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FDI spillovers, New Industry Development, and Economic Growth*

Thanh Tam NGUYEN-HUU[†] Ngoc-Sang PHAM[‡]

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Abstract

The paper investigates the optimal strategy of a small open economy receiving FDI in an optimal growth context. We prove that no domestic firm can enter the new industry when the multinational enterprise's productivity or the fixed entry cost is high. Nevertheless, the host country's investment stock converges to a higher steady state than an economy without FDI. A domestic firm enters the new industry if its productivity is high enough. Moreover, the domestic firm can dominate or even eliminate its foreign counterpart.

Keywords: Optimal growth, FDI spillovers, TFP, fixed cost.

JEL Classifications: O4, O11, F43, F23, C61.

1 Introduction

Over the past few decades, opening up to the global economy and attracting foreign direct investment (FDI) have been policy priorities in developing countries for promoting their economic development. One of the main arguments is that multinational enterprises (MNEs) could boost investment, bring new technologies/management skills, and generate FDI spillovers to domestic firms. However, empirical studies show ambiguous FDI effects on the host country's development.

At the micro-level, MNEs could generate spillovers to domestic competitors in the same industry (horizontal spillovers) or upstream and downstream local firms (vertical spillovers).¹ Empirical evidence shows positive spillovers from downstream

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[†]EM Normandie Business School, Métis Lab (France). Email: tnguyenhuu@em-normandie.fr.

[‡]Corresponding author. EM Normandie Business School, Métis Lab (France). Email: npham@em-normandie.fr. Phone: +33 2 50 32 04 08. Address: EM Normandie (campus Caen), 9 Rue Claude Bloch, 14000 Caen, France.

¹See [Blomstrom and Kokko \(1998\)](#), [Greenaway and Gorg \(2004\)](#), [Crespo and Fontoura \(2007\)](#) for more complete reviews of FDI spillovers, and [Meyer and Sinani \(2009\)](#), [Irsova and Havranek \(2013\)](#) for meta-analyses.

FDI firms (mainly joint venture FDI firms) to domestic suppliers but negative spillovers from upstream FDI firms to downstream domestic producers.² Besides, the literature provides evidence of mixed results regarding FDI horizontal spillovers.³

At the macro-level, the empirical literature finds that the effect of FDI on the host country's economic growth is relatively weak (Carkovic and Levine, 2005). More precisely, whether this effect is significant depends on local conditions such as the host country's human capital (Borensztein et al., 1998; Li and Liu, 2005) and the development of local financial markets (Alfaro et al., 2004, 2010).

The previous conflicting results on the effects of FDI raise a fundamental question on how a host country can benefit from FDI spillovers. This research aims to answer this question by introducing FDI in an optimal growth model to study the optimal allocation of a host country.

We consider a small open economy. In our model, there are three goods: a physical capital, a consumption good, and a so-called new good. The consumption good, taken as *numéraire*, is to serve final consumption. In contrast, the physical capital is served as an input in the production of consumption and new goods. These goods are freely tradable to the rest of the world. There are two agents (a representative consumer of the host country and an MNE) and two production sectors (a traditional sector producing the consumption good and a new sector producing the new good). We assume that only domestic firms in the host country produce the consumption good using physical capital as the sole input. By contrast, producing the new good requires physical capital and so-called *specific labor* (or skilled labor). In the beginning, there was the sole MNE in the new sector. However, a domestic firm can only enter this sector and produce the new good if it holds a critical level of specific labor. This threshold represents a fixed setup cost. By contrast, the MNE does not have to pay that fixed cost, thanks to its parent firm's support.⁴

Our research has three significant contributions to the literature. First, by studying the properties of equilibrium, we explore conditions under which a host country as in our framework should (or should not) invest in the new industry. We prove that if the host country suffers a low initial resource, a high setup cost, or a low level of FDI spillovers, no domestic firm can enter the new industry, regardless of its total factor

²For more discussions on vertical FDI spillover, see Javorcik (2004), Newman et al. (2015), Lu et al. (2017) for the case of Lithuania, Vietnam, China, respectively, and Gorodnichenko et al. (2014) for 17 transition countries.

³Indeed, there are negative or nil impacts of horizontal FDI on domestic firms in developing countries as, for example, Morocco (Haddad and Harrison, 1993), Uruguay (Kokko et al., 1996), Eastern Europe countries (Jude, 2012), Vietnam (Newman et al., 2015). By contrast, evidence of positive horizontal spillovers from FDI in developed countries is reported in Ruane and Ugur (2005) for Ireland, Haskel et al. (2007) for the UK, or Keller and Yeaple (2009) for the US.

⁴Our assumption on the setup cost is in line with several studies in the literature. Indeed, Smith (1987) and Markusen (1995) point out that a potential domestic firm has to pay a firm-specific fixed cost to enter a new industry. By contrast, the MNE has a plant in its home country where it has invested in that specific cost. Hence, this firm does not suffer such expenditure in producing in the host country (Smith, 1987). In another context, Fosfuri et al. (2001) indicate that a domestic firm might access new technologies thanks to worker mobility who initially worked for the MNE. To this end, the domestic firm has to pay a fixed cost that one interprets as an absorptive capability. In our framework, the host country must have at least a critical number of skilled workers to set up the production process.

productivity (TFP, hereafter). Once these necessary conditions hold, productivity plays a primordial role in determining the host country's optimal strategies. It produces the new good if (and only if) the domestic firm in the new industry has a high TFP. Moreover, it can dominate or even eliminate the MNE (in the sense that the MNE stops its production in the host country) if its TFP is high enough. These findings are helpful in policy implications because they could help host developing countries avoid an underdevelopment issue.⁵

Second, we enrich the literature of optimal growth by introducing and studying the effects of the fixed entry cost in an optimal growth model. In such a context, some countries may not invest in the new technology because of high fixed costs (Bruno et al., 2009). Nonetheless, when there is an investment to decrease fixed costs, an underdeveloped economy could catch up with a developing one (Le Van et al., 2016). Our added value is to explore the role of FDI spillovers and productivity.

Third, despite a rich but almost empirical literature on FDI spillovers, there is still a lack of theoretical framework investigation. We contribute to this literature by formalizing mechanisms explaining when, why, and under which conditions FDI can generate positive spillovers. In our optimal growth context, we show that whether FDI spillovers are insignificant or positive depends on the local conditions (such as initial resources, human capital, and productivity) and time. Indeed, in the beginning, no domestic firm could enter a new industry where an MNE has been well installed. However, the impact could become positive after some periods. Our point about the role of timing on the impact of FDI spillovers is consistent with empirical investigations of Merlevede et al. (2014) and references therein. Indeed, Merlevede et al. (2014), by using firm-level data from a panel of Romanian manufacturing firms during 1996-2005, find that the effect of foreign entry is initially negative but will be positive for a longer time.

The paper is structured as follows. Section 2 introduces an optimal growth model with the presence of MNE. We then study conditions under which the host country should invest in a new industry in Section 3. Section 4 concludes. Formal proofs are gathered in the appendix section.

2 Benchmark model

We consider a small open economy (called hereafter *host country*). There are three goods: a consumption good, a physical capital, and a new good. The consumption good, taken as *numéraire*, is produced in a traditional (or old) sector. The prices of the new good and physical capital (in terms of consumption good) are exogenous and denoted by p_n, p , respectively. We assume that the three goods are freely exchangeable with the rest of the world.

In each period, there is an MNE in the host country. It produces the new good in a *new industry* (or *new sector*)⁶ by using physical capital and specific labor (here, we

⁵Some scholars explore the competition between foreign and domestic firms and its impact on the host country's economy (see Ghosh and Saha (2015), Tsai et al. (2016), Amerighi and De Feo (2017), among others). Nonetheless, they rely on a Cournot competition in a static framework while we work in a dynamic model with perfect competition.

⁶In developing countries, this sector may be *Computer and Peripheral equipment*

assume that the specific labor is internationally immobile).

At each date t ($t \geq 0$), the foreign firm (without market power) chooses $K_{e,t}$ units of physical capital and $L_{e,t}^D$ units of specific labor in order to maximize its profit. With these inputs, the firm produces $F^e(K_{e,t}, L_{e,t}^D)$ units of new good, where $F^e : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is its production function. After producing, $K_{e,t}$ units of physical capital are depreciated and give only $(1 - \delta_e)K_{e,t}$ units of physical capital, where δ_e is the capital depreciation rate of the MNE. Hence, the MNE's maximization problem is:

$$(F_t) : \quad \max_{K_{e,t}, L_{e,t}^D \geq 0} \left[p_n F^e(K_{e,t}, L_{e,t}^D) + p(1 - \delta_e)K_{e,t} - pK_{e,t} - w_t L_{e,t}^D \right] \quad (1)$$

where w_t is the wage (in terms of consumption good).

We will work under the following assumption.

Assumption 1. Assume that $\delta_e \in (0, 1]$ and $F^e(K, L) = A_e K^{\alpha_e} L^{1-\alpha_e}$, $\forall K, L \geq 0$, where $\alpha_e \in (0, 1)$ and $A_e > 0$ represents the foreign firm's TFP.

Since the new good is freely exchangeable with the rest of the world, FDI in our framework is likely an export-platform FDI, whose production does not only serve the host country but a global market. It differs from a horizontal FDI, whose production purpose solely serves the local market. According to the literature on export-platform FDI, incentive factors of such investment are access to a global market (e.g., a low export cost to other countries) and a low production cost in the host country. In our framework, we assume that MNEs invest in the host country to benefit from easy access to a global market and a low cost of specific labor.⁷

There is a representative agent in the host country. This agent chooses the allocation of resources to maximize the intertemporal welfare of the whole population. If the country uses $K_{c,t+1}$ units of physical capital for the production process in the traditional sector at date t , then it will get $A_c K_{c,t+1}^\alpha$ units of consumption good at date $t + 1$, where $\alpha > 0$ and $A_c > 0$ represents the capital elasticity and TFP of the old sector.⁸ Besides, if the host country invests H_{t+1} units of consumption good in training specific labor at date t , there will be $A_h H_{t+1}^{\alpha_h}$ units of specific labor at date $t + 1$. Specific labor works for the MNE to get a salary.

Summing up, the representative agent solves the following problem:

$$(P_1) : \quad \max_{(c_t, K_{c,t+1}, L_{e,t+1}, H_{t+1})_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta^t u(c_t) \right] \quad (2a)$$

$$\text{subject to} \quad c_t + pK_{c,t+1} + H_{t+1} \leq A_c K_{c,t}^\alpha + p(1 - \delta)K_{c,t} + w_t L_{e,t}, \forall t \geq 0 \quad (2b)$$

$$L_{e,t} \leq A_h H_t^{\alpha_h}, \forall t \geq 1 \quad (2c)$$

manufacturing, Electrical Equipment manufacturing, Radio, Television and Communication equipment manufacturing, and so forth.

⁷Given the purpose of this research, we skip explaining why the MNE applies an export-platform FDI in the host country instead of exporting or a horizontal FDI. Please refer to Yeaple (2003), Montout and Zitouna (2005), Ekholm et al. (2007), Minda and Nguyen-Huu (2012) among others, for a more in-depth analysis of MNE strategies, including export-platform FDI.

⁸For the sake of simplicity, we assume that physical capital is the sole production factor of the consumption good. Indeed, we may consider a production function as $A_c K_{c,t}^\alpha L_{c,t}^{1-\alpha}$ where $L_{c,t}$ is unskilled labor. Nonetheless, by assuming that unskilled labor supply is exogenous and normalized by $L_{c,t} = 1$, we recover the production function of consumption good given in problem (P_1) .

where δ is the depreciation rate of physical capital in the traditional sector, β is a discount factor, and $K_{c,0}, L_{e,0}$ are given.⁹

We require standard assumptions in our paper.

Assumption 2. $\delta \in (0, 1], \alpha \in (0, 1), \alpha_h \in (0, 1), \beta \in (0, 1)$.

The utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is in C^1 , strictly increasing, strictly concave, and $u'(0) = \infty$.

In our framework, wage w_t is endogenous and determined by the market clearing condition given in Definition 1 below.

Definition 1. Let us define the benchmark economy \mathcal{E}^b as a list of exogenous parameters $(p, p_n, A_e, \alpha_e, \delta_e, A_c, \alpha, \delta, A_h, \alpha_h, \beta, K_{c,0}, L_{e,0})$ and the function $u(\cdot)$. An intertemporal equilibrium in this economy is a positive sequence $(c_t, K_{c,t+1}, H_{t+1}, L_{e,t+1}, L_{e,t}^D, K_{e,t}^D, w_t)_{t=0}^\infty$ satisfying the following conditions:

- (i) Given $(w_t)_{t=0}^\infty$, $(c_t, K_{c,t+1}, H_{t+1}, L_{e,t+1})_{t=0}^\infty$ is a solution to the problem (P_1) .
- (ii) For all $t \geq 0$, given w_t , $(L_{e,t}^D, K_{e,t}^D)$ is a solution to the problem (F_t) .
- (iii) Labor market clears: $L_{e,t}^D = L_{e,t}$, $\forall t \geq 0$.

