



HAL
open science

Efficiency and Stability Trade-offs in School Choice: Implementing and Characterizing Nash Equilibria

Cyril Rouault

► **To cite this version:**

Cyril Rouault. Efficiency and Stability Trade-offs in School Choice: Implementing and Characterizing Nash Equilibria. 2023. hal-04238849v2

HAL Id: hal-04238849

<https://hal.science/hal-04238849v2>

Preprint submitted on 24 Apr 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Copyright

Efficiency and Stability Trade-offs in School Choice: Implementing and Characterizing Nash Equilibria*

Cyril Rouault[†]

Abstract

We explore the school choice problem and examine assignments that Pareto-dominate the student-optimal stable assignment. Utilizing a strategic approach grounded in Nash equilibria, we highlight the connection between stability and Nash equilibria which result in assignments that enhance student allocations through the deferred acceptance mechanism (DA). We demonstrate that for certain assignments, there exists no Nash equilibrium under DA that leads to these enhanced assignments. We identify a strategy profile for a given assignment, enabling determination of whether a Nash equilibrium exists that leads to this assignment. Then, we characterize strategy profiles that are Nash equilibria and lead to assignments of the efficiency adjusted deferred acceptance mechanism (EADA) introduced by Kesten (2010). Furthermore, we demonstrate that for each round of EADA, the considered matching can be achieved with a Nash equilibrium using DA.

JEL Classification: C78, D47, D50, D82.

Keywords: School choice; Matching; Efficiency; Nash equilibrium; Deferred acceptance

1 Introduction

There exists a trade-off between stability and efficiency in one-sided matching markets. Roth (1982) shows that these two properties are incompatible. The deferred acceptance mechanism (DA), introduced by Gale and Shapley (1962), produces a stable assignment that is the most efficient among all stable assignments. However, DA does not always yield an efficient assignment, and unstable assignments may Pareto-dominate

*I am grateful to Olivier Bos, David Delacrétaz, Guillaume Hearinger, Victor Hiller, Vincent Iehlé, Josué Ortega, Morgan Patty, and Olivier Tercieux for their valuable comments.

[†]Université Paris-Saclay, ENS Paris-Saclay, CEPS. *E-mail address:* cyril.rouault@universite-paris-saclay.fr

it. Furthermore, DA is strategy-proof, and improving DA assignment would lead to a non-strategy-proof mechanism. This raises the question: Can agents' strategies lead to an efficient Nash equilibrium outcome that Pareto-dominates the student-optimal stable assignment in DA? We show that the answer is yes.

There are many real-world examples of problems that fit into our framework, but perhaps the most important is public school choice.¹ Many cities worldwide use centralized mechanisms to assign students to schools. Stability is a crucial concern for many schools because they must justify the acceptance and rejection of students. In one-sided markets, stability is a fairness criterion in the sense that a stable assignment eliminates justified envy (Abdulkadiroğlu and Sönmez, 2003). The fairness justification results from school priorities and student preferences.² It is important to note that the standard approach in the literature on school choice problems analyzes efficiency only from the students' perspective. The trade-off is then selecting an appropriate mechanism that meets fairness objectives and maximizes student welfare.

In this paper, we study the preference revelation games under DA and the different strategies for students. School priorities are fixed and known publicly. Although DA is strategy-proof for students,³ they can interact and coordinate their strategies to improve their assignments. While such manipulation may potentially decrease the welfare of some students, our focus is on strategies that lead to Pareto improvements in student assignments. We examine Nash equilibria that result in assignments Pareto-dominating the student-optimal stable assignment under DA. We use the term *implementation* of an assignment to refer to the existence of a Nash equilibrium that yields this assignment under DA.

Some assignments that Pareto-dominate a stable assignment cannot be implemented with DA (Proposition 1). In this paper, we identify the origin of this impossibility. The connection between stability and Nash equilibrium is well-known in the literature. Our study highlights the connection between stability and Nash equilibria that implement assignments that Pareto-dominate the student-optimal stable assignment. To achieve this, we study such Nash equilibria. We show the importance of the school obtained at the student-optimal stable assignment in blocking profitable deviations of other stu-

¹Other problems of allocating doctors to hospitals or children to day-care centers can be considered.

²Priorities are generally designed to respond to social objectives. In the school choice problem, school priorities are determined by law.

³See Dubins and Freedman (1981).

dents. However, for some assignments, these schools are not sufficient and other schools must be ranked in the students' strategy to guarantee Nash equilibrium.

We generalize the strategy profile for all assignments that Pareto-dominate the student-optimal stable assignment. This profile is presented in Theorem 1 and consists of each student reporting a set of schools in her strategy. Therefore, if an assignment is implementable, then this strategy profile is a Nash equilibrium and implements it with DA. Theorem 1 has two major consequences: First, it highlights the importance of certain schools in enabling Nash equilibrium. Secondly, for every assignment that Pareto-dominates the student-optimal stable assignment, if the profile presented is a Nash equilibrium, then the assignment is implementable. Therefore, to determine whether an assignment is implementable or not, we need only consider this strategy profile.

In the remainder of the article, we restrict our analysis to assignments that Pareto-dominate the student-optimal stable assignment and that are considered through the efficiency adjusted deferred acceptance mechanism (EADA) introduced by Kesten (2010). EADA is efficient and Pareto-dominates DA; however, it is not strategy-proof. We show that for any problem, the structure of EADA provides for each round a Nash equilibrium and an assignment that Pareto-dominates the student-optimal stable assignment (Proposition 2). The intuition is that rejection chains are maintained throughout the algorithm and profitable deviations are blocked. We, therefore, study the applications that maintain the Nash equilibrium.

We consider the steps of DA and show that it is possible to identify rejection chains that block deviations from interrupters.⁴ We refer to *blocking applications* as the applications that maintain the existence of this chain. Theorem 2 shows that for each strategy profile such that interrupters are neutralized, and rejection chains are maintained, the profile is a Nash equilibrium and implements the assignment with DA. Theorem 2 emphasizes the connection between stability and the equilibria that implement the assignments that Pareto-dominate the student-optimal stable assignment. Indeed, rejection chains are maintained by respecting the priorities of certain students. It is therefore essential for each student to apply to the school obtained at the student-optimal stable assignment.

⁴Kesten (2010) identifies interrupters as a source of the incompatibility between efficiency and stability. These are student applications, which, when withdrawn, do not change the student's assignment but improve that of others. We define formally interrupters in Section 4.

With Proposition 2 and Theorem 2, it is direct that the reasoning applies to all assignments considered throughout EADA. We can, therefore, characterize Nash equilibria that implement any assignment considered in EADA. To implement the assignment from round t of EADA, the interrupters of rounds 0 to t must be neutralized, and the students apply to the blocking applications identified in rounds 0 to t . Another consequence of Theorem 2 is that if there is no blocking application, then each round of EADA can be implemented with a Nash equilibrium in which each student applies to at most two schools (Corollary 2).

Finally, in Appendix A, we provide a complementary analysis, characterizing strategy profiles that are Nash equilibria and lead to a stable assignment with DA.

Related Literature

Our paper is related to the growing literature studying the trade-off between efficiency and stability. Although many papers propose weaker stability definitions to approach efficiency, we use the existence of Nash equilibria in DA while maintaining the standard definition of stability. Works that consider a weakened form of stability include Morrill (2015) and Troyan et al. (2020), among others. One of the main contributions to the trade-off between efficiency and stability is by Kesten (2010), who suggests an algorithm that yields an efficient assignment that Pareto-dominates the student-optimal stable assignment. Kesten (2010) uses rejection chains to show how applications can be withdrawn, benefiting other students without reducing overall welfare. This solution is widely discussed in the literature, with many papers presenting arguments in favor of EADA outcomes based on fairness criteria. Tang and Yu (2014), Troyan et al. (2020), Ehlers and Morrill (2020), Tang and Zhang (2021), and Reny (2022) explore this aspect. Tang and Yu (2014) suggest a simplified algorithm that produces an outcome equivalent to EADA. However, their algorithm is not based on rejection chains, which are essential for a strategic approach of DA. Although the welfare objectives of these works are similar to ours, none of them includes a strategic aspect.

