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# ON THE EXPANSIVENESS OF COARSE MAPS BETWEEN BANACH SPACES AND GEOMETRY PRESERVATION

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ABSTRACT. We introduce a new notion of embeddability between Banach spaces. By studying the classical Mazur map, we show that it is strictly weaker than the notion of coarse embeddability. We use the techniques from metric cotype introduced by M. Mendel and A. Naor to prove results about cotype preservation and complete our study of embeddability between  $\ell_p$  spaces. We confront our notion with nonlinear invariants introduced by N. Kalton, which are defined in terms of concentration properties for Lipschitz maps defined on countably branching Hamming or interlaced graphs. Finally, we address the problem of the embeddability into  $\ell_\infty$ .

## 1. INTRODUCTION

This article deals with a new notion of nonlinear embeddability between Banach spaces and how their geometries are preserved under this new notion. More precisely, the notion considered herein will be large scale in nature and even weaker than the usual coarse embeddability. Before presenting it, we start by recalling the basics of coarse geometry. Given metric spaces  $(X, d)$  and  $(Y, \partial)$ , and a map  $f: X \rightarrow Y$ , one defines a modulus

$$\omega_f(t) = \sup\{\partial(f(x), f(z)) \mid d(x, z) \leq t\}, \quad \text{for } t \geq 0,$$

and call  $f$  *coarse* if  $\omega_f(t) < \infty$  for all  $t \geq 0$ . In words,  $f$  is coarse if it sends bounded sets to bounded sets in a uniform manner. Coarse maps are the usual morphisms considered in the study of the large scale geometry of metric spaces and, in particular, of Banach spaces. In order to deal with embeddings, one defines a modulus

$$\rho_f(t) = \inf\{\partial(f(x), f(z)) \mid d(x, z) \geq t\}, \quad \text{for } t \geq 0,$$

and call  $f$  *expanding* if  $\lim_{t \rightarrow \infty} \rho_f(t) = \infty$ . In words,  $f$  is expanding if it sends elements far apart to elements likewise uniformly. The map  $f$  is then called a *coarse embedding* if it is both coarse and expanding. Despite its seemingly weak definition, coarse embeddability is known to capture the geometry of Banach spaces in several remarkable ways; to cite a few, we mention the cotype preservation under coarse embeddability into Banach spaces with nontrivial type proved in the seminal paper of M. Mendel and A. Naor ([MN08, Theorem 1.11]) and the preservation of asymptotic- $c_0$ -ness+reflexivity proved by the second named author together with F. Baudier, P. Motakis, and Th. Schlumprecht ([BLS18, Theorem A]).

Functional analysts working in the nonlinear geometry of Banach spaces are interested in knowing the minimal requirements needed for maps between Banach

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spaces to still generate an interesting notion of embeddability; here the word “interesting” should be broadly interpreted as “it is strictly weaker than a previously studied notion of embeddability but still strong enough to impose geometric restrictions”. For instance, C. Rosenthal has started in [Ros17] with the program of weakening the notion of expansiveness of a coarse map  $f$  by properties such as  $f$  being *uncollapsed* in the sense that there are  $\Delta, \delta > 0$  such that

$$\|x - z\| \geq \Delta \text{ implies } \|f(x) - f(z)\| > \delta,$$

or  $f$  being *solvent*, meaning that there is an increasing sequence  $(R_n)_n$  in  $\mathbb{N}$  such that

$$\|x - z\| \in [R_n, R_n + n] \text{ implies } \|f(x) - f(z)\| > n.$$

Even maps  $f$  satisfying only that

$$\|x - z\| = \Delta \text{ implies } \|f(x) - f(z)\| > \delta$$

have already been studied; those are called *almost uncollapsed* (see [Bra17]). Inspired by a recent work by the two authors (see [BL23]), this paper initiates a yet new approach of weakening the expansiveness condition. Indeed, all the weakenings mentioned above are *not* localized: the positions of  $x$  and  $z$  in  $X$  do not matter, but only the distance  $\|x - z\|$ . However, in [BL23], the authors started the study of an equivalence between metric spaces called *asymptotic coarse equivalence* and this takes into account the asymptotic behavior of elements  $x$  in  $X$  as they approach infinity. In particular, those maps are not necessarily expanding anymore, but only satisfy expansiveness as  $x, z \rightarrow \infty$ . This motivates the main definition of these notes:

**Definition 1.1.** Let  $X$  and  $Y$  be Banach spaces and  $\alpha \in [0, 1]$ . A map  $f: X \rightarrow Y$  is called *expanding at rate  $\alpha$*  if for all  $L > 0$  there is a map  $\rho: [0, \infty) \rightarrow [0, \infty)$  with  $\lim_{t \rightarrow \infty} \rho(t) = \infty$  such that

$$\|x - z\| \geq L \max\{\|x\|^\alpha, \|z\|^\alpha\} + L \text{ implies } \|f(x) - f(z)\| \geq \rho(\|x - z\|).$$

In case  $\rho$  can always be chosen to be of the form  $\frac{t}{C} - C$  for some  $C > 0$ , we say that  $f$  is *linearly expanding at rate  $\alpha$* .

A few comments are in place here. Firstly, notice that a coarse map  $X \rightarrow Y$  is expanding if and only if it is expanding at rate 0. Also, we restrict ourselves to  $\alpha \leq 1$  since the condition of  $\|x - z\|$  being at least of the order of  $\max\{\|x\|^\alpha, \|z\|^\alpha\}$  will not happen (up to a bounded subset) if  $\alpha > 1$ . We say that a coarse map  $f$  has *nontrivial coarse expansion* if it is expanding at rate  $\alpha$  for some  $\alpha \in [0, 1]$ . Finally, we recall that if a coarse map  $f$  between Banach spaces satisfies  $\rho_f(t) \geq \frac{t}{C} - C$  for some  $C > 0$  and all  $t \geq 0$ , then  $f$  is called a *coarse Lipschitz embedding*; a stronger notion than coarse embedding. Hence, the notion of a coarse map  $f$  being linearly expanding at some rate should be seen as a weakening of  $f$  being a coarse Lipschitz embedding. We now describe the main findings of this paper.

**1.1. Mazur maps and metric cotype.** We first show that the existence of coarse maps which are expanding at rate  $\alpha$ , for  $\alpha \in (0, 1]$ , is strictly weaker than coarse embeddability. In fact, as it turned out, there are several well-studied maps which witness that: the Mazur maps. Recall, given  $p, q \in [1, \infty)$ , the *Mazur map*  $M_{p,q}: \ell_p \rightarrow \ell_q$  is the homogeneous extension of the canonical map which adjusts elements in the unit sphere of  $\ell_p$  so that they fall into the unit sphere of  $\ell_q$ ; for brevity, we postpone to Section 2 its formal definition. While  $\ell_p$  coarsely embeds

into  $\ell_q$  if and only if either  $p \in [1, 2]$  or  $p \leq q$  ([MN08, Corollary 7.3]), we show that  $M_{p,q}$  is coarse and has nontrivial expansion as long as  $p > q$ . Precisely:

**Theorem 1.2.** *Let  $1 \leq q < p$  and  $\alpha \in (\frac{p-q}{p}, 1]$ . Then  $M_{p,q}: \ell_p \rightarrow \ell_q$  is a coarse map, which is expanding at rate  $\alpha$ . Moreover,  $M_{p,q}$  is linearly expanding at rate 1.*

If  $p > 2$  and  $p > q$ , we know from the aforementioned result of M. Mendel and A. Naor that  $\ell_p$  does not coarsely embed into  $\ell_q$ ; in our terminology just introduced, this means that there is no coarse map  $\ell_p \rightarrow \ell_q$  which is expanding at rate 0. By Theorem 1.2, we are then left to understand what happens for  $\alpha$ 's in the interval  $(0, \frac{p-q}{p}]$ . We show, using techniques from metric cotype of [MN08], that there is no such map for all  $\alpha$ 's in  $(0, \frac{p-q}{p}]$ , when  $2 \leq q < p < \infty$ , and that there is no such map for all  $\alpha$ 's in  $(0, \frac{p-2}{p}]$ , when  $1 \leq q < 2 < p$  (see Corollary 3.4). We refer to Section 3 for all relevant definitions. Let us just say for this introduction that, for a Banach space  $X$ , we denote

$$q_X = \inf\{q \in [2, \infty] \mid X \text{ has cotype } q\}.$$

We also prove the following general result about cotype preservation, which generalizes [MN08, Theorem 1.11]:

**Theorem 1.3.** *Let  $X$  and  $Y$  be Banach spaces and suppose  $Y$  has nontrivial type. Let  $\alpha \in [0, 1]$  and suppose there is a coarse map  $X \rightarrow Y$  which is expanding at rate  $\alpha$ . Then  $q_X \leq \frac{q_Y}{1-\alpha}$ .*

We stress here that our results do more than generalizing [MN08, Theorem 1.11]. Indeed, Theorem 1.3 together with Theorem 1.2 give us optimal results on cotype preservation (see Corollary 3.4).