For each $t \geq 0$, the solution to the MNE's problem (F_t) is determined as follows:

- (a) $K_{e,t}^D = L_{e,t}^D = 0$ if

$$w_t > w^* \text{ where } w^* \equiv \left(\alpha_e^{\alpha_e} (1 - \alpha_e)^{1 - \alpha_e} \frac{p_n A_e}{(\delta_e p)^{\alpha_e}} \right)^{\frac{1}{1 - \alpha_e}} \quad (3)$$

- (b) Any positive pair $(K_{e,t}^D, L_{e,t}^D)$ satisfying condition $\frac{\alpha_e L_{e,t}^D}{(1 - \alpha_e) K_{e,t}^D} = \frac{\delta_e p}{w^*}$ is a solution if $w_t = w^*$.¹⁰

Since $L_{e,t}^D = L_{e,t} = A_h H^{\alpha_h} > 0, \forall t \geq 0$, at equilibrium, we get that $w_t = w^*, \forall t \geq 0$. Wage w_t depends not only on the foreign firm's TFP and capital depreciation rate but also on physical capital and new good prices.

We define the function $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by:

$$F(x) \equiv \max_{(K_c, K): pK_c + H \leq x, K_c \geq 0, H \geq 0} \left\{ A_c K_c^\alpha + p(1 - \delta)K + w^* A_h H^{\alpha_h} \right\}, \forall x \geq 0. \quad (4)$$

Lemma 1. The function F is strictly increasing, strictly concave, smooth, $F(0) = 0$, and satisfies Inada condition $F'(0) = \infty$.

Proof. See Appendix B. □

⁹Constraint $c_t + pK_{c,t+1} + H_{t+1} \leq A_c K_{c,t}^\alpha + p(1 - \delta)K_{c,t+1} + w_t L_{e,t}$ implies that the host country cannot borrow from abroad.

¹⁰If $w_t < w^*$, then the MNE's problem does not have a finite solution.

Denote $S_{t+1} = pK_{c,t+1} + H_{t+1}$ the total savings of the host country at time t ($t \geq 0$). Then, the problem (P_1) can be rewritten as follows:

$$(P'_1) : \max_{(c_t, S_{t+1})_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta^t u(c_t) \right] \text{ subject to } c_t + S_{t+1} \leq F(S_t), c_t \geq 0, S_{t+1} \geq 0, \forall t \geq 0. \quad (5)$$

Without the presence of the MNE, we recover an open economy without FDI. In this case, the problem (P_1) becomes the standard optimal growth model with the budget constraint: $c_t + pK_{c,t+1} \leq A_c K_{c,t}^\alpha + p(1 - \delta)K_{c,t}$, $\forall t \geq 0$. In this case, we can prove that $\lim_{t \rightarrow \infty} S_t = S_a$, where

$$S_a \equiv p \left(\frac{\alpha A_c}{p(\frac{1}{\beta} + \delta - 1)} \right)^{\frac{1}{1-\alpha}}. \quad (6)$$

With the presence of the MNE, we have the following result.

Proposition 1. *Under the above specifications, there is a unique equilibrium. At equilibrium, we have $w_t = w^*$, $\forall t \geq 0$, and S_t converges to S_b defined by $\beta F'(S_b) = 1$, or, equivalently,*

$$S_b = p \left(\frac{\alpha A_c}{p(\frac{1}{\beta} + \delta - 1)} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\alpha_h w^* A_h}{\frac{1}{\beta}} \right)^{\frac{1}{1-\alpha_h}}. \quad (7)$$

S_b increases in A_c, w^*, A_h , and $S_b > S_a$.

Proof. See Appendix B. □

Proposition 1 shows a positive impact of FDI on the host country's economic development. The property $S_b > S_a$ means that the steady state savings of the economy with FDI is higher than that of the economy without FDI. Moreover, the steady state level S_b is increasing in the TFP of domestic firms, wage, and the TFP of foreign firms. It implies that the FDI effect on the steady state output depends on both FDI and the host country's development characteristics. This finding is consistent with several empirical studies mentioned in the Introduction section.

3 FDI spillovers and industrial policy

In Section 2, we assume that only the MNE produces in the new sector, and there are no FDI spillovers. In this section, we develop the benchmark model in Section 2 by adding two elements: (i) the MNE may generate FDI spillovers through labor turnover, and (ii) the host country may create a domestic firm in the new industry.

The problem of the MNE remains unchanged as in (1). We now describe the maximization problem of the representative agent of the host country. At the beginning of each period t ($t \geq 1$), this agent supplies $L_{e,t}$ units of specific labor for the MNE subject to the constraint (2c), i.e.,

$$L_{e,t} \leq A_h H_t^{\alpha_h}. \quad (8)$$

FDI spillovers through labor turnover occur when a domestic firm can hire the former specific labor of the MNE. We assume that FDI spillovers through labor turnover (in terms of specific labor) are given by:

$$Spillovers(A_e, L_{e,t}, S_t) = \frac{BA_e}{1 + S_t} L_{e,t}. \quad (9)$$

where S_t is the savings of the host country and parameter B represents either the absorbability of specific labor or learning-by-doing effects. Our specification (9) is motivated by two observations: (i) FDI spillovers decrease in the host country's development level (proxied by S_t in our framework) and (ii) FDI spillovers increase in the MNE's productivity A_e , the quantity $L_{e,t}$, and the absorbability level B (with $B \geq 0$).¹¹

Given these different elements on the labor market (i.e., specific labor trained at the previous period, specific labor hired by the MNE, and FDI spillovers), the amount of specific labor available for the domestic firm becomes:

$$(A_h H_t^{\alpha_h} - L_{e,t}) + \frac{BA_e}{1 + S_t} L_{e,t}. \quad (10)$$

We now look at the choice of the representative agent. Assume that, at each date $t \geq 1$, this agent can use $K_{d,t}$ units of physical capital and $L_{d,t}$ units of specific labor to produce $F^d(K_{d,t}, L_{d,t})$ units of new good (see Assumption 3 below). The quantity $L_{d,t}$ cannot exceed the amount of specific labor available for the domestic firm, i.e.,

$$L_{d,t} \leq (A_h H_t^{\alpha_h} - L_{e,t}) + \frac{BA_e}{1 + S_t} L_{e,t}. \quad (11)$$

The maximization problem of the representative agent now becomes:

$$(P) : \quad \max_{(c_t, K_{c,t+1}, K_{d,t+1}, L_{d,t+1}, L_{e,t+1}, H_{t+1})_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta^t u(c_t) \right] \quad (12)$$

subject to

$$0 \leq K_{c,t}, K_{d,t}, L_{d,t}, L_{e,t}, H_t, \forall t \geq 1 \quad (13a)$$

$$c_t + S_{t+1} \leq A_c K_{c,t}^\alpha + p(1 - \delta)K_{c,t} + w_t L_{e,t} + p_n F^d(K_{d,t}, L_{d,t}) + p(1 - \delta_d)K_{d,t}, \forall t \geq 0 \quad (13b)$$

$$S_{t+1} = p(K_{c,t+1} + K_{d,t+1}) + H_{t+1}, \forall t \geq 0 \quad (13c)$$

$$L_{e,t} \leq A_h H_t^{\alpha_h}, \forall t \geq 1 \quad (13d)$$

$$L_{d,t} \leq (A_h H_t^{\alpha_h} - L_{e,t}) + \frac{BA_e}{1 + S_t} L_{e,t}, \forall t \geq 1, \quad (13e)$$

¹¹Indeed, Fosfuri et al. (2001) prove, through a static model, that labor turnover can be a channel of FDI spillovers. Furthermore, the degree of such spillovers is increasing with the absorbability of domestic firms. Evidence from Brazil supports heterogeneous impacts of spillovers through labor mobility Poole (2013). High-skilled former foreign firms' workers have a better ability to transfer information, and high-skilled incumbent workers can better absorb information. However, Crespo and Fontoura (2007), Meyer and Sinani (2009) argue that the higher the host country's development level, the fewer FDI spillovers level.

where $K_{d,0}, L_{d,0}, K_{c,0}, L_{e,0} \geq 0$ are given, and δ_d represents the capital depreciation rate in the new sector.

Notice that constraint (13b) implies two potential contributions of the new sector to the host country's economic growth: a payroll from the MNE, $w_t L_{e,t}$, and potential new good sales, $p_n F^d(K_{d,t}, L_{d,t})$, once the domestic firm would enter the new sector.¹²

The domestic firm in the new sector has the following characteristics.

Assumption 3. $\delta_d \in (0, 1]$ and the function $F^d : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is defined by

$$F^d(K, L) = A_d K^{\alpha_d} ((L - \bar{L})^+)^{1-\alpha_d}, \forall K, L \geq 0, \quad (15)$$

where $\alpha_d \in (0, 1)$ and $\bar{L} \geq 0$ is a fixed entry-cost.

Here, we assume that the domestic firm needs an initial investment to enter the new industry. We model this investment by the fixed cost \bar{L} in (15), that represents a minimum quantity of specific labor needed to ensure the functionality of the production process. By contrast, the MNE invested in the home country and did not pay this fixed cost again to produce in the host country.

We now provide the definition of intertemporal equilibrium.

Definition 2. Let us define the economy \mathcal{E} as a list of exogenous parameters

$$(p, p_n, A_e, \alpha_e, \delta_e, B, A_c, \alpha, \delta, A_h, \alpha_h, A_d, \alpha_d, \delta_d, \bar{L}, \beta, K_{c,0}, K_{d,0}, L_{d,0}, L_{e,0})$$

and the utility function $u(\cdot)$. An intertemporal equilibrium in this economy is a positive sequence $(c_t, K_{c,t+1}, K_{d,t+1}, L_{d,t+1}, L_{e,t+1}, H_{t+1}, L_{e,t}^D, K_{e,t}^D, w_t)_{t=0}^\infty$ satisfying the following conditions:

- (i) Given $(w_t)_{t=0}^\infty$, the sequence $(c_t, K_{c,t+1}, K_{d,t+1}, L_{d,t+1}, L_{e,t+1}, H_{t+1})_{t=0}^\infty$ is a solution to the problem (P).
- (ii) For all $t \geq 0$, given w_t , $(L_{e,t}^D, K_{e,t}^D)$ is a solution to the problem (F_t) .
- (iii) $L_{e,t}^D = L_{e,t}, \forall t \geq 0$.

In comparison to the benchmark economy \mathcal{E}^b , the economy \mathcal{E} incorporates two additional elements: there may be a domestic firm that produces the new good in the host country, and there are FDI spillovers through labor turnover. If $A_d = 0, \delta_d = 1, B = 0$, then we recover the benchmark economy.

¹²Here, we assume that the consumer does not consume the new good. However, we can include the new good in the consumption of the representative agent by modifying the problem (P) as follows:

$$(P') : \max_{(c_t, c_t^d, K_{c,t+1}, K_{d,t+1}, L_{d,t+1}, L_{e,t+1}, H_{t+1})_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta^t u(C_t) \right] \quad (14)$$

subject to sequential constraints as in the problem (P) but constraint (13b) is replaced by $c_t + p_n c_t^d + S_{t+1} \leq A_c K_{c,t}^\alpha + p(1-\delta)K_{c,t} + w_t L_{e,t} + p_n F_t^d(K_{d,t}, L_{d,t}) + p(1-\delta)K_{d,t}$, where C_t is a function of c_t and c_t^d . Hence, under mild conditions, problems (P) and (P') would have similar equilibrium outcomes.

3.1 Static analysis

We explore the static analysis. Let us firstly introduce the notion of equilibrium. Given $S > 0$, an equilibrium in a static case is a positive list $(K_c, K_d, L_d, L_e, H, L_e^D, K_e^D, w)$ satisfying the following conditions:

1. Given wage w , the allocation (L_e^D, K_e^D) is a solution to the MNE's problem

$$\max_{K_e^D, L_e^D \geq 0} \{p_n F^e(K_e^D, L_e^D) + p(1 - \delta_e)K_e - pK_e - wL_e^D\}.$$

2. Given wage w , the allocation (K_c, K_d, L_d, L_e, H) is a solution to the following maximization problem

$$(G_S) : \max \left\{ A_c K_c^\alpha + wL_e + p_n A_d K_d^{\alpha_d} ((L_d - \bar{L})^+)^{1-\alpha_d} + p(1 - \delta)K_c + p(1 - \delta_d)K_d \right\} \quad (16a)$$

$$\text{subject to: } K_c, K_d, L_d, L_e, H \geq 0 \quad (16b)$$

$$p(K_c + K_d) + H \leq S \quad (16c)$$

$$L_e \leq A_h H^{\alpha_h} \quad (16d)$$

$$L_d \leq (A_h H^{\alpha_h} - L_e) + \frac{BA_e}{1 + S} L_e. \quad (16e)$$

3. Labor market clears: $L_e^D = L_e$.

We now study the properties of equilibrium. We wonder whether the domestic firm or the MNE produces in the new sector.

Proposition 2. *Let $(K_c, K_d, L_d, L_e, H, L_e^D, K_e^D, w)$ be an equilibrium in a static case.*

1. *If $\frac{BA_e}{1+S} > 1$, then $L_e = A_h H^{\alpha_h}$.*
2. *If $\max(\frac{BA_e}{1+S}, 1) A_h S^{\alpha_h} \leq \bar{L}$, then $A_d K_d^{\alpha_d} ((L_d - \bar{L})^+)^{1-\alpha_d} = 0$.*

Proof. See Appendix B. □

The first point of Proposition 2 indicates that if FDI spillovers are high enough, all specific labor trained by the host country will work for the foreign firm to get a high amount of payroll. The second point shows that when the entry cost is high, the host country is poor and unable to invest in the new industry.