A central motivation of our approach is to characterize the set of Nash equilibria that lead to an outcome that Pareto-dominates the student-optimal stable assignment. Bando (2014) proposes a strategic approach to EADA, introducing an algorithm to identify a strong Nash equilibrium such that the outcome of this algorithm is the EADA outcome. Our contribution differs in two aspects: First, we study the structure of strategy profiles that lead to equilibrium. In particular, this allows us to generalize a strategy profile for

all assignments that can be implemented with DA. Secondly, we characterize the Nash equilibria that yield the same outcome as EADA, including all rounds that compose it, whereas Bando (2014) provides only one Nash equilibrium for each EADA outcome. Dur and Morrill (2020) propose a similar approach. They show that by limiting the number of applications, students can benefit. Our results complete theirs by proposing a characterization of Nash equilibria producing an efficient outcome that is absent from their formulations. Moreover, our results concern all stable assignments and not just the optimal stable assignment. Additionally, we investigate the number of schools to be reported in the strategy profiles. This approach brings new results to the work initiated by Haeringer and Klijn (2009) in a constrained case. However, they do not discuss the efficiency of outcomes.

The remainder of the paper is organized as follows. We introduce the model in Section 2. We present our results on implementing an assignment in a Nash equilibrium in Section 3. In Section 4, we introduce EADA. We characterize the Nash equilibria in EADA in Section 5. Section 6 concludes. We provide additional results on stability in Appendix A. Proofs are collected in Appendix B.

2 Model

2.1 Assignment Problems

We consider school choice problems with a finite set $I = \{i_1, i_2, \dots, i_n\}$ of *students* and a finite set $S = \{s_1, s_2, \dots, s_m\}$ of *schools*. Let i denote a generic element of I , and let s denote a generic element of S . Let q_s denote the *capacity* of school s and $q = (q_s)_{s \in S}$ denote the *capacity vector*. If a student is unassigned, we say she is assigned to the null school \emptyset ,⁵ which has unlimited capacity. Each student $i \in I$ has a *strict preference ordering* P_i over $S \cup \{\emptyset\}$, and R_i represents its symmetric extension. If student i (weakly) prefers school s to school s' , we write $sP_i s'$ ($sR_i s'$). A school s is *acceptable* to i if $sR_i \emptyset$. We denote by $P \equiv (P_i)_{i \in I}$ the *strict preference profile*. The *strict priority order* at school s is denoted by \succ_s . If student i has higher priority than student j at s , we write $i \succ_s j$. We assume that all schools find all students acceptable. We denote by $\succ \equiv (\succ_s)_{s \in S}$ the *priority profile*.

A (*school choice*) *problem* is a 5-tuple (I, S, P, \succ, q) . Throughout I, S , and q are fixed, then we denote a problem by (P, \succ) .

⁵This outside option can be to stay at home or attend a private school, for instance.

An assignment is a mapping $\mu : I \cup S \rightarrow S \cup \{\emptyset\} \cup I$ such that for each student $i \in I$, $\mu(i) \in S \cup \{\emptyset\}$, for each school $s \in S$, $\mu(s) \in 2^I$ with $|\mu(s)| \leq q_s$, and for each student $i \in I$, $\mu(i) = s$ if, and only if $i \in \mu(s)$. If $\mu(i) = \emptyset$, we say that i is *unassigned* at μ .

2.2 Stability and Pareto Domination

We now introduce standard properties of matchings. An assignment μ is *stable* if:

- μ is *individually rational*, i.e., for each $i \in I$, $\mu(i)R_i\emptyset$,
- μ is *non-wasteful*, i.e., for each $i \in I$ and each $s \in S$, $sP_i\mu(i)$ implies $|\mu(s)| = q_s$,
- there is no *justified envy*, i.e., for each $i, j \in I$ with $\mu(j) = s$, $sP_i\mu(i)$ implies $j \succ_s i$.

It is well known that for any problem (P, \succ) , the set of stable assignments is non-empty and forms a lattice, with the student-optimal and the student-pessimal stable assignments. We denote by μ_I the student-optimal stable assignment. Formally, for each student i and each stable assignment μ , $\mu_I(i)R_i\mu(i)$. We say that i *envy* j at μ if $\mu(j)P_i\mu(i)$.

An assignment μ *Pareto-dominates* an assignment μ' if for each student $i \in I$, $\mu(i)R_i\mu'(i)$ and there exists at least one student i such that $\mu(i)P_i\mu'(i)$. An assignment is *efficient* if it is not Pareto-dominated by any other assignment.

2.3 Assignment Game

An *assignment mechanism* φ selects an assignment $\varphi(P, \succ)$ for each problem (P, \succ) . An allocation mechanism is *efficient* if $\varphi(P, \succ)$ is efficient for each (P, \succ) , and is *stable* if $\varphi(P, \succ)$ is stable for each (P, \succ) . Lastly, a mechanism φ *Pareto-dominates* a mechanism φ' if $\varphi(P, \succ)$ Pareto-dominates $\varphi'(P, \succ)$ for each (P, \succ) .

A mechanism φ naturally defines a game in which students are the *players*. Let the *strategy* Q_i be an ordered list of schools corresponding to the preferences of student i over the set of schools. For instance, we denote the strategy of i by $Q_i : s_1, s_2$. This is interpreted as, in her strategy, i ranks s_1 higher than s_2 , and she prefers being unassigned rather than being assigned to any other school. We denote by $A(Q_i)$ the set of schools reported by student i under strategy Q_i . Then, $s \notin A(Q_i)$ must be interpreted as student i ranks s as an unacceptable school under strategy Q_i . Let $Q \equiv (Q_i)_{i \in I}$ be a *strategy profile* of students. The outcome is the student assignment obtained with strategy profile Q as the preference profile. Preferences over assignments determine preferences over outcomes. For strategy profile, we use classic game-theoretic notations; for instance, $Q_{-i} \equiv (Q_{i'})_{i' \in I \setminus \{i\}}$ to denote the strategy profile of students that are not i .

A strategy profile Q is a *Nash equilibrium* if for each $i \in I$, Q_i is student i 's best response to the other students' strategies Q_{-i} . Formally, for each $i \in I$, there is no strategy Q'_i such that $Q'_i \neq Q_i$, and we have:

- $\varphi[((Q'_i, Q_{-i}), \succ)](i) P_i \varphi[((Q_i, Q_{-i}), \succ)](i)$.

We now define the *implementation* in a Nash equilibrium of an outcome, which is central to our paper.

Definition 1. Given an assignment mechanism φ and a problem (P, \succ) , an assignment μ can be *implemented* in a Nash equilibrium (or μ is *implementable*) if there exists a Nash equilibrium Q such that $\varphi(Q, \succ) = \mu$. We say that an assignment μ is *not implementable* with φ if there is no Nash equilibrium Q such that $\varphi(Q, \succ) = \mu$.

We extend the definition of implementation to mechanisms. Formally, a mechanism φ' can be *implemented with the mechanism* φ if for each problem (P, \succ) , there exists a Nash equilibrium Q , for mechanism φ such that $\varphi(Q, \succ) = \varphi'(P, \succ)$.

This paper focuses on implementation using the deferred acceptance algorithm (DA hereafter) introduced by Gale and Shapley (1962). We denote (the student-proposing) DA assignment for problem (P, \succ) by $DA(P, \succ)$. It is well known that for each (P, \succ) , we have $DA(P, \succ) = \mu_I$.⁶

We define DA for problem (P, \succ) as follows:

Step 1: Each student i proposes to the school ranked first in P_i (if there is no such school then i remains unassigned). Each school s tentatively assigns up to q_s seats according to its priority order \succ_s . The remaining students are rejected.

Step l , $l \geq 2$: Each student i that is rejected in Step $l - 1$ proposes to the next school in the ordered list P_i (if there is no such school then i remains unassigned). Each school s considers the new proposers and the students that have a (tentative) seat. School s tentatively assigns up to q_s seats according to its priority order \succ_s . The remaining students are rejected.

