.

We claim that the previous tuples verify the condition (LW) of Theorem 2.5. In fact, it suffices to see that  $\zeta_A(1, 2; 1, 1)$  and  $\zeta_A(2, 1; 1, 1)$  are  $K$ -linearly independent. For the weight  $w' = 3$  (and recall  $q = 2$ ), the set  $\mathcal{AT}'_3$  consists of  $\zeta_A(1, 1, 1; 1, 1, 1)$  and  $\zeta_A(1, 2; 1, 1)$  (see §5.5.1). Then in *loc. cit.* we have proved that  $\zeta_A(1, 1, 1; 1, 1, 1)$  and  $\zeta_A(1, 2; 1, 1)$  are  $K$ -linearly independent. By [30, Theorem 6.1 (1)], we set  $D_1 := \theta^2 - \theta$  and get the following relation

$$D_1 \zeta_A(1, 1, 1; 1, 1, 1) + D_1 \zeta_A(2, 1; 1, 1) + \zeta_A(1, 2; 1, 1) = 0.$$

It follows that  $\zeta_A(1, 2; 1, 1)$  and  $\zeta_A(2, 1; 1, 1)$  are  $K$ -linearly independent as claimed.

Thus we can proceed as done in the discussion preceding Theorem 2.5 to examine the linear independence. We construct  $\Phi'$  with respect to the following tuples in  $I$ :

$$\emptyset, \quad (1; 1), \quad (2; 1), \quad (1, 2; 1, 1), \quad (2, 1; 1, 1).$$

Then,

$$\Phi' = \begin{pmatrix} (t - \theta)^4 & 0 & 0 & 0 & 0 \\ H_1^{(-1)}(t - \theta)^4 & (t - \theta)^3 & 0 & 0 & 0 \\ H_2^{(-1)}(t - \theta)^4 & 0 & (t - \theta)^2 & 0 & 0 \\ 0 & H_2^{(-1)}(t - \theta)^3 & 0 & t - \theta & 0 \\ 0 & 0 & H_1^{(-1)}(t - \theta)^2 & 0 & t - \theta \end{pmatrix},$$

and

$$\mathbf{v} = \left( 0, \quad a_{(1,3;1,1)}(t)H_3^{(-1)}(t - \theta)^3, \quad 0, \quad a_{(1,2,1;1,1,1)}(t)H_1^{(-1)}(t - \theta)^2, \quad a_{(2,1,1;1,1,1)}(t)H_1^{(-1)}(t - \theta) \right).$$

Here we recall that  $H_1 = H_2 = 1$ ,  $H_3 = \theta^2 - t$ .

For some  $\delta_i$ 's in  $\overline{K}[t]$  for corresponding indices  $i$ , Eq. (2.10) induces

$$(5.33) \quad \delta_\emptyset = \left( \delta_\emptyset^{(-1)} + \delta_{(1;1)}^{(-1)}H_1^{(-1)} + \delta_{(2;1)}^{(-1)}H_2^{(-1)} \right) (t - \theta)^4$$

$$(5.34) \quad \delta_{(1;1)} = \left( \delta_{(1;1)}^{(-1)} + \delta_{(1,2;1,1)}^{(-1)}H_2^{(-1)} + a_{(1,3;1,1)}(t)H_3^{(-1)} \right) (t - \theta)^3$$

$$(5.35) \quad \delta_{(2;1)} = \left( \delta_{(2;1)}^{(-1)} + \delta_{(2,1;1,1)}^{(-1)}H_1^{(-1)} \right) (t - \theta)^2$$

$$(5.36) \quad \delta_{(1,2;1,1)} = \left( \delta_{(1,2;1,1)}^{(-1)} + a_{(1,2,1;1,1,1)}(t)H_1^{(-1)} \right) (t - \theta)$$

$$(5.37) \quad \delta_{(2,1;1,1)} = \left( \delta_{(2,1;1,1)}^{(-1)} + a_{(2,1,1;1,1,1)}(t)H_1^{(-1)} \right) (t - \theta).$$

First, for (5.37) and (5.36), we obtain

$$\delta_{(2,1;1,1)} = f(\theta - t)^2, \quad \text{and} \quad a_{(2,1,1;1,1,1)}(t) = f(t^2 - t),$$

$$\delta_{(1,2;1,1)} = g(\theta - t)^2, \quad \text{and} \quad a_{(1,2,1;1,1,1)}(t) = g(t^2 - t).$$

for some  $f, g \in \mathbb{F}_2[t]$ .

Second, for (5.35), by Lemma 4.3 we have  $\delta_{(2;1)} \in \mathbb{F}_2[t]$  with  $\deg_{\theta} \delta_{(2;1)} \leq 4$ . Let  $\delta_{(2;1)} = F(t, \theta) \cdot (t - \theta)^2$  with  $F := F(t, \theta) \in \mathbb{F}_2[t, \theta]$  with  $\deg_{\theta} F \leq 2$ . Then canceling terms and twisting once yields

$$F^{(1)} = (t - \theta)^2 F + f(\theta - t)^2.$$

It follows that  $F \in \mathbb{F}_2[t, \theta^2]$ . We can express  $F = a\theta^2 + b$  for some  $a, b \in \mathbb{F}_2[t]$ . Replacing this expression into the previous equation yields  $b = at^4$  and  $f = b(t^2 - 1)$ .

