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# HYDRODYNAMIC LIMIT FOR THE NON-CUTOFF BOLTZMANN EQUATION

CHUQI CAO AND KLEBER CARRAPATOSO

ABSTRACT. This work deals with the non-cutoff Boltzmann equation for all type of potentials, in both the torus  $\mathbf{T}^3$  and in the whole space  $\mathbf{R}^3$ , under the incompressible Navier-Stokes scaling. We first establish the well-posedness and decay of global mild solutions to this rescaled Boltzmann equation in a perturbative framework, that is for solutions close to the Maxwellian, obtaining in particular integrated-in-time regularization estimates. We then combine these estimates with spectral-type estimates in order to obtain the strong convergence of solutions to the non-cutoff Boltzmann equation towards the incompressible Navier-Stokes-Fourier system.

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## 1. INTRODUCTION

Since Hilbert [50], an important problem in kinetic theory concerns the rigorous link between different scales of description of a gas. More precisely, one is interested in passing rigorously from a mesoscopic description of a gas, modeled by the kinetic Boltzmann equation, towards a macroscopic description, modeled by Euler or Navier-Stokes fluid equations, through a suitable scaling limit. We are interested in this paper on the convergence of solutions to the Boltzmann equation towards the incompressible Navier-Stokes equation, and we refer to the book [66] and the references therein to a detailed description of this type of problem as well as to different scalings and fluid limit equations.

We introduce in Section 1.1 below the (rescaled) Boltzmann equation, and then in Section 1.2 we describe the incompressible Navier-Stokes-Fourier system, which is the expected limit. We finally present our main results in Section 2.

**1.1. The Boltzmann equation.** The Boltzmann equation is a fundamental model in kinetic theory that describes the evolution of a rarefied gaz out of equilibrium by taking into account binary collisions between particles. More precisely, it describes the evolution in time of the unknown  $F(t, x, v) \geq 0$  which represents the density of particles that at time  $t \geq 0$  and position  $x \in \Omega_x = \mathbf{T}^3$  or  $\Omega_x = \mathbf{R}^3$  move with velocity  $v \in \mathbf{R}^3$ . It was introduced by Maxwell [62] and Boltzmann [15] and reads

$$(1.1) \quad \partial F + v \cdot \nabla_x F = \frac{1}{\varepsilon} Q(F, F),$$

which is complemented with an initial data  $F|_{t=0} = F_0$  and where  $\varepsilon \in (0, 1]$  is the Knudsen number, which corresponds to the ration between the mean-free path and the macroscopic length scale.

The Boltzmann collision operator  $Q$  is a bilinear operator acting only on the velocity variable  $v \in \mathbf{R}^3$ , which means that collisions are local in space, and it is given by

$$(1.2) \quad Q(G, F)(v) = \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} B(v - v_*, \sigma) (G'_* F' - G_* F) d\sigma dv_*,$$

where here and below we use the standard short-hand notation  $F = F(v)$ ,  $G_* = G(v_*)$ ,  $F' = F(v')$ , and  $G'_* = G(v'_*)$ , and where the pre- and post-collision velocities  $(v', v'_*)$  and  $(v, v_*)$  are related through

$$(1.3) \quad v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma \quad \text{and} \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma.$$

The above formula is one possible parametrization of the set of solutions of an elastic collision with the physical laws of conservation (momentum and energy)

$$v + v_* = v' + v'_* \quad \text{and} \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

The function  $B(v - v_*, \sigma)$  appearing in (1.2), called the collision kernel, is supposed to be nonnegative and to depend only on the relative velocity  $|v - v_*|$  and the deviation angle  $\theta$  through  $\cos \theta := \frac{v - v_*}{|v - v_*|} \cdot \sigma$ . As it is customary, we may suppose without loss of generality that  $\theta \in [0, \pi/2]$ , for otherwise  $B$  can be replaced by its symmetrized form.

In this paper we shall consider the case of *non-cutoff potentials* that we describe now. The collision kernel  $B$  takes the form

$$B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos \theta),$$

for some nonnegative function  $b$ , called the angular kernel, and some parameter  $\gamma \in (-3, 1]$ . We assume that the angular kernel  $b$  is a locally smooth implicit function which is not locally integrable, more precisely that it satisfies

$$\mathcal{K} \theta^{-1-2s} \leq \sin \theta b(\cos \theta) \leq \mathcal{K}^{-1} \theta^{-1-2s} \quad \text{with} \quad 0 < s < 1,$$

for some constant  $\mathcal{K} > 0$ . Moreover the parameters satisfy the condition

$$(1.4) \quad \max \left\{ -3, -\frac{3}{2} - 2s \right\} < \gamma \leq 1, \quad 0 < s < 1, \quad \gamma + 2s > -1.$$

We shall consider in this paper the full range of parameters  $\gamma$  and  $s$  satisfying (1.4), and we classify them into two cases: When  $\gamma + 2s \geq 0$  we speak of *hard potentials*, and when  $\gamma + 2s < 0$  of *soft potentials*. We also mention that *cutoff kernels* correspond to the case in which we remove the singularity of the angular kernel  $b$  and assume that  $b$  is integrable.

*Remark 1.1.* When particles interact via a repulsive inverse-power law potential  $\phi(r) = r^{-(p-1)}$  with  $p > 2$ , then it holds (see [62, 25]) that  $\gamma = \frac{p-5}{p-1}$  and  $s = \frac{1}{p-1}$ . It is easy to check that  $\gamma + 4s = 1$  which means the above assumption is satisfied for the full range of the inverse power law model.

Formally if  $F$  is a solution to equation (1.1) with the initial data  $F_0$ , then it enjoys the conservation of mass, momentum and the energy, that is,

$$\frac{d}{dt} \int_{\Omega_x \times \mathbf{R}^3} F(t, x, v) \varphi(v) dv dx = 0, \quad \varphi(v) = 1, v, |v|^2,$$

which is a consequence of the collision invariants of the Boltzmann operator

$$\int_{\mathbf{R}^3} Q(F, F)(v) \varphi(v) dv = 0, \quad \varphi(v) = 1, v, |v|^2.$$

Moreover the Boltzmann H-theorem asserts on the one hand that the entropy

$$H(F) = \int_{\Omega_x \times \mathbf{R}^3} F \log F dv dx,$$

is non-increasing in time. Indeed, at least formally, since  $(x-y)(\log x - \log y)$  is nonnegative, we have the following inequality for the entropy dissipation  $D(f)$ :

$$\begin{aligned} D(f) &= -\frac{d}{dt}H(F) = -\int_{\Omega_x \times \mathbf{R}^3} Q(F, F) \, dv \, dx \\ &= \frac{1}{4} \int_{\Omega_x \times \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \sigma) (F' F'_* - F_* F) \log \left( \frac{F' F'_*}{F F_*} \right) \, d\sigma \, dv_* \, dv \, dx \geq 0. \end{aligned}$$

On the other hand, the second part of the H-theorem asserts that local equilibria of the Boltzmann equation are local Maxwellian distributions in velocity, more precisely that

$$D(F) = 0 \quad \Leftrightarrow \quad Q(F, F) = 0 \quad \Leftrightarrow \quad F(t, x, v) = \frac{\rho(t, x)}{(2\pi\theta(t, x))^{3/2}} \exp \left( -\frac{|v - u(t, x)|^2}{2\theta(t, x)} \right),$$

with  $\rho(t, x) > 0$ ,  $u(t, x) \in \mathbf{R}^3$  and  $\theta(t, x) > 0$ . In what follows, we denote by  $\mu = \mu(v)$  the global Maxwellian

$$\mu = (2\pi)^{-3/2} e^{-|v|^2/2}.$$

Observing that the effect of collisions are enhanced when taking small parameter  $\varepsilon \in (0, 1]$ , one can expect from the above H-Theorem that, at least formally, in the limit  $\varepsilon \rightarrow 0$  the solution  $F$  approaches a local Maxwellian equilibrium. One therefore considers, see for instance in [12], a rescaling of the solution  $F$  of (1.1) in which an additional dilatation of the macroscopic time scale has been performed in order to be able to reach the Navier-Stokes equation in the limit. This procedure gives us the following rescaled Boltzmann equation for the new unknown  $F^\varepsilon = F^\varepsilon(t, x, v)$ :

$$(1.5) \quad \partial F^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon^2} Q(F^\varepsilon, F^\varepsilon),$$

with initial data  $F^\varepsilon|_{t=0} = F_0^\varepsilon$ .

In the torus case  $\Omega_x = \mathbf{T}^3$  (normalized as  $|\mathbf{T}^3| = 1$ ), we shall always assume, thanks to the conservation laws, that the initial datum  $F_0^\varepsilon$  satisfies the normalization

$$(1.6) \quad \int_{\mathbf{T}^3} \int_{\mathbf{R}^3} F_0^\varepsilon(x, v) [1, v, |v|^2] \, dv \, dx = [1, 0, 3],$$

that is, the initial data  $F_0^\varepsilon$  has the same mass, momentum and energy as  $\mu$ , and the Maxwellian  $\mu$  is the unique global equilibrium to (1.5).

In order to relate the above rescaled Boltzmann equation (1.5) to the expected incompressible Navier-Stokes-Fourier system (described below in (1.13)) in the limit  $\varepsilon \rightarrow 0$ , we are going to work with the perturbation  $f^\varepsilon$  defined by

$$(1.7) \quad F^\varepsilon = \mu + \varepsilon \sqrt{\mu} f^\varepsilon,$$

which then satisfies the equation

$$(1.8) \quad \partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} L f^\varepsilon + \frac{1}{\varepsilon} \Gamma(f^\varepsilon, f^\varepsilon),$$

with initial data  $f_0^\varepsilon = \frac{F_0^\varepsilon - \mu}{\varepsilon \sqrt{\mu}}$ , and where we denote

$$(1.9) \quad \Gamma(f, g) = \mu^{-1/2} Q(\sqrt{\mu} f, \sqrt{\mu} g),$$

and

$$(1.10) \quad Lf = \Gamma(\sqrt{\mu}, f) + \Gamma(f, \sqrt{\mu}).$$

In the case of the torus  $\Omega_x = \mathbf{T}^3$ , we observe from (1.6) that  $f_0^\varepsilon$  satisfies

$$(1.11) \quad \int_{\mathbf{T}^3} \int_{\mathbf{R}^3} f_0^\varepsilon(x, v) [1, v, |v|^2] \sqrt{\mu}(v) \, dv \, dx = 0,$$

and from the conservation laws recalled above that

$$(1.12) \quad \int_{\mathbf{T}^3} \int_{\mathbf{R}^3} f^\varepsilon(t, x, v) [1, v, |v|^2] \sqrt{\mu}(v) \, dv \, dx = 0.$$

**1.2. The Navier-Stokes-Fourier system.** We recall the Navier-Stokes-Fourier system associated with the Boussinesq equation which writes

$$(1.13) \quad \begin{cases} \partial_t u + u \cdot \nabla_x u - \nu_1 \Delta_x u = \nabla_x p, \\ \partial_t \theta + u \cdot \nabla_x \theta - \nu_2 \Delta_x \theta = 0, \\ \operatorname{div}_x u = 0, \\ \nabla_x(\rho + \theta) = 0, \end{cases}$$

with positive viscosity coefficients  $\nu_1, \nu_2 > 0$ . In this system, the temperature  $\theta = \theta(t, x) : \mathbf{R}_+ \times \Omega_x \rightarrow \mathbf{R}$  of the fluid, the density  $\rho = \rho(t, x) : \mathbf{R}_+ \times \Omega_x \rightarrow \mathbf{R}$  of the fluid, and the pressure  $p = p(t, x) : \mathbf{R}_+ \times \Omega_x \rightarrow \mathbf{R}$  of the fluid are scalar unknowns, whereas the velocity  $u = u(t, x) : \mathbf{R}_+ \times \Omega_x \rightarrow \mathbf{R}^3$  of the fluid is an unknown vector field. The pressure  $p$  can actually be eliminated from the equation by applying to the first equation in (1.13) the Leray projector  $\mathbb{P}$  onto the space of divergence-free vector fields. In other words, for  $u$  we have

$$\partial_t u - \nu_1 \Delta_x u = Q_{\text{NS}}(u, u),$$

where the bilinear operator  $Q_{\text{NS}}$  is defined by

$$Q_{\text{NS}}(v, u) = -\frac{1}{2} \mathbb{P}(\operatorname{div}(v \otimes u) + \operatorname{div}(u \otimes v)), \quad \operatorname{div}(v \otimes u)^j := \sum_{k=1}^3 \partial_k (v^j u^k) = \operatorname{div}(v^j u),$$

and the Leray projector  $\mathbb{P}$  on divergence-free vector fields is as follows, for  $1 \leq j \leq 3$  and all  $\xi \in \Omega'_\xi$ ,

$$\mathcal{F}_x(\mathbb{P}f)^j(\xi) = \mathcal{F}_x(f^j)(\xi) - \frac{1}{|\xi|^2} \sum_{k=1}^3 \xi_j \xi_k \mathcal{F}_x(f^k)(\xi) = \sum_{k=1}^3 (\delta_{j,k} - 1) \frac{\xi_j \xi_k}{|\xi|^2} \mathcal{F}_x(f^k)(\xi),$$

where  $\mathcal{F}_x$  denotes the Fourier transform in the spatial variable  $x \in \Omega_x$ , see for instance [10, Section 5.1].

We therefore consider the system

$$(1.14) \quad \begin{cases} \partial_t u - \nu_1 \Delta_x u = Q_{\text{NS}}(u, u), \\ \partial_t \theta + u \cdot \nabla_x \theta - \nu_2 \Delta_x \theta = 0, \\ \operatorname{div}_x u = 0, \\ \nabla_x(\rho + \theta) = 0, \end{cases}$$

for the unknown  $(\rho, u, \theta)$ , which is complemented with a initial data  $(\rho_0, u_0, \theta_0)$  that we shall always suppose to verify

$$(1.15) \quad \operatorname{div}_x u_0 = 0, \quad \nabla_x(\rho_0 + \theta_0) = 0.$$

In the case of the torus  $\Omega_x = \mathbf{T}^3$ , we suppose moreover that the initial data is mean-free, namely

$$\int_{\mathbf{T}^3} \rho_0(x) \, dx = \int_{\mathbf{T}^3} u_0(x) \, dx = \int_{\mathbf{T}^3} \theta_0(x) \, dx = 0,$$

which then implies that the associated solution  $(\rho, u, \theta)$  also is mean-free

$$(1.16) \quad \int_{\mathbf{T}^3} \rho(t, x) \, dx = \int_{\mathbf{T}^3} u(t, x) \, dx = \int_{\mathbf{T}^3} \theta(t, x) \, dx = 0.$$

## 2. MAIN RESULTS

Before stating our results we introduce some notation. Given a function  $f = f(x, v)$  we denote  $\widehat{f}(\xi, v) = \mathcal{F}_x(f(\cdot, v))(\xi)$  the Fourier transform in the space variable, for  $\xi \in \Omega'_\xi = \mathbf{Z}^3$  (if  $\Omega_x = \mathbf{T}^3$ ) or  $\Omega'_\xi = \mathbf{R}^3$  (if  $\Omega_x = \mathbf{R}^3$ ), more precisely

$$\widehat{f}(\xi, v) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbf{R}^3} e^{-ix \cdot \xi} f(x, v) \, dx.$$

In particular, we observe that if  $f$  satisfies (1.8), then for all  $\xi \in \Omega'_\xi$ , its Fourier transform in space  $\widehat{f}^\varepsilon(\xi)$  satisfies the equation

$$(2.1) \quad \partial_t \widehat{f}^\varepsilon(\xi) = \frac{1}{\varepsilon^2} (L - i\varepsilon v \cdot \xi) \widehat{f}^\varepsilon(\xi) + \frac{1}{\varepsilon} \widehat{\Gamma}(f^\varepsilon, f^\varepsilon)(\xi),$$

where

$$\widehat{\Gamma}(f, g)(\xi) = \sum_{\eta \in \mathbf{Z}^3} \Gamma(\widehat{f}(\xi - \eta), \widehat{g}(\eta)) \quad \text{if } \Omega_x = \mathbf{T}^3,$$

or

$$\widehat{\Gamma}(f, g)(\xi) = \int_{\mathbf{R}^3} \Gamma(\widehat{f}(\xi - \eta), \widehat{g}(\eta)) d\eta \quad \text{if } \Omega_x = \mathbf{R}^3.$$

For functions  $f = f(x, v)$  we write the *micro-macro decomposition*

$$(2.2) \quad f = \mathbf{P}^\perp f + \mathbf{P}f, \quad \mathbf{P}^\perp = I - \mathbf{P},$$

where  $\mathbf{P}$  is the orthogonal projection onto  $\text{Ker}(L) = \{\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}\}$  given by

$$(2.3) \quad \mathbf{P}f(x, v) = \left\{ \rho[f](x) + u[f](x) \cdot v + \theta[f](x) \frac{(|v|^2 - 3)}{2} \right\} \sqrt{\mu}(v),$$

where

$$\begin{aligned} \rho[f](x) &= \int_{\mathbf{R}^3} f(x, v) \sqrt{\mu}(v) dv, \\ u[f](x) &= \int_{\mathbf{R}^3} f(x, v) v \sqrt{\mu}(v) dv, \\ \theta[f](x) &= \int_{\mathbf{R}^3} f(x, v) \frac{(|v|^2 - 3)}{3} \sqrt{\mu}(v) dv. \end{aligned}$$

The function  $\mathbf{P}^\perp f$  is called the *microscopic part* of  $f$ , whereas  $\mathbf{P}f$  is the *macroscopic part* of  $f$ .

We now introduce the functional spaces we work with. For every  $\ell \geq 0$  we denote by  $L_v^2(\langle v \rangle^\ell)$  the weighted Lebesgue space associated to the inner product

$$\langle f, g \rangle_{L_v^2(\langle v \rangle^\ell)} := \langle \langle v \rangle^\ell f, \langle v \rangle^\ell g \rangle_{L_v^2} = \int_{\mathbf{R}^3} f g \langle v \rangle^{2\ell} dv,$$

and the norm

$$\|f\|_{L_v^2(\langle v \rangle^\ell)} := \|\langle v \rangle^\ell f\|_{L_v^2},$$

where  $L_v^2 = L^2(\mathbf{R}_v^3)$  is the standard Lebesgue space. We denote by  $H_v^{s,*}$  the Sobolev-type space associated to the dissipation of the linearized operator  $L$  defined in [4] (see also [44] for the definition of a different but equivalent anisotropic norm), more precisely we denote

$$(2.4) \quad \|f\|_{H_v^{s,*}(\langle v \rangle^\ell)} := \|\langle v \rangle^\ell f\|_{H_v^{s,*}},$$

where

$$(2.5) \quad \begin{aligned} \|f\|_{H_v^{s,*}}^2 &:= \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} b(\cos \theta) |v - v_*|^\gamma \mu(v_*) [f(v') - f(v)]^2 d\sigma dv_* dv \\ &+ \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} b(\cos \theta) |v - v_*|^\gamma f(v_*)^2 [\sqrt{\mu}(v') - \sqrt{\mu}(v)]^2 d\sigma dv_* dv, \end{aligned}$$

which verifies, see [4, 44],

$$\|\langle v \rangle^{\gamma/2+s} f\|_{L_v^2(\langle v \rangle^\ell)} + \|\langle v \rangle^{\gamma/2} f\|_{H_v^s(\langle v \rangle^\ell)} \lesssim \|f\|_{H_v^{s,*}(\langle v \rangle^\ell)} \lesssim \|\langle v \rangle^{\gamma/2+s} f\|_{H_v^s(\langle v \rangle^\ell)}.$$

We also define the space  $(H_v^{s,*})'$  as the dual of  $H_v^{s,*}$ , namely

$$(2.6) \quad \|f\|_{(H_v^{s,*})'} := \sup_{\|\phi\|_{H_v^{s,*}} \leq 1} \langle f, \phi \rangle_{L_v^2}.$$

We further define the space  $H_v^{s,**}(\langle v \rangle^\ell)$  as the space associated to the norm

$$(2.7) \quad \|f\|_{H_v^{s,**}(\langle v \rangle^\ell)}^2 := \|\mathbf{P}^\perp f\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \|a(D_x) \mathbf{P}f\|_{L_v^2}^2,$$

where  $a(D_x)$  is the Fourier multiplier  $a(\xi) = \frac{|\xi|}{\langle \xi \rangle}$ , which gives, in Fourier variable,

$$(2.8) \quad \|\widehat{f}(\xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 = \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2.$$

Finally, given a functional space  $X$  in the variables  $(t, \xi, v)$ , we shall denote by  $\mathcal{F}_x^{-1}(X)$  the Fourier-based space defined as

$$\mathcal{F}_x^{-1}(X) := \left\{ f = f(t, x, v) \mid \widehat{f} \in X \right\}.$$

Hereafter, in order to deal with the torus case  $\Omega_x = \mathbf{T}^3$  and the whole space case  $\Omega_x = \mathbf{R}^3$  simultaneously, we denote  $L_\xi^p = \ell^p(\mathbf{Z}^3)$  in the torus case and  $L_\xi^p = L^p(\mathbf{R}^3)$  in the whole space case, moreover we abuse notation and write

$$\int_{\Omega'_\xi} \phi(\xi) \, d\xi := \begin{cases} \sum_{\xi \in \mathbf{Z}^3} \phi(\xi) & \text{if } \Omega'_\xi = \mathbf{Z}^3, \\ \int_{\mathbf{R}^3} \phi(\xi) \, d\xi & \text{if } \Omega'_\xi = \mathbf{R}^3. \end{cases}$$

In particular, we shall consider below functional spaces of the type  $\mathcal{F}_x^{-1}(L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell))$  and  $\mathcal{F}_x^{-1}(L_\xi^p L_v^2 H_v^{s,*}(\langle v \rangle^\ell))$  (or  $\mathcal{F}_x^{-1}(L_\xi^p L_v^2 H_v^{s,*}(\langle v \rangle^\ell))$ ) and the respective norms, for  $f = f(t, x, v)$ ,

$$\|\widehat{f}\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)} := \left( \int_{\Omega'_\xi} \sup_{t \geq 0} \|\widehat{f}(t, \xi, \cdot)\|_{L_v^2(\langle v \rangle^\ell)}^p \, d\xi \right)^{1/p} \quad \text{for } p \in [1, +\infty),$$

and

$$\|\widehat{f}\|_{L_\xi^p L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} := \left( \int_{\Omega'_\xi} \left\{ \int_0^\infty \|\widehat{f}(t, \xi, \cdot)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 \, dt \right\}^{p/2} \, d\xi \right)^{1/p} \quad \text{for } p \in [1, +\infty),$$

with the usual modification for  $p = +\infty$ .

**2.1. Well-posedness for the rescaled Boltzmann equation.** Our first result concerns the global well-posedness, regularization and decay for equation (1.8) for small initial data.

**Theorem 2.1** (Global well-posedness and decay for the Boltzmann equation). *Let  $\ell \geq 0$ . There is  $\eta_0 > 0$  small enough such that for all  $\varepsilon \in (0, 1]$  the following holds:*

(1) *Torus case  $\Omega_x = \mathbf{T}^3$ : For any initial data  $f_0^\varepsilon \in \mathcal{F}_x^{-1}(L_\xi^1 L_v^2(\langle v \rangle^\ell))$  satisfying (1.12) and  $\|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} \leq \eta_0$ , there exists a unique global mild solution  $f^\varepsilon \in \mathcal{F}_x^{-1}(L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell) \cap L_\xi^1 L_v^2 H_v^{s,*}(\langle v \rangle^\ell))$  to (1.8) satisfying (1.12) and the energy estimate*

$$(2.9) \quad \|\widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|\mathbf{P}^\perp \widehat{f}\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \|\mathbf{P} \widehat{f}\|_{L_\xi^1 L_t^2 L_v^2} \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)}.$$

*Moreover we have the following decay estimates: In the hard potentials case  $\gamma + 2s \geq 0$ , there exists  $\lambda > 0$  such that*

$$(2.10) \quad \|\mathbf{e}_\lambda \widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|\mathbf{e}_\lambda \mathbf{P}^\perp \widehat{f}\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \|\mathbf{e}_\lambda \mathbf{P} \widehat{f}\|_{L_\xi^1 L_t^2 L_v^2} \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)},$$

*where we denote  $\mathbf{e}_\lambda : t \mapsto e^{\lambda t}$ . In the soft potentials case  $\gamma + 2s < 0$ , if  $\ell > 0$  then for any  $0 < \omega < \frac{\ell}{|\gamma + 2s|}$  there holds*

$$(2.11) \quad \|\mathbf{p}_\omega \widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2} + \frac{1}{\varepsilon} \|\mathbf{p}_\omega \mathbf{P}^\perp \widehat{f}\|_{L_\xi^1 L_t^2 H_v^{s,*}} + \|\mathbf{p}_\omega \mathbf{P} \widehat{f}\|_{L_\xi^1 L_t^2 L_v^2} \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)},$$

*where we denote  $\mathbf{p}_\omega : t \mapsto (1 + t)^\omega$ .*

(2) *Whole space case  $\Omega_x = \mathbf{R}^3$ : Let  $p \in (3/2, \infty]$ . For any initial data  $f_0^\varepsilon \in \mathcal{F}_x^{-1}(L_\xi^1 L_v^2(\langle v \rangle^\ell) \cap L_\xi^p L_v^2(\langle v \rangle^\ell))$  satisfying  $\|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}_0^\varepsilon\|_{L_\xi^p L_v^2(\langle v \rangle^\ell)} \leq \eta_0$ , there exists a unique global mild*

solution  $f^\varepsilon \in \mathcal{F}_x^{-1}(L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell) \cap L_\xi^1 L_v^2 H_v^{s,**}(\langle v \rangle^\ell)) \cap \mathcal{F}_x^{-1}(L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell) \cap L_\xi^p L_v^2 H_v^{s,**}(\langle v \rangle^\ell))$  to (1.8) satisfying the energy estimate

$$(2.12) \quad \begin{aligned} & \|\widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|\mathbf{P}^\perp \widehat{f}\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f} \right\|_{L_\xi^1 L_t^2 L_v^2} \\ & + \|\widehat{f}\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|\mathbf{P}^\perp \widehat{f}\|_{L_\xi^p L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f} \right\|_{L_\xi^p L_t^2 L_v^2} \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}_0^\varepsilon\|_{L_\xi^p L_v^2(\langle v \rangle^\ell)}. \end{aligned}$$

Moreover we have the following decay estimates: In the hard potentials case  $\gamma + 2s \geq 0$ , for any  $0 < \vartheta < \frac{3}{2}(1 - \frac{1}{p})$  there holds

$$(2.13) \quad \begin{aligned} & \|\mathfrak{p}_\vartheta \widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|\mathfrak{p}_\vartheta \mathbf{P}^\perp \widehat{f}\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \mathfrak{p}_\vartheta \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f} \right\|_{L_\xi^1 L_t^2 L_v^2} \\ & \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}_0^\varepsilon\|_{L_\xi^p L_v^2(\langle v \rangle^\ell)}. \end{aligned}$$

where we denote  $\mathfrak{p}_\vartheta : t \mapsto (1+t)^\vartheta$ . In the soft potentials case  $\gamma + 2s < 0$ , if  $0 < \vartheta < \frac{3}{2}(1 - \frac{1}{p})$  and  $\ell > \vartheta|\gamma + 2s|$  there holds

$$(2.14) \quad \begin{aligned} & \|\mathfrak{p}_\vartheta \widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2} + \frac{1}{\varepsilon} \|\mathfrak{p}_\vartheta \mathbf{P}^\perp \widehat{f}\|_{L_\xi^1 L_t^2 H_v^{s,*}} + \left\| \mathfrak{p}_\vartheta \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f} \right\|_{L_\xi^1 L_t^2 L_v^2} \\ & \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}_0^\varepsilon\|_{L_\xi^p L_v^2(\langle v \rangle^\ell)}. \end{aligned}$$

The Cauchy theory and the large time behavior for Boltzmann equation for  $\varepsilon = 1$  have been extensively studied. Concerning the theory for large data, we only mention the global existence of renormalized solutions [32] for the cutoff Boltzmann equation, and the global existence of renormalized solutions with defect measure [6] for the non-cutoff Boltzmann equation.

We now give a very brief review for solutions to the Boltzmann equation in a perturbative framework, that is, for solutions near the Maxwellian. For the case of cutoff potentials, we refer to the works [43, 70, 71, 17, 72] as well as the more recent [73, 31] for global solutions in spaces of the form  $L_v^\infty H_x^N$ ; and to [54, 61, 47, 69, 33] for solutions in  $H_{x,v}^N$  or  $H_x^N L_v^2$ . On the other hand, for the non-cutoff Boltzmann equation, we refer to [44, 45] in the torus case and to [4, 2, 3] in the whole space case, for the first global solutions in spaces of the form  $H_{x,v}^N$  by working with anisotropic norms (see (2.5)). The optimal time-decay was obtained in [67] for the whole space, and recently [30] constructed global solutions in the whole space.

All the above results concern solutions with Gaussian decay in velocity, that is, they hold in functional spaces of the type  $H_{x,v}^N$  for the perturbation  $f$  defined in (1.7), which means that  $F - \mu \in H_{x,v}^N(\mu^{-1/2})$ . By developing decay estimates on the resolvents and semigroups of non-symmetric operators in Banach spaces, Gualdani-Mischler-Mouhot [46] proved nonlinear stability for the cutoff Boltzmann equation with hard potentials in  $L_v^1 L_x^\infty(\langle v \rangle^k \mu^{1/2})$ ,  $k > 2$ , that is, in spaces with polynomial decay in velocity ( $f \in L_v^1 L_x^\infty(\langle v \rangle^k \mu^{1/2})$  means  $F - \mu \in L_v^1 L_x^\infty(\langle v \rangle^k)$ ). In the same framework, the case of non-cutoff hard potentials was treated in [49, 7], and that of non-cutoff soft potentials in [22].