The algorithm stops when no student is rejected. Each student is assigned to her final tentative school. Since the set of students is finite, DA terminates in a finite number of steps.

The school-proposing deferred acceptance mechanism is defined analogously, replacing students by schools and schools by students. The resulting assignment is known as the student-pessimal stable assignment.

⁶See Gale and Shapley (1962).

3 Implementability with DA

In this section, we consider assignments that Pareto-dominate the student-optimal stable assignment μ_I for a given problem. From the literature, we know that any stable matching can be implemented under DA (Gale and Sotomayor, 1985). However, as shown by Example 1, there are assignments that Pareto-dominate μ_I and for which there is no Nash equilibrium to implement them with DA.

Example 1. Consider a school choice problem with $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2, s_3, s_4\}$ and for each $s \in S$, $q_s = 1$. The priorities and preferences are given in the following tables. The symbol (\cdot) indicates that priorities are irrelevant to the problem.

\succ_{s_1}	\succ_{s_2}	\succ_{s_3}	\succ_{s_4}	P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}
i_1	i_2	i_3	i_4	s_2	s_1^*	s_2^*	s_1
i_4	i_1	i_1	\cdot	s_3^*	<u>s_2</u>	<u>s_3</u>	<u>s_4^*</u>
i_2	i_3	\cdot	\cdot	<u>s_1</u>			
\cdot	\cdot	\cdot	\cdot				

Table 1: Schools' priorities and students' preferences.

The table on the right indicates two different potential assignments: μ_I , which is underlined in students' preferences, and μ^* (denoted by stars *). It is easy to see that μ^* Pareto-dominates μ_I . In addition, μ^* is Pareto-efficient. However, we show that no Nash equilibrium implements μ^* under DA.⁷

By contradiction, suppose there exists a Nash equilibrium Q such that $DA(Q, \succ) = \mu^*$. We assume that $A(Q_{i_1}) \subseteq \{s_1, s_2, s_3\}$, $A(Q_{i_2}) \subseteq \{s_1, s_2\}$, $A(Q_{i_3}) \subseteq \{s_2, s_3\}$, and $A(Q_{i_4}) \subseteq \{s_1, s_4\}$. We know that $DA(Q, \succ) = \mu^*$, then $s_3 \in A(Q_{i_1})$, $s_1 \in A(Q_{i_2})$, $s_2 \in A(Q_{i_3})$, and $s_4 \in A(Q_{i_4})$.

We show that $A(Q_{i_4}) = \{s_4\}$. Otherwise, if $s_1 \in A(Q_{i_4})$ we know that $s_1 P_{i_4} s_4$ and $\mu^*(s_1) = i_2$ with $i_4 \succ_{s_1} i_2$. In words, we know that if i_4 applies to s_1 , she rejects i_2 which contradicts $DA(Q, \succ) = \mu^*$. By construction, we know that i_2 's strategy is either $Q_{i_2} : s_1, s_2$ or $Q_{i_2} : s_1$. If i_2 ranks s_2 preferred to s_1 in her strategy, then $DA[(Q, \succ)](i_2) = s_2$ which contradicts $DA(Q, \succ) = \mu^*$. Using the same reasoning, we know that i_3 's strategy is either $Q_{i_3} : s_2, s_3$ or $Q_{i_3} : s_2$. If i_3 ranks s_3 preferred to s_2 in her strategy, then $DA[(Q, \succ)](i_2) = s_3 \neq \mu^*(i_3)$.

⁷We use a similar method to that proposed by Bando (2014). However, in his example, the deviation is blocked by the school's capacity: the student is not rejected because the school's maximum capacity has not been reached. In our example, deviation is possible because three students have traded schools, and i_1 has a higher priority than i_3 at s_2 .

Now consider the strategy of student i_1 . We know that $s_2 P_{i_1} \mu^*(i_1)$. Consider a strategy $Q'_{i_1} : s_2$. If $Q_{i_2} : s_1, s_2$, and $Q_{i_3} : s_2, s_3$, $DA((Q'_{i_1}, Q_{-i_1}))$ is given by:

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ s_2 & s_1 & s_3 & s_4 \end{pmatrix}.$$

If $Q_{i_2} : s_1$, and $Q_{i_3} : s_3, s_2$, it is easy to see that DA leads to the same assignment.

Finally, if $Q_{i_2} : s_1, s_2$, and $Q_{i_3} : s_2$, $DA((Q'_{i_1}, Q_{-i_1}))$ is given by:

$DA((Q'_{i_1}, Q_{-i_1}))$ is given by:

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ s_2 & s_1 & \emptyset & s_4 \end{pmatrix},$$

and if $Q_{i_2} : s_1$, DA leads to the same assignment.

Thus, i_1 has a profitable deviation from Q_{i_1} using the strategy Q'_{i_1} . This contradicts Q being a Nash equilibrium.

Proposition 1. There exists problem (P, \succ) such that μ_I is Pareto-dominated by μ' and μ' is not implementable with DA.

Despite this negative result, which leads to the impossibility of implementing specific assignments in DA, we show in the rest of the section that if an assignment can be implemented, then we can design a strategy profile that is a Nash equilibrium and that implements it. Thus, we study the structure of the strategy profile required to implement an assignment that Pareto-dominates the student-optimal stable assignment. For this purpose, we study the schools in the students' strategies that are used to implement an assignment. We now consider a new example in which assignments can be implemented.

Example 2. Let consider a school choice problem (P, \succ) with $I = \{i_1, i_2, i_3, i_4, i_5\}$, $S = \{s_1, s_2, s_3, s_4, s_5\}$ and for each $s \in S, q_s = 1$. The priorities and preferences are given in the following tables:

\succ_{s_1}	\succ_{s_2}	\succ_{s_3}	\succ_{s_4}	\succ_{s_5}	P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}
i_1	i_2	i_3	i_4	\cdot	s_2^\dagger	$s_3^{\bullet\dagger}$	s_2^\bullet	s_1^\dagger	s_4
i_2	i_3	i_4	i_5	\cdot	<u>s_1^\bullet</u>	s_1	s_4^\dagger	s_3	<u>$s_5^{\bullet\dagger}$</u>
i_4	i_1	i_2	i_3	\cdot		<u>s_2</u>	<u>s_3</u>	<u>s_4^\bullet</u>	

Table 2: Schools' priorities and students' preferences.

The table on the right shows three different assignments. An assignment μ_I (underlined), assignment μ^\bullet (denoted by bullet \bullet) and, assignment μ^\dagger (denoted by dagger \dagger). It is easy

to see that μ_I is Pareto-dominated by μ^\bullet and μ^\dagger . In addition, μ^\bullet and μ^\dagger are efficient but are not stable.

Consider μ^\bullet first. Suppose the strategy profile is such that for each student $i \in I$, we have $Q_i^\bullet : \mu^\bullet(i), \mu_I(i)$ if $\mu^\bullet(i) \neq \mu_I(i)$ and $Q_i^\bullet : \mu_I(i)$ otherwise. We denote this strategy profile as Q^\bullet . The resulting assignment of $DA(Q^\bullet, \succ)$ is shown in the table below.

$Q_{i_1}^\bullet$	$Q_{i_2}^\bullet$	$Q_{i_3}^\bullet$	$Q_{i_4}^\bullet$	$Q_{i_5}^\bullet$
s_1^\bullet	s_3^\bullet	s_2^\bullet	s_4^\bullet	s_5^\bullet
	s_2	s_3		

Table 3: A possible Nash equilibrium to implement μ^\bullet .

DA stops at the first round, and $DA(Q^\bullet, \succ) = \mu^\bullet$. Furthermore, by considering each student i choosing their dominant strategy (i.e., P_i), and with strategies of other students Q_{-i}^\bullet fixed, we can observe that there are no profitable deviations for the students. Therefore, the strategy profile Q^\bullet is a Nash equilibrium.

There are two arguments for applying to the school $\mu_I(i)$ for each i . First, if all students apply to that school, then the assignment obtained will be weakly preferred. Second, as illustrated in Example 2, $\mu_I(i)$ for each i blocks profitable deviations.