The following result shows the role of productivity.

Proposition 3. *Let S be given. Assume that $BA_e < 1$ and $A_h S^{\alpha_h} > \bar{L}$. There exists \bar{A}_d such that: for every $A_d \geq \bar{A}_d$, we have $A_d K_d^{\alpha_d} ((L_d - \bar{L})^+)^{1-\alpha_d} > 0$ and $A_e K_e^{\alpha_e} L_e^{1-\alpha_e} = 0$ for any equilibrium $(K_c, K_d, L_d, L_e, H, L_e^D, K_e^D, w)$ in the static case.*

Proof. See Appendix B. □

Proposition 3 implies that even with low FDI spillovers (i.e., $BA_e < 1$), the host country still enables to invest in the new sector if its resource S is high enough ($A_h S^{\alpha_h} > \bar{L}$) or the domestic firm is efficient enough (i.e., its TFP is high).

3.2 Global dynamic analysis

We now investigate the dynamic analysis of equilibrium defined in Definition 2. More precisely, we are interested in the evolution of allocations $S_t, K_{c,t}, K_{d,t}, H_t, L_{e,t}, L_{d,t}$ as well as the aggregate output. First of all, we have the convergence of the optimal path (S_t) .

Proposition 4. (1) *There exists an equilibrium satisfying $w_t = w^*$ for any $t \geq 0$. Such an equilibrium converges.*

(2) *At equilibrium, (S_t) does not converge to zero.*

Proof. See Appendix B. □

We now focus on the transitional dynamics and the steady state of the equilibrium path (S_t) . Denote $X_0 \equiv A_c K_{c,0}^\alpha + p(1 - \delta)K_{c,0} + w_0 L_{e,0}$. Define the sequence $(x_t)_{t \geq 1}$ as $x_1 = X_0, x_{t+1} = F(x_t)$, where the function F is given in (4). Let x^* and \bar{S} be uniquely defined by:

$$F(x^*) = x^* \text{ and } \bar{S} \equiv \max\{X_0, x^*\}. \quad (17)$$

Notice that x^* and \bar{S} depend on the productivity A_c and capital elasticity α of the consumption good sector, the efficiency of specific labor training A_h, α_h , and wage w^* . However, they do not depend on the TFP A_d of the potential domestic firm in the new sector.¹³

Besides, we get that $F(x) \leq F(x^*) = x^*$ for every $x \leq x^*$ and $F(x) \leq x$ for every $x \geq x^*$.¹⁴ At equilibrium, it is easy to prove that $S_t \leq x_t \leq \bar{S}, \forall t \geq 1$.¹⁵ By consequence, we obtain the following result:

Proposition 5 (middle income trap). *Assume that $\max(BA_e, 1)A_h(\bar{S})^{\alpha_h} \leq \bar{L}$, and $K_{d,0} = 0, L_{d,0} = 0$. Then, at equilibrium, we have $Y_{d,t} = 0, \forall t \geq 1$. In this case, S_t converges to S_b ($\lim_{t \rightarrow \infty} S_t = S_b$), where S_b is defined in Proposition 1.*

Proof. See Appendix B. □

Proposition 5 indicates that no domestic firm can be created in a new industry if:

- (i) The host country has a low initial endowment X_0 .
- (ii) The consumption good sector TFP A_c is low.
- (iii) The training sector has a low productivity A_h .

Interestingly, this result holds whatever the level of the TFP A_d of the potential domestic firm.

The following result provides sufficient conditions under which the domestic firm enter the new industry.

¹³If $\alpha_h = \alpha$ and $\delta = 1$, we can explicitly compute that $x^* = (\frac{A_c}{p^\alpha})^{\frac{1}{1-\alpha}} + (wA_h)^{\frac{1}{1-\alpha}}$.

¹⁴Indeed, if $x < x^*$, then $F(x) < F(x^*) = x^*$. If $x > x^*$, then $\frac{F(x)}{x} \leq \frac{F(x^*)}{x^*} = 1$ since F is concave.

¹⁵It is obvious that $S_t \leq x_t, \forall t \geq 1$. We prove $x_t \leq \bar{S}, \forall t \geq 1$, by induction. First, we see that $x_0 \leq \bar{S}$. Second, assume that $x_s \leq \bar{S} \forall s \leq t$. If $X_0 \leq x^*$, then $x_t \leq \bar{S} = x^*$, then $x_{t+1} = F(x_t) \leq F(x^*) = x^* = \bar{S}$. If $X_0 > x^*$, then $x_t \leq \bar{S} = X_0$ and hence $x_{t+1} = F(x_t) = F(x_0) \leq x_1 \leq \bar{S}$.

Proposition 6 (entry of the domestic firm). *Assume that $K_{d,0} = 0, L_{d,0} = 0$ and*

$$\max \left(\frac{BA_e}{1 + X_0}, 1 \right) A_h X_0^{\alpha_h} < \bar{L} \quad (18)$$

$$\max \left(\frac{BA_e}{1 + S_b}, 1 \right) A_h S_b^{\alpha_h} > \bar{L} \quad (19)$$

Then, there exists a threshold $\bar{A}_d > 0$ satisfying the following condition: for any $A_d > \bar{A}_d$, for any equilibrium $(c_t, K_{c,t+1}, K_{d,t+1}, L_{d,t+1}, L_{e,t+1}, H_{t+1}, L_{e,t}^D, K_{e,t}^D, w_t)_{t=0}^\infty$ (which depends on A_d) of the economy \mathcal{E} , there exists a date $t_d > 1$ such that $Y_{d,t} = 0$ for every $t < t_d$ and $Y_{d,t_d} > 0$ for every $t \geq t_d$, where we define $Y_{d,s} \equiv F_s^d(K_{d,s}, L_{d,s}), \forall s$.

Proof. See Appendix B. □

Condition (18) means that the fixed cost \bar{L} is high as compared to the host country's initial resource X_0 . Consequently, the host country cannot produce in the new industry at date t_0 . Condition (19) implies that the host country may overcome the fixed cost \bar{L} if the steady state S_b of the country in the benchmark model is high enough.¹⁶ Proposition 6 indicates that under conditions (18) and (19), the country should invest in the new industry if (and only if) the productivity A_d is high enough.

Propositions 5 and 6 explain the absence or insignificant impact of horizontal FDI spillovers, as reported in several empirical studies.¹⁷ Accordingly, Propositions 5 and 6 indicate that if the local conditions are not sufficiently good, there are no horizontal FDI spillovers (in the sense that no domestic firm can operate in the new industry). However, positive spillovers will appear after some periods (in the sense that $Y_{d,t} > 0, \forall t \geq t_d$ in Proposition 6). Indeed, it takes time for the MNE to improve its involvement in the host country by hiring more local employees ($L_{e,t}$) and so generating more FDI spillovers ($\frac{BA_e}{1+S_t} L_{e,t}$).

Our point about the role of time since foreign entry in the development of domestic firms is in line with the empirical investigation of Merlevede et al. (2014). Indeed, Merlevede et al. (2014) use firm-level data from a panel of Romanian manufacturing firms during 1996-2005 and find that MNEs initially negatively affect local competitors' productivity. However, the effect becomes permanently positive for a longer time.

Besides, Proposition 6 leads to an interesting implication for a low-income country having high productivity in both old sectors (A_c) and new sectors (A_d). The new sectors are underdeveloped owing to high fixed costs. Hence, this country could apply the following optimal development strategy:

- First, the country attracts FDI and trains specific workers for the new sectors.
- Then, those workers work for MNEs (located in the new sectors) to get high salaries and high-skill knowledge (through learning by doing effects or specific training), consequently improving the country's income.
- Once its income reaches a critical threshold that allows the country to cover the fixed costs, new domestic firms can enter the new industries.

¹⁶Notice that (18) and (19) are satisfied if, for instance, $BA_e < 1$ and $A_h X_0^{\alpha_h} < \bar{L} < A_h S_b^{\alpha_h}$.

¹⁷See Blomstrom and Kokko (1998), Greenaway and Gorg (2004), Crespo and Fontoura (2007) for more detailed reviews.

Proposition 6 is consistent with [Markusen and Venables \(1999\)](#) since the authors indicate that FDI may contribute to creating local industrial sectors. However, while [Markusen and Venables \(1999\)](#) provide a static partial equilibrium model, we develop an infinite-horizon growth model. Besides, we rely on a perfect competition framework instead of an imperfect competition, as in [Markusen and Venables \(1999\)](#).

Notice that the domestic firm can even eliminate the MNE, as stated in the following result.

Proposition 7 (steady state). *Assume that $BA_e < 1$. When A_d high enough, there exists an equilibrium steady state with wage w^* , $L_d^s > 0$ and $L_e^s = 0$, where L_d^s, L_e^s are respectively the associated specific labor hire by the domestic firm and the MNE. Moreover, the steady state savings equals S_s where S_s is defined by*

$$S_s \equiv p \left(\frac{\alpha A_c}{p(\frac{1}{\beta} + \delta - 1)} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\alpha_d^{\alpha_d} (1 - \alpha_d)^{1-\alpha_d} p_n A_d (\alpha_h A_h)^{1-\alpha_d}}{(\frac{1}{\beta} + \delta_d - 1)^{\alpha_d} (\frac{1}{\beta})^{1-\alpha_d}} \right)^{\frac{1}{(1-\alpha_h)(1-\alpha_d)}} \quad (20)$$

$$+ p \left(\frac{\alpha_d p_n A_d}{p(\frac{1}{\beta} + \delta_d - 1)} \right)^{\frac{1}{1-\alpha_d}} \left(A_h \left(\frac{\alpha_d^{\alpha_d} (1 - \alpha_d)^{1-\alpha_d} p_n A_d (\alpha_h A_h)^{1-\alpha_d}}{(\frac{1}{\beta} + \delta_d - 1)^{\alpha_d} (\frac{1}{\beta})^{1-\alpha_d}} \right)^{\frac{\alpha_h}{(1-\alpha_h)(1-\alpha_d)}} - \bar{L} \right).$$

For A_d high enough, S_s is strictly higher than S_b in Proposition 5.

Proof. See Appendix B. □

In Proposition 7, we assume that neither the domestic firm absorptability nor the foreign firm TFP is high (in the sense that $BA_e < 1$). Hence, the MNE can be eliminated (i.e., $Y_e = 0$) when the productivity A_d of the domestic firm in the new sector is high enough.¹⁸ In this case, the steady state savings S_s and the income $G(S_s)$ are respectively higher than the values S_b and $F(S_b)$ of the economy \mathcal{E}^b (cf., the economy described in Section 2). Notice that S_s is increasing in the productivity A_d and decreasing in the depreciation rate δ_d of the domestic firm in the new sector.

3.3 Illustration from Vietnamese manufacturing industries

This subsection illustrates our theoretical framework by relying on the case of Vietnamese manufacturing industries. We rely on the data conducted from the Vietnamese Enterprises Survey between 2000 and 2016. It is an annual and one of the most extensive surveys organized by the General Statistics Office of Vietnam since 2000. Each wave gathers information on firm characteristics and activities, such as tax identification, legal status, turnover, capital stock, payroll, raw materials cost, investment, etc.

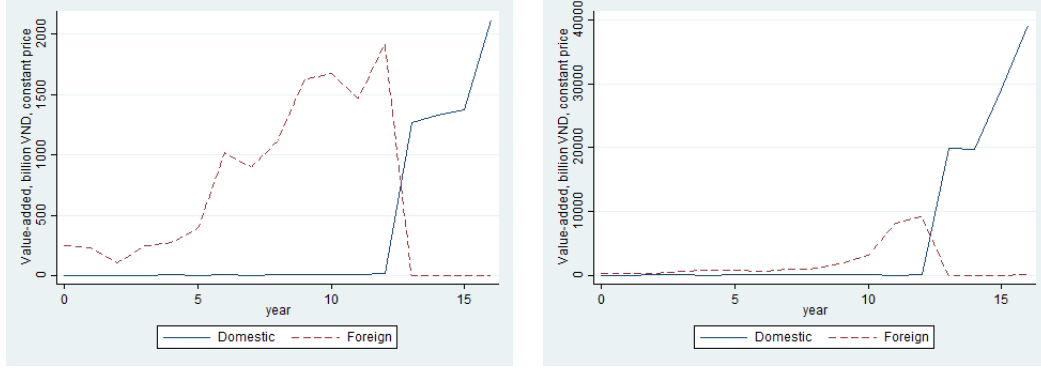
Since the data are at the firm level, we should create a new database at the industrial level to make data compatible with our theoretical framework.¹⁹ To this end, except for the firm TFP, we only need to sum up all related firms (all domestic firms and all foreign firms together) in an industry to get the aggregate level.

¹⁸This is indeed complementary to Proposition 6 by justifying the dominance of the domestic entrant firm on the MNE.

