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NOTE ON LINEAR INDEPENDENCE OF ALTERNATING MULTIPLE ZETA VALUES IN POSITIVE CHARACTERISTIC

BO-HAE IM, HOJIN KIM, KHAC NHUAN LE, TUAN NGO DAC, AND LAN HUONG PHAM

Dedicated to Professor Ngô Việt Trung with admiration

ABSTRACT. We discuss certain results related to the linear independence of alternating multiple zeta values introduced by Harada in 2021.

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1. INTRODUCTION

The present paper was initially conceived in January 2021 as an extension of some results of [22] to the setting of alternating multiple zeta values (AMZV's) introduced by Harada [14]. At that time, we endeavored to explore Zagier-Hoffman's conjectures in positive characteristic for both MZV's and AMZV's, by establishing certain results related to the linear independence of specific sets of AMZV's. Subsequently, in their work [17], the authors discovered a more successful approach by substituting these sets with those of alternating Carlitz multiple polylogarithms (ACMPL's). For the MZV's version of Zagier-Hoffman's conjectures in positive characteristic, we would like to refer the reader to the work [9] which adopts the same approach and yields the same results. Consequently, we made the decision to briefly reference the findings of this paper in Proposition 4.6 of the aforementioned work (*loc. cit.*), deferring the complete exposition of the details to the present article.

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1.1. Background.

Let $A = \mathbb{F}_q[\theta]$ be the polynomial ring in the variable θ over a finite field \mathbb{F}_q of q elements of characteristic $p > 0$. We denote by A_+ the set of monic polynomials in A . Let $K = \mathbb{F}_q(\theta)$ be the fraction field of A equipped with the rational point ∞ . Let K_∞ be the completion of K at ∞ and \mathbb{C}_∞ be the completion of a fixed algebraic closure \overline{K} of K at ∞ . We denote by v_∞ the discrete valuation on K corresponding to the place ∞ normalized such that $v_\infty(\theta) = -1$, and by $|\cdot|_\infty = q^{-v_\infty}$ the associated absolute value on K . The unique valuation of \mathbb{C}_∞ which extends v_∞ will still be denoted by v_∞ .

Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of positive integers and $\mathbb{Z}^{\geq 0} = \{0, 1, 2, \dots\}$ be the set of non-negative integers. In [6] Carlitz introduced the Carlitz zeta values $\zeta_A(n)$ for $n \in \mathbb{N}$ given by

$$\zeta_A(n) := \sum_{a \in A_+} \frac{1}{a^n} \in K_\infty$$

which are analogues of classical special zeta values in the function field setting. For any tuple of positive integers $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, Thakur [24] defined the characteristic p multiple zeta value (MZV for short) $\zeta_A(\mathfrak{s})$ or $\zeta_A(s_1, \dots, s_r)$ by

$$\zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \dots a_r^{s_r}} \in K_\infty$$

where the sum runs through the set of tuples $(a_1, \dots, a_r) \in A_+^r$ with $\deg a_1 > \dots > \deg a_r$. We call r the depth of $\zeta_A(\mathfrak{s})$ and $w(\mathfrak{s}) = s_1 + \dots + s_r$ the weight of $\zeta_A(\mathfrak{s})$. We note that Carlitz zeta values are exactly depth one MZV's. Thakur [25] showed that all the MZV's do not vanish.

For $w \in \mathbb{N}$ we denote by \mathcal{Z}_w the K -vector space spanned by the MZV's of weight w , and by \mathcal{T}_w is the set of $\zeta_A(\mathfrak{s})$ with $\mathfrak{s} = (s_1, \dots, s_r)$ of weight w with $1 \leq s_i \leq q$ for $1 \leq i \leq r-1$ and $s_r < q$. We also define \mathcal{T}_w^0 to be the set of MZV's $\zeta_A(s_1, \dots, s_r)$ such that $1 \leq s_i < q$ for all $1 \leq i \leq r$. We now recall two results of [22]:

Theorem 1.1 ([22, Theorems B]). *Let $w \in \mathbb{N}$. Then the MZV's in \mathcal{T}_w^0 are linearly independent over K .*

Theorem 1.2 ([22, Theorems D]). *Let $w \leq 2q - 2$. Then the MZV's in \mathcal{T}_w are linearly independent over K .*

1.2. Statement of the results.

In [14] Harada introduced the alternating MZV's in positive characteristic. For any tuple of positive integers $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ and $\epsilon = (\epsilon_1, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$, the corresponding AMZV is given by

$$\zeta_A \left(\begin{matrix} \epsilon \\ \mathfrak{s} \end{matrix} \right) := \sum \frac{\epsilon_1^{\deg a_1} \dots \epsilon_r^{\deg a_r}}{a_1^{s_1} \dots a_r^{s_r}} \in K_\infty$$

where the sum runs through the set of tuples $(a_1, \dots, a_r) \in A_+^r$ with $\deg a_1 > \dots > \deg a_r$. We call r the depth of $\zeta_A \left(\begin{matrix} \epsilon \\ \mathfrak{s} \end{matrix} \right)$ and $w(\mathfrak{s}) = s_1 + \dots + s_r$ the weight of $\zeta_A(\mathfrak{s})$. We note all the AMZV's do not vanish [14]. For $w \in \mathbb{N}$ we denote by \mathcal{AZ}_w the K -vector space spanned by the MZV's of weight w . We refer the reader to [12, 14] for more details about these objects. Note that in the classical setting the reader could consult [16, p. 536] for a discussion about the dimensions of the \mathbb{Q} -vector space spanned by all the AMZV's of fixed weight.

We now introduce several sets of AMZV's. First, we denote the set \mathcal{AT}_w by

$$\mathcal{AT}_w := \left\{ \zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} : \zeta_A(\mathfrak{s}) \in \mathcal{T}_w \right\}.$$

Next, we denote by \mathcal{AT}_w^0 the set of AMZV's $\zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix}$ such that $\zeta_A(\mathfrak{s}) \in \mathcal{T}_w^0$.

We also denote by \mathcal{AT}_w^1 the subset of AMZV's $\zeta_A(\mathfrak{s}, \epsilon)$ of \mathcal{AT}_w such that $\epsilon_i = 1$ whenever $s_i = q$ (i.e., the character corresponding to q is always 1). We remark that in general,

$$\mathcal{AT}_w^0 \subsetneq \mathcal{AT}_w^1 \subsetneq \mathcal{AT}_w.$$

Finally, for $w \leq 3q - 2$, we denote by \mathcal{AT}'_w the subset of AMZV's as follows.

- For $1 \leq w \leq 2q - 2$, \mathcal{AT}'_w consists of $\zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix}$ of weight w such that if we write $\mathfrak{s} = (s_1, \dots, s_r)$, then $q - 1 \nmid s_i$.
- For $2q - 1 \leq w \leq 3q - 2$, we consider I_w the set of tuples $\mathfrak{s} = (s_1, \dots, s_r)$ of weight w where $s_i \neq q, 2q - 1, 2q, 3q - 2$ for all i satisfying
 - if $s_{i+1} = q$ or $2q - 1$ for some $i = 1, 2, \dots, r - 1$, then $q - 1 \mid s_i$,
 - $s_i \neq (q - 1, q - 1, q)$ when $w = 3q - 2$,
 and define

$$\mathcal{AT}'_w := \left\{ \zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} : \mathfrak{s} \in I_w, \text{ and } \epsilon_i = 1 \text{ whenever } s_i = q \right\}.$$

Here are some basic properties of these sets that are crucial for linear independence of AMZV's.

Proposition 1.3. *We define a Fibonacci-like sequence $s(w)$ as follows. We put*

$$s(w) = \begin{cases} (q - 1)q^{w-1} & \text{if } 1 \leq w < q, \\ (q - 1)(q^{w-1} - 1) & \text{if } w = q, \end{cases}$$

and for $w > q$, $s(w) = (q - 1) \sum_{i=1}^{q-1} s(w - i) + s(w - q)$. Then for all $w \in \mathbb{N}$,

$$|\mathcal{AT}_w^1| = s(w).$$

Proposition 1.4. *Let $w \leq 3q - 2$. Then*

1) *We have*

$$|\mathcal{AT}'_w| = s(w).$$

2) *For any $(\mathfrak{s}; \epsilon) = (s_1, \dots, s_r; \epsilon_1, \dots, \epsilon_r) \in \mathbb{N}^r \times (\mathbb{F}_q^\times)^r$, if $\zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} \in \mathcal{AT}'_w$,*

then $\zeta_A \begin{pmatrix} \epsilon_1 & \cdots & \epsilon_{r-1} \\ s_1 & \cdots & s_{r-1} \end{pmatrix}$ belongs to \mathcal{AT}'_{w-s_r} .

We are now ready to state some results for linear independence of AMZV's which extend Theorems 1.1 and 1.2:

Theorem 1.5. *Let $w \in \mathbb{N}$. Then the AMZV's in \mathcal{AT}_w^0 are linearly independent over K .*

Theorem 1.6. *Let $w \leq 3q - 2$. Then the AMZV's in \mathcal{AT}'_w are all linearly independent over K . In particular,*

$$\dim_K \mathcal{AZ}_w \geq s(w).$$

We would like to mention several comments. First, Theorem 1.6 is exactly [17, Proposition 4.6]. Second, the proofs of both previous theorems are based on the theory of dual t -motives introduced by Anderson [1] and the Anderson-Brownawell-Papanikolas criterion [2]. Finally, it is proved in [17] that $\dim_K \mathcal{AZ}_w = s(w)$ for all weights $w \in \mathbb{N}$ (see [17, Theorem A]). However, the approach given in *loc. cit.* goes beyond the scope of this paper as we have mentioned in the beginning of the introduction.

1.3. Organization of the paper.

In §2 we recall the notion of dual t -motives following Anderson and the Anderson-Brownawell-Papanikolas criterion for linear independence. Then we state some applications for linear independence of MZV's and AMZV's (see Theorems 2.4 and 2.5). In §3 we prove Propositions 1.3 and 1.4. The main theorems 1.5 and 1.6 will be shown in §4 and §5 respectively.

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2. DUAL t -MOTIVES AND LINEAR INDEPENDENCE

We continue with the notation given in the Introduction and follow closely the presentation given as in [17, 22] (see also [9]). In particular, a large part of this section is line-by-line parallel to [17, §2]. Further, letting t be another independent variable, we denote by \mathbb{T} the Tate algebra in the variable t with coefficients in \mathbb{C}_∞ equipped with the Gauss norm $\|\cdot\|_\infty$.

2.1. Dual t -motives and Anderson-Brownawell-Papanikolas (ABP) criterion.

We recall the notion of dual t -motives due to Anderson (see [5, §4] and [15, §5] for more details). We refer the reader to [1] for the related notion of t -motives.

For $i \in \mathbb{Z}$ we consider the i -fold twisting of $\mathbb{C}_\infty((t))$ defined by

$$\begin{aligned} \mathbb{C}_\infty((t)) &\rightarrow \mathbb{C}_\infty((t)) \\ f = \sum_j a_j t^j &\mapsto f^{(i)} := \sum_j a_j^{q^i} t^j. \end{aligned}$$

We extend i -fold twisting to matrices with entries in $\mathbb{C}_\infty((t))$ by twisting entry-wise.

Let $\overline{K}[t, \sigma]$ be the non-commutative $\overline{K}[t]$ -algebra generated by the new variable σ subject to the relation $\sigma f = f^{(-1)}\sigma$ for all $f \in \overline{K}[t]$.

Definition 2.1. An effective dual t -motive is a $\overline{K}[t, \sigma]$ -module \mathcal{M}' which is free and finitely generated over $\overline{K}[t]$ such that for $\ell \gg 0$ we have

$$(t - \theta)^\ell (\mathcal{M}' / \sigma \mathcal{M}') = \{0\}.$$

We mention that effective dual t -motives are called Frobenius modules in [10, 12, 14, 18]. Note that Hartl and Juschka [15, §4] introduced a more general notion of dual t -motives. In particular, effective dual t -motives are always dual t -motives.

Throughout this paper, we will always work with effective dual t -motives. Therefore, we will sometimes drop the word “effective” where there is no confusion.

Let \mathcal{M} and \mathcal{M}' be two effective dual t -motives. Then a morphism of effective dual t -motives $\mathcal{M} \rightarrow \mathcal{M}'$ is just a homomorphism of left $\overline{K}[t, \sigma]$ -modules. We denote by \mathcal{F} the category of effective dual t -motives equipped with the trivial object $\mathbf{1}$.

We say that an object \mathcal{M} of \mathcal{F} is given by a matrix $\Phi \in \text{Mat}_r(\overline{K}[t])$ if \mathcal{M} is a $\overline{K}[t]$ -module free of rank r and the action of σ is represented by the matrix Φ on a given $\overline{K}[t]$ -basis for \mathcal{M} . We say that an object \mathcal{M} of \mathcal{F} is uniformizable or rigid analytically trivial if there exists a matrix $\Psi \in \text{GL}_r(\mathbb{T})$ satisfying $\Psi^{(-1)} = \Phi\Psi$. The matrix Ψ is called a rigid analytic trivialization of \mathcal{M} .

We now recall the Anderson-Brownawell-Papanikolas criterion which is crucial in the sequel (see [2, Theorem 3.1.1]).

Theorem 2.2 (Anderson-Brownawell-Papanikolas). *Let $\Phi \in \text{Mat}_\ell(\overline{K}[t])$ be a matrix such that $\det \Phi = c(t - \theta)^s$ for some $c \in \overline{K}$ and $s \in \mathbb{Z}^{\geq 0}$. Let $\psi \in \text{Mat}_{\ell \times 1}(\mathcal{E})$ be a vector satisfying $\psi^{(-1)} = \Phi\psi$ and $\rho \in \text{Mat}_{1 \times \ell}(\overline{K})$ such that $\rho\psi(\theta) = 0$. Then there exists a vector $P \in \text{Mat}_{1 \times \ell}(\overline{K}[t])$ such that*

$$P\psi = 0 \quad \text{and} \quad P(\theta) = \rho.$$

2.2. First construction of dual t -motives.

We briefly review some constructions of dual t -motives introduced in [10] (see also [7, 12, 14]). Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ be a tuple of positive integers and $\Omega = (Q_1, \dots, Q_r) \in \overline{K}[t]^r$ satisfying the condition

$$(2.1) \quad (\|Q_1\|_\infty / |\theta|_\infty^{\frac{qs_1}{q-1}})^{q^{i_1}} \dots (\|Q_r\|_\infty / |\theta|_\infty^{\frac{qs_r}{q-1}})^{q^{i_r}} \rightarrow 0$$

as $0 \leq i_r < \dots < i_1$ and $i_1 \rightarrow \infty$.

We consider the dual t -motives $\mathcal{M}_{\mathfrak{s}, \Omega}$ and $\mathcal{M}'_{\mathfrak{s}, \Omega}$ attached to (\mathfrak{s}, Ω) given by the matrices

$$\Phi_{\mathfrak{s}, \Omega} = \begin{pmatrix} (t - \theta)^{s_1 + \dots + s_r} & 0 & 0 & \dots & 0 \\ Q_1^{(-1)}(t - \theta)^{s_1 + \dots + s_r} & (t - \theta)^{s_2 + \dots + s_r} & 0 & \dots & 0 \\ 0 & Q_2^{(-1)}(t - \theta)^{s_2 + \dots + s_r} & \ddots & & \vdots \\ \vdots & & \ddots & (t - \theta)^{s_r} & 0 \\ 0 & \dots & 0 & Q_r^{(-1)}(t - \theta)^{s_r} & 1 \end{pmatrix} \in \text{Mat}_{r+1}(\overline{K}[t]),$$

and $\Phi'_{\mathfrak{s}, \Omega} \in \text{Mat}_r(\overline{K}[t])$ is the upper left $r \times r$ sub-matrix of $\Phi_{\mathfrak{s}, \Omega}$.