A question arises: If an assignment is implementable, is it always possible to implement it with DA with a strategy in which students apply to at most two schools? The answer is no. To illustrate, consider assignment μ^\dagger . We show that μ^\dagger cannot be implemented with a strategy profile similar to the one presented in Table 3 for μ^\bullet .

Consider μ^\dagger from Table 3 and a strategy profile Q^\dagger such that for each $i \in I$, $Q_i^\dagger : \mu^\dagger(i), \mu_I(i)$ and $Q_i^\dagger : \mu_I(i)$ if $\mu^\dagger(i) = \mu_I(i)$. The resulting assignment of $DA(Q^\dagger, \succ)$ is shown in the table below.

$Q_{i_1}^\dagger$	$Q_{i_2}^\dagger$	$Q_{i_3}^\dagger$	$Q_{i_4}^\dagger$	$Q_{i_5}^\dagger$
s_2^\dagger	s_3^\dagger	s_4^\dagger	s_1^\dagger	s_5^\dagger
s_1	s_2	s_3	s_4	

Table 4: Q^\dagger does not implement μ^\dagger .

Even if DA stops at the first step, there exists a profitable deviation for i_3 . By ranking s_2 first in her strategy, for instance $Q'_{i_3} : s_2$, we have $DA[(Q'_{i_3}, Q_{-i_3}^\dagger), \succ](i_3) = s_2$ and $DA[(Q'_{i_3}, Q_{-i_3}^\dagger), \succ](i_4) = s_4$. This is due to the absence of a rejection chain that blocks

the deviation of i_3 . For instance, suppose that i_4 's strategy is now given by $Q'_{i_4} = P_{i_4}$. We have $DA((Q'_{i_4}, Q_{-i_4}^\dagger), \succ) = \mu^\dagger$, and $(Q'_{i_4}, Q_{-i_4}^\dagger)$ is a Nash equilibrium. Considering the strategy profile $(Q'_{i_4}, Q_{-i_4}^\dagger)$, if i_3 ranks s_2 first, then she is reject i_1 from s_2 . Student i_1 then proposes to school s_1 , resulting in the rejection of i_4 . Student i_4 proposes to school s_3 , which in turn rejects i_2 , and by proposing to s_2 rejects i_3 . The rejection chain, therefore, blocks the deviation of i_3 . In the next section, we study the importance of rejection chains and their impact on the strategy profiles we consider.

Theorem 1 generalizes our approach and identifies a subset of schools for each student, leading to a Nash equilibrium that implements an assignment μ . This subset is the set of schools ranked between the school to which the student is assigned in μ and the school to which she is assigned at the student-optimal stable assignment. Note that preferences are weak in (i), so both $\mu(i)$ and $\mu_I(i)$ are in this subset. The condition (ii) is then that all students rank these schools according to their preferences. If μ is implementable, then this profile is a Nash equilibrium.

Theorem 1. Let (P, \succ) be a school choice problem and μ an assignment that Pareto-dominates μ_I . The assignment μ is implementable if, and only if a strategy profile Q such that for each $i \in I$, the following conditions hold:

- (i) $A(Q_i) = \{s \in S | s R_i \mu(i) \text{ and } \mu'(i) R_i s\}$, and
- (ii) schools in $A(Q_i)$ are ranked according to P_i .

is a Nash equilibrium and implements μ with DA.

Theorem 1 has two important consequences. First, the profile designed with (i) and (ii) is applicable to all assignments that are implementable and that Pareto-dominate the student-optimal stable assignment. Hence, to determine whether an assignment is implementable or not, we only need to determine whether the profile described in Theorem 1 is a Nash equilibrium. Note that an alternative proof can be provided for Proposition 1, where we only need to determine if the strategy profile Q is a Nash equilibrium or not. This leads to a profitable deviation for student i_1 in Example 1.

Second, Theorem 1 provides a structure for some Nash equilibria. Although it does not characterize all the strategy profiles that are Nash equilibria implementing μ , it highlights the importance of some schools to prevent profitable deviations. Thus, to characterize Nash equilibria, we restrict our attention for the remainder of the paper to the assignments considered throughout the algorithm proposed by Kesten (2010). Surprisingly, we show that all assignments considered through Kesten's algorithm are implementable with DA and Pareto-dominates μ_I . This restriction allows us to generalize our approach to all problems.

4 Rejection Chains and Kesten's EADA

Rejection chains lead to the incompatibility between stability and efficiency. This can be illustrated with DA. In DA, the emergence of rejection chains results in a decrease in students' preference rankings. However, in our Example 2, we illustrated that these rejection chains can be used to consider a Nash equilibrium. This section aims to identify the rejection chains that enable a Nash equilibrium to exist for each assignment that Pareto-dominates the student-optimal stable assignment. Thus, characterizing Nash equilibria implementing an assignment is possible.

4.1 EADA and Definitions

In this subsection, we introduce the algorithm proposed by Kesten (2010), namely the *efficiency-adjusted DA* (henceforth, EADA).⁸ We first define interrupters, which are essential in EADA. In DA for a problem (P, \succ) , if student i is tentatively assigned to school s at some step l and is later rejected from s at some step $l' > l$, with at least one student rejected from s at some step r such that $l \leq r < l'$, then student i is an *interrupter* for school s , and the pair (i, s) is an *interrupting pair* at step l' .

For any school choice problem (P, \succ) , Kesten's EADA operates as follows:⁹

Round 0: Run DA for the problem (P, \succ) .

Round $t, t \geq 1$: Identify the last step of the round- $(t - 1)$ DA procedure in which interrupter(s) are rejected. Identify all interrupting pairs of this step, and for each pair, remove the respective school from the interrupter's preference. After that, rerun DA (round- t DA) with the new preference profile.

Stop when there are no more interrupters that are rejected.

We denote the EADA assignment for problem (P, \succ) by $K(P, \succ)$. Kesten (2010) shows that the assignment is efficient by neutralizing specific interrupters, and EADA Pareto-dominates DA (Theorem 1 of Kesten (2010)). The term *neutralize* means that a student who is an interrupter for a school no longer applies to it.¹⁰

⁸Tang and Yu (2014) propose a simplified version of EADA. However, even if outcomes are equivalent, we use EADA because it uses rejection chains, which are essential in our approach. Similarly, temporary assignments obtained at each round differ between the simplified EADA and EADA.

⁹We use "round" for EADA and "step" for DA.

¹⁰Kesten differentiates between students who consent to have a modification of their preferences when they are interrupters for a school and those who do not. In our analysis, we assume that all students consent.

In the remainder of this section, we show that EADA can be used to identify Nash equilibrium that implements assignments in DA. We introduce a new notation for this. For each round $t \geq 0$ of EADA, we denote the temporary assignment by $\mu^{\kappa(t)}$ (i.e., the DA outcome for DA run at round t of EADA). We, therefore, have $\mu^{\kappa(0)} = \mu_I$ for any problem (P, \succ) .

A crucial point with the approach proposed by Kesten (2010) is that if a student is an interrupter for school s , then she is indifferent between applying or not to s , given the fixed strategies of other students.¹¹ In our context, by considering Nash equilibria, the assignment is $K(P, \succ)$, and students identified as interrupters have no profitable deviations. In addition, we show that this is true for all rounds of EADA, following modifications in preferences profiles.¹²

Proposition 2. The preference profile used at round t of EADA is a Nash equilibrium and implements $\mu^{\kappa(t)}$ with DA.

Corollary 1. EADA can be implemented with DA.

Corollary 1 directly follows from Proposition 2. As a result, we know that there exists at least one Nash equilibrium that implements EADA in DA.

We denote $\mathbb{K} \equiv \bigcup_{t=0, \dots, p} \{\mu^{\kappa(t)}\}$ as the set of all temporary assignments of EADA, with p being the last round. Similarly, for each round, we consider the set of interrupting pairs. Let \tilde{I}_t be the *set of interrupting pairs* identified by EADA at round t . Note that the order of the sets is important, as Kesten (2010) shows, to guarantee to Pareto-domination of DA.