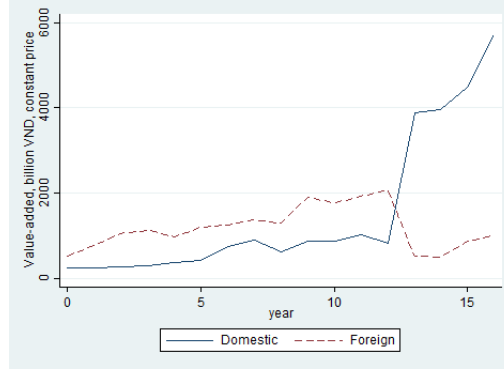
¹⁹There are a representative MNE and a potential domestic firm in the new sector in the above framework.

In our example, the new sector refers to “Computer and Peripheral equipment manufacturing”, “Electrical equipment manufacturing”, or “Radio, Television, and Communication equipment manufacturing”. Figure 1 below displays the domestic and foreign productions in these three sectors during the period 2000-2016.

Figure 1: Domestic and foreign productions during the period 2000-2016



(a) Computer and Peripheral equipment manufacturing (b) Radio, Television, and Communication equipment



(c) Electrical equipment manufacturing

Figure 1 likely supports our above theoretical findings. Indeed, taking a look at the ‘Computer and Peripheral equipment manufacturing’ in Figure 1a, we state that over the period 2000-12, the domestic production (measured by its value-added) was very small (nearly 0). However, between 2013 and 16, domestic production hugely increased. Thus, these findings seem to support our theoretical findings in Proposition 6. The low level of domestic production between 2000-12 seems to match with the statement $Y_{d,t} = 0$ for $t < t_d$ of the proposition 6. Besides, the high level of domestic production from 2013-16 appears to connect to the case $Y_{d,t} > 0$ for $t \geq t_d$. Moreover, Figure 1 also displays a decrease and convergence to zero of foreign production. Hence, domestic firms seem to dominate and eliminate their foreign counterparts in the competition, which is consistent with the case $Y_{d,t} > 0$ and $Y_{e,t} = 0$ in Proposition 7.

The above findings are likely to hold in the other two industries. Figure 1b indicates that the domestic production of ‘Radio, Television, and Communication equipment’ remained very low between 2000 and 2012. However, between 2013-16, there was an increase in domestic production. Moreover, domestic production tended to bring

out foreign production. Hence, these findings are what we state in Propositions 6-7. When it comes to ‘Electrical equipment manufacturing,’ Figure 1 reports the same phenomenon: a very low domestic production in the first period and then a high and overtaking of the former on the foreign production hereafter.

4 Conclusion

We have developed an infinite-horizon optimal growth model to investigate the nexus between FDI spillovers and the transitional dynamics of a host country. In our model with standard assumptions, we prove that whether the host country should (or should not) develop a new industry where MNEs have been well installed depends on different factors, including the setup cost, the productivity of domestic and foreign firms, FDI spillovers, and timing.

The present paper has some limits that could open for further research. First, in our paper, unskilled labor is assumed to be exogenous, and there is no trade-off between unskilled and specific labor supplies. In practice, structural change in developing countries mainly occurs through labor reallocation from the traditional sector toward the modern one. So, it would be interesting to study the interplay between unskilled and skilled labors as well as its impact on the structural change and economic growth in developing countries.

Second, in our paper, we treat the productivity of firms as exogenous. It is important to endogenize productivity by taking into account that (1) MNEs may generate positive spillovers on the domestic firms and (2) the host country may invest or subsidy its R&D activities (see [Hu et al. \(2023a\)](#), [Hu et al. \(2023b\)](#) for instance) which may improve the domestic firms’ productivity. In such a framework, it would be relevant to identify conditions under which these spillovers occur and examine their influence on the optimal policy for the host country.

Appendix

A The optimal growth theory: a preliminary

For a pedagogical purpose, we present general results showing the property of optimal growth paths in models without the concavity of production functions. Although there is a huge literature on the optimal growth theory, we think that Propositions A1, A2 in this section are new, and they are used in the present paper.²⁰

We now introduce a formal optimal growth model. There is one agent who maximizes

²⁰Let us mention some papers which are very closed to ours. [Dechert and Nishimura \(1983\)](#) give a complete characterization of optimal growth paths in a model with convex-concave technologies. [Hung et al. \(2009\)](#) studies an optimal growth model where the aggregate production function is maximum of concave technologies. [Majumdar and Mitra \(1982\)](#), [Kamihigashi and Roy \(2007\)](#) study non-smooth, non-convex models. [Jones and Manuelli \(1990\)](#) work with increasing return to scale technologies.

her intertemporal utility:

$$(P_A) : \max_{(c_t, S_{t+1})_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta^t u(c_t) \right] \quad (\text{A.1})$$

$$\text{subject to: } c_t + S_{t+1} \leq f(S_t), \quad c_t, S_{t+1} \geq 0, \forall t \geq 0, \quad (\text{A.2})$$

where S_0 is given.²¹ For short, we write (x_t) instead of $(x_t)_{t=0}^{\infty}$, where x_t is a vector.

A path $(c_t, S_{t+1})_{t \geq 0}$ is feasible if it satisfies (A.2) for every t . A capital path $(S_t)_{t \geq 1}$ is feasible if there exists a consumption path $(c_t)_{t \geq 0}$ such that $(c_t, S_{t+1})_{t \geq 0}$ is a feasible path. Given a real value S , we say a path $(c_t, S_{t+1})_{t \geq 0}$ (or capital path $(S_t)_{t \geq 1}$) is from S if $S_0 = S$.

A path $(c_t, S_{t+1})_{t \geq 0}$ is optimal from S if it solves problem (P_A) with $S_0 = S$. A path $(c_t, S_{t+1})_{t \geq 0}$ (resp. capital path $(S_t)_{t \geq 1}$) is stationary if $c_t = c$ and $S_{t+1} = S$ for every $t \geq 0$. A pair (c, S) is a steady state if the stationary path $(c_t, S_{t+1})_{t \geq 0}$ with $c_t = c$ and $S_t = S$ is optimal. A capital stock $S \geq 0$ is a steady state if (c, S) is a steady state for some $c \geq 0$.

We require standard assumptions which are maintained throughout this appendix.

Assumption (H1): $u : [0, \infty) \rightarrow \mathbb{R}_+$ is in C^1 , strictly increasing, strictly concave and $u'(0) = \infty$.

Assumption (H2): $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is strictly increasing and $f(0) \geq 0$.²²

Assumption (H3): For every $S > 0$, there exists a feasible path $(c_t, S_{t+1})_{t \geq 0}$ from S such that $\sum_{t=0}^{\infty} \beta^t u(c_t) > -\infty$. We also have $\sum_{t=0}^{\infty} \beta^t u(f^t(S)) < \infty$, where f^t is defined by $f^1 = f, f^{t+1} = f(f^t)$.

The last assumption require that the utility function is well defined and finite.

Let denote $v(S_0)$ be the value function of the problem (P_A) . We have the Bellman equation $v(S_0) = \max_{0 \leq S \leq f(S_0)} \{u(f(S_0) - S) + \beta v(S)\}$. By using this Bellman equation and the argument in Amir (1996), we obtain that:

Lemma 2. Every optimal capital path is monotonic. By consequence, if an optimal path is bounded from above, then it converges.

Following Kamihigashi and Roy (2007), we have Euler condition in the form of inequality

Lemma 3. Let $(c_t, S_{t+1})_{t \geq 0}$ be an optimal path. Then, we have, for every $t \geq 0$,

$$\beta u'(c_{t+1}) D^- f(S_{t+1}) \geq u'(c_t) \geq \beta u'(c_{t+1}) D^+ f(S_{t+1}), \quad (\text{A.3})$$

where the Dini derivatives of function f are defined by $D^+ f(x) = \limsup_{\epsilon \downarrow 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$ and

$$D^- f(x) = \liminf_{\epsilon \downarrow 0} \frac{f(x) - f(x-\epsilon)}{\epsilon}.$$

Proof. Since $u'(0) = \infty$, we have $c_t > 0$ for every $t \geq 0$. For each $t \geq 0$, we define a path $(c'_t, S'_{t+1})_{t \geq 0}$ as follows

$$c'_s = c_s, \quad \forall s \notin \{t, t+1\}, \quad S'_s = S_s, \quad \forall s \neq t+1 \quad (\text{A.4})$$

$$c'_t = c_t - \epsilon, \quad S'_{t+1} = S_{t+1} + \epsilon \quad (\text{A.5})$$

$$c'_{t+1} = c_{t+1} + f(S_{t+1} + \epsilon) - f(S_{t+1}). \quad (\text{A.6})$$

²¹Some studies replace constraint $S_{t+1} \geq 0$ by $S_{t+1} \geq r(S_t)$. The reader is referred to Kamihigashi and Roy (2007), Dimaria et al. (2002), Chapter 5 of Le Van and Dana (2003) for discrete time models, and Romer (1986) for a continuous time model.

²²When $f(0) = 0$, the function $f(S)$ can be interpreted as a gross return of investment S . However, in general, $f(S)$ can contain initial endowment and/or gross interest rate. This setting is general enough to cover concave-convex or convex-concave production functions. It will also be useful when we consider a non-differentiable function, for example, a function in which there is a threshold.

For $\epsilon \in [0, c_t)$, the path $(c'_t, S'_{t+1})_{t \geq 0}$ is feasible. Indeed, we have $c_s \geq 0, S_{s+1} \geq 0, \forall s \geq 0$. Since $c'_s = c_s, \forall s \notin \{t, t+1\}$ and $S'_s = S_s, \forall s \neq t+1$, we also have $c'_s + S'_{s+1} \leq f(S'_t), \forall s \notin \{t, t+1\}$. We now focus on the periods t and $t+1$. We have

$$\begin{aligned} c'_t + S'_{t+1} &= (c_t - \epsilon) + (S_{t+1} + \epsilon) = c_t + S_{t+1} \leq f(S_t) \\ c'_{t+1} + S'_{t+2} &= c_{t+1} + f(S_{t+1} + \epsilon) - f(S_{t+1}) + S_{t+2} \leq f(S_{t+1} + \epsilon) = f(S'_{t+1}) \end{aligned}$$

where the inequality in the second line comes from the fact that $c_{t+1} + S_{t+2} \leq f(S_{t+1})$.

Therefore, the path $(c'_t, S'_{t+1})_{t \geq 0}$ is feasible. By consequence, the optimality of the path $(c_s, S_{s+1})_{s \geq 0}$ implies that

$$\begin{aligned} u(c_t - \epsilon) + \beta u(c_{t+1} + f(S_{t+1} + \epsilon) - f(S_{t+1})) &\leq u(c_t) + \beta u(c_{t+1}) \\ \Leftrightarrow \beta \frac{u(c_{t+1} + f(S_{t+1} + \epsilon) - f(S_{t+1})) - u(c_{t+1})}{\epsilon} &\leq \frac{u(c_t) - u(c_t - \epsilon)}{\epsilon}. \end{aligned}$$

Let ϵ tend to 0, we obtain the right inequality of (A.3). By using the similar argument, we can prove the left inequality of (A.3). \square

Corollary 1. *If $S > 0$ is a steady state, then we have $\beta D^- f(S) \geq 1 \geq \beta D^+ f(S)$.*

Let us start our exposition with the following result which provides a condition under which the optimal capital path cannot converge to zero. The idea is that if productivity is high enough at original, then we will produce.

Proposition A 1. *Assume that there exists $\underline{x} > 0$ such that $\beta D^+ f(S) > 1$ for every $0 \leq S \leq \underline{x}$, then no optimal capital path converges to zero.*

Proof. Since $u'(0) = \infty$, we have $c_t > 0$ for every t , and so is S_t . By the Euler inequality, we get that $u'(c_t) \geq \beta u'(c_{t+1}) D^+ f(S_{t+1})$. According budget constraint, we have $\lim_{t \rightarrow +\infty} c_t = f(0)$.

Case 1: $f(0) = 0$. We have $\lim_{t \rightarrow +\infty} c_t = 0$. Since $\lim_{t \rightarrow +\infty} S_t = 0$, there exists t_0 such that $\beta D^+ f(S_{t+1}) > 1$ for every $t \geq t_0$. Consequently, $c_t \leq c_{t+1}$ for every $t \geq t_0$. Contradiction to the fact that $\lim_{t \rightarrow +\infty} c_t = 0$.

Case 2: $f(0) > 0$. We have $\lim_{t \rightarrow \infty} \frac{u'(c_t)}{u'(c_{t+1})} = 1$. Euler inequality implies that $\limsup_{t \rightarrow +\infty} \beta D^+ f(S_t) \leq 1$, which is a contradiction! \square

Proposition A 2. *Assume that there exist $x_0 \geq x_1 > 0$ and a function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ strictly increasing such that*

(i) $f(x) \leq g(x)$ for every $x \leq x_0$.

(ii) $x \leq g(x) \leq g(x_1) = x_1$ for every $x \leq x_1$, and $g(x) \leq x$ for every $x \geq x_1$.

Then, any optimal capital path of the problem (P_A) with the initial value S_0 , where $S_0 \leq x_1$, is bounded from above by x_1 .