The aforementioned results were obtained in Sobolev-type spaces, very recently Duan, Liu, Sakamoto and Strain [34] obtained the well-posedness of the Boltzmann equation in Fourier-based spaces  $L_\xi^1 L_t^\infty L_x^2$  in the torus case, which was then extended to the whole space case by Duan, Sakamoto and Ueda in [35], see also [23] for the whole space case in polynomial weighted spaces. We also refer to the works [8, 21] for recent results on the well-posedness for non-cutoff Boltzmann using De Giorgi arguments.

In our paper, we establish uniform in  $\varepsilon$  estimates for the rescaled non-cutoff Boltzmann equation (1.8). Our result in Theorem 2.1 is similar to [34, 35], but the proof is quite different. We first investigate the semigroup  $U^\varepsilon$  associated to the linearized operator  $\frac{1}{\varepsilon^2}(L - \varepsilon v \cdot \nabla_x)$  appearing in (1.8). We provide boundedness and integrated-in-time regularization



estimates for  $U^\varepsilon$  (see Proposition 3.2), as well as for its integral in time against a source  $\int_0^t U^\varepsilon(t-s)S(s) ds$  (see Proposition 3.4). Together with nonlinear estimates for  $\Gamma$  (see Lemma 4.1), we are then able to take  $S$  equal to the nonlinear term  $\Gamma(f, f)$  and prove the global well-posedness of mild solutions of (1.8), namely

$$f^\varepsilon(t) = U^\varepsilon(t)f_0^\varepsilon + \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t-s)\Gamma(f^\varepsilon(s), f^\varepsilon(s)) ds,$$

by applying a fixed point argument. The decay estimate is then obtained as a consequence of decay estimates for  $U^\varepsilon$  (see Propositions 3.6 and 3.10) and for  $\int_0^t U^\varepsilon(t-s)S(s) ds$  (see Propositions 3.7 and 3.11). It is important to notice that the fixed point takes place in the space  $\mathcal{F}_x^{-1}(L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell) \cap L_\xi^1 L_v^2 H_v^{s,*}(\langle v \rangle^\ell))$  for the torus case, and in  $\mathcal{F}_x^{-1}(L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell) \cap L_\xi^1 L_v^2 H_v^{s,**}(\langle v \rangle^\ell)) \cap \mathcal{F}_x^{-1}(L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell) \cap L_\xi^p L_v^2 H_v^{s,**}(\langle v \rangle^\ell))$  for the whole space, that is, the integrated-in-time regularization appears in the functional space. Furthermore, the estimate for  $\int_0^t U^\varepsilon(t-s)S(s) ds$  is a key ingredient for our fixed point argument, and on the other hand it is also crucial for establishing the strong convergence in the proof of the hydrodynamic limit established below in Theorem 2.3.

**2.2. Well-posedness for the Navier-Stokes-Fourier system.** Our second result concerns the global well-posedness of the incompressible Navier-Stokes-Fourier system (1.14) for small initial data.

**Theorem 2.2** (Global well-posedness for the Navier-Stokes-Fourier system). *There exists  $\eta_1 > 0$  small enough such that the following holds:*

(1) *Torus case  $\Omega_x = \mathbf{T}^3$ : For any initial data  $(\rho_0, u_0, \theta_0) \in \mathcal{F}_x^{-1}(L_\xi^1)$  satisfying (1.16) and  $\|(\hat{\rho}_0, \hat{u}_0, \hat{\theta}_0)\|_{L_\xi^1} \leq \eta_1$ , there exists a unique global mild solution  $(\rho, u, \theta) \in \mathcal{F}_x^{-1}(L_\xi^1 L_t^\infty \cap L_\xi^1(\langle \xi \rangle) L_t^2)$  to the Navier-Stokes-Fourier system (1.14) satisfying (1.16) and the energy estimate*

$$\|(\hat{\rho}, \hat{u}, \hat{\theta})\|_{L_\xi^1 L_t^\infty} + \|\langle \xi \rangle(\hat{\rho}, \hat{u}, \hat{\theta})\|_{L_\xi^1 L_t^2} \lesssim \|(\hat{\rho}_0, \hat{u}_0, \hat{\theta}_0)\|_{L_\xi^1}.$$

(2) *Whole space case  $\Omega_x = \mathbf{R}^3$ : Let  $p \in (3/2, \infty]$ . For any initial data  $(\rho_0, u_0, \theta_0) \in \mathcal{F}_x^{-1}(L_\xi^1 \cap L_\xi^p)$  satisfying  $\|(\hat{\rho}_0, \hat{u}_0, \hat{\theta}_0)\|_{L_\xi^1} + \|(\hat{\rho}_0, \hat{u}_0, \hat{\theta}_0)\|_{L_\xi^p} \leq \eta_1$ , there exists a unique global mild solution  $(\rho, u, \theta) \in \mathcal{F}_x^{-1}(L_\xi^1 L_t^\infty \cap L_\xi^1(|\xi|) L_t^2 \cap L_\xi^p L_t^\infty \cap L_\xi^p(|\xi|) L_t^2)$  to the Navier-Stokes-Fourier system (1.14) satisfying the energy estimate*

$$\begin{aligned} \|(\hat{\rho}, \hat{u}, \hat{\theta})\|_{L_\xi^1 L_t^\infty} + \|\xi|(\hat{\rho}, \hat{u}, \hat{\theta})\|_{L_\xi^1 L_t^2} + \|(\hat{\rho}, \hat{u}, \hat{\theta})\|_{L_\xi^p L_t^\infty} + \|\xi|(\hat{\rho}, \hat{u}, \hat{\theta})\|_{L_\xi^p L_t^2} \\ \lesssim \|(\hat{\rho}_0, \hat{u}_0, \hat{\theta}_0)\|_{L_\xi^1} + \|(\hat{\rho}_0, \hat{u}_0, \hat{\theta}_0)\|_{L_\xi^p}. \end{aligned}$$

The incompressible Navier-Stokes equation, that is, the first equation in (1.14), possesses a vast literature so we only mention a few works in the three dimensional case below, and we refer the reader to the monographs [57, 10] and the references therein for more details. On the one hand, global weak solutions for large initial data were obtained in the pioneering work [58] (see also [51]). On the other hand, global mild solutions for small initial data were obtained in [37, 53, 28, 19, 20, 38] in different Lebesgue and Sobolev spaces, and we refer again to the book [57] for results in Besov and Morrey spaces. We mention in particular the work of Lei and Lin [56] where global mild solutions in the whole space  $\mathbf{R}^3$  were constructed in the Fourier-based space  $L_\xi^1(|\xi|^{-1})L_t^\infty$ .

Our results in Theorem 2.2 are maybe not completely new, but we do not have a reference for this precise functional setting (observe that the functional spaces in Theorem 2.2 correspond exactly to the same functional setting as in the global well-posedness for the Boltzmann equation in Theorem 2.1). Therefore, and also for the sake of completeness, we shall provide a complete proof of them in Section 5.

Our strategy for obtaining the global solution  $u$  for the incompressible Navier-Stokes equation follows a standard fixed point argument. As in the proof of Theorem 2.1, we first obtain boundedness and integrated-in-time regularization estimates for the semigroup  $V$  associated to the operator  $\nu_1 \Delta_x$  (see Proposition 5.1), as well as for its integral in time

against a source  $\int_0^t V(t-s)S(s)ds$  (see Proposition 5.3). We then combine this with estimates for the nonlinear term  $Q_{\text{NS}}$  (see Lemma 5.4) to obtain, thanks to a fixed point argument, the global well-posedness of mild solutions of the first equation in (1.14), namely

$$u(t) = V(t)u_0 + \int_0^t V(t-s)Q_{\text{NS}}(u(s), u(s)) ds.$$

Once the solution  $u$  is constructed, we can obtain in a similar (and even easier way) the well-posedness of mild solutions of the second equation in (1.14) for the temperature  $\theta$ . Finally we easily obtain the result for the density  $\rho$  thanks to the last equation in (1.14).

**2.3. Hydrodynamic limit.** Our third result regards the hydrodynamic limit of the rescaled Boltzmann equation, that is, we are interested in the behavior of solutions  $(f^\varepsilon)_{\varepsilon \in (0,1]}$  to (1.8) in the limit  $\varepsilon \rightarrow 0$ .

Let  $(\rho_0, u_0, \theta_0)$  be an initial data and consider the associated global solution  $(\rho, u, \theta)$  to the incompressible Navier-Stokes-Fourier system (1.14) given by Theorem 2.2, where the viscosity coefficients  $\nu_1, \nu_2 > 0$  are given as follows (see [12]): Let us introduce the two unique functions  $\Phi$  (which is a matrix-valued function) and  $\Psi$  (which is a vector-valued function) orthogonal to  $\text{Ker } L$  such that

$$\frac{1}{\sqrt{\mu}}L(\sqrt{\mu}\Phi) = \frac{|v|^2}{3}I_{3 \times 3} - v \otimes v, \quad \frac{1}{\sqrt{\mu}}L(\sqrt{\mu}\Psi) = \frac{5 - |v|^2}{2}v,$$

then the viscosity coefficients are defined by

$$\nu_1 = \frac{1}{10} \int_{\mathbf{R}^3} L(\sqrt{\mu}\Phi)\Phi\sqrt{\mu} dv, \quad \nu_2 = \frac{2}{15} \int_{\mathbf{R}^3} \Psi \cdot L(\sqrt{\mu}\Psi)\sqrt{\mu} dv.$$

We define the initial kinetic distribution  $g_0 \in \text{Ker } L$  associated to  $(\rho_0, u_0, \theta_0)$  by

$$(2.15) \quad g_0(x, v) = \mathbf{P}g_0(x, v) = \left[ \rho_0(x) + u_0(x) \cdot v + \theta_0(x) \frac{(|v|^2 - 3)}{2} \right] \sqrt{\mu}(v),$$

and then we consider the kinetic distribution  $g(t) \in \text{Ker } L$  associated to  $(\rho(t), u(t), \theta(t))$  by

$$(2.16) \quad g(t, x, v) = \mathbf{P}g(t, x, v) = \left[ \rho(t, x) + u(t, x) \cdot v + \theta(t, x) \frac{(|v|^2 - 3)}{2} \right] \sqrt{\mu}(v).$$

**Theorem 2.3** (Hydrodynamic limit). *Let  $(f_0^\varepsilon)_{\varepsilon \in (0,1]}$  satisfy the hypotheses of Theorem 2.1 and consider the associated global unique mild solution  $(f^\varepsilon)_{\varepsilon \in (0,1]}$  to (1.8). Let also  $(\rho_0, u_0, \theta_0)$  satisfy the hypotheses of Theorem 2.2 and consider the associated global unique mild solution  $(\rho, u, \theta)$  to (1.14). Finally, let  $g_0 = \mathbf{P}g_0$  be defined by (2.15) and  $g = \mathbf{P}g$  by (2.16). There exists  $0 < \eta_2 < \min(\eta_0, \eta_1)$  such that if*

$$\begin{aligned} \max \left( \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2}, \|\widehat{g}_0\|_{L_\xi^1 L_v^2} \right) &\leq \eta_2 \quad \text{in the case } \Omega_x = \mathbf{T}^3, \\ \max \left( \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2} + \|\widehat{f}_0^\varepsilon\|_{L_\xi^p L_v^2}, \|\widehat{g}_0\|_{L_\xi^1 L_v^2} + \|\widehat{g}_0\|_{L_\xi^p L_v^2} \right) &\leq \eta_2 \quad \text{in the case } \Omega_x = \mathbf{R}^3, \end{aligned}$$

for all  $\varepsilon \in (0, 1]$  and

$$\lim_{\varepsilon \rightarrow 0} \|\widehat{f}_0^\varepsilon - \widehat{g}_0\|_{L_\xi^1 L_v^2} = 0,$$

then there holds

$$(2.17) \quad \lim_{\varepsilon \rightarrow 0} \|\widehat{f}^\varepsilon - \widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2} = 0.$$

*Remark 2.4.* One can get a explicit rate of convergence in (2.17) if we suppose that the initial data  $g_0$  has some additional regularity in  $x$ , namely a rate of  $\varepsilon^\delta$  if the initial data  $g_0$  satisfies

$$\|\langle \xi \rangle^\delta \widehat{g}_0\|_{L_\xi^1 L_v^2} < \infty,$$

for  $\delta \in (0, 1]$ . We refer to (6.25) and (6.26) for a quantitative version of this result.

*Remark 2.5.* Our methods can also be applied to the Landau equation with Coulomb potential, and we obtain similar results as in Theorem 2.1 and in Theorem 2.3.

Before giving some comments on the above result and its strategy, we start by providing a short overview of the existing literature on the problem of deriving incompressible Navier-Stokes fluid equations from the kinetic Boltzmann one, and we refer to the book by Saint-Raymond [66] for a thorough presentation of the topic including other hydrodynamic limits. The first justifications of the link between kinetic and fluid equations were formal and based on asymptotic expansions by Hilbert, Chapman, Cowling and Grad (see [50, 27, 42]). The first rigorous convergence proofs based also on asymptotic expansions were given by Caflisch [18] (see also [55] and [29]). In those papers, the limit is justified up to the first singular time for the fluid equation. Guo [48] has justified the limit towards the Navier-Stokes equation and beyond in Hilbert's expansion for the cutoff Boltzmann and Landau equations.

In the framework of large data solutions, the weak convergence of global renormalized solutions of the cutoff Boltzmann equation of [32] towards global weak solution to the fluid system were obtained in [12, 11, 40, 41, 59, 60, 66]. Moreover, for the case of non-cutoff kernels, we refer to [9] who proved the hydrodynamic limit from global renormalized solutions with defect measure of [6].

We now discuss results in the framework of perturbative solutions, that is, solutions near the Maxwellian. Based on the spectral analysis of the linearized cutoff Boltzmann operator performed in [64, 26, 36], some hydrodynamic results were obtained in [65, 13, 39], see also [24] for the Landau equation. Moreover, for the non-cutoff Boltzmann equation, we refer to [52] where the authors obtained a result of weak-\* convergence in  $L_t^\infty(H_{x,v}^2)$  towards the fluid system by proving uniform in  $\varepsilon$  estimates. Up to our knowledge, our paper is the first to prove a strong convergence towards the incompressible Navier-Stokes-Fourier system for the non-cutoff Boltzmann equation. We also note here that, compared to former hydrodynamical limit results, in our work we do not need any derivative assumption on the initial data.

We now describe our strategy in order to obtain strong convergence results. Our approach is inspired by the one used in [13] for the cutoff Boltzmann equation, which was also used more recently in [16, 39] still for cutoff kernels and in [24] for the Landau equation. Indeed, as in [39, 24], using the spectral analysis performed in [36, 74, 75], in order to prove our main convergence result, we reformulate the fluid equation in a kinetic fashion and we then study the equation satisfied by the difference between the kinetic and the fluid solutions. More precisely, we denote the kinetic solution by

$$f^\varepsilon(t) = U^\varepsilon(t)f_0^\varepsilon + \Psi^\varepsilon[f^\varepsilon, f^\varepsilon](t),$$

and we observe, thanks to [13], that the kinetic distribution  $g$  associated to the fluid solution  $(\rho, u, \theta)$  through (2.16) satisfies

$$g(t) = U(t)g_0 + \Psi[g, g](t),$$

where  $U$  is obtained as the limit of  $U^\varepsilon$  and  $\Psi$  as the limit of  $\Psi^\varepsilon$  when  $\varepsilon \rightarrow 0$ . The idea is then to compute the norm of the difference  $f^\varepsilon - g$  by using convergence estimates from  $U^\varepsilon$  to  $U$  (see Lemma 6.3) and from  $\Psi^\varepsilon$  to  $\Psi$  (see Lemma 6.4), which are based on the spectral study of [74, 75], together with uniform in  $\varepsilon$  estimates for the kinetic solution  $f^\varepsilon$  from Theorem 2.1. This was achieved in [39] for the cutoff Boltzmann equation by applying a fixed point method, however, as explained in [24], this can not be directly applied to the non-cutoff Boltzmann and Landau equations due to the anisotropic loss of regularity in the nonlinear collision operator  $\Gamma$ . To overcome this difficult for the Landau equation, the authors in [24] proved new pointwise-in-time regularization estimates not only for the semigroup  $U^\varepsilon$  but also for the solution to the nonlinear rescaled kinetic equation, which were then used to close the estimates and obtain a result of strong convergence.

In our work, we propose a new method in order to obtain strong convergence using only the integrated-in-time regularization estimates (as opposed to pointwise-in-time regularization estimates in [24]) for the semigroup  $U^\varepsilon$  as well as for  $\int_0^t U^\varepsilon(t-s)S(s)ds$ . More precisely, the fixed point argument in the space  $\mathcal{F}_x^{-1}(L_\xi^1 L_t^\infty L_v^2 \cap L_\xi^1 L_v^2 H_v^{s,*})$  for the torus case, or in  $\mathcal{F}_x^{-1}(L_\xi^1 L_t^\infty L_v^2 \cap L_\xi^1 L_v^2 H_v^{s,**}) \cap \mathcal{F}_x^{-1}(L_\xi^p L_t^\infty L_v^2 \cap L_\xi^p L_v^2 H_v^{s,**})$  for the whole space,

used for the global well-posedness in Theorem 2.1 above together with the corresponding energy estimates are sufficient to estimate the  $\mathcal{F}_x^{-1}(L_\xi^1 L_t^\infty L_v^2)$ -norm of the difference  $f^\varepsilon - g$  and obtain strong convergence.

**2.4. Organization of the paper.** In Section 3, we first establish basic properties for the rescaled linearized non-cutoff Boltzmann collision operator and then compute the basic estimates for the associate semigroup. In Section 4 we prove the well-posedness for the rescaled non-cutoff Boltzmann equation. We establish well-posedness for the Navier-Stokes-Fourier system in Section 5. Finally we obtain the hydrodynamical limit result in Section 6.

### 3. LINEARIZED BOLTZMANN OPERATOR

It is well-known, see for instance [63] and the references therein, that the linearized Boltzmann collision operator  $L$ , defined in (1.10), satisfies the following coercive-type inequality

$$(3.1) \quad \langle Lf, f \rangle_{L_v^2} \leq -\lambda \|\mathbf{P}^\perp f\|_{H_v^{s,*}}^2.$$

where we recall that  $\mathbf{P}^\perp = I - \mathbf{P}$  and  $\mathbf{P}$  is the orthogonal projection onto  $\text{Ker } L$  given by (2.3). For all  $\varepsilon \in (0, 1]$  and all  $\xi \in \Omega'_\xi$ , we denote by  $\Lambda^\varepsilon(\xi)$  the Fourier transform in space of the full linearized operator  $\frac{1}{\varepsilon^2}L - \frac{1}{\varepsilon}v \cdot \nabla_x$ , namely

$$(3.2) \quad \Lambda^\varepsilon(\xi) := \frac{1}{\varepsilon^2}(L - i\varepsilon v \cdot \xi).$$

We first gather dissipativity results for the operator  $\Lambda^\varepsilon(\xi)$  obtained for instance in [68], that we reformulate below as in [23] and inspired from [24, 14] in order to take into account the different scales related to the parameter  $\varepsilon \in (0, 1]$ . For every  $\xi \in \Omega'_\xi$  we define

$$\begin{aligned} B[f, g](\xi) &:= \frac{\delta_1 i}{\langle \xi \rangle^2} \xi \theta[\widehat{f}(\xi)] \cdot M[\mathbf{P}^\perp \widehat{g}(\xi)] + \frac{\delta_1 i}{\langle \xi \rangle^2} \xi \theta[\widehat{g}(\xi)] \cdot M[\mathbf{P}^\perp \widehat{f}(\xi)] \\ &\quad + \frac{\delta_2 i}{\langle \xi \rangle^2} (\xi \otimes u[\widehat{f}(\xi)])^{\text{sym}} : \left\{ \Theta[\mathbf{P}^\perp \widehat{g}(\xi)] + \theta[\widehat{g}(\xi)]I \right\} \\ &\quad + \frac{\delta_2 i}{\langle \xi \rangle^2} (\xi \otimes u[\widehat{g}(\xi)])^{\text{sym}} : \left\{ \Theta[\mathbf{P}^\perp \widehat{f}(\xi)] + \theta[\widehat{f}(\xi)]I \right\} \\ &\quad + \frac{\delta_3 i}{\langle \xi \rangle^2} \xi \rho[\widehat{f}(\xi)] \cdot u[\widehat{g}(\xi)] + \frac{\delta_3 i}{\langle \xi \rangle^2} \xi \rho[\widehat{g}(\xi)] \cdot u[\widehat{f}(\xi)], \end{aligned}$$

with constants  $0 < \delta_3 \ll \delta_2 \ll \delta_1 \ll 1$ , where  $I$  is the  $3 \times 3$  identity matrix and the moments  $M$  and  $\Theta$  are defined by

$$M[f] = \int_{\mathbf{R}^3} f v (|v|^2 - 5) \sqrt{\mu}(v) dv, \quad \Theta[f] = \int_{\mathbf{R}^3} f (v \otimes v - I) \sqrt{\mu}(v) dv,$$

and where for vectors  $a, b \in \mathbf{R}^3$  and matrices  $A, B \in \mathbf{R}^{3 \times 3}$ , we denote

$$(a \otimes b)^{\text{sym}} = \frac{1}{2}(a_j b_k + a_k b_j)_{1 \leq j, k \leq 3}, \quad A : B = \sum_{j, k=1}^3 A_{jk} B_{jk}.$$

We then define the inner product  $\langle \cdot, \cdot \rangle_{L_v^2}$  on  $L_v^2$  (depending on  $\xi$ ) by

$$(3.3) \quad \langle \widehat{f}(\xi), \widehat{g}(\xi) \rangle_{L_v^2} := \langle \widehat{f}(\xi), \widehat{g}(\xi) \rangle_{L_v^2} + \varepsilon B[f, g](\xi),$$

and the associated norm

$$(3.4) \quad \|\widehat{f}(\xi)\|_{L_v^2}^2 := \langle \widehat{f}(\xi), \widehat{f}(\xi) \rangle_{L_v^2}.$$

In a similar fashion, for any  $\ell > 0$ , we define the inner product  $\langle \cdot, \cdot \rangle_{L_v^2(\langle v \rangle^\ell)}$  on  $L_v^2(\langle v \rangle^\ell)$  (depending on  $\xi$ ) by

$$(3.5) \quad \begin{aligned} \langle \widehat{f}(\xi), \widehat{g}(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)} &:= \langle \widehat{f}(\xi), \widehat{g}(\xi) \rangle_{L_v^2} + \delta_0 \langle \mathbf{P}^\perp \widehat{f}(\xi), \mathbf{P}^\perp \widehat{g}(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)} \\ &\quad + \varepsilon B[f, g](\xi), \end{aligned}$$

with  $\delta_1 \ll \delta_0 \ll 1$ , and the associated norm

$$(3.6) \quad \|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 := \langle \widehat{f}(\xi), \widehat{f}(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)}.$$

It is important to notice the factor  $\varepsilon$  in front of the last term in the right-hand side of (3.3) and (3.5).

Arguing as in [68], the main difference being the factor  $\varepsilon$  at the second term of (3.3) and (3.5), we obtain the following dissipativity result.

**Proposition 3.1.** *We can choose  $0 < \delta_3 \ll \delta_2 \ll \delta_1 \ll \delta_0 \ll 1$  appropriately such that:*

(1) *The new norm  $\|\cdot\|_{L_v^2(\langle v \rangle^\ell)}$  is equivalent to the usual norm  $\|\cdot\|_{L_v^2(\langle v \rangle^\ell)}$  on  $L_v^2(\langle v \rangle^\ell)$  with bounds that are independent of  $\xi$  and  $\varepsilon$ .*

(2) *If  $\Omega_x = \mathbf{T}^3$ , for every  $f$  satisfying (1.12) we have, for all  $\xi \in \mathbf{Z}^3$ ,*

$$\operatorname{Re} \langle \Lambda^\varepsilon(\xi) \widehat{f}(\xi), \widehat{f}(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)} \leq -\lambda_0 \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right),$$

for some constant  $\lambda_0 > 0$ .

(3) *If  $\Omega_x = \mathbf{R}^3$ , for every  $f$  we have, for all  $\xi \in \mathbf{R}^3$ ,*

$$\operatorname{Re} \langle \Lambda^\varepsilon(\xi) \widehat{f}(\xi), \widehat{f}(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)} \leq -\lambda_0 \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right),$$

for some constant  $\lambda_0 > 0$ .

The aim of this remainder section is to obtain, using the dissipativity result of Proposition 3.1, decay and regularization estimates for the semigroup associated to the linearized operator  $\widehat{\Lambda}^\varepsilon$ . We denote in the sequel by

$$(3.7) \quad \widehat{U}^\varepsilon(t, \xi) = e^{t\Lambda^\varepsilon(\xi)},$$

the semigroup associated to  $\Lambda^\varepsilon(\xi)$ , and by

$$(3.8) \quad U^\varepsilon(t) = \mathcal{F}_x^{-1} \widehat{U}^\varepsilon(t) \mathcal{F}_x,$$

the semigroup associated to  $\frac{1}{\varepsilon^2}(L - \varepsilon v \cdot \nabla_x)$ .

**3.1. Boundedness and regularization estimates.** We first provide boundedness and integrated-in-time regularization estimates for the semigroup  $U^\varepsilon$  (see Proposition 3.2) as well as its integral in time against a source  $\int_0^t U^\varepsilon(t-s)S(s)ds$  (see Proposition 3.4). These are the key estimates we shall use later in order to prove the well-posedness results for the rescaled Boltzmann equation (1.8) in Theorem 2.1. They are also crucial for establishing the convergence of some of the terms in the proof of the hydrodynamic limit in Theorem 2.3.

**Proposition 3.2.** *Let  $\ell \geq 0$  and  $p \in [1, \infty]$ . Let  $\widehat{f}_0 \in L_\xi^1 L_v^2(\langle v \rangle^\ell)$  and suppose moreover that  $f_0$  verifies (1.12) in the torus case  $\Omega_x = \mathbf{T}^3$ . Then*

$$\|\widehat{U}^\varepsilon(\cdot) \widehat{f}_0\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|\mathbf{P}^\perp \widehat{U}^\varepsilon(\cdot) \widehat{f}_0\|_{L_\xi^p L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{U}^\varepsilon(\cdot) \widehat{f}_0 \right\|_{L_\xi^p L_t^2 L_v^2} \lesssim \|\widehat{f}_0\|_{L_\xi^p L_v^2(\langle v \rangle^\ell)},$$

and moreover, in the torus case, we also have that  $U^\varepsilon(t)f_0$  verifies (1.12) for all  $t \geq 0$ .

*Remark 3.3.* Observe that, in the torus case  $\Omega_x = \mathbf{T}^3$ , one can replace the term  $\frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{U}^\varepsilon(\cdot) \widehat{f}_0$  in above estimate by  $\mathbf{P} \widehat{U}^\varepsilon(\cdot) \widehat{f}_0$  since  $U^\varepsilon(t)f_0$  verifies (1.12).

*Proof.* Let  $f(t) = U^\varepsilon(t)f_0$  for all  $t \geq 0$ , which satisfies the equation

$$(3.9) \quad \partial_t f = \frac{1}{\varepsilon^2}(L - \varepsilon v \cdot \nabla_x) f, \quad f|_{t=0} = f_0.$$

We already observe that in the case of the torus,  $f(t)$  verifies (1.12) thanks to the properties of  $L$ . Moreover, for all  $\xi \in \mathbf{Z}^3$  (if  $\Omega_x = \mathbf{T}^3$ ) or all  $\xi \in \mathbf{R}^3$  (if  $\Omega_x = \mathbf{R}^3$ ), the Fourier transform in space  $\widehat{f}$  satisfies

$$(3.10) \quad \partial_t \widehat{f}(\xi) = \Lambda^\varepsilon(\xi) \widehat{f}(\xi), \quad \widehat{f}(\xi)|_{t=0} = \widehat{f}_0(\xi).$$

Using Proposition 3.1 we have, for all  $t \geq 0$ ,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 &= \operatorname{Re} \langle \Lambda^\varepsilon(\xi) \widehat{f}(\xi), \widehat{f}(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)} \\ &\leq -\lambda_0 \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right), \end{aligned}$$

which implies, for all  $t \geq 0$ ,

$$\|\widehat{f}(t, \xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 + \frac{1}{\varepsilon^2} \int_0^t \|\mathbf{P}^\perp \widehat{f}(s, \xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 ds + \int_0^t \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(s, \xi)\|_{L_v^2}^2 ds \lesssim \|\widehat{f}_0(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2.$$

where we have used that  $\|\widehat{f}(\xi)\|_{L_v^2}$  is equivalent to  $\|\widehat{f}(\xi)\|_{L_v^2}$  independently of  $\xi$  and  $\varepsilon$ . Taking the supremum in time and then taking the square-root of previous estimate yields

$$\|\widehat{f}(\xi)\|_{L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}(\xi) \right\|_{L_t^2 L_v^2} \lesssim \|\widehat{f}_0(\xi)\|_{L_v^2},$$

and we conclude by taking the  $L_\xi^p$  norm.  $\square$

**Proposition 3.4.** *Let  $\ell \geq 0$  and  $p \in [1, \infty]$ . Let  $S = S(t, x, v)$  verify  $\mathbf{P}S = 0$  and  $\langle v \rangle^\ell \widehat{S} \in L_\xi^p L_t^2 (H_v^{s,*})'$ , and denote*

$$g_S(t) = \int_0^t U^\varepsilon(t-s) S(s) ds.$$

Then

$$\|\widehat{g}_S\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|\mathbf{P}^\perp \widehat{g}_S\|_{L_\xi^p L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{g}_S \right\|_{L_\xi^p L_t^2 L_v^2} \lesssim \varepsilon \|\langle v \rangle^\ell \widehat{S}\|_{L_\xi^p L_t^2 (H_v^{s,*})'}.$$

*Remark 3.5.* As in Remark 3.3, we observe that in the torus case  $\Omega_x = \mathbf{T}^3$  one can replace the term  $\frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{g}_S$  in above estimate by  $\mathbf{P} \widehat{g}_S$ .