Each assignment in \mathbb{K} can be obtained by neutralizing interrupters.¹³ For example, $\mu^{\kappa(t)} \in \mathbb{K}$ can be implemented by using a strategy profile such that each student reports her true preferences and the interrupters from rounds 0 to t are neutralized. Throughout EADA, students trade school positions to obtain preferred assignments. We aim to

¹¹This result is given by Proposition 3 of Kesten (2010). The formulation is different. Kesten does not consider indifference, but the assignment of a consenting student does not change.

¹²The modification of the preference profile consists of neutralizing interrupters. This means that the other schools are not removed from the preference profiles. Following Theorem 2, the proof of Proposition 2 is straightforward.

¹³To stick to the terminology of Kesten and student consents, this means that for any partition of consenting students, we can implement the associated assignment in EADA under DA.

identify applications that maintain Nash equilibrium.¹⁴ To achieve this, we define the set of trading students.

Definition 2. Let I_t^T be a set of *trading students* in EADA at round t , where $t \geq 1$. For any pair of students $i, i' \in I_t^T$ with $i \neq i'$, the following conditions hold:

- $\mu^{\kappa(t)}(i) = \mu^{\kappa(t-1)}(i')$, with $\mu^{\kappa(t)}(i) \neq \mu^{\kappa(t-1)}(i)$, and
- $\mu^{\kappa(t)}(i') = \mu^{\kappa(t-1)}(i'')$, with $i'' \in I_t^T$, $i'' \neq i'$, and $\mu^{\kappa(t)}(i') \neq \mu^{\kappa(t-1)}(i')$.

We refer to $\mu^{\kappa(t)}(i)$ as a *blocking application* if, at a later round $t' > t$, student i' is identified as an interrupter for a school s , and $\mu^{\kappa(t)}(i) \neq K[(P, \succ)](i)$.

The intuition behind this definition is the following: If two students trade their assignments at a round t and one of them is later identified as an interrupter, the school they traded will maintain a rejection chain and prevent profitable deviations.

4.2 Illustration of Definitions

Example 2 shows that the strategy Q^\dagger is not a Nash equilibrium. We build a new strategy profile by considering the problem presented in Example 2 and the definitions introduced in the previous subsection.

Example 3. Considering Table 2, EADA proceeds as follows:

- At round 0, the obtained assignment is μ_I .
- At round 1, student i_5 is identified as an interrupter for school s_4 . We have $\tilde{I}_1 = \{(i_5, s_4)\}$. School s_4 is removed from i_5 's preferences (that we denote P'_{i_5}). DA is run with the new preference profile. The obtained assignment is $\mu^{\kappa(1)}$ and is underlined in the following table:

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P'_{i_5}
s_2	s_3	s_2	s_1	
<u>s_1</u>	s_1	<u>s_4</u>	<u>s_3</u>	<u>s_5</u>
	<u>s_2</u>	s_3	s_4	

Table 5: Assignemnt obtained at the end of round 1

We can observe that students i_3 and i_4 traded their assignments, therefore, we have $I_1^T = \{i_3, i_4\}$.

¹⁴Erdil and Ergin (2008) introduce improvement cycles using the same reasoning. However, in their approach, they mainly use these cycles in the set of stable assignments.

- At round 2, student i_3 is identified as an interrupter for school s_2 . We have $\tilde{I}_2 = \{(i_3, s_2)\}$. School s_2 is removed from i_3 's preferences (that we denote P'_{i_3}). DA is run with the new preference profile. The obtained assignment is $\mu^{\kappa(2)}$ and is underlined in the following table:

P_{i_1}	P_{i_2}	P'_{i_3}	P_{i_4}	P'_{i_5}
<u>s_2</u>	<u>s_3</u>		<u>s_1</u>	
s_1	s_1	<u>s_4</u>	s_3	<u>s_5</u>
	s_2	s_3	s_4	

Table 6: Assignemnt obtained at the end of round 2

We can observe that students i_1, i_2 and i_4 traded their assignments, therefore, we have $I_2^T = \{i_1, i_2, i_4\}$.

- There is no other interrupter for the following round, and EADA stops with $K(P, \succ) = \mu^\dagger$.

At round 1, students i_3 and i_4 traded their assignments. In the next round, i_3 is identified as an interrupter. By neutralizing the interrupter, i_4 's assignment changed such that $\mu^{\kappa(1)}(i_4) \neq \mu^\dagger(i_4)$. As a result, $\mu^{\kappa(1)}(i_4) = s_3$ is a blocking application. This example highlights that blocking applications can be identified, and as mentioned with Example 2, the school s_3 allows i_4 to maintain the rejection chain.

5 Characterization of Strategy Profiles with EADA

In this section, we characterize strategies profiles to implement assignments that Pareto-dominate μ_I and are considered through EADA. As illustrated in Example 3, blocking applications are essential for characterizing Nash equilibria. Two conditions must be met to form a rejection chain. The rejection chain is initiated by the rejection of a student at a school. The rejected student then applies to another school, which in turn rejects another student. The process continues until the end of the rejection chain. First, for each rejected student, another student must be rejected when applying to the next school in their preferences. This means that the school s concerned must have at least q_s students temporarily assigned to it. Secondly, the schools constituting this chain must be reported in the students' strategies and ranked according to their preferences. These two conditions correspond to Lemma 1 and Lemma 2, respectively.

Lemma 1. For any problem (P, \succ) , if for $s \in S, |K[(P, \succ)](s)| = q_s$ then $|DA[(P, \succ)](s)| = q_s$. In addition, for each s such that for some $i \in I$ and $t, (i, s) \in \tilde{I}_t$, we have

$$|K[(P, \succ)](s)| = q_s.$$

The main argument for Lemma 1 is stability. It states that if a school s reaches its capacity q_s with EADA, then its capacity q_s is also reached in DA. Moreover, all students involved in an interrupter are involved with schools that have reached their maximum capacity.

Lemma 2. Let Q be a strategy profile such that for each student $i \in I$, the following conditions hold:

- $\mu_I(i) \in A(Q_i)$,
- for each student $i \in I$ and each round t such that $\mu^{\kappa(t)}(i)$ is a blocking application, we have $\mu^{\kappa(t)}(i) \in A(Q_i)$, and
- schools in $A(Q_i)$ are ranked according to P_i ,

Then, there exist rejection chains that block deviations of students involved in an interrupting pair.

Lemma 2 is crucial to characterizing strategy profiles. It identifies, for each student, the schools that maintain rejection chains. First, the school obtained in μ_I and the blocking applications identified throughout EADA. Profitable deviations are, therefore, prevented. If there is no blocking applications in a problem, each student i only needs to apply to $\mu_I(i)$ to block deviations, as formulated in Lemma 2.

Theorem 2. For any problem (P, \succ) , for each strategy profile Q such that the following conditions hold:

- (i) All the interrupters are neutralized (i.e., for each round t and each pair $(i, s) \in \tilde{I}_t$, we have $s \notin A(Q_i)$),
- (ii) for each student $i \in I$, $\mu_I(i), K[(P, \succ)](i) \in A(Q_i)$,
- (iii) for each student $i \in I$ and each round t such that $\mu^{\kappa(t)}(i)$ is a blocking application, we have $\mu^{\kappa(t)}(i) \in A(Q_i)$, and
- (iv) schools in $A(Q_i)$ are ranked according to P_i ,

Q is a Nash equilibrium, and implements $K(P, \succ)$ with DA.

Theorem 2 characterizes all strategy profiles that implement for a problem (P, \succ) , $K(P, \succ)$ with DA. Hence, in all strategy profiles where the interrupters are neutralized, each student applies to the school obtained with $\mu_I, K(P, \succ)$, and all blocking applications with respect to preference order are Nash equilibria and implement EADA.

Theorem 2 has two important implications. First, when there is no blocking application, a strategy profile exists in which each student applies to at most two schools, namely

the school obtained with $\mu_I, K(P, \succ)$, and this profile is a Nash equilibrium and implements $K(P, \succ)$ with DA. Second, we have the same result for every assignment in \mathbb{K} by combining Proposition 2 and Theorem 2.