Throughout this paper, we work with the Carlitz period $\tilde{\pi}$ which is a fundamental period of the Carlitz module (see [13, 24]). We fix a choice of $(q-1)$ st root of $(-\theta)$ and set

$$\Omega(t) := (-\theta)^{-q/(q-1)} \prod_{i \geq 1} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \mathbb{T}^\times$$

so that

$$\Omega^{(-1)} = (t - \theta)\Omega \quad \text{and} \quad \frac{1}{\Omega(\theta)} = \tilde{\pi}.$$

Given $(\mathfrak{s}, \mathfrak{Q})$ as above, Chang introduced the following series (see [7, Lemma 5.3.1] and also [10, Eq. (2.3.2)])

$$(2.2) \quad \mathfrak{L}(\mathfrak{s}; \mathfrak{Q}) = \mathfrak{L}(s_1, \dots, s_r; Q_1, \dots, Q_r) := \sum_{i_1 > \dots > i_r \geq 0} (\Omega^{s_r} Q_r)^{(i_r)} \dots (\Omega^{s_1} Q_1)^{(i_1)}.$$

If we denote \mathcal{E} the ring of series $\sum_{n \geq 0} a_n t^n \in \overline{K}[[t]]_\infty$ such that $\lim_{n \rightarrow +\infty} \sqrt[n]{|a_n|_\infty} = 0$ and $[K_\infty(a_0, a_1, \dots) : K_\infty] < \infty$, then any $f \in \mathcal{E}$ is an entire function. It is proved that $\mathfrak{L}(\mathfrak{s}, \mathfrak{Q}) \in \mathcal{E}$ (see [7, Lemma 5.3.1]). In the sequel, we will use the following crucial property of this series (see [7, Lemma 5.3.5] and [10, Proposition 2.3.3]): for all $j \in \mathbb{Z}^{\geq 0}$, we have

$$(2.3) \quad \mathfrak{L}(\mathfrak{s}; \mathfrak{Q}) \left(\theta^{q^j} \right) = (\mathfrak{L}(\mathfrak{s}; \mathfrak{Q})(\theta))^{q^j}.$$

Then the matrix given by

$$\Psi_{\mathfrak{s}, \mathfrak{Q}} = \begin{pmatrix} \Omega^{s_1 + \dots + s_r} & 0 & 0 & \dots & 0 \\ \mathfrak{L}(s_1; Q_1) \Omega^{s_2 + \dots + s_r} & \Omega^{s_2 + \dots + s_r} & 0 & \dots & 0 \\ \vdots & \mathfrak{L}(s_2; Q_2) \Omega^{s_3 + \dots + s_r} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathfrak{L}(s_1, \dots, s_{r-1}; Q_1, \dots, Q_{r-1}) \Omega^{s_r} & \mathfrak{L}(s_2, \dots, s_{r-1}; Q_2, \dots, Q_{r-1}) \Omega^{s_r} & \dots & \Omega^{s_r} & 0 \\ \mathfrak{L}(s_1, \dots, s_r; Q_1, \dots, Q_r) & \mathfrak{L}(s_2, \dots, s_r; Q_2, \dots, Q_r) & \dots & \mathfrak{L}(s_r; Q_r) & 1 \end{pmatrix} \in \mathrm{GL}_{r+1}(\mathbb{T})$$

satisfies

$$\Psi_{\mathfrak{s}, \mathfrak{Q}}^{(-1)} = \Phi_{\mathfrak{s}, \mathfrak{Q}} \Psi_{\mathfrak{s}, \mathfrak{Q}}.$$

Thus $\Psi_{\mathfrak{s}, \mathfrak{Q}}$ is a rigid analytic trivialization associated to the dual t -motive $\mathcal{M}_{\mathfrak{s}, \mathfrak{Q}}$.

We also denote by $\Psi'_{\mathfrak{s}, \mathfrak{Q}}$ the upper $r \times r$ sub-matrix of $\Psi_{\mathfrak{s}, \mathfrak{Q}}$. It is clear that $\Psi'_{\mathfrak{s}}$ is a rigid analytic trivialization associated to the dual t -motive $\mathcal{M}'_{\mathfrak{s}, \mathfrak{Q}}$.

Further, combined with Eq. (2.3), the above construction of dual t -motives implies that $\tilde{\pi}^w \mathfrak{L}(\mathfrak{s}; \mathfrak{Q})(\theta)$ where $w = s_1 + \dots + s_r$ has the MZ (multizeta) property in the sense of [7, Definition 3.4.1]. By [7, Proposition 4.3.1], we get

Proposition 2.3 ([17, Proposition 2.3]). *Let $(\mathfrak{s}_i; \mathfrak{Q}_i)$ as before for $1 \leq i \leq m$. We suppose that all the tuples of positive integers \mathfrak{s}_i have the same weight, says w . Then the following assertions are equivalent:*

- i) $\mathfrak{L}(\mathfrak{s}_1; \mathfrak{Q}_1)(\theta), \dots, \mathfrak{L}(\mathfrak{s}_m; \mathfrak{Q}_m)(\theta)$ are K -linearly independent.
- ii) $\mathfrak{L}(\mathfrak{s}_1; \mathfrak{Q}_1)(\theta), \dots, \mathfrak{L}(\mathfrak{s}_m; \mathfrak{Q}_m)(\theta)$ are \overline{K} -linearly independent.

2.3. Second construction of dual t -motives.

Based on the first construction, we present another construction which will be useful when working with linear combinations of MZV's or related objects.

Let $w \in \mathbb{N}$ be a positive integer and $n \in \mathbb{N}$. Let $\{(\mathfrak{s}_i; \mathfrak{Q}_i)\}_{1 \leq i \leq n}$ be a collection of pairs satisfying Condition (2.1) such that \mathfrak{s}_i always has weight w . We write $\mathfrak{s}_i = (s_{i1}, \dots, s_{i\ell_i}) \in \mathbb{N}^{\ell_i}$ and $\mathfrak{Q}_i = (Q_{i1}, \dots, Q_{i\ell_i}) \in \overline{K}[t]^{\ell_i}$ so that $s_{i1} + \dots + s_{i\ell_i} = w$. We introduce the set of tuples

$$I(\mathfrak{s}_i; \mathfrak{Q}_i) := \{\emptyset, (s_{i1}; Q_{i1}), \dots, (s_{i1}, \dots, s_{i(\ell_i-1)}; Q_{i1}, \dots, Q_{i(\ell_i-1)})\},$$

and set

$$I := \cup_i I(\mathfrak{s}_i; \mathfrak{Q}_i).$$

Let $a_i \in A \setminus \{0\}$ for all $1 \leq i \leq n$. For all $(\mathbf{t}; \mathbf{\Omega}) \in I$, we set

$$(2.4) \quad f_{\mathbf{t}, \mathbf{\Omega}} := \sum_i a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i}; Q_{i(k+1)}, \dots, Q_{i\ell_i}),$$

where the sum runs through the set of indices i such that $(\mathbf{t}; \mathbf{\Omega}) = (s_{i1}, \dots, s_{ik}; Q_{i1}, \dots, Q_{ik})$ for some $0 \leq k \leq \ell_i - 1$. In particular, $f_\emptyset = \sum_i a_i(t) \mathfrak{L}(s_i; \mathbf{\Omega}_i)$.

We now construct a new matrix Φ' by merging the same rows of $\Phi'_{s_{i1}, \mathbf{\Omega}_1}, \dots, \Phi'_{s_n, \mathbf{\Omega}_n}$ as follows. Then the matrix Φ' will be a matrix indexed by elements of I , says $\Phi' = \left(\Phi'_{(\mathbf{t}; \mathbf{\Omega}), (\mathbf{t}'; \mathbf{\Omega}')} \right)_{(\mathbf{t}; \mathbf{\Omega}), (\mathbf{t}'; \mathbf{\Omega}')} \in \text{Mat}_{|I|}(\overline{K}[t])$. For the row which corresponds to the empty pair \emptyset we put

$$\Phi'_{\emptyset, (\mathbf{t}'; \mathbf{\Omega}')} = \begin{cases} (t - \theta)^w & \text{if } (\mathbf{t}'; \mathbf{\Omega}') = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

For the row indexed by $(\mathbf{t}; \mathbf{\Omega}) = (s_{i1}, \dots, s_{ij}; Q_{i1}, \dots, Q_{ij})$ for some i and $1 \leq j \leq \ell_i - 1$ we put

$$\Phi'_{(\mathbf{t}; \mathbf{\Omega}), (\mathbf{t}'; \mathbf{\Omega}')} = \begin{cases} (t - \theta)^{w-w(\mathbf{t}')} & \text{if } (\mathbf{t}'; \mathbf{\Omega}') = (\mathbf{t}; \mathbf{\Omega}), \\ Q_{ij}^{(-1)} (t - \theta)^{w-w(\mathbf{t}')} & \text{if } (\mathbf{t}'; \mathbf{\Omega}') = (s_{i1}, \dots, s_{i(j-1)}; Q_{i1}, \dots, Q_{i(j-1)}), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\Phi'_{s_i, \mathbf{\Omega}_i} = \left(\Phi'_{(\mathbf{t}; \mathbf{\Omega}), (\mathbf{t}'; \mathbf{\Omega}')} \right)_{(\mathbf{t}; \mathbf{\Omega}), (\mathbf{t}'; \mathbf{\Omega}')} \in I(s_i; \mathbf{\Omega}_i)$ for all i .

We define $\Phi \in \text{Mat}_{|I|+1}(\overline{K}[t])$ by

$$\Phi = \begin{pmatrix} \Phi' & 0 \\ \mathbf{v} & 1 \end{pmatrix} \in \text{Mat}_{|I|+1}(\overline{K}[t]), \quad \mathbf{v} = (v_{\mathbf{t}, \mathbf{\Omega}})_{(\mathbf{t}; \mathbf{\Omega}) \in I} \in \text{Mat}_{1 \times |I|}(\overline{K}[t]),$$

where

$$v_{\mathbf{t}, \mathbf{\Omega}} = \begin{cases} a_i(t) Q_{i\ell_i}^{(-1)} (t - \theta)^{w-w(\mathbf{t})} & \text{if } (\mathbf{t}; \mathbf{\Omega}) = (s_{i1}, \dots, s_{i(\ell_i-1)}; Q_{i1}, \dots, Q_{i(\ell_i-1)}), \\ 0 & \text{otherwise.} \end{cases}$$

We now introduce a rigid analytic trivialization matrix Ψ for Φ . We define $\Psi' = \left(\Psi'_{(\mathbf{t}; \mathbf{\Omega}), (\mathbf{t}'; \mathbf{\Omega}')} \right)_{(\mathbf{t}; \mathbf{\Omega}), (\mathbf{t}'; \mathbf{\Omega}')} \in \text{GL}_{|I|}(\mathbb{T})$ as follows. For the row which corresponds to the empty pair \emptyset we define

$$\Psi'_{\emptyset, (\mathbf{t}'; \mathbf{\Omega}')} = \begin{cases} \Omega^w & \text{if } (\mathbf{t}'; \mathbf{\Omega}') = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

For the row indexed by $(\mathbf{t}; \mathbf{\Omega}) = (s_{i1}, \dots, s_{ij}; Q_{i1}, \dots, Q_{ij})$ for some i and $1 \leq j \leq \ell_i - 1$ we put

$$\Psi'_{(\mathbf{t}; \mathbf{\Omega}), (\mathbf{t}'; \mathbf{\Omega}')} = \begin{cases} \mathfrak{L}(\mathbf{t}; \mathbf{\Omega}) \Omega^{w-w(\mathbf{t}')} & \text{if } (\mathbf{t}'; \mathbf{\Omega}') = \emptyset, \\ \mathfrak{L}(s_{i(k+1)}, \dots, s_{ij}; Q_{i(k+1)}, \dots, Q_{ij}) \Omega^{w-w(\mathbf{t}')} & \text{if } (\mathbf{t}'; \mathbf{\Omega}') = (s_{i1}, \dots, s_{ik}; Q_{i1}, \dots, Q_{ik}) \text{ for some } 1 \leq k \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\Psi'_{s_i, \mathbf{\Omega}_i} = \left(\Psi'_{(\mathbf{t}; \mathbf{\Omega}), (\mathbf{t}'; \mathbf{\Omega}')} \right)_{(\mathbf{t}; \mathbf{\Omega}), (\mathbf{t}'; \mathbf{\Omega}')} \in I(s_i; \mathbf{\Omega}_i)$ for all i .

We define $\Psi \in \mathrm{GL}_{|I|+1}(\mathbb{T})$ by

$$\Psi = \begin{pmatrix} \Psi' & 0 \\ \mathbf{f} & 1 \end{pmatrix} \in \mathrm{GL}_{|I|+1}(\mathbb{T}), \quad \mathbf{f} = (f_{\mathbf{t}, \mathfrak{Q}})_{\mathbf{t} \in I} \in \mathrm{Mat}_{1 \times |I|}(\mathbb{T}).$$

Here we recall (see Eq. (2.4))

$$f_{\mathbf{t}, \mathfrak{Q}} = \sum_i a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i}; Q_{i(k+1)}, \dots, Q_{i\ell_i})$$

where the sum runs through the set of indices i such that $(\mathbf{t}; \mathfrak{Q}) = (s_{i1}, \dots, s_{ik}; Q_{i1}, \dots, Q_{ik})$ for some $0 \leq k \leq \ell_i - 1$. In particular,

$$f_{\emptyset} = \sum_i a_i(t) \mathfrak{L}(\mathfrak{s}_i; \mathfrak{Q}_i).$$

By construction and by §2.2, we get $\Psi^{(-1)} = \Phi\Psi$, that means Ψ is a rigid analytic trivialization for Φ .

2.4. A result for linear independence.

We are now ready to state an application of ABP criterion for linear independence (see [17, 22] and also [9]).

Theorem 2.4. *We keep the above notation. We suppose further that $\{(\mathfrak{s}_i; \mathfrak{Q}_i)\}_{1 \leq i \leq n}$ satisfies the following conditions:*

- (LW) *For any weight $w' < w$, the values $\mathfrak{L}(\mathbf{t}; \mathfrak{Q})(\theta)$ with $(\mathbf{t}; \mathfrak{Q}) \in I$ and $w(\mathbf{t}) = w'$ are all K -linearly independent. In particular, $\mathfrak{L}(\mathbf{t}; \mathfrak{Q})(\theta)$ is always nonzero.*
- (LD) *There exist $a_i \in A \setminus \{0\}$ for $1 \leq i \leq n$ such that*

$$\sum_{i=1}^n a_i \mathfrak{L}(\mathfrak{s}_i; \mathfrak{Q}_i)(\theta) = 0.$$

Then

- $f_{\mathbf{t}, \mathfrak{Q}}(\theta)$ belongs to K where $f_{\mathbf{t}, \mathfrak{Q}}$ is given as in (2.4) for all $(\mathbf{t}; \mathfrak{Q}) \in I$.
- There exist $b \in \mathbb{F}_q[t] \setminus \{0\}$ and $\delta := (\delta_{\mathbf{t}, \mathfrak{Q}})_{(\mathbf{t}; \mathfrak{Q}) \in I} \in \mathrm{Mat}_{1 \times |I|}(\overline{K}[t])$ such that

$$\begin{pmatrix} \mathrm{Id} & 0 \\ \delta & 1 \end{pmatrix}^{(-1)} \begin{pmatrix} \Phi' & 0 \\ b\mathbf{v} & 1 \end{pmatrix} = \begin{pmatrix} \Phi' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ \delta & 1 \end{pmatrix}.$$

Here we note that LW stands for Lower Weights and LD for Linear Dependence. We mention below some comments about the above result.

- (1) In [10] Chang, Papanikolas and Yu studied zeta-like MZV's and proved Theorem 2.4 in the case where $n = 2$, \mathfrak{s}_1 is of weight w , and $\mathfrak{s}_2 = (w)$ (see Theorem 2.5.2 of *loc. cit.*).
- (2) In [18, 21], the authors apply the theorem of Chang-Papanikolas-Yu to obtain several results concerning zeta-like MZV's.
- (3) In [8, 12], the authors generalized the aforementioned theorem of Chang-Papanikolas-Yu and obtain some particular cases of Theorem 2.4. In all these works, Condition (LW) is replaced by a stronger condition: one requires that for any weight $w' < w$, there exists at most one pair $(\mathbf{t}; \mathfrak{Q}) \in I$ such that $w(\mathbf{t}) = w'$ and further $\mathfrak{L}(\mathbf{t}; \mathfrak{Q})(\theta)$ is always nonzero for all $(\mathbf{t}; \mathfrak{Q}) \in I$.

- (4) In [22], the author proved a version of Theorem 2.4 for MZV's (see Theorem 2.5) and applied it to obtain some results towards Zagier-Hoffman's conjectures in positive characteristic. Theorem 2.4 is a straightforward generalization of the previous result and is proved in [17, Theorem 2.4] (see also [9]).
- (5) The curious reader is invited to read the previous works for more details and an exhaustive list of related works.