*Proof.* We first observe that  $g_S$  satisfies the equation

$$(3.11) \quad \partial_t g_S = \frac{1}{\varepsilon^2} (L - \varepsilon v \cdot \nabla_x) g_S + S, \quad g_{|t=0} = 0,$$

thus, for all for all  $\xi \in \mathbf{Z}^3$  (if  $\Omega_x = \mathbf{T}^3$ ) or all  $\xi \in \mathbf{R}^3$  (if  $\Omega_x = \mathbf{R}^3$ ),

$$(3.12) \quad \partial_t \widehat{g}_S(\xi) = \Lambda^\varepsilon(\xi) \widehat{g}_S(\xi) + \widehat{S}(\xi), \quad \widehat{g}(\xi)|_{t=0} = 0,$$

that is, for all  $t \geq 0$ ,

$$(3.13) \quad \widehat{g}_S(t, \xi) = \int_0^t \widehat{U}^\varepsilon(t-s, \xi) \widehat{S}(s, \xi) ds.$$

We remark from (3.3) and the fact that  $\mathbf{P}S = 0$  that

$$\begin{aligned} \langle \widehat{S}(\xi), \widehat{g}_S(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)} &= \langle \widehat{S}(\xi), \widehat{g}_S(\xi) \rangle_{L_v^2} + \delta_0 \langle \mathbf{P}^\perp \widehat{S}(\xi), \mathbf{P}^\perp \widehat{g}_S(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)} + \varepsilon B[S, g_S](\xi) \\ &= \langle \widehat{S}(\xi), \mathbf{P}^\perp \widehat{g}_S(\xi) \rangle_{L_v^2} + \delta_0 \langle \mathbf{P}^\perp \widehat{S}(\xi), \mathbf{P}^\perp \widehat{g}_S(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)} + \varepsilon B[S, g_S](\xi). \end{aligned}$$

Using again that  $\mathbf{P}S = 0$ , so that  $\rho[S] = u[S] = \theta[S] = 0$ , we have

$$B[S, g_S](\xi) = \frac{\delta_{1i}}{1 + |\xi|^2} \xi \theta[\widehat{g}_S(\xi)] \cdot M[\mathbf{P}^\perp \widehat{S}(\xi)] + \frac{\delta_{2i}}{1 + |\xi|^2} (\xi \otimes u[\widehat{g}_S(\xi)])^{\operatorname{sym}} : \Theta[\mathbf{P}^\perp \widehat{S}(\xi)],$$

therefore observing that for any polynomial  $p = p(v)$  there holds

$$\left| \int_{\mathbf{R}^3} \widehat{S}(\xi) p(v) \sqrt{\mu}(v) dv \right| \lesssim \|\widehat{S}(\xi)\|_{(H_v^{s,*})'},$$

we get

$$|B[S, g_S](\xi)| \lesssim \|\mathbf{P}^\perp \widehat{S}(\xi)\|_{(H_v^{s,*})'} \frac{|\xi|}{\langle \xi \rangle} \|\mathbf{P} \widehat{g}_S(\xi)\|_{L_v^2}.$$

Moreover

$$\langle \widehat{S}(\xi), \mathbf{P}^\perp \widehat{g}_S(\xi) \rangle_{L_v^2} \lesssim \|\widehat{S}(\xi)\|_{(H_v^{s,*})'} \|\mathbf{P}^\perp \widehat{g}_S(\xi)\|_{H_v^{s,*}},$$

and

$$\begin{aligned} \langle \widehat{S}(\xi), \mathbf{P}^\perp \widehat{g}_S(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)} &= \langle \langle v \rangle^\ell \widehat{S}(\xi), \langle v \rangle^\ell \mathbf{P}^\perp \widehat{g}_S(\xi) \rangle_{L_v^2} \\ &\lesssim \| \langle v \rangle^\ell \widehat{S}(\xi) \|_{(H_v^{s,*})'} \| \langle v \rangle^\ell \mathbf{P}^\perp \widehat{g}_S(\xi) \|_{H_v^{s,*}}, \end{aligned}$$

Gathering previous estimates yields

$$(3.14) \quad \langle \widehat{S}(\xi), \widehat{g}_S(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)} \lesssim \| \langle v \rangle^\ell \widehat{S}(\xi) \|_{(H_v^{s,*})'} \left( \| \mathbf{P}^\perp \widehat{g}_S(\xi) \|_{H_v^{s,*}(\langle v \rangle^\ell)} + \varepsilon \frac{|\xi|}{\langle \xi \rangle} \| \mathbf{P} \widehat{g}_S(\xi) \|_{L_v^2} \right).$$

Using Proposition 3.1 and arguing as in Proposition 3.2 we have, for all  $t \geq 0$  and all  $\xi \in \Omega'_\xi$ ,

$$(3.15) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \| \widehat{g}_S(\xi) \|_{L_v^2(\langle v \rangle^\ell)}^2 &\leq -\lambda_0 \left( \frac{1}{\varepsilon^2} \| \mathbf{P}^\perp \widehat{g}_S(\xi) \|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \frac{|\xi|^2}{\langle \xi \rangle^2} \| \mathbf{P} \widehat{g}_S(\xi) \|_{L_v^2}^2 \right) \\ &\quad + C \| \langle v \rangle^\ell \widehat{S}(\xi) \|_{(H_v^{s,*})'} \left( \| \mathbf{P}^\perp \widehat{g}_S(\xi) \|_{H_v^{s,*}(\langle v \rangle^\ell)} + \varepsilon \frac{|\xi|}{\langle \xi \rangle} \| \mathbf{P} \widehat{g}_S(\xi) \|_{L_v^2} \right) \\ &\leq -\frac{\lambda_0}{2} \left( \frac{1}{\varepsilon^2} \| \mathbf{P}^\perp \widehat{g}_S(\xi) \|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \frac{|\xi|^2}{\langle \xi \rangle^2} \| \mathbf{P} \widehat{g}_S(\xi) \|_{L_v^2}^2 \right) \\ &\quad + C \varepsilon^2 \| \langle v \rangle^\ell \widehat{S}(\xi) \|_{(H_v^{s,*})'}^2, \end{aligned}$$

where we have used Young's inequality in last line, which implies

$$\begin{aligned} \| \widehat{g}_S(t, \xi) \|_{L_v^2(\langle v \rangle^\ell)}^2 + \frac{1}{\varepsilon^2} \int_0^t \| \mathbf{P}^\perp \widehat{g}_S(s, \xi) \|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 ds + \int_0^t \frac{|\xi|^2}{\langle \xi \rangle^2} \| \mathbf{P} \widehat{g}_S(s, \xi) \|_{L_v^2}^2 ds \\ \lesssim \varepsilon^2 \int_0^t \| \langle v \rangle^\ell \widehat{S}(s, \xi) \|_{(H_v^{s,*})'}^2 ds. \end{aligned}$$

Taking the supremum in time and then taking the square-root of previous estimate yields

$$\| \widehat{g}_S(\xi) \|_{L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \| \mathbf{P}^\perp \widehat{g}_S(\xi) \|_{L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{g}_S(\xi) \right\|_{L_t^2 L_v^2} \lesssim \varepsilon \| \langle v \rangle^\ell \widehat{S}(\xi) \|_{L_t^2 (H_v^{s,*})'},$$

and we conclude by taking the  $L_\xi^p$  norm.  $\square$

**3.2. Decay estimates: Hard potentials in the torus.** In this subsection we shall always assume  $\gamma + 2s \geq 0$  and  $\Omega_x = \mathbf{T}^3$ , and we shall obtain decay estimates for the semigroup  $U^\varepsilon$  (see Proposition 3.6) as well as its integral in time against a source  $\int_0^t U^\varepsilon(t-s)S(s)ds$  (see Proposition 3.7). We recall that given any real number  $\lambda \in \mathbf{R}$  we denote  $e_\lambda : t \mapsto e^{\lambda t}$ .

**Proposition 3.6.** *Let  $\ell \geq 0$ . Let  $\widehat{f}_0 \in L_\xi^1 L_v^2(\langle v \rangle^\ell)$ , then*

$$\begin{aligned} \| e_\lambda \widehat{U}^\varepsilon(\cdot) \widehat{f}_0 \|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \| e_\lambda \mathbf{P}^\perp \widehat{U}^\varepsilon(\cdot) \widehat{f}_0 \|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \| e_\lambda \mathbf{P} \widehat{U}^\varepsilon(\cdot) \widehat{f}_0 \|_{L_\xi^1 L_t^2 L_v^2} \\ \lesssim \| \widehat{f}_0 \|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)}, \end{aligned}$$

for some  $\lambda > 0$  (depending on  $\lambda_0$  of Proposition 3.1).

*Proof.* Let  $f(t) = U^\varepsilon(t)f_0$  for all  $t \geq 0$  which satisfies (3.9), so that  $\widehat{f}(t, \xi) = \widehat{U}^\varepsilon(t, \xi)\widehat{f}(\xi)$  satisfies (3.10) for all  $\xi \in \mathbf{Z}^3$ . Using Proposition 3.1 we have, for all  $t \geq 0$  and some  $\lambda_0 > 0$ ,

$$\frac{1}{2} \frac{d}{dt} \| \widehat{f}(\xi) \|_{L_v^2(\langle v \rangle^\ell)}^2 \leq -\lambda_0 \left( \frac{1}{\varepsilon^2} \| \mathbf{P}^\perp \widehat{f}(\xi) \|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \| \mathbf{P} \widehat{f}(\xi) \|_{L_v^2}^2 \right),$$

which implies, since  $\| \cdot \|_{H_v^{s,*}(\langle v \rangle^\ell)} \geq \| \langle v \rangle^{\gamma/2+2s} \cdot \|_{L_v^2} \geq \| \cdot \|_{L_v^2(\langle v \rangle^\ell)}$  and the fact that  $\| \widehat{f}^\varepsilon(\xi) \|_{L_v^2(\langle v \rangle^\ell)}$  is equivalent to  $\| \widehat{f}^\varepsilon(\xi) \|_{L_v^2(\langle v \rangle^\ell)}$  independently of  $\xi$  and  $\varepsilon$ , that

$$\frac{1}{2} \frac{d}{dt} \| \widehat{f}(\xi) \|_{L_v^2(\langle v \rangle^\ell)}^2 \leq -\lambda \| \widehat{f}^\varepsilon(\xi) \|_{L_v^2(\langle v \rangle^\ell)}^2 - \sigma \left( \frac{1}{\varepsilon^2} \| \mathbf{P}^\perp \widehat{f}(\xi) \|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \| \mathbf{P} \widehat{f}(\xi) \|_{L_v^2}^2 \right),$$

for some positive constants  $\lambda, \sigma > 0$  depending only on the implicit constants in Proposition 3.1–(1) and on  $\lambda_0 > 0$  appearing in Proposition 3.1–(2). We therefore deduce

$$\frac{d}{dt} \left\{ e^{2\lambda t} \|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 \right\} \leq -\sigma e^{2\lambda t} \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right),$$

which implies, for all  $t \geq 0$ ,

$$\begin{aligned} e^{2\lambda t} \|\widehat{f}(t, \xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 + \frac{1}{\varepsilon^2} \int_0^t e^{2\lambda s} \|\mathbf{P}^\perp \widehat{f}(s, \xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 ds + \int_0^t e^{2\lambda s} \|\mathbf{P} \widehat{f}(s, \xi)\|_{L_v^2}^2 ds \\ \lesssim \|\widehat{f}_0(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2. \end{aligned}$$

where we have used again that  $\|\widehat{f}(\xi)\|_{L_v^2}$  is equivalent to  $\|\widehat{f}(\xi)\|_{L_v^2}$  independently of  $\xi$  and  $\varepsilon$ . Taking the supremum in time and then taking the square-root of previous estimate yields

$$\|e_\lambda \widehat{f}(\xi)\|_{L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|e_\lambda \mathbf{P}^\perp \widehat{f}(\xi)\|_{L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \|e_\lambda \mathbf{P} \widehat{f}(\xi)\|_{L_t^2 L_v^2} \lesssim \|\widehat{f}_0(\xi)\|_{L_v^2(\langle v \rangle^\ell)},$$

and we conclude by taking the  $L_\xi^1$  norm.  $\square$

**Proposition 3.7.** *Let  $\ell \geq 0$ . Let  $\lambda > 0$  be given in Proposition 3.6. Let  $S = S(t, x, v)$  verify  $\mathbf{P}S = 0$  and  $e_\lambda \langle v \rangle^\ell \widehat{S} \in L_\xi^1 L_t^2 (H_v^{s,*})'$ , and denote*

$$g_S(t) = \int_0^t U^\varepsilon(t-s) S(s) ds.$$

Then

$$\|e_\lambda \widehat{g}_S\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|e_\lambda \mathbf{P}^\perp \widehat{g}_S\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \|e_\lambda \mathbf{P} \widehat{g}_S\|_{L_\xi^1 L_t^2 L_v^2} \lesssim \varepsilon \|e_\lambda \langle v \rangle^\ell \widehat{S}\|_{L_\xi^1 L_t^2 (H_v^{s,*})'}.$$

*Proof.* Recall that  $g_S$  satisfies equation (3.11) and  $\widehat{g}$  verifies (3.12) for all  $\xi \in \mathbf{Z}^3$  as well as (3.13). Thanks to (3.15) and using that  $\|\cdot\|_{H_v^{s,*}(\langle v \rangle^\ell)} \geq \|\langle v \rangle^{\gamma/2+s} \cdot\|_{L_v^2(\langle v \rangle^\ell)} \geq \|\cdot\|_{L_v^2(\langle v \rangle^\ell)}$  as in the proof of Proposition 3.6, we get for all  $t \geq 0$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\widehat{g}_S(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 &\leq -\lambda \|\widehat{g}_S(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 - \sigma \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{g}_S(\xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \|\mathbf{P} \widehat{g}_S(\xi)\|_{L_v^2}^2 \right) \\ &\quad + C \varepsilon^2 \|\langle v \rangle^\ell \widehat{S}(\xi)\|_{(H_v^{s,*})'}^2, \end{aligned}$$

for some constants  $\lambda, \sigma, C > 0$ . We therefore deduce

$$\begin{aligned} \frac{d}{dt} \left\{ e^{2\lambda t} \|\widehat{g}_S(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 \right\} &\leq -\sigma e^{2\lambda t} \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{g}_S(\xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \|\mathbf{P} \widehat{g}_S(\xi)\|_{L_v^2}^2 \right) \\ &\quad + C \varepsilon^2 e^{2\lambda t} \|\langle v \rangle^\ell \widehat{S}(\xi)\|_{(H_v^{s,*})'}^2, \end{aligned}$$

which implies, for all  $t \geq 0$ ,

$$\begin{aligned} e^{2\lambda t} \|\widehat{g}_S(t, \xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 + \frac{1}{\varepsilon^2} \int_0^t e^{2\lambda s} \|\mathbf{P}^\perp \widehat{g}_S(s, \xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 ds \\ + \int_0^t e^{2\lambda s} \|\mathbf{P} \widehat{g}_S(s, \xi)\|_{L_v^2}^2 ds \lesssim \varepsilon^2 \int_0^t e^{2\lambda s} \|\langle v \rangle^\ell \widehat{S}(s, \xi)\|_{(H_v^{s,*})'}^2 ds. \end{aligned}$$

Taking the supremum in time and then taking the square-root of previous estimate yields

$$\|e_\lambda \widehat{g}_S(\xi)\|_{L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|e_\lambda \mathbf{P}^\perp \widehat{g}_S(\xi)\|_{L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \|e_\lambda \mathbf{P} \widehat{g}_S(\xi)\|_{L_t^2 L_v^2} \lesssim \varepsilon \|e_\lambda \langle v \rangle^\ell \widehat{S}(\xi)\|_{L_t^2 (H_v^{s,*})'},$$

and we conclude by taking the  $L_\xi^1$  norm.  $\square$



**3.3. Decay estimates: Soft potentials in the torus.** In this subsection we shall always assume  $\gamma + 2s < 0$  and  $\Omega_x = \mathbf{T}^3$ , and we shall obtain decay estimates for the semigroup  $U^\varepsilon$  (see Proposition 3.8) as well as its integral in time against a source  $\int_0^t U^\varepsilon(t-s)S(s) ds$  (see Proposition 3.9). We recall that given any real number  $\omega \in \mathbf{R}$  we denote  $p_\omega : t \mapsto (1+t)^\omega$ .

**Proposition 3.8.** *Let  $\ell > 0$  and  $\widehat{f}_0 \in L_\xi^1 L_v^2(\langle v \rangle^\ell)$ , then for any  $0 < \omega < \frac{\ell}{|\gamma+2s|}$  we have*

$$\begin{aligned} & \|p_\omega \widehat{U}^\varepsilon(\cdot) \widehat{f}_0\|_{L_\xi^1 L_t^\infty L_v^2} + \frac{1}{\varepsilon} \|p_\omega \mathbf{P}^\perp \widehat{U}^\varepsilon(\cdot) \widehat{f}_0\|_{L_\xi^1 L_t^2 H_v^{s,*}} + \|p_\omega \mathbf{P}(\widehat{U}^\varepsilon(\cdot) \widehat{f}_0)\|_{L_\xi^1 L_t^2 L_v^2} \\ & \lesssim \|\widehat{f}_0\|_{L_\xi^1 L_v^2} + \|\widehat{U}^\varepsilon(\cdot) \widehat{f}_0\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)}. \end{aligned}$$

*Proof.* Arguing as in the proof of Proposition 3.6, denoting  $f(t) = U^\varepsilon(t)f_0$  and using that  $\|\cdot\|_{H^{s,*}} \geq \|\langle v \rangle^{\gamma/2+s} \cdot\|_{L_v^2}$ , we obtain

$$(3.16) \quad \frac{1}{2} \frac{d}{dt} \|\widehat{f}(\xi)\|_{L_v^2}^2 \leq -\lambda \|\langle v \rangle^{\gamma/2+s} \widehat{f}(\xi)\|_{L_v^2}^2 - \sigma \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}}^2 + \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right),$$

for some positive constants  $\lambda, \sigma > 0$ .

We now observe the following interpolation inequality: for any  $R > 0$  there holds

$$(3.17) \quad \|\widehat{f}(\xi)\|_{L_v^2}^2 \lesssim \langle R \rangle^{|\gamma+2s|} \|\langle v \rangle^{\gamma/2+s} \widehat{f}(\xi)\|_{L_v^2}^2 + \langle R \rangle^{-2\ell} \|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2.$$

Therefore coming back to (3.16) and choosing  $\langle R \rangle = [(\lambda/\omega)(1+t)]^{1/|\gamma+2s|}$  yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\widehat{f}(\xi)\|_{L_v^2}^2 & \leq -\omega(1+t)^{-1} \|\widehat{f}(\xi)\|_{L_v^2}^2 - \sigma \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}}^2 + \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right) \\ & \quad + C(1+t)^{-1-\frac{2\ell}{|\gamma+2s|}} \|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2, \end{aligned}$$

for some constant  $C > 0$  (independent of  $\xi$  and  $\varepsilon$ ). Multiplying both sides by  $(1+t)^{2\omega}$  gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\{ (1+t)^{2\omega} \|\widehat{f}(\xi)\|_{L_v^2}^2 \right\} & \leq -\sigma(1+t)^{2\omega} \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}}^2 + \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right) \\ & \quad + C(1+t)^{2\omega-1-\frac{2\ell}{|\gamma+2s|}} \|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2. \end{aligned}$$

Integrating last estimate in time gives, for all  $t \geq 0$ ,

$$\begin{aligned} (1+t)^{2\omega} \|\widehat{f}(t, \xi)\|_{L_v^2}^2 + \frac{1}{\varepsilon^2} \int_0^t (1+s)^{2\omega} \|\mathbf{P}^\perp \widehat{f}(s, \xi)\|_{H_v^{s,*}}^2 ds + \int_0^t (1+s)^{2\omega} \|\mathbf{P} \widehat{f}(s, \xi)\|_{L_v^2}^2 ds \\ \lesssim \|\widehat{f}_0(\xi)\|_{L_v^2}^2 + \sup_{s \in [0, t]} \|\widehat{f}(s, \xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 \int_0^t (1+s)^{2\omega-1-\frac{2\ell}{|\gamma+2s|}} ds, \end{aligned}$$

where we have used again that  $\|\widehat{f}(\xi)\|_{L_v^2}$  is equivalent to  $\|\widehat{f}(\xi)\|_{L_v^2}$  independently of  $\xi$  and  $\varepsilon$ . Observing that  $(1+t)^{2\omega-1-\frac{2\ell}{|\gamma+2s|}}$  is integrable since  $0 < \omega < \frac{\ell}{|\gamma+2s|}$ , we can take the supremum in time in last estimate and then its square-root to obtain

$$\|p_\omega \widehat{f}(\xi)\|_{L_t^\infty L_v^2} + \frac{1}{\varepsilon} \|p_\omega \mathbf{P}^\perp \widehat{f}(\xi)\|_{L_t^2 H_v^{s,*}} + \|p_\omega \mathbf{P} \widehat{f}(s, \xi)\|_{L_t^2 L_v^2} \lesssim \|\widehat{f}_0(\xi)\|_{L_v^2} + \|\widehat{f}(\xi)\|_{L_t^\infty L_v^2(\langle v \rangle^\ell)},$$

and we conclude the proof by taking the  $L_\xi^1$  norm.  $\square$

**Proposition 3.9.** *Let  $S = S(t, x, v)$  verify  $\mathbf{P}S = 0$  and  $p_\omega \widehat{S} \in L_\xi^1 L_t^2(H_v^{s,*})'$  for some  $0 < \omega < \frac{\ell}{|\gamma+2s|}$  and  $\ell > 0$ , and denote*

$$g_S(t) = \int_0^t U^\varepsilon(t-s)S(s) ds.$$

*Assume that  $\widehat{g}_S \in L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)$ , then we have*

$$\begin{aligned} & \|p_\omega \widehat{g}_S\|_{L_\xi^1 L_t^\infty L_v^2} + \frac{1}{\varepsilon} \|p_\omega \mathbf{P}^\perp \widehat{g}_S\|_{L_\xi^1 L_t^2 H_v^{s,*}} + \|p_\omega \mathbf{P} \widehat{g}_S\|_{L_\xi^1 L_t^2 L_v^2} \\ & \lesssim \varepsilon \|p_\omega \widehat{S}\|_{L_\xi^1 L_t^2(H_v^{s,*})'} + \|\widehat{g}_S\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)}. \end{aligned}$$

*Proof.* Arguing as in the proof of Proposition 3.7, but using now that  $\|\cdot\|_{H^{s,*}(\langle v \rangle^\ell)} \geq \|\langle v \rangle^{\gamma/2+s} \cdot\|_{L_v^2(\langle v \rangle^\ell)}$  as in Proposition 3.8, we have

$$(3.18) \quad \frac{1}{2} \frac{d}{dt} \|\widehat{g}_S(\xi)\|_{L_v^2}^2 \leq -\lambda \|\langle v \rangle^{\gamma/2+s} \widehat{g}_S(\xi)\|_{L_v^2}^2 - \sigma \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{g}_S(\xi)\|_{H_v^{s,*}}^2 + \|\mathbf{P} \widehat{g}_S(\xi)\|_{L_v^2}^2 \right) + C\varepsilon^2 \|\widehat{S}(\xi)\|_{(H_v^{s,*})'}^2.$$

for some constants  $\lambda, \sigma, C > 0$ . Using the interpolation (3.17) as in the proof of Proposition 3.8, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\widehat{g}_S(\xi)\|_{L_v^2}^2 \leq -\omega(1+t)^{-1} \|\widehat{g}_S(\xi)\|_{L_v^2}^2 - \sigma \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{g}_S(\xi)\|_{H_v^{s,*}}^2 + \|\mathbf{P} \widehat{g}_S(\xi)\|_{L_v^2}^2 \right) + C\varepsilon^2 \|\widehat{S}(\xi)\|_{(H_v^{s,*})'}^2 + C(1+t)^{-1-\frac{2\ell}{|\gamma+2s|}} \|\widehat{g}_S(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2,$$

for some constant  $C > 0$  (independent of  $\xi$  and  $\varepsilon$ ). We can then conclude exactly as in the proof of Proposition 3.8.  $\square$

**3.4. Decay estimates: Hard potentials in the whole space.** In this subsection we shall always assume  $\gamma + 2s \geq 0$  and  $\Omega_x = \mathbf{R}^3$ , and we shall obtain decay estimates for the semigroup  $U^\varepsilon$  (see Proposition 3.10) as well as its integral in time against a source  $\int_0^t U^\varepsilon(t-s)S(s) ds$  (see Proposition 3.11). We recall that given any real number  $\omega \in \mathbf{R}$  we denote  $p_\omega : t \mapsto (1+t)^\omega$ .

**Proposition 3.10.** *Let  $\ell \geq 0$ ,  $p \in (3/2, \infty]$  and  $0 < \vartheta < \frac{3}{2}(1 - \frac{1}{p})$ . Let  $\widehat{f}_0 \in L_\xi^1 L_v^2(\langle v \rangle^\ell)$ , then*

$$\begin{aligned} \|\mathbf{p}_\vartheta \widehat{U}^\varepsilon(\cdot) \widehat{f}_0\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|\mathbf{p}_\vartheta \mathbf{P}^\perp \widehat{U}^\varepsilon(\cdot) \widehat{f}_0\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \mathbf{p}_\vartheta \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{U}^\varepsilon(\cdot) \widehat{f}_0 \right\|_{L_\xi^1 L_t^2 L_v^2(\langle v \rangle^\ell)} \\ \lesssim \|\widehat{f}_0\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|\widehat{U}^\varepsilon(\cdot) \widehat{f}_0\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)}. \end{aligned}$$

*Proof.* Let  $f(t) = U^\varepsilon(t)f_0$  for all  $t \geq 0$  which satisfies (3.9), so that  $\widehat{f}(t, \xi) = \widehat{U}^\varepsilon(t, \xi) \widehat{f}(\xi)$  satisfies (3.10) for all  $\xi \in \mathbf{R}^3$ . Using Proposition 3.1 we have, for all  $t \geq 0$  and some  $\lambda_0 > 0$ ,

$$\frac{1}{2} \frac{d}{dt} \|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 \leq -\lambda_0 \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right),$$

and we already observe that, using  $\|\cdot\|_{H_v^{s,*}(\langle v \rangle^\ell)} \geq \|\langle v \rangle^{\gamma/2+s} \cdot\|_{L_v^2(\langle v \rangle^\ell)} \geq \|\cdot\|_{L_v^2(\langle v \rangle^\ell)}$  and  $\varepsilon \in (0, 1]$ ,

$$\frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \gtrsim \frac{|\xi|^2}{\langle \xi \rangle^2} \|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2,$$

where we have used that  $\|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}$  is equivalent to  $\|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}$  independently of  $\xi$  and  $\varepsilon$ . Therefore it follows

$$(3.19) \quad \frac{1}{2} \frac{d}{dt} \|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 \leq -2\lambda \frac{|\xi|^2}{\langle \xi \rangle^2} \|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 - \sigma \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right),$$

for some constants  $\lambda, \sigma > 0$ . We now split our analysis into two cases: high frequencies  $|\xi| \geq 1$  and low frequencies  $|\xi| < 1$ .