Corollary 2. If for a problem (P, \succ) there is no blocking application, then for each assignment $\mu \in \mathbb{K}$ there exists a Q such that for each $i \in I$, $|A(Q_i)| \leq 2$ and Q is a Nash equilibrium that implements μ with DA.

To implement $\mu^{\kappa(t)} \in \mathbb{K}$, it is then sufficient to include every blocking application of the previous t round. This is given by (iii) in the statement of Theorem 2.

We have thus characterized Nash equilibria of DA using EADA. These Nash equilibria lead to an assignment that Pareto-dominates μ_I . In Appendix A, we characterize other Nash equilibria that lead to a stable outcome.

6 Conclusion

This paper studies Nash equilibria that implement assignments that Pareto-dominate the student-optimal stable assignment. We show that implementing assignments that Pareto-dominate a stable assignment may be impossible with DA. However, if an assignment that Pareto-dominates the student-optimal stable assignment is implementable, then we identify a strategy profile that implements it in DA. This profile can be used to determine whether an assignment is implementable with DA: given an assignment, we can construct the profile, and if this profile is a Nash equilibrium, then the assignment is implementable in DA.

We then focus on the assignments considered by EADA. Surprisingly, we show that all assignments considered in rounds of EADA can be implemented with DA. We characterize strategy profiles and argue that achieving efficiency with a stable mechanism at Nash equilibrium is possible using the rejection chain. Through characterization, we identify the assignments that students must maintain to guarantee Nash equilibrium and efficiency of the outcome. This is reflected in our concept of blocking applications. Thus, we show that it is always possible to implement assignments that Pareto-dominate the student-optimal stable assignment in a one-sided market.

Our contribution opens up several opportunities for both theoretical and empirical research. First, the existence of Nash equilibria raises the question of the incentives for students to implement them. Second, with mechanisms limiting the number of student applications, we could analyze whether these Nash equilibria are achieved in practice.

Finally, incorporating incomplete information is another potential extension of our approach. Ehlers and Massó (2015) study a many-to-one matching market with incomplete information. They show a link between stable mechanism Nash equilibrium in incomplete information and a complete information framework.¹⁵

Appendix A: Additional Results on Stability

In this Appendix A, we study Nash equilibrium outcomes to identify conditions that guarantee stability. For this purpose, we define cycles according to Ergin (2002). Let $U_s(j)$ denote the set of students who have a higher priority at school s than student j , i.e., $U_s(j) = \{i \in I \mid i \succ_s j\}$.

Definition 3. Let \succ be a priority structure and q be a vector of quotas. A cycle is constituted by distinct schools $s, s' \in S$ and students $i, j, k \in I$ satisfying the following conditions:

- Cycle condition: $i \succ_s j \succ_s k \succ_{s'} i$.
- Scarcity condition: there exist (possibly empty) disjoint sets of agents $I_s, I_{s'} \subset I \setminus \{i, j, k\}$ such that $I_s \subset U_s(j)$, $I_{s'} \subset U_{s'}(i)$, $|I_s| = q_s - 1$, and $|I_{s'}| = q_{s'} - 1$.

A priority structure is acyclical if it has no cycles.

We know that in the presence of a cycle, there are DA outcomes that are not Pareto-efficient (Ergin, 2002). Nash equilibrium outcomes that are not stable can, therefore, exist in the presence of a cycle, as shown, for instance, by Haeringer and Klijn (2009) using DA. Let us illustrate with an example that considers cycles.

Example 4. Consider $I = \{i_1, i_2, i_3\}$ and $S = \{s_1, s_2, s_3\}$ with for each $s \in S, q_s = 1$. The priorities for the schools and the preferences of the students are given as follows:

\succ_{s_1}	\succ_{s_2}	\succ_{s_3}	P_{i_1}	P_{i_2}	P_{i_3}
i_1	i_2	i_1	$\underline{s_2}$	s_3	$\underline{s_3}$
i_2	i_1	i_3	s_3	$\underline{s_1}$	s_2
i_3	i_3	i_2	s_1	s_2	s_1

There are two cycles in the schools' priorities:

- $i_2 \succ_{s_2} i_1 \succ_{s_2} i_3 \succ_{s_3} i_2$ and $i_1 \succ_{s_3} i_3 \succ_{s_3} i_2 \succ_{s_2} i_1$, for schools s_2 and s_3 .

¹⁵More precisely, they show that a Nash equilibrium characterization under complete information leads to the Nash equilibrium characterization under incomplete information.

Following the definition of a cycle, we refer to the student j as i_1 and i_3 .

The unique stable outcome is underlined in the students' preferences. Let us consider a Nash equilibrium:

Q_{i_1}	Q_{i_2}	Q_{i_3}
<u>s_2</u>	<u>s_3</u>	s_2
s_3	s_2	<u>s_1</u>
s_1		

We can see that in this situation, there are no profitable deviations. Hence, this is a Nash equilibrium, and the assignment is not stable. If i_3 applies to the school s_3 , her assignment does not change, but the strategy of i_2 will be modified following a profitable deviation.

Q_{i_1}	Q_{i_2}	Q_{i_3}
s_2	s_3	s_3
<u>s_3</u>	<u>s_2</u>	s_2
s_1		<u>s_1</u>

The profitable deviation for i_2 is then to apply to s_1 . If the student i_3 applies to s_3 , the obtained assignment is stable at Nash equilibrium.

Let us now consider another preference profile and an associated unstable Nash equilibrium outcome:

P_{i_1}	P_{i_2}	P_{i_3}	Q_{i_1}	Q_{i_2}	Q_{i_3}
<u>s_2</u>	<u>s_3</u>	s_2	<u>s_1</u>	<u>s_3</u>	<u>s_2</u>
s_1	s_2	<u>s_1</u>	s_3	s_2	s_3
s_3	s_1	s_3		s_1	

Similarly, if i_1 applies to s_2 in her strategy Q_{i_1} , then i_3 will have a profitable deviation, which is to apply to s_1 . Following this deviation, the Nash equilibrium outcome obtained will be stable. By routine computation, we obtain that for any preference profile, when the students i_1 and i_3 apply to s_2 and s_3 , respectively, using the correct ranking, all Nash equilibrium outcomes are stable.

Note that we only consider Nash equilibria. Strategy profiles may lead to an unstable outcome where students i_1 and i_3 apply to s_2 and s_3 , respectively. However, this is not a Nash equilibrium.

The following theorem generalizes the result of this example.

Theorem 3. Suppose that for each cycle $i \succ_s j \succ_s k \succ_{s'} i$, we have $s \in A(Q_j)$, and schools in $A(Q_j)$ are ranked according to P_j . Then, each Nash equilibrium outcome is stable.

EADA identifies the interrupters to neutralize, and these interrupters involve the student j in the cycle. Thus, when j is an interrupter for school s , we have if $s \in A(Q_j)$, the assignment is stable at Nash equilibrium.¹⁶ However, if j does not apply, it is possible to have an efficient Nash equilibrium outcome under the condition that all other interrupters are neutralized.

The main consequence of Theorem 3 is that it is sufficient for each student j involved in a cycle to apply to school s to ensure outcome Nash equilibrium stability. It is important to note that $s \notin A(Q_j)$ does not imply that it is impossible to have a stable Nash equilibrium outcome. However, it is possible to guarantee stability by considering the preferences of students i and k . This is possible when i or k cannot block j within the cycle. We illustrate Proposition 3 with the following example:

Example 5. Consider the problem given in Example 4, and as presented, there are two cycles. In the first cycle, i is i_2 and k is i_3 , while in the second cycle, i is i_1 and k is i_2 . One consequence of our next result is that if preference profiles are such that $s_3 P_{i_2} s_2$ and $s_2 P_{i_3} s_3$ or $s_2 P_{i_1} s_3$ and $s_3 P_{i_2} s_2$, there exist unstable Nash equilibrium outcomes. Consider, for instance, the following profile where (\cdot) can be any school:

P_{i_1}	P_{i_2}	P_{i_3}
·	·	·
·	s_2	·
·	s_3	·
	·	

Considering the preference of i_2 , all the Nash equilibrium outcomes are stable for each possible preference of other students P_{-i_2} . Similarly, if $s_3 P_{i_3} s_2$ and $s_3 P_{i_1} s_2$; then all the possible preferences of student i_2 lead to only stable Nash equilibrium outcomes.