**1.2. Embeddings of metric graphs into Banach spaces.** After our study of cotype preservation and the embeddability of the  $\ell_p$ 's, we turn our attention to the embeddability of certain metric graphs into Banach spaces. Recall, given  $k \in \mathbb{N}$ , we let  $[\mathbb{N}]^k$  denote the set of all subsets of  $\mathbb{N}$  with  $k$  elements and, given  $\bar{n} \in [\mathbb{N}]^k$ , we write  $\bar{n} = (n_1, \dots, n_k)$  where  $n_1 < \dots < n_k$ . As initiated by N. Kalton (see [Kal07, Kal13b]), the study of the embeddability of the sequence  $([\mathbb{N}]^k)_k$  (endowed with appropriate metrics) is extremely useful when looking for coarse (or coarse Lipschitz) invariants of Banach spaces. For instance, given  $k \in \mathbb{N}$ , let  $d_{\mathbb{H}} = d_{\mathbb{H},k}$  denote the *Hamming metric* on  $[\mathbb{N}]^k$ , i.e.,

$$d_{\mathbb{H}}(\bar{n}, \bar{m}) = |\{i \in \{1, \dots, k\} \mid n_i \neq m_i\}|, \text{ for } \bar{n}, \bar{m} \in [\mathbb{N}]^k.$$

The following important concentration property was introduced in [KR08] and later formalized by A. Fovelle in [Fov22] in the format presented below:

**Definition 1.4.** Let  $p \in (1, \infty]$ . A Banach space  $X$  is said to have *Hamming full concentration property  $p$* , abbreviated  $HFC_p$ , if there is  $C \geq 1$  such that for all  $k \in \mathbb{N}$  and all 1-Lipschitz maps  $\phi: ([\mathbb{N}]^k, d_{\mathbb{H}}) \rightarrow X$ , there is an infinite  $\mathbb{M} \subseteq \mathbb{N}$  such that

$$\text{diam}(\phi([\mathbb{M}]^k)) \leq Ck^{1/p}$$

(here we use the convention  $1/\infty = 0$  if  $p = \infty$ ).

As shown in [KR08, Theorem 4.2], reflexive spaces with a  $p$ -asymptotically uniformly smooth renorming have the  $HFC_p$ ; see Example 4.4 for definitions. Moreover, as shown in [BLMS21, Theorems A and B], having  $HFC_{\infty}$  is equivalent to  $X$

being asymptotic- $c_0$  and reflexive. We postpone to Example 4.5 the formal definition of asymptotic- $c_0$ -ness, for now, we simply say that  $X$  has such property if copies of the finite dimensional subspaces of  $c_0$  can be found in the finite codimensional subspaces of  $X$  in a uniform manner.

It was known that for  $p \in (1, \infty)$ ,  $\text{HFC}_p$  is stable under coarse-Lipschitz embeddings and that  $\text{HFC}_\infty$  is even stable under coarse embeddings. We show that our weaker notions of embeddability are already enough for the  $\text{HFC}_p$  properties to be preserved in the following sense:

**Theorem 1.5.** *Let  $X$  and  $Y$  be Banach spaces and suppose  $Y$  has  $\text{HFC}_p$  for some  $p \in (1, \infty]$ .*

- (1) *Suppose  $p \in (1, \infty)$ . If there is a coarse map  $f: X \rightarrow Y$  which is linearly expanding at rate 1, then  $X$  must have  $\text{HFC}_p$ .*
- (2) *Suppose  $p = \infty$ . If there is a coarse map  $f: X \rightarrow Y$  which is expanding at rate 1, then  $X$  must have  $\text{HFC}_\infty$ .*

In particular, applying Theorem 1.5(2) together with the characterization of  $\text{HFC}_\infty$  mentioned above ([BLMS21, Theorems B]), yields immediately the following corollary.

**Corollary 1.6.** *If a Banach space  $X$  can be mapped by a coarse map which is also expanding at rate 1 into a reflexive Banach space which is asymptotic- $c_0$ , then  $X$  must be also reflexive and asymptotic- $c_0$ .*

We also study interlacing pairs in Hamming graphs in Subsection 4.2 and use this to obtain results about the embeddability of the James spaces  $J_p$  (Example 4.9); see Theorem 4.10 and Corollary 4.13 for details.

Another important metric we can endow each  $[\mathbb{N}]^k$  with is the *interlaced metric*: we set distinct elements  $\bar{n}, \bar{m} \in [\mathbb{N}]^k$  to be adjacent if either

$$n_1 \leq m_2 \leq \dots \leq n_k \leq m_k \quad \text{or} \quad m_1 \leq n_1 \leq \dots \leq m_k \leq n_k$$

and then we let  $d_{\mathbb{I}} = d_{\mathbb{I},k}$  be the shortest path metric on  $[\mathbb{N}]^k$  given by this graph structure. The study of those metric spaces was fundamental for N. Kalton to rule out the coarse embeddability of  $c_0$  into reflexive spaces and gave rise to the so-called *property  $\mathcal{Q}$ 's*:

**Definition 1.7.** Let  $p \in (1, \infty]$ . A Banach space  $X$  is said to have *property  $\mathcal{Q}_p$* , if there is  $C \geq 1$  such that for all  $k \in \mathbb{N}$  and all 1-Lipschitz maps  $\phi: ([\mathbb{N}]^k, d_{\mathbb{I}}) \rightarrow X$ , there is an infinite  $\mathbb{M} \subseteq \mathbb{N}$  such that

$$\text{diam}(\phi([\mathbb{M}]^k)) \leq Ck^{1/p}$$

(here we use the convention  $1/\infty = 0$  if  $p = \infty$ ). If  $p = \infty$ , we simply say  $X$  has *property  $\mathcal{Q}$* .

We prove that our weakenings of coarse embeddability are also strong enough to ensure the preservation of property  $\mathcal{Q}_p$ . Precisely, we prove the following:

**Theorem 1.8.** *Let  $X$  and  $Y$  be Banach spaces and suppose  $Y$  has property  $\mathcal{Q}_p$  for some  $p \in (1, \infty]$ .*

- (1) *Suppose  $p \in (1, \infty)$ . If there is a coarse map  $f: X \rightarrow Y$  which is linearly expanding at rate 1, then  $X$  must have property  $\mathcal{Q}_p$ .*
- (2) *Suppose  $p = \infty$ . If there is a coarse map  $f: X \rightarrow Y$  which is expanding at rate 1, then  $X$  must have property  $\mathcal{Q}_\infty$ , i.e., property  $\mathcal{Q}$ .*

In Section 5, we characterize Lipschitz embeddability into  $\ell_\infty$  in terms of our new notion of embeddability; this extends a result of N. Kalton (see [Kal11, Theorem 5.3]). Precisely, we prove the following.

**Theorem 1.9.** *The following are equivalent for a Banach space  $X$ :*

- (1)  $X$  Lipschitzly embeds into  $\ell_\infty$ .
- (2) There is a coarse map  $f: X \rightarrow \ell_\infty$  which is linearly expanding at rate  $\alpha$ , for some  $\alpha \in (0, 1)$ .

## 2. REVISITING THE MAZUR MAP

The purpose of this section is to prove some estimates for the classic Mazur map (see Lemma 2.2) and to deduce the proof of Theorem 1.2. Recall, the Mazur map between the unit spheres of two Lebesgue sequence spaces, say  $\ell_p$  and  $\ell_q$ , is the canonical map which adjusts  $p$ -summable sequences so that they are  $q$ -summable. Precisely, given a Banach space  $X$ ,  $B_X$  denotes its closed unit ball and  $\partial B_X$  its unit sphere. Then, given  $p, q \in [1, \infty)$ , the *Mazur map* is the map  $M_{p,q}: \partial B_{\ell_p} \rightarrow \partial B_{\ell_q}$  given by

$$M_{p,q}((x_n)_n) = (\text{sign}(x_n)|x_n|^{p/q})_n$$

for all  $(x_n)_n \in \partial B_{\ell_p}$ . It is evident that  $M_{p,q}$  is a bijection with inverse  $M_{q,p}$ . Moreover, it is well-known that this map is a uniform equivalence. Precisely, the following estimates hold (see [BL00, Theorem 9.1] for a proof and [Maz29] for its first appearance).

**Theorem 2.1** (Mazur Map). *Let  $p, q \in [1, \infty)$  with  $q < p$  and let  $M_{p,q}: \ell_p \rightarrow \ell_q$  denote the Mazur map. There is a constant  $C = C(p, q) > 0$  such that*

$$C\|x - y\|^{p/q} \leq \|M_{p,q}(x) - M_{p,q}(y)\| \leq \frac{p}{q}\|x - y\|$$

for all  $x, y \in \partial B_{\ell_p}$ .

As it is usually done, we extend the map  $M_{p,q}$  to the whole  $\ell_p$  by homogeneity; by an abuse of notation, we still denote this extension by  $M_{p,q}$ . Precisely, for each  $x \in \ell_p$ , we let

$$M_{p,q}(x) = \begin{cases} \|x\| M_{p,q}\left(\frac{x}{\|x\|}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

We emphasize some important properties of the Mazur map below:

- $M_{p,q}^{-1} = M_{q,p}$  for all  $p, q \in [1, \infty)$ ,
- $\|M_{p,q}(x)\| = \|x\|$  for all  $x \in \ell_p$ , and
- $M_{p,q}\left(\frac{x}{\|x\|}\right) = \frac{M_{p,q}(x)}{\|M_{p,q}(x)\|}$  for all  $x \in \ell_p \setminus \{0\}$ .

The following is the main technical result of this section.

