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► **To cite this version:**

Z.E.A. Fellah, M. Fellah, W. Lauriks, C. Depollier, J.-Y. Chapelon, et al.. Solution in time domain of ultrasonic propagation equation in a porous material. *Wave Motion*, 2003, 38 (2), pp.151-163. 10.1016/S0165-2125(03)00045-3 . hal-04239854

HAL Id: hal-04239854

<https://hal.science/hal-04239854v1>

Submitted on 15 Oct 2023

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Solution in time domain of ultrasonic propagation equation in porous material

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This paper provides an analytical solution in time domain for the propagation of transient ultrasonic waves in a homogeneous isotropic porous material having a rigid frame. The coefficient of this propagation equation are constants and depend only of the acoustical parameters of the porous material. This propagation equation contains a fractional derivatives terms in order to describe the viscous and thermal interactions between the two phases of the porous medium. The dynamic response of the medium is obtained using the Laplace transform method. An experimental application to porous plastic foams is given to validate the solution of this propagation equation.

PACS numbers : 43.35+d

I. INTRODUCTION

The ultrasonic characterization of porous materials [1] saturated by air is of a great interest for a large class of industrial applications. These materials are frequently used in the automotive and aeronautics industries or in the building trade.

When a broadband ultrasound pulse passes through a layer of medium, the waveform of the pulse changes as a result of the attenuation and dispersion of the medium. Many media, including porous materials and soft tissues, have been observed to have an attenuation function which increases with frequency. As a result, the higher frequency components of the pulse are attenuated more than the lower frequency components. After passing through the layer, the transmitted pulse is not just a scaled down version of the incident pulse, but will have a different shape. Dispersion refers to the phenomenon that the phase velocity of a propagating wave also changes with frequency [2]. Dispersion causes additional change in the waveform of the propagating pulse because the wave components with different frequencies travel at different speeds. An understanding of the interaction of ultrasound with porous medium in both the time and frequency domains and the ability to determine the waveform change of propagating ultrasound pulses should be valuable in the design of array transducer and in quantitative ultrasound tissues characterization [3, 4].

The classical method for predicting the waveform change of a signal passing through a medium relies on the impulse response of the system. According to the theory of linear systems, the output signal is the convolution of the input signal and the system's impulse response.

Many applications like medical imaging or inverse scattering problems [5] require the study of the behavior of pulses travelling into porous media, it is only recently that the response of these media to such excitations has been fully addressed [6] for some elementary configurations in porous media. To efficiently cope with the specific problems appearing in the transient acoustic field propagation, new approaches are required [7]. At present most analysis of signal propagation

are carried out in the frequency domain using the Fourier transform to translate the results in the time domain and vice-versa. This however has several limitations. The first is that the transformation is difficult to compute numerically with sufficient accuracy for non analytical functions. For example, using Fourier transform to obtain time domain results for a lossy material is a more complicated approach than using a true time domain analysis, and the numerical results are less accurate. The second disadvantage is that by working in the frequency domain some numerical information is lost or hard to recover. For example, in case of noisy data it may be difficult to reconstruct the chronological events of a signal by phase unwrapping. Consequently, it is difficult to obtain a deep understanding to transient signal propagation using frequency domain method.

The time domain response of the material is described by an instantaneous response and a "susceptibility" kernel responsible of the memory effects. A time domain approach differs from the frequency analysis in that the susceptibility functions of the problem are convolution operators acting on the velocity and pressure fields, and therefore a different algebraic formalism has to be applied to solve the wave equation. In the past, many authors have used the fractional calculus as an empirical method to describe the properties of viscoelastic materials, e.g. in Caputo [8], Bagley [9]. The observation that the asymptotic expressions of stiffness and damping in porous materials are proportional to fractional powers of frequency suggests the fact that time derivatives of fractional order might describe the behavior of sound waves in this kind of materials, including relaxation and frequency dependence.

II. MODEL

In the acoustics of porous materials, a distinction can be made between two situations depending on whether the frame is moving or not. In the first case, the dynamics of the waves due to the coupling between the solid frame and the fluid are clearly described by the Biot theory

[10 – 12]. In air-saturated porous media the structure is generally motionless and the waves propagate only in the fluid. This case is described by the model of equivalent fluid which is a particular case in the Biot model, in which the interactions between the fluid and the structure are taken into account in two frequency response factors : the dynamic tortuosity of the medium $\alpha(\omega)$ given by Johnson *et al* [13] and the dynamic compressibility of the air included in the porous material $\beta(\omega)$ given by Allard [14]. In the frequency domain, these factors multiply the density of the fluid and its compressibility respectively and represent the deviation from the behavior of the fluid in free space as the frequency increases. In the time domain, they act as operators and in the high frequency approximation their expressions are given [6] by

$$\tilde{\alpha}(t) = \alpha_{\infty} \left(\delta(t) + \frac{2}{\Lambda} \left(\frac{\eta}{\pi \rho_f} \right)^{1/2} t^{-1/2} \right), \quad (1)$$

$$\tilde{\beta}(t) = \left(\delta(t) + \frac{2(\gamma - 1)}{\Lambda'} \left(\frac{\eta}{\pi Pr \rho_f} \right)^{1/2} t^{-1/2} \right). \quad (2)$$

In these equations, $\delta(t)$ is the Dirac function, Pr is the Prandtl number, η and ρ_f the fluid viscosity and the fluid density respectively and γ is the adiabatic constant. The relevant physical parameters of the model are the tortuosity of the medium α_{∞} initially introduced by Zwikker and Kosten [15] and the viscous and the thermal characteristic lengths Λ and Λ' introduced by Johnson *et al* [13] and Allard [14]. In this model, the time convolution of $t^{-1/2}$ with a function is interpreted as a semi-derivative operator following the definition of the fractional derivative of order ν given in Samko and coll [16] ,

$$D^{\nu}[x(t)] = \frac{1}{\Gamma(-\nu)} \int_0^t (t-u)^{-\nu-1} x(u) du, \quad (3)$$

where $\Gamma(x)$ is the gamma function.