For high frequencies  $|\xi| \geq 1$  we remark that  $\frac{|\xi|^2}{\langle \xi \rangle^2} \geq \frac{1}{2}$ , hence we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mathbf{1}_{|\xi| \geq 1} \|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 &\leq -\lambda \mathbf{1}_{|\xi| \geq 1} \|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 \\ &\quad - \frac{\sigma}{2} \left( \frac{1}{\varepsilon^2} \mathbf{1}_{|\xi| \geq 1} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \mathbf{1}_{|\xi| \geq 1} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right). \end{aligned}$$

Arguing as in the proof of Proposition 3.6 we hence deduce

$$(3.20) \quad \mathbf{1}_{|\xi| \geq 1} \|e_\lambda \widehat{f}(\xi)\|_{L_t^\infty L_v^2((v)^\ell)} + \frac{1}{\varepsilon} \mathbf{1}_{|\xi| \geq 1} \|e_\lambda \mathbf{P}^\perp \widehat{f}(\xi)\|_{L_t^2 H_v^{s,*}((v)^\ell)} + \mathbf{1}_{|\xi| \geq 1} \|e_\lambda \mathbf{P} \widehat{f}(\xi)\|_{L_t^2 L_v^2} \\ \lesssim \mathbf{1}_{|\xi| \geq 1} \|\widehat{f}_0(\xi)\|_{L_v^2((v)^\ell)}.$$

We now investigate the case of low frequencies  $|\xi| < 1$ . We denote by  $p'$  the conjugate exponent of  $p$ , that is  $1/p + 1/p' = 1$  with the convention  $p' = 1$  if  $p = \infty$ , and consider a real number  $r$  verifying  $1 + p'/3 < r < 1 + 1/(2\vartheta)$ , which we observe is possible thanks to the conditions on  $p$  and  $\vartheta$ . Remarking that  $|\xi|^2 \leq 2|\xi|^2/\langle \xi \rangle^2$  if  $|\xi| < 1$ , by Young's inequality we get: for any  $\delta > 0$  there is  $C_\delta > 0$  such that, for all  $|\xi| < 1$  and  $t \geq 0$ , we have

$$(3.21) \quad 1 \leq \delta(1+t) \frac{|\xi|^2}{\langle \xi \rangle^2} + C_\delta(1+t)^{-\frac{1}{r-1}} |\xi|^{-\frac{2}{r-1}}.$$

We therefore obtain, coming back to (3.19) and choosing  $\delta > 0$  appropriately,

$$\frac{1}{2} \frac{d}{dt} \mathbf{1}_{|\xi| < 1} \|\widehat{f}(\xi)\|_{L_v^2((v)^\ell)}^2 \leq -\sigma \left( \frac{1}{\varepsilon^2} \mathbf{1}_{|\xi| < 1} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}((v)^\ell)}^2 + \mathbf{1}_{|\xi| < 1} \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right) \\ - \vartheta(1+t)^{-1} \mathbf{1}_{|\xi| < 1} \|\widehat{f}(\xi)\|_{L_v^2((v)^\ell)}^2 \\ + C(1+t)^{-1-\frac{1}{r-1}} |\xi|^{-\frac{2}{r-1}} \mathbf{1}_{|\xi| < 1} \|\widehat{f}(\xi)\|_{L_v^2((v)^\ell)}^2.$$

for some constant  $C > 0$ . Multiplying both sides by  $(1+t)^{2\vartheta}$  gives

$$\frac{1}{2} \frac{d}{dt} \left\{ (1+t)^{2\vartheta} \mathbf{1}_{|\xi| < 1} \|\widehat{f}(\xi)\|_{L_v^2((v)^\ell)}^2 \right\} \\ \leq -\sigma(1+t)^{2\vartheta} \left( \frac{1}{\varepsilon^2} \mathbf{1}_{|\xi| < 1} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}((v)^\ell)}^2 + \mathbf{1}_{|\xi| < 1} \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right) \\ + C(1+t)^{2\vartheta-1-\frac{1}{r-1}} |\xi|^{-\frac{2}{r-1}} \mathbf{1}_{|\xi| < 1} \|\widehat{f}(\xi)\|_{L_v^2((v)^\ell)}^2.$$

Integrating in time implies, for all  $t \geq 0$ ,

$$(1+t)^{2\vartheta} \mathbf{1}_{|\xi| < 1} \|\widehat{f}(t, \xi)\|_{L_v^2((v)^\ell)}^2 + \frac{1}{\varepsilon^2} \int_0^t (1+s)^{2\vartheta} \mathbf{1}_{|\xi| < 1} \|\mathbf{P}^\perp \widehat{f}(s, \xi)\|_{H_v^{s,*}((v)^\ell)}^2 ds \\ + \int_0^t (1+s)^{2\vartheta} \mathbf{1}_{|\xi| < 1} \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(s, \xi)\|_{L_v^2}^2 ds \\ \lesssim \mathbf{1}_{|\xi| < 1} \|\widehat{f}_0(\xi)\|_{L_v^2((v)^\ell)}^2 + \mathbf{1}_{|\xi| < 1} |\xi|^{-\frac{2}{r-1}} \|\widehat{f}(\xi)\|_{L_t^\infty L_v^2((v)^\ell)}^2,$$

where we have used that  $(1+t)^{2\vartheta-1-\frac{1}{r-1}}$  is integrable since  $r < 1 + 1/(2\vartheta)$ . We now take the supremum in time and finally the square-root of the resulting estimate, which gives

$$(3.22) \quad \mathbf{1}_{|\xi| < 1} \|\mathbf{P} \widehat{f}(\xi)\|_{L_t^\infty L_v^2((v)^\ell)} + \frac{1}{\varepsilon} \mathbf{1}_{|\xi| < 1} \|\mathbf{P} \mathbf{P}^\perp \widehat{f}(\xi)\|_{L_t^2 H_v^{s,*}((v)^\ell)} + \mathbf{1}_{|\xi| < 1} \left\| \mathbf{P} \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}(\xi) \right\|_{L_t^2 L_v^2} \\ \lesssim \mathbf{1}_{|\xi| < 1} \|\widehat{f}_0(\xi)\|_{L_v^2((v)^\ell)} + \mathbf{1}_{|\xi| < 1} |\xi|^{-\frac{1}{r-1}} \|\widehat{f}(\xi)\|_{L_t^\infty L_v^2((v)^\ell)}.$$

Gathering the estimate for high frequencies (3.20) together with the one for low frequencies (3.22), it follows

$$\|\mathbf{P} \widehat{f}(\xi)\|_{L_t^\infty L_v^2((v)^\ell)} + \frac{1}{\varepsilon} \|\mathbf{P} \mathbf{P}^\perp \widehat{f}(\xi)\|_{L_t^2 H_v^{s,*}((v)^\ell)} + \left\| \mathbf{P} \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}(\xi) \right\|_{L_t^2 L_v^2} \\ \lesssim \|\widehat{f}_0(\xi)\|_{L_v^2((v)^\ell)} + \mathbf{1}_{|\xi| < 1} |\xi|^{-\frac{1}{r-1}} \|\widehat{f}(\xi)\|_{L_t^\infty L_v^2((v)^\ell)}.$$

Taking the  $L_\xi^1$  norm above, we use Hölder's inequality to obtain

$$\begin{aligned} \int_{\mathbf{R}^3} \mathbf{1}_{|\xi|<1} |\xi|^{-\frac{1}{r-1}} \|\widehat{f}(\xi)\|_{L_t^\infty L_v^2(\langle v \rangle^\ell)} d\xi &\lesssim \left( \int_{\mathbf{R}^3} \mathbf{1}_{|\xi|<1} |\xi|^{-\frac{p'}{r-1}} d\xi \right)^{1/p'} \|\widehat{f}\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)} \\ &\lesssim \|\widehat{f}\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)}, \end{aligned}$$

since  $r > 1 + p'/3$ , which implies

$$\begin{aligned} \|\mathbf{p}_\vartheta \widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|\mathbf{p}_\vartheta \mathbf{P}^\perp \widehat{f}\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \mathbf{p}_\vartheta \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f} \right\|_{L_\xi^1 L_t^2 L_v^2} \\ \lesssim \|\widehat{f}_0\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)}, \end{aligned}$$

and concludes the proof.  $\square$

**Proposition 3.11.** *Let  $\ell \geq 0$ ,  $p \in (3/2, \infty]$  and  $0 < \vartheta < \frac{3}{2}(1 - \frac{1}{p})$ . Let  $S = S(t, x, v)$  verify  $\mathbf{P}S = 0$  and  $\mathbf{p}_\vartheta \langle v \rangle^\ell \widehat{S} \in L_\xi^1 L_t^2 (H_v^{s,*})'$ , and denote*

$$g_S(t) = \int_0^t U^\varepsilon(t-s) S(s) ds.$$

Assume that  $\widehat{g}_S \in L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)$ , then

$$\begin{aligned} \|\mathbf{p}_\vartheta \widehat{g}_S\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|\mathbf{p}_\vartheta \mathbf{P}^\perp \widehat{g}_S\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \mathbf{p}_\vartheta \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{g}_S \right\|_{L_\xi^1 L_t^2 L_v^2} \\ \lesssim \varepsilon \|\mathbf{p}_\vartheta \langle v \rangle^\ell \widehat{S}\|_{L_\xi^1 L_t^2 (H_v^{s,*})'} + \|\widehat{g}_S\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)}. \end{aligned}$$

*Proof.* Recalling that  $\widehat{g}_S$  satisfies (3.12), we can argue as for obtaining (3.19) to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\widehat{g}_S(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 &\leq -2\lambda \frac{|\xi|^2}{\langle \xi \rangle^2} \|\widehat{g}_S(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 - \sigma \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{g}_S(\xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{g}_S(\xi)\|_{L_v^2}^2 \right) \\ &\quad + C\varepsilon^2 \|\langle v \rangle^\ell \widehat{S}(\xi)\|_{(H_v^{s,*})'}^2, \end{aligned}$$

for some constants  $\lambda, \sigma, C > 0$ . By separating the cases of high and low frequencies, we can conclude exactly as in the proof of Proposition 3.10.  $\square$

**3.5. Decay estimates: Soft potentials in the whole space.** In this subsection we shall always assume  $\gamma + 2s < 0$  and  $\Omega_x = \mathbf{R}^3$ , and we shall obtain decay estimates for the semigroup  $U^\varepsilon$  (see Proposition 3.12) as well as its integral in time against a source  $\int_0^t U^\varepsilon(t-s) S(s) ds$  (see Proposition 3.13). We recall that given any real number  $\omega \in \mathbf{R}$  we denote  $\mathbf{p}_\omega : t \mapsto (1+t)^\omega$ .

**Proposition 3.12.** *Let  $p \in (3/2, \infty]$  and  $0 < \vartheta < \frac{3}{2}(1 - \frac{1}{p})$ . Let  $f_0 \in \mathcal{F}_x^{-1}(L_\xi^1 L_v^2(\langle v \rangle^\ell) \cap L_\xi^p L_v^2)$  with  $\ell > \vartheta|\gamma + 2s|$ , then we have*

$$\begin{aligned} \|\mathbf{p}_\vartheta \widehat{U}^\varepsilon(\cdot) \widehat{f}_0\|_{L_\xi^1 L_t^\infty L_v^2} + \frac{1}{\varepsilon} \|\mathbf{p}_\vartheta \mathbf{P}^\perp \widehat{U}^\varepsilon(\cdot) \widehat{f}_0\|_{L_\xi^1 L_t^2 H_v^{s,*}} + \left\| \mathbf{p}_\vartheta \frac{|\xi|}{\langle \xi \rangle} \mathbf{P}(\widehat{U}^\varepsilon(\cdot) \widehat{f}_0) \right\|_{L_\xi^1 L_t^2 L_v^2} \\ \lesssim \|\widehat{f}_0\|_{L_\xi^1 L_v^2} + \|\widehat{U}^\varepsilon(\cdot) \widehat{f}_0\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\widehat{U}^\varepsilon(\cdot) \widehat{f}_0\|_{L_\xi^p L_t^\infty L_v^2}. \end{aligned}$$

*Proof.* Arguing as in the proof of Proposition 3.10, denoting  $f(t) = U^\varepsilon(t) f_0$  and using that  $\|\cdot\|_{H_v^{s,*}(\langle v \rangle^\ell)} \geq \|\langle v \rangle^{\gamma/2+s} \cdot\|_{L_v^2(\langle v \rangle^\ell)}$ , we first obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\widehat{f}(\xi)\|_{L_v^2}^2 &\leq -\lambda \|\langle v \rangle^{\gamma/2+s} \mathbf{P}^\perp \widehat{f}(\xi)\|_{L_v^2}^2 - \lambda \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \\ (3.23) \quad &\quad - \sigma \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}}^2 + \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right), \end{aligned}$$

for some positive constants  $\lambda, \sigma > 0$ . We now split the analysis into high frequencies and low frequencies.

For high frequencies  $|\xi| \geq 1$  we observe that  $\frac{|\xi|^2}{\langle \xi \rangle^2} \geq \frac{1}{2}$ , which yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mathbf{1}_{|\xi| \geq 1} \|\widehat{f}(\xi)\|_{L_v^2}^2 &\leq -\lambda \mathbf{1}_{|\xi| \geq 1} \|\langle v \rangle^{\gamma/2+s} \mathbf{P}^\perp \widehat{f}(\xi)\|_{L_v^2}^2 - \lambda \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \\ &\quad - \sigma \left( \frac{1}{\varepsilon^2} \mathbf{1}_{|\xi| \geq 1} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}}^2 + \mathbf{1}_{|\xi| \geq 1} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right) \\ &\leq -\lambda \mathbf{1}_{|\xi| \geq 1} \|\langle v \rangle^{\gamma/2+s} \widehat{f}(\xi)\|_{L_v^2}^2 \\ &\quad - \sigma \left( \frac{1}{\varepsilon^2} \mathbf{1}_{|\xi| \geq 1} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}}^2 + \mathbf{1}_{|\xi| \geq 1} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right), \end{aligned}$$

for some other constants  $\lambda, \sigma > 0$ . Thanks to the interpolation inequality (3.17) of the proof of Proposition 3.8, we hence deduce

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mathbf{1}_{|\xi| \geq 1} \|\widehat{f}(\xi)\|_{L_v^2}^2 &\leq -\omega(1+t)^{-1} \mathbf{1}_{|\xi| \geq 1} \|\widehat{f}(\xi)\|_{L_v^2}^2 \\ &\quad - \sigma \left( \frac{1}{\varepsilon^2} \mathbf{1}_{|\xi| \geq 1} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}}^2 + \mathbf{1}_{|\xi| \geq 1} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right) \\ &\quad + C(1+t)^{-1-\frac{2\ell}{|\gamma+2s|}} \|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2, \end{aligned}$$

for any  $\vartheta < \omega < \frac{\ell}{|\gamma+2s|}$  and some constant  $C > 0$ . With this inequality we can thus argue as in the proof of Proposition 3.8, which gives, recalling that  $(1+t)^{2\omega-1-\frac{2\ell}{|\gamma+2s|}}$  is integrable since  $0 < \omega < \frac{\ell}{|\gamma+2s|}$ ,

$$(3.24) \quad \begin{aligned} \mathbf{1}_{|\xi| \geq 1} \|\mathbf{p}_\omega \widehat{f}(\xi)\|_{L_t^\infty L_v^2} + \frac{1}{\varepsilon} \mathbf{1}_{|\xi| \geq 1} \|\mathbf{p}_\omega \mathbf{P}^\perp \widehat{f}(\xi)\|_{L_t^2 H_v^{s,*}} + \mathbf{1}_{|\xi| \geq 1} \|\mathbf{p}_\omega \mathbf{P} \widehat{f}(\xi)\|_{L_t^2 L_v^2} \\ \lesssim \mathbf{1}_{|\xi| \geq 1} \|\widehat{f}_0(\xi)\|_{L_v^2} + \mathbf{1}_{|\xi| \geq 1} \|\widehat{f}(\xi)\|_{L_t^\infty L_v^2(\langle v \rangle^\ell)}. \end{aligned}$$

We now turn our attention to the low frequencies case  $|\xi| < 1$ . First of all, from (3.23), we use the interpolation inequality (3.17) of the proof of Proposition 3.8 to deduce

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mathbf{1}_{|\xi| < 1} \|\widehat{f}(\xi)\|_{L_v^2}^2 &\leq -\omega(1+t)^{-1} \mathbf{1}_{|\xi| < 1} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{L_v^2}^2 - \lambda \mathbf{1}_{|\xi| < 1} \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \\ &\quad - \sigma \left( \frac{1}{\varepsilon^2} \mathbf{1}_{|\xi| < 1} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}}^2 + \mathbf{1}_{|\xi| < 1} \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right) \\ &\quad + C(1+t)^{-1-\frac{2\ell}{|\gamma+2s|}} \mathbf{1}_{|\xi| < 1} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2, \end{aligned}$$

for any  $\vartheta < \omega < \frac{\ell}{|\gamma+2s|}$  and some constant  $C > 0$ . As in the proof of Proposition 3.10, we denote by  $p'$  the conjugate exponent of  $p$ , and consider a real number  $r$  verifying  $1 + p'/3 < r < 1 + 1/(2\vartheta)$ . Using inequality (3.21) we hence deduce

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mathbf{1}_{|\xi| < 1} \|\widehat{f}(\xi)\|_{L_v^2}^2 &\leq -\vartheta(1+t)^{-1} \mathbf{1}_{|\xi| < 1} \|\widehat{f}(\xi)\|_{L_v^2}^2 \\ &\quad - \sigma \left( \frac{1}{\varepsilon^2} \mathbf{1}_{|\xi| < 1} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}}^2 + \mathbf{1}_{|\xi| < 1} \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right) \\ &\quad + C(1+t)^{-1-\frac{2\ell}{|\gamma+2s|}} \mathbf{1}_{|\xi| < 1} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 \\ &\quad + C(1+t)^{-1-\frac{1}{r-1}} |\xi|^{-\frac{2}{r-1}} \mathbf{1}_{|\xi| < 1} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2, \end{aligned}$$

for some constant  $C > 0$ . Multiplying both sides by  $(1+t)^{2\vartheta}$  gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\{ (1+t)^{2\vartheta} \mathbf{1}_{|\xi| < 1} \|\widehat{f}(\xi)\|_{L_v^2}^2 \right\} &\leq -\sigma(1+t)^{2\vartheta} \left( \frac{1}{\varepsilon^2} \mathbf{1}_{|\xi| < 1} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}}^2 + \mathbf{1}_{|\xi| < 1} \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right) \\ &\quad + C(1+t)^{2\vartheta-1-\frac{2\ell}{|\gamma+2s|}} \mathbf{1}_{|\xi| < 1} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 \\ &\quad + C(1+t)^{2\vartheta-1-\frac{1}{r-1}} |\xi|^{-\frac{2}{r-1}} \mathbf{1}_{|\xi| < 1} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2. \end{aligned}$$

Integrating in time implies, for all  $t \geq 0$ ,

$$\begin{aligned} (1+t)^{2\vartheta} \mathbf{1}_{|\xi|<1} \|\widehat{f}(t, \xi)\|_{L_v^2}^2 &+ \frac{1}{\varepsilon^2} \int_0^t (1+s)^{2\vartheta} \mathbf{1}_{|\xi|<1} \|\mathbf{P}^\perp \widehat{f}(s, \xi)\|_{H_v^{s,*}}^2 ds \\ &+ \int_0^t (1+s)^{2\vartheta} \mathbf{1}_{|\xi|<1} \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{f}(s, \xi)\|_{L_v^2}^2 ds \\ &\lesssim \mathbf{1}_{|\xi|<1} \|\widehat{f}_0(\xi)\|_{L_v^2}^2 + \mathbf{1}_{|\xi|<1} \|\widehat{f}(\xi)\|_{L_t^\infty L_v^2(\langle v \rangle^\ell)}^2 + \mathbf{1}_{|\xi|<1} |\xi|^{-\frac{2}{r-1}} \|\widehat{f}(\xi)\|_{L_t^\infty L_v^2}^2, \end{aligned}$$

where we have used that  $(1+t)^{2\vartheta-1-\frac{2\ell}{|\gamma+2s|}}$  and  $(1+t)^{2\vartheta-1-\frac{1}{r-1}}$  are integrable since  $0 < \vartheta < \omega < \frac{\ell}{|\gamma+2s|}$  and  $r < 1 + 1/(2\vartheta)$ , respectively. We can now take the supremum in time and then the square-root of the resulting estimate, which gives

$$\begin{aligned} (3.25) \quad \mathbf{1}_{|\xi|<1} \|\mathbf{p}_\vartheta \widehat{f}(\xi)\|_{L_t^\infty L_v^2} &+ \frac{1}{\varepsilon} \mathbf{1}_{|\xi|<1} \|\mathbf{p}_\vartheta \mathbf{P}^\perp \widehat{f}(\xi)\|_{L_t^2 H_v^{s,*}} + \mathbf{1}_{|\xi|<1} \left\| \mathbf{p}_\vartheta \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}(\xi) \right\|_{L_t^2 L_v^2} \\ &\lesssim \mathbf{1}_{|\xi|<1} \|\widehat{f}_0(\xi)\|_{L_v^2} + \mathbf{1}_{|\xi|<1} \|\widehat{f}(\xi)\|_{L_t^\infty L_v^2(\langle v \rangle^\ell)} + \mathbf{1}_{|\xi|<1} |\xi|^{-\frac{1}{r-1}} \|\widehat{f}(\xi)\|_{L_t^\infty L_v^2}. \end{aligned}$$

Gathering the estimate for high frequencies (3.24) together with the one for low frequencies (3.25) and observing that  $\vartheta < \omega$ , it follows

$$\begin{aligned} \|\mathbf{p}_\vartheta \widehat{f}(\xi)\|_{L_t^\infty L_v^2(\langle v \rangle^\ell)} &+ \frac{1}{\varepsilon} \|\mathbf{p}_\vartheta \mathbf{P}^\perp \widehat{f}(\xi)\|_{L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \mathbf{p}_\vartheta \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}(\xi) \right\|_{L_t^2 L_v^2} \\ &\lesssim \|\widehat{f}_0(\xi)\|_{L_v^2} + \|\widehat{f}(\xi)\|_{L_t^\infty L_v^2(\langle v \rangle^\ell)} + \mathbf{1}_{|\xi|<1} |\xi|^{-\frac{1}{r-1}} \|\widehat{f}(\xi)\|_{L_t^\infty L_v^2}. \end{aligned}$$

Taking the  $L_\xi^1$  norm above, we use Hölder's inequality to control the last term in the right-hand side as in the proof of Proposition 3.10, to obtain

$$\int_{\mathbf{R}^3} \mathbf{1}_{|\xi|<1} |\xi|^{-\frac{1}{r-1}} \|\widehat{f}(\xi)\|_{L_t^\infty L_v^2(\langle v \rangle^\ell)} d\xi \lesssim \|\widehat{f}\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)},$$

since  $r > 1 + p'/3$ , which implies

$$\begin{aligned} \|\mathbf{p}_\vartheta \widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} &+ \frac{1}{\varepsilon} \|\mathbf{p}_\vartheta \mathbf{P}^\perp \widehat{f}\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \mathbf{p}_\vartheta \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f} \right\|_{L_\xi^1 L_t^2 L_v^2} \\ &\lesssim \|\widehat{f}_0\|_{L_\xi^1 L_v^2} + \|\widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)} \end{aligned}$$

and concludes the proof.  $\square$

**Proposition 3.13.** *Let  $p \in (3/2, \infty]$  and  $0 < \vartheta < \frac{3}{2}(1 - \frac{1}{p})$ . Let  $S = S(t, x, v)$  verify  $\mathbf{P}S = 0$  and  $\mathbf{p}_\vartheta \widehat{S} \in L_\xi^1 L_t^2(H_v^{s,*})'$ , and denote*

$$g_S(t) = \int_0^t U^\varepsilon(t-s)S(s) ds.$$

Assume that  $g_S \in \mathcal{F}_x^{-1}(L_\xi^1 L_v^2(\langle v \rangle^\ell) \cap L_\xi^p L_v^2)$  with  $\ell > \vartheta|\gamma + 2s|$ , then

$$\begin{aligned} \|\mathbf{p}_\vartheta \widehat{g}_S\|_{L_\xi^1 L_t^\infty L_v^2} &+ \frac{1}{\varepsilon} \|\mathbf{p}_\vartheta \mathbf{P}^\perp \widehat{g}_S\|_{L_\xi^1 L_t^2 H_v^{s,*}} + \left\| \mathbf{p}_\vartheta \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{g}_S \right\|_{L_\xi^1 L_t^2 L_v^2} \\ &\lesssim \varepsilon \|\mathbf{p}_\vartheta \widehat{S}\|_{L_\xi^1 L_t^2(H_v^{s,*})'} + \|\widehat{g}_S\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\widehat{g}_S\|_{L_\xi^p L_t^\infty L_v^2}. \end{aligned}$$

*Proof.* Recalling that  $\widehat{g}_S$  satisfies (3.12), we can argue as for obtaining (3.23) to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\widehat{g}_S(\xi)\|_{L_v^2}^2 &\leq -\lambda \|\langle v \rangle^{\gamma/2+s} \mathbf{P}^\perp \widehat{g}_S(\xi)\|_{L_v^2}^2 - \lambda \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{g}_S(\xi)\|_{L_v^2}^2 \\ &\quad - \sigma \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{g}_S(\xi)\|_{H_v^{s,*}}^2 + \frac{|\xi|^2}{\langle \xi \rangle^2} \|\mathbf{P} \widehat{g}_S(\xi)\|_{L_v^2}^2 \right) + C\varepsilon^2 \|\widehat{S}(\xi)\|_{(H_v^{s,*})'}^2, \end{aligned}$$

for some constants  $\lambda, \sigma, C > 0$ . By separating the cases of high and low frequencies, we can conclude exactly as in the proof of Proposition 3.12.  $\square$

4. WELL-POSEDNESS AND REGULARIZATION FOR THE RESCALED BOLTZMANN EQUATION

Consider the equation (1.8) that we rewrite here

$$\begin{cases} \partial_t f^\varepsilon = \frac{1}{\varepsilon^2}(L - \varepsilon v \cdot \nabla_x) f^\varepsilon + \frac{1}{\varepsilon} \Gamma(f^\varepsilon, f^\varepsilon) \\ f^\varepsilon_{t=0} = f_0^\varepsilon. \end{cases}$$

We shall consider mild solutions of (1.8), that is, we shall prove the well-posedness of a solution  $f^\varepsilon$  to (1.8) in Duhamel's form

$$(4.1) \quad f^\varepsilon(t) = U^\varepsilon(t) f_0^\varepsilon + \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t-s) \Gamma(f^\varepsilon(s), f^\varepsilon(s)) ds.$$

Taking the Fourier transform in space of (1.8), we have

$$(4.2) \quad \begin{cases} \partial_t \widehat{f}^\varepsilon(\xi) = \Lambda^\varepsilon(\xi) \widehat{f}^\varepsilon(\xi) + \frac{1}{\varepsilon} \widehat{\Gamma}(f^\varepsilon, f^\varepsilon)(\xi) \\ \widehat{f}^\varepsilon(\xi)_{t=0} = \widehat{f}_0^\varepsilon(\xi), \end{cases}$$

and by Duhamel's formula

$$(4.3) \quad \widehat{f}^\varepsilon(t, \xi) = \widehat{U}^\varepsilon(t, \xi) \widehat{f}_0^\varepsilon(\xi) + \frac{1}{\varepsilon} \int_0^t \widehat{U}^\varepsilon(t-s, \xi) \widehat{\Gamma}(f^\varepsilon(s), f^\varepsilon(s))(\xi) ds.$$

**4.1. Nonlinear estimates.** We start by recalling some well-known trilinear estimates on the collision operator  $\Gamma$ . We start with estimates without velocity weight. From [44, 5], for the hard potentials case  $\gamma + 2s \geq 0$  there holds

$$(4.4) \quad \left| \langle \Gamma(f, g), h \rangle_{L_v^2} \right| \lesssim \|f\|_{L_v^2} \|g\|_{H_v^{s,*}} \|h\|_{H_v^{s,*}}.$$

Moreover from [1], for the soft potentials case  $\gamma + 2s < 0$  one has

$$(4.5) \quad \begin{aligned} & \left| \langle \Gamma(f, g), h \rangle_{L_v^2 \langle v \rangle^\ell} \right| \\ & \lesssim \left( \|\langle v \rangle^{\gamma/2+s} f\|_{L_v^2} \|g\|_{H_v^{s,*}} + \|f\|_{H_v^{s,*}} \|\langle v \rangle^{\gamma/2+s} g\|_{L_v^2} \right) \|h\|_{H_v^{s,*}} \\ & \quad + \min \left\{ \|\langle v \rangle^{\gamma/2+s} f\|_{L_v^2} \|g\|_{L_v^2}, \|f\|_{L_v^2} \|\langle v \rangle^{\gamma/2+s} g\|_{L_v^2} \right\} \|h\|_{H_v^{s,*}}. \end{aligned}$$

From these estimates we already obtain

$$(4.6) \quad \begin{aligned} \|\Gamma(f, g)\|_{(H_v^{s,*})'} &= \sup_{\|\phi\|_{H_v^{s,*} \langle v \rangle^\ell} \leq 1} \langle \Gamma(f, g), \phi \rangle_{L_v^2 \langle v \rangle^\ell} \\ &\lesssim \|\langle v \rangle^{(\gamma/2+s)-} f\|_{L_v^2} \|g\|_{H_v^{s,*}} + \|f\|_{H_v^{s,*}} \|\langle v \rangle^{(\gamma/2+s)-} g\|_{L_v^2} \\ &\quad + \min \left\{ \|\langle v \rangle^{(\gamma/2+s)-} f\|_{L_v^2} \|g\|_{L_v^2}, \|f\|_{L_v^2} \|\langle v \rangle^{(\gamma/2+s)-} g\|_{L_v^2} \right\}, \end{aligned}$$

which holds for both hard and soft potentials.

Furhtermore, we also have estimates when adding velocity weight  $\langle v \rangle^\ell$ . For any  $\ell > 0$ , from [44, 1, 5] (see for instance [34, Lemma 4.1] for a summary) for the hard potentials case  $\gamma + 2s \geq 0$  there holds

$$(4.7) \quad \left| \langle \Gamma(f, g), h \rangle_{L_v^2 \langle v \rangle^\ell} \right| \lesssim \left( \|f\|_{L_v^2 \langle v \rangle^\ell} \|g\|_{H_v^{s,*} \langle v \rangle^\ell} + \|f\|_{H_v^{s,*} \langle v \rangle^\ell} \|g\|_{L_v^2 \langle v \rangle^\ell} \right) \|h\|_{H_v^{s,*} \langle v \rangle^\ell},$$

and for the soft potentials case  $\gamma + 2s < 0$  one has

$$(4.8) \quad \begin{aligned} & \left| \langle \Gamma(f, g), h \rangle_{L_v^2 \langle v \rangle^\ell} \right| \\ & \lesssim \left( \|\langle v \rangle^{\gamma/2+s} f\|_{L_v^2 \langle v \rangle^\ell} \|g\|_{H_v^{s,*} \langle v \rangle^\ell} + \|f\|_{H_v^{s,*} \langle v \rangle^\ell} \|\langle v \rangle^{\gamma/2+s} g\|_{L_v^2 \langle v \rangle^\ell} \right) \|h\|_{H_v^{s,*} \langle v \rangle^\ell} \\ & \quad + \|\langle v \rangle^{\gamma/2+s} f\|_{L_v^2 \langle v \rangle^\ell} \|g\|_{L_v^2 \langle v \rangle^\ell} \|h\|_{H_v^{s,*} \langle v \rangle^\ell}. \end{aligned}$$

Therefore we also deduce

$$(4.9) \quad \begin{aligned} \|\langle v \rangle^\ell \Gamma(f, g)\|_{(H_v^{s,*})'} &= \sup_{\|\phi\|_{H_v^{s,*}(\langle v \rangle^\ell)} \leq 1} \langle \Gamma(f, g), \phi \rangle_{L_v^2(\langle v \rangle^\ell)} \\ &\lesssim \|\langle v \rangle^{(\gamma/2+s)-} f\|_{L_v^2(\langle v \rangle^\ell)} \|g\|_{H_v^{s,*}(\langle v \rangle^\ell)} + \|f\|_{H_v^{s,*}(\langle v \rangle^\ell)} \|\langle v \rangle^{(\gamma/2+s)-} g\|_{L_v^2(\langle v \rangle^\ell)} \\ &\quad + \|\langle v \rangle^{(\gamma/2+s)-} f\|_{L_v^2(\langle v \rangle^\ell)} \|g\|_{L_v^2(\langle v \rangle^\ell)}, \end{aligned}$$

which again gathers both hard and soft potentials cases.