Proof. We have $c_0 + S_1 \leq f(S_0) \leq f(x_1) \leq G(x_1) = x_1$. Thus, $S_1 \leq x_1$ and $f(S_1) \leq f(x_1) \leq x_1$. By induction argument, we get $f(S_t) \leq x_1$ for every t . \square

The intuition of Proposition A2 is the following: If the return function f is dominated by a function g whose the optimal capital path is bounded from above, then the optimal capital path associated with function f is also bounded from above. When $f(x) = g(x) = Ax^\alpha + (1 - \delta)x$, with δ is the depreciation rate, $A > 0$ is TFP, $\alpha \in (0, 1)$, we recover the standard Ramsey model.

Proposition A2 also complements Proposition 4.1 in [Kamihigashi and Roy \(2007\)](#) because our result covers the following function while Proposition 4.1 in [Kamihigashi and Roy \(2007\)](#) does not: $f(x) = Ax^\alpha$, if $x \leq a$ and $= (A + x - a)x^\alpha$ if $x \geq a$ where $A > 0$, $a \geq 0$ and $\alpha \in (0, 1)$.

We end this section by presenting a condition for unbounded growth with arbitrary initial capital stock S_0 . This result is a consequence of Proposition 4.6 in [Kamihigashi and Roy \(2007\)](#).

Proposition A 3. *Assume that $\beta D^+ f(x) > 1$ for every $x > 0$. Then every optimal capital path goes to infinity.*

B Appendix for Sections 2-3

Proof of Lemma 1. For a pedagogical reason, we present an elementary proof.

First, it is easy to see that F is strictly increasing and $F(0) = 0$.

Second, we consider the maximization problem: $\max_{pK_c + H \leq x, K_c \geq 0, H \geq 0} \{A_c K_c^\alpha + p(1 - \delta)K_c + w^* A_h H^{\alpha_h}\}$ with $x > 0$. Observe that the objective function $A_c K_c^\alpha + p(1 - \delta)K_c + w^* A_h H^{\alpha_h}$ is strictly concave, continuously differentiable, strictly increasing. Recall that $\alpha \in (0, 1)$ and $\alpha_h \in (0, 1)$. So, by using a standard argument, we can prove that: for $x > 0$ given, there is a unique pair (K_c, H) determined by the first-order condition $\alpha A_c K_c^{\alpha-1} + p(1 - \delta) = p\alpha_h w^* A_h H^{\alpha_h-1}$ and $pK_c + H = x$ such that $F(x) = A_c K_c^\alpha + p(1 - \delta)K_c + w^* A_h H^{\alpha_h}$.

By the Implicit Theorem, such H, K_c are differentiable in x . By consequence, the function F is differentiable. Moreover, we find that $F'(x) = \alpha_h w^* A_h H^{\alpha_h-1}$, $\forall x > 0$. Since the optimal value H is strictly increasing in x and $\alpha_h \in (0, 1)$, the function F' is strictly decreasing in x . By consequence, F is strictly concave. We can also see that $F'(0) = \infty$. \square

Proof of Proposition 1. The problem (P_1) can be rewritten as follows

$$(P'_1) : \max_{(c_t, S_{t+1})_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta^t u(c_t) \right] \text{ subject to } c_t + S_{t+1} \leq F(S_t), c_t, S_{t+1} \geq 0, \quad (\text{B.1})$$

where S_0 is given and the function F is defined by

$$F(S) \equiv \max_{K_c, H, L_e \geq 0} \{A_c K_c^\alpha + p(1 - \delta)K_c + w^* L_e\} \text{ subject to: } pK_c + H \leq S, L_e \leq A_h H^{\alpha_h}.$$

We see that F is continuous, strictly increasing (notice that if $\alpha = \alpha_h, \delta = 1$, then $F(S) = AS^\alpha$). According to Lemma 2, S_t is monotonic. Since $\alpha < 1$, $\alpha_h < 1$ and $(1 - \delta) < 1$, we can prove that S_t is bounded from above. Hence, as in the standard Ramsey model, there exists the limit $\lim_{t \rightarrow \infty} S_t \equiv S_b$. We now check that $S_b > S_a$. Indeed, we have $\beta F'(S_b) = 1 = \beta \left(\frac{\alpha A_c S_a^{\alpha-1}}{p^\alpha} + 1 - \delta \right)$. We observe that

$$F(S) \equiv \max_{K_c \geq 0, H \geq 0, pK_c + H \leq S} \{A_c K_c^\alpha + p(1 - \delta)K_c + w^* A_h H^{\alpha_h}\}$$

We can see that F is differentiable on $(0, \infty)$. Given $S > 0$, the FOCs of this maximization problem are $\alpha A_c K_c^{\alpha-1} + p(1-\delta) = \lambda p$ and $\lambda = \alpha_h w^* A_h H^{\alpha_h-1}$. So, λ is uniquely determined by

$$p \left(\frac{\alpha A_c}{p(\lambda + \delta - 1)} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\alpha_h w^* A_h}{\lambda} \right)^{\frac{1}{1-\alpha_h}} = S$$

We can also verify that $F'(S) = \lambda$. At the steady state, we have $\beta F'(S) = 1$ which implies that $\lambda = 1/\beta$. So, the steady state S_b is determined by

$$p \left(\frac{\alpha A_c}{p(\frac{1}{\beta} + \delta - 1)} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\alpha_h w^* A_h}{\frac{1}{\beta}} \right)^{\frac{1}{1-\alpha_h}} = S_b.$$

It is easy to see that $S_b > S_a$. □

Proof of Proposition 2. Point (i) Assume that $\frac{BA_e}{1+S} > 1$. If $L_e < A_h H^{\alpha_h}$, then consider $L'_e = L_e + \epsilon$ such that $L'_e < A_h H^{\alpha_h}$ and $L'_d \equiv (A_h H^{\alpha_h} - L'_e) + \frac{BA_e}{1+S} L'_e$. Then, $L'_d > L_d$, $L'_e > L_e$. Notice that $w \geq w^* > 0$. We see that the allocation $(K_c, K_d, L'_d, L'_e, H)$ strictly dominates the allocation (K_c, K_d, L_d, L_e, H) , which is a contradiction. Therefore, we have $L_e = A_h H^{\alpha_h}$.

Point (ii). We consider two cases.

If $\frac{BA_e}{1+S} \leq 1$, we have $L_d \leq A_h H^{\alpha_h} \leq A_h S^{\alpha_h} \leq \bar{L}$. By consequence, $Y_d = 0$.

If $\frac{BA_e}{1+S} \geq 1$, we have

$$\begin{aligned} L_d &\leq A_h H^{\alpha_h} - L_e + \frac{BA_e}{1+S} L_e = A_h H^{\alpha_h} + \left(\frac{BA_e}{1+S} - 1 \right) L_e \\ &\leq A_h H^{\alpha_h} + \left(\frac{BA_e}{1+S} - 1 \right) A_h H^{\alpha_h} = \frac{BA_e}{1+S} A_h H^{\alpha_h} \leq \frac{BA_e}{1+S} A_h S^{\alpha_h} \leq \bar{L}. \end{aligned}$$

This implies that $Y_d = 0$. □

Proof of Proposition 3. Notation: If $(K_c, K_d, L_d, L_e, H, L_e^D, K_e^D, w)$ is an equilibrium, we denote $Y_d \equiv A_d K_d^{\alpha_d} ((L_d - \bar{L})^+)^{1-\alpha_d}$ and $Y_e \equiv A_e K_e^{\alpha_e} L_e^{1-\alpha_e}$.

Step 1. We claim that there exists a threshold A_1 (depending on S) such that $Y_d > 0, \forall A_d > A_1$, for any equilibrium.

Given S , denote $G(S, w)$ the maximum value of the problem (G_S) associate to the wage w . Then, we define $G(S) \equiv \sup_w \{G(S, w) : w \text{ is in the set of all equilibrium wages}\}$.

Now, consider an equilibrium $(K_c, K_d, L_d, L_e, H, L_e^D, K_e^D, w)$.

If $Y_d = 0$, then $G(S)$ does not depend on A_d . Indeed, consider an equilibrium where $Y_d = 0$. There are two cases. First, if $L_e^D = 0$, then the market clearing condition implies that $L_e = 0$. So, $G(S, w)$ does not depend on w and then on A_d . Second, if $L_e^D > 0$, then $w = w^*$. In this case, we also have that $G(S, w) = G(S, w^*)$ which does not depend on A_d .

Since $A_h S^{\alpha_h} > \bar{L}$, we can find s so that $s < S$ and $A_h s^{\alpha_h} > \bar{L}$. Consequently, we can choose H and K_d so that $0 < H < s$, $A_h H^{\alpha_h} > \bar{L}$ and $K_d = (s - H)/p$. Then we choose $L_d = A_h H^{\alpha_h}$. With these choices, we can see that $L_d > \bar{L}$ and $G(S)$ tends continuously to infinity when A_d tends to infinity. This contradicts the optimality of allocation.

Therefore, there exists a threshold A_1 such that $Y_d > 0$ for every $A_d > A_1$, for any equilibrium.

Step 2. We prove that there exists \bar{A}_d (depending on S) such that: for every $A_d \geq \bar{A}_d$, we have $A_d K_d^{\alpha_d} ((L_d - \bar{L})^+)^{1-\alpha_d} > 0$ and $A_e K_e^{\alpha_e} L_e^{1-\alpha_e} = 0$ for any equilibrium $(K_c, K_d, L_d, L_e, H, L_e^D, K_e^D, w)$.

We consider A_d with $A_d > A_1$ (where A_1 is in Step 1). In this case, we have $Y_d > 0$ for any equilibrium. Suppose that there exists an equilibrium with $Y_e > 0$. We can compute that

$$w = w^* \equiv \left(\alpha_e^{\alpha_e} (1 - \alpha_e)^{1 - \alpha_e} \frac{p_n A_e}{(\delta_e p)^{\alpha_e}} \right)^{\frac{1}{1 - \alpha_e}}. \quad (\text{B.2})$$

Denote $\lambda, \lambda_1, \lambda_2$ Lagrange multipliers associated to constraints (16c), (16d), (16e), and $L_e \geq 0$ respectively. We have

$$K_c : \quad \alpha A_c K_c^{\alpha-1} + p(1 - \delta) = \lambda p \quad (\text{B.3})$$

$$K_d : \quad \alpha_d p_n A_d K_d^{\alpha_d-1} (L_d - \bar{L})^{1 - \alpha_d} + p(1 - \delta_d) = \lambda p \quad (\text{B.4})$$

$$L_d : \quad (1 - \alpha_d) p_n A_d K_d^{\alpha_d} (L_d - \bar{L})^{-\alpha_d} = \lambda_2 \quad (\text{B.5})$$

$$H : \quad \lambda = (\lambda_1 + \lambda_2) \alpha_h A_h H^{\alpha_h-1} \quad (\text{B.6})$$

$$L_e : \quad w = \lambda_1 + \left(1 - \frac{B A_e}{1 + S} \right) \lambda_2 \geq \left(1 - \frac{B A_e}{1 + S} \right) \lambda_2. \quad (\text{B.7})$$

From the FOC with respect to L_d , we can express $\frac{L_d - \bar{L}}{K_d} = \left(\frac{(1 - \alpha_d) p_n A_d}{\lambda_2} \right)^{\frac{1}{\alpha_d}}$. Then, we substitute this in the FOC with respect to K_d , and notice that $p(1 - \delta_d) \geq 0$, to get that

$$\frac{p_n A_d}{p^{\alpha_d}} \alpha_d^{\alpha_d} (1 - \alpha_d)^{1 - \alpha_d} \leq \lambda^{\alpha_d} \lambda_2^{1 - \alpha_d} \leq \lambda^{\alpha_d} \left(\frac{w}{1 - \frac{B A_e}{1 + S}} \right)^{1 - \alpha_d}$$

where the last inequality is from (B.7). Combining this with (B.2), we obtain

$$\lambda^{\alpha_d} \geq \left(1 - \frac{B A_e}{1 + S} \right)^{1 - \alpha_d} \frac{p_n A_d}{p^{\alpha_d}} \alpha_d^{\alpha_d} (1 - \alpha_d)^{1 - \alpha_d} \left(\alpha_e^{\alpha_e} (1 - \alpha_e)^{1 - \alpha_e} \frac{p_n A_e}{(\delta_e p)^{\alpha_e}} \right)^{\frac{-(1 - \alpha_d)}{1 - \alpha_e}}.$$

This implies that $\lambda \geq D_1$, where

$$D_1 \equiv \left[\left(1 - \frac{B A_e}{1 + S} \right)^{1 - \alpha_d} \frac{p_n A_d}{p^{\alpha_d}} \alpha_d^{\alpha_d} (1 - \alpha_d)^{1 - \alpha_d} \left(\alpha_e^{\alpha_e} (1 - \alpha_e)^{1 - \alpha_e} \frac{p_n A_e}{(\delta_e p)^{\alpha_e}} \right)^{\frac{-(1 - \alpha_d)}{1 - \alpha_e}} \right]^{\frac{1}{\alpha_d}}.$$

Observe that D_1 does not depend on S .