In this framework, the basic equations of our model can be expressed as follows

$$\rho_f \tilde{\alpha}(t) * \frac{\partial v_i}{\partial t} = -\nabla_i p \quad \text{and} \quad \frac{\tilde{\beta}(t)}{K_a} * \frac{\partial p}{\partial t} = -\nabla \cdot v, \quad (4)$$

where $*$ denotes the time convolution operation, p is the acoustic pressure, v is the particle velocity and K_a is the bulk modulus of the air. The first equation is the Euler equation, the

second is a constitutive equation obtained from the equation of mass conservation associated with the behavior (or adiabatic) equation .

For a wave propagating along the x -axis , these equations become :

$$\rho_f \alpha_\infty \frac{\partial v}{\partial t} + 2 \frac{\rho_f \alpha_\infty}{\Lambda} \left(\frac{\eta}{\pi \rho_f} \right)^{1/2} \int_0^t \frac{\partial v / \partial t'}{\sqrt{t-t'}} dt' = - \frac{\partial p}{\partial x}, \quad (5)$$

$$\frac{1}{K_a} \frac{\partial p}{\partial t} + 2 \frac{\gamma - 1}{K_a \Lambda'} \left(\frac{\eta}{\pi P r \rho_f} \right)^{1/2} \int_0^t \frac{\partial p / \partial t'}{\sqrt{t-t'}} dt' = - \frac{\partial v}{\partial x}. \quad (6)$$

In these equations, the convolutions express the dispersive nature of the porous material. They take into account the memory effects due to the fact that the response of the medium to the wave excitation is not instantaneous but needs more time to take effect. The retarding force is no longer proportional to the time derivative of the acoustic velocity but is found to be proportional to the fractional derivative of order $1/2$ of this quantity. This occurs because the volume of fluid involved in the motion is not the throughout the whole length of the signal as it is in the case of a fully developed steady flow. The phenomenon may be understood by considering such a volume of fluid in a pore to be in harmonic motion. At high frequencies, only a thin layer of fluid is excited and so the average shear stress is high. At a lower frequency, the same amplitude of fluid motion allows a thicker layer of fluid to participate in the motion and consequently the shear stress is lower. The penetration distance of the viscous forces and therefore the excitation of the fluid depends on frequency. In the time domain, such dependence is associated with a fractional derivative.

III. DIRECT PROBLEM

The direct scattering problem is that of determining the scattered field as well as the internal field, that arises when a known incident field impinges on the porous material with known physical properties. To compute the solution of the direct problem one need to know the solution of the modified wave propagation equation in the porous medium. In that case, it is easy to deduce the internal field knowing the incident signal and the boundary conditions.

The generalized lossy wave equation in the time domain is derived from the basic equations (5, 6) by elementary calculation in the following form :

$$\frac{\partial^2 p}{\partial x^2} - A \frac{\partial^2 p}{\partial t^2} - B \int_0^t \frac{\partial^2 p / \partial t'^2}{\sqrt{t-t'}} dt' - C \frac{\partial p}{\partial t} = 0, \quad (7)$$

where the coefficients A , B and C are constants expressed respectively by :

$$A = \frac{1}{c^2} = \frac{\rho_f \alpha_\infty}{K_a}, \quad B = \frac{2\alpha_\infty}{K_a} \sqrt{\frac{\rho_f \eta}{\pi}} \left(\frac{1}{\Lambda} + \frac{\gamma-1}{\sqrt{Pr\Lambda'}} \right) \quad C = \frac{4\alpha_\infty(\gamma-1)\eta}{K_a \Lambda \Lambda' \sqrt{Pr}}. \quad (8)$$

The first one is related to the velocity $c = 1/\sqrt{\rho_f \alpha_\infty / K_a}$ of the wave in the air included in the porous material. α_∞ is the refractive index of the medium which changes the wave velocity from $c_0 = \sqrt{K_a / \rho_f}$ in free space to $c = c_0 / \sqrt{\alpha_\infty}$ in the porous medium. The other coefficients are essentially dependent on the characteristic lengths Λ and Λ' and express the viscous and thermal interactions between the fluid and the structure.

A. General solution of the propagation equation

Consider a homogeneous porous medium which fills the half space $x \geq 0$. This medium is assumed to be isotropic and to have a rigid frame. An incident signal $g(t)$ impinges normally on the surface $x = 0$ from the left at time $t = 0$. For porous media having a high porosity like plastic foams, the reflected signal can be neglected. These materials have such a small amount of rigid frame that the incident wave does not feel its effects. In that case, the direct problem lies in finding the solution of equation (7) written also as :

$$\frac{\partial^2 p(x, t)}{\partial x^2} - \left(\frac{1}{c^2} \delta(t) + CH(t) + \frac{B}{\sqrt{t}} \right) * \frac{\partial^2 p(x, t)}{\partial t^2} = C \frac{\partial p}{\partial t}(x, 0). \quad (9)$$

where $H(t)$ is the Heaviside function [17] : $H(t) = 0$ for $t \leq 0$ and $H(t) = 1$, for $t > 0$.