Thanks to (4.6) we deduce our main nonlinear estimate without weight.

**Lemma 4.1.** *Let  $p \in [1, \infty]$ . For any smooth enough functions  $f, g$  there holds*

$$\|\widehat{\Gamma}(f, g)\|_{L_\xi^p L_t^2(H_v^{s,*})'} \lesssim \Gamma_1 + \Gamma_2 + \min\{\Gamma_3, \Gamma_4\}$$

where

$$\begin{aligned} \Gamma_1 &= \min \left\{ \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^p L_t^\infty L_v^2} \|\widehat{g}\|_{L_\xi^1 L_t^2 H_v^{s,*}}, \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^p L_t^2 L_v^2} \|\widehat{g}\|_{L_\xi^1 L_t^\infty H_v^{s,*}} \right. \\ &\quad \left. \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2} \|\widehat{g}\|_{L_\xi^p L_t^2 H_v^{s,*}}, \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^1 L_t^2 L_v^2} \|\widehat{g}\|_{L_\xi^p L_t^\infty H_v^{s,*}} \right\}, \\ \Gamma_2 &= \min \left\{ \|\widehat{f}\|_{L_\xi^p L_t^2 H_v^{s,*}} \|\langle v \rangle^{(\gamma/2+s)-} \widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2}, \|\widehat{f}\|_{L_\xi^p L_t^\infty H_v^{s,*}} \|\langle v \rangle^{(\gamma/2+s)-} \widehat{g}\|_{L_\xi^1 L_t^2 L_v^2} \right. \\ &\quad \left. \|\widehat{f}\|_{L_\xi^1 L_t^2 H_v^{s,*}} \|\langle v \rangle^{(\gamma/2+s)-} \widehat{g}\|_{L_\xi^p L_t^\infty L_v^2}, \|\widehat{f}\|_{L_\xi^1 L_t^\infty H_v^{s,*}} \|\langle v \rangle^{(\gamma/2+s)-} \widehat{g}\|_{L_\xi^p L_t^2 L_v^2} \right\}, \\ \Gamma_3 &= \min \left\{ \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^p L_t^\infty L_v^2} \|\widehat{g}\|_{L_\xi^1 L_t^2 L_v^2}, \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^p L_t^2 L_v^2} \|\widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2} \right. \\ &\quad \left. \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2} \|\widehat{g}\|_{L_\xi^p L_t^2 L_v^2}, \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^1 L_t^2 L_v^2} \|\widehat{g}\|_{L_\xi^p L_t^\infty L_v^2} \right\}, \end{aligned}$$

and

$$\Gamma_4 = \min \left\{ \|\widehat{f}\|_{L_\xi^p L_t^2 L_v^2} \|\langle v \rangle^{(\gamma/2+s)-} \widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2}, \|\widehat{f}\|_{L_\xi^p L_t^\infty L_v^2} \|\langle v \rangle^{(\gamma/2+s)-} \widehat{g}\|_{L_\xi^1 L_t^2 L_v^2} \right. \\ \left. \|\widehat{f}\|_{L_\xi^1 L_t^2 L_v^2} \|\langle v \rangle^{(\gamma/2+s)-} \widehat{g}\|_{L_\xi^p L_t^\infty L_v^2}, \|\widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2} \|\langle v \rangle^{(\gamma/2+s)-} \widehat{g}\|_{L_\xi^p L_t^2 L_v^2} \right\}.$$

*Proof.* Using (4.6) we write

$$\left\{ \int_0^\infty \|\langle v \rangle^\ell \widehat{\Gamma}(f(t), g(t))(\xi)\|_{(H_v^{s,*})'}^2 dt \right\}^{1/2} \lesssim I_1 + I_2 + \min\{I_3, I_4\}$$

with

$$\begin{aligned} I_1 &= \left\{ \int_0^\infty \left( \int_{\Omega'_\eta} \|\langle v \rangle^{(\gamma/2+s)-} f(t, \xi - \eta)\|_{L_v^2} \|\widehat{g}(t, \eta)\|_{H_v^{s,*}} d\eta \right)^2 dt \right\}^{1/2}, \\ I_2 &= \left\{ \int_0^\infty \left( \int_{\Omega'_\eta} \|f(t, \xi - \eta)\|_{H_v^{s,*}} \|\langle v \rangle^{(\gamma/2+s)-} \widehat{g}(t, \eta)\|_{L_v^2} d\eta \right)^2 dt \right\}^{1/2}, \\ I_3 &= \left\{ \int_0^\infty \left( \int_{\Omega'_\eta} \|\langle v \rangle^{(\gamma/2+s)-} f(t, \xi - \eta)\|_{L_v^2} \|\widehat{g}(t, \eta)\|_{L_v^2} d\eta \right)^2 dt \right\}^{1/2}, \end{aligned}$$

and

$$I_4 = \left\{ \int_0^\infty \left( \int_{\Omega'_\eta} \|f(t, \xi - \eta)\|_{L_v^2} \|\langle v \rangle^{(\gamma/2+s)-} \widehat{g}(t, \eta)\|_{L_v^2} d\eta \right)^2 dt \right\}^{1/2}.$$

We now investigate the term  $I_1$ . Thanks to Minkowski and Hölder inequalities we then obtain

$$\begin{aligned} I_1 &\lesssim \int_{\Omega'_\eta} \left( \int_0^\infty \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}(t, \xi - \eta)\|_{L_v^2}^2 \|\widehat{g}(t, \eta)\|_{H_v^{s,*}}^2 dt \right)^{1/2} d\eta \\ &\lesssim \int_{\Omega'_\eta} \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}(\xi - \eta)\|_{L_t^\infty L_v^2} \|\widehat{g}(\eta)\|_{L_t^2 H_v^{s,*}} d\eta. \end{aligned}$$



Taking the  $L_\xi^p$  norm in above estimate and using Young's inequality for convolution we first obtain

$$I_1 \lesssim \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^p L_t^\infty L_v^2} \|\widehat{g}\|_{L_\xi^1 L_t^2 H_v^{s,*}} \quad \text{and} \quad I_1 \lesssim \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2} \|\widehat{g}\|_{L_\xi^p L_t^2 H_v^{s,*}}.$$

Arguing exactly as above but exchanging the role of  $f$  and  $g$  when performing Hölder's inequality, we also obtain

$$I_1 \lesssim \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^p L_t^2 L_v^2} \|\widehat{g}\|_{L_\xi^1 L_t^\infty H_v^{s,*}} \quad \text{and} \quad I_1 \lesssim \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^1 L_t^2 L_v^2} \|\widehat{g}\|_{L_\xi^p L_t^\infty H_v^{s,*}},$$

that is  $I_1 \lesssim \Gamma_1$ .

The estimates for the other terms  $I_2$ ,  $I_3$  and  $I_4$  can be obtained exactly as for  $I_1$ , so we omit it.  $\square$

Arguing exactly as in the proof of Lemma 4.1 but using the weighted estimate (4.9), we also obtain the main weighted nonlinear estimate below, the proof of which we omit for simplicity.

**Lemma 4.2.** *Let  $\ell > 0$  and  $p \in [1, \infty]$ . For any smooth enough functions  $f, g$  there holds*

$$\|\langle v \rangle^\ell \widehat{\Gamma}(f, g)\|_{L_\xi^p L_t^2 (H_v^{s,*})'} \lesssim \widetilde{\Gamma}_1 + \widetilde{\Gamma}_2 + \widetilde{\Gamma}_3,$$

where

$$\widetilde{\Gamma}_1 = \min \left\{ \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)} \|\widehat{g}\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)}, \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^p L_t^2 L_v^2(\langle v \rangle^\ell)} \|\widehat{g}\|_{L_\xi^1 L_t^\infty H_v^{s,*}(\langle v \rangle^\ell)}, \right. \\ \left. \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} \|\widehat{g}\|_{L_\xi^p L_t^2 H_v^{s,*}(\langle v \rangle^\ell)}, \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^1 L_t^2 L_v^2(\langle v \rangle^\ell)} \|\widehat{g}\|_{L_\xi^p L_t^\infty H_v^{s,*}(\langle v \rangle^\ell)} \right\},$$

$$\widetilde{\Gamma}_2 = \min \left\{ \|\widehat{f}\|_{L_\xi^p L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} \|\langle v \rangle^{(\gamma/2+s)-} \widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)}, \|\widehat{f}\|_{L_\xi^p L_t^\infty H_v^{s,*}(\langle v \rangle^\ell)} \|\langle v \rangle^{(\gamma/2+s)-} \widehat{g}\|_{L_\xi^1 L_t^2 L_v^2(\langle v \rangle^\ell)}, \right. \\ \left. \|\widehat{f}\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} \|\langle v \rangle^{(\gamma/2+s)-} \widehat{g}\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)}, \|\widehat{f}\|_{L_\xi^1 L_t^\infty H_v^{s,*}(\langle v \rangle^\ell)} \|\langle v \rangle^{(\gamma/2+s)-} \widehat{g}\|_{L_\xi^p L_t^2 L_v^2(\langle v \rangle^\ell)} \right\},$$

and

$$\widetilde{\Gamma}_3 = \min \left\{ \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)} \|\widehat{g}\|_{L_\xi^1 L_t^2 L_v^2(\langle v \rangle^\ell)}, \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^p L_t^2 L_v^2(\langle v \rangle^\ell)} \|\widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)}, \right. \\ \left. \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} \|\widehat{g}\|_{L_\xi^p L_t^2 L_v^2(\langle v \rangle^\ell)}, \|\langle v \rangle^{(\gamma/2+s)-} \widehat{f}\|_{L_\xi^1 L_t^2 L_v^2(\langle v \rangle^\ell)} \|\widehat{g}\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)} \right\}.$$

**4.2. Proof of Theorem 2.1–(1).** We consider the torus case  $\Omega_x = \mathbf{T}^3$ .

**4.2.1. Global existence.** Let  $\ell \geq 0$  be fixed and define the space

$$\mathcal{X} = \left\{ f \in \mathcal{F}_x^{-1}(L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell) \cap L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)) \mid f \text{ satisfies (1.12)}, \|f\|_{\mathcal{X}} < \infty \right\}$$

with

$$\|f\|_{\mathcal{X}} := \|\widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|\mathbf{P}^\perp \widehat{f}\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \|\mathbf{P} \widehat{f}\|_{L_\xi^1 L_t^2 L_v^2}.$$

Let  $f_0^\varepsilon \in \mathcal{F}_x^{-1}(L_\xi^1 L_v^2(\langle v \rangle^\ell))$  verify

$$\|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2} \leq \eta_0,$$

and consider the map  $\Phi : \mathcal{X} \rightarrow \mathcal{X}$ ,  $f^\varepsilon \mapsto \Phi[f^\varepsilon]$  defined by, for all  $t \geq 0$ ,

$$(4.10) \quad \Phi[f^\varepsilon](t) = U^\varepsilon(t) f_0^\varepsilon + \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t-s) \Gamma(f^\varepsilon(s), f^\varepsilon(s)) ds,$$

thus, for all  $\xi \in \mathbf{Z}^3$ ,

$$(4.11) \quad \widehat{\Phi}[f^\varepsilon](t, \xi) = \widehat{U}^\varepsilon(t, \xi) \widehat{f}_0^\varepsilon(\xi) + \frac{1}{\varepsilon} \int_0^t \widehat{U}^\varepsilon(t-s, \xi) \widehat{\Gamma}(f^\varepsilon(s), f^\varepsilon(s))(\xi) ds.$$

Thanks to Proposition 3.2 we deduce, for some constant  $C_0 > 0$  independent of  $\varepsilon$ , that

$$\|U^\varepsilon(\cdot) f_0^\varepsilon\|_{\mathcal{X}} \leq C_0 \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2}.$$

Moreover thanks to Proposition 3.4 we get, for some constant  $C_1 > 0$  independent of  $\varepsilon$ ,

$$\begin{aligned} \frac{1}{\varepsilon} \left\| \int_0^t U^\varepsilon(t-s) \Gamma(f^\varepsilon(s), f^\varepsilon(s)) ds \right\|_{\mathcal{X}} &\leq C_1 \|\widehat{\Gamma}(f^\varepsilon, f^\varepsilon)\|_{L_\xi^1 L_t^2(H_v^{s,*})'} \\ &\leq C_1 \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2} \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}} \\ &\leq C_1 \|f^\varepsilon\|_{\mathcal{X}}^2, \end{aligned}$$

where we have used Lemma 4.2 in the second line. Gathering previous estimates yields

$$(4.12) \quad \|\Phi[f^\varepsilon]\|_{\mathcal{X}} \leq C_0 \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2} + C_1 \|f^\varepsilon\|_{\mathcal{X}}^2.$$

Moreover for  $f^\varepsilon, g^\varepsilon \in \mathcal{X}$  we observe that

$$\begin{aligned} \Phi[f^\varepsilon](t) - \Phi[g^\varepsilon](t) &= \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t-s) \Gamma(f^\varepsilon(s), f^\varepsilon(s) - g^\varepsilon(s)) ds \\ &\quad + \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t-s) \Gamma(f^\varepsilon(s) - g^\varepsilon(s), g^\varepsilon(s)) ds, \end{aligned}$$

hence Proposition 3.4 and Lemma 4.2 yields, for some constant  $C_1 > 0$  independent of  $\varepsilon$ ,

$$(4.13) \quad \begin{aligned} \|\Phi[f^\varepsilon] - \Phi[g^\varepsilon]\|_{\mathcal{X}} &\leq C_1 \|\langle v \rangle^\ell \widehat{\Gamma}(f^\varepsilon, f^\varepsilon - g^\varepsilon)\|_{L_\xi^1 L_t^2(H_v^{s,*})'} + C_1 \|\langle v \rangle^\ell \widehat{\Gamma}(f^\varepsilon - g^\varepsilon, g^\varepsilon)\|_{L_\xi^1 L_t^2(H_v^{s,*})'} \\ &\leq C_1 \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} \|\widehat{f}^\varepsilon - \widehat{g}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + C_1 \|\widehat{f}^\varepsilon - g^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} \|\widehat{g}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} \\ &\leq C_1 (\|f^\varepsilon\|_{\mathcal{X}} + \|g^\varepsilon\|_{\mathcal{X}}) \|f^\varepsilon - g^\varepsilon\|_{\mathcal{X}}. \end{aligned}$$

As a consequence of estimates (4.12)–(4.13) we can construct a global solution  $f^\varepsilon \in \mathcal{X}$  to the equation (4.1) if  $\eta_0 > 0$  is small enough. Indeed let  $B_{\mathcal{X}}(\eta) = \{f \in \mathcal{X} \mid \|f\|_{\mathcal{X}} \leq \eta\}$  for  $\eta > 0$  be the closed ball in  $\mathcal{X}$  of radius  $\eta$ . Choose

$$\eta = 2C_0\eta_0 \quad \text{and} \quad \eta_0 \leq \frac{1}{8C_0C_1},$$

and observe that  $\eta_0$  does not depend on  $\varepsilon$ . Then for any  $f^\varepsilon \in B_{\mathcal{X}}(\eta)$  we have from (4.12) that

$$\|\Phi[f^\varepsilon]\|_{\mathcal{X}} \leq 2C_0\eta_0 = \eta,$$

and for any  $f^\varepsilon, g^\varepsilon \in B_{\mathcal{X}}(\eta)$  we have from (4.13) that

$$\|\Phi[f^\varepsilon] - \Phi[g^\varepsilon]\|_{\mathcal{X}} \leq 4C_0C_1\eta_0 \|f^\varepsilon - g^\varepsilon\|_{\mathcal{X}} \leq \frac{1}{2} \|f^\varepsilon - g^\varepsilon\|_{\mathcal{X}}.$$

Thus  $\Phi : B_{\mathcal{X}}(\eta) \rightarrow B_{\mathcal{X}}(\eta)$  is a contraction and therefore there is a unique  $f^\varepsilon \in B_{\mathcal{X}}(\eta)$  such that  $\Phi[f^\varepsilon] = f^\varepsilon$ , which is then a solution to (4.1). This completes the proof of global existence in Theorem 2.1–(1) together with estimate (2.9).

4.2.2. *Uniqueness.* Consider two solutions  $f^\varepsilon, g^\varepsilon \in \mathcal{F}_x^{-1}(L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell) \cap L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell))$  to (4.1) associated to the same initial data  $f_0^\varepsilon \in \mathcal{F}_x^{-1}(L_\xi^1 L_v^2(\langle v \rangle^\ell))$  satisfying  $\|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} \leq \eta_0$  with  $\eta_0 > 0$  small enough and

$$\begin{aligned} \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} &\lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)}, \\ \|\widehat{g}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\widehat{g}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} &\lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)}. \end{aligned}$$

Arguing as in the existence proof above, we obtain

$$\begin{aligned} \|f^\varepsilon - g^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|f^\varepsilon - g^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} &\lesssim \left( \|f^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|g^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} \right) \left( \|f^\varepsilon - g^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|f^\varepsilon - g^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} \right). \end{aligned}$$

Using that  $\|f^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|g^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} \lesssim \eta_0$  is small enough we conclude the proof of uniqueness in Theorem 2.1–(1).

4.2.3. *Decay for hard potentials.* Let  $f^\varepsilon$  be the solution to (4.1) constructed in Theorem 2.1–(1) associated to the initial data  $f_0^\varepsilon$ , and let  $\lambda > 0$  be given by Proposition 3.2. Using Proposition 3.6 and Proposition 3.7 we obtain

$$\begin{aligned} & \|e_\lambda \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|e_\lambda \mathbf{P}^\perp \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \|e_\lambda \mathbf{P} \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 L_v^2} \\ & \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|e_\lambda \langle v \rangle^\ell \widehat{\Gamma}(f^\varepsilon, f^\varepsilon)\|_{L_\xi^1 L_t^2(H_v^{s,*})}. \end{aligned}$$

Thanks to Lemma 4.2 we have

$$\|e_\lambda \langle v \rangle^\ell \widehat{\Gamma}(f^\varepsilon, f^\varepsilon)\|_{L_\xi^1 L_t^2(H_v^{s,*})} \lesssim \|e_\lambda \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)},$$

therefore using that  $\|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)}$  from the existence result in Theorem 2.1–(1), we obtain

$$\begin{aligned} & \|e_\lambda \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|e_\lambda \mathbf{P}^\perp \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \|e_\lambda \mathbf{P} \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 L_v^2} \\ & \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|e_\lambda \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} \\ & \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} \|e_\lambda \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)}. \end{aligned}$$

Since  $\|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} \leq \eta_0$  is small enough, the last term in the right-hand side can be absorbed into the left-hand side, which thus concludes the proof of the decay estimate (2.10) in Theorem 2.1–(1).

4.2.4. *Decay for soft potentials.* Let  $f^\varepsilon$  be the solution to (4.1) constructed in Theorem 2.1–(1) associated to the initial data  $f_0^\varepsilon$  with  $\ell > 0$ , and let  $0 < \omega < \frac{\ell}{|\gamma+2s|}$ .

Using Proposition 3.8 and Proposition 3.9 we obtain

$$\begin{aligned} & \|\mathbf{p}_\omega \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2} + \frac{1}{\varepsilon} \|\mathbf{p}_\omega \mathbf{P}^\perp \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}} + \|\mathbf{p}_\omega \mathbf{P} \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 L_v^2} \\ & \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2} + \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\mathbf{p}_\omega \widehat{\Gamma}(f^\varepsilon, f^\varepsilon)\|_{L_\xi^1 L_t^2(H_v^{s,*})}, \end{aligned}$$

and from Lemma 4.2 we have

$$\|\mathbf{p}_\omega \widehat{\Gamma}(f^\varepsilon, f^\varepsilon)\|_{L_\xi^1 L_t^2(H_v^{s,*})} \lesssim \|\mathbf{p}_\omega \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2} \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}}.$$

Using that  $\|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}} \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)}$  from the existence result in Theorem 2.1–(1), we deduce

$$\begin{aligned} & \|\mathbf{p}_\omega \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2} + \frac{1}{\varepsilon} \|\mathbf{p}_\omega \mathbf{P}^\perp \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}} + \|\mathbf{p}_\omega \mathbf{P} \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 L_v^2} \\ & \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} \|\mathbf{p}_\omega \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2}. \end{aligned}$$

Since  $\|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} \leq \eta_0$  is small enough, the last term in the right-hand side can be absorbed into the left-hand side, which thus concludes the proof of the decay estimate (2.11) in Theorem 2.1–(1).

4.3. **Proof of Theorem 2.1–(2).** We consider the whole space case  $\Omega_x = \mathbf{R}^3$ .

4.3.1. *Global existence.* Recall that  $p \in (3/2, \infty]$  and define the space, with  $\ell \geq 0$ ,

$$\mathscr{Y} = \left\{ f \in \mathcal{F}_x^{-1}(L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell) \cap L_\xi^1 L_t^2 H_v^{s,**}(\langle v \rangle^\ell)) \cap L_\xi^p L_t^1 L_v^2(\langle v \rangle^\ell) \cap L_\xi^p L_t^2 H_v^{s,**}(\langle v \rangle^\ell) \mid \|f\|_{\mathscr{Y}} < \infty \right\},$$

with

$$\begin{aligned} \|f\|_{\mathscr{Y}} & := \|\widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|\mathbf{P}^\perp \widehat{f}\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f} \right\|_{L_\xi^1 L_t^2 L_v^2} \\ & + \|\widehat{f}\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|\mathbf{P}^\perp \widehat{f}\|_{L_\xi^p L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f} \right\|_{L_\xi^p L_t^2 L_v^2}. \end{aligned}$$

Let  $f_0^\varepsilon \in \mathcal{F}_x^{-1}(L_\xi^1 L_v^2(\langle v \rangle^\ell) \cap L_\xi^p L_v^2(\langle v \rangle^\ell))$  verify

$$\|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}_0^\varepsilon\|_{L_\xi^p L_v^2(\langle v \rangle^\ell)} \leq \eta_0,$$

and consider the map  $\Phi : \mathcal{Y} \rightarrow \mathcal{Y}$ ,  $f^\varepsilon \mapsto \Phi[f^\varepsilon]$  given by (4.10), which in particular satisfies (4.11) for all  $\xi \in \mathbf{R}^3$ .

Thanks to Proposition 3.2 we deduce, for some constant  $C_0 > 0$  independent of  $\varepsilon$ , that

$$\|U^\varepsilon(\cdot) f_0^\varepsilon\|_{\mathcal{Y}} \leq C_0 \left( \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}_0^\varepsilon\|_{L_\xi^p L_v^2(\langle v \rangle^\ell)} \right).$$

Moreover thanks to Proposition 3.4 we get

$$\begin{aligned} & \frac{1}{\varepsilon} \left\| \int_0^t U^\varepsilon(t-s) \Gamma(f^\varepsilon(s), f^\varepsilon(s)) ds \right\|_{\mathcal{Y}} \\ & \lesssim \|\langle v \rangle^\ell \widehat{\Gamma}(f^\varepsilon, f^\varepsilon)\|_{L_\xi^1 L_t^2(H_v^{s,*})'} + \|\langle v \rangle^\ell \widehat{\Gamma}(f^\varepsilon, f^\varepsilon)\|_{L_\xi^p L_t^2(H_v^{s,*})'} \\ & \lesssim \left( \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}^\varepsilon\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)} \right) \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)}, \end{aligned}$$

where we have used Lemma 4.2 in the second line. We now observe that, splitting  $\widehat{f}^\varepsilon = \mathbf{P}^\perp \widehat{f}^\varepsilon + \mathbf{P} \widehat{f}^\varepsilon$ , on the one hand we have

$$\|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} \lesssim \|\mathbf{P}^\perp \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \|\mathbf{P} \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 L_v^2}.$$

On the other hand

$$\begin{aligned} \|\mathbf{P} \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 L_v^2} & \lesssim \|\mathbf{1}_{|\xi| \geq 1} \mathbf{P} \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 L_v^2} + \|\mathbf{1}_{|\xi| < 1} \mathbf{P} \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 L_v^2} \\ & \lesssim \left\| \mathbf{1}_{|\xi| \geq 1} \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}^\varepsilon \right\|_{L_\xi^1 L_t^2 L_v^2} + \left\| \mathbf{1}_{|\xi| < 1} |\xi|^{-1} \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}^\varepsilon \right\|_{L_\xi^1 L_t^2 L_v^2} \\ & \lesssim \left\| \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}^\varepsilon \right\|_{L_\xi^1 L_t^2 L_v^2} + \left\| \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}^\varepsilon \right\|_{L_\xi^p L_t^2 L_v^2}, \end{aligned}$$

where we have used Hölder's inequality in last line, using that  $p > 3/2$  so that  $\mathbf{1}_{|\xi| < 1} |\xi|^{-1} \in L_\xi^{p'}$ . Putting together the two last estimates, we have

$$(4.14) \quad \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} \lesssim \|\mathbf{P}^\perp \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}^\varepsilon \right\|_{L_\xi^1 L_t^2 L_v^2} + \left\| \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}^\varepsilon \right\|_{L_\xi^p L_t^2 L_v^2}.$$

We hence deduce that there is some constant  $C_1 > 0$ , independent of  $\varepsilon$ , such that

$$\begin{aligned} & \frac{1}{\varepsilon} \left\| \int_0^t U^\varepsilon(t-s) \Gamma(f^\varepsilon(s), f^\varepsilon(s)) ds \right\|_{\mathcal{Y}} \\ & \leq C_1 \left( \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}^\varepsilon\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)} \right) \\ & \quad \times \left( \|\mathbf{P}^\perp \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}^\varepsilon \right\|_{L_\xi^1 L_t^2 L_v^2} + \left\| \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}^\varepsilon \right\|_{L_\xi^p L_t^2 L_v^2} \right). \end{aligned}$$

Therefore, gathering previous estimates, we obtain

$$(4.15) \quad \|\Phi[f^\varepsilon]\|_{\mathcal{Y}} \leq C_0 \left( \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2} + \|\widehat{f}_0^\varepsilon\|_{L_\xi^p L_v^2} \right) + C_1 \|f^\varepsilon\|_{\mathcal{Y}}^2.$$

Moreover, for  $f^\varepsilon, g^\varepsilon \in \mathcal{Y}$  we obtain arguing as above, thanks to Proposition 3.4 and Lemma 4.2, that

$$\begin{aligned} & \frac{1}{\varepsilon} \left\| \int_0^t U^\varepsilon(t-s) \Gamma(f^\varepsilon(s), f^\varepsilon(s) - g^\varepsilon) ds \right\|_{\mathcal{Y}} \\ & \leq C_1 \left( \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \|\widehat{f}^\varepsilon\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}^\varepsilon\|_{L_\xi^p L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} \right) \\ & \quad \times \left( \|\widehat{f}^\varepsilon - \widehat{g}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}^\varepsilon - \widehat{g}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} \right), \end{aligned}$$

as well as

$$\begin{aligned} & \frac{1}{\varepsilon} \left\| \int_0^t U^\varepsilon(t-s) \Gamma(f^\varepsilon(s) - g^\varepsilon(s), g^\varepsilon) ds \right\|_{\mathcal{X}} \\ & \leq C_1 \left( \|\widehat{g}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\widehat{g}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \|\widehat{g}^\varepsilon\|_{L_\xi^2 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\widehat{g}^\varepsilon\|_{L_\xi^2 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} \right) \\ & \quad \times \left( \|\widehat{f}^\varepsilon - \widehat{g}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}^\varepsilon - \widehat{g}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} \right). \end{aligned}$$

Together with (4.14) for the term  $\|\widehat{f}^\varepsilon - \widehat{g}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)}$ , this implies that

$$(4.16) \quad \|\Phi[f^\varepsilon] - \Phi[g^\varepsilon]\|_{\mathcal{X}} \leq C_1 (\|f\|_{\mathcal{X}} + \|g\|_{\mathcal{X}}) \|f - g\|_{\mathcal{X}}.$$

As a consequence of estimates (4.15)–(4.16) we can construct a global solution  $f^\varepsilon \in \mathcal{X}$  to the equation (4.1) if  $\eta_0 > 0$  is small enough by arguing as in Section 4.2.1. This completes the proof of global existence in Theorem 2.1–(2) together with estimate (2.12).

4.3.2. *Uniqueness.* Using the above estimates, we can argue as in Section 4.2.2.

4.3.3. *Decay for hard potentials.* Let  $f^\varepsilon$  be the solution to (4.1) constructed in Theorem 2.1–(2) associated to the initial data  $f_0^\varepsilon$ , and let  $0 < \vartheta < \frac{3}{2}(1 - \frac{1}{p})$ . Arguing as above, using Proposition 3.10 and Proposition 3.11 we obtain

$$\begin{aligned} & \|\mathbf{p}_\vartheta \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|\mathbf{p}_\vartheta \mathbf{P}^\perp \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \mathbf{p}_\vartheta \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}^\varepsilon \right\|_{L_\xi^1 L_t^2 L_v^2} \\ & \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}^\varepsilon\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\mathbf{p}_\vartheta \langle v \rangle^\ell \widehat{\Gamma}(f^\varepsilon, f^\varepsilon)\|_{L_\xi^1 L_t^2(H_v^{s,*})}. \end{aligned}$$

Thanks to Lemma 4.2 we have

$$\|\mathbf{p}_\vartheta \langle v \rangle^\ell \widehat{\Gamma}(f^\varepsilon, f^\varepsilon)\|_{L_\xi^1 L_t^2(H_v^{s,*})} \lesssim \|\mathbf{p}_\vartheta \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)},$$

and by (4.14) we have

$$\begin{aligned} & \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} \lesssim \|\mathbf{P}^\perp \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \frac{|\xi|}{\langle \xi \rangle} \widehat{\mathbf{P}} f^\varepsilon \right\|_{L_\xi^1 L_t^2 L_v^2} + \left\| \frac{|\xi|}{\langle \xi \rangle} \widehat{\mathbf{P}} f^\varepsilon \right\|_{L_\xi^p L_t^2 L_v^2} \\ & \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}_0^\varepsilon\|_{L_\xi^p L_v^2(\langle v \rangle^\ell)}, \end{aligned}$$

where we have used the estimate of Theorem 2.1–(2) in last line. Observing that we also have  $\|\widehat{f}^\varepsilon\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)} \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}_0^\varepsilon\|_{L_\xi^p L_v^2(\langle v \rangle^\ell)}$ , it follows

$$\begin{aligned} & \|\mathbf{p}_\vartheta \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \frac{1}{\varepsilon} \|\mathbf{p}_\vartheta \mathbf{P}^\perp \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}(\langle v \rangle^\ell)} + \left\| \mathbf{p}_\vartheta \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}^\varepsilon \right\|_{L_\xi^1 L_t^2 L_v^2} \\ & \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}_0^\varepsilon\|_{L_\xi^p L_v^2(\langle v \rangle^\ell)} \\ & \quad + \|\mathbf{p}_\vartheta \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} \left( \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}_0^\varepsilon\|_{L_\xi^p L_v^2(\langle v \rangle^\ell)} \right). \end{aligned}$$

Since  $\|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}_0^\varepsilon\|_{L_\xi^p L_v^2(\langle v \rangle^\ell)} \leq \eta_0$  is small enough, the last term in the right-hand side can be absorbed into the left-hand side, which thus concludes the proof of the decay estimate (2.13) in Theorem 2.1–(2).