Condition (B.7) implies that $w \geq (\lambda_1 + \lambda_2) \left(1 - \frac{B A_e}{1 + S} \right)$. Therefore, condition (B.6) and condition $\lambda \geq D$ imply that

$$H^{1 - \alpha_h} = \alpha_h A_h \frac{\lambda_1 + \lambda_2}{\lambda} \leq \alpha_h A_h \frac{w}{1 - \frac{B A_e}{1 + S}} \frac{1}{D} \leq \alpha_h A_h \frac{w}{1 - B A_e} \frac{1}{D_1}$$

In other words, we have $H \leq \bar{H}$, where \bar{H} is defined by $\bar{H} \equiv \left(\alpha_h A_h \frac{w}{1 - B A_e} \frac{1}{D_1} \right)^{\frac{1}{1 - \alpha_h}}$.

Since $A_h H^{\alpha_h} > \bar{L}$, we have $\bar{L} < A_h (\bar{H})^{\alpha_h}$. Observe that \bar{H} depends on A_d . We write $\bar{H} = \bar{H}(A_d)$. We define the threshold \bar{A} by $\bar{L} = A_h (\bar{H}(\bar{A}))^{\alpha_h}$. Notice that \bar{A} does not depend on S .

If we choose $A_d \geq \max(A_1, \bar{A})$, then $\bar{L} < L_d \leq A_h H^{\alpha_h} \leq A_h \bar{H}^{\alpha_h}$, which is a contradiction to $\bar{L} = A_h (\bar{H})^{\alpha_h}$. By consequence, we get that $Y_e = 0$. \square

Remark 1. According to the definition of \bar{A} , we get the following result: If $A > \bar{A}$, then there does not exist an equilibrium with $Y_d > 0$ and $Y_e > 0$.

Proof of Proposition 4. Part 1. Let $(c_t, K_{c,t+1}, K_{d,t+1}, L_{d,t+1}, L_{e,t+1}, H_{t+1}, L_{e,t}^D, K_{e,t}^D, w_t)_{t=0}^\infty$ be an equilibrium with $w_t = w^*, \forall t \geq 0$.

Then $(c_t, S_{t+1})_{t \geq 0}$ is a solution to the following problem

$$\max_{(c_t, S_{t+1})_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta^t u(c_t) \right] \text{ subject to } c_t + S_{t+1} \leq G(S_t), c_t \geq 0, S_t \geq 0 \quad (\text{B.8})$$

where the function G is defined by

$$G(S) \equiv \max \left\{ A_c K_c^\alpha + w^* L_e + p_n A_d K_d^{\alpha_d} ((L_d - \bar{L})^+)^{1-\alpha_d} + p(1-\delta)K_c + p(1-\delta_d)K_d \right\} \quad (\text{B.9})$$

subject to $(K_c, K_d, L_d, L_e, H) \in \mathbb{G}$, where the set \mathbb{G} is defined by

$$\mathbb{G} \equiv \left\{ (K_c, K_d, L_d, L_e, H) \in \mathbb{R}_+^5 : p(K_c + K_d) + H \leq S, \quad L_e \leq A_h H^{\alpha_h}, \right. \\ \left. L_d \leq (A_h H^{\alpha_h} - L_e) + \frac{B A_e}{1+S} L_e \right\}.$$

We have $G(0) = 0$. Since the function G is strictly increasing, the optimal path (S_t) is monotonic. Moreover, we have

$$G(S) \leq \frac{A_c}{p^\alpha} S^\alpha + w^* A_h S^{\alpha_h} + \frac{p_n A_d A_h^{1-\alpha_d}}{p^\alpha} S^{\alpha_d} S^{\alpha_h(1-\alpha_d)} + (1-\bar{\delta})S \\ \leq \max \left(\frac{A_c}{p^\alpha}, w^* A_h, \frac{p_n A_d A_h^{1-\alpha_d}}{p^\alpha} \right) (S^\alpha + S^{\alpha_h} + S^{\alpha_h+\alpha_d(1-\alpha_h)}) + (1-\bar{\delta})S.$$

where $\bar{\delta} \equiv \min(\delta_c, \delta_d)$. Define the real function G_2 by $G_2(S) \equiv \max \left(\frac{A_c}{p^\alpha}, w^* A_h, \frac{p_n A_d A_h^{1-\alpha_d}}{p^\alpha} \right) (S^\alpha + S^{\alpha_h} + S^{\alpha_h+\alpha_d(1-\alpha_h)}) + (1-\bar{\delta})S$. Since $\alpha_d < 1$, we have $\alpha_h + \alpha_d(1-\alpha_h) < \alpha_h + (1-\alpha_h) = 1$. So, $\lim_{S \rightarrow \infty} G_2'(S) = 1 - \bar{\delta} < 1$. By consequence, there exists $x_1 > 0$ such that $x \leq F_2(x) \leq F_2(x_1) = x_1$ for every $x \leq x_1$, and $F_2(x) \leq x$ for every $x \geq x_1$. So, we can apply Proposition A2 to get that the sequence S_t is bounded from above.

Moreover, Lemma 2 and the assumption $X_0 < S_a$ imply that S_t is increasing in t . Hence, the sequence S_t increasingly converges and hence $(c_t, K_{c,t}, K_{d,t}, H_t)$ converges.

Part 2. Consider an equilibrium. Assume that S_t converges to zero. Then $H_t, L_{e,t}, L_{d,t}$ converge to zero. By consequence, there exists a date t_0 such that $L_{d,t} < \bar{L}, \forall t \geq t_0$. In this case, we have $L_{e,t} = A_h H_t^{\alpha_h} > 0, \forall t \geq t_0$. So, the wage equals $w_t = w^*, \forall t \geq t_0$. By consequence, the sequence $(c_t, S_{t+1})_{t \geq 0}$ is a solution to the following problem

$$\max_{(c_t, S_{t+1})_{t \geq 0}} \left[\sum_{t=t_0+1}^{+\infty} \beta^t u(c_t) \right] \text{ subject to } c_t + S_{t+1} \leq G_1(S_t) \quad (\text{B.10})$$

where G_0 is defined by

$$(G_S) : G_0(S) = \max_{K_c, L_e, H \geq 0} A_c K_c^\alpha + p(1-\delta)K_c + w^* A_h H^{\alpha_h} \\ \text{subject to: } pK_c + H \leq S.$$

By consequence, we can apply Proposition A1 in Appendix A to get that S_t does not converge to zero, which is a contraction. \square

Proof of Proposition 5. We assume that $\max(BA_e, 1)A_h\bar{S}^{\alpha_h} \leq \bar{L}$.

As in the proof of Proposition 2, we have, for every t ,

$$L_{d,t} \leq \max\left(\frac{BA_e}{1+S_t}, 1\right)A_hS_t^{\alpha_h} \leq \max(BA_e, 1)A_hS_t^{\alpha_h}$$

Recall that $S_t \leq x_t \leq \bar{S}$, $\forall t \geq 1$. So, we have $L_{d,t} \leq \max(BA_e, 1)A_h\bar{S}^{\alpha_h} \leq \bar{L}$, $\forall t \geq 1$. This implies that $F_1^d(K_{d,1}, L_{d,1}) = A_dK_{d,1}^{\alpha_d}((L_{d,1} - \bar{L})^+)^{1-\alpha_d}$. As $L_{d,1} \leq \bar{L}$, we get that $Y_{d,1} = F_1^d(K_{d,1}, L_{d,1}) = 0$. By induction, we obtain that $Y_{d,t} = 0$, $\forall t \geq 1$. \square

Proof of Proposition 6. Suppose that for any A_d , there exists an equilibrium

$$(c_t, K_{c,t+1}, K_{d,t+1}, L_{d,t+1}, L_{e,t+1}, H_{t+1}, L_{e,t}^D, K_{e,t}^D, w_t)_{t=0}^\infty$$

such that $Y_{d,t} \equiv F^d(K_{d,t}, L_{d,t}) = 0$ for every t .

Denote W the welfare of the host country in this equilibrium. Observe that W is uniformly bounded from above for any equilibrium satisfying $Y_{d,t} \equiv F^d(K_{d,t}, L_{d,t}) = 0$, $\forall t \geq 1$. Moreover, since $Y_{d,t} = 0$ for every t , the welfare does not depend on A_d and we have that $\lim_{t \rightarrow \infty} S_t = S_b$. Since $\max\left(\frac{BA_e}{1+S_b}, 1\right)A_hS_b^{\alpha_h} > \bar{L}$, there exists t such that $\max\left(\frac{BA_e}{1+S_t}, 1\right)A_hS_t^{\alpha_h} > \bar{L}$. We consider two cases.

1. If $\frac{BA_e}{1+S_t} < 1$. Let $L'_{e,t} = 0$ and $L'_{d,t} = A_h(H'_t)^{\alpha_h}$. Choose H'_t closed to S_t such that $\max\left(\frac{BA_e}{1+S_t}, 1\right)A_h(H'_t)^{\alpha_h} > \bar{L}$. Let A_d be high enough, the new welfare of the country (associated to this new allocation) will be higher than W . This violates the optimality of the country's choice.
2. If $\frac{BA_e}{1+S_t} \geq 1$. Let $L'_{e,t} = A_hH_t^{\alpha_h}$ and $L'_{d,t} = \frac{BA_e}{1+S_t}A_hH_t^{\alpha_h}$. Choose H'_t is closed to S_t such that $\max\left(\frac{BA_e}{1+S_t}, 1\right)A_h(H'_t)^{\alpha_h} > \bar{L}$. Let A_d be high enough, the new welfare of the country will be higher than W . This violates the optimality of the country's choice.

The above arguments imply that there exists \bar{A}_d such that: for any $A_d > \bar{A}_d$, for any equilibrium (which depends on parameter A_d), there is a date t_0 such that $Y_{d,t_0} > 0$. Then t_d will be determined by $t_d = \inf\{t_0 \text{ is integer such that } Y_{d,t_0} > 0\}$.

It is easy to see that $t_d > 1$ because $\max\left(\frac{BA_e}{1+S_0}, 1\right)A_hX_0^{\alpha_h} < \bar{L}$ which implies that $Y_{d,1} = 0$.

Remark 2. *There exists A_d^* such that for any equilibrium, there exists a date t_d such that $Y_{d,t} = 0$, $Y_{e,t} > 0$ for any $t < t_d$ and $Y_{d,t_d} > 0$ and $Y_{e,t_d} = 0$.*

Proof of Remark 2. Let $A_d > \max(\bar{A}, \bar{A}_d)$, where \bar{A} is defined in the proof of Proposition 3 and \bar{A}_d is defined in the proof of Proposition 6. Then, for any equilibrium, there exists a date t_d such that $Y_{d,t_d} > 0$ and $Y_{e,t_d} = 0$. Indeed, let t_d be defined as in the proof of Proposition 6. It is obvious that $Y_{d,t_d} > 0$. According to the proof of Proposition 3, condition $A_d > \bar{A}$ implies that we cannot have $Y_{d,t_d} > 0$ and $Y_{e,t_d} > 0$ at the same time. By consequence, we get that $Y_{e,t_d} = 0$. \square

\square

Proof of Proposition 7. Our proof consists of different steps. Let us start by the following result.

Lemma 4. A positive allocation (K_c, K_d, L_d, L_e, H) with $L_d > \bar{L}$ is a solution to the maximization problem (B.9) if there are positive real values $\lambda, \lambda_1, \lambda_2, \lambda_e$,²³ satisfying the following conditions:

$$(K_c) : \quad \alpha A_c K_c^{\alpha-1} + p(1 - \delta) = \lambda p \quad (\text{B.12})$$

$$(K_d) : \quad \alpha_d p_n A_d K_d^{\alpha_d-1} (L_d - \bar{L})^{1-\alpha_d} + p(1 - \delta_d) = \lambda p \quad (\text{B.13})$$

$$(L_d) : \quad (1 - \alpha_d) p_n A_d K_d^{\alpha_d} (L_d - \bar{L})^{-\alpha_d} = \lambda_2 \quad (\text{B.14})$$

$$(H) : \quad \lambda = (\lambda_1 + \lambda_2) \alpha_h A_h H^{\alpha_h-1}, \quad \lambda_1 (A_h H^{\alpha_h} - L_e) = 0 \quad (\text{B.15})$$

$$(L_e) : \quad w^* + \lambda_e = \lambda_1 + \left(1 - \frac{B A_e}{1 + S}\right) \lambda_2, \quad \lambda_e L_e = 0, \quad \lambda_e \geq 0 \quad (\text{B.16})$$

$$p(K_c + K_d) + H = S \quad (\text{B.17})$$

$$L_d = (A_h H^{\alpha_h} - L_e) + \frac{B A_e}{1 + S} L_e \quad (\text{B.18})$$

and $A_c K_c^\alpha + w^* L_e + p_n A_d K_d^{\alpha_d} ((L_d - \bar{L})^+)^{1-\alpha_d} + p(1 - \delta) K_c + p(1 - \delta_d) K_d > F(S)$, where recall that the function F is defined by (4).