It is assumed the initials conditions :

$$p(0, t) = g(t) \quad \text{and} \quad \lim_{\substack{t \rightarrow 0 \\ t > 0}} p(x, t) = \lim_{\substack{t \rightarrow 0 \\ t > 0}} \frac{\partial p}{\partial t}(x, t) = 0. \quad (10)$$

which means that the medium is idle for $t = 0$.

We will try to solve Eq. 9 by the Laplace transform method by taking into account to the conditions (10). We note $P(x, z)$ the Laplace transform of $p(x, t)$ defined by

$$P(x, z) = \mathcal{L}[p(x, t)] = \int_0^{\infty} \exp(-zt)p(x, t)dt \quad (11)$$

and the inverse Laplace transform by

$$p(x, t) = \mathcal{L}^{-1}[P(x, z)] = \int_0^{\infty} \exp(zt)P(x, z)dz \quad (12)$$

Using the following relations

$$\mathcal{L}[\delta(t)] = 1, \quad \mathcal{L}[H(t)] = \frac{1}{z} \quad \text{and} \quad \mathcal{L}\left[\frac{1}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{z}}, \quad (13)$$

the Laplace transform of the wave equation (9) gives :

$$\frac{\partial^2 P(x, z)}{\partial x^2} - z^2 \left(\frac{1}{c^2} + \frac{C}{z} + B\sqrt{\frac{\pi}{z}} \right) P(x, z) = - \left(\frac{1}{c^2} + B\sqrt{\frac{\pi}{z}} \right) \cdot \left(zp(x, 0) + \frac{\partial p(x, 0)}{\partial t} \right) - Cp(x, 0). \quad (14)$$

Taking into account the initials conditions (10), Eq.14 is simplified to :

$$\frac{\partial^2 P(x, z)}{\partial x^2} - z^2 \left(\frac{1}{c^2} + \frac{C}{z} + B\sqrt{\frac{\pi}{z}} \right) P(x, z) = 0. \quad (15)$$

which is a differential equation of second order with real constants coefficients. The characteristic equation of (15) is

$$r^2 - z^2 \left(\frac{1}{c^2} + \frac{C}{z} + B\sqrt{\frac{\pi}{z}} \right) = 0. \quad (16)$$

with a general solution in the form

$$P(x, z) = e^{-\frac{x}{c}\sqrt{f(z)}}\varphi(z) + e^{\frac{x}{c}\sqrt{f(z)}}\psi(z). \quad (17)$$

with :

$$f(z) = z^2 \left(1 + \frac{C \cdot c^2}{z} + Bc^2\sqrt{\frac{\pi}{z}} \right) = z^2 + Bc^2\sqrt{\pi}z\sqrt{z} + Cz c^2 = z(z + b'\sqrt{z} + c'). \quad (18)$$

and

$$b' = Bc^2\sqrt{\pi} \quad \text{and} \quad c' = Cc^2. \quad (19)$$

$\varphi(z)$ and $\psi(z)$ are two functions independent of x .

By taking the finite solution at infinity which corresponds to the physical solution of our problem :

$$P(x, z) = e^{-\frac{x}{c}\sqrt{f(z)}}\varphi(z) \quad (20)$$

The solution of the Eq. 9 ($p(x, t)$) is the inverse Laplace transform of $P(x, z)$:

$$p(x, t) = \mathcal{L}^{-1} \left(e^{-\frac{x}{c}\sqrt{f(z)}}\varphi(z) \right) = \mathcal{L}^{-1} \left(e^{-\frac{x}{c}\sqrt{f(z)}} \right) * \mathcal{L}^{-1} (\varphi(z)). \quad (21)$$

In the next section, we will try to calculate the term : $\mathcal{L}^{-1} \left(e^{-\frac{x}{c}\sqrt{f(z)}} \right)$ which is the core of the problem.

A. Calculus of $\mathcal{L}^{-1} \left(e^{-\frac{x}{c}\sqrt{f(z)}} \right)$

By putting $\Delta^2 = b'^2 - 4c'$. It is easy to verify that Δ is always positive (Appendix. A), in this case, we can factorize $f(z)$ as :

$$f(z) = z \left(\left(\sqrt{z} + \frac{b'}{2} \right)^2 - \frac{\Delta^2}{4} \right), \quad (22)$$

$f(z)$ is always positive because $c' > 0$ and $z > 0$, we have then :

$$P(x, z) = \varphi(z) \exp \left(-\frac{x}{c} \sqrt{\left(z + \frac{b'}{2}\sqrt{z} \right)^2 - \left(\frac{\Delta\sqrt{z}}{2} \right)^2} \right), \quad (23)$$

we distinguish two cases, when $\Delta^2 = 0$, or $\Delta > 0$

Case when $\Delta^2 = 0$:

When $\Delta^2 = 0$, the solution $P(x, z)$ can be written as :

$$P(x, z) = \varphi(z) \exp \left(-\frac{x}{c} \left(z + \frac{b'}{2}\sqrt{z} \right) \right). \quad (24)$$