**Lemma 2.2.** *Let  $p, q \in [1, \infty)$  with  $q < p$ .*

- (1) *For all  $x, y \in \ell_p$ , we have*

$$\|M_{p,q}(x) - M_{p,q}(y)\| \leq \left(\frac{2p}{q} + 1\right)\|x - y\|.$$

- (2) *For all  $\varepsilon > 0$  and all  $\alpha \in (0, 1]$ , there is  $L = L(p, q, \varepsilon, \alpha) > 0$  so that, for all  $x, y \in \ell_p$  with  $\|x - y\| \geq \varepsilon \max\{\|x\|^\alpha, \|y\|^\alpha\}$ , we have*

$$\frac{1}{L}\|x - y\|^{\frac{p}{q} - \frac{1}{\alpha}\left(\frac{p}{q} - 1\right)} \leq \|M_{p,q}(x) - M_{p,q}(y)\|.$$

Before proving Lemma 2.2, we isolate some simple estimates for future use (cf. [Kal13a, Lemma 3.1]).

**Lemma 2.3.** *Let  $X$  be a Banach space and  $x, y \in X$  with  $\|x\| \geq \|y\| > 0$ . The following holds*

$$(1) \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq 2 \frac{\|x-y\|}{\|x\|} \text{ and}$$

$$(2) \|x-y\| \leq \|x\| - \|y\| + \|y\| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|. \quad \square$$

*Proof of Lemma 2.2.* (1) Fix  $x, y \in \ell_p$ . Without loss of generality, suppose  $\|x\| \geq \|y\| > 0$ . Then Theorem 2.1 and Lemma 2.3 imply

$$\begin{aligned} \|M_{p,q}(x) - M_{p,q}(y)\| &= \left\| \|x\| M_{p,q}\left(\frac{x}{\|x\|}\right) - \|y\| M_{p,q}\left(\frac{y}{\|y\|}\right) \right\| \\ &\leq \|x\| \left\| M_{p,q}\left(\frac{x}{\|x\|}\right) - M_{p,q}\left(\frac{y}{\|y\|}\right) \right\| \\ &\quad + (\|x\| - \|y\|) \left\| M_{p,q}\left(\frac{y}{\|y\|}\right) \right\| \\ &\leq \left(\frac{2p}{q} + 1\right) \|x-y\|. \end{aligned}$$

(2) Fix  $\varepsilon > 0$ ,  $\alpha \in (0, 1]$ , and  $x, y \in \ell_p$  with  $\|x-y\| \geq \varepsilon \max\{\|x\|^\alpha, \|y\|^\alpha\}$ . It is easily checked that we may assume that  $\|x\| \geq \|y\| > 0$ .

Suppose first that  $\|x\| - \|y\| > \frac{\|x-y\|}{2}$ , then

$$\begin{aligned} \|M_{p,q}(x) - M_{p,q}(y)\| &\geq \|M_{p,q}(x)\| - \|M_{p,q}(y)\| \\ &= \|x\| - \|y\| \\ &\geq \frac{\|x-y\|}{2} \end{aligned}$$

(here we use that  $\|M_{p,q}(z)\| = \|z\|$  for all  $z \in \ell_p$ ).

Suppose now that  $0 \leq \|x\| - \|y\| \leq \frac{\|x-y\|}{2}$ . Then, by Lemma 2.3, we have

$$(2.1) \quad \|x-y\| \leq 2\|y\| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq 2\|x\| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

Using Theorem 2.1 and the fact that  $\|M_{p,q}(z)\| = \|z\|$  for all  $z \in \ell_p$  again, we have

$$\begin{aligned} \|x\| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| &\leq \frac{\|x\|}{C^{q/p}} \left\| M_{p,q}\left(\frac{x}{\|x\|}\right) - M_{p,q}\left(\frac{y}{\|y\|}\right) \right\|^{q/p} \\ &= \frac{\|x\|}{C^{q/p}} \left\| \frac{M_{p,q}(x)}{\|M_{p,q}(x)\|} - \frac{M_{p,q}(y)}{\|M_{p,q}(y)\|} \right\|^{q/p} \\ &\leq \left(\frac{2}{C}\right)^{q/p} \|x\|^{1-q/p} \|M_{p,q}(x) - M_{p,q}(y)\|^{q/p}. \end{aligned}$$

As  $q < p$  and  $\|x-y\| \geq \varepsilon \max\{\|x\|^\alpha, \|y\|^\alpha\} = \varepsilon \|x\|^\alpha$ , (2.1) and the inequality above imply that

$$\|x-y\| \leq 2 \left(\frac{2}{C}\right)^{q/p} \varepsilon^{\frac{q/p-1}{\alpha}} \|x-y\|^{\frac{1-q/p}{\alpha}} \|M_{p,q}(x) - M_{p,q}(y)\|^{q/p}.$$

Simplifying the above, we conclude that

$$\|x-y\|^{\frac{p}{q} - \frac{1}{\alpha} \left(\frac{p}{q} - 1\right)} \leq \frac{2^{1+p/q}}{C} \varepsilon^{\frac{1-p/q}{\alpha}} \|M_{p,q}(x) - M_{p,q}(y)\|.$$

The result then follows taking  $L$  to be the maximum of 2 and  $C^{-1}2^{1+p/q}\varepsilon^{(1-p/q)/\alpha}$ .  $\square$

*Proof of Theorem 1.2.* The first claim is immediate from Lemma 2.2 since  $\alpha$  being larger than  $\frac{p-q}{p}$  implies that  $\frac{p}{q} - \frac{1}{\alpha}(\frac{p}{q} - 1)$  is positive. The second claim follows immediately from Lemma 2.2 also since  $\frac{p}{q} - \frac{1}{\alpha}(\frac{p}{q} - 1) = 1$  if  $\alpha = 1$ .  $\square$

As a consequence of Theorem 1.2, we obtain that the existence of a coarse map between Banach spaces which is expanding at rate  $\alpha$ , for  $\alpha \in (0, 1]$ , is strictly weaker than coarse embeddability. Indeed, it contrasts with the well-known fact that  $\ell_p$  coarsely embeds into  $\ell_q$  if and only if either  $p \leq q$  or  $p, q \in [1, 2]$  (see [MN08, Corollary 7.3]).

### 3. EXPANSION AND COTYPE PRESERVATION

In Section 2, we showed that, for  $p, q \in [1, \infty)$  with  $q < p$ , there are coarse maps with nontrivial expanding properties from  $\ell_p$  to  $\ell_q$ ; which contrasts with the known results about the coarse embeddability of  $\ell_p$  into  $\ell_q$ . Precisely, Theorem 1.2 shows there are coarse maps  $\ell_p \rightarrow \ell_q$  which are expanding at rate  $\alpha$  as long as  $\alpha > \frac{p-q}{p}$ . Since we know that  $\ell_p$  coarsely embeds into  $\ell_q$  if and only if either  $p \leq q$  or  $p, q \in [1, 2]$  (see [MN08, Corollary 7.3]), this result is not true for  $\alpha = 0$ . The main goal of the current section is to understand what happens for  $\alpha$ 's in the interval  $(0, \frac{p-q}{p}]$ . Using techniques developed by M. Mendel and A. Naor in their seminal paper about metric cotype ([MN08]), we show that the results of Section 2 do not hold for  $2 \leq q < p < \infty$  and  $\alpha \leq \frac{p-q}{p}$ , nor for  $1 \leq q < 2 < p$  and  $\alpha \leq \frac{p-2}{p}$ .

We start this section recalling the necessary background on metric cotype. Given  $m \in \mathbb{N}$ ,  $\mathbb{Z}_m$  denotes the set of integers modulo  $m$ . Given  $n, m \in \mathbb{N}$ ,  $\mu = \mu_{m,n}$  denotes the normalized counting measure on  $\mathbb{Z}_m^n$  and  $\sigma = \sigma_n$  denotes the normalized counting measure on  $\{-1, 0, 1\}^n$ . For each  $j \in \{1, \dots, n\}$ ,  $e_j$  denotes the vector in  $\mathbb{Z}_m^n$  whose  $j$ -th coordinate is 1 and all others are 0.

**Definition 3.1** (Metric cotype). Let  $(X, d)$  be a metric space and  $q, \Gamma > 0$ . We say that  $X$  has *metric cotype  $q$  with constant  $\Gamma$*  if for all  $n \in \mathbb{N}$  there is an even  $m \in \mathbb{N}$  such that for all  $f: \mathbb{Z}_m^n \rightarrow X$  we have

$$(3.1) \quad \sum_{j=1}^n \int_{\mathbb{Z}_m^n} d\left(f\left(x + \frac{m}{2}e_j\right), f(x)\right)^q d\mu(x) \leq \Gamma^q m^q \int_{\{-1, 0, 1\}^n} \int_{\mathbb{Z}_m^n} d(f(x + \varepsilon), f(x))^q d\mu(x) d\sigma(\varepsilon).$$

Given  $n \in \mathbb{N}$  and  $\Gamma > 0$ , we let  $m_q(X, n, \Gamma)$  be the smallest even integer  $m$  such that (3.1) holds for all  $f: \mathbb{Z}_m^n \rightarrow X$ . If no such  $m$  exists, we set  $m_q(X, n, \Gamma) = \infty$ .

The next lemma is the main technical result of this section and is a refinement of [MN08, Lemma 7.1].