2.5. Example: dual t -motives connected to MZV's.

2.5.1. First construction.

Following Anderson and Thakur [3, 4] we introduce dual t -motives connected to MZV's and AMZV's. We briefly review Anderson-Thakur polynomials introduced in the seminal work of Anderson and Thakur [3]. For $k \geq 0$, we set $[k] := \theta^{q^k} - \theta$ and $D_k := \prod_{\ell=1}^k [\ell]^{q^{k-\ell}}$. For $n \in \mathbb{N}$ we write $n-1 = \sum_{j \geq 0} n_j q^j$ with $0 \leq n_j \leq q-1$ and define

$$\Gamma_n := \prod_{j \geq 0} D_j^{n_j}.$$

We set $\gamma_0(t) := 1$ and $\gamma_j(t) := \prod_{\ell=1}^j (\theta^{q^\ell} - t^{q^\ell})$ for $j \geq 1$. Then Anderson-Thakur polynomials $\alpha_n(t) \in A[t]$ are given by the generating series

$$\sum_{n \geq 1} \frac{\alpha_n(t)}{\Gamma_n} x^n := x \left(1 - \sum_{j \geq 0} \frac{\gamma_j(t)}{D_j} x^{q^j} \right)^{-1}.$$

Finally, we define $H_n(t)$ by switching θ and t

$$(2.5) \quad H_n(t) = \alpha_n(t) \Big|_{t=\theta, \theta=t}.$$

By [3, Eq. (3.7.3)] we get

$$(2.6) \quad \deg_\theta H_n \leq \frac{(n-1)q}{q-1} < \frac{nq}{q-1}.$$

Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ be a tuple and $\epsilon = (\epsilon_1, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$. For all $1 \leq i \leq r$ we fix a fixed $(q-1)$ -th root γ_i of $\epsilon_i \in \mathbb{F}_q^\times$ and set $Q_{s_i, \epsilon_i} := \gamma_i H_{s_i}$. Then we set $\Omega_{\mathfrak{s}, \epsilon} := (Q_{s_1, \epsilon_1}, \dots, Q_{s_r, \epsilon_r})$ and put $\mathfrak{L}(\mathfrak{s}; \epsilon) := \mathfrak{L}(\mathfrak{s}; \Omega_{\mathfrak{s}, \epsilon})$. By (2.6) we know that $\|H_n\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$ for all $n \in \mathbb{N}$, thus $\Omega_{\mathfrak{s}, \epsilon}$ satisfies Condition (2.1). Thus we can define the dual t -motives $\mathcal{M}_{\mathfrak{s}, \epsilon} = \mathcal{M}_{\mathfrak{s}, \Omega_{\mathfrak{s}, \epsilon}}$ and $\mathcal{M}'_{\mathfrak{s}, \epsilon} = \mathcal{M}'_{\mathfrak{s}, \Omega_{\mathfrak{s}, \epsilon}}$ attached to \mathfrak{s} whose matrices and rigid analytic trivializations will be denoted by $(\Phi_{\mathfrak{s}, \epsilon}, \Psi_{\mathfrak{s}, \epsilon})$ and $(\Phi'_{\mathfrak{s}, \epsilon}, \Psi'_{\mathfrak{s}, \epsilon})$, respectively. These dual t -motives are connected to MZV's and AMZV's by the following result (see [12, Proposition 2.12] for more details):

$$(2.7) \quad \mathfrak{L}(\mathfrak{s}; \epsilon)(\theta) = \frac{\gamma_1 \dots \gamma_r \Gamma_{s_1} \dots \Gamma_{s_r} \zeta_A \left(\begin{smallmatrix} \epsilon \\ \mathfrak{s} \end{smallmatrix} \right)}{\widetilde{\pi}^{s_1 + \dots + s_r}}.$$

By a result of Thakur [26], one can show (see [14, Theorem 2.1]) that $\zeta_A \left(\begin{smallmatrix} \epsilon \\ \mathfrak{s} \end{smallmatrix} \right) \neq 0$.

Thus $\mathfrak{L}(\mathfrak{s}; \epsilon)(\theta) \neq 0$.

2.5.2. Second construction.

Let $w \in \mathbb{N}$ be a positive integer and $n \in \mathbb{N}$. Let $\{(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)\}_{1 \leq i \leq n}$ be a collection of pairs such that \mathfrak{s}_i always has weight w and $a_i \in A \setminus \{0\}$ for all $1 \leq i \leq n$. Recall from §2.5 that the collection $\{(\mathfrak{s}_i; \boldsymbol{\Omega}_{\mathfrak{s}_i; \boldsymbol{\epsilon}_i})\}_{1 \leq i \leq n}$ satisfies Condition (2.1). We write $\mathfrak{s}_i = (s_{i1}, \dots, s_{i\ell_i}) \in \mathbb{N}^{\ell_i}$ and $\boldsymbol{\epsilon}_i = (\epsilon_{i1}, \dots, \epsilon_{i\ell_i}) \in (\mathbb{F}_q^\times)^{\ell_i}$ so that $s_{i1} + \dots + s_{i\ell_i} = w$.

We introduce the set of tuples

$$\begin{aligned} I(\mathfrak{s}_i; \boldsymbol{\epsilon}_i) &:= \{\emptyset, (s_{i1}; \epsilon_{i1}), \dots, (s_{i1}, \dots, s_{i(\ell_i-1)}; \epsilon_{i1}, \dots, \epsilon_{i(\ell_i-1)})\}, \\ I^+(\mathfrak{s}_i; \boldsymbol{\epsilon}_i) &:= I(\mathfrak{s}_i; \boldsymbol{\epsilon}_i) \cup \{(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)\}, \end{aligned}$$

and set

$$I := \cup_i I(\mathfrak{s}_i; \boldsymbol{\epsilon}_i), \quad I^+ := \cup_i I^+(\mathfrak{s}_i; \boldsymbol{\epsilon}_i).$$

For all $(\mathbf{t}; \boldsymbol{\epsilon}) \in I$, we set

$$(2.8) \quad f_{\mathbf{t}, \boldsymbol{\epsilon}} := \sum_i a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i}; \epsilon_{i(k+1)}, \dots, \epsilon_{i\ell_i}),$$

where the sum runs through the set of indices i such that $(\mathbf{t}; \boldsymbol{\epsilon}) = (s_{i1}, \dots, s_{ik}; \epsilon_{i1}, \dots, \epsilon_{ik})$ for some $0 \leq k \leq \ell_i - 1$. In particular,

$$f_\emptyset = \sum_i a_i(t) \mathfrak{L}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i).$$

By §2.3 we get a new matrix Φ' by merging the same rows of $\Phi'_{\mathfrak{s}_1, \boldsymbol{\epsilon}_1}, \dots, \Phi'_{\mathfrak{s}_n, \boldsymbol{\epsilon}_n}$. Then we obtain $\Phi \in \text{Mat}_{|I|+1}(\overline{K}[t])$ by

$$(2.9) \quad \Phi = \begin{pmatrix} \Phi' & 0 \\ \mathbf{v} & 1 \end{pmatrix} \in \text{Mat}_{|I|+1}(\overline{K}[t]), \quad \mathbf{v} = (v_{\mathbf{t}, \boldsymbol{\epsilon}})_{(\mathbf{t}; \boldsymbol{\epsilon}) \in I} \in \text{Mat}_{1 \times |I|}(\overline{K}[t]),$$

where

$$v_{\mathbf{t}, \boldsymbol{\epsilon}} := \begin{cases} a_i(t) Q_{\mathfrak{s}_i; \boldsymbol{\epsilon}_i}^{(-1)}(t - \theta)^{w-w(\mathbf{t})} & \text{if } (\mathbf{t}; \boldsymbol{\epsilon}) = (s_{i1}, \dots, s_{i(\ell_i-1)}; \epsilon_{i1}, \dots, \epsilon_{i(\ell_i-1)}), \\ 0 & \text{otherwise.} \end{cases}$$

Next we get a rigid analytic trivialization matrix $\Psi \in \text{GL}_{|I|+1}(\mathbb{T})$ for Φ which is of the form

$$\Psi = \begin{pmatrix} \Psi' & 0 \\ \mathbf{f} & 1 \end{pmatrix} \in \text{GL}_{|I|+1}(\mathbb{T})$$

where $\Psi' = \left(\Psi'_{(\mathbf{t}; \boldsymbol{\epsilon}), (\mathbf{t}'; \boldsymbol{\epsilon}')} \right)_{(\mathbf{t}; \boldsymbol{\epsilon}), (\mathbf{t}'; \boldsymbol{\epsilon}') \in I} \in \text{GL}_{|I|}(\mathbb{T})$ and $\mathbf{f} = (f_{\mathbf{t}, \boldsymbol{\epsilon}})_{\mathbf{t} \in I} \in \text{Mat}_{1 \times |I|}(\mathbb{T})$ with $f_{\mathbf{t}, \boldsymbol{\epsilon}}$ given by Eq. (2.8).

2.5.3. Linear independence for MZV's.

Theorem 2.4 applied to this case yields

Theorem 2.5. *We keep the above notation. We suppose further that $\{(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)\}_{1 \leq i \leq n}$ satisfies the following conditions:*

(LW) *For any weight $w' < w$, the values $\mathfrak{L}(\mathbf{t}; \boldsymbol{\epsilon})(\theta)$ with $(\mathbf{t}; \boldsymbol{\epsilon}) \in I$ and $w(\mathbf{t}) = w'$ are all K -linearly independent.*

(LD) *There exist $a_i \in A \setminus \{0\}$ for $1 \leq i \leq n$ such that*

$$\sum_{i=1}^n a_i \mathfrak{L}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0.$$

Then

- 1) $f_{\mathbf{t}, \epsilon}(\theta)$ belongs to K where $f_{\mathbf{t}, \epsilon}$ is given as in (2.8) for all $(\mathbf{t}; \epsilon) \in I$.
 2) There exist $b \in \mathbb{F}_q[t] \setminus \{0\}$ and $\delta := (\delta_{\mathbf{t}, \epsilon})_{(\mathbf{t}; \epsilon) \in I} \in \text{Mat}_{1 \times |I|}(\overline{K}[t])$ such that

$$(2.10) \quad \begin{pmatrix} Id & 0 \\ \delta & 1 \end{pmatrix}^{(-1)} \begin{pmatrix} \Phi' & 0 \\ b\mathbf{v} & 1 \end{pmatrix} = \begin{pmatrix} \Phi' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Id & 0 \\ \delta & 1 \end{pmatrix}.$$

From now on, as we will work with MZV's and AMZV's, we will use freely the notation given as in this section.

3. PROOF OF PROPOSITIONS 1.3 AND 1.4

3.1. Proof of Proposition 1.3.

The proof of Proposition 1.3 follows exactly the same line as that of [17, §1.4]. In fact, we consider the set \mathcal{AT}_w^2 consisting of positive arrays $\begin{pmatrix} \epsilon_1 & \cdots & \epsilon_n \\ s_1 & \cdots & s_n \end{pmatrix}$ of weight w such that $q \nmid s_i$ for all i and define a map

$$\varphi: \mathcal{AT}_w^2 \longrightarrow \mathcal{AT}_w^1$$

as follows: for each array $\begin{pmatrix} \epsilon \\ \mathbf{s} \end{pmatrix} = \begin{pmatrix} \epsilon_1 & \cdots & \epsilon_n \\ s_1 & \cdots & s_n \end{pmatrix} \in \mathcal{AT}_w^2$, since $q \nmid s_i$, we can write $s_i = h_i q + r_i$ where $0 < r_i < q$ and $h_i \in \mathbb{Z}^{\geq 0}$. The image $\varphi \begin{pmatrix} \epsilon \\ \mathbf{s} \end{pmatrix}$ is the array

$$\varphi \begin{pmatrix} \epsilon \\ \mathbf{s} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \cdots & 1 \\ q & \cdots & q \end{pmatrix}}_{h_1 \text{ times}} \begin{pmatrix} \epsilon_1 \\ r_1 \end{pmatrix} \cdots \underbrace{\begin{pmatrix} 1 & \cdots & 1 \\ q & \cdots & q \end{pmatrix}}_{h_n \text{ times}} \begin{pmatrix} \epsilon_n \\ r_n \end{pmatrix}.$$

Then one sees that φ is a bijection, hence $|\mathcal{AT}_w^1| = |\mathcal{AT}_w^2|$. It is easy to see that $|\mathcal{AT}_w^2| = s(w)$ for all $w \in \mathbb{N}$. The proposition follows.

3.2. Proof of Proposition 1.4.

Part 2 follows immediately from the construction of \mathcal{AT}'_w . For Part 1, it is easy to construct a bijection between \mathcal{AT}'_w and \mathcal{AT}_w^2 . We leave it as an exercise for the reader.

4. PROOF OF THEOREM 1.5

4.1. Preliminaries.

We begin this section by proving several auxiliary lemmas which will be useful in the sequel. Here we can refer to the readers [17, §3].

Lemma 4.1 ([17, Lemma 3.1]). *Let $\epsilon_i \in \mathbb{F}_q^\times$ be different elements. We denote by $\gamma_i \in \overline{\mathbb{F}_q}$ a $(q-1)$ -th root of ϵ_i . Then γ_i are all \mathbb{F}_q -linearly independent.*

Proof. We refer the reader to [17, Lemma 3.1] for a proof of this lemma. We give here another proof using the Moore determinant (see [13, Section 1.3]). Suppose that $\epsilon_1, \dots, \epsilon_n$ are distinct elements in \mathbb{F}_q^\times . The Moore determinant associated to $\gamma_1, \dots, \gamma_n$ is defined as

$$\Delta(\gamma_1, \dots, \gamma_n) := \det \begin{pmatrix} \gamma_1 & \cdots & \gamma_n \\ \gamma_1^q & \cdots & \gamma_n^q \\ \vdots & & \vdots \\ \gamma_1^{q^{n-1}} & \cdots & \gamma_n^{q^{n-1}} \end{pmatrix}.$$

Since γ_i is a $(q-1)$ -th root of ϵ_i , one verifies by induction that $\gamma_i^{q^k} = \gamma_i \epsilon_i^k$ for $0 \leq k \leq n-1$. It follows that

$$\begin{aligned} \Delta(\gamma_1, \dots, \gamma_n) &= \det \begin{pmatrix} \gamma_1 & \dots & \gamma_n \\ \gamma_1 \epsilon_1 & \dots & \gamma_n \epsilon_n \\ \vdots & & \vdots \\ \gamma_1 \epsilon_1^{n-1} & \dots & \gamma_n \epsilon_n^{n-1} \end{pmatrix} = \gamma_1 \cdots \gamma_n \det \begin{pmatrix} 1 & \dots & 1 \\ \epsilon_1 & \dots & \epsilon_n \\ \vdots & & \vdots \\ \epsilon_1^{n-1} & \dots & \epsilon_n^{n-1} \end{pmatrix} \\ &= \gamma_1 \cdots \gamma_n \prod_{1 \leq j < i \leq n} (\epsilon_i - \epsilon_j). \end{aligned}$$

The last equality follows from the Vandermonde determinant. We thus deduce that $\Delta(\gamma_1, \dots, \gamma_n) \neq 0$ since $\epsilon_1, \dots, \epsilon_n$ are distinct elements in \mathbb{F}_q^\times . From [13, Lemma 1.3.3], we conclude that $\gamma_1, \dots, \gamma_n$ are linearly independent over \mathbb{F}_q , as desired. \square

Lemma 4.2. *Let $\zeta_A \begin{pmatrix} \epsilon_i \\ \mathfrak{s}_i \end{pmatrix} \in \mathcal{AT}_w^0$ and $a_i \in A \setminus \{0\}$ satisfying*

$$\sum_i a_i \mathfrak{L}(\mathfrak{s}_i; \epsilon_i)(\theta) = 0.$$

For $\epsilon \in \mathbb{F}_q^\times$ we denote by $I(\epsilon) = \{i : \chi(\epsilon_i) = \epsilon\}$ the set of pairs such that the corresponding character equals ϵ . Then for all $\epsilon \in \mathbb{F}_q^\times$,

$$\sum_{i \in I(\epsilon)} a_i \mathfrak{L}(\mathfrak{s}_i; \epsilon_i)(\theta) = 0.$$

Proof. See [17, Lemma 3.2]. \square

Lemma 4.3. *Let $m \in \mathbb{N}$, $\delta \in \overline{K}[t]$ and $F(t, \theta) \in \overline{\mathbb{F}}_q[t, \theta]$ (resp. $F(t, \theta) \in \mathbb{F}_q[t, \theta]$) satisfying*

$$\delta = \delta^{(-1)}(t - \theta)^m + F^{(-1)}(t, \theta).$$

Then $\delta \in \overline{\mathbb{F}}_q[t, \theta]$ (resp. $\delta \in \mathbb{F}_q[t, \theta]$) and

$$\deg_\theta \delta \leq \max \left\{ \frac{qm}{q-1}, \frac{\deg_\theta F(t, \theta)}{q} \right\}.$$

Proof. The proof follows the same line as that of [18, Theorem 2]. See also [17, Lemma 3.3]. \square

4.2. Proof of Theorem 1.5.

The proof of Theorem 1.5 follows the same line as that of [22, Theorem B].

By (2.7) it suffices to show that all the values $\mathfrak{L}(\mathfrak{s}; \epsilon)(\theta)$ such that $\zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} \in \mathcal{AT}_w^0$ are K -linearly independent. The proof is by induction on the weight $w \in \mathbb{N}$.

For $w = 1$ suppose that there exist $\zeta_A \begin{pmatrix} \epsilon_i \\ 1 \end{pmatrix} \in \mathcal{AT}_1^0$ and $a_i \in A \setminus \{0\}$ such that

$$\sum_i a_i \mathfrak{L}(1; \epsilon_i)(\theta) = 0.$$

Then

$$\sum_i a_i \gamma_i \zeta_A \begin{pmatrix} \epsilon_i \\ 1 \end{pmatrix} = 0.$$

Since $\zeta_A \begin{pmatrix} \epsilon_i \\ 1 \end{pmatrix} \in K_\infty^\times$, we get a contradiction.