Using the relation

$$\mathcal{L}^{-1} \exp\left(-\frac{x}{c}z\right) = \delta(t - x/c), \quad (25)$$

and the relation :

$$\mathcal{L}^{-1} \left[\exp\left(-\frac{b'x}{2c}\sqrt{z}\right) \right] = \frac{1}{4\sqrt{\pi}} \frac{b'x}{c} \frac{1}{t^{3/2}} \exp\left(-\frac{b'^2x^2}{16c^2t}\right) \quad t > 0, \quad (26)$$

we obtain :

$$\begin{aligned} \mathcal{L}^{-1} \left[\exp\left(-\frac{x}{c}\left(z + \frac{b'}{2}\sqrt{z}\right)\right) \right] &= \mathcal{L}^{-1} \left[\exp\left(-\frac{x}{c}z\right) \right] * \mathcal{L}^{-1} \left[\exp\left(-\frac{b'x}{2c}\sqrt{z}\right) \right] \\ &= \frac{1}{4\sqrt{\pi}} \frac{b'x}{c} \frac{1}{(t - x/c)^{3/2}} \exp\left(-\frac{b'^2x^2}{16c^2(t - x/c)}\right), \quad t \geq x/c. \end{aligned} \quad (27)$$

The solution of the propagation equation (9) in this case is given by :

$$p(x, t) = \begin{cases} 0, & \text{if } 0 \leq t \leq x/c, \\ \frac{1}{4\sqrt{\pi}} \frac{b'x}{c} \int_{x/c}^t \frac{1}{(\tau - x/c)^{3/2}} \exp\left(-\frac{b'^2x^2}{16c^2(\tau - x/c)}\right) g(t - \tau) d\tau, & \text{if } t > x/c. \end{cases} \quad (28)$$

where $g(t) = \mathcal{L}^{-1}(\varphi(z))$.

Case when $\Delta^2 > 0$

Using the relation :

$$\int_0^\infty \frac{\exp\left(-a\sqrt{x^2 + y^2}\right)}{\sqrt{x^2 + y^2}} I_0(bx) x dx = \frac{\exp\left(-y\sqrt{a^2 - b^2}\right)}{\sqrt{a^2 - b^2}} \quad (29)$$

where $I_0(x)$: is the modified Bessel function, and a, b are constants. By making the variables change $\zeta = \sqrt{x^2 + y^2}$, we get :

$$\int_y^{+\infty} \exp(-a\zeta) I_0\left(b\sqrt{\zeta^2 - y^2}\right) d\zeta = \frac{\exp\left(-y\sqrt{a^2 - b^2}\right)}{\sqrt{a^2 - b^2}}. \quad (30)$$

By deriving the two sides of the Eq. 30 by y and by taking into account the fact that $I_0(0) = 1$ and $I_0'(x) = I_1(x)$ we obtain the relation :

$$\exp\left(-y\sqrt{a^2 - b^2}\right) = \exp(-ay) + by \int_y^{+\infty} \exp(-a\zeta) \frac{I_1\left(b\sqrt{\zeta^2 - y^2}\right)}{\sqrt{\zeta^2 - y^2}} d\zeta, \quad (31)$$

by putting in Eq. 31 : $y = \frac{x}{c}$, $a = z + \frac{b'}{2}\sqrt{z}$ and $b = \frac{\Delta\sqrt{z}}{2}$, we get then :

$$\exp\left(-\frac{x}{c}\sqrt{f(z)}\right) = \exp\left(-\left(z + \frac{b'}{2}\sqrt{z}\right)\frac{x}{c}\right) + \frac{\Delta\sqrt{z}}{2} \frac{x}{c} \int_{x/c}^{+\infty} \exp\left(-\left(z + \frac{b'}{2}\sqrt{z}\right)\zeta\right) \frac{I_1\left(b\sqrt{\zeta^2 - \frac{x^2}{c^2}}\right)}{\sqrt{\zeta^2 - \frac{x^2}{c^2}}} d\zeta. \quad (32)$$

The integral representation of Bessel integral $I_\nu(z)$ is :

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + 1/2)\Gamma(1/2)} \int_{-1}^1 (1-t^2)^{\nu-1/2} \exp(\pm zt) dt, \quad (33)$$

when $\nu = 0$, we obtain the relation :

$$I_0(z) = \frac{1}{\pi} \int_{-1}^1 \frac{\exp(\pm zt) dt}{\sqrt{1-t^2}} = \frac{2}{\pi} \int_0^1 \frac{\exp(-zt)}{\sqrt{1-t^2}} dt, \quad (34)$$

by deriving the two members by z :

$$I_0'(z) = I_1(z) = -\frac{2}{\pi} \int_0^1 \frac{\exp(-zt) t dt}{\sqrt{1-t^2}}, \quad (35)$$

for any real φ , and by substituting z by φz , we obtain :

$$I_1(\varphi z) = -\frac{2}{\pi} \int_0^1 \exp(-\varphi zt) \frac{t dt}{\sqrt{1-t^2}}, \quad (36)$$

and with the variable change $y = \varphi t$:

$$I_1(\varphi z) = -\frac{2}{\pi\alpha} \int_0^\varphi \exp(-zy) \frac{y}{\sqrt{\varphi^2 - y^2}} dy, \quad (37)$$

by multiplying the two sides of Eq. 37 by $\exp(-rz)$, for any positive real r

$$\exp(-rz) I_1(\varphi z) = -\frac{2}{\pi\varphi} \int_0^\varphi \exp-(z(y+r)) \frac{y}{\sqrt{\varphi^2 - y^2}} y dy, \quad (38)$$

and with the variable change : $y+r = \tau$, we get :

$$\exp(-rz) I_1(\varphi z) = -\frac{2}{\pi\varphi} \int_r^{r+\varphi} \exp(-z\tau) \frac{\tau-r}{\sqrt{(\varphi-\tau+r)(\varphi+\tau-r)}}, \quad (39)$$

by introducing the Heaviside distribution H :