4.3.4. *Decay for soft potentials.* Let  $0 < \vartheta < \frac{3}{2}(1 - \frac{1}{p})$ . Let  $f^\varepsilon$  be the solution to (4.1) constructed in Theorem 2.1–(2) associated to the initial data  $f_0^\varepsilon$  with  $\ell > \vartheta|\gamma + 2s|$ . Arguing as above, using Proposition 3.12 and Proposition 3.13 we obtain

$$\begin{aligned} & \|\mathbf{p}_\vartheta \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2} + \frac{1}{\varepsilon} \|\mathbf{p}_\vartheta \mathbf{P}^\perp \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}} + \left\| \mathbf{p}_\vartheta \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}^\varepsilon \right\|_{L_\xi^1 L_t^2 L_v^2} \\ & \lesssim \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2} + \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} + \|\widehat{f}^\varepsilon\|_{L_\xi^p L_t^\infty L_v^2} + \|\mathbf{p}_\vartheta \widehat{\Gamma}(f^\varepsilon, f^\varepsilon)\|_{L_\xi^1 L_t^2(H_v^{s,*})}. \end{aligned}$$

For the nonlinear term above, we argue as in Section 4.3.3 so that

$$\begin{aligned} & \| \mathbf{p}_\vartheta \widehat{\Gamma}(f^\varepsilon, f^\varepsilon) \|_{L_\xi^1 L_t^2(H_v^{s,*})} \\ & \lesssim \| \mathbf{p}_\vartheta \widehat{f}^\varepsilon \|_{L_\xi^1 L_t^\infty L_v^2} \left( \| \mathbf{P}^\perp \widehat{f}^\varepsilon \|_{L_\xi^1 L_t^2 H_v^{s,*}} + \left\| \frac{|\xi|}{\langle \xi \rangle} \widehat{\mathbf{P}} f^\varepsilon \right\|_{L_\xi^1 L_t^2 L_v^2} + \left\| \frac{|\xi|}{\langle \xi \rangle} \widehat{\mathbf{P}} f^\varepsilon \right\|_{L_\xi^p L_t^2 L_v^2} \right), \end{aligned}$$

Therefore, using the estimate of Theorem 2.1–(2), we obtain

$$\begin{aligned} & \| \mathbf{p}_\vartheta \widehat{f}^\varepsilon \|_{L_\xi^1 L_t^\infty L_v^2} + \frac{1}{\varepsilon} \| \mathbf{p}_\vartheta \mathbf{P}^\perp \widehat{f}^\varepsilon \|_{L_\xi^1 L_t^2 H_v^{s,*}} + \left\| \mathbf{p}_\vartheta \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} \widehat{f}^\varepsilon \right\|_{L_\xi^1 L_t^2 L_v^2} \\ & \lesssim \| \widehat{f}_0^\varepsilon \|_{L_\xi^1 L_v^2((v)^\ell)} + \| \widehat{f}_0^\varepsilon \|_{L_\xi^p L_v^2((v)^\ell)} \\ & \quad + \| \mathbf{p}_\vartheta \widehat{f}^\varepsilon \|_{L_\xi^1 L_t^\infty L_v^2} \left( \| \widehat{f}_0^\varepsilon \|_{L_\xi^1 L_v^2((v)^\ell)} + \| \widehat{f}_0^\varepsilon \|_{L_\xi^p L_v^2((v)^\ell)} \right). \end{aligned}$$

Since  $\| \widehat{f}_0^\varepsilon \|_{L_\xi^1 L_v^2((v)^\ell)} + \| \widehat{f}_0^\varepsilon \|_{L_\xi^p L_v^2((v)^\ell)} \leq \eta_0$  is small enough, the last term in the right-hand side can be absorbed into the left-hand side, which thus concludes the proof of the decay estimate (2.14) in Theorem 2.1–(2).

## 5. WELL-POSEDNESS FOR THE NAVIER-STOKES-FOURIER SYSTEM

We start by considering the incompressible Navier-Stokes equation, that is, the first equation in (1.14). We denote by  $V$  the semigroup associated to the operator  $\nu_1 \Delta_x$ , and we also denote, for all  $t \geq 0$  and  $\xi \in \Omega'_\xi$ ,

$$\widehat{V}(t, \xi) = \mathcal{F}_x(V(t)\mathcal{F}_x^{-1})(\xi) = e^{-\nu_1|\xi|^2 t}.$$

We shall obtain below boundedness and integrated-in-time regularization estimates for  $V$  as well as for its integral in time against a source  $\int_0^t V(t-s)S(s) ds$ .

**Proposition 5.1.** *Let  $p \in [1, \infty]$ . Let  $u_0 \in \mathcal{F}_x^{-1}(L_\xi^p)$  and suppose moreover that  $u_0$  satisfies (1.16) in the torus case  $\Omega_x = \mathbf{T}^3$ . Then*

$$\| \widehat{V}(\cdot) \widehat{u}_0 \|_{L_\xi^p L_t^\infty} + \| |\xi| \widehat{V}(\cdot) \widehat{u}_0 \|_{L_\xi^p L_t^2} \lesssim \| \widehat{u}_0 \|_{L_\xi^p},$$

and moreover  $V(t)u_0$  also satisfies (1.16) for all  $t \geq 0$  in the torus case.

*Remark 5.2.* Observe that, in the torus case  $\Omega_x = \mathbf{T}^3$ , one can replace  $|\xi| \widehat{V}(\cdot) \widehat{u}_0$  in above estimate by  $\langle \xi \rangle \widehat{V}(\cdot) \widehat{u}_0$  since  $V(t)u_0$  is mean-free.

*Proof.* Let  $u(t) = V(t)u_0$ , which satisfies

$$\partial_t u = -\nu_1 \Delta_x u, \quad u|_{t=0} = u_0.$$

We already observe that, in the torus case, the solution  $u(t)$  is also mean-free, that is satisfies (1.16). For all  $\xi \in \Omega'_\xi$  we thus have

$$\partial_t \widehat{u}(t, \xi) = -\nu_1 |\xi|^2 \widehat{u}(t, \xi), \quad \widehat{u}(\xi)|_{t=0} = \widehat{u}_0(\xi),$$

thus for any  $t \geq 0$  we have

$$|\widehat{u}(t, \xi)|^2 + \int_0^t |\xi|^2 |\widehat{u}(s, \xi)|^2 dt \lesssim |\widehat{u}_0(\xi)|^2.$$

Taking the supremum in time and then taking the square-root of previous estimate yields

$$\| \widehat{u}(\xi) \|_{L_t^\infty} + \| |\xi| \widehat{u}(\xi) \|_{L_t^2} \lesssim |\widehat{u}_0(\xi)|,$$

and we conclude the proof by taking the  $L_\xi^p$  norm.  $\square$

**Proposition 5.3.** *Suppose  $p \in [1, \infty]$ . Let  $S = S(t, \xi)$  satisfies  $\| \langle \xi \rangle^{-1} \widehat{S} \|_{L_\xi^p L_t^2}$  in the torus case  $\Omega_x = \mathbf{T}^3$  as well as (1.16), and  $|\xi|^{-1} \widehat{S} \in L_\xi^p L_t^2$  in the whole space case  $\Omega_x = \mathbf{R}^3$ . Denote*

$$u_S(t) = \int_0^t V(t-s)S(s) ds.$$

Then in the torus case we have

$$\|\widehat{u}_S\|_{L_\xi^p L_t^\infty} + \|\langle \xi \rangle \widehat{u}_S\|_{L_\xi^p L_t^2} \lesssim \|\langle \xi \rangle^{-1} \widehat{S}\|_{L_\xi^p L_t^2}.$$

and in the whole space case

$$\|\widehat{u}_S\|_{L_\xi^p L_t^\infty} + \|\|\xi\| \widehat{u}_S\|_{L_\xi^p L_t^2} \lesssim \|\|\xi\|^{-1} \widehat{S}\|_{L_\xi^p L_t^2}.$$

*Proof.* We first observe that  $u_S$  satisfies

$$\partial_t u_S + \nu_1 \Delta_x u_S = S, \quad u_S|_{t=0} = 0.$$

We only prove the whole space case, the case of the torus being similar by observing that  $u_S$  is mean-free, that is verifies (1.16).

For all  $\xi \in \mathbf{R}^3$  and all  $t \geq 0$  we have

$$\partial_t \widehat{u}_S(t, \xi) + \nu_1 |\xi|^2 \widehat{u}_S(t, \xi) = \widehat{S}(t, \xi), \quad \widehat{u}_S(\xi)|_{t=0} = 0.$$

We can compute

$$\partial_t \frac{1}{2} |\widehat{u}_S(t, \xi)|^2 + \nu_1 |\xi|^2 |\widehat{u}_S(t, \xi)|^2 \leq (\widehat{S}(t, \xi), \widehat{u}_S(t, \xi)),$$

which implies, for all  $t \geq 0$ ,

$$|\widehat{u}_S(t, \xi)|^2 + \int_0^t |\xi|^2 |\widehat{u}_S(s, \xi)|^2 ds \lesssim \int_0^t \|\|\xi\|^{-1} S(s, \xi)\|^2 ds.$$

Taking the supremum in time, then taking the square-root of the estimate, and taking the  $L_\xi^p$  norm, the proof is thus finished.  $\square$

We now obtain bilinear estimates for the operator  $Q_{\text{NS}}$ .

**Lemma 5.4.** *Let  $p \in [1, \infty]$ . Let  $u, v \in \mathcal{F}_x^{-1}(L_\xi^1 L_t^\infty \cap L_\xi^p L_t^\infty)$ , then*

$$(5.1) \quad \|\|\xi\|^{-1} \widehat{Q}_{\text{NS}}(v, u)\|_{L_\xi^p L_t^2} \lesssim \|v\|_{L_\xi^p L_t^2} \|u\|_{L_\xi^1 L_t^\infty},$$

and also

$$(5.2) \quad \|\|\xi\|^{-1} \widehat{Q}_{\text{NS}}(v, u)\|_{L_\xi^p L_t^2} \lesssim \|v\|_{L_\xi^p L_t^\infty} \|u\|_{L_\xi^1 L_t^2}.$$

*Proof.* From the definition of  $Q_{\text{NS}}$ , we first observe that for all  $\xi \in \Omega'_\xi$  we have

$$|\widehat{Q}_{\text{NS}}(v, u)(\xi)| \lesssim |\xi| \int_{\Omega'_\eta} |\widehat{v}(\eta)| |\widehat{u}(\xi - \eta)| d\eta,$$

thus by Minkowski's inequality and then Hölder's inequality

$$\begin{aligned} \|\|\xi\|^{-1} \widehat{Q}_{\text{NS}}(v, u)(\xi)\|_{L_t^2} &\lesssim \int_{\Omega'_\eta} \left( \int_0^\infty |\widehat{v}(t, \eta)|^2 |\widehat{u}(t, \xi - \eta)|^2 dt \right)^{1/2} d\eta \\ &\lesssim \int_{\Omega'_\eta} \|\widehat{v}(\eta)\|_{L_t^2} \|\widehat{u}(\xi - \eta)\|_{L_t^\infty} d\eta. \end{aligned}$$

We then conclude the proof of (5.1) by taking the  $L_\xi^p$  norm above and applying Young's convolution inequality. The proof of (5.2) can be obtained in a similar way, by exchanging the role of  $u$  and  $v$  when applying Hölder's inequality with respect to the time variable.  $\square$

**5.1. Global existence in the torus  $\Omega_x = \mathbf{T}^3$ .** We shall construct mild solutions to the first equation in (1.14), namely

$$(5.3) \quad u(t) = V(t)u_0 + \int_0^t V(t-s)Q_{\text{NS}}(u(s), u(s)) ds.$$

We define the space

$$\mathcal{X} = \left\{ u \in \mathcal{F}_x^{-1}(L_\xi^1 L_t^\infty \cap L_\xi^1(\langle \xi \rangle) L_t^2) \mid u \text{ satisfies (1.16), } \|u\|_{\mathcal{X}} < \infty \right\},$$

with

$$\|u\|_{\mathcal{X}} := \|\widehat{u}\|_{L_\xi^1 L_t^\infty} + \|\langle \xi \rangle \widehat{u}\|_{L_\xi^1 L_t^2}.$$

Let  $u_0 \in \mathcal{F}_x^{-1}(L_\xi^1)$  satisfy (1.16) and

$$\|\widehat{u}_0\|_{L_\xi^1} \leq \eta_1.$$

Consider the map  $\Phi : \mathcal{X} \rightarrow \mathcal{X}$ ,  $u \mapsto \Phi[u]$  defined by, for all  $t \geq 0$ ,

$$(5.4) \quad \Phi[u](t) = V(t)u_0 + \int_0^t V(t-s)Q_{\text{NS}}(u(s), u(s)) ds.$$

thus, for all  $\xi \in \mathbf{Z}^3$ ,

$$(5.5) \quad \widehat{\Phi}[u](t, \xi) = \widehat{V}(t, \xi)\widehat{u}_0(\xi) + \int_0^t \widehat{V}(t-s, \xi)\widehat{Q}_{\text{NS}}(u(s), u(s))(\xi) ds.$$

For the first term we have from Proposition 5.1 that

$$\|\widehat{V}(t, \xi)\widehat{u}_0(\xi)\|_{\mathcal{X}} \leq C_0\|\widehat{u}_0\|_{L_\xi^1},$$

and by Proposition 5.3 we have

$$\begin{aligned} \left\| \int_0^t \widehat{V}(t-s, \xi)\widehat{Q}_{\text{NS}}(u(s), u(s))(\xi) ds \right\|_{\mathcal{X}} &\lesssim \| |\xi|^{-1}\widehat{Q}_{\text{NS}}(u, u) \|_{L_\xi^1 L_t^2} \\ &\lesssim \|\widehat{u}\|_{L_\xi^1 L_t^2} \|\widehat{u}\|_{L_\xi^1 L_t^\infty} \\ &\lesssim \|\langle \xi \rangle \widehat{u}\|_{L_\xi^1 L_t^2} \|\widehat{u}\|_{L_\xi^1 L_t^\infty} \\ &\lesssim \|u\|_{\mathcal{X}}^2, \end{aligned}$$

where we have used Lemma 5.4. Thus we obtain

$$\|\Phi[u]\|_{\mathcal{X}} \lesssim C_0\|\widehat{u}_0\|_{L_\xi^1} + C_1\|u\|_{\mathcal{X}}^2.$$

Moreover for  $u, v \in \mathcal{X}$  we can also compute, using again Proposition 5.3 and Lemma 5.4, that

$$\begin{aligned} &\left\| \int_0^t \widehat{V}(t-s, \xi)\widehat{Q}_{\text{NS}}((u-v)(s), v(s))(\xi) ds \right\|_{\mathcal{X}} + \left\| \int_0^t \widehat{V}(t-s, \xi)\widehat{Q}_{\text{NS}}(u(s), (u-v)(s))(\xi) ds \right\|_{\mathcal{X}} \\ &\lesssim \| |\xi|^{-1}\widehat{Q}_{\text{NS}}(u-v, v) \|_{L_\xi^1 L_t^2} + \| |\xi|^{-1}\widehat{Q}_{\text{NS}}(u, u-v) \|_{L_\xi^1 L_t^2} \\ &\lesssim \|\widehat{u} - \widehat{v}\|_{L_\xi^1 L_t^\infty} \|\widehat{v}\|_{L_\xi^1 L_t^2} + \|\widehat{u}\|_{L_\xi^1 L_t^2} \|\widehat{u} - \widehat{v}\|_{L_\xi^1 L_t^\infty}. \end{aligned}$$

Therefore there is  $C_1 > 0$  such that

$$\|\Phi[u] - \Phi[v]\|_{\mathcal{X}} \leq C_1(\|u\|_{\mathcal{X}} + \|v\|_{\mathcal{X}})\|u - v\|_{\mathcal{X}}.$$

Gathering the two inequalities and arguing as in Sections 4.2.1 and 4.2.2, we can construct a global unique solution  $u \in \mathcal{X}$  to the equation (5.3) if  $\eta_1 > 0$  is small enough, which moreover satisfies

$$\|u\|_{\mathcal{X}} \lesssim \|\widehat{u}_0\|_{L_\xi^1}.$$

Once  $u$  have been constructed, we can then argue in a similar and even simpler way in order to construct a global unique mild solution  $\theta$  for the second equations in (1.14) if  $\eta_1 > 0$  is small enough, namely

$$\theta(t) = \overline{V}(t)\theta_0 + \int_0^t \overline{V}(t-s)[- \operatorname{div}_x(u(s)\theta(s))] ds,$$

where  $\overline{V}$  denotes the semigroup associated to the operator  $\nu_2 \Delta_x$ , and which satisfies moreover

$$\|\theta\|_{\mathcal{X}} \lesssim \|\widehat{u}_0\|_{L_\xi^1} + \|\widehat{\theta}_0\|_{L_\xi^1}.$$

We finally obtain the solution  $\rho$  by using the last equation in (1.14) and observing that we consider mean-free solutions, so that  $\widehat{\rho}(t, 0) = \widehat{\theta}(t, 0) = 0$ . This completes the proof of Theorem 2.2-(1).



**5.2. Global existence in the whole space  $\Omega_x = \mathbf{R}^3$ .** Similarly as before we define the space, recalling that  $p \in (3/2, +\infty]$ ,

$$\mathcal{Y} = \left\{ u \in \mathcal{F}_x^{-1}(L_\xi^1 L_t^\infty \cap L_\xi^1(|\xi|)L_t^2) \cap \mathcal{F}_x^{-1}(L_\xi^p L_t^\infty \cap L_\xi^p(|\xi|)L_t^2) \mid \|u\|_{\mathcal{Y}} < \infty \right\},$$

with

$$\|u\|_{\mathcal{Y}} := \|\widehat{u}\|_{L_\xi^1 L_t^\infty} + \|\xi|\widehat{u}\|_{L_\xi^1 L_t^2} + \|\widehat{u}\|_{L_\xi^p L_t^\infty} + \|\xi|\widehat{u}\|_{L_\xi^p L_t^2}.$$

Let  $u_0 \in \mathcal{F}_x^{-1}(L_\xi^1 \cap L_\xi^p)$  satisfy

$$\|\widehat{u}_0\|_{L_\xi^1} + \|\widehat{u}_0\|_{L_\xi^p} \leq \eta_1,$$

and consider the map  $\Phi : \mathcal{Y} \rightarrow \mathcal{Y}$ ,  $u \mapsto \Phi[u]$  defined by (5.4), in particular (5.5) is verified for all  $\xi \in \mathbf{R}^3$ .

For the first term in (5.5) we have from Proposition 5.1 that

$$\|\widehat{V}(t, \xi)\widehat{u}_0(\xi)\|_{\mathcal{Y}} \leq C_0(\|\widehat{u}_0\|_{L_\xi^1} + \|\widehat{u}_0\|_{L_\xi^p}).$$

Furthermore, by Proposition 5.3 we have

$$\begin{aligned} \left\| \int_0^t \widehat{V}(t-s, \xi) \widehat{Q}_{\text{NS}}(u(s), u(s))(\xi) ds \right\|_{\mathcal{Y}} &\lesssim \|\xi|^{-1} \widehat{Q}_{\text{NS}}(u, u)\|_{L_\xi^1 L_t^2} + \|\xi|^{-1} \widehat{Q}_{\text{NS}}(u, u)\|_{L_\xi^p L_t^2} \\ &\lesssim \|\widehat{u}\|_{L_\xi^1 L_t^2} \left( \|\widehat{u}\|_{L_\xi^1 L_t^\infty} + \|\widehat{u}\|_{L_\xi^p L_t^\infty} \right). \end{aligned}$$

where we have used Lemma 5.4. We now observe that

$$\|\widehat{u}\|_{L_\xi^1 L_t^2} \lesssim \|\mathbf{1}_{|\xi| \geq 1} \widehat{u}\|_{L_\xi^1 L_t^2} + \|\mathbf{1}_{|\xi| < 1} \widehat{u}\|_{L_\xi^1 L_t^2},$$

and for the first term we easily have

$$\|\mathbf{1}_{|\xi| \geq 1} \widehat{u}\|_{L_\xi^1 L_t^2} \lesssim \|\xi|\widehat{u}\|_{L_\xi^1 L_t^2}.$$

For the second term we use Hölder's inequality to obtain

$$\|\mathbf{1}_{|\xi| < 1} \widehat{u}\|_{L_\xi^1 L_t^2} \lesssim \|\mathbf{1}_{|\xi| < 1} |\xi|^{-1}\|_{L_\xi^{p'}} \|\mathbf{1}_{|\xi| < 1} |\xi|\widehat{u}\|_{L_\xi^p L_t^2} \lesssim \|\xi|\widehat{u}\|_{L_\xi^p L_t^2},$$

where we have used that  $\|\mathbf{1}_{|\xi| < 1} |\xi|^{-1}\|_{L_\xi^{p'}} < \infty$  since  $p > 3/2$ . Therefore we get

$$(5.6) \quad \|\widehat{u}\|_{L_\xi^1 L_t^2} \lesssim \|\xi|\widehat{u}\|_{L_\xi^1 L_t^2} + \|\xi|\widehat{u}\|_{L_\xi^p L_t^2}.$$

Gathering previous estimates, we have hence obtained

$$\|\Phi[u]\|_{\mathcal{Y}} \leq C_0 \left( \|\widehat{u}_0\|_{L_\xi^1} + \|\widehat{u}_0\|_{L_\xi^p} \right) + C_1 \|u\|_{\mathcal{Y}}^2.$$

Moreover for  $u, v \in \mathcal{Y}$  we can also compute, using again Proposition 5.3 and Lemma 5.4, that

$$\begin{aligned} &\left\| \int_0^t \widehat{V}(t-s, \xi) \widehat{Q}_{\text{NS}}((u-v)(s), v(s))(\xi) ds \right\|_{\mathcal{Y}} + \left\| \int_0^t \widehat{V}(t-s, \xi) \widehat{Q}_{\text{NS}}(u(s), (u-v)(s))(\xi) ds \right\|_{\mathcal{Y}} \\ &\lesssim \|\xi|^{-1} \widehat{Q}_{\text{NS}}(u-v, v)\|_{L_\xi^1 L_t^2} + \|\xi|^{-1} \widehat{Q}_{\text{NS}}(u, u-v)\|_{L_\xi^1 L_t^2} \\ &\quad + \|\xi|^{-1} \widehat{Q}_{\text{NS}}(u-v, v)\|_{L_\xi^p L_t^2} + \|\xi|^{-1} \widehat{Q}_{\text{NS}}(u, u-v)\|_{L_\xi^p L_t^2} \\ &\lesssim \|\widehat{u} - \widehat{v}\|_{L_\xi^1 L_t^\infty} \|\widehat{v}\|_{L_\xi^1 L_t^2} + \|\widehat{u}\|_{L_\xi^1 L_t^2} \|\widehat{u} - \widehat{v}\|_{L_\xi^1 L_t^\infty} + \|\widehat{u} - \widehat{v}\|_{L_\xi^p L_t^\infty} \|\widehat{v}\|_{L_\xi^1 L_t^2} + \|\widehat{u}\|_{L_\xi^1 L_t^2} \|\widehat{u} - \widehat{v}\|_{L_\xi^p L_t^\infty} \\ &\lesssim \left( \|\widehat{u}\|_{L_\xi^1 L_t^2} + \|\widehat{v}\|_{L_\xi^1 L_t^2} \right) \left( \|\widehat{u} - \widehat{v}\|_{L_\xi^1 L_t^\infty} + \|\widehat{u} - \widehat{v}\|_{L_\xi^p L_t^\infty} \right). \end{aligned}$$

Using inequality (5.6) we therefore get, for some constant  $C_1 > 0$ ,

$$\|\Phi[u] - \Phi[v]\|_{\mathcal{Y}} \leq C_1 (\|u\|_{\mathcal{Y}} + \|v\|_{\mathcal{Y}}) \|u - v\|_{\mathcal{Y}}.$$

Gathering these two inequalities together, the proof of Theorem 2.2–(2) is completed by arguing as in Section 5.1 above.

## 6. HYDRODYNAMIC LIMIT

Recalling that the semigroup  $U^\varepsilon$  is defined in (3.8), and also  $\widehat{U}^\varepsilon$  in (3.7), we also define, for all  $t \geq 0$ ,

$$(6.1) \quad \Psi^\varepsilon[f, g](t) = \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t-s) \Gamma(f(s), g(s)) ds,$$

as well as its Fourier transform in space, for all  $\xi \in \Omega'_\xi$ ,

$$(6.2) \quad \widehat{\Psi}^\varepsilon[f, g](t, \xi) = \frac{1}{\varepsilon} \int_0^t \widehat{U}^\varepsilon(t-s, \xi) \widehat{\Gamma}(f(s), g(s))(\xi) ds.$$

**6.1. Estimates on  $\widehat{U}^\varepsilon$ .** We denote that  $0 \leq \chi \leq 1$  is a fixed compactly supported function of  $B_1$  equal to one on  $B_{\frac{1}{2}}$ , where  $B_R$  is the ball with radius  $R$  centered at zero.

Arguing as in [13, 39] but using the spectral estimates of [74, 75] for the non-cutoff Boltzmann equation, we then have:

**Lemma 6.1.** *There exist  $\kappa > 0$  such that one can write*

$$U^\varepsilon(t) = \sum_{j=1}^4 U_j^\varepsilon(t) + U^{\varepsilon\#}(t),$$

with

$$\widehat{U}_j^\varepsilon(t, \xi) := \widehat{U}_j\left(\frac{t}{\varepsilon^2}, \varepsilon\xi\right), \quad \widehat{U}^{\varepsilon\#}(t, \xi) = \widehat{U}^\# \left(\frac{t}{\varepsilon^2}, \varepsilon\xi\right),$$

where we have the following properties:

(1) For  $1 \leq j \leq 4$ ,

$$\widehat{U}_j(t, \xi) = \chi\left(\frac{|\xi|}{\kappa}\right) e^{t\lambda_j(\xi)} P_j(\xi),$$

with  $\lambda_j$  satisfying

$$\lambda_j(\xi) = i\alpha_j(\xi) - \beta_j|\xi|^2 + \gamma_j(|\xi|),$$

with

$$\alpha_1 > 0, \quad \alpha_2 < 0, \quad \alpha_3 = \alpha_4 = 0, \quad \beta_j > 0,$$

and

$$\gamma_j(|\xi|) = O(|\xi|^3), \quad \text{as } \xi \rightarrow 0, \quad \gamma_j(\xi) \leq \beta_j|\xi|^2/2, \quad \forall |\xi| \leq \kappa,$$

as well as

$$P_j(\xi) = P_j^0\left(\frac{\xi}{|\xi|}\right) + |\xi|P_j^1\left(\frac{\xi}{|\xi|}\right) + |\xi|^2P_j^2(\xi),$$

with  $P_j^n$  bounded linear operators on  $L_v^2$  with operator norms uniformly bounded for  $|\xi| \leq \kappa$ .

(2) We also have that the orthogonal projector  $\mathbf{P}$  onto  $\text{Ker } L$  satisfies

$$\mathbf{P} = \sum_{j=1}^4 P_j^0\left(\frac{\xi}{|\xi|}\right).$$

Moreover  $P_j^0\left(\frac{\xi}{|\xi|}\right)$ ,  $P_j^1\left(\frac{\xi}{|\xi|}\right)$  and  $P_j^2(\xi)$  are bounded from  $L_v^2$  to  $L^2(\langle v \rangle^l)$  uniformly in  $|\xi| \leq \kappa$  for all  $l \geq 0$ .

(3) In the hard potentials case  $\gamma + 2s \geq 0$ , for all  $t \geq 0$  and all  $\xi \in \mathbf{R}^3$  there holds, for any  $\ell \geq 0$ ,

$$(6.3) \quad \|\widehat{U}^{\varepsilon\#}(t, \xi) \widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)} \leq C e^{-\lambda_0 \frac{t}{\varepsilon^2}} \|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)},$$

for any  $f$  satisfying moreover (1.11) in the torus case, where  $\lambda_0, C > 0$  are independent of  $t, \xi, \varepsilon$ .

(4) In the soft potential case  $\gamma + 2s < 0$ , for all  $t \geq 0$  and all  $\xi \in \mathbf{R}^3$  there holds, for any  $k, \ell \geq 0$ ,

$$(6.4) \quad \|\widehat{U}^{\varepsilon\#}(t, \xi) \mathbf{P}^\perp \widehat{f}(\xi)\|_{L_v^2((v)^k)} \leq C \left(1 + \frac{t}{\varepsilon^2}\right)^{-\frac{\ell}{|\gamma+2s|}} \|\widehat{f}(\xi)\|_{L_v^2((v)^{k+\ell})},$$

for any  $f$  satisfying moreover (1.11) in the torus case, where  $C > 0$  is independent of  $t, \xi, \varepsilon$ .