Now suppose that for $w' < w$ the values $\mathfrak{L}(\mathfrak{s}; \epsilon)(\theta)$ such that $\zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} \in \mathcal{AT}_{w'}^0$ are all linearly independent over K . We claim that the values $\mathfrak{L}(\mathfrak{s}; \epsilon)(\theta)$ such that $\zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} \in \mathcal{AT}_w^0$ are all linearly independent over K . Suppose that there exist $\zeta_A \begin{pmatrix} \epsilon_i \\ \mathfrak{s}_i \end{pmatrix} \in \mathcal{AT}_w^0$ and $a_i \in A \setminus \{0\}$ such that

$$(4.1) \quad \sum_i a_i \mathfrak{L}(\mathfrak{s}_i; \epsilon_i)(\theta) = 0.$$

We write $\mathfrak{s}_i = (s_{i1}, \dots, s_{i\ell_i}) \in \mathbb{N}^{\ell_i}$ and $\epsilon_i = (\epsilon_{i1}, \dots, \epsilon_{i\ell_i}) \in (\mathbb{F}_q^\times)^{\ell_i}$. Then by (2.7) and the fact that \mathfrak{s}_i has the same weight w , we get

$$\sum_i a_i \gamma_{i1} \dots \gamma_{i\ell_i} \Gamma_{s_{i1}} \dots \Gamma_{s_{i\ell_i}} \zeta_A \begin{pmatrix} \epsilon_i \\ \mathfrak{s}_i \end{pmatrix} = 0.$$

By Lemma 4.2 we can suppose further that ϵ_i has the same character, i.e., there exists $\epsilon \in \mathbb{F}_q^\times$ such that for all i ,

$$(4.2) \quad \chi(\epsilon_i) = \epsilon_{i1} \dots \epsilon_{i\ell_i} = \epsilon.$$

We show that the relation (4.1) leads to a contradiction.

By the previous discussion, we can apply Theorem 2.5 for the above tuples $(\mathfrak{s}_i; \epsilon_i)$. With the notation of Theorem 2.5, we deduce that that for all $(\mathfrak{t}; \epsilon) \in I$, $f_{\mathfrak{t}, \epsilon}(\theta)$ belongs to K . Here we recall that $f_{\mathfrak{t}, \epsilon}$ is given by

$$f_{\mathfrak{t}, \epsilon} := \sum_i a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i}; \epsilon_{i(k+1)}, \dots, \epsilon_{i\ell_i}),$$

where the sum runs through the set of indices i such that $(\mathfrak{t}; \epsilon) = (s_{i1}, \dots, s_{ik}; \epsilon_{i1}, \dots, \epsilon_{ik})$ for some $0 \leq k \leq \ell_i - 1$.

We now deduce a contradiction from this fact. For $(\mathfrak{s}_i; \epsilon_i)$ we denote by $m_i \in \mathbb{Z}^{\geq 0}$ the biggest index such that $(s_{i1}, \dots, s_{im_i}; \epsilon_{i1}, \dots, \epsilon_{im_i})$ belongs to $I(\mathfrak{s}_j; \epsilon_j)$ for some $j \neq i$. Note that $s_{i(m_i+1)} \neq 0$ since the weights of s_i and s_j ($j \neq i$) are the same. For $m_i + 1 \leq k < \ell_i$ we consider $(\mathfrak{t}; \epsilon) = (s_{i1}, \dots, s_{ik}; \epsilon_{i1}, \dots, \epsilon_{ik})$. Thus (2.8) gives $f_{\mathfrak{t}, \epsilon} = a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i}; \epsilon_{i(k+1)}, \dots, \epsilon_{i\ell_i})$ and we have seen that $f_{\mathfrak{t}, \epsilon}$ belongs to K . By (2.7) we get

$$a_i \gamma_{i(k+1)} \dots \gamma_{i\ell_i} \Gamma_{i(k+1)} \dots \Gamma_{i\ell_i} \zeta_A \begin{pmatrix} \epsilon_{i(k+1)} & \dots & \epsilon_{i\ell_i} \\ s_{i(k+1)} & \dots & s_{i\ell_i} \end{pmatrix} / \tilde{\pi}^{s_{i(k+1)} + \dots + s_{i\ell_i}} \in K.$$

Since $a_i \in K^*$, this forces that $s_{i(k+1)} + \dots + s_{i\ell_i}$ is divisible by $q - 1$ and $\epsilon_{i(k+1)} \dots \epsilon_{i\ell_i} = 1$. Since this condition holds for all $m_i + 1 \leq k < \ell_i$, it follows that $s_{i(k+1)}$ is divisible by $q - 1$ and $\epsilon_{i(k+1)} = 1$ for all $m_i + 1 \leq k < \ell_i$. Since $1 \leq s_{i(k+1)} \leq q - 1$, we conclude that

$$s_{i(k+1)} = q - 1, \quad \epsilon_{i(k+1)} = 1, \quad \text{for all } m_i + 1 \leq k < \ell_i.$$

Let $m := \max_i m_i$. Then we can find two different tuples, says $(\mathfrak{s}_1; \epsilon_1)$ and $(\mathfrak{s}_2; \epsilon_2)$ be such that $m_1 = m_2 = m$. Thus

$$(s_{11}, \dots, s_{1m}; \epsilon_{11}, \dots, \epsilon_{1m}) = (s_{21}, \dots, s_{2m}; \epsilon_{21}, \dots, \epsilon_{2m})$$

and $(s_{1(m+1)}, \epsilon_{1(m+1)}) \neq (s_{2(m+1)}, \epsilon_{2(m+1)})$. The previous discussion shows that $s_{1(m+2)} = \cdots = q-1$, $\epsilon_{1(m+2)} = \cdots = 1$ and $s_{2(m+2)} = \cdots = q-1$, $\epsilon_{2(m+2)} = \cdots = 1$. Combining with the fact that \mathfrak{s}_1 and \mathfrak{s}_2 have the same weight yields $s_{1(m+1)} \equiv s_{2(m+1)} \pmod{q-1}$. Since $1 \leq s_{1(m+1)}, s_{2(m+1)} \leq q-1$, we deduce $s_{1(m+1)} = s_{2(m+1)}$. Finally, combining with the fact that ϵ_1 and ϵ_2 have the same character (see (4.2)), we obtain $\epsilon_{1(m+1)} = \epsilon_{2(m+1)}$. Thus we get a contradiction since $(s_{1(m+1)}; \epsilon_{1(m+1)}) = (s_{2(m+1)}; \epsilon_{2(m+1)})$.

The proof of Theorem 1.5 is finished.

5. PROOF OF THEOREM 1.6

5.1. Strategy of the proof.

Recall that $w \leq 3q-2$. We use Proposition 1.4 and prove Theorem 1.6 by induction on w .

For $w \leq q$, Theorem 1.6 follows immediately from Theorem 1.5 (see §5.2).

We suppose that for $w' < w$ AMZV's in $\mathcal{AT}'_{w'}$ are all linearly independent over K . We claim that AMZV's in \mathcal{AT}'_w are all linearly independent over K .

By (2.7) it suffices to show that all the values $\mathfrak{L}(\mathfrak{s}; \epsilon)(\theta)$ such that $\zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} \in \mathcal{AT}'_w$ are all linearly independent over K . Suppose that there exist $\zeta_A \begin{pmatrix} \epsilon_i \\ \mathfrak{s}_i \end{pmatrix} \in \mathcal{AT}'_w$ and $a_i \in A \setminus \{0\}$ such that

$$\sum_i a_i \mathfrak{L}(\mathfrak{s}_i; \epsilon_i)(\theta) = 0.$$

As Proposition 1.4 implies that Condition (LW) holds, we apply Theorem 2.5 for the tuples $(\mathfrak{s}_i; \epsilon_i)$. Part 1 of Theorem 2.5 gives a shortlist of tuples $(\mathfrak{s}_i; \epsilon_i)$ which often contains few elements. Then we write down explicitly the system of equations (2.10) given in Part 2 of Theorem 2.5 (we note that in this system we can suppose that $b=1$) and manage to show that it does not have a nontrivial solution $\delta \neq 0$. This completes the proof.

In what follows we carry out this strategy in detail.

5.2. Weight $w \leq q$.

Proposition 5.1. *Suppose that $w \leq q$. Then AMZV's in \mathcal{AT}'_w are all linearly independent over K .*

Proof. As a direct consequence of Theorem 1.5, we deduce that Theorem 1.6 holds when $w \leq q$. In fact, we note that $\mathcal{AT}'_w = \mathcal{AT}^1_w = \mathcal{AT}^0_w$ when $w \leq q$. \square

5.3. Weight $q+1 \leq w \leq 2q-2$.

Proposition 5.2. *Suppose that $q > 2$ and $q+1 \leq w \leq 2q-2$. Then AMZV's in \mathcal{AT}'_w are all linearly independent over K .*

Proof. The proof is done by induction on w . For $w \leq 2q-2$, we suppose that for $w' < w$ AMZV's in $\mathcal{AT}'_{w'}$ are all linearly independent over K . We claim that AMZV's in \mathcal{AT}'_w are all linearly independent over K .

By (2.7) it suffices to show that all the values $\mathfrak{L}(\mathfrak{s}; \epsilon)(\theta)$ such that $\zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} \in \mathcal{AT}'_w$ are all linearly independent over K . Suppose that there exist $\zeta_A \begin{pmatrix} \epsilon_i \\ \mathfrak{s}_i \end{pmatrix} \in \mathcal{AT}'_w$ and

$a_i \in A \setminus \{0\}$ such that

$$\sum_i a_i \mathfrak{L}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0.$$

We write $\mathfrak{s}_i = (s_{i1}, \dots, s_{i\ell_i}) \in \mathbb{N}^{\ell_i}$ and $\boldsymbol{\epsilon}_i = (\epsilon_{i1}, \dots, \epsilon_{i\ell_i}) \in (\mathbb{F}_q^\times)^{\ell_i}$. Then by (2.7) and the fact that \mathfrak{s}_i has the same weight w , we get

$$\sum_i a_i \gamma_{i1} \dots \gamma_{i\ell_i} \Gamma_{s_{i1}} \dots \Gamma_{s_{i\ell_i}} \zeta_A \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{s}_i \end{pmatrix} = 0.$$

As before, by Lemma 4.2 we can suppose further that $\boldsymbol{\epsilon}_i$ has the same character, i.e., there exists $\epsilon \in \mathbb{F}_q^\times$ such that for all i ,

$$\chi(\boldsymbol{\epsilon}_i) = \epsilon_{i1} \dots \epsilon_{i\ell_i} = \epsilon.$$

We show that this relation leads to a contradiction.

We apply Theorem 2.5 for the above tuples $(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)$. With the notation of Theorem 2.5, we deduce that for all $(\mathbf{t}; \boldsymbol{\epsilon}) \in I$, $f_{\mathbf{t}, \boldsymbol{\epsilon}}(\theta)$ belongs to K . Here we recall that $f_{\mathbf{t}, \boldsymbol{\epsilon}}$ is given by

$$f_{\mathbf{t}, \boldsymbol{\epsilon}} = \sum_i a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i}; \epsilon_{i(k+1)}, \dots, \epsilon_{i\ell_i}),$$

where the sum runs through the set of indices i such that $(\mathbf{t}; \boldsymbol{\epsilon}) = (s_{i1}, \dots, s_{ik}; \epsilon_{i1}, \dots, \epsilon_{ik})$ for some $0 \leq k \leq \ell_i - 1$.

We have $(q-1)|(w-w(\mathbf{t}))$; assume not, then $\tilde{\pi}^{w-w(\mathbf{t})} \notin K_\infty$, so

$$\sum_{i \in J(\mathbf{t})} a_i(t) \gamma_{i(k+1)} \dots \gamma_{i\ell_i} \Gamma_{s_{i(k+1)}} \dots \Gamma_{s_{i\ell_i}} \zeta_A \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{s} \end{pmatrix} = 0,$$

which contradicts to the induction hypothesis on $\mathcal{A}\mathcal{J}'_{w-w(\mathbf{t})}$.

Thus, we have $(q-1)|s_{i2}, s_{i3}, \dots, s_{i\ell_i}$. With our assumption $w \leq 2q-2$, it suffices to consider either $\mathfrak{s} = (w)$ or $\mathfrak{s} = (k, q-1)$ where $k = w - (q-1)$ with $1 < k \leq q-1$. We need to consider only $(w; \epsilon)$ or $(k, q-1; \epsilon, 1)$ as in the proof of Theorem 1.5. Thus we have the following relation,

$$(5.1) \quad a_1 \zeta_A \begin{pmatrix} \epsilon \\ w \end{pmatrix} + a_2 \zeta_A \begin{pmatrix} \epsilon & 1 \\ k & q-1 \end{pmatrix} = 0 \text{ for some } a_1, a_2 \in A \setminus \{0\}.$$

We claim that (5.1) leads to a contradiction. In fact, we put $s_1 = k$ and $s_2 = q-1$. Also fix γ a $(q-1)$ th root of ϵ . By Theorem 2.5, we recall that the matrix Φ' is given by

$$\Phi' = \begin{pmatrix} (t-\theta)^w & 0 \\ (\gamma H_{s_1})^{(-1)}(t-\theta)^w & (t-\theta)^{s_2} \end{pmatrix}$$

for $I = \{\emptyset, (k; \epsilon)\}$. For some $\delta_1, \delta_2 \in \overline{K}[t]$, Eq. (2.10) induces

$$(5.2) \quad \delta_1 = \left(\delta_1^{(-1)} + \delta_2^{(-1)} (\gamma H_{s_1})^{(-1)} + a_1(t) \cdot (\gamma H_w)^{(-1)} \right) (t-\theta)^w,$$

$$(5.3) \quad \delta_2 = \left(\delta_2^{(-1)} + a_2(t) \cdot H_{s_2}^{(-1)} \right) (t-\theta)^{s_2}.$$

We know that $H_{s_1} = H_{s_2} = 1$ and $H_w = (t^q - t) + (k-1)(t^q - \theta^q)$. So (5.3) is reduced to

$$\delta_2 = \left(\delta_2^{(-1)} + a_2(t) \right) (t-\theta)^{s_2}.$$

By Lemma 4.3 and the fact that $\deg_{\theta} H_n < \frac{nq}{q-1}$ for all $n \in \mathbb{N}$ (see (2.6)), we deduce that $\delta_2 \in \mathbb{F}_q[t, \theta]$ and $\deg_{\theta} \delta_2 \leq \frac{q(q-1)}{q-1} = q$. We put $\delta_2 = (a\theta + b)(t - \theta)^{q-1}$ with $a, b \in \mathbb{F}_q[t]$. This yields

$$\delta_2 = f(\theta^q - t^q), \quad a_2(t) = f(t^q - t) \text{ for some } f \in \mathbb{F}_q[t] \setminus \{0\}$$

after calculations (twisting once and comparing the coefficients of θ^{q-1} -terms).

Now for δ_1 , by Lemma 4.3 again, $\delta_1 \in \overline{\mathbb{F}}_q[t, \theta]$ and $\deg_{\theta} \delta_1 \leq \frac{qw}{q-1}$. We treat the cases when $1 < k < q-1$ and $k = q-1$ separately.

If $1 < k < q-1$, we have $\deg_{\theta} \delta_1 \leq w+1$, i.e., $\delta_1 = (a\theta + b)(t - \theta)^w$ for some $a, b \in \overline{\mathbb{F}}_q[t]$. After twisting, comparing the coefficients of θ^{k-1} - and θ^k - terms yields $a = b = 0$, and then comparing the coefficients of θ^q -terms gives $a_1(t)(t^q - t) = 0$. Since $a_1 \neq 0$, we have a contradiction.