$$H(r+\varphi-\tau) = \begin{cases} 1 & \text{if } \tau \leq r+\varphi \\ 0 & \text{if } \tau > r+\varphi \end{cases}$$

and

$$H(r - \tau) = \begin{cases} 1 & \text{if } \tau \leq r \\ 0 & \text{if } \tau > r \end{cases}$$

we get than :

$$\exp(-rz)I_1(\wp z) = -\frac{2}{\wp\pi} \int_0^\infty \exp(-z\tau) \frac{(\tau - r)(H(r + \wp - \tau) - H(r - \tau))}{\sqrt{(\wp - \tau + r)(\wp + \tau - r)}} d\tau \quad (40)$$

and :

$$L^{-1}(\exp(-rz)I_1(\wp z)) = \mathfrak{S}(t) = -\frac{2}{\pi\wp} \frac{(t - r)(H(r + \wp - t) - H(r - t))}{\sqrt{(\wp - t + r)(\wp + t - r)}} \quad (41)$$

Let be $F(z)$ the Laplace-Carson transform of $\mathfrak{S}(t)$ defined by :

$$F(z) = z \int_0^\infty \exp(-\tau z) \mathfrak{S}(\tau) d\tau, \quad (42)$$

we have than :

$$F(z) = z \exp(-rz)I_1(\wp z), \quad (43)$$

we know also that :

$$L^{-1}(F(\sqrt{z})) = \frac{1}{\sqrt{\pi y}} \int_0^\infty \exp\left(-\frac{\tau^2}{4y}\right) \mathfrak{S}(\tau) d\tau, \quad (44)$$

for a real constant M

$$L^{-1}(MF(\sqrt{z})) = d(\bar{y}) = \frac{M}{\sqrt{\pi y}} \int_0^\infty \exp\left(-\frac{\tau^2}{4y}\right) \mathfrak{S}(\tau) d\tau, \quad (45)$$

by substituting $\mathfrak{S}(t)$ by its expression (45), $\bar{d}(y)$ becomes :

$$\begin{aligned} \bar{d}(y) &= -\frac{2B}{\wp\pi\sqrt{\pi y}} \int_0^\infty \exp\left(-\frac{\tau^2}{4y}\right) \frac{(\tau - r)(H(r + \wp - \tau) - H(r - \tau))}{\sqrt{(\wp - \tau + r)(\wp + \tau - r)}} d\tau \\ &= -\frac{2B}{\wp\pi\sqrt{\pi y}} \int_r^{r+\wp} \exp\left(-\frac{\tau^2}{4y}\right) \frac{(\tau - r)}{\sqrt{(\wp - \tau + r)(\wp + \tau - r)}} d\tau, \end{aligned} \quad (46)$$

by putting $\tau - r = u$, the integral (49) becomes :

$$d(\bar{y}) = \frac{-2B}{\wp\pi\sqrt{\pi y}} \int_0^\wp \exp\left(-\frac{(u+r)^2}{4y}\right) \frac{udu}{\sqrt{\wp^2 - u^2}}, \quad (47)$$

and by the variable change $u = \wp\tau$

$$d(\bar{y}) = \frac{-2B}{\pi^{3/2}\sqrt{y}} \int_0^1 \exp\left(-\frac{(\wp\tau + r)^2}{4y}\right) \frac{\tau d\tau}{\sqrt{1 - \tau^2}}, \quad (48)$$

and by putting :

$$M = M(\zeta) = \frac{\Delta x}{2c} \frac{1}{\sqrt{\zeta^2 - x^2/c^2}}, \quad \wp = \wp(\zeta) = \frac{\Delta}{2} \sqrt{\zeta^2 - x^2/c^2}, \quad r = r(\zeta) = b'/2\zeta, \quad (49)$$

Consequently :

$$L^{-1} \left(\frac{\Delta \sqrt{z} x}{2c} \exp \left(-\frac{b'}{2} \sqrt{z} \right) \frac{I_1 \left(\frac{\Delta \sqrt{z}}{2} \sqrt{\zeta^2 - \frac{x^2}{c^2}} \right)}{\sqrt{\zeta^2 - \frac{x^2}{c^2}}} \right) = \bar{d}(y), \quad (50)$$

using the fact that : $L^{-1}(\exp(-\zeta z)) = s(y) = \delta(y - \zeta)$, we will have then :

$$L^{-1} \left(\frac{\Delta \sqrt{z} x}{2c} \left(\exp - \left(z + \frac{b'}{2} \sqrt{z} \right) \zeta \right) \frac{I_1 \left(\frac{\Delta \sqrt{z}}{2} \sqrt{\zeta^2 - \frac{x^2}{c^2}} \right)}{\sqrt{\zeta^2 - \frac{x^2}{c^2}}} \right) = s * \bar{d} \quad (51)$$

$$= \int_0^\tau s(y) \bar{d}(\tau - y) dy = \int_0^\tau \delta(y - \zeta) \bar{d}(\tau - y) dy = \bar{d}(\tau - \zeta) \quad (52)$$

Let us now to explicit $\bar{d}(\tau - \zeta)$, by replacing M, \wp and r (given by Eq. (49)) in Eq. (48), we get then :

$$\bar{d}(y) = -\frac{\Delta x}{\pi^{3/2} c} \frac{1}{\sqrt{\zeta^2 - x^2/c^2}} \frac{1}{\sqrt{y}} \int_0^1 \exp \left(-\frac{\left(\mu \frac{\Delta}{2} \sqrt{\zeta^2 - x^2/c^2} + \frac{b'}{2} \zeta \right)^2}{4y} \right) \frac{\mu d\mu}{\sqrt{1 - \mu^2}} \quad (53)$$