**Lemma 3.2.** *Let  $(X, d)$  be a metric space,  $n \in \mathbb{N}$ ,  $q, s, \Gamma > 0$ , and  $r \in (0, \infty]$ . Let  $\alpha \in [0, \frac{rs}{rs+1}]$ ,  $\rho: [0, \infty) \rightarrow [0, \infty)$ , and maps  $f_n: \ell_r^n(\mathbb{C}) \rightarrow X$ , for  $n \in \mathbb{N}$ , be such that*

$$\|x - z\| \geq \max\{\|x\|^\alpha, \|z\|^\alpha\} + 1 \text{ implies } \|f_n(x) - f_n(z)\| \geq \rho(\|x - z\|),$$



for all  $n \in \mathbb{N}$  and  $x, z \in \ell_r^n(\mathbb{C})$ . Then, we have that for all  $n \in \mathbb{N}$ :

$$n^{1/q} \rho(2n^s) \leq \Gamma \cdot m_q(X, n, \Gamma) \cdot \omega_{f_n} \left( \frac{2\pi n^{s+1/r}}{m_q(X, n, \Gamma)} \right)$$

(if  $r = \infty$ , we use the conventions  $1/\infty = 0$  and  $\infty/\infty = 1$ ).

*Proof.* To simplify notation, let  $m = m_q(X, n, \Gamma)$ . By a slight abuse of notation, we let  $e_1, \dots, e_n$  denote the standard basis of both  $\mathbb{Z}_m^n$  and  $\ell_r^n(\mathbb{C})$ . Define a map  $h: \mathbb{Z}_m^n \rightarrow \ell_r^n(\mathbb{C})$  by letting

$$h(x) = n^s \cdot \sum_{j=1}^n e^{\frac{2\pi i x_j}{m}} e_j$$

for all  $x = (x_j)_{j=1}^n \in \mathbb{Z}_m^n$ . Note first that, since  $t \mapsto e^{it}$  is 1-Lipschitz on  $\mathbb{R}$ ,

$$\|h(x + \varepsilon) - h(x)\|_r \leq n^s \left\| \left( \frac{2\pi |\varepsilon_j|}{m} \right)_{j=1}^n \right\|_r \leq \frac{2\pi n^{s+1/r}}{m},$$

for all  $\varepsilon = (\varepsilon_j)_{j=1}^n \in \{-1, 0, 1\}^n$  and all  $x = (x_j)_{j=1}^n \in \mathbb{Z}_m^n$ . Let  $g_n = f_n \circ h$ , then

$$\|g_n(x + \varepsilon) - g_n(x)\| \leq \omega_{f_n} \left( \frac{2\pi n^{s+1/r}}{m} \right)$$

for all  $\varepsilon = (\varepsilon_j)_{j=1}^n \in \{-1, 0, 1\}^n$  and all  $x = (x_j)_{j=1}^n \in \mathbb{Z}_m^n$ . Therefore, integrating with respect to  $x$  and  $\varepsilon$ , we have

$$(3.2) \quad \int_{\{-1, 0, 1\}^n} \int_{\mathbb{Z}_m^n} \|g_n(x + \varepsilon) - g_n(x)\|^q d\mu(x) d\sigma(\varepsilon) \leq \omega_{f_n} \left( \frac{2\pi n^{s+1/r}}{m} \right)^q.$$

We now use the hypothesis on  $f_n$ . As  $\alpha \leq \frac{rs}{rs+1}$ , we have that

$$2n^s = 2(n^{s+1/r})^{\frac{rs}{rs+1}} \geq (n^{s+1/r})^\alpha + 1,$$

for all  $n \in \mathbb{N}$ . Hence,

$$\left\| h\left(x + \frac{m}{2} e_j\right) - h(x) \right\|_r = 2n^s \geq (n^{s+1/r})^\alpha + 1,$$

for all  $x \in \mathbb{Z}_m^n$  and all  $j \in \{1, \dots, n\}$ . Therefore, as  $\|h(y)\|_r = n^{s+1/r}$  for all  $y \in \mathbb{Z}_m^n$ , it follows that

$$\left\| g_n\left(x + \frac{m}{2} e_j\right) - g_n(x) \right\| \geq \rho(2n^s)$$

for all  $x \in \mathbb{Z}_m^n$  and all  $j \in \{1, \dots, n\}$ . Hence,

$$(3.3) \quad \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\| g_n\left(x + \frac{m}{2} e_j\right) - g_n(x) \right\|^q d\mu(x) \geq n\rho(2n^s)^q.$$

By the definition of  $m = m_q(M, n, \Gamma)$ , (3.2) and (3.3) show that

$$n\rho(2n^s)^q \leq \Gamma^q m^q \omega_{f_n} \left( \frac{2\pi n^{s+1/r}}{m} \right)^q.$$

Taking the  $q$ -th root at both sides above finishes the proof.  $\square$

We now turn to our result about preservation of cotype by coarse maps satisfying some weak expanding conditions. For completeness, we quickly recall the notions of type and cotype. Let  $X$  be a Banach space and  $p \in (1, 2]$ . We say that  $X$  has

type  $p$  if there is  $C > 0$  such that

$$\frac{1}{2^n} \sum_{(\varepsilon_i)_{i=1}^n \in \{-1,1\}^n} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^p \leq C \sum_{i=1}^n \|x_i\|^p$$

for all  $n \in \mathbb{N}$  and all  $x_1, \dots, x_n \in X$ . If  $X$  does not have type  $p$  for any  $p \in (1, 2]$ ,  $X$  is said to have *trivial type*. If  $q \in [2, \infty)$ , we say that  $X$  has *cotype  $q$*  if there is  $C > 0$  such that

$$\frac{1}{2^n} \sum_{(\varepsilon_i)_{i=1}^n \in \{-1,1\}^n} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^q \geq C \sum_{i=1}^n \|x_i\|^q$$

for all  $n \in \mathbb{N}$  and all  $x_1, \dots, x_n \in X$ . We let

$$q_X = \inf\{q \in [2, \infty] \mid X \text{ has cotype } q\},$$

where  $q_X$  is taken to be infinity if  $X$  has no cotype in  $[2, \infty)$ .

Let  $p \in [1, \infty]$  and  $C \geq 1$ . We say that a Banach space  $X$  *contains the  $\ell_p^n$ 's  $C$ -uniformly* if for all  $n \in \mathbb{N}$ ,  $\ell_p^n$  linearly embeds into  $X$  with distortion at most  $C$  and that  $X$  *uniformly contains the  $\ell_p^n$ 's* if it contains the  $\ell_p^n$ 's  $C$ -uniformly, for some  $C \geq 1$ . It is known that if  $X$  uniformly contains the  $\ell_p^n$ 's, then it contains the  $\ell_p^n$ 's  $C$ -uniformly for all  $C > 1$ . We shall also use the following fundamental result of B. Maurey and G. Pisier [MP76]: a Banach space  $X$  uniformly contains the  $\ell_{q_X}^n$ 's. We refer the reader to [Mau03, Theorem 6] for a precise statement and a self contained proof of this result, as well as for a complete survey on these notions.

We are now ready to prove the following statement, which is slightly more precise than Theorem 1.3.

**Theorem 3.3.** *Let  $2 \leq q < p < \infty$ . Suppose that  $X$  is a Banach space uniformly containing the  $\ell_p^n$ 's and that  $Y$  has cotype  $q$  and nontrivial type and suppose also that  $0 \leq \alpha \leq \frac{p-q}{p}$ . Then there is no coarse map from  $X$  to  $Y$  that is expanding at rate  $\alpha$ .*

*Proof.* Fix a coarse map  $f: X \rightarrow Y$  which is expanding at rate  $\alpha$ . In particular, there exists  $\rho: [0, \infty) \rightarrow [0, \infty)$  increasing with  $\lim_{t \rightarrow \infty} \rho(t) = \infty$  for which

$$(3.4) \quad \|x - z\| \geq \frac{1}{2} \max\{\|x\|^\alpha, \|z\|^\alpha\} + 1 \quad \text{implies} \quad \|f(x) - f(z)\| \geq \rho(\|x - z\|).$$

Suppose towards a contradiction that  $0 < \alpha \leq \frac{p-q}{p}$ , that is,  $p \geq \frac{q}{1-\alpha}$ . Treating  $\ell_p^n(\mathbb{C})$  as a real Banach space, we have, by assumption, that for each  $n \in \mathbb{N}$ , there exists an isomorphic embedding  $g_n: \ell_p^n(\mathbb{C}) \rightarrow X$  with distortion at most 2. Without loss of generality, assume

$$(3.5) \quad \|x\| \leq \|g_n(x)\| \leq 2\|x\|$$

for all  $n \in \mathbb{N}$  and all  $x \in \ell_p^n(\mathbb{C})$ . For each  $n \in \mathbb{N}$ , let  $h_n = f \circ g_n$ . It easily follows from (3.4) and (3.5) that for  $x, z \in \ell_p^n(\mathbb{C})$ ,

$$\|x - z\| \geq \max\{\|x\|^\alpha, \|z\|^\alpha\} + 1 \quad \text{implies} \quad \|h_n(x) - h_n(z)\| \geq \rho(\|x - z\|).$$

Note also that

$$(3.6) \quad \omega_{h_n}(t) \leq \omega_f(2t) \quad \text{for all } n \in \mathbb{N} \text{ and all } t \geq 0.$$