Proposition 3. Suppose that for each cycle $i \succ_s j \succ_s k \succ_{s'} i$, and for each student i and k , i ranks $s P_i s'$ or student k ranks $s' P_k s$, or both, in their true preferences. Then, all Nash equilibrium outcomes are stable.

The stability of Nash equilibrium outcomes can be identified in two situations, and we can generalize these results as follows:

¹⁶This must be true for each cycle. Note that if s is not acceptable to j , then the school should not be included in j 's strategy.

- If at least one student among i and k for each cycle has preferences such that $sP_i s'$ or $s'P_k s$, from Proposition 3, then all Nash equilibrium outcomes are stable.
- Otherwise, for the cycles concerned, if $s \in A(Q_j)$ and schools are ranked according to P_j , then by Theorem 3, all Nash equilibrium outcomes are stable.

We have identified the applications in the student strategies that guarantee the stability of the Nash equilibrium outcomes. One of our conditions coincides with the interrupter notion mentioned above.

Appendix B: Proofs

Proof of Theorem 1

Proof. Let μ_S denote a stable assignment, μ^* denote an assignment that Pareto-dominates μ_S . We know that there exists $I' \subseteq I$ such that for each $i \in I'$, $\mu^*(i)P_i\mu_S(i)$ by the definition of Pareto-domination. Then, there exist $i, i' \in I'$ such that $\mu^*(i) = \mu_S(i')$, by stability of μ_S .

(if): Suppose μ^* is implementable. Construct the following preference profile Q : For each $i \in I$, let

- $A(Q_i) = \{s \in S \mid sR_i\mu_S(i) \text{ and } \mu^*(i)R_i s\}$, and
- schools in $A(Q_i)$ are ranked according to P_i .

For students such that $\mu^*(i) = \mu_S(i)$, $A(Q_i)$ is a singleton. It is easy to check that under the constructed Q , we have $DA(Q, \succ) = \mu^*$, and that DA stops at step 1. We have to show that the strategy profile Q is a Nash equilibrium.

By contradiction, suppose that Q is not a Nash equilibrium. Then, there exists $i_1 \in I$ such that $Q'_{i_1} \neq Q_{i_1}$ and $DA[((Q'_{i_1}, Q_{-i_1}), \succ)](i_1)P_{i_1}DA[(Q, \succ)](i_1)$, and we denote s^\dagger the school to which student i_1 is assigned under $DA((Q'_{i_1}, Q_{-i_1}), \succ)$.

We know that $s^\dagger \neq \emptyset$ because $DA[((Q'_{i_1}, Q_{-i_1}), \succ)](i_1)P_{i_1}DA[(Q, \succ)](i_1)R_{i_1}\mu_S(i_1)$ as μ_S is individually rational. Then, by stability of μ_S , there exists $i_2 \in I$ such that $i_2 \succ_{s^\dagger} i_1$ and $\mu_S(i_2) = s^\dagger$. We have two cases:

- Case 1: $\mu^*(i_2) = s^\dagger$, then by stability of μ_S we know that $i_2 \succ_{s^\dagger} i_1$, the profitable deviation of i_1 is blocked by i_2 since $A(Q_{i_2}) = \{s^\dagger\}$. Hence, $DA[((Q'_{i_1}, Q_{-i_1}), \succ)](i_1) \neq s^\dagger$.
- Case 2: $\mu^*(i_2) \neq s^\dagger$, we have that $\mu^*(i_2)P_{i_2}s^\dagger$ by Pareto-domination. We know that there exists i_3 such that $\mu^*(i_3) = s^\dagger$. If $i_3 = i_1$, the deviation is not profitable since $\mu^*(i_1) = s^\dagger$. If $i_3 \neq i_1$, we have two cases:

- Case 2.1: $i_3 \succ_{s^\dagger} i_1$, the deviation is not profitable since it is blocked by i_3 as $s^\dagger \in A(Q_{i_3})$. Hence, $DA[(Q'_{i_1}, Q_{-i_1}), \succ](i_1) \neq s^\dagger$.
- Case 2.2: $i_1 \succ_{s^\dagger} i_3$. We know that μ^* is implementable, then there exists a Nash equilibrium Q^* such that $DA(Q^*, \succ) = \mu^*$, meaning that the deviation of i_1 is blocked under Q^* but not under Q . To be blocked under Q^* we know that a student $i_4 \in I$ rejects i_1 from s^\dagger . We therefore have $i_4 \succ_{s^\dagger} i_1 \succ_{s^\dagger} i_3$. We have to show that $\mu^*(i_4)R_{i_4}s^\dagger R_{i_4}\mu_I(i_4)$.

If s^\dagger is ranked before $\mu^*(i_4)$ in Q^* , we know that i_4 rejects i_3 from s^\dagger , and Q^* does not implement μ^* . If s^\dagger is ranked after $\mu_I(i_4)$ this means that i_4 is rejected from $\mu_I(i_4)$ which contradict the stability of μ_I . Then $\mu^*(i_4)R_{i_4}s^\dagger R_{i_4}\mu_I(i_4)$ and by construction $s^\dagger \in A(Q_{i_4})$ meaning that the deviation of i_1 is blocked under Q and Q is a Nash equilibrium.

(only if): The proof is direct because Q is a Nash equilibrium and implements μ^* , therefore, μ^* is implementable. ■

Proof of Lemma 1

Proof. By contradiction, suppose that for s we have $|K[(P, \succ)](s)| = q_s$ and $|DA[(P, \succ)](s)| < q_s$ and there exists at least one $i \in I$ such that $K[(P, \succ)](i) = s \neq DA[(P, \succ)](i)$. Since EADA Pareto-dominates DA, $DA(P, \succ)$ is not stable, as there exists a student i who prefers s to $DA[(P, \succ)](i)$. This violates the non-wastefulness property of the assignment. This leads to a contradiction since $DA(P, \succ)$ is stable.

Consider an interrupting pair (j, s) and by contradiction, suppose that $|K[(P, \succ)](s)| < q_s$. From the definition of an interrupter, another student's application rejected j from s . Then, either $|K[(P, \succ)](s)| = q_s$ or (j, s) is not an interrupting pair, leading to a contradiction. ■

Proof of Lemma 2

Proof. Consider a student i who is assigned to school s under EADA, (i.e., $K[(P, \succ)](i) = s$), and suppose there exists $j \in I \setminus \{i\}$ such that (j, s) is an interrupting pair. We know that $j \succ_s i$ and $sP_jK[(P, \succ)](j)$ (if the school was not involved in an interrupting pair, then there is no student i' such that $sP_{i'}K[(P, \succ)](i')$ and $i' \succ_s i$). Consider that $s \in A(Q_j)$, thus j triggers a rejection chain leading to the rejection of i from school s . Then, i applies to a school ranked after s in Q_i . We denote that school s' :

- If $s' \neq \mu_I(i)$, then this leads to the rejection of a student i' who was assigned to s' with EADA, such that $K[(P, \succ)](i') = s'$. Then, i' rejects i'' from the school ranks after s' in $Q_{i'}$, and this process continues until j is rejected from s .
- If $s' = \mu_I(i)$, then this leads to the rejection of a student i' who was assigned to s' with EADA, such that $K[(P, \succ)](i') = \mu_I(i)$. Then, i' rejects i'' from the school ranks after s' in $Q_{i'}$, and this process continues until j is rejected from s .

By contradiction, suppose that i does not reject anyone or does not generate a rejection chain that leads to the rejection of j from s .