$\bar{d}(\tau - \zeta)$: is then obtained by replacing y by $\tau - \zeta$, then

$$\bar{d}(\tau - \zeta) = -\frac{\Delta x}{\pi^{3/2} c} \frac{1}{\sqrt{\zeta^2 - x^2/c^2}} \frac{1}{\sqrt{\tau - \zeta}} \int_0^1 \exp \left(-\frac{\left(\mu \frac{\Delta}{2} \sqrt{\zeta^2 - x^2/c^2} + \frac{b'}{2} \zeta \right)^2}{4(\tau - \zeta)} \right) \frac{\mu d\mu}{\sqrt{1 - \mu^2}} \quad \tau > \zeta \quad (54)$$

the inverse Laplace transform of second term of right side in Eq. (32) is :

$$G(\tau) = \int_{x/c}^\tau \bar{d}(\tau - \zeta) d\zeta, \quad \text{because} \quad \tau > \zeta \quad (55)$$

in this integral, we put the variable change $\tau - \zeta = \xi$, and we get ;

$$G(\tau) = \int_0^{\tau - x/c} \bar{d}(\xi) d\xi \quad (56)$$

with :

$$\bar{d}(\xi) = -\frac{\Delta x}{\pi^{3/2} c} \frac{1}{\sqrt{(\tau - \xi)^2 - x^2/c^2}} \frac{1}{\sqrt{\xi}} \int_0^1 \exp \left(-\frac{\left(\mu \frac{\Delta}{2} \sqrt{(\tau - \xi)^2 - x^2/c^2} + \frac{b'}{2} (\tau - \xi) \right)^2}{4\xi} \right) \frac{\mu d\mu}{\sqrt{1 - \mu^2}}. \quad (57)$$

by putting : $\bar{d}(\xi) = \Delta \frac{x}{c} h(\xi)$ with :

$$h(\xi) = -\frac{1}{\pi^{3/2}} \frac{1}{\sqrt{(\tau - \xi)^2 - x^2/c^2}} \frac{1}{\sqrt{\xi}} \int_0^1 \exp\left(-\frac{\left(\mu \frac{\Delta}{2} \sqrt{(\tau - \xi)^2 - x^2/c^2} + \frac{b'}{2}(\tau - \xi)\right)^2}{4\xi}\right) \frac{\mu d\mu}{\sqrt{1 - \mu^2}}. \quad (58)$$

$G(\tau)$ becomes then :

$$G(\tau) = \Delta \frac{x}{c} \int_0^{\tau - x/c} h(\xi) d\xi \quad (59)$$

the general solution $p(x,t)$ is :

$$p(x,t) = \begin{cases} 0 & \text{if } 0 \leq t \leq x/c \\ u(x,t) + \int_{x/c}^t G(\tau) g(t - \tau) d\tau, & \text{if } t > x/c \end{cases} \quad (60)$$

Where $u(x,t)$ is given by the solution when $\Delta = 0$ in Eq. 28

$$u(x,t) = \frac{1}{4\sqrt{\pi}} \frac{b'x}{c} \int_{x/c}^t \frac{1}{(\tau - x/c)^{3/2}} \exp\left(-\frac{b'^2 x^2}{16c^2(\tau - x/c)}\right) g(t - \tau) d\tau \quad (61)$$

and finally we write $p(x,t)$ as this :

$$p(x,t) = \begin{cases} 0 & \text{if } 0 \leq t \leq x/c \\ \frac{x}{c} \int_{x/c}^t \left(\frac{b'}{4\sqrt{\pi}} \frac{1}{(\tau - x/c)^{3/2}} \exp\left(-\frac{b'^2 x^2}{16c^2(\tau - x/c)}\right) + \Delta \int_0^{\tau - x/c} h(\xi) d\xi \right) g(t - \tau) d\tau, & t > x/c \end{cases} \quad (62)$$

It is easy to show that this solution is continuous on the surface $x = 0$ of the porous material (Appendix. B)

$$\lim_{x \rightarrow 0} p(x,t) = p(0,t) = g(t). \quad (63)$$

In the next section, we will try to give an experimental validation of the solution of the propagation equation for a plastic foam which is a porous material saturated by air.

IV. ULTRASONIC MEASUREMENTS

As an application of this model, some numerical simulations are compared to experimental results. Experiments are done in air with two broadband Panametrics V389 piezoelectric transducers having a 250 kHz central frequency in air and a bandwidth at 6 dB extending from 60 kHz

to 420 kHz. Pulses of 900 V are provided by a 5058PR Panametrics pulser/receiver. The received signals are amplified up to 90 dB and filtered above 1 MHz to avoid high frequency noise (energy is totally filtered by the sample in this upper frequency domain). Electronics perturbations are removed by 1000 acquisition averages. The experimental setup is showed on Fig.1

The simulated signals are computed from (62) in which $g(t)$ is the signal generated by the transducer given in Fig. 2. The experimental data are deduced from the transmitted field scattered by a slab of plastic foam of finite depth $0 \leq x \leq L$. In dealing with a slab of high porosity foam, as already mentioned above, the signals reflected by the front wall ($x = 0$) and by the back wall ($x = L$) of the slab can be neglected. Thus, near the back wall, the signal propagating in the foam is nearly identical to the transmitted one $p(L - \epsilon, t) = g^t(L + \epsilon, t)$.