It follows from [MN08, Theorem 4.1] that there exists  $\Gamma > 0$  such that  $m_q(Y, n, \Gamma) = O(n^{1/q})$ , i.e., there exists  $A > 0$  such that  $m_q(Y, n, \Gamma) \leq An^{1/q}$  for all  $n \in \mathbb{N}$  (here

is there the hypothesis of  $Y$  having nontrivial type in used). On the other hand, by [MN08, Lemma 2.3], we have that  $m_q(Y, n, \Gamma) \geq \frac{n^{1/q}}{\Gamma}$ . We now apply Lemma 3.2 with  $q = q$ ,  $r = p$  and  $s = \frac{1}{q} - \frac{1}{p}$ . Then  $0 \leq \alpha \leq \frac{p-q}{p} = \frac{rs}{rs+1}$ . It then follows from Lemma 3.2 and (3.6) that

$$\rho\left(2n^{\frac{1}{q}-\frac{1}{p}}\right) \leq \Gamma A\omega_{h_n}\left(\frac{2\pi n^{1/q}}{m_q(Y, n, \Gamma)}\right) \leq \Gamma A\omega_f(4\pi\Gamma)$$

for all  $n \in \mathbb{N}$ . As  $\frac{1}{q} - \frac{1}{p} > 0$  and  $\lim_{t \rightarrow \infty} \rho(t) = \infty$ , we obtain the expected contradiction.  $\square$

**Corollary 3.4.** *Let  $1 \leq q < p < \infty$  and  $\alpha \in [0, 1]$ .*

- (1) *If  $q \geq 2$ , then there is a coarse map  $f: \ell_p \rightarrow \ell_q$  which is also expanding at rate  $\alpha$  if and only if  $\alpha > \frac{p-q}{p}$ .*
- (2) *If  $q \leq 2 < p$ , then there is a coarse map  $f: \ell_p \rightarrow \ell_q$  which is also expanding at rate  $\alpha$  if and only if  $\alpha > \frac{p-2}{p}$ .*

*Proof.* (1) Assume first that  $\alpha \in (\frac{p-q}{p}, 1]$ . The existence of a coarse map  $f: \ell_p \rightarrow \ell_q$  which is expanding at rate  $\alpha$  is ensured by Theorem 1.2.

Assume now that  $\alpha \in [0, \frac{p-q}{p}]$ . Since  $\ell_p$  obviously contains the  $\ell_p^n$ 's uniformly and  $\ell_q$  has cotype  $q$ , Theorem 3.3 implies that there is no coarse map from  $\ell_p$  to  $\ell_q$  that is expanding at rate  $\alpha$ .

(2) Since  $q \leq 2$ ,  $\ell_q$  has cotype 2. Hence, proceeding as in the previous item, we have that there is no coarse map  $f: \ell_p \rightarrow \ell_q$  which is also expanding at rate  $\alpha$  for  $\alpha \in [0, \frac{p-2}{p}]$ . If  $\alpha \geq \frac{p-2}{p}$ , the Mazur map  $M_{p,q}: \ell_p \rightarrow \ell_q$  is also expanding at rate  $\alpha$  (Theorem 1.2). Finally, if  $q \in (\frac{p-2}{p}, \frac{p-q}{p}]$ , the result follows since the Mazur map  $M_{p,2}: \ell_p \rightarrow \ell_2$  is expanding at rate  $\alpha$ . Therefore, if  $f: \ell_2 \rightarrow \ell_q$  is a coarse embedding (see [Now06, Theorem 5]), then  $f \circ M_{p,2}$  has the required property.  $\square$

*Proof of Theorem 1.3.* If  $\alpha = 0$ , this is [MN08, Theorem 1.11] and if  $\alpha = 1$ , this is obvious. Assume now that  $\alpha \in (0, 1)$ . We may also assume that  $q_Y < \infty$ . As we have recalled, by [MP76] (alternatively, see [Mau03, Theorem 6]),  $X$  uniformly contains the  $\ell_{q_X}^n$ 's. On the other hand, for any  $q > q_Y$ ,  $Y$  has cotype  $q$ . It then follows from Theorem 3.3 that  $\alpha > \frac{q_X - q}{q_X}$ . Taking the infimum over all  $q > q_Y$  yields the conclusion.  $\square$

#### 4. EMBEDDING OF CERTAIN GRAPHS AND EXPANSION

In this section we will show how to use some classical countably branching metric graphs to rule out the existence of coarse maps which are linearly expanding (or just expanding) at rate 1 or less between Banach spaces. Our statements will actually be about the stability of some concentration properties for Lipschitz maps on certain graphs under these generalizations of coarse Lipschitz (or just coarse) embeddings. We will deal with two different distances that are defined on the following sets.

**Definition 4.1.** Let  $k \in \mathbb{N}$ . We denote the set of all subsets of  $\mathbb{N}$  with  $k$  elements by  $[\mathbb{N}]^k$  and, given  $\bar{n} \in [\mathbb{N}]^k$ , we write  $\bar{n} = (n_1, \dots, n_k)$  where  $n_1 < \dots < n_k$ .

**4.1. Hamming graphs.** We start this section by recalling the definition of the Hamming metric and of the associated concentration property  $\text{HFC}_p$ .

**Definition 4.2.** Let  $k \in \mathbb{N}$ . The *Hamming metric*  $d_{\mathbb{H}} = d_{\mathbb{H},k}$  on  $[\mathbb{N}]^k$  is defined as follows: given  $\bar{n} = (n_1, \dots, n_k), \bar{m} = (m_1, \dots, m_k) \in [\mathbb{N}]^k$ , we let

$$d_{\mathbb{H}}(\bar{n}, \bar{m}) = |\{i \in \{1, \dots, k\} \mid n_i \neq m_i\}|.$$

Note that  $([\mathbb{N}]^k, d_{\mathbb{H}})$  is a metric graph in the sense that if two elements of  $[\mathbb{N}]^k$  are declared to be adjacent if and only if they are at distance 1, then the distance between arbitrary elements of  $[\mathbb{N}]^k$  is the length of the shortest path joining them. In particular, if  $X$  is a Banach space and  $\phi : ([\mathbb{N}]^k, d_{\mathbb{H}}) \rightarrow X$  is a Lipschitz map, then the Lipschitz constant of  $\phi$  is  $\text{Lip}(\phi) = \omega_{\phi}(1)$ . We now recall the following definition, due to A. Fovelle (see [Fov22, Subsection 2.5]).

**Definition 4.3** (Definition 1.4). Let  $p \in (1, \infty]$ . We say that a Banach space  $X$  has the *Hamming full concentration property for  $p$*  (abbreviated as  $\text{HFC}_p$ ) if there exists a constant  $C \geq 1$  such that for any  $k \in \mathbb{N}$  and any 1-Lipschitz map  $\phi : ([\mathbb{N}]^k, d_{\mathbb{H}}) \rightarrow X$ , there exists an infinite subset  $\mathbb{M}$  of  $\mathbb{N}$  such that

$$\text{diam}(\phi([\mathbb{N}]^k)) \leq Ck^{1/p}.$$

In the above situation, we say that  $X$  has the *HFC<sub>p</sub> with constant  $C$* .

**Example 4.4.** N. Kalton and L. Randrianarivony proved in [KR08, Theorem 4.2] that, given  $p \in (1, \infty)$ , any reflexive Banach space which is also  $p$ -asymptotically uniformly smooth (abbreviated as  $p$ -AUS) must have  $\text{HFC}_p$ . Although this will not play an important role in these notes, we recall the definition of  $p$ -AUSness here for the readers convenience: the *modulus of asymptotic uniform smoothness* of a Banach space  $X$  is given by

$$\bar{\rho}_X(t) = \sup_{x \in \partial B_X} \inf_{Y \in \text{cof}(X)} \sup_{y \in \partial B_Y} \|x + ty\| - 1,$$

where  $\text{cof}(X)$  denotes the set of all closed finite codimensional subspaces of  $X$ . Then, for  $p \in (1, \infty)$ ,  $X$  is called  $p$ -AUS if there is  $C \geq 1$  such that  $\bar{\rho}_X(t) \leq Ct^p$  for all  $t \geq 0$ . Clearly, for  $p \in (1, \infty)$ ,  $\ell_p$  is  $p$ -AUS, hence, the cited result above implies that  $\ell_p$  has  $\text{HFC}_p$ . But, if  $q < p$ ,  $\ell_q$  fails  $\text{HFC}_p$  as it is witnessed by the maps

$$\bar{n} = (n_1, \dots, n_k) \in [\mathbb{N}]^k \mapsto \sum_{i=1}^k e_i \in \ell_q, \quad k \in \mathbb{N},$$

where  $(e_i)_{i=1}^{\infty}$  is the canonical unit basis of  $\ell_q$ .

**Example 4.5.** A Banach space  $X$  is said to be *asymptotic- $c_0$*  if the following happens:

$$\exists C \geq 1, \forall n \in \mathbb{N}, \exists X_1 \in \text{cof}(X), \forall x_1 \in B_{X_1}, \dots, \exists X_n \in \text{cof}(X), \forall x_n \in B_{X_n}$$

$$\text{such that } \left\| \sum_{i=1}^n a_i x_i \right\| \leq C \max_{1 \leq i \leq n} |a_i| \text{ for all } (a_i)_{i=1}^n \in \mathbb{R}^{\mathbb{N}}.$$

The second named author together with F. Baudier, P. Motakis, and Th. Schlumprecht proved that the property of a Banach space having  $\text{HFC}_{\infty}$  is equivalent to it being reflexive and asymptotic- $c_0$  (see [BLMS21, Theorem B]).