Next consider when $k = q-1$, i.e., when $w = 2q-2$. Since $\deg_{\theta} \delta_1 \leq w+2$, we let

$$\delta_1 = (a\theta^2 + b\theta + c)(t - \theta)^w \quad \text{for some } a, b, c \in \overline{\mathbb{F}}_q[t].$$

Then twisting once and comparing the coefficients of θ^{2q-1} - and θ^{2q-2} terms give $a\theta^2 + b\theta + c = a(t - \theta)^2$, i.e.,

$$a^{(1)}(t - \theta^q)^2 = a(t - \theta)^{2q} + \gamma f(\theta^q - t^q) + a_1(t)\gamma((t^q - t) - 2(t^q - \theta^q)).$$

Comparing the coefficients of θ^{2q} yields $a^{(1)} = a$, which implies $a \in \mathbb{F}_q[t]$. Hence

$$a(t - \theta^q)^2 = a(t - \theta)^{2q} + \gamma f(\theta^q - t^q) + a_1(t)\gamma((t^q - t) - 2(t^q - \theta^q)).$$

Multiplying by $(t^q - t)$, we get

$$\gamma f(\theta^q - t^q)(t^q - t) = ((t - \theta^q)^2 - (t - \theta)^{2q})(a(t^q - t) - a_1(t)\gamma)$$

by using the identity $(t^q - t)[(t^q - t) - 2(t^q - \theta^q)] = (t - \theta^q)^2 - (t^q - \theta^q)^2$. Since the right-hand side is not divided by $t - \theta$ unless it is zero, we have $f = 0$, which is a contradiction. This completes the proof. \square

5.4. Weight $2q-1 \leq w \leq 3q-2$.

In this subsection, we consider the case when $2q-1 \leq w \leq 3q-2$. We recall that I_w is the set of MZV's of weight w of the form $\zeta_A(s_1, \dots, s_r)$ where $s_i \neq q, 2q-1, 2q, 3q-2$ for all i satisfying

- if $s_{i+1} = q$ or $2q-1$ for some $i = 1, 2, \dots, r-1$, then $q-1 | s_i$,
- $s_i \neq (q-1, q-1, q)$ when $w = 3q-2$,

and that the set \mathcal{AT}'_w is defined by

$$\mathcal{AT}'_w = \left\{ \zeta_A \begin{pmatrix} \epsilon \\ \mathfrak{s} \end{pmatrix} : \zeta_A(\mathfrak{s}) \in I_w, \text{ and } \epsilon_i = 1 \text{ whenever } s_i = q \right\}.$$

Proposition 5.3. *Suppose that $q > 2$ and $2q-1 \leq w \leq 3q-2$. Then AMZV's in \mathcal{AT}'_w are all linearly independent over K .*

Proof. Suppose that there exist $\zeta_A \begin{pmatrix} \epsilon_i \\ \mathfrak{s}_i \end{pmatrix} \in \mathcal{AT}'_w$ and $a_i \in A \setminus \{0\}$ such that

$$(5.4) \quad \sum_i a_i \gamma_{i1} \dots \gamma_{i\ell_i} \Gamma_{s_{i1}} \dots \Gamma_{s_{i\ell_i}} \zeta_A \begin{pmatrix} \epsilon_i \\ \mathfrak{s}_i \end{pmatrix} = 0.$$

We set $a_i(t) := a_i|_{\theta=t} \in \mathbb{F}_q[t] \setminus \{0\}$. We will proceed by induction to lead a contradiction. We write $\mathfrak{s}_i = (s_{i1}, \dots, s_{il_i})$ and $\epsilon_i = (\epsilon_{i1}, \dots, \epsilon_{il_i})$, and recall

$$\begin{aligned} I(\mathfrak{s}_i; \epsilon_i) &= \{\emptyset, (s_{i1}; \epsilon_{i1}), \dots, (s_{i1}, \dots, s_{i(\ell_i-1)}; \epsilon_{i1}, \dots, \epsilon_{i(\ell_i-1)})\}, \\ I^+(\mathfrak{s}_i; \epsilon_i) &= I(\mathfrak{s}_i; \epsilon_i) \cup \{(\mathfrak{s}_i; \epsilon_i)\}, \end{aligned}$$

and

$$\begin{aligned} I &= \cup_i I(\mathfrak{s}_i; \epsilon_i), \\ I^+ &= \cup_i I^+(\mathfrak{s}_i; \epsilon_i). \end{aligned}$$

As done previously in (2.8) for $(\mathfrak{t}; \epsilon) \in I$, we set

$$f_{\mathfrak{t}; \epsilon} = \sum_i a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{il_i}; \epsilon_{i(k+1)}, \dots, \epsilon_{il_i})$$

where the sum runs over indices i with $(\mathfrak{t}; \epsilon) = (s_{i1}, \dots, s_{ik}; \epsilon_{i1}, \dots, \epsilon_{ik})$ for some $0 \leq k < \ell_i$.

Let $(\mathfrak{t}; \lambda) \in I$ and $\mathfrak{t} \neq \emptyset$. Let $J(\mathfrak{t}, \lambda)$ be a set of all tuples with $\zeta_A \begin{pmatrix} \epsilon_1 & \cdots & \epsilon_\ell \\ s_1 & \cdots & s_\ell \end{pmatrix} \in \mathcal{AT}'_w$ such that $(\mathfrak{t}; \lambda) = (s_{k+1}, \dots, s_\ell; \epsilon_{k+1}, \dots, \epsilon_\ell)$. Then

$$f_{\mathfrak{t}; \epsilon} = \sum_{i \in J(\mathfrak{t}; \epsilon)} a_i \mathfrak{L}(s_{i(k+1)}, \dots, s_{il_i}; \epsilon_{i(k+1)}, \dots, \epsilon_{il_i}).$$

By Theorem 2.5 and referring to (2.7) and (2.8),

(5.5)

$$f_{\mathfrak{t}; \epsilon}(\theta) = \sum_{i \in J(\mathfrak{t}; \epsilon)} a_i \gamma_{i(k+1)} \cdots \gamma_{il_i} \Gamma_{s_{i(k+1)}} \cdots \Gamma_{s_{il_i}} \zeta_A \begin{pmatrix} s_{i(k+1)} & \cdots & s_{il_i} \\ \epsilon_{i(k+1)} & \cdots & \epsilon_{il_i} \end{pmatrix} / \tilde{\pi}^{w-w(\mathfrak{t})} \in K.$$

Proposition 5.3 follows from Lemma 5.4 and Propositions 5.7, 5.8, and 5.9 which will be proved in the sequel. \square

Lemma 5.4. *Recall that $(\mathfrak{t}; \epsilon) \in I$ and $\mathfrak{t} \neq \emptyset$. Then for each $\epsilon_i \in \mathbb{F}_q^\times$ we can choose a $(q-1)$ th root γ_i , which satisfy*

- either the tuple $(\mathfrak{t}, q; \epsilon, 1)$ or $(\mathfrak{t}, 2q-1; \epsilon, \epsilon)$ belongs to I^+ ,
- or $w - w(\mathfrak{t})$ is divisible by $q-1$ and there exists $a \in K$ such that

$$\begin{aligned} \sum_{i \in J(\mathfrak{t}; \epsilon)} a_i \gamma_{i(k+1)} \cdots \gamma_{il_i} \Gamma_{s_{i(k+1)}} \cdots \Gamma_{s_{il_i}} \zeta_A \begin{pmatrix} \epsilon_{i(k+1)} & \cdots & \epsilon_{il_i} \\ s_{i(k+1)} & \cdots & s_{il_i} \end{pmatrix} \\ + a \Gamma_{w-w(\mathfrak{t})} \zeta_A \begin{pmatrix} 1 \\ w - w(\mathfrak{t}) \end{pmatrix} = 0. \end{aligned}$$

Proof. Suppose that the tuples $(\mathfrak{t}, q; \epsilon, 1)$ and $(\mathfrak{t}, 2q-1; \epsilon, \epsilon)$ do not belong to I^+ and $w - w(\mathfrak{t})$ is not divisible by $q-1$. Then $\tilde{\pi}^{w-w(\mathfrak{t})} \notin K_\infty$ as $w - w(\mathfrak{t})$ is not divisible by $q-1$. Referring to (2.7) and Steps 1 and 2 above, since $f_{\mathfrak{t}; \epsilon}(\theta) \in K$ by (5.5), this implies that

$$(5.6) \quad \sum_{i \in J(\mathfrak{t}; \epsilon)} a_i \gamma_{i(k+1)} \cdots \gamma_{il_i} \Gamma_{s_{i(k+1)}} \cdots \Gamma_{s_{il_i}} \zeta_A \begin{pmatrix} \epsilon_{i(k+1)} & \cdots & \epsilon_{il_i} \\ s_{i(k+1)} & \cdots & s_{il_i} \end{pmatrix} = 0.$$

Since $(\mathfrak{t}, q; \epsilon, 1)$ and $(\mathfrak{t}, 2q-1; \epsilon, \epsilon)$ do not belong to I^+ , the AMZV's appearing in (5.6) belong to $\mathcal{AT}'_{w-w(\mathfrak{t})}$. Then, recalling that $a_i \neq 0$ for $i \in J(\mathfrak{t})$, we obtain a

contradiction by the induction hypothesis. Hence, we have shown that if $(\mathbf{t}, q; \boldsymbol{\epsilon}, 1)$ and $(\mathbf{t}, 2q - 1; \boldsymbol{\epsilon}, \epsilon)$ do not belong to I^+ , then $w - w(\mathbf{t})$ is divisible by $q - 1$.

If $q - 1$ divides $w - w(\mathbf{t})$, then since $\zeta_A \binom{1}{n} / \tilde{\pi}^{w-w(\mathbf{t})} \in K_\infty$ for any integer n such that $(q - 1) \mid n$, we conclude that there exists some $a \in K$ such that

$$\sum_{i \in J(\mathbf{t}; \boldsymbol{\epsilon})} a_i \gamma_{i(k+1)} \cdots \gamma_{i\ell_i} \Gamma_{s_{i(k+1)}} \cdots \Gamma_{s_{i\ell_i}} \zeta_A \begin{pmatrix} \epsilon_{i(k+1)} & \cdots & \epsilon_{i\ell_i} \\ s_{i(k+1)} & \cdots & s_{i\ell_i} \end{pmatrix} + a \Gamma_{w-w(\mathbf{t})} \zeta_A \begin{pmatrix} 1 \\ w - w(\mathbf{t}) \end{pmatrix} = 0.$$

□

Remark 5.5. 1) Note that if $w = 2q - 1$, then $(\mathbf{t}, 2q - 1; \boldsymbol{\epsilon}, \epsilon) \notin I^+$ and the condition $(\mathbf{t}, q; \boldsymbol{\epsilon}, 1) \in I^+$ forces $s_{ik} = q - 1$ and so $(\mathbf{t}; \boldsymbol{\epsilon}) = (q - 1; \epsilon_{i1})$ and $(\mathbf{s}_i; \boldsymbol{\epsilon}_i) = (q - 1, q; \epsilon_{i1}, 1)$.

2) If $2q \leq w \leq 3q - 3$, then $(\mathbf{t}, 2q - 1; \boldsymbol{\epsilon}, \epsilon) \notin I^+$ since $w \leq 3q - 3$, and the condition $(\mathbf{t}, q; \boldsymbol{\epsilon}, 1) \in I^+$ forces $s_{ik} = q - 1$ and $\sum_{j=1}^{k-1} \mathbf{s}_{jk} \leq w - (2q - 1) \leq q - 2$.

3) If $w = 3q - 2$, then the condition $(\mathbf{t}, 2q - 1; \boldsymbol{\epsilon}, \epsilon) \in I^+$ forces $s_{ik} = q - 1$, so $(\mathbf{t}; \boldsymbol{\epsilon}) = (q - 1; \epsilon_{i1})$ and $(\mathbf{s}_i; \boldsymbol{\epsilon}_i) = (q - 1, 2q - 1; \epsilon_{i1}, 1)$. The condition $(\mathbf{t}, q; \boldsymbol{\epsilon}, 1) \in I^+$ forces $s_{ik} = q - 1$ or $2q - 2$.

Remark 5.6. As shown in the proof of Theorem 1.5, whenever $q - 1 \mid w - w(\mathbf{t})$ for some $\mathbf{t} = (s_{i1}, \dots, s_{ik}) \neq \emptyset$ with $k < \ell_i$, we have $\epsilon_{ik} = \cdots = \epsilon_{i\ell_i} = 1$. This can be also seen by successive applications of Lemma 5.4.

5.4.1. *The case $w = 2q - 1$.*

Consider $w = 2q - 1$ and the set I_w consisting of MZV's of weight w of the form

- $\zeta_A(s_1, \dots, s_r)$ where $s_i \neq q, 2q - 1$ for all i ,
- $\zeta_A(q - 1, q)$,

then from our setting,

$$\mathcal{AT}'_w := \left\{ \zeta_A \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathbf{s} \end{pmatrix} : \zeta_A(\mathbf{s}) \in I_w, \text{ and } \epsilon_i = 1 \text{ whenever } s_i = q \right\}.$$

Proposition 5.7. *Suppose that $q > 2$ and $w = 2q - 1$. Then AMZV's in \mathcal{AT}'_w are all linearly independent over K .*

Proof. By Lemma 5.4, the first shortlist of tuples $(\mathbf{s}_i, \boldsymbol{\epsilon}_i)$ appearing in (5.4) has the integer tuples,

$$\mathbf{s}_1 = (1, 2q - 2), \mathbf{s}_2 = (1, q - 1, q - 1), \mathbf{s}_3 = (q - 1, q), \mathbf{s}_4 = (q - 1, 1, q - 1),$$

and the character tuples over \mathbb{F}_q^\times (and characters corresponding to q is 1).

As in Remark 5.6 and Lemma 4.2, we have the first shortlist as follows: for each $\epsilon \in \mathbb{F}_q^\times$,

$$(1, 2q - 2; \epsilon, 1), (1, q - 1, q - 1; \epsilon, 1, 1), (q - 1, q; \epsilon, 1), (q - 1, 1, q - 1; \epsilon_1, \epsilon_2, 1),$$

with $\epsilon_1, \epsilon_2 \in \mathbb{F}_q^\times$ such that $\epsilon_1 \epsilon_2 = \epsilon$.

Applying Lemma 5.4 with $(\mathbf{t}; \boldsymbol{\epsilon}) = (1; \epsilon)$, there exists some $a \in K$ such that

$$a_{(1, 2q-2; \epsilon, 1)} \Gamma_{2q-2} \zeta_A \begin{pmatrix} 1 \\ 2q - 1 \end{pmatrix} + a_{(1, q-1, q-1; \epsilon, 1, 1)} \Gamma_{q-1} \Gamma_{q-1} \zeta_A \begin{pmatrix} 1 & 1 \\ q - 1 & q - 1 \end{pmatrix}$$

$$+ a\Gamma_{2q-2}\zeta_A \left(\frac{1}{2q-2} \right) = 0.$$

Recalling that $\zeta_A(q-1, q-1)$ is not zeta-like, we can deduce that $a_{(1, q-1, q-1; \epsilon, 1, 1)} = 0$. So we reduce to the second shortlist as follows:

$$(1, 2q-2; \epsilon, 1), \quad (q-1, q; \epsilon, 1), \quad (q-1, 1, q-1; \epsilon_1, \epsilon_2, 1)$$

with $\epsilon_1, \epsilon_2 \in \mathbb{F}_q^\times$ such that $\epsilon_1\epsilon_2 = \epsilon$ for each given $\epsilon \in \mathbb{F}_q^\times$.

From now on, we fix ϵ and proceed as done in the discussion preceding Theorem 2.5 to examine the linear independence. We construct Φ' with respect to the following tuples in I :

$$\emptyset, (1; \epsilon), (q-1; \epsilon_1), (q-1, 1; \epsilon_1, \epsilon_2),$$

where $\epsilon = \epsilon_1\epsilon_2$. Note that in the following ϵ is given, ϵ_1 is free, and $\epsilon_2 = \epsilon\epsilon_1^{-1}$ is determined by ϵ_1 . Also $\gamma, \gamma_1, \gamma_2$ are $(q-1)$ th roots of $\epsilon, \epsilon_1, \epsilon_2$, respectively as chosen in Lemma 5.4.