For foams having a low porosity, this approximation breaks down and in that case, reflected signals must be taken into account. In Fig.3, experimental and simulated results are presented for two plastic foams F_1 and F_2 having different flow resistivities. The parameters of the foam F_1 are : thickness 5 cm, $\alpha_\infty = 1.055$, $\Lambda = 234 \mu\text{ m}$, $\Lambda' = 702 \mu\text{ m}$, flow resistivity $\sigma = 9000 \text{ Nm}^{-4}\text{s}$ and porosity $\phi = 0.97$, those of the foam F_2 are : thickness 1.1 cm, $\alpha_\infty = 1.26$, $\Lambda = 60 \mu\text{ m}$, $\Lambda' = 180 \mu\text{ m}$, $\sigma = 38000 \text{ Nm}^{-4}\text{s}$ and $\phi = 0.98$. The good agreement for foams with low or high flow resistivity, especially for the maximum value of their amplitudes, may be regarded as being in support of the quite realistic assumption about the replacement of the transmitted signal by the internal one. The slight difference observed between the two curves is probably due to experimental measurements rather than to the lack of reflection on the walls of the slab.

V. CONCLUSION

In this paper, an analytical solution in time domain for the propagation of ultrasonic waves in porous media having a rigid frame is established. The direct problem is solved using the concept of Laplace transform and an experimental validation of the solution of the propagation

equation is given. The attraction of a time domain based approach is that analysis is naturally bounded by the finite duration of ultrasonic pressures and is consequently the most appropriate approach for the transient signal. On the basis of the solution of the direct problem, we will try in the future treat the inverse problem in order to characterize the porous medium via ultrasonic measurements in time domain.

APPENDIX A : Sign of Δ

In the section III. A, we have put $\Delta^2 = b'^2 - 4c'$, with $b' = Bc^2\sqrt{\pi}$ and $c' = Cc^2$. The coefficients A , B and C are given by Eq. 8 :

$$A = \frac{1}{c^2} = \frac{\rho_f \alpha_\infty}{K_a}, \quad B = \frac{2\alpha_\infty}{K_a} \sqrt{\frac{\rho_f \eta}{\pi}} \left(\frac{1}{\Lambda} + \frac{\gamma - 1}{\sqrt{Pr} \Lambda'} \right) \quad C = \frac{4\alpha_\infty (\gamma - 1) \eta}{K_a \Lambda \Lambda' \sqrt{Pr}}. \quad (64)$$

It is possible to write :

$$\Delta = (Bc^2\sqrt{\pi})^2 - 4Cc^2 = c^2(B^2c^2\pi - 4C), \quad (65)$$

we want to see the sign of Δ , it is enough to see the sign of $W = B^2c^2\pi - 4C$ because c^2 is always positive.

By substituting A , B and C by their values given in (64), we obtain

$$W = \frac{4\alpha_\infty^2}{K_a^2} \left(\frac{\rho_f \eta}{\pi} \right) \left(\frac{1}{\Lambda} + \frac{\gamma - 1}{\Lambda' \sqrt{Pr}} \right)^2 \left(\frac{K_a \pi}{\rho_f \alpha_\infty} \right) - 4 \left(\frac{4\alpha_\infty (\gamma - 1) \eta}{K_a \Lambda \Lambda' \sqrt{Pr}} \right), \quad (66)$$

$W \geq 0$, means that

$$\left(\frac{1}{\Lambda} + \frac{\gamma - 1}{\Lambda' \sqrt{Pr}} \right)^2 - \frac{4(\gamma - 1)}{\Lambda \Lambda' \sqrt{Pr}} \geq 0 \quad (67)$$

By putting $x = \frac{1}{\Lambda}$ and $y = \frac{\gamma - 1}{\Lambda' \sqrt{Pr}}$, x and y are always positives, we obtain then the relation :

$$(x + y)^2 - 4xy \geq 0 \quad (68)$$

and then

$$|\sqrt{x} - \sqrt{y}| \geq 0 \quad (69)$$

with $|\sqrt{x} - \sqrt{y}|$ is the absolute value of $\sqrt{x} - \sqrt{y}$. This quantity is always positive and then the condition $\Delta \geq 0$ is well verified.

APPENDIX B : Calculus of $\lim_{x \rightarrow 0} p(x, t)$

Consider the limit :

$$\lim_{x \rightarrow 0} p(x, t) = \lim_{x \rightarrow 0} \frac{x}{c} \frac{b'}{4\sqrt{\pi}} \int_{x/c}^t \frac{1}{(\tau - x/c)^{3/2}} \exp\left(-\frac{b'^2 x^2}{16c^2(\tau - x/c)}\right) g(t - \tau) d\tau \quad (70)$$

$$+ \lim_{x \rightarrow 0} \Delta \frac{x}{c} \int_{x/c}^t \left(\int_0^{\tau - x/c} h(\xi) d\xi \right) g(t - \tau) d\tau \quad \text{for } t \geq 0. \quad (71)$$

The limit of the term : $\Delta \frac{x}{c} \int_{x/c}^t \left(\int_0^{\tau - x/c} h(\xi) d\xi \right) g(t - \tau) d\tau$ when x tends to 0 is equal to 0.