*Proof.* The proof is the same as in [24, Lemma 5.10]. For the soft potentials case, we need to replace the use of [74, Theorem 3.2 and Remark 5.2] in the proof by [75, Theorem 1.1 and Section 4], in particular the decay estimate (6.4) comes from [75, Equation (2.46)] and the fact that  $B_0(\xi) \mathbf{P}^\perp = B(\xi) \mathbf{P}^\perp$ , where  $B_0(\xi)$  and  $B(\xi)$  are defined in [75, Equation (1.18)] and satisfy  $B_0(\xi) = B(\xi) - \mathbf{P}$ .  $\square$

Denoting

$$\widetilde{P}_j \left( \xi, \frac{\xi}{|\xi|} \right) := P_j^1 \left( \frac{\xi}{|\xi|} \right) + \xi P_j^2(\xi),$$

for  $1 \leq j \leq 4$ , we can further split  $\widehat{U}_j^\varepsilon$  into four parts (one main part and three remainder terms):

$$(6.5) \quad U_j^\varepsilon = U_{j0}^\varepsilon + U_{j0}^{\varepsilon\#} + U_{j1}^\varepsilon + U_{j2}^\varepsilon,$$

where

$$\begin{aligned} \widehat{U}_{j0}^\varepsilon(t, \xi) &= e^{i\alpha_j |\xi| \frac{t}{\varepsilon} - \beta_j t |\xi|^2} P_j^0 \left( \frac{\xi}{|\xi|} \right), \\ \widehat{U}_{j0}^{\varepsilon\#}(t, \xi) &= \left( \chi \left( \frac{\varepsilon |\xi|}{\kappa} \right) - 1 \right) e^{i\alpha_j |\xi| \frac{t}{\varepsilon} - \beta_j t |\xi|^2} P_j^0 \left( \frac{\xi}{|\xi|} \right), \\ \widehat{U}_{j1}^\varepsilon(t, \xi) &= \chi \left( \frac{\varepsilon |\xi|}{\kappa} \right) e^{i\alpha_j |\xi| \frac{t}{\varepsilon} - \beta_j t |\xi|^2} \left( e^{t \frac{\gamma_j(\varepsilon |\xi|)}{\varepsilon^2}} - 1 \right) P_j^0 \left( \frac{\xi}{|\xi|} \right), \\ \widehat{U}_{j2}^\varepsilon(t, \xi) &= \chi \left( \frac{\varepsilon |\xi|}{\kappa} \right) e^{i\alpha_j |\xi| \frac{t}{\varepsilon} - \beta_j t |\xi|^2} e^{t \frac{\gamma_j(\varepsilon |\xi|)}{\varepsilon^2}} \varepsilon |\xi| \widetilde{P}_j \left( \varepsilon \xi, \frac{\xi}{|\xi|} \right). \end{aligned}$$

In particular we observe that  $\widehat{U}_{30}^\varepsilon$  and  $\widehat{U}_{40}^\varepsilon$  are independent of  $\varepsilon$ , so that we define

$$(6.6) \quad \widehat{U}(t, \xi) := \widehat{U}_{30}^\varepsilon(t, \xi) + \widehat{U}_{40}^\varepsilon(t, \xi),$$

which is then independent of  $\varepsilon$ . We finally define

$$(6.7) \quad U(t) = \mathcal{F}_x^{-1} \widehat{U}(t) \mathcal{F}_x.$$

**Lemma 6.2.** ([39], Proposition A.3) *We have that  $U(0)$  is the projection on the subset of  $\text{Ker } L$  consisting of functions  $f$  satisfying  $\text{div } u_f = 0$  and also  $\rho_f + \theta_f = 0$ . We also have*

$$U(t)f = U(t)U(0)f, \quad \forall t \geq 0,$$

and

$$\text{div } u_f = 0 \quad \text{and} \quad \rho_f + \theta_f = 0 \quad \Rightarrow \quad P_j^0 \left( \frac{\xi}{|\xi|} \right) f = 0, \quad j = 1, 2.$$

The following lemma studies the limit of  $U^\varepsilon(t)$  as  $\varepsilon$  goes to 0.

**Lemma 6.3.** *Let  $f = f(x, v) \in \text{Ker } L$  then we have*

$$\|(U^\varepsilon(\cdot) - U(\cdot))f\|_{L_x^1 L_t^\infty L_v^2} \lesssim \|f\|_{L_x^1 L_v^2},$$

and

$$\|(U^\varepsilon(\cdot) - U(\cdot))f\|_{L_x^1 L_t^\infty L_v^2} \lesssim \varepsilon \|\xi\| f\|_{L_x^1 L_v^2}.$$

*Proof.* The proof follows the idea of [39, Lemma 3.5], that we shall adapt since we work in different functional spaces.

First of all we observe that from the decomposition of  $U^\varepsilon$  in (6.5) we can write, for all  $t \geq 0$  and  $\xi \in \Omega'_\xi$ ,

$$\begin{aligned} \widehat{U}^\varepsilon(t, \xi) \widehat{f}(\xi) - \widehat{U}^\varepsilon(t, \xi) \widehat{f}(\xi) &= \sum_{j=1}^4 \widehat{U}_{j0}^{\varepsilon\#}(t, \xi) \widehat{f}(\xi) + \widehat{U}_{j1}^\varepsilon(t, \xi) \widehat{f}(\xi) + \widehat{U}_{j1}^\varepsilon(t, \xi) \widehat{f}(\xi) \\ &\quad + \sum_{j=1}^2 \widehat{U}_{j0}^\varepsilon(t, \xi) \widehat{f}(\xi) + \widehat{U}^{\varepsilon\#}(t, \xi) \widehat{f}(\xi), \end{aligned}$$

and we shall estimate each term separately below.

We first compute the term  $U_{jm}^\varepsilon(t)f$  for  $j = 1, 2, 3, 4$  and  $m = 1, 2$ . For the  $U_{j1}$  term, using Lemma 6.1 together with the inequality  $|e^a - 1| \leq |a|e^{|a|}$  for any  $a \in \mathbf{R}^+$ , we have

$$(6.8) \quad \chi \left( \frac{\varepsilon|\xi|}{\kappa} \right) e^{-\beta_j t |\xi|^2} \left| e^{t \frac{\gamma_j(\varepsilon|\xi)}{\varepsilon^2}} - 1 \right| \leq \chi \left( \frac{\varepsilon|\xi|}{\kappa} \right) e^{-\frac{\beta_j}{2} t |\xi|^2} t \varepsilon |\xi|^3 \lesssim \chi \left( \frac{\varepsilon|\xi|}{\kappa} \right) \varepsilon |\xi| \lesssim \min\{1, \varepsilon|\xi|\}.$$

Then we can compute, for all  $t \geq 0$  and  $\xi \in \mathbf{R}^3$ ,

$$\begin{aligned} \|\widehat{U}_{j1}^\varepsilon(t, \xi) \widehat{f}(\xi)\|_{L_v^2} &\leq \chi \left( \frac{\varepsilon|\xi|}{\kappa} \right) \left| e^{i\alpha_j |\xi| \frac{t}{\varepsilon} - \beta_j t |\xi|^2} \right| \left( e^{t \frac{\gamma_j(\varepsilon|\xi)}{\varepsilon^2}} - 1 \right) \|P_j^0 \left( \frac{\xi}{|\xi|} \right) \widehat{f}(\xi)\|_{L_v^2} \\ &\lesssim \min\{1, \varepsilon|\xi|\} \|\widehat{f}(\xi)\|_{L_v^2}. \end{aligned}$$

For the  $\widehat{U}_{j2}^\varepsilon(t, \xi)$  term we have

$$\begin{aligned} \|\widehat{U}_{j1}^\varepsilon(t, \xi) \widehat{f}(\xi)\|_{L_v^2} &\leq \chi \left( \frac{\varepsilon|\xi|}{\kappa} \right) \left| e^{i\alpha_j |\xi| \frac{t}{\varepsilon} - \beta_j t |\xi|^2} e^{t \frac{\gamma_j(\varepsilon|\xi)}{\varepsilon^2}} \right| \varepsilon |\xi| \|\tilde{P}_j(\varepsilon\xi, \frac{\xi}{|\xi|}) \widehat{f}(\xi)\|_{L_v^2} \\ &\lesssim \min\{1, \varepsilon|\xi|\} \|\widehat{f}(\xi)\|_{L_v^2}. \end{aligned}$$

For the term  $\widehat{U}_{j0}^{\varepsilon\#}(t, \xi)$ , using the fact that

$$(6.9) \quad \left| \chi \left( \frac{\varepsilon|\xi|}{\kappa} \right) - 1 \right| \lesssim \min\{1, \varepsilon|\xi|\},$$

we have

$$\begin{aligned} \|\widehat{U}_{j0}^{\varepsilon\#}(t, \xi) \widehat{f}(\xi)\|_{L_v^2} &\leq \left( \chi \left( \frac{\varepsilon|\xi|}{\kappa} \right) - 1 \right) \left| e^{i\alpha_j |\xi| \frac{t}{\varepsilon} - \beta_j t |\xi|^2} \right| \|P_j^0 \left( \frac{\xi}{|\xi|} \right) \widehat{f}(\xi)\|_{L_v^2} \\ &\lesssim \min\{1, \varepsilon|\xi|\} \|\widehat{f}(\xi)\|_{L_v^2}. \end{aligned}$$

Taking the  $L_\xi^1 L_t^\infty$  norm in both sides yields, for all  $j = 1, 2, 3, 4$ ,

$$\|\widehat{U}_{j1}^\varepsilon(\cdot) \widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2} + \|\widehat{U}_{j2}^\varepsilon(\cdot) \widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2} + \|\widehat{U}_{j0}^{\varepsilon\#}(\cdot) \widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \min\{\|\widehat{f}\|_{L_\xi^1 L_v^2}, \varepsilon \|\|\xi| \widehat{f}\|_{L_\xi^1 L_v^2}\}.$$

By Lemma 6.2 we have, if  $f \in \text{Ker } L$  is a well-prepared data, then

$$U_{10}^\varepsilon f + U_{20}^\varepsilon f = 0.$$

Finally we compute the term  $U^{\varepsilon\#}(t, \xi)$ , noticing that

$$\widehat{U}^{\varepsilon\#}(t, \xi) \widehat{f}(\xi, v) = \widehat{U}^\varepsilon(t, \xi) \widehat{U}^{\varepsilon\#}(0, \xi) \widehat{f}(\xi, v) = \widehat{U}^\varepsilon(t, \xi) \left( 1 - \chi \left( \frac{\varepsilon|\xi|}{\kappa} \right) \sum_{j=1}^4 P_j(\varepsilon\xi) \right) \widehat{f}(\xi, v).$$

Since  $f$  belongs to  $\text{Ker } L$ , we have

$$\widehat{U}^{\varepsilon\#}(t, \xi) \widehat{f}(\xi, v) = \widehat{U}^\varepsilon(t, \xi) \left( 1 - \chi \left( \frac{\varepsilon|\xi|}{\kappa} \right) - \varepsilon |\xi| \chi \left( \frac{\varepsilon|\xi|}{\kappa} \right) \sum_{j=1}^4 \tilde{P}_j(\varepsilon\xi) \right) \widehat{f}(\xi, v).$$

By Proposition 3.2 we have

$$\begin{aligned} \|\widehat{U}^{\varepsilon\#}(\cdot) \widehat{f}\|_{L_\xi^1 L_t^\infty L_v^2} &\lesssim \left\| \left( 1 - \chi \left( \frac{\varepsilon|\xi|}{\kappa} \right) - \varepsilon |\xi| \chi \left( \frac{\varepsilon|\xi|}{\kappa} \right) \sum_{j=1}^4 \tilde{P}_j(\varepsilon\xi) \right) \widehat{f}(\xi) \right\|_{L_\xi^1 L_v^2} \\ &\lesssim \min\{\|\widehat{f}\|_{L_\xi^1 L_v^2}, \varepsilon \|\|\xi| \widehat{f}\|_{L_\xi^1 L_v^2}\}, \end{aligned}$$

thus the proof is finished by gathering together the two previous estimates.  $\square$

**6.2. Estimates on  $\widehat{\Psi}^\varepsilon$ .** The decomposition of the semigroup  $U^\varepsilon(t)$  in (6.5) also gives us a decomposition of the operator  $\Psi^\varepsilon(t)$  defined in (6.1).

**Lemma 6.4.** *The following decomposition holds*

$$\Psi^\varepsilon = \sum_{j=1}^4 \Psi_j^\varepsilon + \Psi^{\varepsilon\#},$$

with

$$\widehat{\Psi}^{\varepsilon\#}[f_1, f_2](t, \xi) := \frac{1}{\varepsilon} \int_0^t \widehat{U}^{\varepsilon\#}(t-s, \xi) \widehat{\Gamma}(f_1(s), f_2(s))(\xi) ds,$$

and, for all  $1 \leq j \leq 4$ ,

$$\Psi_j^\varepsilon = \Psi_{j0}^\varepsilon + \Psi_{j0}^{\varepsilon\#} + \Psi_{j1}^\varepsilon + \Psi_{j2}^\varepsilon,$$

where

$$\begin{aligned} \widehat{\Psi}_{j0}^\varepsilon[f_1, f_2](t, \xi) &= \int_0^t e^{i\alpha_j|\xi| \frac{t-s}{\varepsilon} - \beta_j(t-s)|\xi|^2} |\xi| P_j^1\left(\frac{\xi}{|\xi|}\right) \widehat{\Gamma}(f_1(s), f_2(s))(\xi) ds, \\ \widehat{\Psi}_{j0}^{\varepsilon\#}[f_1, f_2](t, \xi) &= \left( \chi\left(\frac{\varepsilon|\xi|}{\kappa}\right) - 1 \right) \int_0^t e^{i\alpha_j|\xi| \frac{t-s}{\varepsilon} - \beta_j(t-s)|\xi|^2} |\xi| P_j^1\left(\frac{\xi}{|\xi|}\right) \widehat{\Gamma}(f_1(s), f_2(s))(\xi) ds, \\ \widehat{\Psi}_{j1}^\varepsilon[f_1, f_2](t, \xi) &= \chi\left(\frac{\varepsilon|\xi|}{\kappa}\right) \int_0^t e^{i\alpha_j|\xi| \frac{t-s}{\varepsilon} - \beta_j(t-s)|\xi|^2} \left( e^{(t-s)\frac{\gamma_j(\varepsilon|\xi|)}{\varepsilon^2}} - 1 \right) |\xi| P_j^1\left(\frac{\xi}{|\xi|}\right) \widehat{\Gamma}(f_1(s), f_2(s))(\xi) ds, \\ \widehat{\Psi}_{j2}^\varepsilon[f_1, f_2](t, \xi) &= \chi\left(\frac{\varepsilon|\xi|}{\kappa}\right) \int_0^t e^{i\alpha_j|\xi| \frac{t-s}{\varepsilon} - \beta_j(t-s)|\xi|^2} e^{(t-s)\frac{\gamma_j(\varepsilon|\xi|)}{\varepsilon^2}} \varepsilon |\xi|^2 P_j^2(\varepsilon\xi) \widehat{\Gamma}(f_1(s), f_2(s))(\xi) ds. \end{aligned}$$

Similarly as above, we observe again that that  $\widehat{\Psi}_{30}^\varepsilon$  and  $\widehat{\Psi}_{40}^\varepsilon$  are independent of  $\varepsilon$ , so that we define

$$(6.10) \quad \widehat{\Psi}[f, g](t, \xi) := \widehat{\Psi}_{30}^\varepsilon[f, g](t, \xi) + \widehat{\Psi}_{40}^\varepsilon[f, g](t, \xi)$$

which is then independent of  $\varepsilon$ . We finally define

$$(6.11) \quad \Psi[f, g](t) = \mathcal{F}_x^{-1} \widehat{\Psi}[f, g](t) \mathcal{F}_x.$$

We are now able to prove the following result on the convergence of  $\Psi^\varepsilon$  towards  $\Psi$ .

**Lemma 6.5.** *Let  $(\rho_0, u_0, \theta_0)$  satisfy the hypotheses of Theorem 2.2 and consider the associated global unique solution  $(\rho, u, \theta)$  to (1.14). Let also  $g_0 = g_0(x, v) \in \text{Ker } L$  be defined by (2.15) and  $g = g(t, x, v) \in \text{Ker } L$  by (2.16). Then we have:*

(1) *Torus case  $\Omega_x = \mathbf{T}^3$ : There holds*

$$\|\Psi^\varepsilon[g, g] - \Psi[g, g]\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \varepsilon \left( \|\widehat{g}_0\|_{L_\xi^1 L_v^2}^2 + \|\widehat{g}_0\|_{L_\xi^1 L_v^2}^3 \right).$$

(2) *Whole space case  $\Omega_x = \mathbf{R}^3$ : For any  $p \in (3/2, \infty]$  there holds*

$$\|\Psi^\varepsilon[g, g] - \Psi[g, g]\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \varepsilon \left( \|\widehat{g}_0\|_{L_\xi^1 L_v^2}^2 + \|\widehat{g}_0\|_{L_\xi^1 L_v^2}^3 + \|\widehat{g}_0\|_{L_\xi^p L_v^2}^2 + \|\widehat{g}_0\|_{L_\xi^p L_v^2}^3 \right).$$

*Proof.* We adapt the proof of [39, Lemma 4.1] for the cutoff Boltzmann equation with hard potentials. Thanks to the decomposition of  $\Psi^\varepsilon$  in Lemma 6.4 we write, for all  $t \geq 0$  and  $\xi \in \Omega'_\xi$ ,

$$\begin{aligned} \widehat{\Psi}^\varepsilon[g, g](t, \xi) - \widehat{\Psi}[g, g](t, \xi) &= \sum_{j=1}^4 \widehat{\Psi}_{j0}^{\varepsilon\#}[g, g](t, \xi) + \widehat{\Psi}_{j1}^\varepsilon[g, g](t, \xi) + \widehat{\Psi}_{j2}^\varepsilon[g, g](t, \xi) \\ &\quad + \sum_{j=1}^2 \widehat{\Psi}_{j0}^\varepsilon[g, g](t, \xi) + \widehat{\Psi}^{\varepsilon\#}[g, g](t, \xi). \end{aligned}$$

We remark that for the zero frequency  $\xi = 0$  we have

$$\widehat{\Psi}^\varepsilon[g, g](t, 0) = \widehat{\Psi}^{\varepsilon\#}[g, g](t, 0).$$

We split the proof into several steps and estimate each term separately below.

*Step 1.* By Lemma 6.4, (6.8) and Minkowski inequality, for the term  $\widehat{\Psi}_{j0}^{\varepsilon\#}[g, g]$  with  $j = 1, 2, 3, 4$ , for all  $t \geq 0$  and all  $\xi \in \Omega'_\xi \setminus \{0\}$  we have

$$\begin{aligned} \left\| \widehat{\Psi}_{j0}^{\varepsilon\#}[g, g](t, \xi) \right\|_{L_v^2} &\lesssim \left| \chi \left( \frac{\varepsilon|\xi|}{\kappa} \right) - 1 \right| \int_0^t e^{-\beta_j(t-s)|\xi|^2} |\xi| \left\| P_j^1 \left( \frac{\xi}{|\xi|} \right) \widehat{\Gamma}(g(s), g(s))(\xi) \right\|_{L_v^2} ds \\ &\lesssim \varepsilon \int_0^t e^{-\beta_j(t-s)|\xi|^2} |\xi|^2 \|\widehat{\Gamma}(g(s), g(s))(\xi)\|_{L_v^2} ds \\ &\lesssim \varepsilon \|\widehat{\Gamma}(g, g)(\xi)\|_{L_t^\infty L_v^2}. \end{aligned}$$

Similarly for the term  $\widehat{\Psi}_{j1}^\varepsilon[g, g]$ , by Lemma 6.4, (6.9) and Minkowski inequality we have, for all  $j = 1, 2, 3, 4$ ,

$$\begin{aligned} \left\| \widehat{\Psi}_{j1}^\varepsilon[g, g](t, \xi) \right\|_{L_v^2} &\lesssim \chi \left( \frac{\varepsilon|\xi|}{\kappa} \right) \int_0^t e^{-\beta_j(t-s)|\xi|^2} \left| e^{(t-s)\frac{\gamma_j(\varepsilon|\xi|)}{\varepsilon^2}} - 1 \right| |\xi| \left\| P_j^1 \left( \frac{\xi}{|\xi|} \right) \widehat{\Gamma}(f_1(s), f_2(s)) \right\|_{L_v^2} ds \\ &\lesssim \varepsilon \int_0^t e^{-\frac{\beta_j}{2}(t-s)|\xi|^2} |\xi|^2 \|\widehat{\Gamma}(g(s), g(s))(\xi)\|_{L_v^2} ds \\ &\lesssim \varepsilon \|\widehat{\Gamma}(g, g)(\xi)\|_{L_t^\infty L_v^2}. \end{aligned}$$

Similarly for the term  $\widehat{\Psi}_{j2}^\varepsilon[g, g]$ , by Lemma 6.4 and Minkowski inequality we have, for all  $j = 1, 2, 3, 4$ ,

$$\begin{aligned} \left\| \widehat{\Psi}_{j2}^\varepsilon[g, g](t, \xi) \right\|_{L_v^2} &\lesssim \chi \left( \frac{\varepsilon|\xi|}{\kappa} \right) \int_0^t e^{-\beta_j(t-s)|\xi|^2} \left| e^{(t-s)\frac{\gamma_j(\varepsilon|\xi|)}{\varepsilon^2}} \right| \varepsilon |\xi|^2 \left\| P_j^2(\varepsilon\xi) \widehat{\Gamma}(g(s), g(s)) \right\|_{L_v^2} ds \\ &\lesssim \varepsilon \int_0^t e^{-\frac{\beta_j}{2}(t-s)|\xi|^2} |\xi|^2 \|\widehat{\Gamma}(g(s), g(s))(\xi)\|_{L_v^2} ds \\ &\lesssim \varepsilon \|\widehat{\Gamma}(g, g)(\xi)\|_{L_t^\infty L_v^2}. \end{aligned}$$

Taking  $L_\xi^1 L_t^\infty$  norm both side and using Young's inequality for convolution we have, for all  $j = 1, 2, 3, 4$ ,

$$\left\| \widehat{\Psi}_{j0}^{\varepsilon\#}[g, g] \right\|_{L_\xi^1 L_t^\infty L_v^2} + \left\| \widehat{\Psi}_{j1}^\varepsilon[g, g] \right\|_{L_\xi^1 L_t^\infty L_v^2} + \left\| \widehat{\Psi}_{j2}^\varepsilon[g, g] \right\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \varepsilon \|\widehat{\Gamma}(g, g)(\xi)\|_{L_\xi^1 L_t^\infty L_v^2}.$$

Thanks to [68] and the fact that  $\|\langle v \rangle^\ell \mathbf{P}\phi\|_{H_v^m} \lesssim \|\mathbf{P}\phi\|_{L_v^2}$  for all  $m, \ell \geq 0$ , we have

$$(6.12) \quad \|\Gamma(\mathbf{P}g_1, \mathbf{P}g_2)\|_{L_v^2} \lesssim \|\mathbf{P}g_1\|_{L_v^2} \|\mathbf{P}g_2\|_{L_v^2},$$

therefore arguing as in Lemma 4.1 it follows, for any  $p \in [1, \infty]$  and  $\ell \geq 0$ ,

$$(6.13) \quad \|\widehat{\Gamma}(g, g)\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)} \lesssim \|g\|_{L_\xi^1 L_t^\infty L_v^2} \|g\|_{L_\xi^p L_t^\infty L_v^2}.$$

We therefore obtain, for all  $j = 1, 2, 3, 4$ ,

$$(6.14) \quad \left\| \widehat{\Psi}_{j0}^{\varepsilon\#}[g, g] \right\|_{L_\xi^1 L_t^\infty L_v^2} + \left\| \widehat{\Psi}_{j1}^\varepsilon[g, g] \right\|_{L_\xi^1 L_t^\infty L_v^2} + \left\| \widehat{\Psi}_{j2}^\varepsilon[g, g] \right\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \varepsilon \|g\|_{L_\xi^1 L_t^\infty L_v^2}^2.$$

*Step 2.* We now focus on the term  $\widehat{\Psi}_{j0}^\varepsilon[g, g]$  with  $j = 1, 2$ , and recall that  $\alpha_j > 0$  for  $j = 1, 2$ . We denote

$$\widehat{H}_j(t, s, \xi) = e^{-\beta_j(t-s)|\xi|^2} |\xi| P_j^1 \left( \frac{\xi}{|\xi|} \right) \widehat{\Gamma}(g(s), g(s))(\xi),$$

and thus, using integration by parts, for all  $t \geq 0$  and all  $\xi \in \Omega'_\xi \setminus \{0\}$  we have

$$\begin{aligned}
(6.15) \quad \widehat{\Psi}_{j0}^\varepsilon[g, g](t, \xi) &= \int_0^t e^{i\alpha_j|\xi|\frac{t-s}{\varepsilon} - \beta_j(t-s)|\xi|^2} |\xi| P_j^1\left(\frac{\xi}{|\xi|}\right) \widehat{\Gamma}(g(s), g(s))(\xi) ds \\
&= \frac{\varepsilon}{i\alpha_j|\xi|} \left( \int_0^t e^{i\alpha_j|\xi|\frac{t-s}{\varepsilon}} \partial_t \widehat{H}_j(t, s, \xi) ds - \widehat{H}_j(t, t, \xi) + e^{i\alpha_j|\xi|\frac{t}{\varepsilon}} \widehat{H}_j(t, 0, \xi) \right) \\
&= \frac{\varepsilon}{i\alpha_j} \left( \int_0^t e^{i\alpha_j|\xi|\frac{t-s}{\varepsilon}} \beta_j |\xi|^2 e^{-\beta_j(t-s)|\xi|^2} P_j^1\left(\frac{\xi}{|\xi|}\right) \widehat{\Gamma}(g(s), g(s))(\xi) ds \right) \\
&\quad + \frac{\varepsilon}{i\alpha_j} \left( \int_0^t e^{i\alpha_j|\xi|\frac{t-s}{\varepsilon}} e^{-\beta_j(t-s)|\xi|^2} P_j^1\left(\frac{\xi}{|\xi|}\right) \partial_t \widehat{\Gamma}(g(s), g(s))(\xi) ds \right) \\
&\quad - \frac{\varepsilon}{i\alpha_j} P_j^1\left(\frac{\xi}{|\xi|}\right) \widehat{\Gamma}(g(t), g(t))(\xi) \\
&\quad + \frac{\varepsilon}{i\alpha_j} e^{i\alpha_j|\xi|\frac{t}{\varepsilon}} e^{-\beta_j t |\xi|^2} P_j^1\left(\frac{\xi}{|\xi|}\right) \widehat{\Gamma}(g(0), g(0))(\xi) \\
&=: I_1(t, \xi) + I_2(t, \xi) + I_3(t, \xi) + I_4(t, \xi).
\end{aligned}$$

For the first term in (6.15) we have for all  $t \geq 0$  and  $\xi \in \Omega'_\xi$ , using Lemma 6.4,

$$\begin{aligned}
\|I_1(t, \xi)\|_{L_v^2} &\lesssim \varepsilon \int_0^t \beta_j |\xi|^2 e^{-\beta_j(t-s)|\xi|^2} \|\widehat{\Gamma}(g(s), g(s))(\xi)\|_{L_v^2} ds \\
&\lesssim \varepsilon \|\widehat{\Gamma}(g, g)(\xi)\|_{L_t^\infty L_v^2}.
\end{aligned}$$

Similarly, for the third term in (6.15) there holds

$$\begin{aligned}
\|I_3(t, \xi)\|_{L_v^2} &\lesssim \varepsilon \|\widehat{\Gamma}(g(t), g(t))(\xi)\|_{L_v^2} \\
&\lesssim \varepsilon \|\widehat{\Gamma}(g, g)(\xi)\|_{L_t^\infty L_v^2}.
\end{aligned}$$

and for the fourth one

$$\begin{aligned}
\|I_4(t, \xi)\|_{L_v^2} &\lesssim \varepsilon e^{-\beta_j t |\xi|^2} \|\widehat{\Gamma}(g(0), g(0))(\xi)\|_{L_v^2} \\
&\lesssim \varepsilon \|\widehat{\Gamma}(g, g)(\xi)\|_{L_t^\infty L_v^2}.
\end{aligned}$$

This yields

$$(6.16) \quad \|I_1\|_{L_\xi^1 L_t^\infty L_v^2} + \|I_3\|_{L_\xi^1 L_t^\infty L_v^2} + \|I_4\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \varepsilon \|\widehat{\Gamma}(g, g)\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \varepsilon \|\widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2}^2,$$

where we have used (6.13) in last line

For the second term in (6.15) we first write, for all  $t \geq 0$  and  $\xi \in \Omega'_\xi$ ,

$$\|I_2(t, \xi)\|_{L_v^2} \lesssim \varepsilon \int_0^t e^{-\beta_j(t-s)|\xi|^2} \|\partial_t \widehat{\Gamma}(g(s), g(s))(\xi)\|_{L_v^2} ds.$$

Since  $\partial_t \widehat{\Gamma}(g, g) = \widehat{\Gamma}(\partial_t g, g) + \widehat{\Gamma}(g, \partial_t g)$ , from (6.12) we get

$$\|\partial_t \widehat{\Gamma}(g(s), g(s))(\xi)\|_{L_v^2} \lesssim \int_{\Omega'_\eta} \|\widehat{g}(s, \xi - \eta)\|_{L_v^2} \|\partial_t g(s, \eta)\|_{L_v^2} d\eta.$$

As  $g$  is defined through  $(u, \theta, \rho)$  which satisfies the Navier-Stokes-Fourier system (1.14), we have for all  $s \geq 0$  and all  $\eta \in \Omega'_\eta$

$$\|\partial_t \widehat{g}(s, \eta)\|_{L_v^2} \lesssim |\eta|^2 \|\widehat{g}(s, \eta)\|_{L_v^2} + |\eta| \int_{\Omega'_\zeta} \|\widehat{g}(s, \eta - \zeta)\|_{L_v^2} \|\widehat{g}(s, \zeta)\|_{L_v^2} d\zeta.$$