Recalling the constructions of Φ' and \mathbf{v} in Theorem 2.5, we get

$$\begin{aligned} \Phi'_{\emptyset, \emptyset} &= (t - \theta)^{2q-1}, \\ \Phi'_{(1; \epsilon), \emptyset} &= (\gamma H_1)^{(-1)} (t - \theta)^{2q-1}, \\ \Phi'_{(1; \epsilon), (1; \epsilon)} &= (t - \theta)^{2q-2}, \\ \Phi'_{(q-1; \epsilon_1), \emptyset} &= (\gamma_1 H_{q-1})^{(-1)} (t - \theta)^{2q-1}, \\ \Phi'_{(q-1; \epsilon_1), (q-1; \epsilon_1)} &= (t - \theta)^q, \\ \Phi'_{(q-1, 1; \epsilon_1, \epsilon_2), (q-1; \epsilon_1)} &= (\gamma_2 H_1)^{(-1)} (t - \theta)^q, \\ \Phi'_{(q-1, 1; \epsilon_1, \epsilon_2), (q-1, 1; \epsilon_1, \epsilon_2)} &= (t - \theta)^{q-1}, \\ \text{others} &= 0, \end{aligned}$$

and

$$\begin{aligned} v_{\emptyset} &= 0, \\ v_{(1; \epsilon)} &= a_{(1, 2q-2; \epsilon, 1)} H_{2q-2}^{(-1)} (t - \theta)^{2q-2}, \\ v_{(q-1; \epsilon)} &= a_{(q-1, q; \epsilon, 1)} H_q^{(-1)} (t - \theta)^q, \\ v_{(q-1; \epsilon_1)} &= 0, \quad \text{for } \epsilon_1 \neq \epsilon, \\ v_{(q-1, 1; \epsilon_1, \epsilon_2)} &= a_{(q-1, 1, q-1; \epsilon_1, \epsilon_2, 1)} H_{q-1}^{(-1)} (t - \theta)^{q-1}. \end{aligned}$$

Hence, for some δ_i 's in $\overline{K}[t]$ for corresponding indices i , Eq. (2.10) induces

$$(5.7) \quad \delta_{\emptyset} = \left(\delta_{\emptyset}^{(-1)} + \delta_{(1; \epsilon)}^{(-1)} (\gamma H_1)^{(-1)} + \sum_{\epsilon_1} \delta_{(q-1; \epsilon_1)}^{(-1)} (\gamma_1 H_{q-1})^{(-1)} \right) (t - \theta)^{2q-1}$$

$$(5.8) \quad \delta_{(1; \epsilon)} = \left(\delta_{(1; \epsilon)}^{(-1)} + a_{(1, 2q-2; \epsilon, 1)} H_{2q-2}^{(-1)} \right) (t - \theta)^{2q-2}$$

$$(5.9) \quad \delta_{(q-1; \epsilon)} = \left(\delta_{(q-1; \epsilon)}^{(-1)} + \delta_{(q-1, 1; \epsilon, 1)}^{(-1)} H_1^{(-1)} + a_{(q-1, q; \epsilon, 1)} H_q^{(-1)} \right) (t - \theta)^q$$

$$(5.10) \quad \delta_{(q-1; \epsilon_1)} = \left(\delta_{(q-1; \epsilon_1)}^{(-1)} + \delta_{(q-1, 1; \epsilon_1, \epsilon_2)}^{(-1)} (\gamma_2 H_1)^{(-1)} \right) (t - \theta)^q, \quad \text{for } \epsilon_1 \neq \epsilon$$

(5.11)

$$\delta_{(q-1,1;\epsilon_1,\epsilon_2)} = \left(\delta_{(q-1,1;\epsilon_1,\epsilon_2)}^{(-1)} + a_{(q-1,1,q-1;\epsilon_1,\epsilon_2,1)} H_{q-1}^{(-1)} \right) (t-\theta)^{q-1}$$

Note that if $(1 \leq n \leq q)$, $H_n = 1$, and if $q+1 \leq n \leq q^2$, then

$$H_n(t) = \sum_{j=0}^k \binom{(n-1)-jq+j}{j} (t^q - t)^{k-j} (t^q - \theta^q)^j$$

where $k = \lfloor \frac{n-1}{q} \rfloor$ (see [21]). Thus $H_{2q-2} = 2\theta^q - t - t^q$.

Consider (5.11). Lemma 4.3 shows that $\delta_{(q-1,1;\epsilon_1,\epsilon_2)} \in \mathbb{F}_q[t, \theta]$ and $\deg_{\theta} \delta_{(q-1,1;\epsilon_1,\epsilon_2)} \leq q$. We can put $\delta_{(q-1,1;\epsilon_1,\epsilon_2)} = (f\theta + g)(t - \theta)^{q-1}$ with $f, g \in \mathbb{F}_q[t]$, and plug it into (5.11). Canceling terms and twisting once yield

$$f\theta^q + g = (f\theta + g)(t - \theta)^{q-1} + a_{(q-1,1,q-1;\epsilon_1,\epsilon_2,1)}.$$

Comparing the θ^{q-1} -terms yields $g = -tf$, i.e., $\delta_{(q-1,1;\epsilon_1,\epsilon_2)} = f(\theta^q - t^q)$ and $a_{(q-1,1,q-1;\epsilon_1,\epsilon_2,1)} = f(t^q - t)$. Since f here is parametrized by ϵ_1, ϵ_2 , we write $f_{\epsilon_1, \epsilon_2} = f \in \mathbb{F}_q[t]$, to have

$$\delta_{(q-1,1;\epsilon_1,\epsilon_2)} = f_{\epsilon_1, \epsilon_2}(\theta^q - t^q), \quad \text{and} \quad a_{(q-1,1,q-1;\epsilon_1,\epsilon_2,1)} = f_{\epsilon_1, \epsilon_2}(t^q - t).$$

Next, for (5.10), Lemma 4.3 shows that $\delta_{(q-1;\epsilon_1)} \in \overline{\mathbb{F}}_q[t, \theta]$ and $\deg_{\theta} \delta_{(q-1;\epsilon_1)} \leq \frac{q^2}{q-1} = (q+1) + \frac{1}{q-1}$, i.e., $\delta_{(q-1;\epsilon_1)} = (f'\theta + g')(t - \theta)^q$ with $f', g' \in \overline{\mathbb{F}}_q[t]$. We plug it into (5.10), cancel terms, twist and compare the θ^{q+1} -terms to have $f' = 0$, and compare the θ^q -terms to have $g' = \gamma_2 f_{\epsilon_1, \epsilon_2}$. Further, comparing the θ^0 -terms yields $g'^{(1)} = (g' - \gamma_2 f_{\epsilon_1, \epsilon_2})t^q = 0$, i.e., $g' = f_{\epsilon_1, \epsilon_2} = 0$, and

$$(5.12) \quad \delta_{(q-1;\epsilon_1)} = 0 \quad \text{for } \epsilon_1 \neq \epsilon.$$

For (5.9), we proceed similarly to have

$$\delta_{(q-1;\epsilon)} = f_{\epsilon,1}(t - \theta)^q, \quad \text{and} \quad a_{(q-1,q;\epsilon,1)} = f_{\epsilon,1}, \quad \text{for some } f_{\epsilon,1} \in \mathbb{F}_q[t].$$

For (5.8), recall $H_{2q-2}^{(-1)} = \frac{(t-\theta)^2 - (t^q - \theta)^2}{t^q - t} = -t - t^q + 2\theta$. Then, Lemma 4.3 shows that $\delta_{(1;\epsilon)} \in \mathbb{F}_q[t, \theta]$ with $\deg_{\theta} \delta_{(1;\epsilon)} \leq 2q$. Let $\delta_{(1;\epsilon)} = (c\theta^2 + d\theta + e)(t - \theta)^{2q-2}$ for some $c, d, e \in \mathbb{F}_q[t]$; in a similar process, we have $\delta_{(1;\epsilon)} = c(t - \theta)^{2q}$ for some $c \in \mathbb{F}_q[t]$. This c is parametrized by ϵ , so we write $c_{\epsilon} = c \in \mathbb{F}_q[t]$ to have

$$\delta_{(1;\epsilon)} = c_{\epsilon}(t - \theta)^{2q} = c_{\epsilon}(t^q - \theta^q)^2, \quad \text{and} \quad a_{(1,2q-2;\epsilon,1)} = c_{\epsilon}(t^q - t).$$

Finally for (5.7), Lemma 4.3 shows that $\delta_{\emptyset} \in \overline{\mathbb{F}}_q[t]$ with $\deg_{\theta} \leq 2q+1$, i.e., $\delta_{\emptyset} = (c'\theta^2 - d'\theta + e')(t - \theta)^{2q-1}$ for some $c', d', e' \in \overline{\mathbb{F}}_q[t]$. Proceeding similarly, comparing the θ^{2q+1} - and the θ^{2q-1} - coefficients yields $c' = 0$ and $e' = d't$, i.e., $\delta_{\emptyset} = d'(t - \theta)^{2q}$ for some $d' \in \overline{\mathbb{F}}_q[t]$. Thus, recalling (5.12),

$$d'^{(1)}(t - \theta)^q = d'(t - \theta)^{2q} + \gamma c_{\epsilon}(t - \theta)^{2q} + \gamma(t - \theta)^q f_{\epsilon,1}.$$

The right hand side is divisible by $t - \theta$, which forces $d'^{(1)} = 0$, i.e., $d' = 0$. By canceling common factors of the remaining terms, we have

$$f_{\epsilon,1} = -c_{\epsilon}(t - \theta)^q.$$

Again the right hand side of this is divisible by $t - \theta$, i.e., $f_{\epsilon,1} = c_{\epsilon} = 0$. Thus we have $a_{(1,2q-2;\epsilon,1)} = c_{\epsilon}(t^q - t) = 0$, $a_{(q-1,q;\epsilon,1)} = f_{\epsilon,1} = 0$, $a_{(q-1,1,q-1;\epsilon_1,\epsilon_2,1)} = f_{\epsilon_1, \epsilon_2}(t^q - t) = 0$. This completes the proof. \square

5.4.2. *The case $2q \leq w \leq 3q - 3$.*

Proposition 5.8. *Suppose that $q > 2$ and $2q \leq w \leq 3q - 3$. Then AMZV's in \mathcal{AT}'_w are all linearly independent over K .*

Proof. By Lemma 5.4, the first shortlist of tuples $(\mathfrak{s}_i, \epsilon_i)$ appearing (5.4) has get the integer tuples,

$$\mathfrak{s}_1 = (w - (2q - 2), q - 1, q - 1), \mathfrak{s}_2 = (w - (2q - 2), 2q - 2), \mathfrak{s}_3 = (w - (q - 1), q - 1)$$

for $2q \leq w \leq 3q - 3$, and additional $\mathfrak{s}_4 = (w)$ when $w \neq 2q$, with characters in \mathbb{F}_q^\times (and characters corresponding to q is 1). With Remark 5.6 and Lemma 4.2, for fixed $\epsilon \in \mathbb{F}_q^\times$, the first shortlist is

$$\begin{aligned} \text{if } w \neq 2q, \quad & (w - (2q - 2), q - 1, q - 1; \epsilon, 1, 1), (w - (2q - 2), 2q - 2; \epsilon, 1), \\ & (w - (q - 1), q - 1; \epsilon, 1), (w; \epsilon), \end{aligned}$$

$$\text{if } w = 2q, \quad (2, q - 1, q - 1; \epsilon, 1, 1), (2, 2q - 2; \epsilon, 1), (q + 1, q - 1; \epsilon, 1).$$

Note that any component of integer parts in the above list is not q when $2q \leq w \leq 3q - 3$.

Applying Lemma 5.4 with $(\mathfrak{t}; \epsilon) = (w - (2q - 2); \epsilon)$, we obtain that there exists some $a \in K$ such that

$$\begin{aligned} & a_{(w-(2q-2), q-1, q-1; \epsilon, 1, 1)} \Gamma_{q-1} \Gamma_{q-1} \zeta_A \begin{pmatrix} 1 & 1 \\ q-1 & q-1 \end{pmatrix} \\ & + a_{(w-(2q-2), 2q-2; \epsilon, 1)} \Gamma_{2q-2} \zeta_A \begin{pmatrix} 1 \\ 2q-2 \end{pmatrix} + a \Gamma_{2q-2} \zeta_A \begin{pmatrix} 1 \\ 2q-2 \end{pmatrix} = 0, \end{aligned}$$

Since $\zeta(q - 1, q - 1)$ is not zeta-like, we deduce that $a_{(w-(2q-2), q-1, q-1; \epsilon, 1, 1)} = 0$. Thus we reduce to the following second shortlist:

$$\begin{aligned} \text{if } w \neq 2q, \quad & (w - (2q - 2), 2q - 2; \epsilon, 1), (w - (q - 1), q - 1; \epsilon, 1), (w; \epsilon), \\ \text{if } w = 2q, \quad & (2, 2q - 2; \epsilon, 1), (q + 1, q - 1; \epsilon, 1). \end{aligned}$$

First consider the cases when $2q + 1 \leq w \leq 3q - 3$. Note that we can assume $q > 3$. (If $q = 3$, this case is empty.) It is enough to show the linear independence of the following linear combination for each $\epsilon \in \mathbb{F}_q^\times$:

$$\begin{aligned} & a_{(w-(2q-2), 2q-2; \epsilon, 1)} \gamma \Gamma_{w-(2q-2)} \Gamma_{2q-2} \zeta_A \begin{pmatrix} \epsilon & 1 \\ w - (2q - 2) & 2q - 2 \end{pmatrix} \\ & + a_{(w-(q-1), q-1; \epsilon, 1)} \gamma \Gamma_{w-(q-1)} \Gamma_{q-1} \zeta_A \begin{pmatrix} \epsilon & 1 \\ w - (q - 1) & q - 1 \end{pmatrix} \\ & + a_{(w; \epsilon)} \gamma \Gamma_w \zeta_A \begin{pmatrix} \epsilon \\ w \end{pmatrix} = 0, \end{aligned}$$

From this, we proceed as done in the discussion preceding Theorem 2.5 to examine the linear independence. We construct Φ' with respect to the following tuples in I :

$$\emptyset, (w - (2q - 2); \epsilon), (w - (q - 1); \epsilon).$$

Then, we get

$$\Phi' = \begin{pmatrix} (t-\theta)^w & 0 & 0 \\ (\gamma H_{w-(2q-2)})^{(-1)}(t-\theta)^w & (t-\theta)^{2q-2} & 0 \\ (\gamma H_{w-(q-1)})^{(-1)}(t-\theta)^w & 0 & (t-\theta)^{q-1} \end{pmatrix},$$

and

$$\mathbf{v} = \left(a_{(w;\epsilon)}(\gamma H_w)^{(-1)}, \quad a_{(w-(2q-2),2q-2;\epsilon,1)}H_{2q-2}^{(-1)}, \quad a_{(w-(q-1),q-1;\epsilon,1)}H_{q-1}^{(-1)} \right).$$

For some δ_i 's in $\overline{K}[t]$ for corresponding indices i , Eq. (2.10) induces:

$$(5.13) \quad \delta_\emptyset = \left(\delta_\emptyset^{(-1)} + \delta_{(w-(2q-2);\epsilon)}^{(-1)}(\gamma H_{w-(2q-2)})^{(-1)} \right. \\ \left. + \delta_{(w-(q-1);\epsilon)}^{(-1)}(\gamma H_{w-(q-1)})^{(-1)} + a_{(w;\epsilon)}(\gamma H_w)^{(-1)} \right) (t-\theta)^w$$

$$(5.14) \quad \delta_{(w-(2q-2);\epsilon)} = \left(\delta_{(w-(2q-2);\epsilon)}^{(-1)} + a_{(w-(2q-2),2q-2;\epsilon,1)}H_{2q-2}^{(-1)} \right) (t-\theta)^{2q-2}$$

$$(5.15) \quad \delta_{(w-(q-1);\epsilon)} = \left(\delta_{(w-(q-1);\epsilon)}^{(-1)} + a_{(w-(q-1),q-1;\epsilon,1)}H_{q-1}^{(-1)} \right) (t-\theta)^{q-1}$$

Recall $H_{q-1} = 1$, $H_{2q-2} = 2\theta^q - t - t^q$, and for $1 \leq k \leq q$ (noting that $q \geq 3$), $H_{q+k} = (t^q - t) + k(t^q - \theta^q)$, and

$$H_{2q+k} = (t^q - t)^2 + k(t^q - t)(t^q - \theta^q) + \frac{k(k+1)}{2}(t^q - \theta^q)^2.$$

Consider (5.15). Lemma 4.3 shows that $\delta_{(w-(q-1);\epsilon)} \in \mathbb{F}_q[t, \theta]$ with $\deg_\theta \delta_{(w-(q-1);\epsilon)} \leq q$. Let $\delta_{(w-(q-1);\epsilon)} = (a\theta + b)(t-\theta)^{q-1}$ for some $a, b \in \mathbb{F}_q[t]$, cancel terms and twist once to have

$$(a\theta^q + b) = (a\theta + b)(t-\theta)^{q-1} + a_{(w-(q-1),q-1;\epsilon,1)}.$$

Comparing the θ^q -coefficients yields $at + b = 0$, so rewriting a as $f_{w;\epsilon} \in \mathbb{F}_q[t]$, we have

$$\delta_{(w-(q-1);\epsilon)} = f_{w;\epsilon}(\theta - t)^q, \quad \text{and} \quad a_{(w-(q-1),q-1;\epsilon,1)} = f_{w;\epsilon}(t^q - t).$$

Similarly for (5.14), we have $\delta_{(w-(2q-2);\epsilon)} \in \mathbb{F}_q[t]$ with $\deg_\theta \delta_{(w-(2q-2);\epsilon)} \leq 2q$. Let $\delta_{(w-(2q-2);\epsilon)} = (a'\theta^2 + b'\theta + c')(t-\theta)^{2q-2}$ then it is reduced to

$$\delta_{(w-(2q-2);\epsilon)} = f'_{w;\epsilon}(\theta - t)^{2q}, \quad \text{and} \quad a_{(w-(2q-2),2q-2;\epsilon,1)} = f'_{w;\epsilon}(t^q - t)$$

for some $a' = f'_{w;\epsilon} \in \mathbb{F}_q[t]$.