Consider the stability of $DA(P, \succ)$ and DA. We know that j is an interrupter for s , then there exists a student k such that $s \in A(Q_k)$ with $\mu_I(k) = s$ (otherwise k is also an interrupter for s and is the last interrupter for s to be rejected. Consequently, EADA identifies k as the interrupter to be neutralized) and causes j to be rejected at a later step. If k is not in the rejection chain generated by j , then k will apply to s at the same step of DA at which j is rejected from s . Suppose that (j, s) is neutralized, then i is rejected at a later step by k and $\mu_I(k) = s$. Since we have $k \succ_s j$ and $j \succ_s i$, it implies $k \succ_s i$. Therefore, i cannot be assigned to s if i does not generate the rejection chain that rejects j from s . This means that k is rejected from a school by j 's application to school s , and k rejects j from s .

Consequently, there exists a school, let us denote it as s^* , to which i applied such that $sP_i s^*$ and $s^* R_i \mu_I(i)$, which generated a rejection chain that rejected j from s . We consider three cases:

- $s^* \neq \mu_I(i)$ and $s^* \neq K[(P, \succ)](i)$: i is an interrupter for s^* because i is not assigned to s^* and leads to the rejection of a student from s^* (which then leads to the rejection of j from s). According to Lemma 1, i is indifferent between applying to s^* or not.
- $s^* \neq \mu_I(i)$ and $s^* = K[(P, \succ)](i)$: i can generate the same rejection chain if $K[(P, \succ)](i) \in A(Q_i)$.
- $s^* = \mu_I(i)$: i can generate the same rejection chain if $\mu_I(i) \in A(Q_i)$.

Hence, we conclude that i generates its rejection chain (which leads to the rejection of j from s) by applying to $K[(P, \succ)](i)$ if such a school exists, and $\mu_I(i)$. ■

Proof of Theorem 2

Proof. Let Q be a strategy profile that satisfies (i)-(iv). We need to show that there are no profitable deviations for students.

Let us first show that there are no profitable deviations for students identified as interrupters for a school at some round t . We know that for all $s \in S$ such that there exists $i \in I$ with $\mu_I(i) = s$ then there exists $i' \in I$ such that $K[(P, \succ)](i') = s$ by Lemma 1. By Lemma 2, we know that a rejection chain leads students to be rejected from the school for which they are an interrupter. For the other students, we know that the obtained assignment is Pareto-efficient, and there are no profitable deviations.

Therefore, students identified as interrupters are indifferent between applying to schools with which they are interrupters or not if a strategy profile satisfies (i)-(iv). The assignment is Pareto-efficient because, for each student $i \in I$, we have $K[(P, \succ)](i) \in A(Q_i)$. Moreover, it is a Nash equilibrium. ■

.23

Proof of Proposition 3

Proof. Suppose there is a cycle $i \succ_s j \succ_s k \succ_{s'} i$. Let us denote μ a Nash equilibrium outcome.

Consider three cases:

Case 1: Suppose $sP_i s'$ and $sP_k s'$:

- If $\mu(i)P_i s$, then i does not block k . Then, if $\mu(k) = s$, we have $\mu(j)P_j s$; otherwise, $\mu(k) \neq s$.
- If $sP_i \mu(i)$, then i can block k , and i is blocked by a student, denoted as i' , such that $i' \succ_s i$ and therefore $i' \succ_s k$.
- If $\mu(i) = s$, then i blocks k if $sP_k \mu(k)$.

Case 2: Suppose $s'P_i s$ and $s'P_k s$:

- If $\mu(k)P_k s'$, then k does not block i . Then, if $\mu(i) = s'$, we have no student k' such that $k' \succ_{s'} i$ and $s'P_{k'} \mu(k')$.
- If $s'P_k \mu(k)$, then k can block i , and k is blocked by a student, denoted as k' , such that $k' \succ_{s'} k$ and therefore $k' \succ_{s'} i$.
- If $\mu(k) = s'$, then k blocks i if $s'P_i \mu(i)$.

Case 3: Suppose $sP_i s'$ and $s'P_k s$:

- If $\mu(i) = s$ and $\mu(k) = s'$. Then it is immediate, and μ is stable.

- If $\mu(i) = s$ and $\mu(k) \neq s'$. Then, if $\mu(k)P_k s'$, μ is stable, and if $s'P_k \mu(k)$, we know that there exists $i' \in I$ such that $\mu(i') = s'$ and $i' \succ_{s'} k$. We, therefore, have $i' \succ_{s'} i$; then i cannot block i' .
- If $\mu(i) \neq s$ and $\mu(k) = s'$. Then, if $\mu(i)P_i s$, μ is stable, and if $sP_i \mu(i)$, we know that there exists $i'' \in I$ such that $\mu(i'') = s$ and $i'' \succ_s i$. We, therefore, have $i'' \succ_s k$; then k cannot block i'' .
- If $\mu(i) \neq s$ and $\mu(k) \neq s'$. The reasoning is the same as in the previous cases.

In each case, μ is stable. ■

Proof of Theorem 3

Proof. From Theorem 1 by Ergin (2002), we know that if a cycle exists in schools' priorities, then a preference profile exists that leads to unstable outcomes at Nash equilibrium. From Proposition 3, we can consider a specific case only such that $s'P_i s$ and $sP_k s'$. Suppose that $s \in A(Q_j)$ with a ranking according to P_j .

Let us denote a stable outcome in this proof by $\mu_S(i)$.

Case 1: If $\mu_S(j)P_j s$: i and k can never benefit from the cycle, as the rejection chain that blocks j 's application is impossible.

Case 2: If $sR_j \mu_S(j)$: by definition, $i \succ_s j \succ_s k$. As $s \in A(Q_j)$, we know that j prevents potential deviations of k using the rejection chain.

In case 2, k is rejected by the application of j , which guarantees the stability of the outcome. In case 1, j is indifferent between applying to s or not because, at Nash equilibrium, she is assigned to a preferred school. Using the reasoning for each cycle completes the proof. ■

References

- Abdulkadiroğlu, A. and Sönmez, T. (2003). School choice: A mechanism design approach. *American economic review*, 93(3):729–747.
- Bando, K. (2014). On the existence of a strictly strong nash equilibrium under the student-optimal deferred acceptance algorithm. *Games and Economic Behavior*, 87:269–287.
- Dubins, L. E. and Freedman, D. A. (1981). Machiavelli and the gale-shapley algorithm. *The American Mathematical Monthly*, 88(7):485–494.

- Dur, U. M. and Morrill, T. (2020). What you don't know can help you in school assignment. *Games and Economic Behavior*, 120:246–256.
- Ehlers, L. and Massó, J. (2015). Matching markets under (in) complete information. *Journal of economic theory*, 157:295–314.
- Ehlers, L. and Morrill, T. (2020). (il) legal assignments in school choice. *The Review of Economic Studies*, 87(4):1837–1875.
- Erdil, A. and Ergin, H. (2008). What's the matter with tie-breaking? improving efficiency in school choice. *American Economic Review*, 98(3):669–689.
- Ergin, H. (2002). Efficient resource allocation on the basis of priorities. *Econometrica*, 70(6):2489–2497.
- Gale, D. and Shapley, L. S. (1962). College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15.
- Gale, D. and Sotomayor, M. (1985). Ms. machiavelli and the stable matching problem. *The American Mathematical Monthly*, 92(4):261–268.
- Haeringer, G. and Klijn, F. (2009). Constrained school choice. *Journal of Economic theory*, 144(5):1921–1947.
- Kesten, O. (2010). School choice with consent. *The Quarterly Journal of Economics*, 125(3):1297–1348.
- Morrill, T. (2015). Making just school assignments. *Games and Economic Behavior*, 92:18–27.
- Reny, P. J. (2022). Efficient matching in the school choice problem. *American Economic Review*, 112(6):2025–2043.
- Roth, A. E. (1982). Incentive compatibility in a market with indivisible goods. *Economics letters*, 9(2):127–132.
- Tang, Q. and Yu, J. (2014). A new perspective on kesten's school choice with consent idea. *Journal of Economic Theory*, 154:543–561.
- Tang, Q. and Zhang, Y. (2021). Weak stability and pareto efficiency in school choice. *Economic Theory*, 71:533–552.
- Troyan, P., Delacrétaz, D., and Kloosterman, A. (2020). Essentially stable matchings. *Games and Economic Behavior*, 120:370–390.