This implies

$$\begin{aligned}
\|I_2(t, \xi)\|_{L_v^2} &\lesssim \varepsilon \int_0^t e^{-\beta_j(t-s)|\xi|^2} \int_{\Omega'_\eta} \|\widehat{g}(s, \xi - \eta)\|_{L_v^2} |\eta|^2 \|\widehat{g}(s, \eta)\|_{L_v^2} d\eta ds \\
&\quad + \varepsilon \int_0^t e^{-\beta_j(t-s)|\xi|^2} \int_{\Omega'_\eta} \|\widehat{g}(s, \xi - \eta)\|_{L_v^2} |\eta| \int_{\Omega'_\zeta} \|\widehat{g}(s, \eta - \zeta)\|_{L_v^2} \|\widehat{g}(s, \zeta)\|_{L_v^2} d\zeta d\eta ds \\
&=: R_1(t, \xi) + R_2(t, \xi).
\end{aligned}$$

For the term  $R_1$  we split the integral on  $\eta$  into two parts: the region  $2|\xi| > |\eta|$  in which we have  $|\eta|^2 \leq 4|\xi|^2$ ; and the region  $2|\xi| \leq |\eta|$  where we have  $|\eta - \xi| \sim |\eta|$ , which yields

$$\begin{aligned} R_1(t, \xi) &\lesssim \varepsilon \int_0^t e^{-\beta_j(t-s)|\xi|^2} \int_{\Omega'_\eta} \mathbf{1}_{|\eta| < 2|\xi|} \|\widehat{g}(s, \xi - \eta)\|_{L_v^2} |\eta|^2 \|\widehat{g}(s, \eta)\|_{L_v^2} d\eta ds \\ &\quad + \varepsilon \int_0^t e^{-\beta_j(t-s)|\xi|^2} \int_{\Omega'_\eta} \mathbf{1}_{|\eta| \geq 2|\xi|} \|\widehat{g}(s, \xi - \eta)\|_{L_v^2} |\eta|^2 \|\widehat{g}(s, \eta)\|_{L_v^2} d\eta ds \\ &\lesssim \varepsilon \int_0^t |\xi|^2 e^{-\beta_j(t-s)|\xi|^2} \int_{\Omega'_\eta} \|\widehat{g}(s, \xi - \eta)\|_{L_v^2} \|\widehat{g}(s, \eta)\|_{L_v^2} d\eta ds \\ &\quad + \varepsilon \int_0^t e^{-\beta_j(t-s)|\xi|^2} \int_{\Omega'_\eta} |\xi - \eta| \|\widehat{g}(s, \xi - \eta)\|_{L_v^2} |\eta| \|\widehat{g}(s, \eta)\|_{L_v^2} d\eta ds. \end{aligned}$$

Thanks to Hölder's inequality in the time variable, it follows

$$\begin{aligned} \|R_1(\xi)\|_{L_t^\infty} &\lesssim \varepsilon \int_{\Omega'_\eta} \|\widehat{g}(\xi - \eta)\|_{L_t^\infty L_v^2} \|\widehat{g}(\eta)\|_{L_t^\infty L_v^2} d\eta \\ &\quad + \varepsilon \int_{\Omega'_\eta} |\xi - \eta| \|\widehat{g}(\xi - \eta)\|_{L_t^2 L_v^2} |\eta| \|\widehat{g}(\eta)\|_{L_t^2 L_v^2} d\eta, \end{aligned}$$

therefore taking the  $L_\xi^1$  norm and using Young's convolution inequality we obtain

$$(6.17) \quad \|R_1\|_{L_\xi^1 L_t^\infty} \lesssim \varepsilon \|\widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2}^2 + \varepsilon \|\xi\widehat{g}\|_{L_\xi^1 L_t^2 L_v^2}^2.$$

For the term  $R_2$  we write

$$\begin{aligned} \|R_2(\xi)\|_{L_t^\infty} &\lesssim \varepsilon \sup_{t \geq 0} \int_0^t |\xi| e^{-\beta_j(t-s)|\xi|^2} |\xi|^{-1} \int_{\Omega'_\eta} \|\widehat{g}(s, \xi - \eta)\|_{L_v^2} |\eta| \int_{\Omega'_\zeta} \|\widehat{g}(s, \eta - \zeta)\|_{L_v^2} \|\widehat{g}(s, \zeta)\|_{L_v^2} d\zeta d\eta ds \\ &\lesssim \varepsilon \sup_{t \geq 0} \left( \int_0^t |\xi|^2 e^{-\beta_j(t-s)|\xi|^2} ds \right)^{1/2} |\xi|^{-1} \left( \int_0^\infty G(s, \xi)^2 ds \right)^{1/2}, \end{aligned}$$

where we denote

$$G(s, \xi) = \int_{\Omega'_\eta} \|\widehat{g}(s, \xi - \eta)\|_{L_v^2} H(s, \eta) d\eta, \quad H(s, \eta) = |\eta| \int_{\Omega'_\zeta} \|\widehat{g}(s, \eta - \zeta)\|_{L_v^2} \|\widehat{g}(s, \zeta)\|_{L_v^2} d\zeta.$$

By Minkowski and Hölder inequalities

$$\begin{aligned} \|G(\xi)\|_{L_t^2} &\lesssim \int_{\Omega'_\eta} \left( \int_0^\infty \|\widehat{g}(s, \xi - \eta)\|_{L_v^2}^2 |H(s, \eta)|^2 ds \right)^{1/2} d\eta \\ &\lesssim \int_{\Omega'_\eta} \|\widehat{g}(\xi - \eta)\|_{L_t^\infty L_v^2} \|H(\eta)\|_{L_t^2} d\eta. \end{aligned}$$

Moreover

$$H(s, \eta) \lesssim \int_{\Omega'_\zeta} |\eta - \zeta| \|\widehat{g}(s, \eta - \zeta)\|_{L_v^2} \|\widehat{g}(s, \zeta)\|_{L_v^2} d\zeta + \int_{\Omega'_\zeta} \|\widehat{g}(s, \eta - \zeta)\|_{L_v^2} |\zeta| \|\widehat{g}(s, \zeta)\|_{L_v^2} d\zeta.$$

Thus again by Minkowski and Hölder inequalities,

$$\begin{aligned} \|H(\eta)\|_{L_t^2} &\lesssim \int_{\Omega'_\zeta} \left( \int_0^\infty \|\eta - \zeta\widehat{g}(s, \eta - \zeta)\|_{L_v^2}^2 \|\widehat{g}(s, \zeta)\|_{L_v^2}^2 ds \right)^{1/2} d\zeta \\ &\quad + \int_{\Omega'_\zeta} \left( \int_0^\infty \|\widehat{g}(s, \eta - \zeta)\|_{L_v^2}^2 \|\zeta\widehat{g}(s, \zeta)\|_{L_v^2}^2 ds \right)^{1/2} d\zeta \\ &\lesssim \int_{\Omega'_\zeta} \|\widehat{g}(\eta - \zeta)\|_{L_t^\infty L_v^2} \|\zeta\widehat{g}(s, \zeta)\|_{L_t^2 L_v^2} d\zeta. \end{aligned}$$

Hence we get

$$\|R_2(\xi)\|_{L_t^\infty} \lesssim \varepsilon |\xi|^{-1} \int_{\Omega'_\eta} \int_{\Omega'_\zeta} \|\widehat{g}(\xi - \eta)\|_{L_t^\infty L_v^2} \|\widehat{g}(\eta - \zeta)\|_{L_t^\infty L_v^2} \|\zeta\widehat{g}(\zeta)\|_{L_t^2 L_v^2} d\zeta d\eta.$$



Taking the  $L_\xi^1$  norm and distinguishing between high and low frequencies yields

$$(6.18) \quad \|\mathbf{1}_{|\xi| \geq 1} R_2\|_{L_\xi^1 L_t^\infty} \lesssim \varepsilon \|\widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2}^2 \|\xi|\widehat{g}\|_{L_\xi^1 L_t^2 L_v^2},$$

and, in the whole space case  $\Omega_x = \mathbf{R}^3$  and  $\Omega'_\xi = \mathbf{R}^3$ ,

$$(6.19)$$

$$\begin{aligned} \|\mathbf{1}_{|\xi| < 1} R_2\|_{L_\xi^1 L_t^\infty} &\lesssim \varepsilon \|\mathbf{1}_{|\xi| < 1} |\xi|^{-1}\|_{L_\xi^{p'}} \left\| \int_{\Omega'_\eta} \int_{\Omega'_\zeta} \|\widehat{g}(\xi - \eta)\|_{L_t^\infty L_v^2} \|\widehat{g}(\eta - \zeta)\|_{L_t^\infty L_v^2} \|\zeta|\widehat{g}(\zeta)\|_{L_t^2 L_v^2} d\zeta d\eta \right\|_{L_\xi^p} \\ &\lesssim \varepsilon \|\widehat{g}\|_{L_\xi^p L_t^\infty L_v^2} \|\widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2} \|\xi|\widehat{g}\|_{L_\xi^1 L_t^2 L_v^2}, \end{aligned}$$

where we have used that  $\mathbf{1}_{|\xi| < 1} |\xi|^{-1} \in L_\xi^{p'}$  since  $p > 3/2$ .

*Step 3.* It only remains to compute the term  $\widehat{\Psi}^{\varepsilon\#}$ , for which we first write, for all  $t \geq 0$  and  $\xi \in \Omega'_\xi$ ,

$$\|\widehat{\Psi}^{\varepsilon\#}[g, g](t, \xi)\|_{L_v^2} \lesssim \frac{1}{\varepsilon} \int_0^t \|\widehat{U}^{\varepsilon\#}(t-s, \xi) \widehat{\Gamma}(g(s), g(s))(\xi)\|_{L_v^2} ds.$$

In the hard potentials case  $\gamma + 2s \geq 0$ , thanks to (6.3) we have, for all  $t \geq 0$  and  $\xi \in \Omega'_\xi$ ,

$$\begin{aligned} \|\widehat{\Psi}^{\varepsilon\#}[g, g](t, \xi)\|_{L_v^2} &\lesssim \frac{1}{\varepsilon} \int_0^t e^{-\lambda \frac{(t-s)}{\varepsilon^2}} \|\widehat{\Gamma}(g(s), g(s))(\xi)\|_{L_v^2} ds \\ &\lesssim \frac{1}{\varepsilon} \|\widehat{\Gamma}(g, g)(\xi)\|_{L_t^\infty L_v^2} \int_0^t e^{-\lambda \frac{(t-s)}{\varepsilon^2}} ds \\ &\lesssim \varepsilon \|\widehat{\Gamma}(g, g)(\xi)\|_{L_t^\infty L_v^2}. \end{aligned}$$

For the soft potentials case  $\gamma + 2s < 0$ , observing that  $\mathbf{P}\widehat{\Gamma}(g, g) = 0$  we fix  $\ell > 0$  such that  $\frac{\ell}{|\gamma+2s|} > 1$  then we use (6.3) to obtain, for all  $t \geq 0$  and  $\xi \in \Omega'_\xi$ ,

$$\begin{aligned} \|\widehat{\Psi}^{\varepsilon\#}[g, g](t, \xi)\|_{L_v^2} &\lesssim \frac{1}{\varepsilon} \int_0^t \left(1 + \frac{(t-s)}{\varepsilon^2}\right)^{-\frac{\ell}{|\gamma+2s|}} \|\widehat{\Gamma}(g(s), g(s))(\xi)\|_{L_v^{2((v)^\ell)}} ds \\ &\lesssim \frac{1}{\varepsilon} \|\widehat{\Gamma}(g, g)(\xi)\|_{L_t^\infty L_v^{2((v)^\ell)}} \int_0^t \left(1 + \frac{(t-s)}{\varepsilon^2}\right)^{-\frac{\ell}{|\gamma+2s|}} ds \\ &\lesssim \varepsilon \|\widehat{\Gamma}(g, g)(\xi)\|_{L_t^\infty L_v^{2((v)^\ell)}}. \end{aligned}$$

Taking the  $L_\xi^1 L_t^\infty$  norm in above estimates and using (6.13) yields, for both hard potentials and soft potentials cases,

$$(6.20) \quad \|\widehat{\Psi}^{\varepsilon\#}[g, g]\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \varepsilon \|\widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2}^2.$$

*Step 4: Conclusion.* We conclude the proof by gathering estimates (6.14), (6.16), (6.17), (6.18), (6.19), and (6.20) together with the bounds for  $g$  from Theorem 2.2.  $\square$

**6.3. Proof of Theorem 2.3.** Let  $f^\varepsilon$ , for any  $\varepsilon \in (0, 1]$ , be the unique global mild solution to (1.8) associated to the initial data  $f_0^\varepsilon$  constructed in Theorem 2.1.

Let  $g = \mathbf{P}g$  be the kinetic distribution defined by (2.16) through the unique global mild solution  $(\rho, u, \theta)$  to (1.14) associated to the initial data  $(\rho_0, u_0, \theta_0)$  constructed in Theorem 2.2, and denote also  $g_0 = \mathbf{P}g_0$  the initial kinetic distribution defined by (2.15) through the initial data  $(\rho_0, u_0, \theta_0)$ .

We now from [13, 39] for instance, that  $g$  verifies the equation

$$(6.21) \quad g(t) = U(t)g_0 + \Psi[g, g](t),$$

where we recall that  $U(t)$  is defined in (6.7), and  $\Psi(t)$  in (6.11). Taking the Fourier transform in  $x \in \Omega_x$ , we then have

$$(6.22) \quad \widehat{g}(t, \xi) = \widehat{U}(t, \xi) \widehat{g}_0(\xi) + \widehat{\Psi}[g, g](t, \xi).$$

for all  $\xi \in \Omega'_\xi$ , and where we recall that  $\widehat{U}$  is defined in (6.6), and  $\widehat{\Psi}$  in (6.10).

We first observe that the difference  $f^\varepsilon - g$  satisfies

$$\begin{aligned}
\widehat{f}^\varepsilon(\xi) - \widehat{g}(\xi) &= \widehat{U}^\varepsilon(t, \xi) \widehat{f}_0^\varepsilon(\xi) - \widehat{U}(t, \xi) \widehat{g}_0(\xi) + \widehat{\Psi}^\varepsilon[f^\varepsilon, f^\varepsilon](t, \xi) - \widehat{\Psi}[g, g](t, \xi) \\
&= \widehat{U}^\varepsilon(t, \xi) \left\{ \widehat{f}_0^\varepsilon(\xi) - \widehat{g}_0(\xi) \right\} + \left\{ \widehat{U}^\varepsilon(t, \xi) - \widehat{U}(t, \xi) \right\} \widehat{g}_0(\xi) \\
(6.23) \quad &+ \left\{ \widehat{\Psi}^\varepsilon[g, g](t, \xi) - \widehat{\Psi}[g, g](t, \xi) \right\} + \left\{ \widehat{\Psi}^\varepsilon[f^\varepsilon, f^\varepsilon](t, \xi) - \widehat{\Psi}^\varepsilon[g, g](t, \xi) \right\} \\
&=: T_1 + T_2 + T_3 + T_4,
\end{aligned}$$

and we estimate each one of these terms separately.

For the first term, from Lemma 6.1 we have

$$\|\widehat{U}^\varepsilon(\cdot) \{ \widehat{f}_0^\varepsilon - \widehat{g}_0 \}\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \|\widehat{f}_0^\varepsilon - \widehat{g}_0\|_{L_\xi^1 L_v^2}.$$

Thanks to Lemma 6.3 and an interpolation argument, we obtain for the second term, for any  $\delta \in [0, 1]$ ,

$$\|\{\widehat{U}^\varepsilon(\cdot) - \widehat{U}(\cdot)\} \widehat{g}_0\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \varepsilon^\delta \|\langle \xi \rangle^\delta \widehat{g}_0\|_{L_\xi^1 L_v^2}.$$

For the third term we use Lemma 6.5, which yields

$$\|\widehat{\Psi}^\varepsilon[g, g] - \widehat{\Psi}[g, g]\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \varepsilon \left( \|\widehat{g}_0\|_{L_\xi^1 L_v^2}^2 + \|\widehat{g}_0\|_{L_\xi^1 L_v^2}^3 \right),$$

in the case  $\Omega_x = \mathbf{T}^3$ , and

$$\|\widehat{\Psi}^\varepsilon[g, g] - \widehat{\Psi}[g, g]\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \varepsilon \left( \|\widehat{g}_0\|_{L_\xi^1 L_v^2}^2 + \|\widehat{g}_0\|_{L_\xi^1 L_v^2}^3 + \|\widehat{g}_0\|_{L_\xi^2 L_v^2}^2 + \|\widehat{g}_0\|_{L_\xi^2 L_v^2}^3 \right),$$

in the case  $\Omega_x = \mathbf{R}^3$ .

For the fourth term  $T_4$ , we first decompose  $f^\varepsilon = \mathbf{P}^\perp f^\varepsilon + \mathbf{P} f^\varepsilon$  and use that  $g = \mathbf{P} g$  to write

$$\begin{aligned}
T_4 &= \widehat{\Psi}^\varepsilon[f^\varepsilon, f^\varepsilon](t, \xi) - \widehat{\Psi}^\varepsilon[g, g](t, \xi) \\
&= \widehat{\Psi}^\varepsilon[\mathbf{P}^\perp f^\varepsilon, \mathbf{P}^\perp f^\varepsilon](t, \xi) + \widehat{\Psi}^\varepsilon[\mathbf{P} f^\varepsilon, \mathbf{P} f^\varepsilon](t, \xi) + \widehat{\Psi}^\varepsilon[\mathbf{P}^\perp f^\varepsilon, \mathbf{P} f^\varepsilon](t, \xi) \\
&\quad + \widehat{\Psi}^\varepsilon[\mathbf{P} f^\varepsilon, \mathbf{P} f^\varepsilon](t, \xi) - \widehat{\Psi}^\varepsilon[g, g](t, \xi) \\
&= \widehat{\Psi}^\varepsilon[\mathbf{P}^\perp f^\varepsilon, \mathbf{P}^\perp f^\varepsilon](t, \xi) + \widehat{\Psi}^\varepsilon[\mathbf{P} f^\varepsilon, \mathbf{P} f^\varepsilon](t, \xi) + \widehat{\Psi}^\varepsilon[\mathbf{P}^\perp f^\varepsilon, \mathbf{P} f^\varepsilon](t, \xi) \\
&\quad + \widehat{\Psi}^\varepsilon[\mathbf{P}(f^\varepsilon - g), \mathbf{P} f^\varepsilon](t, \xi) + \widehat{\Psi}^\varepsilon[\mathbf{P} g, \mathbf{P}(f^\varepsilon - g)](t, \xi).
\end{aligned}$$

Thanks to Proposition 3.4 and Lemma 4.1 we have

$$\begin{aligned}
\|\widehat{\Psi}^\varepsilon[\mathbf{P}^\perp f^\varepsilon, \mathbf{P}^\perp f^\varepsilon]\|_{L_\xi^1 L_t^\infty L_v^2} &\lesssim \|\widehat{\Gamma}[\mathbf{P}^\perp f^\varepsilon, \mathbf{P}^\perp f^\varepsilon]\|_{L_\xi^1 L_t^2(H_v^{s,*})'} \\
&\lesssim \|\mathbf{P}^\perp \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2} \|\mathbf{P}^\perp \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}},
\end{aligned}$$

moreover

$$\begin{aligned}
\|\widehat{\Psi}^\varepsilon[\mathbf{P} f^\varepsilon, \mathbf{P} f^\varepsilon]\|_{L_\xi^1 L_t^\infty L_v^2} &\lesssim \|\widehat{\Gamma}[\mathbf{P} f^\varepsilon, \mathbf{P} f^\varepsilon]\|_{L_\xi^1 L_t^2(H_v^{s,*})'} \\
&\lesssim \|\mathbf{P} \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2} \|\mathbf{P} \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}},
\end{aligned}$$

and also

$$\begin{aligned}
\|\widehat{\Psi}^\varepsilon[\mathbf{P}^\perp f^\varepsilon, \mathbf{P} f^\varepsilon]\|_{L_\xi^1 L_t^\infty L_v^2} &\lesssim \|\widehat{\Gamma}[\mathbf{P}^\perp f^\varepsilon, \mathbf{P} f^\varepsilon]\|_{L_\xi^1 L_t^2(H_v^{s,*})'} \\
&\lesssim \|\mathbf{P}^\perp \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}} \|\mathbf{P} \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2},
\end{aligned}$$

where we have used that  $\|\mathbf{P}\phi\|_{H_v^{s,*}} \lesssim \|\mathbf{P}\phi\|_{L_v^2}$  and  $\|\langle v \rangle^{(\gamma/2+s)-}\phi\|_{L_v^2} \lesssim \|\phi\|_{H_v^{s,*}}$ . This implies

$$\begin{aligned}
(6.24) \quad &\|\widehat{\Psi}^\varepsilon[\mathbf{P}^\perp f^\varepsilon, \mathbf{P}^\perp f^\varepsilon]\|_{L_\xi^1 L_t^\infty L_v^2} + \|\widehat{\Psi}^\varepsilon[\mathbf{P} f^\varepsilon, \mathbf{P} f^\varepsilon]\|_{L_\xi^1 L_t^\infty L_v^2} + \|\widehat{\Psi}^\varepsilon[\mathbf{P}^\perp f^\varepsilon, \mathbf{P} f^\varepsilon]\|_{L_\xi^1 L_t^\infty L_v^2} \\
&\lesssim \|\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^\infty L_v^2} \|\mathbf{P}^\perp \widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 H_v^{s,*}}.
\end{aligned}$$

Therefore, using the bounds of Theorem 2.1, we deduce from (6.24) that

$$\begin{aligned}
&\|\widehat{\Psi}^\varepsilon[\mathbf{P}^\perp f^\varepsilon, \mathbf{P}^\perp f^\varepsilon]\|_{L_\xi^1 L_t^\infty L_v^2} + \|\widehat{\Psi}^\varepsilon[\mathbf{P} f^\varepsilon, \mathbf{P} f^\varepsilon]\|_{L_\xi^1 L_t^\infty L_v^2} + \|\widehat{\Psi}^\varepsilon[\mathbf{P}^\perp f^\varepsilon, \mathbf{P} f^\varepsilon]\|_{L_\xi^1 L_t^\infty L_v^2} \\
&\lesssim \varepsilon \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2}^2,
\end{aligned}$$

in the case  $\Omega_x = \mathbf{T}^3$ , and

$$\begin{aligned} & \|\widehat{\Psi}^\varepsilon[\mathbf{P}^\perp f^\varepsilon, \mathbf{P}^\perp f^\varepsilon]\|_{L_\xi^1 L_t^\infty L_v^2} + \|\widehat{\Psi}^\varepsilon[\mathbf{P} f^\varepsilon, \mathbf{P}^\perp f^\varepsilon]\|_{L_\xi^1 L_t^\infty L_v^2} + \|\widehat{\Psi}^\varepsilon[\mathbf{P}^\perp f^\varepsilon, \mathbf{P} f^\varepsilon]\|_{L_\xi^1 L_t^\infty L_v^2} \\ & \lesssim \varepsilon \left( \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2}^2 + \|\widehat{f}_0^\varepsilon\|_{L_\xi^p L_v^2}^2 \right), \end{aligned}$$

in the case  $\Omega_x = \mathbf{R}^3$ .

Furthermore, from Proposition 3.4 and Lemma 4.1, and also using that  $\|\mathbf{P}\phi\|_{H_v^{s,*}} \lesssim \|\mathbf{P}\phi\|_{L_v^2}$ , we have

$$\begin{aligned} \|\widehat{\Psi}^\varepsilon[\mathbf{P}(f^\varepsilon - g), \mathbf{P} f^\varepsilon]\|_{L_\xi^1 L_t^\infty L_v^2} & \lesssim \|\widehat{\Gamma}(\mathbf{P}(f^\varepsilon - g), \mathbf{P} f^\varepsilon)\|_{L_\xi^1 L_t^2 (H_v^{s,*})'} \\ & \lesssim \|\mathbf{P}(\widehat{f}^\varepsilon - \widehat{g})\|_{L_\xi^1 L_t^\infty L_v^2} \|\mathbf{P}\widehat{f}^\varepsilon\|_{L_\xi^1 L_t^2 L_v^2}, \end{aligned}$$

also

$$\begin{aligned} \|\widehat{\Psi}^\varepsilon[\mathbf{P}g, \mathbf{P}(f^\varepsilon - g)]\|_{L_\xi^1 L_t^\infty L_v^2} & \lesssim \|\widehat{\Gamma}(\mathbf{P}g, \mathbf{P}(f^\varepsilon - g))\|_{L_\xi^1 L_t^2 (H_v^{s,*})'} \\ & \lesssim \|\mathbf{P}(\widehat{f}^\varepsilon - \widehat{g})\|_{L_\xi^1 L_t^\infty L_v^2} \|\mathbf{P}\widehat{g}\|_{L_\xi^1 L_t^2 L_v^2}. \end{aligned}$$

In the case of the torus  $\Omega_x = \mathbf{T}^3$ , we can use the bounds of Theorem 2.1–(1) and Theorem 2.2–(1) to obtain

$$\begin{aligned} & \|\widehat{\Psi}^\varepsilon[\mathbf{P}(f^\varepsilon - g), \mathbf{P} f^\varepsilon]\|_{L_\xi^1 L_t^\infty L_v^2} + \|\widehat{\Psi}^\varepsilon[\mathbf{P}g, \mathbf{P}(f^\varepsilon - g)]\|_{L_\xi^1 L_t^\infty L_v^2} \\ & \lesssim \left( \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2} + \|\widehat{g}_0\|_{L_\xi^1 L_v^2} \right) \|\widehat{f}^\varepsilon - \widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2} \\ & \lesssim \eta_2 \|\widehat{f}^\varepsilon - \widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2}. \end{aligned}$$

In the case of the whole space  $\Omega_x = \mathbf{R}^3$ , we first use (4.14) to write

$$\|\mathbf{P} f^\varepsilon\|_{L_\xi^1 L_t^2 L_v^2} \lesssim \left\| \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} f^\varepsilon \right\|_{L_\xi^1 L_t^2 L_v^2} + \left\| \frac{|\xi|}{\langle \xi \rangle} \mathbf{P} f^\varepsilon \right\|_{L_\xi^p L_t^2 L_v^2},$$

then we use the bounds of Theorem 2.1–(2) and Theorem 2.2–(2) to get

$$\begin{aligned} & \|\widehat{\Psi}^\varepsilon[\mathbf{P}(f^\varepsilon - g), \mathbf{P} f^\varepsilon]\|_{L_\xi^1 L_t^\infty L_v^2} + \|\widehat{\Psi}^\varepsilon[\mathbf{P}g, \mathbf{P}(f^\varepsilon - g)]\|_{L_\xi^1 L_t^\infty L_v^2} \\ & \lesssim \left( \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2} + \|\widehat{f}_0^\varepsilon\|_{L_\xi^p L_v^2} + \|\widehat{g}_0\|_{L_\xi^1 L_v^2} + \|\widehat{g}_0\|_{L_\xi^p L_v^2} \right) \|\widehat{f}^\varepsilon - \widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2} \\ & \lesssim \eta_2 \|\widehat{f}^\varepsilon - \widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2}. \end{aligned}$$

Gathering previous estimates and using that  $\eta_2 > 0$  is small enough, so that when taking the  $L_\xi^1 L_t^\infty L_v^2$  of (6.23) the fourth and fifth terms in the right-hand side of (6.23) can be absorbed by the left-hand side, we deduce

$$\begin{aligned} (6.25) \quad & \|\widehat{f}^\varepsilon - \widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \|\widehat{f}_0^\varepsilon - \widehat{g}_0\|_{L_\xi^1 L_v^2} + \varepsilon^\delta \|\langle \xi \rangle^\delta \widehat{g}_0\|_{L_\xi^1 L_v^2} \\ & + \varepsilon \left( \|\widehat{g}_0\|_{L_\xi^1 L_v^2}^2 + \|\widehat{g}_0\|_{L_\xi^1 L_v^2}^3 \right) + \varepsilon \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2}^2, \end{aligned}$$

in the case  $\Omega_x = \mathbf{T}^3$ , and

$$\begin{aligned} (6.26) \quad & \|\widehat{f}^\varepsilon - \widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \|\widehat{f}_0^\varepsilon - \widehat{g}_0\|_{L_\xi^1 L_v^2} + \varepsilon^\delta \|\langle \xi \rangle^\delta \widehat{g}_0\|_{L_\xi^1 L_v^2} \\ & + \varepsilon \left( \|\widehat{g}_0\|_{L_\xi^1 L_v^2}^2 + \|\widehat{g}_0\|_{L_\xi^1 L_v^2}^3 + \|\widehat{g}_0\|_{L_\xi^p L_v^2}^2 + \|\widehat{g}_0\|_{L_\xi^p L_v^2}^3 \right) \\ & + \varepsilon \left( \|\widehat{f}_0^\varepsilon\|_{L_\xi^1 L_v^2}^2 + \|\widehat{f}_0^\varepsilon\|_{L_\xi^p L_v^2}^2 \right), \end{aligned}$$

in the case  $\Omega_x = \mathbf{R}^3$ . From this estimate, we first conclude that

$$\lim_{\varepsilon \rightarrow 0} \|\widehat{f}^\varepsilon - \widehat{g}\|_{L_\xi^1 L_t^\infty L_v^2} = 0,$$

assuming moreover that  $\langle \xi \rangle^\delta \widehat{g}_0 \in L_\xi^1 L_v^2$  for some  $\delta \in (0, 1]$ . We can finally prove Theorem 2.3, where we only assume  $\widehat{g}_0 \in L_\xi^1 L_v^2$ , by using the previous convergence and arguing by density as in [24]. This completes the proof of Theorem 2.3.

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