Now for (5.13), we have $\delta_\emptyset \in \overline{\mathbb{F}_q}[t]$ with $\deg_\theta \delta_\emptyset \leq w + 3$. Let $\delta_\emptyset = (t-\theta)^w F$ for some $F \in \overline{\mathbb{F}_q}[t, \theta]$ with $\deg_\theta F \leq 3$. Then by direct calculation,

$$\begin{aligned} F^{(1)} &= (t-\theta)^w F + \gamma f'_{w;\epsilon}(\theta - t)^{2q} H_{w-(2q-2)} + \gamma f_{w;\epsilon}(\theta - t)^q H_{w-(q-1)} + \gamma a_{(w;\epsilon)} H_w \\ &= (t-\theta)^w F + \gamma f'_{w;\epsilon}(\theta - t)^{2q} + \gamma f_{w;\epsilon}(\theta - t)^q H_{q+(w-2q+1)} + \gamma a_{(w;\epsilon)} H_{2q+(w-2q)} \\ &= (t-\theta)^w F + \gamma f'_{w;\epsilon}(\theta^q - t^q)^2 \\ &\quad + \gamma f_{w;\epsilon}(\theta^q - t^q)(t^q - t + (w+1)(t^q - \theta^q)) \\ &\quad + \gamma a_{(w;\epsilon)} \left((t^q - t)^2 + w(t^q - t)(t^q - \theta^q) + \frac{(w-2q)(w-2q+1)}{2}(t^q - \theta^q)^2 \right) \end{aligned}$$

It follows that that $(t-\theta)^w F \in \overline{\mathbb{F}_q}[t, \theta^q]$. We claim that $F \neq 0$; otherwise, by comparing the coefficients of powers of the $(t-\theta)$ -term, $F = 0$ implies $a_{(w;\epsilon)} = 0$, and then $f_{w;\epsilon} = f'_{w;\epsilon} = 0$, i.e., $a_{(w-(q-1),q-1;\epsilon,1)} = a_{(w-(2q-2),2q-2;\epsilon,1)} = 0$, so

$a_{(w;\epsilon)} = 0$ by (5.4), which is a contradiction, so $F \neq 0$. Thus, the only possibility arises for $w = 3q - 3$, $F = \gamma(t - \theta)^3 g$ for some $g \in \overline{\mathbb{F}_q}[t]$. Since $w = 3q - 3$, we have

$$\begin{aligned} \epsilon(t - \theta^q)^3 g^{(1)} &= (t^q - \theta^q)^3 g + f'_{3q-3;\epsilon}(\theta^q - t^q)^2 + f_{3q-3;\epsilon}(\theta^q - t^q)(2\theta^q - t - t^q) \\ &\quad + a_{(3q-3;\epsilon)} \frac{(t^q - \theta^q)^3 - (t - \theta^q)^3}{t^q - t} \end{aligned}$$

and by comparing the coefficients of powers of the $(t^q - \theta^q)$ -term, we have $a_{(3q-3;\epsilon)} = -(t^q - t)g$, and $f_{3q-3;\epsilon} = f'_{3q-3;\epsilon} = 0$. Thus, $a_{(2q-2,q-1;\epsilon,1)} = a_{(q-1,2q-2;\epsilon,1)} = 0$, so $a_{(3q-3;\epsilon)} = 0$, which is a contradiction. This completes this case.

When $w = 2q$, there are two candidates, $\zeta_A \begin{pmatrix} \epsilon & 1 \\ 2 & 2q-2 \end{pmatrix}$ and $\zeta_A \begin{pmatrix} \epsilon & 1 \\ q+1 & q-1 \end{pmatrix}$ for each given ϵ for (5.4). We construct Φ' with respect to the following tuples in I :

$$\emptyset, (2; \epsilon), (q+1; \epsilon).$$

Then,

$$\Phi' = \begin{pmatrix} (t - \theta)^{2q} & 0 & 0 \\ (\gamma H_2)^{(-1)}(t - \theta)^w & (t - \theta)^{2q-2} & 0 \\ (\gamma H_{q+1})^{(-1)}(t - \theta)^w & 0 & (t - \theta)^{q-1} \end{pmatrix},$$

and

$$\mathbf{v} = \left(0, \quad a_{(2,2q-2;\epsilon,1)} H_{2q-2}^{(-1)}, \quad a_{(q+1,q-1;\epsilon,1)} H_{q-1}^{(-1)} \right).$$

For some δ_i 's in $\overline{K}[t]$ for corresponding indices i , Eq. (2.10) induces

$$(5.16) \quad \delta_\emptyset = \left(\delta_\emptyset^{(-1)} + \delta_{(2;\epsilon)}^{(-1)} (\gamma H_2)^{(-1)} + \delta_{(q+1;\epsilon)}^{(-1)} (\gamma H_{q+1})^{(-1)} \right) (t - \theta)^w$$

$$(5.17) \quad \delta_{(2;\epsilon)} = \left(\delta_{(2;\epsilon)}^{(-1)} + a_{(2,2q-2;\epsilon,1)} H_{2q-2}^{(-1)} \right) (t - \theta)^{2q-2}$$

$$(5.18) \quad \delta_{(q+1;\epsilon)} = \left(\delta_{(q+1;\epsilon)}^{(-1)} + a_{(q+1,q-1;\epsilon,1)} H_{q-1}^{(-1)} \right) (t - \theta)^{q-1}.$$

Recall $H_2 = H_{q-1} = 1$, $H_{q+1} = 2t^q - t - \theta^q$ and $H_{2q-2} = 2\theta^q - t - t^q$.

In the very same way, by analyzing (5.18) and (5.17), we obtain

$$\begin{aligned} \delta_{(q+1;\epsilon)} &= f_\epsilon(\theta - t)^q, \quad \text{and} \quad a_{(q+1,q-1;\epsilon,1)} = f_\epsilon(t^q - t), \\ \delta_{(2;\epsilon)} &= f'_\epsilon(\theta - t)^{2q}, \quad \text{and} \quad a_{(2,2q-2;\epsilon,1)} = f'_\epsilon(t^q - t), \end{aligned}$$

for some $f_\epsilon, f'_\epsilon \in \mathbb{F}_q[t]$.

Next, for (5.16), we have $\delta_\emptyset \in \overline{\mathbb{F}_q}[t]$ with $\deg_\theta \delta_\emptyset \leq 2q+3$. Let $\delta_\emptyset = F(t, \theta) \cdot (t - \theta)^{2q}$ with $F := F(t, \theta) \in \overline{\mathbb{F}_q}[t, \theta]$ with $\deg_\theta F \leq 3$. Then by direct calculation,

$$\begin{aligned} F^{(1)} &= (t - \theta)^{2q} F + \gamma f'_\epsilon(\theta - t)^{2q} H_2 + \gamma f_\epsilon(\theta - t)^q H_{q+1} \\ &= (\theta - t)^{2q} F + \gamma f'_\epsilon(\theta - t)^{2q} + \gamma f_\epsilon(\theta - t)^q (2t^q - t - \theta^q) \end{aligned}$$

We claim that $F \neq 0$; if $F = 0$ then comparing the θ^q - and θ^{2q} -coefficients yield $f_\epsilon = f'_\epsilon = 0$, i.e., $a_{(q+1,q-1;\epsilon,1)} = a_{(2,2q-2;\epsilon,1)} = 0$, which is a contradiction.

Since $(\theta - t)^q \mid F^{(1)}$, we have $F = (\theta - t^q)G(t, \theta)$ for some $G(t, \theta) \in \overline{\mathbb{F}_q}[t]$. Note that $\deg_\theta G \leq 2$. Then we have

$$G^{(1)} = (\theta - t^q)(\theta - t)^q G + \gamma f'_\epsilon(\theta - t)^q + \gamma f_\epsilon(2t^q - t - \theta^q).$$

Let $d = \deg_\theta G$. Then $dq = q + 1 + d$, and $0 \leq d \leq 2$, which leads to that $d = 2$ and $q = 3$. Let $G(t, \theta) = a\theta^2 + b\theta + c$ with $a, b, c \in \overline{\mathbb{F}_q}[t]$. Then comparing the θ^{2q-1} -

θ^{2q-2} -, θ^0 -coefficients yields $c^{(1)} = ct^{2q}$, $c = bt^q$, $b = at^q$, i.e., $a = b = c = 0$. This leads to $F = 0$, which is a contradiction, so this completes the proof. \square

5.4.3. *The case $w = 3q - 2$.*

Proposition 5.9. *Suppose that $q > 2$ and $w = 3q - 2$. Then AMZV's in \mathcal{AT}'_w are all linearly independent over K .*

Proof. By Lemma 5.4, and by considering Remark 5.6 and Lemma 4.2, the first shortlist of tuples $(\mathfrak{s}_i, \epsilon_i)$ appearing (5.4) has the integer tuples; for each given $\epsilon \in \mathbb{F}_q^\times$,

$$\begin{array}{ll} (2q-2, q; \epsilon, 1), & (1, q-1, q-1, q-1; \epsilon, 1, 1, 1), \\ (1, q-1, 2q-2; \epsilon, 1, 1), & (1, 2q-2, q-1; \epsilon, 1, 1), \\ (1, 3q-3; \epsilon, 1), & (q-1, 2q-1; \epsilon_1, \epsilon_2), \\ (q-1, 1, q-1, q-1; \epsilon'_1, \epsilon'_2, 1, 1), & (q-1, 1, 2q-2; \epsilon''_1, \epsilon''_2, 1), \\ (q-1, q, q-1; \epsilon, 1, 1), & (2q-2, 1, q-1; \epsilon'''_1, \epsilon'''_2, 1), \end{array}$$

with $\epsilon_1 \epsilon_2 = \epsilon'_1 \epsilon'_2 = \epsilon''_1 \epsilon''_2 = \epsilon'''_1 \epsilon'''_2 = \epsilon$.

Applying Lemma 5.4 with $(\mathfrak{t}; \epsilon) = (1, q-1; \epsilon, 1)$, there exist $a \in K$ such that

$$\begin{aligned} & a_{(1, q-1, q-1, q-1; \epsilon, 1, 1, 1)} \Gamma_1 \Gamma_1 \zeta_A \begin{pmatrix} 1 & 1 \\ q-1 & q-1 \end{pmatrix} + a_{(1, q-1, 2q-2; \epsilon, 1, 1)} \Gamma_1 \zeta_A \begin{pmatrix} 1 \\ 2q-2 \end{pmatrix} \\ & + a \Gamma_{2q-2} \zeta_A \begin{pmatrix} 1 \\ 2q-2 \end{pmatrix} = 0. \end{aligned}$$

Since $\zeta_A(q-1, q-1)$ is not zetalike, we deduce that $a_{(1, q-1, q-1, q-1; \epsilon, 1, 1, 1)} = 0$. Similarly with $(\mathfrak{t}; \epsilon) = (1, q-1; \epsilon'_1, \epsilon'_2)$ where $\epsilon'_1 \epsilon'_2 = \epsilon$, we deduce that $a_{(q-1, 1, q-1, q-1; \epsilon'_1, \epsilon'_2, 1, 1)} = 0$; since the choice of ϵ'_1 is free, all candidates of the form are deleted.

With $(\mathfrak{t}; \epsilon) = (1; \epsilon)$, Lemma 5.4 asserts that there exists $b \in K$ such that

$$\begin{aligned} & a_{(1, q-1, 2q-2; \epsilon, 1, 1)} \Gamma_{q-1} \Gamma_{2q-2} \zeta_A \begin{pmatrix} 1 & 1 \\ q-1 & 2q-2 \end{pmatrix} \\ & + a_{(1, 2q-2, q-1; \epsilon, 1, 1)} \Gamma_{2q-2} \Gamma_{q-1} \zeta_A \begin{pmatrix} 1 & 1 \\ 2q-2 & q-1 \end{pmatrix} \\ & + a_{(1, 3q-3; \epsilon, 1)} \Gamma_{3q-3} \zeta_A \begin{pmatrix} 1 \\ 3q-3 \end{pmatrix} + b \Gamma_{3q-3} \zeta_A \begin{pmatrix} 1 \\ 3q-3 \end{pmatrix} = 0, \end{aligned}$$

and since $\zeta_A(q-1, 2q-2)$, $\zeta_A(2q-2, q-1)$, $\zeta_A(3q-3) \in \mathcal{J}'_{3q-3}$, we deduce that $a_{(1, q-1, 2q-2; \epsilon, 1, 1)} = a_{(1, 2q-2, q-1; \epsilon, 1, 1)} = 0$.

Next, we consider $(\mathfrak{t}; \epsilon) = (q-1; \epsilon)$. By (5.5), $f_{(q-1; \epsilon)} \in K$, so

$$\begin{aligned} & a_{(q-1, 2q-1; \epsilon, 1)} \Gamma_{2q-1} \zeta_A \begin{pmatrix} 1 \\ 2q-1 \end{pmatrix} + a_{(q-1, 1, 2q-2; \epsilon, 1, 1)} \Gamma_1 \Gamma_{2q-2} \zeta_A \begin{pmatrix} 1 & 1 \\ 1 & 2q-2 \end{pmatrix} \\ & + a_{(q-1, q, q-1; \epsilon, 1, 1)} \Gamma_q \Gamma_{q-1} \zeta_A \begin{pmatrix} 1 & 1 \\ q & q-1 \end{pmatrix} = 0. \end{aligned}$$

Since $\zeta_A(2q-1)$ and $\zeta_A(1, 2q-2)$ are colinear, but $\zeta_A(2q-1)$ and $\zeta_A(q, q-1)$ are not colinear, and $\zeta(q, q-1)$ is not zeta-like, we have $a_{(q-1, q, q-1; \epsilon, 1, 1)} = 0$. Further, similarly for $(\mathfrak{t}; \epsilon) = (q-1; \epsilon_1)$ for arbitrary $\epsilon_1 \in \mathbb{F}_q^\times$, by (5.5) $f_{(q-1; \epsilon_1)} \in K$, so

$$a_{(q-1, 2q-1; \epsilon_1, \epsilon \epsilon^{-1})} \gamma \gamma_1^{-1} \Gamma_{2q-1} \zeta_A \begin{pmatrix} 1 \\ 2q-1 \end{pmatrix}$$

$$+ a_{(q-1,1,2q-2;\epsilon_1,\epsilon_1^{-1},1,1)} \gamma \gamma_1^{-1} \Gamma_1 \Gamma_{2q-2} \zeta_A \begin{pmatrix} 1 & 1 \\ 1 & 2q-2 \end{pmatrix} = 0$$

Note that $\Gamma_1 = 1$, $\Gamma_{2q-1} = \Gamma_{2q-2} = (\theta^q - \theta)$, and $\zeta_A(2q-1) = -(\theta^q - \theta)\zeta_A(1, 2q-2)$ is transcendental over K . Thus we have

$$(5.19) \quad a_{(q-1,1,2q-2;\epsilon_1,\epsilon_1^{-1},1)} - a_{(q-1,2q-1;\epsilon_1,\epsilon_1^{-1})}(t^q - t) = 0.$$

Thus we reduce to the following second shortlist:

$$(2q-2, q; \epsilon, 1), \quad (1, 3q-3; \epsilon, 1), \quad (q-1, 2q-1; \epsilon_1, \epsilon_1^{-1}), \\ (q-1, 1, 2q-2; \epsilon_2, \epsilon_2^{-1}, 1), \quad (2q-2, 1, q-1; \epsilon_3, \epsilon_3^{-1}, 1),$$

with each given $\epsilon \in \mathbb{F}_q^\times$ and $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{F}_q^\times$. We note that we have reindexed the characters for the sake of convenience.