*Proof of Theorem 1.5(1).* Let  $f : X \rightarrow Y$  be a coarse map which is linearly expanding at rate 1. We start by fixing some constants: As  $f$  is coarse,  $\omega_f(1) < \infty$  and it

follows that there is  $K > 1$  such that

$$\omega_f(t) \leq Kt + K, \text{ for all } t \geq 0$$

(this easy consequence of the triangle inequality and the metric convexity of Banach spaces can be found, for instance, in [Kal08, Lemma 1.4]). As  $f$  is linearly expanding at rate 1, there is  $L > 1$  such that

$$(4.1) \quad \|x - z\| \geq \frac{1}{8} \max\{\|x\|, \|z\|\} + 1 \text{ implies } \|f(x) - f(z)\| \geq \frac{1}{L}\|x - z\| - L.$$

Let  $C_Y \geq 1$  be a constant such that  $Y$  has  $\text{HFC}_p$  with constant  $C_Y$  and fix  $C > 6KLC_Y + 6L^2$ .

Assume towards a contradiction that  $X$  does not have  $\text{HFC}_p$ . Then there exist  $k \in \mathbb{N}$  and a 1-Lipschitz map  $\phi: ([\mathbb{N}]^k, d_{\mathbb{H}}) \rightarrow X$  such that

$$\text{diam}(\phi([\mathbb{M}]^k)) \geq Ck^{1/p}$$

for all infinite  $\mathbb{M} \subseteq \mathbb{N}$ . Let

$$(4.2) \quad \lambda = \inf\{k^{-1/p} \text{diam}(\phi([\mathbb{M}]^k)) \mid \mathbb{M} \subseteq \mathbb{N} \text{ and } |\mathbb{M}| = \infty\};$$

so,  $\lambda \geq C$  (in particular,  $\lambda > 12$ ). Pick now an infinite  $\mathbb{M} \subseteq \mathbb{N}$  such that

$$\lambda \leq \text{diam}(\phi([\mathbb{M}]^k))k^{-1/p} \leq 2\lambda.$$

In particular, there exists  $x \in X$  such that  $\phi([\mathbb{M}]^k)$  is included in the closed ball of radius  $2\lambda k^{1/p}$  centered at  $x$ . By replacing  $\phi$  by  $\phi - x$ , we may assume that  $\phi([\mathbb{M}]^k)$  is included in  $2\lambda k^{1/p}B_X$ .

By the definition of  $\lambda$ ,  $\text{diam}(\phi([\mathbb{D}]^k)) \geq \lambda k^{1/p}$  for all infinite  $\mathbb{D} \subseteq \mathbb{M}$ . So, for any such  $\mathbb{D}$ , we can find  $\bar{n}, \bar{m} \in [\mathbb{D}]^k$  with  $\bar{n} < \bar{m}$  such that  $\|\phi(\bar{n}) - \phi(\bar{m})\| \geq \frac{\lambda}{3}k^{1/p}$ . Indeed, for all  $\bar{n}, \bar{m} \in [\mathbb{D}]^k$ , we can find  $\bar{p} \in [\mathbb{D}]^k$  such that  $\bar{n} < \bar{p}$  and  $\bar{m} < \bar{p}$ . Therefore, if  $\|\phi(\bar{n}) - \phi(\bar{m})\|$  were smaller than  $\frac{\lambda}{3}k^{1/p}$  for all  $\bar{n} < \bar{m}$  in  $[\mathbb{D}]^k$ , the triangle inequality would imply that  $\text{diam}(\phi([\mathbb{D}]^k)) < \lambda k^{1/p}$ ; contradiction. Moreover, identifying a pair  $(\bar{n}, \bar{m})$ , where  $\bar{n}, \bar{m} \in [\mathbb{D}]^k$  and  $\bar{n} < \bar{m}$ , with an element of  $[\mathbb{M}]^{2k}$  and applying Ramsey's theorem, we may furthermore assume, by passing to an infinite subset of  $\mathbb{M}$  if necessary, that

$$(4.3) \quad \|\phi(\bar{n}) - \phi(\bar{m})\| \geq \frac{\lambda}{3}k^{1/p} \text{ for all } \bar{n}, \bar{m} \in [\mathbb{M}]^k \text{ with } \bar{n} < \bar{m}.$$

As  $\phi([\mathbb{M}]^k)$  is included in  $2\lambda k^{1/p}B_X$ , (4.3) and the fact that  $\lambda > 12$  imply that

$$\frac{1}{8} \max\{\|\phi(\bar{n})\|, \|\phi(\bar{m})\|\} + 1 \leq \frac{\lambda k^{1/p}}{4} + 1 \leq \frac{\lambda k^{1/p}}{3} \leq \|\phi(\bar{n}) - \phi(\bar{m})\|$$

for all  $\bar{n}, \bar{m} \in [\mathbb{M}]^k$  with  $\bar{n} < \bar{m}$ . Therefore, by (4.1), we must have

$$(4.4) \quad \|(f \circ \phi)(\bar{n}) - (f \circ \phi)(\bar{m})\| \geq \frac{\lambda k^{1/p}}{3L} - L \geq \frac{Ck^{1/p}}{3L} - L$$

for all  $\bar{n}, \bar{m} \in [\mathbb{M}]^k$  with  $\bar{n} < \bar{m}$ .

On the other hand, as  $\phi$  is 1-Lipschitz and  $[\mathbb{M}]^k$  is a metric graph, our choice of  $K$  implies that  $\omega_{f \circ \phi}(1) \leq 2K$  and therefore that  $f \circ \phi$  is  $2K$ -Lipschitz. As  $Y$  has  $\text{HFC}_p$  with constant  $C_Y$ , an homogeneity argument implies the existence of an infinite subset  $\mathbb{M}'$  of  $\mathbb{M}$  satisfying

$$(4.5) \quad \text{diam}(f(\phi([\mathbb{M}']^k))) \leq 2KC_Y k^{1/p}.$$

Therefore, (4.4) and (4.5) imply that

$$(4.6) \quad \frac{C}{3L} \leq 2KC_Y + L.$$

This contradicts our choice of  $C$ .  $\square$

Combining this result and Example 4.4, we obtain immediately the following.

**Corollary 4.6.** *Let  $p, q \in [1, \infty)$  and assume that  $p < q$ . Then there is no coarse map from  $\ell_p$  to  $\ell_q$  that is linearly expanding at rate 1.*

**Remark 4.7.** It is important to recall here that if  $p > q$ , then the Mazur map  $M_{p,q} : \ell_p \rightarrow \ell_q$  is coarse and linearly expanding at rate 1 (Theorem 1.2).

In case  $p = \infty$ , a stronger version of Theorem 1.5 holds. Precisely,  $\text{HFC}_\infty$  is preserved by coarse maps which are expanding at rate 1; no need for linear expansion here. Since the argument is completely analogous, we only indicate the mild differences in the proof below.

*Proof of Theorem 1.5(2).* The proof follows almost verbatim the one of Theorem 1.5(1). Precisely, let  $f : X \rightarrow Y$  be a coarse map which is expanding at rate 1 and  $K$  and  $C_Y$  be chosen as in the proof of Theorem 1.5(1). The expansion property of  $f$  gives us  $\rho : [0, \infty) \rightarrow [0, \infty)$  with  $\lim_{t \rightarrow \infty} \rho(t) = \infty$  and such that

$$\|x - z\| \geq \frac{1}{8} \max\{\|x\|, \|z\|\} + 1 \quad \text{implies} \quad \|f(x) - f(z)\| \geq \rho(\|x - z\|).$$

Since  $\lim_{t \rightarrow \infty} \rho(t) = \infty$ , we can choose  $C > 0$  such that  $\rho(C/3) > 2KC_Y$  and, assuming that  $X$  does not have  $\text{HFC}_\infty$ , we obtain  $k \in \mathbb{N}$  and a 1-Lipschitz map  $\phi : ([\mathbb{N}]^k, d_{\text{H}}) \rightarrow X$  such that

$$\text{diam}(\phi([\mathbb{M}]^k)) \geq C$$

for all infinite  $\mathbb{M} \subseteq \mathbb{N}$ . Defining  $\lambda$  as in (4.2) (here  $1/\infty = 0$ ) and letting  $\mathbb{M} \subseteq \mathbb{N}$  be as in the proof of Theorem 1.5, the proof then proceeds verbatim until (4.4) which in this case is replaced by

$$(4.7) \quad \|(f \circ \phi)(\bar{n}) - (f \circ \phi)(\bar{m})\| \geq \rho\left(\frac{\lambda}{3}\right) > 2KC_Y$$

for all  $\bar{n}, \bar{m} \in [\mathbb{M}]^k$  with  $\bar{n} < \bar{m}$ . Then, (4.5) becomes

$$\text{diam}(f(\phi([\mathbb{M}]^k))) \leq 2KC_Y.$$

and both those inequalities put together give us a contradiction.  $\square$

*Proof of Corollary 1.6.* By [BLMS21, Theorem B], we know that having  $\text{HFC}_\infty$  is equivalent to being reflexive and asymptotic- $c_0$ . The result is then an immediate consequence of Theorem 1.5(2).  $\square$

**4.2. Interlacing pairs in Hamming graphs.** Property  $\text{HFC}_p$ , for  $p \in (1, \infty]$ , implies reflexivity (this follows from [BKL10, Theorem 4.1]). Therefore this property is not relevant to study embeddings between non reflexive Banach spaces. With the goal of addressing this problem, the following weakening of the  $\text{HFC}_p$  was introduced in [LR18] and formalized in [Fov22, Subsection 2.5].