Proof. Denote $g(K_c, K_d, L_d, L_e, H) \equiv A_c K_c^\alpha + w^* L_e + p_n A_d K_d^{\alpha_d} ((L_d - \bar{L})^+)^{1-\alpha_d} + p(1 - \delta) K_c + p(1 - \delta_d) K_d$. The result stated in Lemma 4 is a direct consequence of the following observations.

$$\begin{aligned} G(S) &= \max(G_a(S), G_b(S)), \text{ where} \\ G_a(S) &\equiv \{g(K_c, K_d, L_d, L_e, H) : (K_c, K_d, L_d, L_e, H) \in \mathbb{G} \text{ and } L_d \leq \bar{L}, \} \\ G_b(S) &\equiv \{g(K_c, K_d, L_d, L_e, H) : (K_c, K_d, L_d, L_e, H) \in \mathbb{G} \text{ and } L_d \geq \bar{L}, \}. \end{aligned} \quad (\text{B.19})$$

Note that $G_a(S) = F(S)$. □

We then define real functions $K_c(\cdot), H(\cdot), K_d(\cdot), L_d(\cdot), f_s(\cdot)$ from $[\max(1 - \delta, 1 - \delta_d), +\infty)$ to \mathbb{R} by the following:

$$\begin{aligned} K_c(x) &\equiv \left(\frac{\alpha A_c}{p(x + \delta - 1)} \right)^{\frac{1}{1-\alpha}} \\ H(x) &\equiv \left(\frac{\alpha_d^{\alpha_d} (1 - \alpha_d)^{1-\alpha_d} p_n A_d (\alpha_h A_h)^{1-\alpha_d}}{p^{\alpha_d} (x + \delta_d - 1)^{\alpha_d} x^{1-\alpha_d}} \right)^{\frac{1}{(1-\alpha_h)(1-\alpha_d)}} \\ K_d(x) &\equiv \left(\frac{\alpha_d p_n A_d}{p(x + \delta_d - 1)} \right)^{\frac{1}{1-\alpha_d}} \left(A_h \left(\frac{\alpha_d^{\alpha_d} (1 - \alpha_d)^{1-\alpha_d} p_n A_d (\alpha_h A_h)^{1-\alpha_d}}{p^{\alpha_d} (x + \delta_d - 1)^{\alpha_d} x^{1-\alpha_d}} \right)^{\frac{\alpha_h}{(1-\alpha_h)(1-\alpha_d)}} - \bar{L} \right) \\ L_d(x) &\equiv A_h (H(x))^{\alpha_h} \\ f_s(x) &\equiv p(K_c(x) + K_d(x)) + H(x). \end{aligned}$$

Denote

$$S_s \equiv f_s(\beta). \quad (\text{B.20})$$

For A_d high enough, we see that $K_d(\beta)$ and $L_d(\beta)$ are strictly positive.

Lemma 5. For A_d high enough, we have $G_b(S_s) = g(K_c(\beta), K_d(\beta), L_d(\beta), 0, H(\beta))$.

²³ $\lambda, \lambda_1, \lambda_2, \lambda_e$ are indeed the Lagrange multipliers associated to constraints (16c), (16d), (16e) and $L_e \geq 0$ respectively.

Proof. It is sufficient to check all conditions in Lemma 4. It is clear, by construction, that $S_s = p(K_c(\beta) + K_d(\beta)) + H(\beta)$ and $L_d(\beta) = A_h(H(\beta))^{\alpha_h}$. We now define the Lagrangian multipliers. We choose $\lambda_1 = 0$, and non-negative multipliers $\lambda, \lambda_2, \lambda_e$ by

$$\alpha A_c(K_c(\beta))^{\alpha-1} + p(1 - \delta) = \lambda p \quad (\text{B.21})$$

$$\lambda = \lambda_2 \alpha_h A_h(H(\beta))^{\alpha_h-1} \quad (\text{B.22})$$

$$w^* + \lambda_e = \left(1 - \frac{BA_e}{1 + S_s}\right) \lambda_2. \quad (\text{B.23})$$

Observe, by construction, that

$$\begin{aligned} \lambda &= \frac{1}{\beta} \\ \lambda_2 &= \lambda \frac{(H(\beta))^{1-\alpha_h}}{\alpha_h A_h} = \frac{1}{\beta \alpha_h A_h} \left(\frac{\alpha_d^{\alpha_d} (1 - \alpha_d)^{1-\alpha_d} p_n A_d (\alpha_h A_h)^{1-\alpha_d}}{p^{\alpha_d} (\frac{1}{\beta} + \delta_d - 1)^{\alpha_d} (\frac{1}{\beta})^{1-\alpha_d}} \right)^{\frac{1}{(1-\alpha_d)}} \\ &= \left(\frac{\alpha_d^{\alpha_d} (1 - \alpha_d)^{1-\alpha_d} p_n A_d}{p^{\alpha_d} (\frac{1}{\beta} + \delta_d - 1)^{\alpha_d}} \right)^{\frac{1}{(1-\alpha_d)}}. \end{aligned}$$

Notice that we can choose A_d high enough so that $(1 - BA_e)\lambda_2 > w^*$. So, $\lambda_e = \left(1 - \frac{BA_e}{1 + S_s}\right) \lambda_2 - w^* \geq (1 - BA_e)\lambda_2 - w^* > 0$. \square

Lemma 6. For A_d high enough, we have $G(S_s) = G_b(S_s) > F(S_s)$.

Proof. As in the proof of Proposition 1, $F(S_s) = A_c K_c^\alpha + p(1 - \delta)K_c + w^* A_h H^{\alpha_h}$. where $K_c = \left(\frac{\alpha A_c}{p(\mu + \delta - 1)}\right)^{\frac{1}{1-\alpha}}$, $H = \left(\frac{\alpha_h w^* A_h}{\mu}\right)^{\frac{1}{1-\alpha_h}}$, and μ is uniquely defined by

$$p \left(\frac{\alpha A_c}{p(\mu + \delta - 1)} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\alpha_h w^* A_h}{\mu} \right)^{\frac{1}{1-\alpha_h}} = S_s.$$

So, we have

$$\begin{aligned} S_s &= p \left(\frac{\alpha A_c}{p(\mu + \delta - 1)} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\alpha_h w^* A_h}{\mu} \right)^{\frac{1}{1-\alpha_h}} \\ &= p \left(\frac{\alpha A_c}{p(\frac{1}{\beta} + \delta - 1)} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\alpha_d^{\alpha_d} (1 - \alpha_d)^{1-\alpha_d} p_n A_d (\alpha_h A_h)^{1-\alpha_d}}{p^{\alpha_d} (\frac{1}{\beta} + \delta_d - 1)^{\alpha_d} (\frac{1}{\beta})^{1-\alpha_d}} \right)^{\frac{1}{(1-\alpha_h)(1-\alpha_d)}} \\ &\quad + p \left(\frac{\alpha_d p_n A_d}{p(\frac{1}{\beta} + \delta_d - 1)} \right)^{\frac{1}{1-\alpha_d}} \left(A_h \left(\frac{\alpha_d^{\alpha_d} (1 - \alpha_d)^{1-\alpha_d} p_n A_d (\alpha_h A_h)^{1-\alpha_d}}{p^{\alpha_d} (\frac{1}{\beta} + \delta_d - 1)^{\alpha_d} (\frac{1}{\beta})^{1-\alpha_d}} \right)^{\frac{\alpha_h}{(1-\alpha_h)(1-\alpha_d)}} - \bar{L} \right) \end{aligned} \quad (\text{B.24})$$

We have

$$\begin{aligned} F(S_s) &= A_c \left(\frac{\alpha A_c}{p(\mu + \delta - 1)} \right)^{\frac{\alpha}{1-\alpha}} + p(1 - \delta) \left(\frac{\alpha A_c}{p(\mu + \delta - 1)} \right)^{\frac{1}{1-\alpha}} + A_h \left(\frac{\alpha_h w^* A_h}{\mu} \right)^{\frac{\alpha_h}{1-\alpha_h}} \\ G_b(S_s) &= A_c (K_c(\beta))^\alpha + p(1 - \delta) K_c(\beta) + p_n A_d (K_d(\beta))^{\alpha_d} ((L_d(\beta) - \bar{L})^+)^{1-\alpha_d} + p(1 - \delta_d) K_d(\beta) \\ &= A_c \left(\frac{\alpha A_c}{p(\frac{1}{\beta} + \delta - 1)} \right)^{\frac{\alpha}{1-\alpha}} + p(1 - \delta) \left(\frac{\alpha A_c}{p(\frac{1}{\beta} + \delta - 1)} \right)^{\frac{1}{1-\alpha}} \\ &\quad + p_n A_d \left(\frac{\alpha_d p_n A_d}{p(\frac{1}{\beta} + \delta_d - 1)} \right)^{\frac{\alpha_d}{1-\alpha_d}} \left(A_h \left(\frac{\alpha_d^{\alpha_d} (1 - \alpha_d)^{1-\alpha_d} p_n A_d (\alpha_h A_h)^{1-\alpha_d}}{p^{\alpha_d} (\frac{1}{\beta} + \delta_d - 1)^{\alpha_d} (\frac{1}{\beta})^{1-\alpha_d}} \right)^{\frac{\alpha_h}{(1-\alpha_h)(1-\alpha_d)}} - \bar{L} \right) \\ &\quad + p(1 - \delta_d) \left(\frac{\alpha_d p_n A_d}{p(\frac{1}{\beta} + \delta_d - 1)} \right)^{\frac{1}{1-\alpha_d}} \left(A_h \left(\frac{\alpha_d^{\alpha_d} (1 - \alpha_d)^{1-\alpha_d} p_n A_d (\alpha_h A_h)^{1-\alpha_d}}{p^{\alpha_d} (\frac{1}{\beta} + \delta_d - 1)^{\alpha_d} (\frac{1}{\beta})^{1-\alpha_d}} \right)^{\frac{\alpha_h}{(1-\alpha_h)(1-\alpha_d)}} - \bar{L} \right) \end{aligned}$$

Next, we find

$$\begin{aligned} \lim_{A_d \rightarrow \infty} \frac{G_b(S_s)}{A_d^{\frac{1}{(1-\alpha_d)(1-\alpha_h)}}} &= p_n \left(\frac{\alpha_d p_n}{p(\frac{1}{\beta} + \delta_d - 1)} \right)^{\frac{\alpha_d}{1-\alpha_d}} A_h \left(\frac{\alpha_d^{\alpha_d} (1-\alpha_d)^{1-\alpha_d} p_n (\alpha_h A_h)^{1-\alpha_d}}{p^{\alpha_d} (\frac{1}{\beta} + \delta_d - 1)^{\alpha_d} (\frac{1}{\beta})^{1-\alpha_d}} \right)^{\frac{\alpha_h}{(1-\alpha_h)(1-\alpha_d)}} \\ &\quad + p(1-\delta_d) \left(\frac{\alpha_d p_n}{p(\frac{1}{\beta} + \delta_d - 1)} \right)^{\frac{1}{1-\alpha_d}} A_h \left(\frac{\alpha_d^{\alpha_d} (1-\alpha_d)^{1-\alpha_d} p_n (\alpha_h A_h)^{1-\alpha_d}}{p^{\alpha_d} (\frac{1}{\beta} + \delta_d - 1)^{\alpha_d} (\frac{1}{\beta})^{1-\alpha_d}} \right)^{\frac{\alpha_h}{(1-\alpha_h)(1-\alpha_d)}}. \end{aligned}$$

Denote

$$M \equiv \left(\frac{\alpha_d^{\alpha_d} (1-\alpha_d)^{1-\alpha_d} p_n (\alpha_h)^{1-\alpha_d}}{p^{\alpha_d} (\frac{1}{\beta} + \delta_d - 1)^{\alpha_d} (\frac{1}{\beta})^{1-\alpha_d}} \right)^{\frac{1}{(1-\alpha_h)(1-\alpha_d)}}.$$

We have

$$\begin{aligned} \lim_{A_d \rightarrow \infty} \frac{G_b(S_s)}{A_d^{\frac{1}{(1-\alpha_d)(1-\alpha_h)}}} &= A_h^{\frac{1}{1-\alpha_h}} M^{\alpha_h} \left[p_n \left(\frac{\alpha_d p_n}{p(\frac{1}{\beta} + \delta_d - 1)} \right)^{\frac{\alpha_d}{1-\alpha_d}} + p(1-\delta_d) \left(\frac{\alpha_d p_n}{p(\frac{1}{\beta} + \delta_d - 1)} \right)^{\frac{1}{1-\alpha_d}} \right] \\ &= A_h^{\frac{1}{1-\alpha_h}} M^{\alpha_h} \frac{p_n^{\frac{1}{1-\alpha_d}}}{p^{\frac{\alpha_d}{1-\alpha_d}}} \left[\left(\frac{\alpha_d}{\frac{1}{\beta} + \delta_d - 1} \right)^{\frac{\alpha_d}{1-\alpha_d}} + (1-\delta_d) \left(\frac{\alpha_d}{\frac{1}{\beta} + \delta_d - 1} \right)^{\frac{1}{1-\alpha_d}} \right] \end{aligned}$$

and

$$\begin{aligned} \lim_{A_d \rightarrow \infty} \frac{S_s}{A_d^{\frac{1}{(1-\alpha_d)(1-\alpha_h)}}} &= \left(\frac{\alpha_d^{\alpha_d} (1-\alpha_d)^{1-\alpha_d} p_n (\alpha_h A_h)^{1-\alpha_d}}{p^{\alpha_d} (\frac{1}{\beta} + \delta_d - 1)^{\alpha_d} (\frac{1}{\beta})^{1-\alpha_d}} \right)^{\frac{1}{(1-\alpha_h)(1-\alpha_d)}} \\ &\quad + p \left(\frac{\alpha_d p_n}{p(\frac{1}{\beta} + \delta_d - 1)} \right)^{\frac{1}{1-\alpha_d}} A_h \left(\frac{\alpha_d^{\alpha_d} (1-\alpha_d)^{1-\alpha_d} p_n (\alpha_h A_h)^{1-\alpha_d}}{p^{\alpha_d} (\frac{1}{\beta} + \delta_d - 1)^{\alpha_d} (\frac{1}{\beta})^{1-\alpha_d}} \right)^{\frac{\alpha_h}{(1-\alpha_h)(1-\alpha_d)}} \\ &= A_h^{\frac{1}{1-\alpha_h}} M^{\alpha_h} \left[M^{1-\alpha_h} + p \left(\frac{\alpha_d p_n}{p(\frac{1}{\beta} + \delta_d - 1)} \right)^{\frac{1}{1-\alpha_d}} \right] \end{aligned}$$