From here, we proceed as done in the discussion preceding Theorem 2.5 to examine the linear independency. We construct Φ' with respect to the following tuples in I . Note that $\epsilon \in \mathbb{F}_q^\times$ is given throughout the remaining part of the proof, and $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \mathbb{F}_q^\times$ are free.

$$\emptyset, (1; \epsilon), (q-1; \epsilon_1), (q-1, 1; \epsilon_2, \epsilon_2^{-1}), (2q-2; \epsilon_3), (2q-2, 1; \epsilon_4, \epsilon_4^{-1}).$$

Recalling the construction of Φ' and \mathbf{v} , we get

$$\begin{aligned} \Phi'_{\emptyset, \emptyset} &= (t - \theta)^{3q-2}, \\ \Phi'_{(1; \epsilon), \emptyset} &= (\gamma H_1)^{(-1)}(t - \theta)^{3q-2}, \\ \Phi'_{(1; \epsilon), (1; \epsilon)} &= (t - \theta)^{3q-3}, \\ \Phi'_{(q-1; \epsilon_1), \emptyset} &= (\gamma_1 H_{q-1})^{(-1)}(t - \theta)^{3q-2}, \\ \Phi'_{(q-1; \epsilon_1), (q-1; \epsilon_1)} &= (t - \theta)^{2q-1}, \\ \Phi'_{(q-1, 1; \epsilon_2, \epsilon_2^{-1}), (q-1; \epsilon_2)} &= (\gamma \gamma_2^{-1} H_1)^{(-1)}(t - \theta)^{2q-1}, \\ \Phi'_{(q-1, 1; \epsilon_2, \epsilon_2^{-1}), (q-1, 1; \epsilon_2, \epsilon_2^{-1})} &= (t - \theta)^{2q-2}, \\ \Phi'_{(2q-2; \epsilon_3), \emptyset} &= (\gamma_3 H_{2q-2})^{(-1)}(t - \theta)^{3q-2}, \\ \Phi'_{(2q-2; \epsilon_3), (2q-2; \epsilon_3)} &= (t - \theta)^q, \\ \Phi'_{(2q-2, 1; \epsilon_4, \epsilon_4^{-1}), (2q-2; \epsilon_4)} &= (\gamma \gamma_4^{-1} H_1)^{(-1)}(t - \theta)^q, \\ \Phi'_{(2q-2, 1; \epsilon_4, \epsilon_4^{-1}), (2q-2, 1; \epsilon_4, \epsilon_4^{-1})} &= (t - \theta)^{q-1}, \\ \text{others} &= 0, \end{aligned}$$

and

$$\begin{aligned} v_{(1; \epsilon)} &= a_{(1, 3q-3; \epsilon, 1)} H_{3q-3}^{(-1)}(t - \theta)^{3q-3} \\ v_{(q-1; \epsilon_1)} &= a_{(q-1, 2q-1; \epsilon_1, \epsilon_1^{-1})} (\gamma \gamma_1^{-1} H_{2q-1})^{(-1)}(t - \theta)^{2q-1} \\ v_{(q-1, 1; \epsilon_2, \epsilon_2^{-1})} &= a_{(q-1, 1, 2q-2; \epsilon_2, \epsilon_2^{-1}, 1)} H_{2q-2}^{(-1)}(t - \theta)^{2q-2} \\ v_{(2q-2; \epsilon)} &= a_{(2q-2, q; \epsilon, 1)} H_q^{(-1)}(t - \theta)^q \\ v_{(2q-2, 1; \epsilon_3, \epsilon_3^{-1})} &= a_{(2q-2, 1, q-1; \epsilon_3, \epsilon_3^{-1}, 1)} H_{q-1}^{(-1)}(t - \theta)^{q-1}, \\ \text{others} &= 0, \end{aligned}$$

where γ and γ_j are $(q-1)$ th roots of unity of ϵ and ϵ_j , for $j = 1, \dots, 4$, respectively. For some δ_i 's in $\overline{K}[t]$ for corresponding indices i , Eq. (2.10) induces: for given $\epsilon \in \mathbb{F}_q^\times$ and for free $\epsilon_1 \in \mathbb{F}_q^\times$,

$$(5.20) \quad \delta_\emptyset = \left(\delta_\emptyset^{(-1)} + \sum_{\epsilon_1 \in \mathbb{F}_q^\times} \left((\gamma_1 H_{2q-2} \delta_{(2q-2; \epsilon_1)})^{(-1)} + (\gamma_1 H_{q-1} \delta_{(q-1; \epsilon_1)})^{(-1)} \right) + (\gamma H_1)^{(-1)} \delta_{(1; \epsilon)}^{(-1)} \right) (t - \theta)^{3q-2}$$

$$(5.21) \quad \delta_{(1; \epsilon)} = \left(\delta_{(1; \epsilon)}^{(-1)} + a_{(1, 3q-3; \epsilon, 1)} H_{3q-3}^{(-1)} \right) (t - \theta)^{3q-3}$$

(5.22)

$$\delta_{(2q-2; \epsilon)} = \left(\delta_{(2q-2; \epsilon)}^{(-1)} + \delta_{(2q-2, 1; \epsilon, 1)}^{(-1)} H_1^{(-1)} + a_{(2q-2, q; \epsilon, 1)} H_q^{(-1)} \right) (t - \theta)^q$$

(5.23)

$$\delta_{(2q-2; \epsilon_1)} = \left(\delta_{(2q-2; \epsilon_1)}^{(-1)} + \delta_{(2q-2, 1; \epsilon_1, \epsilon_1^{-1})}^{(-1)} (\gamma \gamma_1^{-1} H_1)^{(-1)} \right) (t - \theta)^q \quad \text{for } \epsilon_1 \neq \epsilon$$

(5.24)

$$\delta_{(q-1; \epsilon_1)} = \left(\delta_{(q-1; \epsilon_1)}^{(-1)} + \delta_{(q-1, 1; \epsilon_1, \epsilon_1^{-1})}^{(-1)} (\gamma \gamma_1^{-1} H_1)^{(-1)} + a_{(q-1, 2q-1; \epsilon_1, \epsilon_1^{-1})} (\gamma \gamma_1^{-1} H_{2q-1})^{(-1)} \right) (t - \theta)^{2q-1}$$

(5.25)

$$\delta_{(q-1, 1; \epsilon_1, \epsilon_1^{-1})} = \left(\delta_{(q-1, 1; \epsilon_1, \epsilon_1^{-1})}^{(-1)} + a_{(q-1, 1, 2q-2; \epsilon_1, \epsilon_1^{-1}, 1)} H_{2q-2}^{(-1)} \right) (t - \theta)^{2q-2}$$

(5.26)

$$\delta_{(2q-2, 1; \epsilon_1, \epsilon_1^{-1})} = \left(\delta_{(2q-2, 1; \epsilon_1, \epsilon_1^{-1})}^{(-1)} + a_{(2q-2, 1, q-1; \epsilon_1, \epsilon_1^{-1}, 1)} H_{q-1}^{(-1)} \right) (t - \theta)^{q-1}$$

Recall again that $H_1 = H_{q-1} = H_q = 1$, $H_{2q-1} = \theta^q - t$, $H_{2q-2} = 2\theta^q - t - t^q$, and

$$H_{3q-3} = \begin{cases} (t - t^q)^2 + 3(t^q - \theta^q)(t - \theta^q), & \text{if } q > 3, \\ (t^q - t), & \text{if } q = 3. \end{cases}$$

Consider (5.26). Lemma 4.3 shows that $\delta_{(2q-2, 1; \epsilon_1, \epsilon_1^{-1})} \in \mathbb{F}_q[t, \theta]$ and $\deg_\theta \delta_{(2q-2, 1; \epsilon_1, \epsilon_1^{-1})} \leq q$. We let $\delta_{(2q-2, 1; \epsilon_1, \epsilon_1^{-1})} = (f_{\epsilon_1} \theta + g_{\epsilon_1})(t - \theta)^{q-1}$ with $f_{\epsilon_1}, g_{\epsilon_1} \in \mathbb{F}_q[t]$. Then, we plug it into (5.26) and compare the coefficients of powers of θ -terms, to obtain

$$\delta_{(2q-2, 1; \epsilon_1, \epsilon_1^{-1})} = f_{\epsilon_1} (\theta - t)^q, \text{ and } a_{(2q-2, 1, q-1; \epsilon_1, \epsilon_1^{-1}, 1)} = f_{\epsilon_1} (t^q - t).$$

Similarly for (5.25), $\delta_{(q-1, 1; \epsilon_1, \epsilon_1^{-1})} \in \mathbb{F}_q[t, \theta]$ and $\deg_\theta \delta_{(q-1, 1; \epsilon_1, \epsilon_1^{-1})} \leq 2q$. We let $\delta_{(q-1, 1; \epsilon_1, \epsilon_1^{-1})} = (f'_{\epsilon_1} \theta^2 + g'_{\epsilon_1} \theta + h'_{\epsilon_1})(t - \theta)^{2q-2}$ with $f'_{\epsilon_1}, g'_{\epsilon_1}, h'_{\epsilon_1} \in \mathbb{F}_q[t]$. Then, we plug it into (5.25) and compare the coefficients of powers of θ -terms, to obtain

$$\delta_{(q-1, 1; \epsilon_1, \epsilon_1^{-1})} = f'_{\epsilon_1} (\theta - t)^{2q}, \text{ and } a_{(q-1, 1, 2q-2; \epsilon_1, \epsilon_1^{-1}, 1)} = f'_{\epsilon_1} (t^q - t).$$

For (5.24), Lemma 4.3 shows that $\delta_{(q-1; \epsilon_1)} \in \overline{\mathbb{F}_q}[t, \theta]$ with $\deg_\theta \delta_{(q-1; \epsilon_1)} \leq 2q+1$. We let $\delta_{(q-1; \epsilon_1)} = (a\theta^2 + b\theta + c)(t - \theta)^{2q-1}$ for $a, b, c \in \overline{\mathbb{F}_q}[t]$ to obtain

$$\begin{aligned} a^{(1)} \theta^{2q} + b^{(1)} \theta^q + c^{(1)} &= (a\theta^2 + b\theta + c) (t - \theta)^{2q-1} \\ &\quad + \gamma_2 f'_{\epsilon_1} (\theta^q - t^q)^2 + \gamma_2 a_{(q-1, 2q-1; \epsilon_1, \epsilon_1^{-1})} (\theta^q - t). \end{aligned}$$

Here $\gamma_2 = \gamma\gamma_1^{-1}$. Comparing the coefficients of the θ^{2q-1} - and θ^{2q-2} -terms yields $a = 0$, $b = \gamma_2 f'_{\epsilon_1}$, $c = -\gamma_2 t f'_{\epsilon_1}$.

In summary,

$$\begin{aligned}\delta_{(q-1;\epsilon_1)} &= -\gamma\gamma_1^{-1} f'_{\epsilon_1} (\theta - t)^{2q}, \\ a_{(q-1,2q-1;\epsilon_1,\epsilon\epsilon_1^{-1})} &= \gamma_2^{-1} \gamma_2^q f'_{\epsilon_1} = \epsilon\epsilon_1^{-1} f'_{\epsilon_1}.\end{aligned}$$

Further, recalling (5.19), the above equations yields $(t^q - t)f'_{\epsilon_1} = (t^q - t)\epsilon\epsilon_1^{-1} f'_{\epsilon_1}$, i.e., $(\epsilon - \epsilon_1)f'_{\epsilon_1} = 0$. Thus $f'_{\epsilon_1} = 0$ for all $\epsilon_1 \neq \epsilon$, and

$$(5.27) \quad \begin{aligned}\delta_{(q-1;\epsilon_1)} &= \begin{cases} -f'_{\epsilon} (\theta - t)^{2q}, & \text{if } \epsilon_1 = \epsilon, \\ 0, & \text{if } \epsilon_1 \neq \epsilon. \end{cases} \\ a_{(q-1,2q-1;\epsilon_1,\epsilon\epsilon_1^{-1})} &= \begin{cases} f'_{\epsilon}, & \text{if } \epsilon_1 = \epsilon, \\ 0, & \text{if } \epsilon_1 \neq \epsilon. \end{cases}\end{aligned}$$

For (5.23), Lemma 4.3 shows that when $\epsilon_1 \neq \epsilon$, $\delta_{(2q-2;\epsilon_1)} \in \overline{\mathbb{F}_q}[t, \theta]$ with $\deg_{\theta} \delta_{(2q-2;\epsilon_1)} \leq q + 1$. Letting $\delta_{(2q-2;\epsilon)} = (a\theta + b)(t - \theta)^q$ for $a, b \in \overline{\mathbb{F}_q}[t]$, cancelling, twisting once and comparing the θ^q - and θ^{q+1} -terms yields $a = 0$, $b = \gamma\gamma_1^{-1} f'_{\epsilon_1}$, i.e., $\delta_{(2q-2;\epsilon_1)} = -\gamma\gamma_1^{-1} f'_{\epsilon_1} (\theta - t)^q$, and plugging it into (5.23) yields $f'_{\epsilon_1} = 0$, for $\epsilon_1 \neq \epsilon$. Note that this implies that

$$(5.28) \quad \delta_{(2q-2;\epsilon_1)} = 0 \text{ and } a_{(2q-2,1,q-1;\epsilon_1,\epsilon\epsilon_1^{-1},1)} = 0 \text{ if } \epsilon_1 \neq \epsilon.$$

Similarly for (5.22), Lemma 4.3 shows that $\delta_{(2q-2;\epsilon)} \in \mathbb{F}_q[t, \theta]$ with $\deg_{\theta} \delta_{(2q-2;\epsilon)} \leq q + 1$. Letting $\delta_{(2q-2;\epsilon)} = (a\theta + b)(t - \theta)^q$ for $a, b \in \mathbb{F}_q[t]$, cancelling, twisting once and comparing the θ^q - and θ^{q+1} -terms yield $a = 0$, $b = f_{\epsilon}$, i.e.,

$$\delta_{(2q-2;\epsilon)} = -f_{\epsilon} (\theta - t)^q, \text{ and } a_{(2q-2,q;\epsilon,1)} = f_{\epsilon}.$$

Now, for (5.21), Lemma 4.3 shows that $\delta_{(1;\epsilon)} \in \mathbb{F}_q[t, \theta]$ with $\deg_{\theta} \delta_{(1;\epsilon)} \leq 3q$. Letting $\delta_{(1;\epsilon)} = (a\theta^3 + b\theta^2 + c\theta + d)(t - \theta)^{3q-3}$ with some $a, b, c, d \in \mathbb{F}_q[t]$, cancelling, twisting once and comparing the θ^{3q-2} - and θ^{3q-1} -terms yield $b = -3at$, $c = 3at^2$. When $q > 3$, comparing the θ^{3q-3} -terms yields $d = -at^3$, i.e., $\delta_{(1;\epsilon)} = a(\theta - t)^{3q}$; when $q = 3$, comparing the θ^{3q-3} -terms yield $d = -at^3$, i.e., $\delta_{(1;\epsilon)} = a(\theta - t)^{3q}$. So in any case $\delta_{(1;\epsilon)} = f''(\theta - t)^{3q}$ for $f'' = a \in \mathbb{F}_q[t]$. Again, when $q > 3$, one can put this into (5.21), cancel terms, twist once and compare the θ^0 -terms to have $a_{(1,3q-3;\epsilon,1)} = f''(t^q - t)$; when $q = 3$, one have $a_{(1,3q-3;\epsilon,1)} = f''(t^q - t)^2$. In summary,

$$\delta_{(1;\epsilon)} = f''(\theta - t)^{3q}, \text{ and } a_{(1,3q-3;\epsilon,1)} = \begin{cases} f''(t^q - t), & \text{if } q > 3, \\ f''(t^q - t)^2, & \text{if } q = 3. \end{cases}$$

Finally, with (5.27) and (5.28), (5.20) is written as

$$\delta_{\emptyset} = \left(\delta_{\emptyset}^{(-1)} + (\gamma(2\theta^q - t - t^q)\delta_{(2q-2;\epsilon)})^{(-1)} + (\gamma\delta_{(q-1;\epsilon)})^{(-1)} + \gamma^{(-1)}\delta_{(1;\epsilon)}^{(-1)} \right) (t - \theta)^{3q-2}.$$

Lemma 4.3 shows that $\delta_{\emptyset} \in \overline{\mathbb{F}_q}[t, \theta]$ with $\deg_{\theta} \delta_{\emptyset} \leq 3q + 1$. Let $\delta_{\emptyset} = \gamma(t - \theta)^{3q-2}G$ for some $G \in \overline{\mathbb{F}_q}[t, \theta]$ with $\deg_{\theta} G \leq 3$, then by canceling and twisting,

$$\epsilon G^{(1)} = (t - \theta)^{3q-2}G - (\theta - t)^q (f_{\epsilon}(2\theta^q - t - t^q) + f'_{\epsilon}(\theta - t)^q) + f''(\theta - t)^{3q},$$

so $(t - \theta)^{3q-2}G \in \overline{\mathbb{F}_q}[t, \theta^q]$, i.e., $G = (t - \theta)^2g$ with $g \in \overline{\mathbb{F}_q}[t]$. Then

$$(t - \theta)^{3q}(g - f'') - \epsilon(t - \theta^q)^2g^{(1)}$$

$$\begin{aligned}
&= (\theta - t)^q (f_\epsilon(2\theta^q - t - t^q) + f'_\epsilon(\theta - t)^q) \\
&= (\theta - t)^q(\theta^q - t)f_\epsilon + (\theta - t)^{2q}f_\epsilon + (\theta - t)^{2q}f_\epsilon.
\end{aligned}$$

Considering the factors of powers of the $(t - \theta^q)$ - and $(t - \theta)$ -terms, we have $g^{(1)} = 0$, i.e., $g = 0$, $f_\epsilon = 0$, $f'_\epsilon = 0$, and $f'' = 0$, so $a_{(2q-2,q;\epsilon,1)} = a_{(1,3q-3;\epsilon,1)} = a_{(2q-2,1;\epsilon_1,\epsilon_1^{-1})} = a_{(q-1,2q-1;\epsilon_1,\epsilon_1^{-1})} = a_{(q-1,1,2q-2;\epsilon_1,\epsilon_1^{-1},1)} = 0$ for arbitrary $\epsilon_1 \in \mathbb{F}_q^\times$, which is a contradiction. This completes the proof. \square

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