**Definition 4.8.** Let  $\mathbb{M}$  be an infinite subset of  $\mathbb{N}$  and let  $k \in \mathbb{N}$ . The set of *strictly interlacing* pairs in  $[\mathbb{M}]^k$  is given by

$$I_k(\mathbb{M}) = \{(\bar{n}, \bar{m}) \in [\mathbb{M}]^k \times [\mathbb{M}]^k \mid n_1 < m_1 < n_2 < m_2 < \dots < n_k < m_k\}.$$

Given a Banach space  $X$  and  $p \in (1, \infty]$ , we say that  $X$  has the *Hamming interlaced concentration property for  $p$* , abbreviated as  $HIC_p$ , if there exists a constant  $C \geq 1$  such that for any  $k \in \mathbb{N}$  and any 1-Lipschitz map  $\phi: ([\mathbb{N}]^k, d_{\mathbb{H}}) \rightarrow X$ , there exists an infinite subset  $\mathbb{M}$  of  $\mathbb{N}$  such that

$$\|f(\bar{n}) - f(\bar{m})\| \leq Ck^{1/p}, \text{ for all } (\bar{n}, \bar{m}) \in I_k(\mathbb{M}).$$

**Example 4.9.** Recall, given  $p \in (1, \infty)$ , that the James sequence space  $J_p$  is defined by

$$J_p = \left\{ (x(n))_n \in \mathbb{R}^{\mathbb{N}} \mid \lim_n x(n) = 0 \text{ and } \|x\|_{J_p} = \sup_{p_1 < \dots < p_n} \left( \sum_{i=2}^n |x(p_i) - x(p_{i-1})|^p \right)^{1/p} < \infty \right\}.$$

So, the classic James sequence space  $J$  is simply  $J_2$  and, just as  $J$ , each  $J_p$  has codimension 1 in its bidual; which makes them quasi-reflexive Banach spaces. M. Raja and the second named author proved in [LR18, Theorem 2.2] that, for  $p \in (1, \infty)$ , James space  $J_p$  has  $HIC_p$  and that, for  $q < p$ ,  $J_q$  fails  $HIC_p$ .

We can prove the following variant of Theorem 1.5. Notice that the assumption on the rate of expansion has been weakened.

**Theorem 4.10.** *Let  $X$  and  $Y$  be Banach spaces and suppose  $Y$  has  $HIC_p$  for some  $p \in (1, \infty)$ . If there is a coarse map  $f: X \rightarrow Y$  which is linearly expanding at rate  $\frac{1}{p}$ , then  $X$  must have  $HIC_p$ .*

*Proof.* The proof is similar to the one of Theorem 1.5, but we need to detail some of the minor modifications. So, let  $f: X \rightarrow Y$  be as in the statement and  $C_Y \geq 1$  be such that  $Y$  has  $HIC_p$  with constant  $C_Y$ . There exists  $K \geq 1$  so that  $\omega_f(t) \leq Kt + K$  and  $L \geq 1$  such that

$$(4.8) \quad \|x - z\| \geq \max\{\|x\|^{1/p}, \|z\|^{1/p}\} + 1 \text{ implies } \|f(x) - f(z)\| \geq \frac{1}{L}\|x - z\| - L.$$

Assume that  $X$  fails  $HIC_p$  and fix  $C > 3$  (to be precised later). Then there exist  $k \in \mathbb{N}$  and a 1-Lipschitz map  $\phi: ([\mathbb{N}]^k, d_{\mathbb{H}}) \rightarrow X$  such that for any infinite  $\mathbb{M} \subseteq \mathbb{N}$  there exists  $(\bar{n}, \bar{m}) \in I_k(\mathbb{M})$  so that  $\|\phi(\bar{n}) - \phi(\bar{m})\| > Ck^{1/p}$ . Since  $\Phi$  is 1-Lipschitz and  $\text{diam}([\mathbb{N}]^k) = k$ , we may assume that  $\phi([\mathbb{N}]^k) \subseteq kB_X$ . Let

$$(4.9) \quad \lambda = \inf \left\{ \sup_{(\bar{n}, \bar{m}) \in I_k(\mathbb{M})} \|\phi(\bar{n}) - \phi(\bar{m})\| k^{-1/p} \mid \mathbb{M} \subseteq \mathbb{N} \text{ and } |\mathbb{M}| = \infty \right\}.$$

So,  $\lambda \geq C$ . Pick now an infinite  $\mathbb{M} \subseteq \mathbb{N}$  such that

$$\lambda \leq \sup_{(\bar{n}, \bar{m}) \in I_k(\mathbb{M})} \|\phi(\bar{n}) - \phi(\bar{m})\| k^{-1/p} < 2\lambda.$$

By the definition of  $\lambda$ , for all infinite  $\mathbb{D} \subseteq \mathbb{M}$ , there exists  $(\bar{n}, \bar{m}) \in I_k(\mathbb{D})$  such that  $\|\phi(\bar{n}) - \phi(\bar{m})\| > \frac{2}{3}\lambda k^{1/p}$ . Identifying  $I_k(\mathbb{D})$  with  $[\mathbb{D}]^{2k}$  and using Ramsey's theorem, we can therefore assume that  $\|\phi(\bar{n}) - \phi(\bar{m})\| > \frac{2}{3}\lambda k^{1/p}$  for all  $(\bar{n}, \bar{m}) \in I_k(\mathbb{M})$ . It follows that for all  $(\bar{n}, \bar{m}) \in I_k(\mathbb{M})$ :

$$\|\phi(\bar{n}) - \phi(\bar{m})\| \geq \frac{2C}{3}k^{1/p} \geq \frac{C}{3}k^{1/p} + 1 \geq \max\{\|\phi(\bar{n})\|^{1/p}, \|\phi(\bar{m})\|^{1/p}\} + 1.$$

It then follows from (4.8) that

$$\|(f \circ \phi)(\bar{n}) - (f \circ \phi)(\bar{m})\| \geq \frac{2C}{3L}k^{1/p} - L,$$

for all  $(\bar{n}, \bar{m}) \in I_k(\mathbb{M})$ .

On the other hand  $f \circ \phi$  is  $2K$ -Lipschitz and  $Y$  has property  $\text{HIC}_p$  with constant  $C_Y$ , so there exists an infinite subset  $\mathbb{M}'$  of  $\mathbb{M}$  so that  $\|(f \circ \phi)(\bar{n}) - (f \circ \phi)(\bar{m})\| \leq 2KC_Y k^{1/p}$ , for all  $(\bar{n}, \bar{m}) \in I_k(\mathbb{M}')$ . This yields a contradiction for an initial large enough choice of  $C$  (depending on  $K, L, C_Y$ ).  $\square$

**Remark 4.11.** In the case  $p = \infty$ , the analogous statement is that  $\text{HIC}_\infty$  is preserved by coarse maps expanding at rate 0, in other words by coarse embeddings. This was already noticed in [Fov22].

**Problem 4.12.** Let  $p \in (1, \infty]$ . We do not know if the property  $\text{HIC}_p$  is preserved by coarse maps that are linearly expanding at rate 1.

As an immediate application of Theorem 4.10 and Example 4.9, we get the following.

**Corollary 4.13.** *Let  $p, q \in [1, \infty)$  and assume that  $p < q$ . Then there is no coarse map from  $J_p$  to  $J_q$  that is linearly expanding at rate  $1/q$ .*  $\square$

**4.3. Interlacing graphs.** We now deal with the interlacing metric of  $[\mathbb{N}]^k$ , which was introduced by N. Kalton in [Kal07] to rule out the coarse embeddability of  $c_0$  into reflexive spaces. We start by recalling the definition of the interlaced metric and of property  $\mathcal{Q}_p$ .

**Definition 4.14.** Let  $k \in \mathbb{N}$ . The *interlacing metric*  $d_{\mathbb{I}} = d_{\mathbb{I}, k}$  on  $[\mathbb{N}]^k$  is defined as follows: we endow  $[\mathbb{N}]^k$  with a graph structure by letting distinct  $\bar{n} = (n_1, \dots, n_k), \bar{m} = (m_1, \dots, m_k) \in [\mathbb{N}]^k$  be adjacent if either

$$n_1 \leq m_1 \leq \dots \leq n_k \leq m_k \text{ or } m_1 \leq n_1 \leq \dots \leq m_k \leq n_k.$$

The metric  $d_{\mathbb{I}}$  is then the shortest path metric on  $[\mathbb{N}]^k$  with respect to this graph structure.

**Definition 4.15** (Definition 1.7). Let  $p \in (1, \infty]$ . We say that a Banach space  $X$  has property  $\mathcal{Q}_p$  if there exists a constant  $C \geq 1$  such that for any  $k \in \mathbb{N}$  and any 1-Lipschitz map  $\phi: ([\mathbb{N}]^k, d_{\mathbb{I}}) \rightarrow X$ , there exists an infinite subset  $\mathbb{M}$  of  $\mathbb{N}$  such that

$$\text{diam}(\phi([\mathbb{N}]^k)) \leq Ck^{1/p}.$$

(here if  $p = \infty$ , we use the convention  $1/\infty = 0$ ). If  $p = \infty$ , we simply say, following [Kal07], that  $X$  has *property  $\mathcal{Q}$* .