Thus

$$\begin{aligned} f &= at^2(t^2 - 1), \\ \delta_{(2;1)} &= a(t - \theta)^2(t^2 - \theta)^2. \end{aligned}$$

Next, for (5.34), by Lemma 4.3 we have  $\delta_{(1;1)} \in \mathbb{F}_2[t]$  with  $\deg_{\theta} \delta_{(1;1)} \leq 6$ . Let  $\delta_{(1;1)} = G(t, \theta) \cdot (t - \theta)^2$  with  $G := G(t, \theta) \in \mathbb{F}_2[t, \theta]$  with  $\deg_{\theta} G \leq 3$ . Then canceling terms and twisting once yields

$$G^{(1)} = (t - \theta)^3 G + g_c(\theta - t)^2 + a_{(1,3;1,1)}(t)(\theta^2 - t).$$

It follows that  $(t - \theta)^3 G \in \mathbb{F}_2[t, \theta^2]$ . We can express  $G = (t - \theta)(c\theta^2 + d)$  for some  $c, d \in \mathbb{F}_2[t]$ . We replace this expression into the previous equation and compare the coefficients to conclude that  $g = 0$ ,  $d = ct$  and  $a_{(1,3;1,1)}(t) = c(t^4 - t)$ . In particular,

$$\begin{aligned} \delta_{(1,2;1,1)} &= a_{(1,2,1;1,1,1)}(t) = 0, \\ \delta_{(1;1)} &= c(t - \theta)^4(\theta^2 - t), \\ a_{(1,3;1,1)}(t) &= c(t^4 - t). \end{aligned}$$

Finally, for (5.33), by Lemma 4.3 we have  $\delta_{\emptyset} \in \mathbb{F}_2[t]$  with  $\deg_{\theta} \delta_{\emptyset} \leq 8$ . Let  $\delta_{\emptyset} = H(t, \theta) \cdot (t - \theta)^4$  with  $H := H(t, \theta) \in \mathbb{F}_2[t, \theta]$  with  $\deg_{\theta} H \leq 4$ . Then

$$H^{(1)} = (t - \theta)^4 H + c(t - \theta)^4(\theta^2 - t) + a(t - \theta)^2(t^2 - \theta)^2.$$

Then  $H \in \mathbb{F}_2[t, \theta^2]$  and  $H^{(1)}$  is divisible by  $(t - \theta)^2$ . Thus we can express  $H = a'(\theta^4 - t^8) + b'(\theta^2 - t^4)$  with  $a', b' \in \mathbb{F}_2[t]$ . Replacing this expression into the previous equation gets

$$a'(t - \theta)^8 + b'(t - \theta)^4 = (t - \theta)^4(a'(\theta^4 - t^8) + b'(\theta^2 - t^4)) + c(t - \theta)^4(\theta^2 - t) + a(t - \theta)^2(t^2 - \theta)^2.$$

Comparing the powers of  $t - \theta$  yields  $a = 0$ . It implies  $f = 0$  and  $a_{(2,1,1;1,1,1)}(t) = 0$ .

We have shown that  $a_{(1,2,1;1,1,1)} = 0$  and  $a_{(2,1,1;1,1,1)} = 0$ . It follows that  $a_{(1,3;1,1)} = 0$  and we get a contradiction.

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DEPT. OF MATHEMATICAL SCIENCES, KAIST, 291 DAEHAK-RO, YUSEONG-GU, DAEJEON 34141,  
SOUTH KOREA

*Email address:* `bhim@kaist.ac.kr`

DEPT. OF MATHEMATICAL SCIENCES, KAIST, 291 DAEHAK-RO, YUSEONG-GU, DAEJEON 34141,  
SOUTH KOREA

*Email address:* `hojinkim@kaist.ac.kr`

NORMANDIE UNIVERSITÉ, UNIVERSITÉ DE CAEN NORMANDIE - CNRS, LABORATOIRE DE MATHÉMATIQUES  
NICOLAS ORESME (LMNO), UMR 6139, 14000 CAEN, FRANCE.

*Email address:* `khac-nhuan.le@unicaen.fr`

NORMANDIE UNIVERSITÉ, UNIVERSITÉ DE CAEN NORMANDIE - CNRS, LABORATOIRE DE MATHÉMATIQUES  
NICOLAS ORESME (LMNO), UMR 6139, 14000 CAEN, FRANCE.

*Email address:* `tuan.ngodac@unicaen.fr`

INSTITUTE OF MATHEMATICS, VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY, 18 HOANG  
QUOC VIET, 10307 HANOI, VIET NAM

*Email address:* `p Nhung@math.ac.vn`