Consider the limit of the term : $\frac{x}{c} \frac{b'}{4\sqrt{\pi}} \int_{x/c}^t \frac{1}{(\tau - x/c)^{3/2}} \exp\left(-\frac{b'^2 x^2}{16c^2(\tau - x/c)}\right) g(t - \tau) d\tau$. Let be

the variables change

$$\aleph^2 = \frac{b'^2 x^2}{16c^2(\tau - x/c)} \implies \aleph = \frac{b'x}{4c\sqrt{\tau - x/c}} \implies d\aleph = -\frac{b'xd\tau}{8c(\tau - x/c)^{3/2}} \quad (72)$$

$$\implies \frac{d\tau}{(\tau - x/c)^{3/2}} = -\frac{8c}{b'x} d\aleph \quad (73)$$

The equation becomes :

$$\frac{x}{c} \frac{b'}{4\sqrt{\pi}} \int_{x/c}^t \frac{1}{(\tau - x/c)^{3/2}} \exp\left(-\frac{b'^2 x^2}{16c^2(\tau - x/c)}\right) g(t - \tau) d\tau = \quad (74)$$

$$\frac{2}{\sqrt{\pi}} \int_{\frac{b'x}{4c\sqrt{t-x/c}}}^{\infty} \exp(-\aleph^2) g\left(t - \frac{x}{c} - \frac{b'^2 x^2}{16c^2 \xi^2}\right) d\aleph \quad (75)$$

when $x \rightarrow 0$, the integral (75) tends to

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} \exp(-\aleph^2) g(t) d\aleph = g(t) \left(\frac{2}{\sqrt{\pi}} \int_0^{\infty} \exp(-\aleph^2) d\aleph \right), \quad (76)$$

we know that :

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} \exp(-\aleph^2) d\aleph = 1 \quad \text{when } x \rightarrow 0, \quad (77)$$

we have then :

$$\lim_{x \rightarrow 0} v(x, t) = g(t) \quad (78)$$

This means that the solution is continuous on the surface $x = 0$ of the porous material.

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FIGURE CAPTIONS

Fig. 1. Experimental set-up of the ultrasonic measurements

Fig. 2. Incident signal generated by the transducer.

Fig.3-a Experimental (solid line) and simulated signals (dashed line) for the foam F1.

Fig.3-b Experimental (solid line) and simulated signals (dashed line) for the foam F2.

FIG. 1

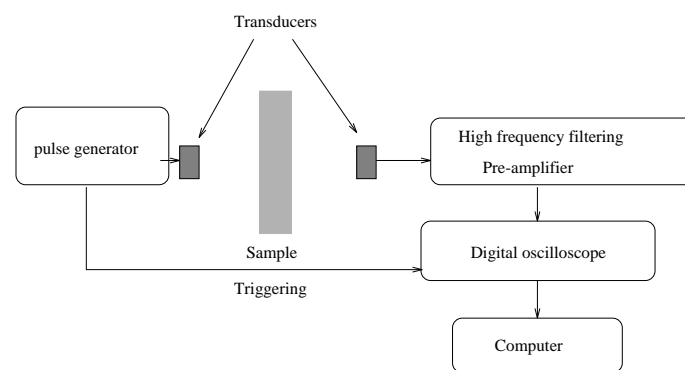


FIG. 2

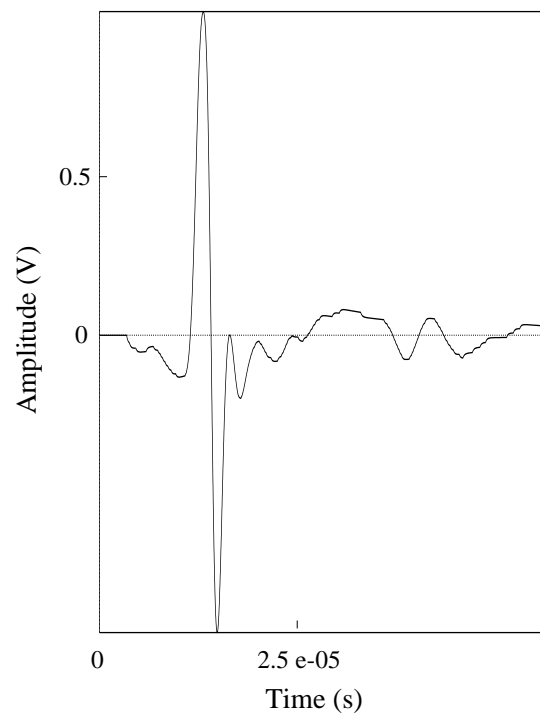


FIG. 3-A

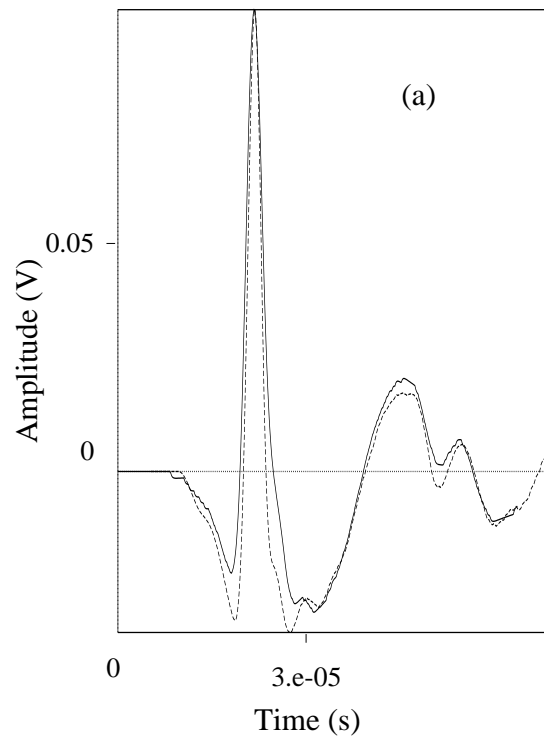


FIG. 3-B