**Example 4.16.** N. Kalton proved in [Kal07, Corollary 4.3] that reflexive Banach spaces have property  $\mathcal{Q}$ . In [BLPP23], it is shown that, for  $p \in (1, \infty)$ , a dual of a  $p$ -AUS Banach space has  $\mathcal{Q}_{p'}$ , where  $p'$  is the conjugate of  $p$  and that the dual of an asymptotic- $c_0$  space has  $\mathcal{Q}$  ([BLPP23, Theorem 4.1]).

We now extend the result insuring the stability of property  $\mathcal{Q}_p$  under coarse Lipschitz embeddings.

*Proof of Theorem 1.8(1).* Again the proof follows the lines of the argument for Theorem 1.5(1). let  $f: X \rightarrow Y$  be as in the statement and  $C_Y$  be such that  $Y$  has  $\mathcal{Q}_p$  with constant  $C_Y$ . Then the constants  $K, L, C$  are defined as in the proof of Theorem 1.5(1) and assume that  $X$  fails  $\mathcal{Q}_p$ . Then there exist  $k \in \mathbb{N}$  and a 1-Lipschitz map  $\phi: ([\mathbb{N}]^k, d_{\mathbb{I}}) \rightarrow X$  such that for any infinite  $\mathbb{M} \subseteq \mathbb{N}$ ,

$$\text{diam}(\phi([\mathbb{M}]^k)) \geq C$$



Let

$$(4.10) \quad \lambda = \inf\{\text{diam}(\phi([\mathbb{M}]^k))k^{-1/p} \mid \mathbb{M} \subseteq \mathbb{N} \text{ and } |\mathbb{M}| = \infty\};$$

and we pick an infinite  $\mathbb{M} \subseteq \mathbb{N}$  such that

$$\lambda \leq \text{diam}(\phi([\mathbb{M}]^k))k^{-1/p} < 2\lambda.$$

So assume, as we may, that  $\phi([\mathbb{M}]^k) \subseteq 2\lambda k^{1/p} B_X$ . By the definition of  $\lambda$  and arguing as in the proof of Theorem 1.5(1), we may assume that for all  $\bar{n} < \bar{m} \in [\mathbb{M}]^k$ ,  $\|\phi(\bar{n}) - \phi(\bar{m})\| \geq \frac{\lambda}{3} k^{1/p}$ . As  $\phi([\mathbb{M}]^k)$  is included in  $2\lambda k^{1/p} \cdot B_X$ , we deduce similarly that

$$(4.11) \quad \|(f \circ \phi)(\bar{n}) - (f \circ \phi)(\bar{m})\| \geq \frac{\lambda k^{1/p}}{3L} - L \geq \frac{Ck^{1/p}}{3L} - L$$

for all  $\bar{n}, \bar{m} \in [\mathbb{M}]^k$  with  $\bar{n} < \bar{m}$ . The contradiction then follows exactly as in the proof of Theorem 1.5.  $\square$

*Proof of Theorem 1.8(2).* We just have to adapt similarly the proof of Theorem 1.5(2). We leave the details to the reader.  $\square$

Asymptotic uniform convexity is often used, together with the approximate midpoint principle to find obstructions to the coarse Lipschitz embeddability. This is how it is shown that  $\ell_p$  does not coarse Lipschitz embed into  $\ell_q$  for  $p > q$  (see [JLS96]). We cannot find an approximate midpoint principle for coarse maps that are linearly expanding at rate 1 and, as we already emphasized, there is a good reason for that: it follows from Theorem 1.2 that  $M_{p,q} : \ell_p \rightarrow \ell_q$  is a coarse map from that is linearly expanding at rate 1 when  $p > q$ . However, the situation is different for the family of James spaces  $(J_p)_p$  and the use of property  $Q$  and its variants can serve as a substitute to obtain some preservation of asymptotic uniform convexity.

**Corollary 4.17.** *Let  $p, q \in (1, \infty)$  and assume that  $p > q$ . Then there is no coarse map from  $J_p$  to  $J_q$  that is linearly expanding at rate 1.*

*Proof.* It is proved by the authors, C. Petitjean and A. Procházka in [BLPP23, Corollaries 3.4 and 3.8] that, for  $p \in (1, \infty)$ ,  $J_p$  has property  $\mathcal{Q}_{p'}$ , where  $p'$  is the conjugate exponent of  $p$ , but fails  $\mathcal{Q}_r$  for all  $r > p$ . Then the conclusion follows from Theorem 1.8(2).  $\square$

## 5. EMBEDDINGS INTO $\ell_\infty$

In this last section, we prove that Lipschitz embeddability into  $\ell_\infty$  is equivalent to the existence of a coarse map which is linearly expanding at some rate strictly smaller than 1. We start with an intermediate result.

**Proposition 5.1.** *Let  $\alpha \in (0, 1)$ . Let  $X$  be a Banach space and suppose there is a Lipschitz map  $f : X \rightarrow \ell_\infty$  which is also linearly expanding at rate  $\alpha$ . Then  $X$  Lipschitzly embeds into  $\ell_\infty$ .*

*Proof.* Fix  $L > 0$  such that, for all  $x, z \in X$ ,

$$\|x - z\| \geq L(\max\{\|x\|, \|z\|\})^\alpha + L \text{ implies } \|f(x) - f(z)\| \geq \frac{1}{L}\|x - z\| - L.$$

Denote  $\mathbb{Q}_+$  the set of positive rational numbers and define a map  $F : X \rightarrow \ell_\infty(\mathbb{Q}_+, \ell_\infty)$  by letting

$$F(x) = (q^{-1}f(qx))_{q \in \mathbb{Q}_+}, \quad x \in X.$$

Since,

$$\|F(x) - F(z)\| = \sup_{q \in \mathbb{Q}_+} q^{-1} \|f(qx) - f(qz)\| \leq \text{Lip}(f) \|x - z\|,$$

for all  $x, z \in X$ , we have that  $\text{Lip}(F) \leq \text{Lip}(f)$ .

Fix now  $x, z \in X$  with  $x \neq z$ . As  $\alpha \in (0, 1)$ , there is  $t > 0$  large enough so that

$$\left\| \frac{tx}{\|x - z\|} - \frac{tz}{\|x - z\|} \right\| = t \geq L \left( \max \left\{ \frac{\|tx\|}{\|x - z\|}, \frac{\|tz\|}{\|x - z\|} \right\} \right)^\alpha + L.$$

Taking an even larger  $t$  if necessary, we can also assume that  $L < \frac{t}{2L}$ . We may also assume that  $q = \frac{t}{\|x - z\|} \in \mathbb{Q}_+$ . We obtain that

$$\begin{aligned} \|F(x) - F(z)\| &\geq \frac{\|x - z\|}{t} \left\| f\left(\frac{tx}{\|x - z\|}\right) - f\left(\frac{tz}{\|x - z\|}\right) \right\| \\ &\geq \frac{\|x - z\|}{t} \left( \frac{t}{L} - L \right) \geq \frac{1}{2L} \|x - z\| \end{aligned}$$

As  $x$  and  $z$  were arbitrary, this shows that  $F$  is a Lipschitz embedding from  $X$  into  $\ell_\infty(\mathbb{Q}_+, \ell_\infty)$ , which is clearly isometric to  $\ell_\infty$ .  $\square$

*Proof of Theorem 1.9.* The implication (1) $\Rightarrow$ (2) is immediate. Suppose (2) holds and let  $f: X \rightarrow \ell_\infty$  be such a map. Let  $N \subseteq X$  be a net of  $X$ , i.e., for some  $\delta, \varepsilon > 0$  the set  $N$  is  $\delta$ -separated and  $\varepsilon$ -dense in  $X$ . Since  $f$  is coarse and  $X$  is metrically convex, as we have already seen (cf [Kal08, Lemma 1.4]),  $f$  is coarse-Lipschitz, in fact, we have that

$$\|f(x) - f(z)\| \leq \omega_f(1) \|x - z\| + \omega_f(1)$$

for all  $x, z \in X$ . Therefore, as  $N$  is  $\delta$ -separated,  $f \upharpoonright N$ , the restriction of  $f$  to  $N$ , is Lipschitz with  $\text{Lip}(f \upharpoonright N) \leq \omega_f(1)(1 + 1/\delta) = C$ . Since any Lipschitz map into  $\ell_\infty$  can be extended to larger subsets without increasing its Lipschitz constant (see [Kal08, Section 3.3]), there is a Lipschitz map  $h: X \rightarrow \ell_\infty$  such that  $h \upharpoonright N = f \upharpoonright N$  and  $\text{Lip}(h) = C$ . As  $N$  is  $\varepsilon$ -dense in  $X$  and  $f$  and  $h$  coincide on  $N$ , we easily deduce that  $\|f - h\| \leq C\varepsilon + \omega_f(\varepsilon)$  on  $X$ . It then follows that  $h$  is also linearly expanding at rate  $\alpha$ . Then, by Proposition 5.1, there exists a Lipschitz embedding from  $X$  into  $\ell_\infty$  and (1) follows.  $\square$

**Problem 5.2.** We do not know whether we can take  $\alpha = 1$  in the statement of Theorem 1.9.

**Problem 5.3.** It is proved in [Kal11, Theorem 5.3] that the Lipschitz embeddability of a Banach space  $X$  into  $\ell_\infty$  is in fact equivalent to its coarse embeddability. We do not know whether it is also equivalent to the existence of coarse map from  $X$  to  $\ell_\infty$  that is expanding at some nontrivial rate  $\alpha \in (0, 1]$ ; notice that Theorem 1.9 assumes *linear* expansion at some rate  $\alpha \in (0, 1)$ .

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