We now find

$$\begin{aligned} \lim_{A_d \rightarrow \infty} \frac{G_b(S_s)}{F(S_s)} &= \frac{\lim_{A_d \rightarrow \infty} \frac{G_b(S_s)}{A_d^{\frac{1}{(1-\alpha_d)(1-\alpha_h)}}}}{\lim_{A_d \rightarrow \infty} \frac{F(S_s)}{S_s} \lim_{A_d \rightarrow \infty} \frac{S_s}{A_d^{\frac{1}{(1-\alpha_d)(1-\alpha_h)}}}} \\ &= \frac{1}{\lim_{A_d \rightarrow \infty} \frac{F(S_s)}{S_s}} \frac{\frac{p_n^{\frac{1}{1-\alpha_d}}}{p^{\frac{\alpha_d}{1-\alpha_d}}} \left[\left(\frac{\alpha_d}{\frac{1}{\beta} + \delta_d - 1} \right)^{\frac{\alpha_d}{1-\alpha_d}} + (1-\delta_d) \left(\frac{\alpha_d}{\frac{1}{\beta} + \delta_d - 1} \right)^{\frac{1}{1-\alpha_d}} \right]}{\left[M^{1-\alpha_h} + p \left(\frac{\alpha_d p_n}{p(\frac{1}{\beta} + \delta_d - 1)} \right)^{\frac{1}{1-\alpha_d}} \right]} \\ &= \frac{1}{\lim_{A_d \rightarrow \infty} \frac{F(S_s)}{S_s}} \frac{\frac{p_n^{\frac{1}{1-\alpha_d}}}{p^{\frac{\alpha_d}{1-\alpha_d}}} \left[\left(\frac{\alpha_d}{\frac{1}{\beta} + \delta_d - 1} \right)^{\frac{\alpha_d}{1-\alpha_d}} + (1-\delta_d) \left(\frac{\alpha_d}{\frac{1}{\beta} + \delta_d - 1} \right)^{\frac{1}{1-\alpha_d}} \right]}{\left[\left(\frac{\alpha_d^{\alpha_d} (1-\alpha_d)^{1-\alpha_d} p_n (\alpha_h)^{1-\alpha_d}}{p^{\alpha_d} (\frac{1}{\beta} + \delta_d - 1)^{\alpha_d} (\frac{1}{\beta})^{1-\alpha_d}} \right)^{\frac{1}{1-\alpha_d}} + p \left(\frac{\alpha_d p_n}{p(\frac{1}{\beta} + \delta_d - 1)} \right)^{\frac{1}{1-\alpha_d}} \right]} \\ &= \frac{1}{\lim_{A_d \rightarrow \infty} \frac{F(S_s)}{S_s}} \frac{\left(\frac{\alpha_d}{\frac{1}{\beta} + \delta_d - 1} \right)^{\frac{\alpha_d}{1-\alpha_d}} + (1-\delta_d) \left(\frac{\alpha_d}{\frac{1}{\beta} + \delta_d - 1} \right)^{\frac{1}{1-\alpha_d}}}{\left(\frac{\alpha_d^{\alpha_d} (1-\alpha_d)^{1-\alpha_d} \alpha_h^{1-\alpha_d}}{(\frac{1}{\beta} + \delta_d - 1)^{\alpha_d} (\frac{1}{\beta})^{1-\alpha_d}} \right)^{\frac{1}{1-\alpha_d}} + \left(\frac{\alpha_d}{\frac{1}{\beta} + \delta_d - 1} \right)^{\frac{1}{1-\alpha_d}}} \\ &= \frac{1}{\lim_{A_d \rightarrow \infty} \frac{F(S_s)}{S_s}} \frac{1 + (1-\delta_d) \left(\frac{\alpha_d}{\frac{1}{\beta} + \delta_d - 1} \right)}{(1-\alpha_d) \alpha_h \beta + \frac{\alpha_d}{\frac{1}{\beta} + \delta_d - 1}}. \end{aligned}$$

We will prove that $\lim_{A_d \rightarrow \infty} \frac{G_b(S_s)}{F(S_s)}$ is strictly higher than 1.

Let us denote $M_1(\delta_d) \equiv \frac{1+(1-\delta_d)\left(\frac{\alpha_d}{\frac{1}{\beta}+\delta_d-1}\right)}{(1-\alpha_d)\alpha_h\beta+\frac{\alpha_d}{\frac{1}{\beta}+\delta_d-1}}$. We observe that

$$\begin{aligned} M_1(\delta_d) &= \frac{1}{(1-\alpha_d)\alpha_h\beta} \frac{\frac{1}{\beta} + (\delta_d - 1) - \delta_d\alpha_d + \alpha_d}{\frac{1}{\beta} + (\delta_d - 1) + \frac{\alpha_d}{(1-\alpha_d)\alpha_h\beta}} \\ &= \frac{1}{\alpha_h\beta} \frac{\delta_d(1-\alpha_d) + (\frac{1}{\beta} - 1) + \alpha_d}{\delta_d(1-\alpha_d) + (\frac{1}{\beta} - 1)(1-\alpha_d) + \frac{\alpha_d}{\alpha_h\beta}} \\ &= \frac{1}{\alpha_h\beta} \frac{\delta_d(1-\alpha_d) + (\frac{1}{\beta} - 1) + \alpha_d}{\delta_d(1-\alpha_d) + (\frac{1}{\beta} - 1) + \alpha_d + \frac{\alpha_d}{\alpha_h\beta}(1-\alpha_h)} \end{aligned}$$

It is easy to check that $M_1(\delta_d)$ is increasing in δ_d . Consequently, we get $M_1(\delta_d) \geq M_1(0) = \frac{1+\frac{\alpha_d}{\frac{1}{\beta}-1}}{(1-\alpha_d)\alpha_h\beta+\frac{\alpha_d}{\frac{1}{\beta}-1}} > 1$ because $1 > (1-\alpha_d)\alpha_h\beta$.

Therefore, $\lim_{A_d \rightarrow \infty} \frac{G_b(S_s)}{F(S_s)} > 1$ if $\lim_{A_d \rightarrow \infty} \frac{F(S_s)}{S_s} \leq 1$. We now prove that $\lim_{A_d \rightarrow \infty} \frac{F(S_s)}{S_s} \leq 1$. We observe that $\lim_{A_d \rightarrow \infty} S_s = \infty$, $\lim_{A_d \rightarrow \infty} \mu = 1 - \delta$.

1. $\delta < 1$. In this case, condition $\lim_{A_d \rightarrow \infty} \mu = 1 - \delta > 0$ and (B.24) imply that $\lim_{A_d \rightarrow \infty} \frac{pK_c}{S_s} = 1$ and

$$\lim_{A_d \rightarrow \infty} \frac{F(S_s)}{S_s} = 1 - \delta \leq 1.$$

2. $\delta = 1$. In this case, we have $\lim_{A_d \rightarrow \infty} \mu = 1 - \delta = 0$. Observe that

$$S_s = p \left(\frac{\alpha A_c}{p\mu} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\alpha_h w^* A_h}{\mu} \right)^{\frac{1}{1-\alpha_h}} \quad (\text{B.25})$$

$$F(S_s) = A_c \left(\frac{\alpha A_c}{p\mu} \right)^{\frac{\alpha}{1-\alpha}} + A_h \left(\frac{\alpha_h w^* A_h}{\mu} \right)^{\frac{\alpha_h}{1-\alpha_h}}. \quad (\text{B.26})$$

We consider three cases.

- (a) $\alpha = \alpha_h$. In this case, we have

$$S_s = \frac{p \left(\frac{\alpha A_c}{p} \right)^{\frac{1}{1-\alpha}} + (\alpha_h w^* A_h)^{\frac{1}{1-\alpha_h}}}{\mu^{\frac{1}{1-\alpha}}} \quad (\text{B.27})$$

$$\lim_{A_d \rightarrow \infty} \frac{F(S_s)}{S_s} = \lim_{A_d \rightarrow \infty} \frac{A_c \left(\frac{\alpha A_c}{p} \right)^{\frac{\alpha}{1-\alpha}} + A_h (\alpha_h w^* A_h)^{\frac{\alpha}{1-\alpha}}}{p \left(\frac{\alpha A_c}{p} \right)^{\frac{1}{1-\alpha}} + (\alpha w^* A_h)^{\frac{1}{1-\alpha}}} \frac{\mu^{\frac{1}{1-\alpha}}}{\mu^{\frac{\alpha}{1-\alpha}}} = 0 \quad (\text{B.28})$$

- (b) $\alpha > \alpha_h$. In this case, we have $\frac{1}{1-\alpha} > \frac{1}{1-\alpha_h}$. Condition (B.25) and $\lim_{A_d \rightarrow \infty} \mu = 1 - \delta = 0$ imply that

$$\lim_{A_d \rightarrow \infty} S_s \mu^{\frac{1}{1-\alpha}} = p \left(\frac{\alpha A_c}{p} \right)^{\frac{1}{1-\alpha}} + \alpha_h w^* A_h^{\frac{1}{1-\alpha_h}} \mu^{\frac{1}{1-\alpha} - \frac{1}{1-\alpha_h}} = p \left(\frac{\alpha A_c}{p} \right)^{\frac{1}{1-\alpha}} \quad (\text{B.29})$$

By consequence, we have

$$\lim_{A_d \rightarrow \infty} \frac{F(S_s)}{S_s} = \lim_{A_d \rightarrow \infty} \frac{F(S_s) \mu^{\frac{1}{1-\alpha}}}{S_s \mu^{\frac{1}{1-\alpha}}} \quad (\text{B.30})$$

$$= \lim_{A_d \rightarrow \infty} \frac{A_c \left(\frac{\alpha A_c}{p \mu} \right)^{\frac{\alpha}{1-\alpha}} \mu^{\frac{1}{1-\alpha}} + A_h \left(\frac{\alpha_h w^* A_h}{\mu} \right)^{\frac{\alpha_h}{1-\alpha_h}} \mu^{\frac{1}{1-\alpha}}}{S_s \mu^{\frac{1}{1-\alpha}}} = 0. \quad (\text{B.31})$$

(c) When $\alpha < \alpha_h$, by using the same argument as in the previous case, we get that $\lim_{A_d \rightarrow \infty} \frac{F(S_s)}{S_s} = 0$.

To sum up, we obtain that $\lim_{A_d \rightarrow \infty} \frac{G_b(S_s)}{F(S_s)} > 1$ which implies that $G_b(S_s) > F(S_s)$ when A_d is high enough. By combining with condition (B.19), we get that: for A_d high enough, we have $G(S_s) = G_b(S_s) > F(S_s)$. \square

Lemma 7. For A_d high enough, the function G is differentiable at S_s and $\beta G'(S_s) = 1$.

Proof. For A_d high enough, according to Lemma 6, we have $G(S_s) = G_b(S_s) > F(S_s)$. Since the functions G_b and F are continuous, there is an open neighborhood $\mathbf{B}(S_s)$ of S_s so that we have $G_b(S) > F(S)$ for any $S \in \mathbf{B}(S_s)$. Combining this with (B.19), we get that $G(S) = G_b(S) > F(S)$ for any $S \in \mathbf{B}(S_s)$. Therefore, $G(S)$ is differentiable in this $\mathbf{B}(S_s)$. Hence, we can easily obtain that $G'(S_s) = \lambda = \frac{1}{\beta}$, where λ is determined in Lemma 4. \square

By summing up our above steps, the stationary sequence $(c(\beta), K_c(\beta), K_d(\beta), L_d(\beta), 0, H(\beta))_{t=0}^{\infty}$, where $c(\beta) \equiv G(S_s) - S_s$, is a solution to the problem (P) with $w_t = w^*$. Moreover, for any $t \geq 0$, the allocation $(0, 0)$ is a solution to the problem (F_t) with $w_t = w^*$.

We can conclude, according to Definition 2 that the path $(S_t)_{t \geq 1}$ with $S_t = S_s, \forall t \geq 1$, is an equilibrium path. Moreover, $L_d > 0, L_e = 0$, where L_d^s, L_e^s are respectively specific labor hire by the domestic and the MNE at the steady state. We have just finished our proof of Proposition 7 \square

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